

Article

Rota-Baxter Systems for BiHom-Type Algebras

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Abstract: The purpose of this paper is to study Rota–Baxter systems for BiHom-type algebras such as BiHom analogues of associative, dendriform, quadri algebras. It is shown that BiHom-dendriform structures of a particular kind are equivalent to Rota–Baxter systems. It is shown further that a Rota–Baxter system induces a weak BiHom-pseudotwistor which can be held responsible for the existence of a new product on the underlying BiHom-associative algebra. Moreover, we study the relationship between BiHom-quadri-algebras and Rota–Baxter systems for BiHom-dendriform algebras.

Keywords: Rota–Baxter system; BiHom-dendriform algebra; weak BiHom-pseudotwistor; BiHom-quadri-algebra

MSC: 17D30; 17A01; 16T05

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1. Introduction

Rota–Baxter operators first appeared in G. Baxter’s work in probability study to understand Spitzer’s identity in fluctuation theory [1]. Afterwards, G. C. Rota brought the subject into the areas of algebras and combinatorics [2,3]. Rota–Baxter operators have appeared in a wide range of areas in mathematics and mathematical physics, especially related to the associative analog of classical Yang–Baxter equation, see, for example, [4–10]. Aguiar further showed that a Rota–Baxter algebra of weight zero naturally carries a structure of a dendriform algebra which has been introduced by Loday [11] in the study of K -theory. It turns out that dendriform algebras are connected to several areas in mathematics and physics. Moreover, Rota–Baxter algebras are related to dendriform algebras via a pair of adjoint functors [8,12].

A generalisation of the notion of a Rota–Baxter operator was proposed in [13], called Rota–Baxter system. This generalisation consisted of two operators acting on an associative algebra and satisfying equations similar to the Rota–Baxter equation. Rota–Baxter operators of any weights and twisted Rota–Baxter operators were solutions of the proposed system. It was shown that in the case of a non-degenerate algebra the Rota–Baxter system was equivalent to a particular dendriform algebra.

Algebras of the Hom-type were introduced in the Physics literature of the 1990s in the context of quantum deformations of some algebras of vector fields, which satisfy a modified Jacobi identity involving a homomorphism (this kind of algebras were called Hom-Lie algebras and studied in [14,15]). Hom-analogues of other algebraic structures have been introduced afterwards, such as Hom-(co)associative (co)algebras, Hom-bialgebras, Hom-pre-Lie algebras, etc. Recently, structures of a more general type have been introduced in [16], called BiHom-type algebras, where a construction of a Hom-category including a group action led to concepts of BiHom-type algebras. Hence, BiHom-associative algebras and BiHom-Lie algebras, involving two linear maps (called structure maps), were introduced and studied. Examples of (Bi)Hom-type algebras can be obtained from classical types of algebras by “Yau twisting”.

The BiHom analogue of dendriform algebras and Rota–Baxter operators has been introduced in [17] and the result a Rota–Baxter algebra of weight zero naturally carries a structure of a dendriform algebra mentioned above also hold in BiHom-case. As a non-trivial generalisation of the Rota–Baxter operators, we begin the study of Rota–Baxter systems on BiHom-associative algebras and construct large kinds of examples of such Rota–Baxter systems. The following main question is considered from the classical Rota–Baxter operators to the Rota–Baxter systems on BiHom-case: if (A, R, S) is a Rota–Baxter system on the BiHom-associative algebra (A, μ, α, β) , then can $(A, \prec, \succ, \alpha, \beta)$ form a BiHom-dendriform algebra with the structures $x \prec y = xS(y)$ and $x \succ y = R(x)y$? Along the way, we will make a detailed study of various structures that may be related to Rota–Baxter systems for BiHom-type algebras, for example a Rota–Baxter system induces a weak pseudotwistor which can be held responsible for the existence of a new associative product on the underlying BiHom-associative algebra and how a Rota–Baxter system on a BiHom-dendriform algebra induces a BiHom-quadri-algebra.

The paper is organized as follows. In Section 2, we recall the definitions and some properties of BiHom-associative algebra, BiHom-dendriform algebra, (left and right) BiHom-pre-Lie algebra, Rota–Baxter operator and Rota–Baxter system. In Section 3, we introduce the notion of Rota–Baxter systems for a BiHom-associative algebra and construct several concrete examples. In Section 4, a connection between Rota–Baxter systems for BiHom-associative algebras and BiHom-dendriform algebras is provided and by this connection we give an example of 2-dimensional BiHom-dendriform algebra. It is shown further that a Rota–Baxter system induces a weak BiHom-pseudotwistor. Finally, we define a Rota–Baxter system for a BiHom-dendriform algebra and study the relationship between Rota–Baxter systems and BiHom-quadri-algebras in Section 5.

2. Preliminaries

We work over a base field \mathbb{k} . All algebras, linear spaces, etc., will be over \mathbb{k} ; unadorned \otimes means $\otimes_{\mathbb{k}}$. Unless otherwise specified, the algebras (associative or not) that will appear in what follows are *not* supposed to be unital, and a multiplication $\mu : A \otimes A \rightarrow A$ on a linear space A is denoted by juxtaposition: $\mu(v \otimes v') = vv'$. For the composition of two maps f and g , we write either $g \circ f$ or simply gf . For the identity map on a linear space A we use the notation id_A .

Definition 1 ([16]). *A BiHom-associative algebra over \mathbb{k} is a 4-tuple (A, μ, α, β) , where A is a \mathbb{k} -linear space, $\alpha : A \rightarrow A$, $\beta : A \rightarrow A$ and $\mu : A \otimes A \rightarrow A$ are linear maps, with notation $\mu(x \otimes y) = xy$, for all $x, y \in A$, satisfying the following conditions, for all $x, y, z \in A$:*

$$\alpha \circ \beta = \beta \circ \alpha, \tag{1}$$

$$\alpha(xy) = \alpha(x)\alpha(y) \text{ and } \beta(xy) = \beta(x)\beta(y), \quad (\text{multiplicativity}) \tag{2}$$

$$\alpha(x)(yz) = (xy)\beta(z). \quad (\text{BiHom-associativity}) \tag{3}$$

We call α and β (in this order) the structure maps of A .

A morphism $f : (A, \mu_A, \alpha_A, \beta_A) \rightarrow (B, \mu_B, \alpha_B, \beta_B)$ of BiHom-associative algebras is a linear map $f : A \rightarrow B$ such that $\alpha_B \circ f = f \circ \alpha_A$, $\beta_B \circ f = f \circ \beta_A$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

Definition 2 ([17]). A BiHom-dendriform algebra is a 5-tuple $(A, \prec, \succ, \alpha, \beta)$ consisting of a linear space A and linear maps $\prec, \succ: A \otimes A \rightarrow A$ and $\alpha, \beta: A \rightarrow A$ satisfying the conditions

$$\alpha \circ \beta = \beta \circ \alpha, \tag{4}$$

$$\alpha(x \prec y) = \alpha(x) \prec \alpha(y), \alpha(x \succ y) = \alpha(x) \succ \alpha(y), \tag{5}$$

$$\beta(x \prec y) = \beta(x) \prec \beta(y), \beta(x \succ y) = \beta(x) \succ \beta(y), \tag{6}$$

$$(x \prec y) \prec \beta(z) = \alpha(x) \prec (y \prec z + y \succ z), \tag{7}$$

$$(x \succ y) \prec \beta(z) = \alpha(x) \succ (y \prec z), \tag{8}$$

$$\alpha(x) \succ (y \succ z) = (x \prec y + x \succ y) \succ \beta(z), \tag{9}$$

for all $x, y, z \in A$. We call α and β (in this order) the structure maps of A .

A morphism $f: (A, \prec, \succ, \alpha, \beta) \rightarrow (A', \prec', \succ', \alpha', \beta')$ of BiHom-dendriform algebras is a linear map $f: A \rightarrow A'$ satisfying $f(x \prec y) = f(x) \prec' f(y)$ and $f(x \succ y) = f(x) \succ' f(y)$, for all $x, y \in A$, as well as $f \circ \alpha = \alpha' \circ f$ and $f \circ \beta = \beta' \circ f$.

Proposition 1 ([17]). Let $(A, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra. Define a multiplication $*$: $A \otimes A \rightarrow A$ by $x * y = x \prec y + x \succ y$, for all $x, y \in A$. Then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra.

Definition 3 ([18]). A left (respectively, right) BiHom-pre-Lie algebra $(A, \cdot, \alpha, \beta)$ is a 4-tuple in which A is a linear space and $\cdot: A \otimes A \rightarrow A$ and $\alpha, \beta: A \rightarrow A$ are linear maps satisfying $\alpha \circ \beta = \beta \circ \alpha, \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \beta(x \cdot y) = \beta(x) \cdot \beta(y)$ and

$$\alpha\beta(x) \cdot (\alpha(y) \cdot z) - (\beta(x) \cdot \alpha(y)) \cdot \beta(z) = \alpha\beta(y) \cdot (\alpha(x) \cdot z) - (\beta(y) \cdot \alpha(x)) \cdot \beta(z),$$

respectively,

$$\alpha(x) \cdot (\beta(y) \cdot \alpha(z)) - (x \cdot \beta(y)) \cdot \alpha\beta(z) = \alpha(x) \cdot (\beta(z) \cdot \alpha(y)) - (x \cdot \beta(z)) \cdot \alpha\beta(y),$$

for all $x, y, z \in A$. The maps α and β (in this order) are called the structure maps of A .

Proposition 2 ([18]). Let $(A, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra, such that α and β are bijective. Let $\triangleright, \triangleleft: A \otimes A \rightarrow A$ be linear maps defined for all $x, y \in A$ by

$$x \triangleright y = x \succ y - (\alpha^{-1}\beta(y)) \prec (\alpha\beta^{-1}(x)), \quad x \triangleleft y = x \prec y - (\alpha^{-1}\beta(y)) \succ (\alpha\beta^{-1}(x)).$$

Then $(A, \triangleright, \alpha, \beta)$ (respectively, $(A, \triangleleft, \alpha, \beta)$) is a left (respectively, right) BiHom-pre-Lie algebra.

A Rota–Baxter structure on an algebra of a given type is defined as follows.

Definition 4. Let A be a linear space and $\mu: A \otimes A \rightarrow A, \mu(x \otimes y) = xy$, for all $x, y \in A$, a linear multiplication on A and let $\lambda \in \mathbb{k}$. A Rota–Baxter operator of weight λ for (A, μ) is a linear map $R: A \rightarrow A$ satisfying the so-called Rota–Baxter condition

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy), \quad \forall x, y \in A. \tag{10}$$

As a generalisation of the Rota–Baxter operator, a Rota–Baxter system is defined as follows:

Definition 5 ([13]). A triple (A, R, S) consisting of an algebra A and two \mathbb{k} -linear operators $R, S: A \rightarrow A$ is called a Rota–Baxter system if, for all $a, b \in A$,

$$R(a)R(b) = R(R(a)b + aS(b)), \tag{11}$$

$$S(a)S(b) = S(R(a)b + aS(b)). \tag{12}$$

3. Rota–Baxter Systems for BiHom-Associative Algebras

In this section, we first introduce the notion of Rota–Baxter systems for BiHom-associative algebras. Then we construct several concrete examples.

The definition of a Rota–Baxter system for a BiHom-associative algebra is exactly the same as in Definition 5. Notice that it only adds the relations of the structure maps.

Definition 6. Let (A, μ, α, β) be a BiHom-associative algebra and $R, S : A \rightarrow A$ two \mathbb{k} -linear operators. We call (A, α, β, R, S) a Rota–Baxter system if,

$$\alpha \circ R = R \circ \alpha, R \circ \beta = \beta \circ R, \alpha \circ S = S \circ \alpha, S \circ \beta = \beta \circ S, \tag{13}$$

$$R(a)R(b) = R(R(a)b + aS(b)), \tag{14}$$

$$S(a)S(b) = S(R(a)b + aS(b)), \tag{15}$$

for all $a, b \in A$.

Proposition 3. Let (A, μ) be an associative algebra and $\alpha, \beta : A \rightarrow A$ two commuting algebra endomorphisms. Assume that $R, S : A \rightarrow A$ are linear maps commuting with α, β , such that (A, R, S) is a Rota–Baxter system. Define a new multiplication on A by $x * y = \alpha(x) \cdot \beta(y)$ for all $x, y \in A$, then $A_{\alpha, \beta} = (A, *, \alpha, \beta)$ is a BiHom-associative algebra, called the Yau twist of (A, μ) , and $(A_{\alpha, \beta}, \alpha, \beta, R, S)$ also forms a Rota–Baxter system.

Proof. We only prove (14) and leave the rest to the reader:

$$\begin{aligned} R(a) * R(b) &= \alpha(R(a))\beta(R(b)) = R(\alpha(a))R(\beta(b)) \\ &\stackrel{(11)}{=} R(R(\alpha(a))\beta(b) + \alpha(a)S(\beta(b))) \\ &= R(\alpha(R(a))\beta(b) + \alpha(a)\beta(S(b))) \\ &= R(R(a) * b + a * S(b)), \end{aligned}$$

finishing the proof. \square

Rota–Baxter operators (10) together with BiHom-associative algebras which they operate are examples of Rota–Baxter systems as explained in the following.

Proposition 4. Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \rightarrow A$ a Rota–Baxter operator of weight λ commuting with α and β . Then, $(A, \alpha, \beta, R, R + \lambda id)$ and $(A, \alpha, \beta, R + \lambda id, R)$ are Rota–Baxter systems, where $R + \lambda id$ is a map from A to A , defined as $R + \lambda id(x) = R(x) + \lambda x$ for all $x \in A$.

Proof. We only prove $(A, \alpha, \beta, R, R + \lambda id)$ is a Rota–Baxter system and the other is similar. It is easy to prove (13). For (14), we compute:

$$\begin{aligned} R(x)R(y) &\stackrel{(10)}{=} R(R(x)y + xR(y) + \lambda xy) \\ &= R(R(x)y + x(R + \lambda id)(y)). \end{aligned}$$

For (15), we compute:

$$\begin{aligned} (R + \lambda id)(x)(R + \lambda id)(y) &= (R(x) + \lambda x)(R(y) + \lambda y) \\ &= R(x)R(y) + \lambda R(x)y + \lambda xR(y) + \lambda^2 xy \\ &\stackrel{(10)}{=} R(R(x)y + xR(y) + \lambda xy) + \lambda(R(x)y + xR(y) + \lambda xy) \\ &= (R + \lambda id)(R(x)y + xR(y) + \lambda xy) \\ &= (R + \lambda id)(R(x)y + x(R + \lambda id)(y)), \end{aligned}$$

finishing the proof. \square

Now we construct a non-travel example of Rota–Baxter system on a 2-dimensional BiHom-associative algebra.

Example 1. We consider the following 2-dimensional BiHom-associative algebra (introduced in [16]), where the multiplication and the structure maps α, β are defined, with respect to a basis $\{e_1, e_2\}$, by

$$\begin{aligned} \mu(e_1, e_1) &= e_1, & \mu(e_1, e_2) &= be_1 + (1 - a)e_2, \\ \mu(e_2, e_1) &= \frac{b(1 - a)}{a}e_1 + ae_2, & \mu(e_2, e_2) &= \frac{b}{a}e_2, \\ \alpha(e_1) &= e_1, & \alpha(e_2) &= \frac{b(1 - a)}{a}e_1 + ae_2, \\ \beta(e_1) &= e_1, & \beta(e_2) &= be_1 + (1 - a)e_2, \end{aligned}$$

where a, b are parameters in \mathbb{k} with $a \neq 0$.

Now we define the \mathbb{k} -linear operators R, S with respect to the basis by

$$R(e_1) = -e_1, \quad R(e_2) = -\frac{b}{a}e_1,$$

and

$$S(e_1) = 0, \quad S(e_2) = -\frac{b}{a}e_1 + e_2.$$

In these formulae, (A, α, β, R, S) is a Rota–Baxter system.

Proof. For (13), we compute on the basis $\{e_1, e_2\}$:

$$\begin{aligned} (R \circ \alpha)(e_1) &= R(\alpha(e_1)) = R(e_1) = -e_1, \\ (\alpha \circ R)(e_1) &= \alpha(R(e_1)) = \alpha(-e_1) = -e_1, \\ (R \circ \alpha)(e_2) &= R(\alpha(e_2)) = R\left(\frac{b(1 - a)}{a}e_1 + ae_2\right) = \frac{b(a - 1)}{a}e_1 - be_2 = -\frac{b}{a}e_1, \\ (\alpha \circ R)(e_2) &= \alpha(R(e_2)) = \alpha\left(-\frac{b}{a}e_1\right) = -\frac{b}{a}e_1. \end{aligned}$$

It follows $R \circ \alpha = \alpha \circ R$, and

$$\begin{aligned} (R \circ \beta)(e_1) &= R(\beta(e_1)) = R(e_1) = -e_1, \\ (\beta \circ R)(e_1) &= \beta(R(e_1)) = \beta(-e_1) = -e_1, \\ (R \circ \beta)(e_2) &= R(\beta(e_2)) = R(be_1 + (1 - a)e_2) = \frac{b(a - 1)}{a}e_1 - be_2 = -\frac{b}{a}e_1, \\ (\beta \circ R)(e_2) &= \beta(R(e_2)) = \beta\left(-\frac{b}{a}e_1\right) = -\frac{b}{a}e_1 \end{aligned}$$

means $R \circ \beta = \beta \circ R$. Similarly we get $S \circ \alpha = \alpha \circ S$ and $S \circ \beta = \beta \circ S$.

For (14), we compute on the basis $\{e_1, e_2\}$:

$$\begin{aligned}
 R(e_1)R(e_1) &= (-e_1)(-e_1) = e_1, \\
 R(R(e_1)e_1 + e_1S(e_1)) &= R((-e_1)e_1 + e_10) = R(-e_1) = e_1. \\
 R(e_1)R(e_2) &= (-e_1)\left(-\frac{b}{a}e_1\right) = \frac{b}{a}e_1, \\
 R(R(e_1)e_2 + e_1S(e_2)) &= R((-e_1)e_2 + e_1\left(-\frac{b}{a}e_1 + e_2\right)) = R(-e_1e_2 - \frac{b}{a}e_1 + e_1e_2) \\
 &= R\left(-\frac{b}{a}e_1\right) = \frac{b}{a}e_1. \\
 R(e_2)R(e_1) &= \left(-\frac{b}{a}e_1\right)(-e_1) = \frac{b}{a}e_1, \\
 R(R(e_2)e_1 + e_2S(e_1)) &= R\left(\left(-\frac{b}{a}e_1\right)e_1 + e_20\right) = R\left(-\frac{b}{a}e_1\right) = \frac{b}{a}e_1. \\
 R(e_2)R(e_2) &= \left(-\frac{b}{a}e_1\right)\left(-\frac{b}{a}e_1\right) = \frac{b^2}{a^2}e_1, \\
 R(R(e_2)e_2 + e_2S(e_2)) &= R\left(\left(-\frac{b}{a}e_1\right)e_2 + e_2\left(-\frac{b}{a}e_1 + e_2\right)\right) \\
 &= R\left(-\frac{b}{a}(be_1 + (1-a)e_2) - \frac{b}{a}\left(\frac{b(1-a)}{a}e_1 + ae_2\right) + \frac{b}{a}e_2\right) \\
 &= R\left(-\frac{b^2}{a}e_1 - \frac{b(1-a)}{a}e_2 - \frac{b^2(1-a)}{a^2}e_1 - be_2 + \frac{b}{a}e_2\right) = R\left(-\frac{b^2}{a^2}e_1\right) = \frac{b^2}{a^2}e_1.
 \end{aligned}$$

It follows that $R(x)R(y) = R(R(x)y + xS(y))$ for all $x, y \in \{e_1, e_2\}$. Similarly we can prove (15) holds. \square

Definition 7. Let (A, μ, α, β) be a BiHom-associative algebra. The element $a \in A$ says central if for all $b \in A$, we have

$$\beta(a)\alpha(b) = \beta(b)\alpha(a). \tag{16}$$

Using the central element in A we can construct an example of Rota–Baxter system.

Example 2. Let (A, μ, α, β) be a BiHom-associative algebra, such that α and β are bijective. Suppose that $r, s \in A$ with $\alpha(r) = \beta(r) = r, \alpha(s) = \beta(s) = s, rs = 0$ and one of them, says s , being central. Define $R, S : A \rightarrow A$ by

$$R : a \mapsto \alpha^{-1}(a)r, \quad S : a \mapsto s\beta^{-1}(a).$$

Then (A, α, β, R, S) is a Rota–Baxter system.

Proof. For all $a \in A$, we have

$$(R \circ \alpha)(a) = R(\alpha(a)) = ar = a\alpha(r) = \alpha(\alpha^{-1}(a)r) = \alpha(R(a)) = (\alpha \circ R)(a).$$

Similarly, we can prove the rest formulae in (13).

For (14), since $rs = 0$ and s is central, we get $rs = 0 \Rightarrow \beta(r)\alpha(s) = 0 \Rightarrow \beta(s)\alpha(r) = 0 \Rightarrow sr = 0$. Now we compute:

$$\begin{aligned}
 & R(R(a)b + aS(b)) \\
 = & R((\alpha^{-1}(a)r)b + a(s\beta^{-1}(b))) \\
 = & \alpha^{-1}[(\alpha^{-1}(a)r)b]r + \alpha^{-1}[a(s\beta^{-1}(b))]r \\
 = & [\alpha^{-1}(\alpha^{-1}(a)r)\alpha^{-1}(b)]\beta(r) + [\alpha^{-1}(a)\alpha^{-1}(s\beta^{-1}(b))]\beta(r) \\
 \stackrel{(3)}{=} & (\alpha^{-1}(a)r)(\alpha^{-1}(b)r) + a[(\alpha^{-1}(s)\alpha^{-1}\beta^{-1}(b))r] \\
 = & (\alpha^{-1}(a)r)(\alpha^{-1}(b)r) + a[(\beta(s)\alpha(\alpha^{-2}\beta^{-1}(b)))r] \\
 \stackrel{(16)}{=} & (\alpha^{-1}(a)r)(\alpha^{-1}(b)r) + a[(\alpha^{-2}(b)s)\beta(r)] \\
 \stackrel{(3)}{=} & (\alpha^{-1}(a)r)(\alpha^{-1}(b)r) + a[\alpha^{-1}(b)(sr)] \\
 = & (\alpha^{-1}(a)r)(\alpha^{-1}(b)r) + 0 = (\alpha^{-1}(a)r)(\alpha^{-1}(b)r) = R(a)R(b).
 \end{aligned}$$

The proof of (15) is similar and leave to the reader. \square

4. BiHom-Dendriform Algebras and Weak BiHom-Pseudotwistors

In this section, we relate the Rota–Baxter systems to BiHom-dendriform algebras and weak BiHom-pseudotwistors.

Theorem 1. Let (A, μ, α, β) be a BiHom-associative algebra and $R, S : A \rightarrow A$ be \mathbb{k} -linear maps commuting with α and β . Define operations \prec and \succ on A by

$$x \prec y = xS(y) \quad \text{and} \quad x \succ y = R(x)y, \tag{17}$$

for all $x, y \in A$. Then

- (1) If (A, α, β, R, S) is a Rota–Baxter system, then $(A, \prec, \succ, \alpha, \beta)$ is a BiHom-dendriform algebra;
- (2) If (A, μ, α, β) is non-degenerate (which means that for all $a \in A, \alpha(a)r = 0$ implies $r = 0$ and for all $b \in A, r\beta(b) = 0$ implies $r = 0$) and $(A, \prec, \succ, \alpha, \beta)$ is a BiHom-dendriform algebra, then (A, α, β, R, S) is a Rota–Baxter system.

Proof. (1) We only prove (7)–(9) and leave the rest to the reader. We have:

$$\begin{aligned}
 & (x \prec y) \prec \beta(z) - \alpha(x) \prec (y \prec z) - \alpha(x) \prec (y \succ z) \\
 = & (xS(y)) \prec \beta(z) - \alpha(x) \prec (yS(z)) - \alpha(x) \prec (R(y)z) \\
 = & (xS(y))S(\beta(z)) - \alpha(x)S(yS(z)) - \alpha(x)S(R(y)z) \\
 = & (xS(y))\beta(S(z)) - \alpha(x)S(R(y)z + yS(z)) \\
 \stackrel{(3)}{=} & \alpha(x)(S(y)S(z)) - \alpha(x)S(R(y)z + yS(z)) \\
 \stackrel{(15)}{=} & \alpha(x)(S(y)S(z)) - \alpha(x)(S(y)S(z)) = 0,
 \end{aligned}$$

and

$$\begin{aligned}
 & (x \succ y) \prec \beta(z) - \alpha(x) \succ (y \prec z) \\
 = & (R(x)y) \prec \beta(z) - \alpha(x) \succ (yS(z)) \\
 = & (R(x)y)S(\beta(z)) - R(\alpha(x))(yS(z)) \\
 = & (R(x)y)\beta(S(z)) - \alpha(R(x))(yS(z)) \\
 \stackrel{(3)}{=} & \alpha(R(x))(yS(z)) - \alpha(R(x))(yS(z)) = 0,
 \end{aligned}$$

and

$$\begin{aligned}
 & \alpha(x) \succ (y \succ z) - (x \prec y) \succ \beta(z) - (x \succ y) \succ \beta(z) \\
 = & \alpha(x) \succ (R(y)z) - (xS(y)) \succ \beta(z) - (R(x)y) \succ \beta(z) \\
 = & R(\alpha(x))(R(y)z) - R(xS(y))\beta(z) - R(R(x)y)\beta(z) \\
 \stackrel{(14)}{=} & \alpha(R(x))(R(y)z) - (R(x)R(y))\beta(z) \\
 \stackrel{(3)}{=} & (R(x)R(y))\beta(z) - (R(x)R(y))\beta(z) = 0.
 \end{aligned}$$

- (2) In the converse direction, let us assume that $(A, \prec, \succ, \alpha, \beta)$ is a BiHom-dendriform algebra, with \prec, \succ defined in (17). From the proof in (1), we get the relation (7) comes out as $\alpha(x)[S(y)S(z) - S(R(y)z + yS(z))] = 0$, hence it gives (15) by the non-degeneracy of the multiplication in A . In a similar way, (9) gives (14). \square

We have some particular cases of Theorem 1.

Corollary 1 ([13]). *Let A be an associative algebra and $R, S : A \rightarrow A$ be \mathbb{k} -linear homomorphisms. Define new operations \prec and \succ on A by*

$$x \prec y = xS(y) \quad \text{and} \quad x \succ y = R(x)y,$$

for all $x, y \in A$. If (A, R, S) is a Rota–Baxter system, then (A, \prec, \succ) is a dendriform algebra.

Proof. Take in Theorem 1 $\alpha = \beta = id_A$. \square

Corollary 2 ([17]). *Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \rightarrow A$ a Rota–Baxter operator of weight 0, such that $R \circ \alpha = \alpha \circ R, R \circ \beta = \beta \circ R$. Define operations \prec and \succ on A by*

$$x \prec y = xR(y) \quad \text{and} \quad x \succ y = R(x)y,$$

for all $x, y \in A$. Then $(A, \prec, \succ, \alpha, \beta)$ is a BiHom-dendriform algebra.

Proof. Take in Theorem 1 $R = S$. It follows that R is a Rota–Baxter operator of weight 0. \square

Using Theorem 1 to the Rota–Baxter system in Example 1, we can construct an example of 2-dimensional BiHom-dendriform algebra.

Example 3. *Let (A, μ, α, β) be a 2-dimensional BiHom-associative algebra as in Example 1. \mathbb{k} -linear maps $\prec_R, \succ_R : A \otimes A \rightarrow A$ are defined by*

$$\begin{aligned}
 e_1 \prec_R e_1 &= 0, & e_1 \prec_R e_2 &= \frac{b(a-1)}{a}e_1 + (1-a)e_2, \\
 e_2 \prec_R e_1 &= 0, & e_2 \prec_R e_2 &= \frac{b^2(a-1)}{a^2}e_1 + \frac{b(1-a)}{a}e_2, \\
 e_1 \succ_R e_1 &= -e_1, & e_1 \succ_R e_2 &= -be_1 + (a-1)e_2, \\
 e_2 \succ_R e_1 &= -\frac{b}{a}e_1, & e_2 \succ_R e_2 &= -\frac{b^2}{a}e_1 + \frac{b(a-1)}{a}e_2,
 \end{aligned}$$

where a, b are parameters in \mathbb{k} with $a \neq 0$. Then $(A, \prec_R, \succ_R, \alpha, \beta)$ is a BiHom-dendriform algebra.

As a consequence of Proposition 1 and Theorem 1 we obtain:

Corollary 3. *Let (A, α, β, R, S) be a Rota–Baxter system. If we define on A a new multiplication by*

$$x * y = R(x)y + xS(y), \tag{18}$$

for all $x, y \in A$, then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra.

Additionally, as a consequence of Proposition 2 and Theorem 1 we obtain:

Corollary 4. Let (A, α, β, R, S) be a Rota–Baxter system, such that α and β are bijective. Let $\triangleright, \triangleleft : A \otimes A \rightarrow A$ be linear maps defined for all $x, y \in A$ by

$$x \triangleright y = R(x)y - (\alpha^{-1}\beta(y))S(\alpha\beta^{-1}(x)), \quad x \triangleleft y = xS(y) - R(\alpha^{-1}\beta(y))(\alpha\beta^{-1}(x)).$$

Then $(A, \triangleright, \alpha, \beta)$ (respectively, $(A, \triangleleft, \alpha, \beta)$) is a left (respectively, right) BiHom-pre-Lie algebra.

Another way of understanding the BiHom-associativity of product (18) is by connecting Rota–Baxter systems with the weak (BiHom-)pseudotwistors in [17,19]. We begin by recalling the following concept of the weak BiHom-pseudotwistors from [17]:

Definition 8. Let (D, μ, α, β) be a BiHom-associative algebra and $T : D \otimes D \rightarrow D \otimes D$ be a linear map and assume that there exists a linear map $\mathcal{T} : D \otimes D \otimes D \rightarrow D \otimes D \otimes D$, such that T commutes with $\alpha \otimes \alpha, \beta \otimes \beta$, and the following relations hold:

$$T \circ (\alpha \otimes (\mu \circ T)) = (\alpha \otimes \mu) \circ \mathcal{T}, \tag{19}$$

$$T \circ ((\mu \circ T) \otimes \beta) = (\mu \otimes \beta) \circ \mathcal{T}. \tag{20}$$

The map T is called a weak BiHom-pseudotwistor and \mathcal{T} is called the weak companion of T .

We can give now another proof for Corollary 3.

Proposition 5. Let (A, α, β, R, S) be a Rota–Baxter system. Then the linear map

$$T : A \otimes A \rightarrow A \otimes A, \quad T(a \otimes b) = R(a) \otimes b + a \otimes S(b), \tag{21}$$

for all $a, b \in A$, is a weak BiHom-pseudotwistor with weak companion $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$,

$$\mathcal{T}(a \otimes b \otimes c) = R(a) \otimes R(b) \otimes c + R(a) \otimes b \otimes S(c) + a \otimes S(b) \otimes S(c). \tag{22}$$

Proof. We only prove that $T \circ (\alpha \otimes (\mu \circ T)) = (\alpha \otimes \mu) \circ \mathcal{T}$ and leave the rest to the reader:

$$\begin{aligned} & T \circ (\alpha \otimes (\mu \circ T))(a \otimes b \otimes c) \\ &= R(\alpha(a)) \otimes R(b)c + R(\alpha(a)) \otimes bS(c) + \alpha(a) \otimes S(R(b)c) + \alpha(a) \otimes S(bS(c)) \\ &\stackrel{(15)}{=} \alpha(R(a)) \otimes R(b)c + \alpha(R(a)) \otimes bS(c) + \alpha(a) \otimes S(b)S(c) \\ &= (\alpha \otimes \mu)(R(a) \otimes R(b) \otimes c + R(a) \otimes b \otimes S(c) + a \otimes S(b) \otimes S(c)) \\ &= (\alpha \otimes \mu) \circ \mathcal{T}(a \otimes b \otimes c), \end{aligned}$$

finishing the proof. \square

Let us note that the product $*$ defined by (18) is simply equal to $\mu \circ T$, where T is given in (21). Hence, $*$ is BiHom-associative by Proposition 8.4 in [17]. Furthermore, in a way similar to the case of BiHom-dendriform algebras, if (A, μ, α, β) is a non-degenerate BiHom-associative algebra, then T given by (21) is a weak BiHom-pseudotwistor with the companion (22) if, and only if, (A, α, β, R, S) is a Rota–Baxter system.

5. Rota–Baxter Systems for BiHom-Dendriform Algebras

In this section, we study how to construct BiHom-quadri-algebras from Rota–Baxter systems for BiHom-dendriform algebras.

We now introduce the concept of BiHom-quadri-algebra (for the Hom version see [20]).

Definition 9 ([17]). A BiHom-quadri-algebra is a 7-tuple $(Q, \nwarrow, \swarrow, \nearrow, \searrow, \alpha, \beta)$ where Q is a linear space and $\nwarrow, \swarrow, \nearrow, \searrow: Q \otimes Q \rightarrow Q$ and $\alpha, \beta: Q \rightarrow Q$ are linear maps satisfying the axioms below (28)–(37) (for all $x, y, z \in Q$). To state them, define the following operations:

$$x \succ y := x \nearrow y + x \searrow y \tag{23}$$

$$x \prec y := x \nwarrow y + x \swarrow y, \tag{24}$$

$$x \vee y := x \searrow y + x \swarrow y, \tag{25}$$

$$x \wedge y := x \nearrow y + x \nwarrow y, \tag{26}$$

$$\begin{aligned} x * y &:= x \searrow y + x \nearrow y + x \swarrow y + x \nwarrow y \\ &:= x \succ y + x \prec y = x \vee y + x \wedge y. \end{aligned} \tag{27}$$

The axioms are

$$\alpha \circ \beta = \beta \circ \alpha, \tag{28}$$

$$\alpha(x \nearrow y) = \alpha(x) \nearrow \alpha(y), \quad \alpha(x \searrow y) = \alpha(x) \searrow \alpha(y), \tag{29}$$

$$\alpha(x \nwarrow y) = \alpha(x) \nwarrow \alpha(y), \quad \alpha(x \swarrow y) = \alpha(x) \swarrow \alpha(y), \tag{30}$$

$$\beta(x \nearrow y) = \beta(x) \nearrow \beta(y), \quad \beta(x \searrow y) = \beta(x) \searrow \beta(y) \tag{31}$$

$$\beta(x \nwarrow y) = \beta(x) \nwarrow \beta(y), \quad \beta(x \swarrow y) = \beta(x) \swarrow \beta(y) \tag{32}$$

$$(x \nwarrow y) \nwarrow \beta(z) = \alpha(x) \nwarrow (y * z), \quad (x \nearrow y) \nwarrow \beta(z) = \alpha(x) \nearrow (y \prec z), \tag{33}$$

$$(x \wedge y) \nearrow \beta(z) = \alpha(x) \nearrow (y \succ z), \quad (x \swarrow y) \nwarrow \beta(z) = \alpha(x) \swarrow (y \wedge z), \tag{34}$$

$$(x \searrow y) \nwarrow \beta(z) = \alpha(x) \searrow (y \nwarrow z), \quad (x \vee y) \nearrow \beta(z) = \alpha(x) \searrow (y \nearrow z), \tag{35}$$

$$(x \prec y) \swarrow \beta(z) = \alpha(x) \swarrow (y \vee z), \quad (x \succ y) \swarrow \beta(z) = \alpha(x) \searrow (y \swarrow z), \tag{36}$$

$$(x * y) \searrow \beta(z) = \alpha(x) \searrow (y \searrow z). \tag{37}$$

Similarly to the definition of Rota–Baxter systems for the BiHom-associative algebras, we introduce the following concept:

Definition 10. Let $(D, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra and $R, S: A \rightarrow A$ two \mathbb{k} -linear operators. We call $(D, \prec, \succ, \alpha, \beta, R, S)$ a Rota–Baxter system on D if $R \circ \alpha = \alpha \circ R, R \circ \beta = \beta \circ R, S \circ \alpha = \alpha \circ S, S \circ \beta = \beta \circ S$ and the following conditions are satisfied, for all $x, y \in D$:

$$R(x) \succ R(y) = R(x \succ S(y) + R(x) \succ y), \tag{38}$$

$$R(x) \prec R(y) = R(x \prec S(y) + R(x) \prec y), \tag{39}$$

$$S(x) \succ S(y) = S(x \succ S(y) + R(x) \succ y), \tag{40}$$

$$S(x) \prec S(y) = S(x \prec S(y) + R(x) \prec y). \tag{41}$$

Proposition 6. Let $(D, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra and (D, R, S) a Rota–Baxter system. Define new operations on D by $x \searrow_R y = R(x) \succ y, x \nearrow_R y = x \succ S(y), x \swarrow_R y = R(x) \prec y$ and $x \nwarrow_R y = x \prec S(y)$, for all $x, y \in D$. Then $(D, \nwarrow_R, \swarrow_R, \nearrow_R, \searrow_R, \alpha, \beta)$ is a BiHom-quadri-algebra.

Proof. The operations $\succ_R, \prec_R, \vee_R, \wedge_R, *_R$ defined on D by (23)–(27) corresponding to the operations $\nwarrow_R, \swarrow_R, \nearrow_R, \searrow_R$ are

$$\begin{aligned} x \prec_R y &= x \nwarrow_R y + x \swarrow_R y = x \prec S(y) + R(x) \prec y, \\ x \succ_R y &= x \searrow_R y + x \nearrow_R y = R(x) \succ y + x \succ S(y), \\ x \wedge_R y &= x \nearrow_R y + x \nwarrow_R y = x \succ S(y) + x \prec S(y), \\ x \vee_R y &= x \searrow_R y + x \swarrow_R y = R(x) \succ y + R(x) \prec y, \\ x *_R y &= x \nwarrow_R y + x \swarrow_R y + x \searrow_R y + x \nearrow_R y \\ &= x \prec S(y) + R(x) \prec y + R(x) \succ y + x \succ S(y). \end{aligned}$$

The proof of (28)–(32) is straightforward. For all $x, y, z \in D$ we only check axioms (33), (34), and (37).

$$\begin{aligned} \alpha(x) \nwarrow_R (y *_R z) &= \alpha(x) \prec S(y *_R z) \\ &= \alpha(x) \prec S(y \prec S(z) + R(y) \prec z + R(y) \succ z + y \succ S(z)) \\ &= \alpha(x) \prec [S(R(y) \succ z + y \succ S(z)) + S(y \prec S(z) + R(y) \prec z)] \\ &\stackrel{(40)(41)}{=} \alpha(x) \prec [S(y) \succ S(z) + S(y) \prec S(z)] \\ &\stackrel{(7)}{=} (x \prec S(y)) \prec \beta(S(z)) = (x \prec S(y)) \prec S(\beta(z)) \\ &= (x \nwarrow_R y) \nwarrow_R \beta(z), \end{aligned}$$

$$\begin{aligned} \alpha(x) \nearrow_R (y \prec_R z) &= \alpha(x) \nearrow_R (y \prec S(z) + R(y) \prec z) \\ &= \alpha(x) \succ S(y \prec S(z) + R(y) \prec z) \\ &\stackrel{(41)}{=} \alpha(x) \succ (S(y) \prec S(z)) \\ &\stackrel{(8)}{=} (x \succ S(y)) \prec \beta(S(z)) = (x \succ S(y)) \prec S(\beta(z)) \\ &= (x \nearrow_R y) \nwarrow_R \beta(z). \end{aligned}$$

$$\begin{aligned} (x \wedge_R y) \nearrow_R \beta(z) &= (x \succ S(y) + x \prec S(y)) \succ S(\beta(z)) \\ &= (x \succ S(y) + x \prec S(y)) \succ \beta(S(z)) \\ &\stackrel{(9)}{=} \alpha(x) \succ (S(y) \succ S(z)) \\ &\stackrel{(40)}{=} \alpha(x) \succ S(R(y) \succ z + y \succ S(z)) \\ &= \alpha(x) \nearrow_R (R(y) \succ z + y \succ S(z)) \\ &= \alpha(x) \nearrow_R (y \succ_R z), \end{aligned}$$

$$\begin{aligned} (x \swarrow_R y) \nwarrow_R \beta(z) &= (R(x) \prec y) \prec S(\beta(z)) = (R(x) \prec y) \prec \beta(S(z)) \\ &\stackrel{(7)}{=} \alpha(R(x)) \prec (y \prec S(z) + y \succ S(z)) \\ &= R(\alpha(x)) \prec (y \succ S(z) + y \prec S(z)) \\ &= \alpha(x) \swarrow_R (y \wedge_R z). \end{aligned}$$

$$\begin{aligned}
 (x *_R y) \searrow_R \beta(z) &= R(x \prec S(y) + R(x) \prec y + R(x) \succ y + x \succ S(y)) \succ \beta(z) \\
 &= [R(R(x) \succ y + x \succ S(y)) + R(R(x) \prec y + x \prec S(y))] \succ \beta(z) \\
 &\stackrel{(38)(39)}{=} (R(x) \succ R(y) + R(x) \prec R(y)) \succ \beta(z) \\
 &\stackrel{(9)}{=} \alpha(R(x)) \succ (R(y) \succ z) = R(\alpha(x)) \succ (R(y) \succ z) \\
 &= \alpha(x) \searrow_R (y \searrow_R z).
 \end{aligned}$$

We leave the rest to the reader. \square

Remark 1. In the setting of Proposition 6, the axioms (38)–(41) can be rewritten as $R(x) \succ R(y) = R(x \succ_R y)$, $R(x) \prec R(y) = R(x \prec_R y)$ and $S(x) \succ S(y) = S(x \succ_R y)$, $S(x) \prec S(y) = S(x \prec_R y)$. Thus, R, S are morphisms of BiHom-dendriform algebras from $D_h = (D, \prec_R, \succ_R, \alpha, \beta)$ to $(D, \prec, \succ, \alpha, \beta)$.

On the other hand, if we denote by $(D, *, \alpha, \beta)$ the BiHom-associative algebra obtained from D as in Proposition 1, it is obvious that we have $x \wedge_R y = x * S(y)$ and $x \vee_R y = R(x) * y$, for all $x, y \in D$. Thus, the BiHom-dendriform algebra structure obtained on D by applying Theorem 1 for the Rota–Baxter system (R, S) on the BiHom-associative algebra $(D, *, \alpha, \beta)$ is exactly the vertical BiHom-dendriform algebra $D_v = (D, \wedge_R, \vee_R, \alpha, \beta)$ obtained from the BiHom-quadruple algebra $(D, \searrow_R, \swarrow_R, \nearrow_R, \nwarrow_R, \alpha, \beta)$.

Proposition 7. Let (A, μ, α, β) be a BiHom-associative algebra and $(A, \alpha, \beta, R, S), (A, \alpha, \beta, R', S')$ be two Rota–Baxter systems, such that $R \circ R' = R' \circ R, S \circ R' = R' \circ S, R \circ S' = S' \circ R, S \circ S' = S' \circ S$. Then (R', S') is a Rota–Baxter system on the BiHom-dendriform algebra $(A, \prec_R, \succ_R, \alpha, \beta)$, here $x \prec_R y = xS(y)$ and $x \succ_R y = R(x)y$ for all $x, y \in A$.

Proof. We need check the axioms (38)–(41). For all $x, y \in A$, we have

$$\begin{aligned}
 R'(x) \succ_R R'(y) &= R(R'(x))R'(y) = R'(R(x))R'(y) \\
 &\stackrel{(14)}{=} R'(R'(R(x))y + R(x)S'(y)) = R'(R(R'(x))y + R(x)S'(y)) \\
 &= R'(R'(x) \succ_R y + x \succ_R S'(y)),
 \end{aligned}$$

$$\begin{aligned}
 R'(x) \prec_R R'(y) &= R'(x)S(R'(y)) = R'(x)R'(S(y)) \\
 &\stackrel{(14)}{=} R'(R'(x)S(y) + xS'(S(y))) = R'(R'(x)S(y) + xS(S'(y))) \\
 &= R'(R'(x) \prec_R y + x \prec_R S'(y)),
 \end{aligned}$$

$$\begin{aligned}
 S'(x) \succ_R S'(y) &= R(S'(x))S'(y) = S'(R(x))S'(y) \\
 &\stackrel{(15)}{=} S'(R'(R(x))y + R(x)S'(y)) = S'(R(R'(x))y + R(x)S'(y)) \\
 &= S'(R'(x) \succ_R y + x \succ_R S'(y)),
 \end{aligned}$$

$$\begin{aligned}
 S'(x) \prec_R S'(y) &= S'(x)S(S'(y)) = S'(x)S'(S(y)) \\
 &\stackrel{(15)}{=} S'(R'(x)S(y) + xS'(S(y))) = S'(R'(x)S(y) + xS(S'(y))) \\
 &= S'(R'(x) \prec_R y + x \prec_R S'(y)),
 \end{aligned}$$

finishing the proof. \square

Applying Proposition 6 to the Rota–Baxter system (R', S') on the BiHom-dendriform algebra $(A, \prec_R, \succ_R, \alpha, \beta)$ in Proposition 7, we get the following result.

Corollary 5. In the setting of Proposition 7, there exists a BiHom-quadri-algebra $(A, \curvearrowright_R, \curvearrowleft_R, \curvearrowright_R, \curvearrowleft_R, \alpha, \beta)$ where the operations are defined by

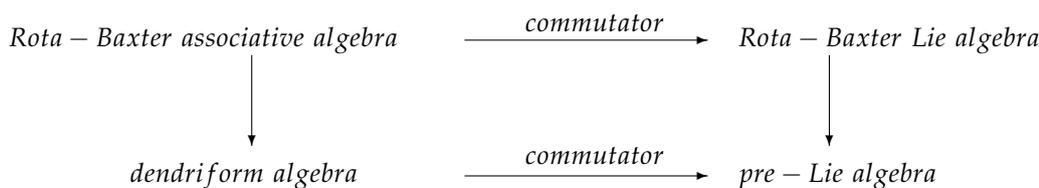
$$\begin{aligned} x \curvearrowright_R y &= R'(x) \succ_R y = R(R'(x))y, \\ x \curvearrowleft_R y &= x \succ_R S'(y) = R(x)S'(y), \\ x \curvearrowleft_R y &= R'(x) \prec_R y = R'(x)S(y), \\ x \curvearrowright_R y &= x \prec_R S'(y) = xS(S'(y)) \end{aligned}$$

for all $x, y \in A$.

In particular, there is a new BiHom-associative algebra $(A, *, \alpha, \beta)$, with $a * b = R(R'(a))b + R(a)S'(b) + R'(a)S(b) + aS(S'(b))$, for all $a, b \in A$.

6. Conclusions

In this paper we study the Rota–Baxter system for BiHom-type algebras, such as BiHom-associative algebras and BiHom-dendriform algebras, especially the connections of these objects are given. Applying Rota–Baxter system to a BiHom-associative algebra there forms a BiHom-dendriform algebra, furthermore, we can get a left (right) BiHom-pre-Lie algebra. A connection between Rota–Baxter systems on BiHom-associative algebras and so-called weak BiHom-pseudotwistors is provided. Similarly, we can define the Rota–Baxter system for BiHom-dendriform algebras and construct BiHom-quadri-algebras. For Rota–Baxter operators, there is a commutative diagram of categories as follows:



Here, the horizontal arrows are taking commutator brackets and the vertical arrows are applying the Rota–Baxter operators or splitting of operations. We generalize the left and down arrows of the diagram to the Rota–Baxter system for BiHom-case in this paper. The further directions of the research is to study the right and up arrows of the diagram for Rota–Baxter system BiHom-Lie algebra.

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