Modeling Volatility in Option Pricing with Applications

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ABSTRACT

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The focus of this dissertation is modeling volatility in option pricing by the Black-Scholes formula. A major drawback of the formula is that the returns from assets are assumed to have constant volatility over time. The empirical evidence is overwhelmingly against it. In this dissertation, we allow random volatility for estimating call option prices by Black-Scholes formula and by Monte Carlo simulation.

The Black-Scholes formula follows from an assumption that assets evolve according to a Geometric Brownian Motion with constant volatility. This dissertation allows time-varying random volatility in the Geometric Brownian Motion to outline a proof of the formula, thus addressing this drawback. To estimate option prices with the Black-Scholes, the dissertation considers its expectation with respect to two potential probability models of random volatility. Unfortunately, a closed form expression of the expectation of the formula for computing the option prices is intractable. Then the dissertation settles with using an approximation which to its credit incorporates in it the kurtosis of the probability model of random volatility. To our knowledge, option pricing methods in literature do not incorporate kurtosis information.

The option pricing with random volatility is pursued for two stochastic volatility models. One model is a member of generalized auto regressive conditional heteroscedasticity (GARCH). The second is a member of Stochastic

Volatility models. For each model, estimation of their parameters is outlined. Two real financial series data are then used to illustrate estimation of the option prices, and compared them with those from the Black-Scholes formula with constant volatility.

Motivated by a Monte Carlo procedure in the literature for option pricing when the volatility follows a GARCH model, this dissertation lays a foundation for future research to simulate option prices when the random volatility is assumed to follow a Stochastic Volatility model instead of GARCH.

Chapter 1 reviews the basic concepts of the log returns from assets and the corresponding probability model. It uses a financial series to illustrate some crucial aspects of the log returns by descriptive summary statistics. It continues to review the GARCH and Stochastic Volatility models for modeling the log returns. Their relevant statistical aspects are pointed out.

Chapter 2 outlines a proof of the Black-Scholes formula allowing the volatility of assets to be random varying with time as assets evolve following a Geometric Brownian Motion. For computing option prices approximate expressions for the expectation of the Black-Scholes formula are derived when volatility follows a GARCH model and next when it follows a Stochastic Volatility model. These approximate expressions are computed in Chapters 3 and 4 for option pricing. Both these chapters use two financial series for illustration.

Chapter 5 reviews briefly a Monte Carlo procedure for option pricing when volatility follows a GARCH model. Motivated by it, it lays a foundation for future research to simulate option prices when the random volatility is assumed to follow a Stochastic Volatility model instead of the GARCH. Chapter 6 outlines a few ideas for future research.

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To my beloved parents,

Juying and Quansen,

the eternal source of my life and strength.

To myself,

Hui Gong,

the only person worthy of my company.

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CHAPTER 1

Introduction and Literature Review

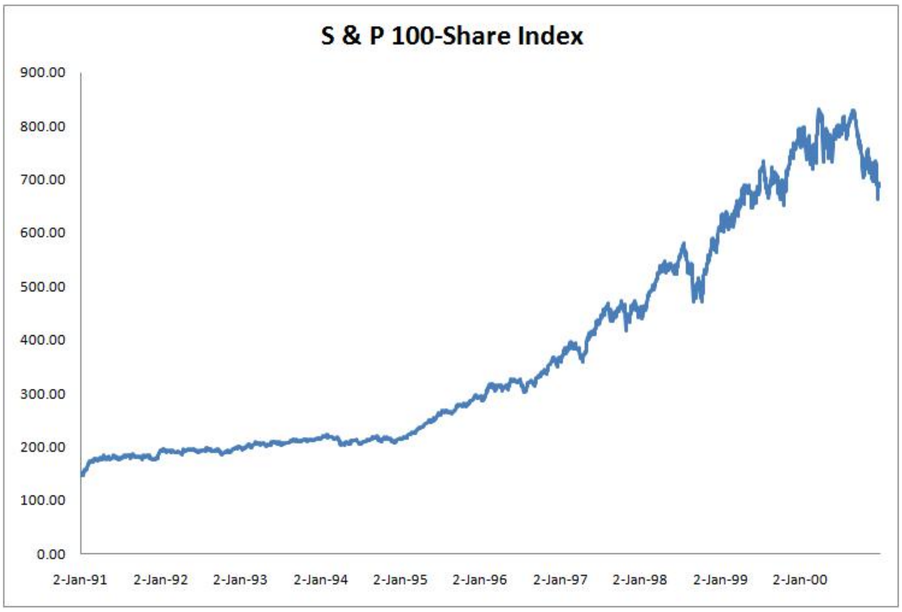
1.1 Introduction

This chapter will introduce the notion of the log returns. It will be seen that the log return over a period of t units of time is the sum of t consecutive log returns from one unit time periods. If the consecutive one unit time period returns are assumed independent and identically distributed (iid), then their sum is a random walk. Empirical evidence will show that the log returns are not iid because their volatility (standard deviation) is not constant over time. This chapter will highlight empirically such statistical properties including the heavy-tails of the log returns from financial assets for a real finance series. For modeling the random volatility of the log returns and their heavy tails, this chapter will review the generalized auto regressive conditional heteroscedasticity (GARCH) and the Stochastic Volatility models. Finally it will end with a review of the Black-Scholes formula.

1.2 Basic Concepts

This dissertation is motivated by the fact that returns from financial assets experience time varying random volatility induced by many unpredictable factors. While the assumption of constant volatility is usefully employed in regression and popular time series models, it is not realistic when modeling financial series and derivatives. A financial series when experiencing significant volatility continues to be volatile for a time in immediate future before it settles back to be less volatile until again unexpected events will cause it to become volatile. The stocks, options, Standard and Poor's indices, the interest rates etc all experience such volatility movements. The S&P index of 100 shares from January 2, 1991 to Nov 2, 2009 graphed in Figure 1.1 is seen to be volatile in its movements. It is significantly more volatile during some time periods than others as it moves around its long term trend.

Figure 1.1: S & P 100 Data from 2-Jan-91 to 2-Nov-09



The volatile periods create outliers in the distribution of assets and their returns. The outliers produce mixture distributions with long tails. At different time points, the returns from investing in assets can be viewed conditionally distributed with different time varying random volatility. Then the unconditional distribution of the returns from investments is a mixture of infinitely many such conditional distributions. Even when the conditional distributions are normal with varying variances, their mixture is not a normal distribution. Instead it will have long tails caused by few relatively large variances of the conditional distribution.

The distributions which incorporate time dependent volatility of the residuals are the family of generalized autoregressive conditional heteroscedasticity (GARCH) models. This chapter will review it briefly. Another family for modeling the volatility is the class of Stochastic Volatility (SV) models. It will also be reviewed briefly in this chapter. Their relevant statistical properties will be reviewed here for their role in modeling random volatility in the Black-Scholes formula in Chapter 2.

For a review of literature and to introduce notations, we begin with the definition of the returns from a financial asset series. Let S_t be a financial time series, such as the price or cost at time t of the assets. Over the consecutive k periods from time $(t-k)$ to time t, the return (gross) $R_t(k)$ from the asset series S_t is defined by,

$$R_t(k) = \frac{S_t}{S_{t-k}}.$$

For only one time period $(t-1)$ to time t, the return is simply $R_t = \dfrac{S_t}{S_{t-1}}$. Thus the return $R_t(k)$ from the recent k consecutive periods starting from time $(t-k)$ to time t is the product of the k one period returns during this time. That is,

$$R_t(k) = \frac{S_t}{S_{t-k}} = \left(\frac{S_t}{S_{t-1}}\right)\left(\frac{S_{t-1}}{S_{t-2}}\right)\dots\left(\frac{S_{t-k+1}}{S_{t-k}}\right) = R_t R_{t-1} \dots R_{t-k+1}.$$

Needless to write, the returns are stochastic. Many factors beyond investors' control determine the distribution of the returns. The future returns cannot be predicted with any certainty. But the statistical modeling provides tools for making predictions about expected returns with some degree of confidence. The foundation of statistical modeling of the returns rests on the assumption of the probability distribution of one-time period returns. So which reasonable probability model is suitable for modeling the returns? The normal probability distributions would be candidate for their ease of manipulations and their central role in statistical modeling. But normal distributions will not be suitable for modeling the returns since they are non negative. This hurdle can be overcome if we were to model the natural logarithms of the returns. Hence, consider the natural log return r_t defined as,

$$r_t = \log R_t, \qquad t = 1, 2, \ldots, k.$$

Then the log return from the recent consecutive k periods returns, $R_t(k)$, is the sum of k one period log returns. That is,

$$\log R_t(k) = \log \frac{S_t}{S_{t-k}} = \log R_t + \log R_{t-1} + \ldots + \log R_{t-k+1}.$$

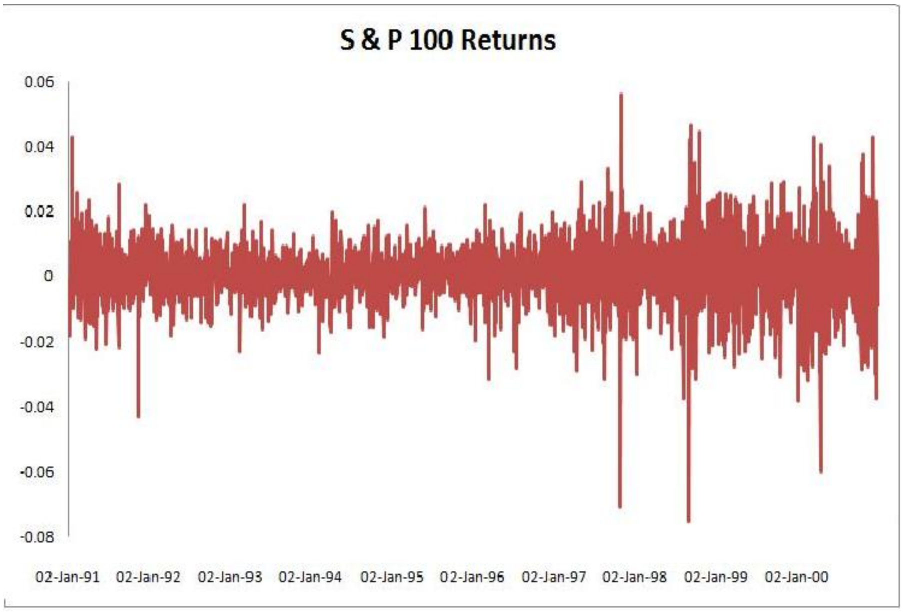
Letting $k = t$, we can write,

$$\begin{aligned} \log \frac{S_t}{S} &= r_1 + r_2 + \ldots + r_t, && \text{or} \\ \frac{S_t}{S} &= \exp(r_1 + r_2 + \ldots + r_t), && \text{or} \\ S_t &= S \exp(r_1 + r_2 + \ldots + r_t), && \text{or} \\ \log S_t &= \log S + (r_1 + r_2 + \ldots + r_t). \end{aligned}$$

Now if we assume the log returns $r_1, r_2, \ldots, r_t$ are independent and identically distributed with mean μ and a constant volatility parameter (standard deviation) σ, then $\log S_t$ is a random walk and the corresponding financial series S_t is an exponential random walk. Should the one period log returns are also assumed normally distributed, then S_t will then follow the lognormal geometric random walk with drift parameter μ and the volatility parameter σ.

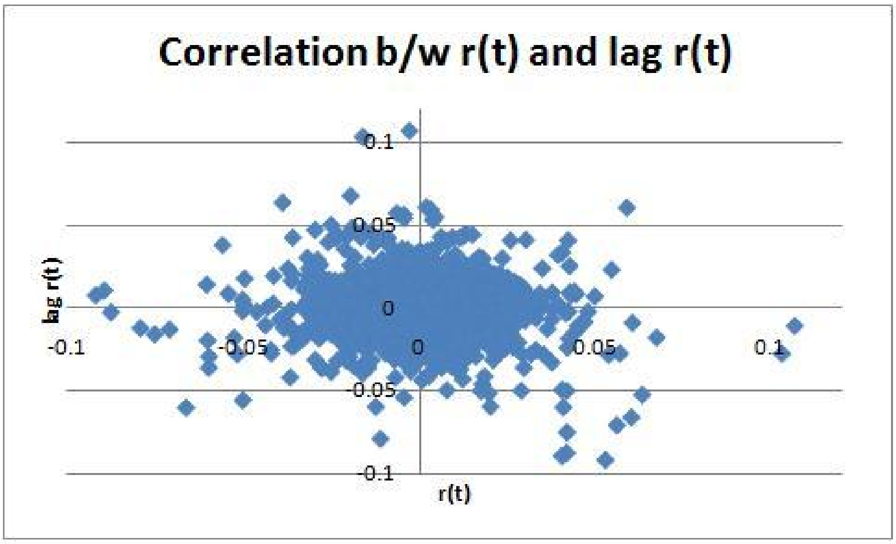
But can we realistically assume the consecutive log returns $r_1, r_2, \ldots, r_t$ i.i.d. normal variables? As stated earlier, financial series exhibit time dependent random volatility. It would mean that the log returns are neither independent nor identically distributed. The daily movements of the S&P index of 100 shares from January 2, 1991 to November 2, 2009 are shown in Figure 1.1, and its log returns in Figure 1.2. Clearly the volatility of the log returns is not constant. It tends to vary with time and persists for a while to levels off only to increase again. The sort of non constant volatility phenomenon is heteroscedasticity. Let the volatility of the log return r_t be quantified by the standard deviation θ_t of its distribution varying with time. We will assume the log returns distributed conditionally normal with time varying standard deviation θ_t.

Figure 1.2: Log Returns $r_t = \log \dfrac{S_t}{S_{t-1}}$



Though the log returns are not identically distributed, Figure 1.3 suggests their one lag auto correlation is insignificant. Their mixture distribution is not normal distribution; see its normal probability plot in Figure 1.4. It has heavy tails. A good number of log returns are in the tails as will not be expected in the corresponding tails of the normal distribution. Also its kurtosis of the mixture is 8.36, a number much in excess of 3 for the normal distribution. These empirical statistics indicate the existence of fat tails for mixture distribution of log returns. The skewness of the log returns as a measure of symmetry is 0.15, an insignificant amount. It implies that the mixture of the log returns is symmetrically distributed around their trend. In short, the log returns are not normally distributed but their mixture distribution is symmetric having high peak at the mean, thin midrange and fat tails; see also histogram in Figure 1.5.

Figure 1.3: One Lag Auto Correlation between Log Returns



1.3 Generalized Autoregressive Conditional Heteroscedasticity (GARCH) Models

Heteroscedasticity is a statistical language to mean non constant volatility. To model the non constant conditional volatility of the log returns or of their resid-

Figure 1.4: Normal Probability Plot of the Log Returns

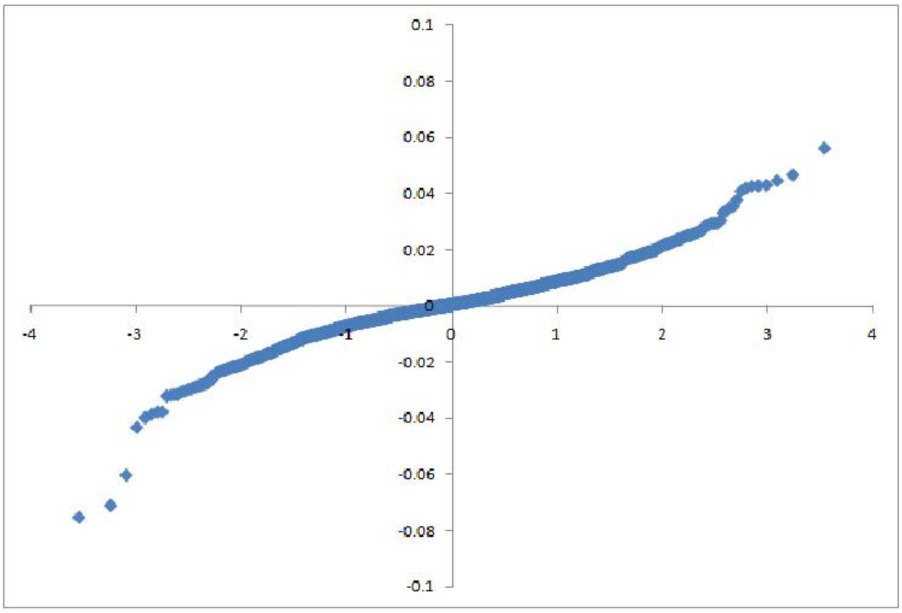
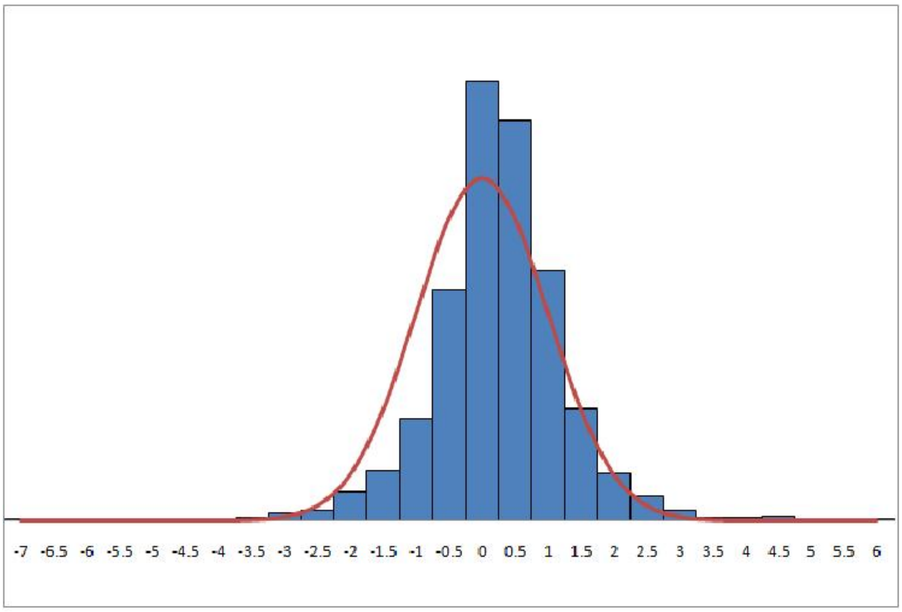


Figure 1.5: Marginal Distribution of the Log Returns



uals, we briefly review two classes of models in quantitative finance. One class consists of generalized auto regressive conditional heteroscedasticity (GARCH) models. It was first proposed by Bollerslev (1986). The other is the class of Stochastic Volatility models proposed among others by Taylor (1982b). First we introduce the class of GARCH models.

An autoregressive conditional heteroscedasticity (ARCH) model is defined as,

$$r_t = \mu + \theta_t Z_t, \quad \text{and} \quad \theta_t = \sqrt{\alpha_0 + \alpha_1 \theta_{t-1}^2 Z_{t-1}^2},\ \alpha_0 \geq 0,\ \alpha_1 \geq 0.$$

The normal process Z_t with mean 0, variance 1 is independent of the conditional standard deviation process θ_t. For the process r_t to be stationary, $\alpha_1 < 1$. Note that the terms of r_t are uncorrelated with conditional variances,

$$\mathrm{Var}(r_t) = \sigma_t^2 = \theta_t^2 = \alpha_0 + \alpha_1 \theta_{t-1}^2 Z_{t-1}^2.$$

The unconditional variance of the return series r_t is,

$$\sigma^2 = \frac{\alpha_0}{1-\alpha_1}.$$

Clearly, an ARCH model captures a behavior such that a large return r_t leads to large conditional variance of the next return r_{t+1}. This feature will persist before the return process stabilizes for a while until occurrence of an unexpected event renders it volatile once again. It is this feature which is neglected by the assumption of constant volatility.

An ARCH(p) model further assumes the conditional variance θ_t^2 depends upon previous p returns r_{t-i}. That is,

$$r_t = \mu + \theta_t Z_t, \quad \text{and} \quad \theta_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \theta_{t-i}^2 Z_{t-i}^2,\ \omega \geq 0,\ \alpha_i \geq 0.$$

The GARCH(1, 1) model generalizes ARCH(1) by incorporating the conditional variance of r_{t-1}. That is,

$$r_t = \mu + \theta_t Z_t, \quad \text{and} \quad \theta_t^2 = \omega + \alpha\theta_{t-1}^2 Z_{t-1}^2 + \beta\theta_{t-1}^2, \ \omega \geq 0, \ \alpha \geq 0.$$

As before Z_t is normally distributed with mean 0, variance 1 and independent of the volatility process θ_t so that $E(r_t) = \mu$.

The GARCH(p, q) model is a further generalization. It assumes that the conditional variance θ_t^2 to depend upon not only previous returns r_{t-i} but also previous conditional variances θ_{t-j}^2. That is,

$$r_t = \mu + \theta_t Z_t, \quad \text{and} \quad \theta_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \theta_{t-i}^2 Z_{t-i}^2 + \sum_{j=1}^{q} \beta_j \theta_{t-j}^2, \ \omega \geq 0, \ \alpha_i \geq 0.$$

To ensure the stationary of r_t, the unconditional variance of the return process r_t is assumed to be finite. That is,

$$\text{Var}(r_t) = E[r_t^2] = \frac{\omega}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j} < \infty.$$

The method of maximum likelihood can be used to estimate the parameters of the GARCH model. Denoted by $\nu = (\omega, \alpha_1, \ldots, \alpha_p, \beta_1 \ldots, \beta_q)'$ the vector of parameters of GARCH(p, q) model. Since the process Z_t is normal with mean 0, variance 1 and independent of θ_t, r_t is normal with mean μ and variance θ_t^2. The likelihood function $L(\nu)$ of the returns is,

$$L = \sum_{t=1}^{T} \Big[\log \frac{1}{\sqrt{2\pi}\theta_t} - \frac{(r_t - \mu)^2}{2\theta_t^2} \Big] = -\frac{1}{2} \sum_{t=1}^{T} [\log(2\pi) + \log \theta_t^2 + Z_t^2].$$

Its score vector is

$$\frac{\partial L}{\partial \nu} = -\frac{1}{2} \sum_{t=1}^{T} \frac{1}{\theta_t^2} \Big(\frac{\partial \theta_t^2(\theta)}{\partial \nu} \Big) - \frac{1}{2} \sum_{t=1}^{T} \frac{\partial Z_t^2}{\partial \nu} = \sum_{t=1}^{T} \frac{Z_t^2 - 1}{2\theta_t^2} \Big(\frac{\partial \theta_t^2}{\partial \nu} \Big),$$

Since $Z_t^2 = \dfrac{(r_t-\mu)^2}{\theta_t^2}$, we have $\dfrac{\partial Z_t^2}{\partial \nu} = -\dfrac{(r_t-\mu)^2}{\theta_t^4}\Big(\dfrac{\partial \theta_t^2}{\partial \nu}\Big) = -\dfrac{\theta_t^2 Z_t^2}{\theta_t^4}\Big(\dfrac{\partial \theta_t^2}{\partial \nu}\Big)$.

In particular for GARCH(1, 1) model where $\theta_t^2 = \omega + \alpha\theta_{t-1}^2 Z_{t-1}^2 + \beta\theta_{t-1}^2$, $\nu = (\omega, \alpha, \beta)'$, and the partial derivatives are

$$\frac{\partial \theta_t^2}{\partial \omega} = 1 + \beta\frac{\partial \theta_{t-1}^2}{\partial \omega}, \qquad \frac{\partial \theta_t^2}{\partial \alpha} = (r_{t-1}-\mu)^2 + \beta\frac{\partial \theta_{t-1}^2}{\partial \alpha}, \qquad \frac{\partial \theta_t^2}{\partial \beta} = \theta_{t-1}^2 + \beta\frac{\partial \theta_{t-1}^2}{\partial \beta},$$

1.4 Stochastic Volatility (SV) Models

Alternative to the GARCH models, Stochastic Volatility (SV) models have been proposed for volatility modeling. A SV model differs from the GARCH in that the conditional volatility θ_t^2 or a function of it follows an autoregressive model with a noise term η_t which is normally distributed with mean 0, variance σ_η^2. A simple Stochastic Volatility model proposed by Taylor (1982b) is,

$$r_t = \mu + \theta_t Z_t, \quad \text{and} \quad \log\theta_t^2 = u + \phi(\log\theta_{t-1}^2 - u) + \eta.$$

The coefficient ϕ represents volatility persistence, $-1 < \phi < 1$. A general representation of Stochastic Volatility models is,

$$f(\theta_t^2, \delta) = u + \phi[f(\theta_{t-1}^2, \delta) - u] + k(\theta_{t-1}^2)\eta_t$$

For choice of $f(\theta_t^2, \delta)$, u, ϕ and $k(\theta_{t-1}^2)$, we can get a particular SV model such as $\log\theta_t^2 = u + \eta_t$, Clark (1973); $\theta_t = u + \phi(\theta_{t-1} - u) + \eta_t$, Scott (1987), Anderson (1994) and Stein and Stein (1991); $\theta_t^2 = u + \phi(\theta_{t-1}^2 - u) + \sigma_\eta\eta_t$, Anderson (1994); $\dfrac{(\theta_t^2)^\delta - 1}{\delta} = u + \phi\Big[\dfrac{(\theta_{t-1}^2)^\delta - 1}{\delta} - u\Big] + \eta_t$, Yu, Yang, and Zhang (2006) etc. There are many more SV models in the literatures.

Unlike the GARCH models, the Stochastic Volatility models do not lend themselves to maximum likelihood method. Dufour, J. and Valéry (2005) used the method of moments for some Stochastic Volatility models. For illustration, consider a Stochastic Volatility model,

$$\begin{aligned} r_t &= \mu + \theta_t Z_t, \\ \theta_t^2 &= \alpha \theta_t^2 + \sigma_\eta \eta_t. \end{aligned}$$

For estimating $\nu = (\alpha, \sigma_\eta)'$, use the moments of the volatility process. Since Z_t and η_t are independent and both normal with mean 0 and variance 1, then

$$\begin{aligned} \mu_k \equiv E(r_t^k) &= \beta^k \frac{k!}{2^{(k/2)}(k/2)!} \exp\left[\frac{k^2\sigma_\eta^2}{8(1-\alpha^2)}\right], \qquad \text{if k is even,} \\ &= 0, \qquad \text{if k is odd,} \end{aligned}$$

$$\mu_{k,l} \equiv E(r_t^k r_{t+m}^l) = \beta^{k+l} \frac{k!}{2^{(k/2)}(k/2)!} \frac{l!}{2^{(l/2)}(l/2)!} \exp\left[\frac{\sigma_\eta^2(k^2 + l^2 + 2kl\alpha^m)}{8(1-\alpha^2)}\right],$$

if both k and l are even nonnegative integers, and $\mu_{k,l} = 0$ if either k or l is odd integer.

For $k = 2$, $k = 4$, $k = l = 2$ and $m = 1$,

$$\begin{aligned} \mu_2 &= E(r_t^2) = \exp[\sigma_\eta^2/2(1-\alpha^2)], \\ \mu_4 &= E(r_t^4) = 3\exp[2\sigma_\eta^2/2(1-\alpha^2)], \\ \mu_{2,2} &= E(r_t^2 r_{t-1}^2) = \exp[\sigma_\eta^2/2(1-\alpha)], \end{aligned}$$

The corresponding empirical moments μ_2, μ_4 and $\mu_{2,2}$ are

$$\hat{\mu}_2 = \frac{1}{T}\sum_{t=1}^{T} \hat{r}_t^2, \quad \hat{\mu}_4 = \frac{1}{T}\sum_{t=1}^{T} \hat{r}_t^4, \quad \hat{\mu}_2(1) = \frac{1}{T}\sum_{t=1}^{T} \hat{r}_t^2 \hat{r}_{t-1}^2.$$

Setting the theoretical moments μ_2, μ_4 and $\mu_{2,2}$ equal to corresponding theoretical moments, we get

$$
\begin{aligned}
\hat{\alpha} &= \left[\frac{\log[\hat{\mu}_2(1)] + \log[\hat{\mu}_4/(3\hat{\mu}_2^4)]}{\log[\hat{\mu}_4/(3\hat{\mu}_2^2)]}\right] - 1, \\
\hat{\sigma}_\eta &= \left[(1-\hat{\alpha}^2)\log[\hat{\mu}_4/(3\hat{\mu}_2^2)]\right]^{1/2}.
\end{aligned}
$$

For the Anderson (1994) Model

$$
\begin{aligned}
r_t &= \mu + \theta_t Z_t \\
\theta_t^2 &= u + \phi(\theta_{t-1}^2 - u) + \sigma_\eta \eta_t
\end{aligned}
$$

where the parameters to be estimated are u, ϕ, σ_η, we can find

$$
u_2 = E(r_t^2) = u, \quad u_4 = E(r_t^4) = \frac{\sigma_\eta^2}{1-\phi^2} + u^2, \quad u_{2,2} = E(r_t^2 r_{t+1}^2) = \frac{\phi\sigma_\eta^2}{1-\phi^2} + u^2.
$$

The parameters in terms of the moments and cross-moments are

$$
u = u_2, \qquad \phi = \frac{u_{2,2} - u_2^2}{u_4 - u_2^2}, \qquad \sigma_\eta^2 = \frac{[u_4 - u_{2,2}][u_4 + u_{2,2} - 2u_2^2]}{u_4 - u_2^2}.
$$

1.5 Option Pricing

An option is a contract between a buyer and a seller. It gives buyer the right – but not the obligation – to buy or to sell an asset at a later expiry time T at an agreed price (exercise price) K. A call option gives buyer the right to buy the asset, while a put option gives buyer the right to sell the asset. If the buyer chooses to exercise this right, the seller of the option is obliged to sell or buy the asset at the agreed price. In return for granting the right, the seller collects a payment (the premium) C for call option and P for put option from the buyer of the option. An option is exercised if the buyer can gain from exercising this right. Thus the buyer will choose to exercise if the discounted (present) value of the difference between the future asset price S_T and the exercise price K at time T exceeds the premium paid for the option. Hence, given the risk-free interest rate r, the buyer of an option will exercise the right to buy it if the discounted price $e^{-r(T-t)}(S_T - K)$ exceeds C for a call option, and to sell if if the discounted price $e^{-r(T-t)}(K - S_T)$ exceeds P for a put option. Otherwise the buyer will suffer loss and even decline the right and lose the premium. Needless to say, the mechanism for pricing options is important. The Black-Scholes formula, and the Monte Carlo Simulation method proposed by Duan (1995) are two important methods for option pricing.

Fischer Black and Myron Scholes (1970) derived in a closed-form formula for pricing the European option which can be exercised only at the expiry date T. The Black-Scholes formula is a function of the current asset price S, the expiry date T, the exercise price K, the risk-free interest rate r, and a volatility parameter θ assumed to be constant. Let, $d_1 = \dfrac{\log(S/K) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}}$, $d_2 = d_1 - \theta\sqrt{T}$, and $\Phi(.)$ denotes the standard normal cumulative distribution function, then the Black-Scholes formula for C is,

$$C(S, T, K, r, \theta) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

When an asset's dividend yield is q, then d_1 is modified as,

$$d_1 = \frac{\log(S/K) + (r - q + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}}, \qquad \text{and}$$

$$C_{BS}(S, T, K, r, q, \theta) = Se^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2).$$

In the derivation of their formula, Black-Scholes treats the volatility θ as constant. It handicaps the formula seriously since in reality asset's volatility is not constant. It varies with time. There is an extensive literature on GARCH and SV models devoted to modeling of the time-dependent volatility. This dissertation incorporates information embedded in time-dependent volatility by means of using GARCH and SV modeling in the Black-Scholes option pricing formula. While incorporating information embedded in time-dependent volatility using GARCH and SV models, the dissertation will also incorporate kurtosis of returns in improving the option pricing formula.

Chapter 2 is devoted to GARCH Volatility and its incorporation in the option pricing formula. We review the Black-Scholes formula and then present a method to incorporate GARCH volatility in the option pricing formula. We propose to replace the volatility parameter θ in the Black-Scholes formula by expectation of the GARCH volatility. To also incorporate the kurtosis information of the returns in the Black-Scholes formula, the proposed method uses the Taylor's expansion of the volatility expectation. Then using SV models of Volatility instead of GARCH models for the volatility parameter also incorporates volatility information in the Black-Scholes formula. The improvement resulting in option prices from the methods of Chapter 2 is illustrated in Chapter 3 and 4 empirically for GARCH and SV volatility. The empirical evidence shows that when the non-constant volatility information is incorporated, it leads option prices which are much closer to the observed prices. Chapter 5 reviews a method by Duan's (1995) for option pricing. We will review his Monte Carlo Simulation method for GARCH volatility to extract information for option pricing. The simulation method is extended to SV models for option pricing. Chapter 6 summarizes briefly the dissertation and discusses further research.

CHAPTER 2

Black-Scholes Formula With Varying Volatility

2.1 Introduction

In this chapter we recall the very basics of the call option and its pricing by the Black-Scholes formula. It will give a derivation of the Black-Scholes formula allowing the volatility of the log returns to be stochastic. Since volatility of the returns is random, it is proposed to estimate the call option price by the expectation of the Black-Scholes formula with respect to the probability model of volatility. Since the exact expectation is intractable, an approximate expression is derived. The approximate expression, however, incorporates the kurtosis information of the log returns.

An option entitles a buyer of the option to buy or sell a specific asset from the option seller at the set price in the future. Since it guarantees a purchase right not an obligation in the future, a payment has to be made to reserve the right. This payment is coined as call price in a call option where a right of buying is reserved while it is called a put price in a put option where a right of selling is reserved. Considering the occurrence of payment at current moment while a right to buy or sell executed in the future time, one would have to determine the amount of this current payment to avoid any potential loss in the future.

Many methods have been proposed to estimate option price. The Black-Scholes formula is one of the better developed methods. It has several unbeatable advantages with the characteristic of being straightforward, simple for computation and closed form. On the downside, it is restricted by the assumption of a constant volatility θ over time period set for the option to expire. Hence, taking advantage of the strength of the Black-Scholes formula, we propose to relax the constant volatility assumption to allow varying volatility θ over time. Additionally, we will incorporate the kurtosis information in option pricing. The kurtosis quantifies the excess of outliers prevalent in financial time series. Information from outliers is invariably unaccounted by all other option pricing methods.

The Black-Scholes formula was originally designed for pricing the European options, which are exercised at the expiry date. Besides the European options, the second group of options is American options. These options allow the holder of the option the ability to exercise the option at any point in time up to expiry. This characteristic renders solutions to value them somewhat difficult. Only American options with no dividends or with a single dividend can be priced by Black-Scholes formula. In this dissertation, our focus is on the Black-Scholes formula.

2.2 A Derivation of Black-Scholes Formula

Underlying the Black-Scholes formula is the evolution of the asset prices S_t which follows the stochastic Geometric Brownian Process:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \theta S_t \mathrm{d}W_t. \tag{2.1}$$

The constant μ is the drift rate of the asset prices S_t and θ is the volatility (standard deviation) of $\log(S_t/S_{t-1})$, and W_t is the Wiener (Brownian) process. If r is the risk free rate of return, then the call price C of option on asset S_t by the Black-Scholes formula is,

$$C = S\Phi(d) - Ke^{-rT}\Phi(d - \theta\sqrt{T})$$

where $d = \dfrac{\log(S/K) + (r + \frac{1}{2}\theta^2)T}{\theta\sqrt{T}}$, $\Phi(.)$ is the standard normal distribution function, $S = S_t$ when $t = 0$, K is the strike price at expiry time T.

A call option is exercised only when at expiry time T, $(S_T - K)$ is positive. In other words, the call option at expiry time T is worth $Max(S_T - K, 0)$. At the risk free rate r, the option at time $t = 0$ is worth only $e^{-rt}Max(S_T - K, 0)$. Thus the call price of the option is defined as,

$$C = e^{-rt}E[\text{Max}(S_T - K, 0)] \tag{2.2}$$

where the expectation is with respect to risk neutral probability model.

The asset prices among many factors depend also crucially on the market risk factor. Investors demand premium for taking the risk. Therefore, in order to avoid arbitrage (risk free gain), the expectation in (2.2) is carried out adjusting for the risk factor. The mathematics of finance adjusts the Wiener process of the stochastic differential equation (2.1) for the market price of risk. The risk-neutral probabilities derived from the risk adjusted Wiener process are then used to carry out the expectation in (2.2).

As stated previously, our aim is to allow the volatility θ to be random and varying with time in the Black-Scholes formula. In doing so we will also incorporate information from the kurtosis of the returns $r_t = \log\left(\dfrac{S_t}{S_{t-1}}\right)$ of the log returns. First we outline a derivation of the Black-Scholes formula allowing the volatility parameter θ to vary with time. Consider the stochastic Geometric Brownian Motion (2.1), but now replacing θ in it by θ_t. That is,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \theta_t S_t \mathrm{d}W_t. \tag{2.3}$$

Under no arbitrage opportunity, the return from the evolution of S_t process must equal to return from a risk free investment having return rate r per period. Therefore we seek a Wiener process (risk neutral probabilities) so that the expected log return $E\Big[\log\Big(\dfrac{S_t}{S_{t-1}}\Big)\Big]$ with respect to risk neutral probabilities is independent of μ. This task is accomplished by the Wiener process $\widetilde{W}_t$ defined as,

$$\widetilde{W}_t = W_t + \frac{\mu - r}{\theta_t} t$$

The expression $\dfrac{\mu - r}{\theta_t}$ is the market price of risk. The stochastic differential equation (2.3) under the market price of risk can be written as

$$\mathrm{d}S_t = rS_t \mathrm{d}t + \theta_t S_t \mathrm{d}\widetilde{W}_t. \tag{2.4}$$

For a twice differential continuous function $\log S_t$. an application of Itô's lemma leads to,

$$\begin{aligned}
d\log S_t &= \frac{d}{dS_t}(\log S_t)dS_t + (\theta_t S_t)^2 \frac{1}{2}\frac{d^2}{dS_t^2}(\log S_t)dt \\
&= \frac{1}{S_t}(rS_t dt + \theta_t S_t d\widetilde{W}_t) + (\theta_t S_t)^2 \frac{1}{2}\Big(-\frac{1}{S_t^2}\Big)dt \\
&= \Big(r - \frac{1}{2}\theta_t^2\Big)dt + \theta_t d\widetilde{W}_t
\end{aligned}$$

Hence, $S_t = S\exp\Big[\Big(r - \dfrac{\theta_t^2}{2}\Big)t + \theta_t \widetilde{W}_t\Big]$, where $S_t = S$ at time $t = 0$.

Note that $\widetilde{W}_t$ is a Wiener process with variance $t\theta_t^2$. To outline our proof of the Black-Scholes formula, substitute $S_t = S\exp\Big[\Big(r - \dfrac{\theta_t^2}{2}\Big)t + \theta_t \widetilde{W}_t\Big]$ in (2.2) and take the expectation with respect to risk free probability measure induced by $\widetilde{W}_t$. Hence,

$$\begin{aligned} C &= e^{-rt}E\Big[\text{Max}\Big(S\exp\Big[\Big(r-\frac{\theta_t^2}{2}\Big)t+\theta_t\widetilde{W}_t\Big]-K,0\Big)\Big] \\ &= Se^{-rt}E\Big[\text{Max}\Big(\exp\Big[\Big(r-\frac{\theta_t^2}{2}\Big)t+\theta_t\sqrt{t}Z_t\Big]-\frac{K}{S},0\Big)\Big]. \end{aligned}$$

where Z_t are i.i.d. normal. If $A = \Big\{\Big(r-\frac{\theta_t^2}{2}\Big)t+\theta_t\sqrt{t}Z_t > \log\frac{K}{S}\Big\}$, and I_A is its indicator function, then

$$C = Se^{-rt}\Big[E\exp\Big[\Big(r-\frac{\theta_t^2}{2}\Big)t+\theta_t\sqrt{t}Z_t\Big]I_A - \frac{K}{S}E(I_A)\Big]. \tag{2.5}$$

Note that,

$$\begin{aligned} \frac{K}{S}E[I_A] &= \frac{K}{S}P\Big[Z_t \geq -\frac{\log(S/K)+(r-\frac{1}{2}\theta^2)t}{\theta\sqrt{t}}\Big] \\ &= \frac{K}{S}P\Big[Z_t < \frac{\log(S/K)+(r-\frac{1}{2}\theta^2)t}{\theta\sqrt{t}}\Big] \\ &= \frac{K}{S}\Phi(d-\theta_t\sqrt{t}), \end{aligned}$$

where $d = \dfrac{\log(S/K)+(r+\frac{1}{2}\theta_t^2)t}{\theta_t\sqrt{t}}$. Also

$$\begin{aligned} E\exp\Big[\Big(r-\frac{\theta_t^2}{2}\Big)t+\theta_t\sqrt{t}Z_t\Big]I_A &= \int_A \exp\Big[\Big(r-\frac{\theta_t^2}{2}\Big)t+\theta_t\sqrt{t}Z_t\Big]\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{Z_t^2}{2}\Big)dZ_t \\ &= \exp(rt)\int_A \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{(Z_t-\theta_t\sqrt{t})^2}{2}\Big)dZ_t \\ &= \exp(rt)\int_{-\infty}^{\frac{\log(S/K)+(r-\frac{1}{2}\theta_t^2)t}{\theta_t\sqrt{t}}} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{Y_t^2}{2}\Big)dY_t \\ &= \exp(rt)\Phi\Big(\frac{\log(S/K)+(r-\frac{1}{2}\theta_t^2)t}{\theta_t\sqrt{t}}\Big) = \exp(rt)\Phi(d). \end{aligned}$$

Substitution of these two expectations in (2.5) yields the Black-Scholes formula where volatility parameter θ_t is allowed to vary with time.

2.3 Option Prices Allowing Random Volatility

How may we use the Black-Scholes formula with random volatility θ_t to compute the option prices? A straightforward approach would be to calculate expected Black-Scholes option prices with respect to the distribution of θ_t. Since the calculation of exact expectations will be nearly impossible with respect to distribution of θ_t, we will be limited to using an approximation. Let $f(\theta_t^2)$ and $g(\theta_t^2)$ be defined as $f(\theta_t^2) = \Phi\Big(\frac{A + \frac{1}{2}\theta_t^2 T}{\sqrt{\theta_t^2 T}}\Big)$ and $g(\theta_t^2) = \Phi\Big(\frac{A - \frac{1}{2}\theta_t^2 T}{\sqrt{\theta_t^2 T}}\Big)$, where $A = \log\frac{S}{K} + rT$, then the expected call prices is

$$\begin{aligned}
E(C) &= SE_{\theta_t}[f(\theta_t)] - Ke^{-rT}E_{\theta_t}[g(\theta_t)] \\
&\approx S_t\Big(f[E(\theta_t^2)] + \frac{1}{2}f''[E(\theta_t^2)]E[\theta_t^2 - E(\theta_t^2)]^2\Big) \\
&\quad -Ke^{-rT}\Big(g[E(\theta_t^2)] + \frac{1}{2}g''[E(\theta_t^2)]E[\theta_t^2 - E(\theta_t^2)]^2\Big), \\
&= S_t\bigg(f[E(\theta_t^2)] + \frac{1}{2}f''[E(\theta_t^2)]E(\theta_t^2)(\kappa^{(\theta)} - 1)\bigg) \\
&\quad -Ke^{-rT}\bigg(g[E(\theta_t^2)] + \frac{1}{2}g''[E(\theta_t^2)]E(\theta_t^2)(\kappa^{(\theta)} - 1)\bigg),
\end{aligned}$$

where $\kappa^{(\theta)}$ is the kurtosis of θ and

$$\begin{aligned}
f''[E(\theta_t^2)] &= \frac{1}{\sqrt{2\pi}}\Big[-\Big(\frac{2A - TE(\theta_t^2)}{4\sqrt{T}E(\theta_t^2)\sqrt{E(\theta_t^2)}}\Big)\Big(\frac{4A^2 - [TE(\theta_t^2)]^2}{8T[E(\theta_t^2)]^2}\Big) \\
&\quad +\Big(\frac{-6A - TE(\theta_t^2)}{8\sqrt{T}[E(\theta_t^2)]^2\sqrt{E(\theta_t^2)}}\Big)\Big]\exp\Big[-\frac{(2A + TE(\theta_t^2))^2}{8TE(\theta_t^2)}\Big],
\end{aligned}$$

$$
\begin{aligned}
g''[E(\theta_t^2)] \quad = \quad & \frac{1}{\sqrt{2\pi}}\Big[-\Big(\frac{2A+TE(\theta_t^2)}{4\sqrt{T}E(\theta_t^2)\sqrt{E(\theta_t^2)}}\Big)\Big(\frac{4A^2-[TE(\theta_t^2)]^2}{8T[E(\theta_t^2)]^2}\Big) \\
& +\Big(\frac{-6A+TE(\theta_t^2)}{8\sqrt{T}[E(\theta_t^2)]^2\sqrt{E(\theta_t^2)}}\Big)\Big]\exp\Big[-\frac{(2A-TE(\theta_t^2))^2}{8TE(\theta_t^2)}\Big],
\end{aligned}
$$

An important step in the proposed procedure, allowing random volatility to estimate the call option prices, requires to incorporate $E(\theta_t^2)$, and kurtosis $\kappa^{(\theta)}$. In the next two chapters we use GARCH model (Chapter 3), and a stochastic volatility model (Chapter 4) illustrate all the steps. In both Chapters 3 and 4, two financial series are used for illustrations and comparisons of our approaches with the estimated Black-Scholes formula prices.

CHAPTER 3

Estimating Black-Scholes Call Prices With GARCH Volatility and Illustration

3.1 Introduction

This chapter considers estimation of call prices when volatility in Black-Scholes formula follows a GARCH model. We outline estimation of the GARCH parameters for our procedure to compute the expected call prices. To illustrate, we use two finance series. Their call prices are computed and compared with those from the Black-Scholes formula when volatility is treated as constant. One series consists of the S & P 100 daily indices from January 2, 1991 to November 2, 2009 with 4752 observations. The source is: http://www.nasdaq.com/aspx/historical quotes.aspx? symbol=OEX&selected=OEX

The second series consists of DRYS stock prices from February 3, 2005 to November 20, 2009. It has 1209 observations. The source is: http://finance.yahoo.com/q/hp?s=DRYS.

Recall the progress of a series S_t underlying the Black-Scholes formula is

$$dS_t = rS_t dt + \theta_t S_t d\widetilde{W}_t,$$

where volatility θ_t is the standard deviation of the return series $r_t = \log \dfrac{S_t}{S_{t-1}}$. We are proposing that the return series r_t is function of the volatility θ_t, that is $r_t = \mu + \theta_t Z_t$, where $Z_t \sim N(0, 1)$, while the volatility θ_t follows a GARCH(p, q) process, that is

$$\theta_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \theta_{t-i}^2 Z_{t-i}^2 + \sum_{j=1}^{q} \beta_j \theta_{t-j}^2.$$

In this chapter, for illustrating empirically modified Black-Scholes call prices, we assume that θ_t follow a GARCH(1, 1) process. That is,

$$\theta_t^2 = \omega + \alpha\theta_{t-1}^2 Z_{t-1}^2 + \beta\theta_{t-1}^2.$$

Following Taylor (2005), Microsoft Excel's Solver tool is used to estimate the GARCH (1, 1) model parameters, by maximum likelihood method.

$$\frac{\partial\theta_t^2}{\partial\omega} = 1 + \beta\frac{\partial\theta_{t-1}^2}{\partial\omega}, \qquad \frac{\partial\theta_t^2}{\partial\alpha} = (r_{t-1} - \mu)^2 + \beta\frac{\partial\theta_{t-1}^2}{\partial\alpha}, \qquad \frac{\partial\theta_t^2}{\partial\beta} = \theta_{t-1}^2 + \beta\frac{\partial\theta_{t-1}^2}{\partial\beta},$$

where, $\hat{\mu} = \bar{r_t}$ and the initial partial derivatives can be set to their unconditional expectation, $\dfrac{\partial\theta_1^2}{\partial\omega} = \dfrac{\partial\theta_1^2}{\partial\alpha} = \dfrac{\partial\theta_1^2}{\partial\beta} = \dfrac{\omega}{1-\alpha-\beta}$.

3.2 Illustration with GARCH Model, S & P 100 Index Series

Recall this series from Chapter 1, the summary and descriptive statistics are given below.

The parameter estimates are listed in the following table.

Table 3.1: S & P Data Description

Date	Index S_t	Return $r_t = \log \dfrac{S_t}{S_{t-1}}$
2-Jan-91	153.36	.
3-Jan-91	151.44	-0.012598592
4-Jan-91	151.28	-0.001057083
⋮	⋮	⋮
2-Nov-09	485.15	0.005871

Table 3.2: S & P Data Return Series Statistics

Statistic	Value
Mean	0.000242355
Standard Deviation	0.012011318
Skewness	-0.150007498
Kurtosis	8.357838966

Table 3.3: The MLE for GARCH Model

Parameters	GARCH
μ	0.000248673
ω	6.23546×10^{-7}
α	0.059097217
β	0.937395784

The call prices obtained from the standard Black-Scholes formula assume a constant volatility θ, which is the standard deviation of the returns series r_t that is 0.012011318. For our proposed modified Black-Scholes formula, applying $\hat{E}(\theta_t^2) = \dfrac{\hat{\omega}}{1-\hat{\alpha}-\hat{\beta}} = 0.0001778$, and kurtosis $\hat{\kappa}^{(r)} = 8.357838966$, the expected call prices can be obtained with respect to each initial price S, strike price K and expiry date T.

Note: For the simplicity, we have considered the risk-free interest $r = 0.1/365$ and $r = 0.03/365$ in our analysis.

The estimated call prices from the two methods mentioned above are listed in the following two tables, with two different assuming risk-free interest rates. The observed call prices are listed too. The difference between the observed and estimated call prices is measured by the %Error which is calculated by

$$\%\text{Error} = \frac{|\text{Observed Call Price} \ - \ \text{Estimated Call Price}|}{\text{Observed Call Price}} \times 100\%.$$

Note: The values in the table above are truncated for those that are estimated as negative. i.e. For $T = 24$, $S = 425.19$, $K = 450$, $r = 0.01/365$, the estimated call price by our proposed method is negative, since the call price is not negative in practice, they will be truncated to 0.

The following graph is a visual representation of the table above to show the difference between the observed call prices and the estimated call prices for the two methods. Each panel shows one expiry date T with one risk-free interest rate r. The x-axis is the strike price range K while the y-axis is the call prices C. At each strike price, the observed call price, the call price from the standard Black-Scholes formula and the call price from our proposed modified Black-Scholes formula are compared.

There exist another three indicators which can be used to measure the model's performance: (i) the root mean squared error (RMSE), (ii) the average

Table 3.4: Comparison of Call Prices from Black-Scholes Formula and our Proposed Modified Black-Scholes Formula

T	S	K	Observed Call Price	Call Price (r=0.1/365)			
				BS	% Error	Proposed	% Error
24	425.73	395	30.75	32.58	-6%	32.08	-4%
	425.73	400	25.88	28.51	-10%	27.54	-6%
	425.73	405	21.00	24.66	-17%	23.00	-10%
	425.67	410	16.50	21.02	-27%	18.40	-11%
	425.68	415	11.88	17.68	-49%	13.86	-17%
	425.65	420	7.69	14.58	-90%	9.29	-21%
	425.65	425	4.44	11.76	-165%	4.75	-7%
	425.68	430	2.10	9.23	-341%	0.23	89%
	425.65	435	0.78	6.93	-789%	0.00	100%
	425.16	440	0.25	4.71	-1785%	0.00	100%
	424.78	445	0.10	2.84	-2885%	0.00	100%
	425.19	450	0.10	1.47	-1446%	0.00	100%
87	425.73	380	46.75	55.00	-18%	51.62	-10%
	425.73	385	42.00	51.10	-22%	47.16	-12%
	425.73	390	37.50	47.33	-26%	42.69	-14%
	425.73	395	33.00	43.69	-32%	38.23	-16%
	425.73	400	28.50	40.17	-41%	33.76	-18%
	425.73	405	24.13	36.79	-52%	29.29	-21%
	425.26	410	20.38	33.22	-63%	24.40	-20%
	425.86	415	16.13	30.53	-89%	20.48	-27%
	425.68	420	12.82	27.44	-114%	15.85	-24%
	425.42	425	9.32	24.44	-162%	11.15	-20%
	425.62	430	6.51	21.87	-236%	6.86	-6%
	425.82	435	4.51	19.43	-331%	2.58	43%
	425.68	440	2.75	16.93	-515%	0.00	100%
	425.75	445	1.60	14.67	-820%	0.00	100%
	425.78	450	0.85	12.52	-1382%	0.00	100%
	425.39	455	0.44	10.32	-2245%	0.00	100%
115	425.73	380	47.25	58.99	-25%	54.22	-15%
	425.73	390	38.13	51.56	-35%	45.35	-19%
	425.73	400	29.38	44.58	-52%	36.49	-24%
	425.73	410	21.19	38.06	-80%	27.63	-30%
	425.41	420	13.88	31.81	-129%	18.47	-33%
	425.63	430	8.13	26.38	-225%	9.81	-21%
	425.28	440	3.88	21.09	-444%	0.62	84%
	425.13	450	1.50	16.39	-993%	0.00	100%

T	S	K	Observed	Call Price (r=0.03/365)			
			Call Price	BS	% Error	Proposed	% Error
24	425.73	395	30.75	30.82	0%	29.69	3%
	425.73	400	25.88	26.85	-4%	25.36	2%
	425.73	405	21.00	23.11	-10%	21.02	0%
	425.67	410	16.50	19.58	-19%	16.63	-1%
	425.68	415	11.88	16.37	-38%	12.30	-4%
	425.65	420	7.69	13.41	-74%	7.94	-3%
	425.65	425	4.44	10.73	-142%	3.60	19%
	425.68	430	2.10	8.34	-298%	0.00	100%
	425.65	435	0.78	6.19	-694%	0.00	100%
	425.16	440	0.25	4.13	-1552%	0.00	100%
	424.78	445	0.10	2.40	-2426%	0.00	100%
	425.19	450	0.10	1.15	-1111%	0.00	100%
87	425.73	380	46.75	49.35	-6%	44.40	5%
	425.73	385	42.00	45.62	-9%	40.09	5%
	425.73	390	37.50	42.02	-12%	35.78	5%
	425.73	395	33.00	38.57	-17%	31.46	5%
	425.73	400	28.50	35.26	-24%	27.15	5%
	425.73	405	24.13	32.09	-33%	22.83	5%
	425.26	410	20.38	28.77	-41%	18.11	11%
	425.86	415	16.13	26.27	-63%	14.32	11%
	425.68	420	12.82	23.43	-83%	9.85	23%
	425.42	425	9.32	20.70	-122%	5.30	43%
	425.62	430	6.51	18.37	-182%	1.16	82%
	425.82	435	4.51	16.16	-259%	0.00	100%
	425.68	440	2.75	13.92	-406%	0.00	100%
	425.75	445	1.60	11.91	-647%	0.00	100%
	425.78	450	0.85	10.02	-1086%	0.00	100%
	425.39	455	0.44	8.09	-1738%	0.00	100%
115	425.73	380	47.25	51.87	-10%	45.16	4%
	425.73	390	38.13	44.84	-18%	36.55	4%
	425.73	400	29.38	38.31	-30%	27.94	5%
	425.73	410	21.19	32.28	-52%	19.33	9%
	425.41	420	13.88	26.57	-92%	10.44	25%
	425.63	430	8.13	21.68	-167%	2.03	75%
	425.28	440	3.88	16.98	-338%	0.00	100%
	425.13	450	1.50	12.86	-757%	0.00	100%

Figure 3.1: S&P Option Prices Comparison of GARCH Volatility at r=.1/365

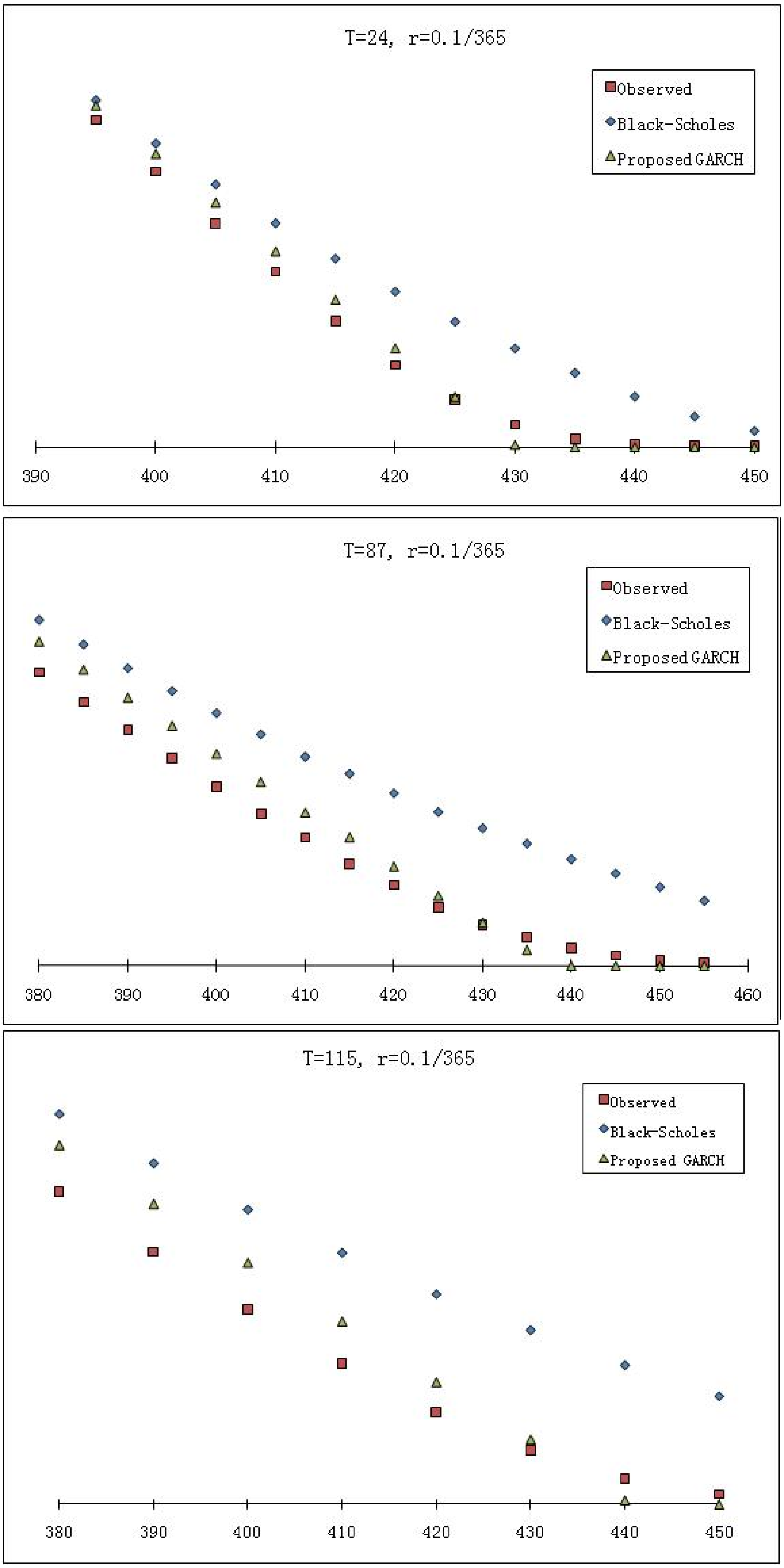
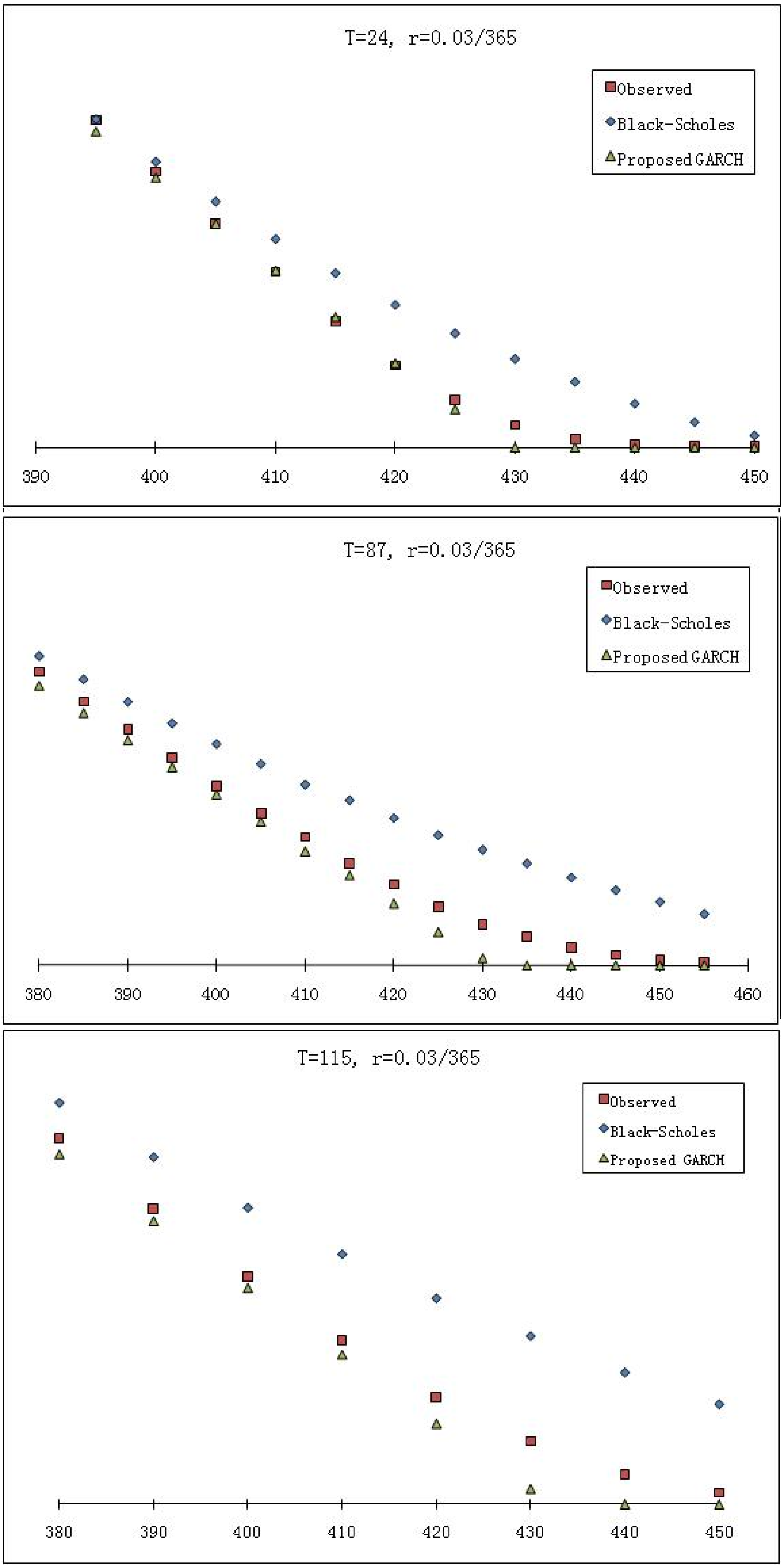


Figure 3.2: S&P Option Prices Comparison of GARCH Volatility at r=.03/365



absolute error (APE) and (iii) the average relative pricing error (ARPE). These indicators provide the deviation from the estimated call prices to the observed call prices. The calculation formulas are given below for the three indicators, respectively.

$$\text{RMSE}\,(\$) = \sqrt{\sum_{j=1}^{NO} \frac{(C_j^{observed} - C_j^{model})^2}{NO}},$$

$$\text{APE}\,(\%) = \frac{1}{NO \times \bar{C}^{observed}} \sum_{j=1}^{NO} |C_j^{observed} - C_j^{model}| \times 100,$$

$$\text{ARPE}\,(\%) = \frac{1}{NO} \sum_{j=1}^{NO} \frac{|C_j^{observed} - C_j^{model}|}{C_j^{observed}} \times 100,$$

where NO represents the total number of options, and $\bar{C}^{observed}$ is the average option price.

Table 3.5: Performance Comparison

		$r = 0.1/365$		
Maturity	Model	RMSE(%)	APE(%)	ARPE(%)
24	Black-Scholes	4.974	44.888	634.151
	Proposed GARCH	1.381	11.416	47.098
87	Black-Scholes	12.595	69.058	384.369
	Proposed GARCH	3.720	18.117	39.408
115	Black-Scholes	15.838	76.861	247.820
	Proposed GARCH	5.355	23.734	40.751
		$r = 0.03/365$		
Maturity	Model	RMSE(%)	APE(%)	ARPE(%)
24	Black-Scholes	4.054	34.287	530.638
	Proposed GARCH	0.787	5.400	44.337
87	Black-Scholes	8.827	46.455	295.443
	Proposed GARCH	2.636	12.780	43.725
115	Black-Scholes	10.692	50.242	182.952
	Proposed GARCH	3.135	13.394	40.255

The tables, the graph, and the three indicators illustrate the fact that our proposed method outperforms the Black-Scholes for different risk-free interest rates on the short, middle and long term expiry dates.

3.3 Illustration with GARCH Model, DRYS Stock Prices

The summary and descriptive statistics of this series are listed in the following two tables.

Table 3.6: DRYS Data Description

3-Feb-05	17.63	.
4-Feb-05	18.14	0.028517
7-Feb-05	17.89	-0.01388
$\vdots$	$\vdots$	$\vdots$
20-Nov-09	6.29	-0.3745

Table 3.7: DRYS Data Return Series Statistics

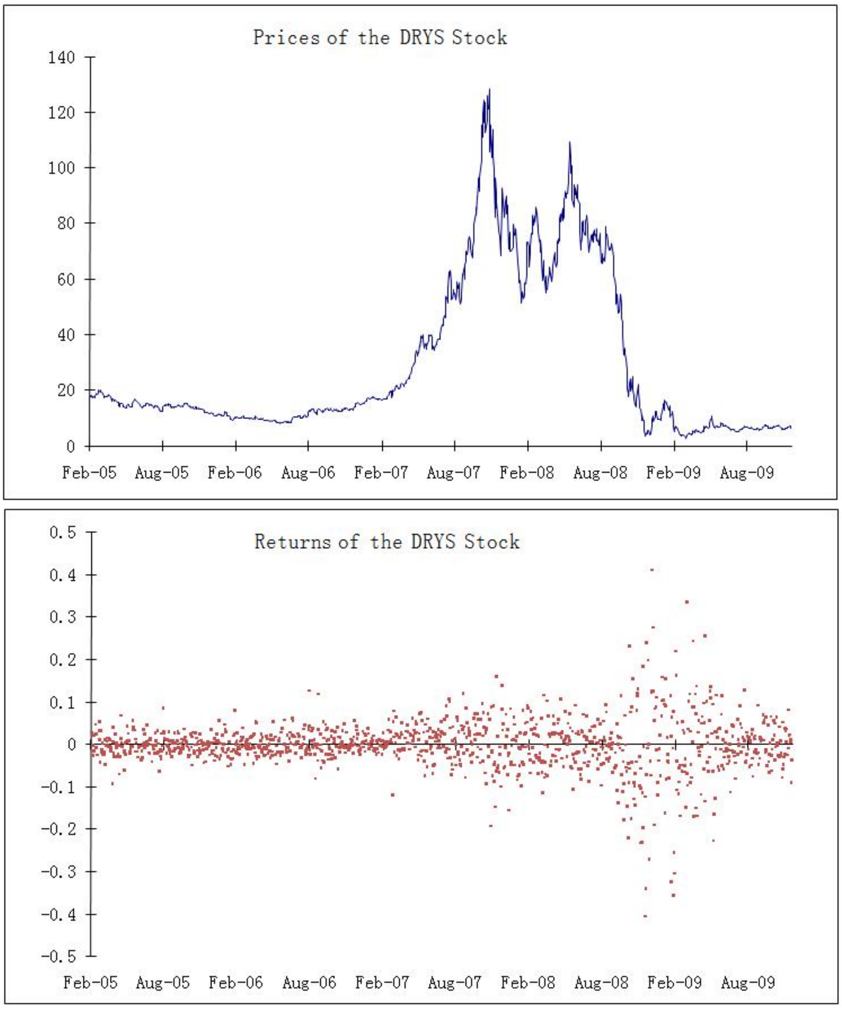
Statistic	Value
Mean	-0.00852474
Standard Deviation	0.060327929
Skewness	-0.403168699
Kurtosis	9.188521645

This data series exhibits high degree of heteroscedasticity. The following figures of the stock prices S_t and the return r_t agree with the statement.

By using maximum likelihood estimation demonstrated in the previous section, the parameter estimates are listed in the table below.

With finding $\hat{E}(\theta_t^2) = \dfrac{\hat{\omega}}{1 - \hat{\alpha} - \hat{\beta}} = 0.003312865$, and kurtosis $\kappa^{(r)} = 9.18852165$, the comparison of call prices from the standard Black-Scholes

Figure 3.3: DRYS Series Distribution



[htbp]

Table 3.8: The MLE for GARCH Model

Parameters	GARCH
μ	0.000821993
ω	1.36569×10^{-5}
α	0.077302356
β	0.918575261

Table 3.9: Comparison of Call Prices from Black-Scholes Formula and our Proposed Modified Black-Scholes Formula

T	S	K	Observed Call Price	Call Price			
				BS	% Error	Proposed	% Error
17	6.29	2	4.5	4.25	6%	4.29	5%
	6.29	3	3.65	3.17	13%	3.29	10%
	6.29	4	2.46	2.17	12%	2.26	8%
	6.29	5	1.32	1.37	-3%	1.32	0%
	6.29	6	0.54	0.77	-42%	0.28	49%
	6.29	7	0.19	0.35	-82%	0.14	26%
	6.29	8	0.07	0.07	0%	0.02	78%
	6.29	9	0.03	0.00	100%	0.00	100%
	6.29	10	0.02	0.00	100%	0.00	100%
	6.29	11	0.02	0.00	100%	0.00	100%
	6.29	12	0.03	0.00	100%	0.00	100%
	6.29	13	0.02	0.00	100%	0.00	100%
	6.29	14	0.05	0.00	100%	0.00	100%
36	6.29	1	5.52	5.26	5%	5.29	4%
	6.29	2.5	4.18	3.62	13%	3.80	9%
	6.29	4	2.53	2.26	11%	2.26	10%
	6.29	5	1.49	1.57	-6%	1.32	11%
	6.29	6	0.82	1.04	-27%	0.29	64%
	6.29	7.5	0.28	0.47	-67%	0.21	24%
	6.29	9	0.1	0.09	6%	0.00	100%
	6.29	10	0.06	0.00	100%	0.00	100%
	6.29	11	0.03	0.00	100%	0.00	100%
	6.29	12.5	0.02	0.00	100%	0.00	100%

formula and our proposed modified Black-Scholes formula is listed in the following table and figure, with two expiry dates.

Note: The values in the table above are truncated for those that are estimated as negative. i.e. For $T = 17$, $S = 6.29$, $K = 14$, the estimated call prices by the Black-Scholes formula and our proposed method are both negative, since the call prices are not negative in practice, they will be truncated to 0.

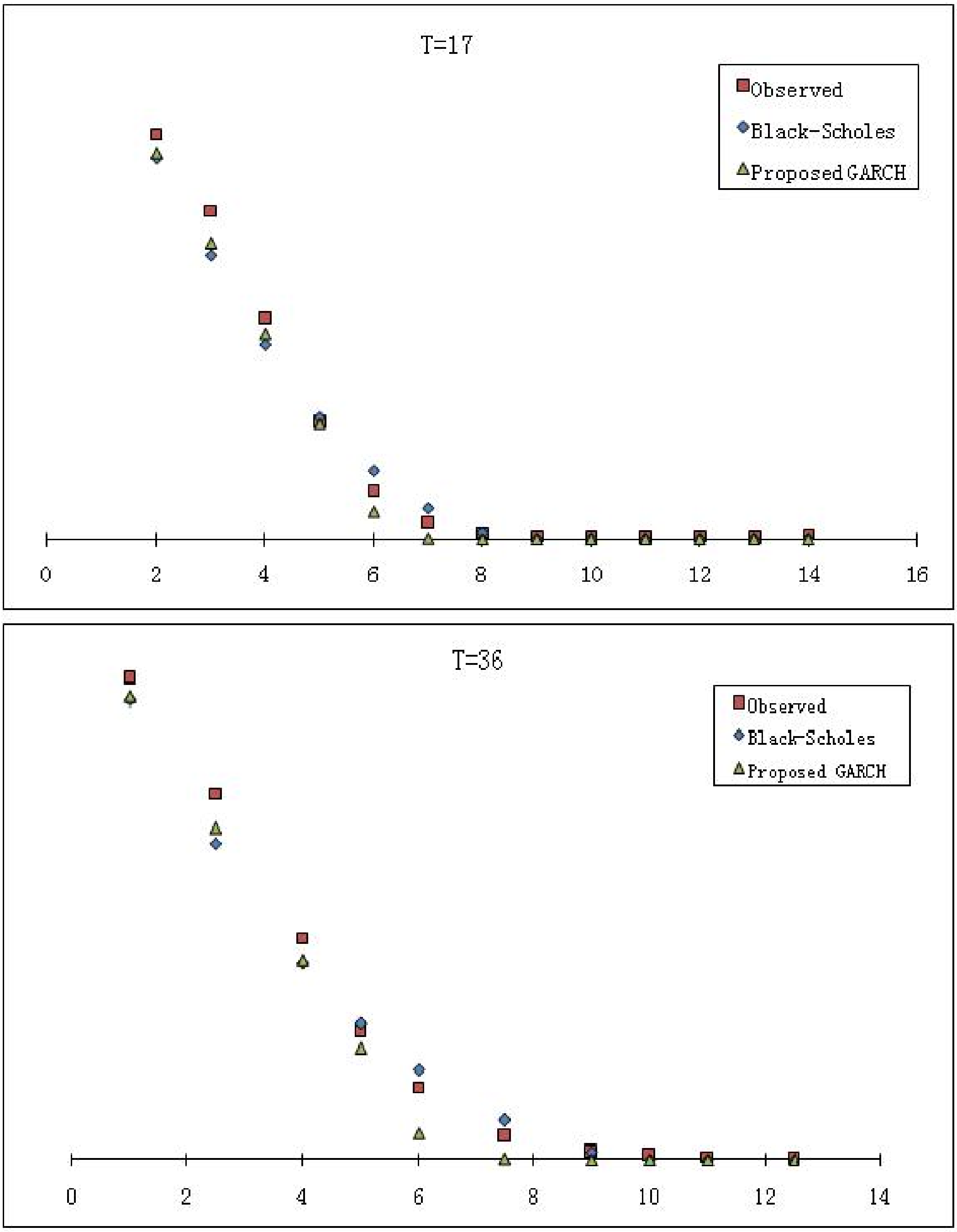
The performance of the two models measured by three indicators: (i) the root mean squared error (RMSE), (ii) the average absolute error (APE) and (iii) the average relative pricing error (ARPE) are listed in the following table.

Table 3.10: Performance Comparison

		$r = .03/365$		
Maturity	Model	RMSE(%)	APE(%)	ARPE(%)
17	BS	0.259	23.095	473.728
	GARCH	0.150	10.124	59.585
36	BS	0.273	15.408	292.391
	GARCH	0.244	12.305	52.383

From Tables and graph above when comparing the call prices with the observed call prices, our proposed modified Black-Scholes formula outperforms the standard Black-Scholes formula.

Figure 3.4: DRYS Option Prices Comparison of GARCH Volatility



CHAPTER 4

Estimating Black-Scholes Call Prices Varying with Stochastic Volatility and Illustration

4.1 Introduction

This chapter assumes stochastic volatility model for the volatility in Black-Scholes formula. Under this assumption, as was done in Chapter 3, we outline the procedure for computing the estimated call prices. Here also we illustrate the same two real financial series for illustration. Again, the call prices resulting from the assumption of stochastic volatility are compared with those of the Black-Scholes formula with constant volatility.

The two series S & P 100 daily index series and DryShips, Inc. (DRYS) stock prices have been illustrated in the Chapter 1 and Chapter 3.

First still recall the progress of a series S_t underlying the Black-Scholes formula is

$$dS_t = rS_t dt + \theta_t S_t d\widetilde{W}_t,$$

where volatility θ_t is the standard deviation of the return series $r_t = \log \frac{S_t}{S_{t-1}}$. We are proposing that the return series r_t is function of the volatility θ_t, that is $r_t = \mu + \theta_t Z_t$, where $Z_t \sim N(0,1)$, and we are proposing that the volatility θ_t follows a Stochastic Volatility process. A general representation of Stochastic Volatility models is,

$$f(\theta_t^2, \delta) = u + \phi[f(\theta_{t-1}^2, \delta) - u] + k(\theta_{t-1}^2)\eta_t$$

In this chapter, for illustrating empirically modified Black-Scholes call prices, we assume that θ_t follow the Anderson (1994) Stochastic Volatility process. That is,

$$\theta_t^2 = u + \phi(\theta_{t-1}^2 - u) + \sigma\eta_t.$$

Moment of Estimation proposed by Dufour, J. and Valery (2005) is applied for the parameter estimation of the Anderson model.

$$u = u_2, \quad \phi = \frac{u_{2,2} - u_2^2}{u_4 - u_2^2}, \quad \sigma^2 = \frac{[u_4 - u_{2,2}][u_4 + u_{2,2} - 2u_2^2]}{u_4 - u_2^2},$$

where the corresponding empirical moments u_2, u_4 and $u_{2,2}$ are

$$\hat{u}_2 = \frac{1}{T}\sum_{i=1}^{T} r_i^2, \quad \hat{u}_4 = \frac{1}{T}\sum_{i=1}^{T} r_i^4, \quad \hat{u}_{2,2} = \frac{1}{T}\sum_{i=1}^{T} r_i^2 r_{i-1}^2.$$

4.2 Illustration with Stochastic Volatility Model, S & P 100 Index Series

The summary and descriptive statistics have been illustrated in the previous chapter. The parameter estimates for the Anderson Stochastic Volatility process are listed in the following table.

The call prices obtained from the standard Black-Scholes formula assume a constant volatility θ, and this parameter is the standard deviation of the residuals series r_t that is 0.012011318. For our proposed modified Black-Scholes

Table 4.1: The Moment Estimate of Anderson Model

Parameters	Anderson
μ	0.000242355
u	0.000144241
ϕ	0.212388388
σ^2	2.0558×10^{-7}

formula, applying $\hat{E}(\theta_t^2) = \hat{\mu} = 0.000144241$, and kurtosis $\hat{\kappa}^{(r)} = 8.357838966$, the expected call prices can be obtained with respect to each initial price S, strike price K and expiry date T. The detail is listed in the following table.

Note: The values in the table above are truncated for those who are estimated as negative. i.e. For $T = 24$, $S = 425.19$, $K = 450$, $r = 0.01/365$, the estimated call price by our proposed method is negative, since the call price is not negative in practice, they will be truncated to 0.

The following graph converts the above tables into a visual representation of the data to show the difference between the observed call prices and the estimated call prices for the two methods. Each panel shows one expiry date T with one risk-free interest rate r. The x-axis is the strike price range K while the y-axis is the call prices C. At each strike price, the observed call price, the call price from the standard Black-Scholes formula and the call price from our proposed modified Black-Scholes formula are compared.

We also use the three indicators as second way to measure the performance:(i) the root mean squared error (RMSE), (ii) the average absolute error (APE) and (iii) the average relative pricing error (ARPE).

The graph shows that the estimated call prices by our proposed modified Black-Scholes formula are much closer to the observed call prices than the standard Black-Scholes formula. While in the above table, the smaller numbers for our proposed modified Black-Scholes formula also indicates that our proposed modified Black-Scholes formula with Anderson Stochastic Volatility process outperforms the standard Black-Scholes formula.

Table 4.2: Comparison of Call Prices from Black-Scholes Formula and our Proposed Modified Black-Scholes Formula with Anderson Stochastic Volatility Process

T	S	K	Observed Call Price	Call Price (r=0.1/365)			
				BS	% Error	Proposed	% Error
24	425.73	395	30.75	32.58	-6%	32.61	6%
	425.73	400	25.88	28.51	-10%	27.88	8%
	425.73	405	21	24.66	-17%	23.14	10%
	425.67	410	16.5	21.02	-27%	18.35	11%
	425.68	415	11.875	17.68	-49%	13.63	15%
	425.65	420	7.69	14.58	-90%	8.86	15%
	425.65	425	4.44	11.76	-165%	4.13	7%
	425.68	430	2.095	9.23	-341%	0.00	100%
	425.65	435	0.78	6.93	-789%	0.00	100%
	425.16	440	0.25	4.71	-1785%	0.00	100%
	424.78	445	0.095	2.84	-2885%	0.00	100%
	425.19	450	0.095	1.47	-1446%	0.00	100%
87	425.73	380	46.75	55.00	-18%	52.97	13%
	425.73	385	42	51.10	-22%	48.32	15%
	425.73	390	37.5	47.33	-26%	43.66	16%
	425.73	395	33	43.69	-32%	39.01	18%
	425.73	400	28.5	40.17	-41%	34.36	21%
	425.73	405	24.13	36.79	-52%	29.70	23%
	425.26	410	20.375	33.22	-63%	24.60	21%
	425.86	415	16.125	30.53	-89%	20.52	27%
	425.68	420	12.815	27.44	-114%	15.70	22%
	425.42	425	9.315	24.44	-162%	10.80	16%
	425.62	430	6.505	21.87	-236%	6.33	3%
	425.82	435	4.505	19.43	-331%	1.87	58%
	425.68	440	2.75	16.93	-515%	0.00	100%
	425.75	445	1.595	14.67	-820%	0.00	100%
	425.78	450	0.845	12.52	-1382%	0.00	100%
	425.39	455	0.44	10.32	-2245%	0.00	100%
115	425.73	380	47.25	58.99	-25%	55.67	18%
	425.73	390	38.13	51.56	-35%	46.44	22%
	425.73	400	29.38	44.58	-52%	37.20	27%
	425.73	410	21.19	38.06	-80%	27.97	32%
	425.41	420	13.875	31.81	-129%	18.43	33%
	425.63	430	8.125	26.38	-225%	9.40	16%
	425.28	440	3.875	21.09	-444%	0.00	100%
	425.13	450	1.5	16.39	-993%	0.00	100%

T	S	K	Observed	Call Price (r=0.03/365)			
			Call Price	BS	% Error	Proposed	% Error
24	425.73	395	30.75	30.82	0%	30.14	2%
	425.73	400	25.88	26.85	-4%	25.61	1%
	425.73	405	21	23.11	-10%	21.09	0%
	425.67	410	16.5	19.58	-19%	16.51	0%
	425.68	415	11.875	16.37	-38%	11.99	1%
	425.65	420	7.69	13.41	-74%	7.44	3%
	425.65	425	4.44	10.73	-142%	2.91	34%
	425.68	430	2.095	8.34	-298%	0.00	100%
	425.65	435	0.78	6.19	-694%	0.00	100%
	425.16	440	0.25	4.13	-1552%	0.00	100%
	424.78	445	0.095	2.40	-2426%	0.00	100%
	425.19	450	0.095	1.15	-1111%	0.00	100%
87	425.73	380	46.75	49.35	-6%	45.49	3%
	425.73	385	42	45.62	-9%	40.98	2%
	425.73	390	37.5	42.02	-12%	36.48	3%
	425.73	395	33	38.57	-17%	31.98	3%
	425.73	400	28.5	35.26	-24%	27.48	4%
	425.73	405	24.13	32.09	-33%	22.98	5%
	425.26	410	20.375	28.77	-41%	18.05	11%
	425.86	415	16.125	26.27	-63%	14.09	13%
	425.68	420	12.815	23.43	-83%	9.43	26%
	425.42	425	9.315	20.70	-122%	4.69	50%
	425.62	430	6.505	18.37	-182%	0.37	94%
	425.82	435	4.505	16.16	-259%	0.00	100%
	425.68	440	2.75	13.92	-406%	0.00	100%
	425.75	445	1.595	11.91	-647%	0.00	100%
	425.78	450	0.845	10.02	-1086%	0.00	100%
	425.39	455	0.44	8.09	-1738%	0.00	100%
115	425.73	380	47.25	51.87	-10%	46.27	2%
	425.73	390	38.13	44.84	-18%	37.29	2%
	425.73	400	29.38	38.31	-30%	28.31	4%
	425.73	410	21.19	32.28	-52%	19.33	9%
	425.41	420	13.875	26.57	-92%	10.05	28%
	425.63	430	8.125	21.68	-167%	1.27	84%
	425.28	440	3.875	16.98	-338%	0.00	100%
	425.13	450	1.5	12.86	-757%	0.00	100%

Figure 4.1: S&P Option Prices Comparison of SV at r=.1/365

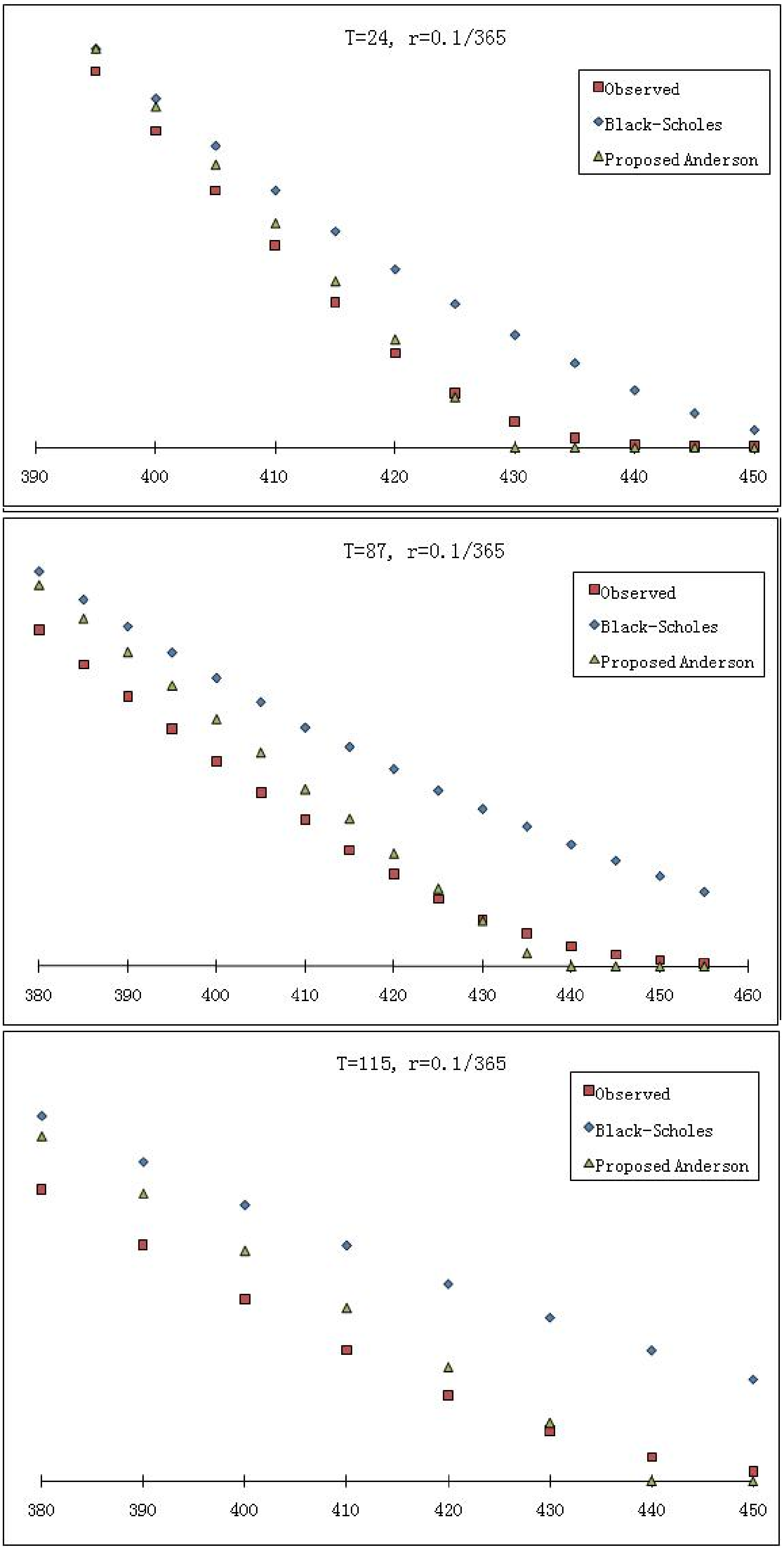


Figure 4.2: S&P Option Prices Comparison of SV at r=.03/365

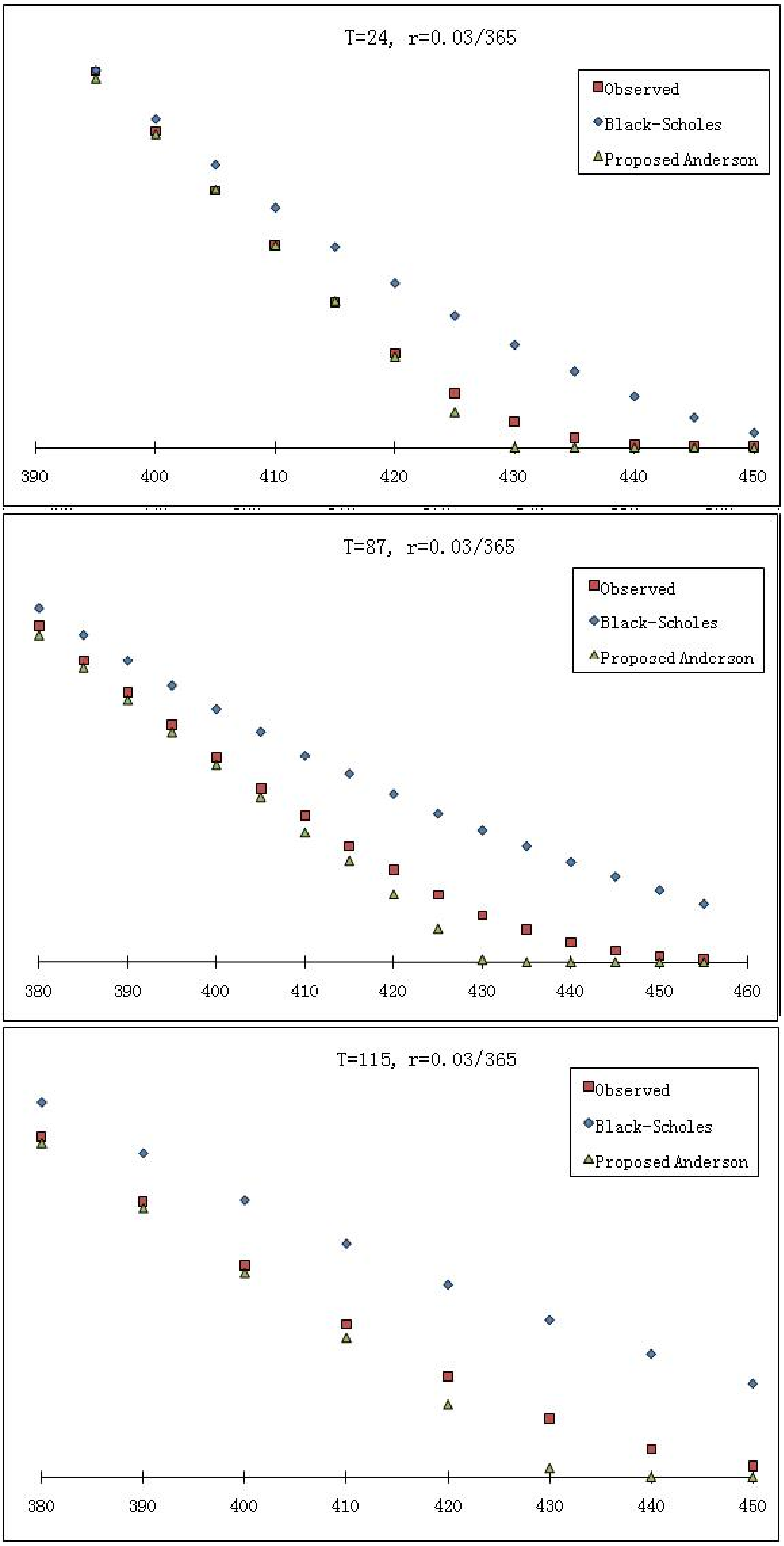


Table 4.3: Performance Comparison

Maturity	Model	RMSE(%)	APE(%)	ARPE(%)
		$r = 0.1/365$		
24	Black-Scholes	4.974	44.888	634.151
	Proposed Anderson	1.446	11.858	47.683
87	Black-Scholes	12.595	69.058	384.369
	Proposed Anderson	4.212	20.048	40.886
115	Black-Scholes	15.838	76.861	247.820
	Proposed Anderson	5.986	26.042	43.341
		$r = 0.03/365$		
Maturity	Model	RMSE(%)	APE(%)	ARPE(%)
24	Black-Scholes	4.054	34.287	530.638
	Proposed Anderson	0.813	5.092	45.174
87	Black-Scholes	8.827	46.455	295.443
	Proposed Anderson	2.720	12.231	44.600
115	Black-Scholes	10.692	50.242	182.952
	Proposed Anderson	3.262	12.739	41.078

4.3 Illustration with Stochastic Volatility Model, DRYS Stock Prices

Recall this series in the previous chapter, the summary and descriptive statistics have been demonstrated. The parameters in the Anderson Stochastic process are estimated and listed in the following table.

Table 4.4: The Moment Estimates of Anderson Model

Parameters	Anderson
μ	-0.000852474
u	0.003636449
ϕ	0.392512262
σ^2	0.00012468

Finding $\hat{E}(\theta_t^2) = \hat{u} = 0.003636449$, and kurtosis $\kappa^{(r)} = 9.18852165$, the comparison of call prices from the standard Black-Scholes formula and our proposed modified Black-Scholes formula is listed in the following table and figure, with two expiry dates. A visual representation of the table is provided too.

Table 4.5: Comparison of Black-Scholes Call Prices and our Proposed Method Call Prices based on Anderson Model

T	S	K	Observed	Call Price (r=0.03/365)			
			Call Price	BS	% Error	Proposed	% Error
17	6.29	2	4.5	4.25	6%	4.29	5%
	6.29	3	3.65	3.17	13%	3.29	10%
	6.29	4	2.46	2.17	12%	2.24	9%
	6.29	5	1.32	1.37	-3%	1.35	3%
	6.29	6	0.54	0.77	-42%	0.27	50%
	6.29	7	0.19	0.35	-82%	0.13	30%
	6.29	8	0.07	0.07	0%	0.06	20%
	6.29	9	0.03	0.00	100%	0.00	100%
	6.29	10	0.02	0.00	100%	0.00	100%
	6.29	11	0.02	0.00	100%	0.00	100%
	6.29	12	0.03	0.00	100%	0.00	100%
	6.29	13	0.02	0.00	100%	0.00	100%
	6.29	14	0.05	0.00	100%	0.00	100%
36	6.29	1	5.52	5.26	5%	5.29	4%
	6.29	2.5	4.18	3.62	13%	3.80	9%
	6.29	4	2.53	2.26	11%	2.25	11%
	6.29	5	1.49	1.57	-6%	1.35	9%
	6.29	6	0.82	1.04	-27%	0.29	65%
	6.29	7.5	0.28	0.47	-67%	0.23	18%
	6.29	9	0.1	0.09	6%	0.00	100%
	6.29	10	0.06	0.00	100%	0.00	100%
	6.29	11	0.03	0.00	100%	0.00	100%
	6.29	12.5	0.02	0.00	100%	0.00	100%

Note: The values in the table above are truncated for those who are estimated as negative. i.e. For $T = 17$, $S = 6.29$, $K = 14$, the estimated call prices by the Black-Scholes formula and our proposed method are both negative, since the call prices are not negative in practice, they will be truncated to 0.

The three indicators are still served to measure the performance:(i) the root mean squared error (RMSE), (ii) the average absolute error (APE) and (iii) the average relative pricing error (ARPE).

Figure 4.3: DRYS Option Prices Comparison of SV Volatility

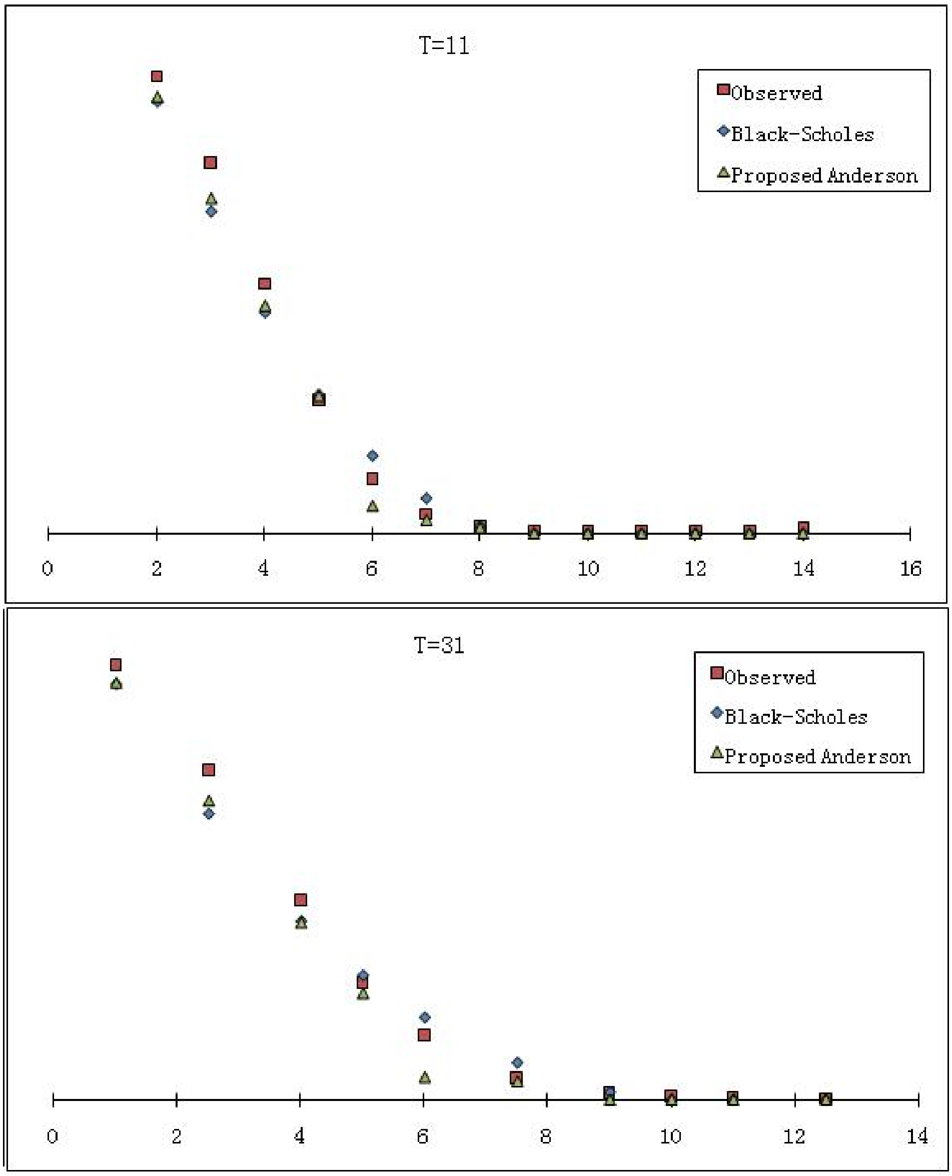


Table 4.6: Performance Comparison

		$r = .03/365$		
Maturity	Model	RMSE(%)	APE(%)	ARPE(%)
17	BS	0.259	23.059	473.728
	Anderson	0.152	10.286	55.782
36	BS	0.273	15.408	292.391
	Anderson	0.244	12.114	51.618

In most cases, the call prices from our proposed modified Black-Scholes are closer to the observed call prices. The graph provides a visual exhibition of this closeness. The performance comparison table offers an alternative to measure the performance between the standard Black-Scholes formula and our proposed modified Black-Scholes formula and the smaller number also verifies our conclusion that our proposed modified Black-Scholes formula outperforms the traditional Black-Scholes formula.

CHAPTER 5

Estimating Log Returns Recursively with Stochastic Volatility Models

5.1 Introduction

An alternative to the Black-Scholes formula for estimating the call prices is a Monte Carlo simulation proposed by Duan (1995) of the log returns $r_t = \log \dfrac{S_t}{S_{t-1}}$ from assets S_t. He assumed that the residuals volatility θ_t has the following GARCH (1,1) model,

$$\begin{aligned} r_t &= r - \frac{1}{2}\theta_t^2 + \theta_t Z_t, \\ \theta_t^2 &= \omega + \alpha\theta_{t-1}^2(Z_{t-1} + \lambda)^2 + \beta\theta_{t-1}^2, \end{aligned}$$

where $\omega > 0$, $\alpha \geq 0$ and $\beta \geq 0$. In the model Z_t are i.i.d. normal independent of θ_t and r is the rate of return per period from a risk free investment. In simulating the log returns r_t from the above model, Duan assumed the market price of the risk, denoted in the model by λ, is constant. Having estimated of ω, α and β by historical log returns r_t data, $1 \leq t \leq s$, the asset price at expiry time T can be simulated by,

$$S_T = S_s \exp(\sum_{t=s+1}^{T} r_t),$$

where S_s is the last historical asset price. Having simulated N asset prices $S_{i,T}$, $i = 1, \ldots, N$, then for specified strike price K, the call price is estimated by,

$$\frac{e^{-rT}}{N} \sum_{i=1}^{N} \max(S_{i,T} - K, 0).$$

This chapter lays the foundation for Monte Carlo simulation of the log returns when their volatility follows a stochastic volatility model instead of a GARCH model. It is needed because the stochastic volatility models differ from the GARCH models in that they incorporate two random processes to model the volatility of the log returns. We propose two tracks for simulating the log returns recursively. One approach uses a transformation to facilitate the conditional expectation of volatility θ_t to estimate it recursively to be fed into the expression for the log returns. It is based on an important assumption that the two random components in a stochastic volatility model are normally distributed. The second approach does not assume that the two random components are normally distributed. Instead it uses the minimum mean squared error for recursively simulating the log returns.

5.2 Estimating Log Returns Recursively: Transformation Approach

Modeling of the log returns by stochastic volatility models incorporates two random components. They assume the log returns to depend on a random component Z_t as well as their volatility parameter θ_t which, in turn, depends on yet another random component η_t and possibly on previous volatilities. This section outlines a transformation which will allow the conditional expectation of volatility θ_t to be expressed recursively for iteratively simulating it into the equation for the log returns to estimate them. The transformation rests on the assumption that both random components Z_t and η_t are normally distributed.

To pursue this approach consider the following class of stochastic volatility models.

$$\begin{aligned} r_t &= A\theta_t^2 + K(\theta_t^2)Z_t + C \\ \theta_t^2 &= a\theta_{t-1}^2 + k(\theta_{t-1}^2)\eta_t + c \end{aligned} \tag{5.1}$$

where $Z_t \sim N(0, \sigma_Z^2)$, $\eta_t \sim N(0, \sigma_\eta^2)$, and $\text{Cov}(\eta_t, Z_t) = \rho\sigma_\eta\sigma_Z$. For the bivariate normal distribution (Z_t, η_t), it is true that conditional expectation and variance of η_t given Z_t fixed are,

$$E[\eta_t|Z_t] = E(\eta_t) + \frac{\rho\sigma_\eta}{\sigma_Z}(Z_t - E(Z_t)), \quad \text{and}\,\text{Var}(\eta_t|Z_t) = \sigma_\eta^2(1-\rho^2), \quad \text{respectively.}$$

Conditionally, given θ_{t-1}, both r_t and θ_t^2 are normally distributed. Since $\text{Var}(r_t)$ is a function of θ_t^2, the above result for conditional mean and variance is not applicable. However, the following transformation due to Shiryaev (1995) makes it possible to get around this problem.

$$\tilde{Z}_t = \frac{K(\theta_t^2)}{\sqrt{E[K(\theta_t^2)]^2}} Z_t, \qquad \tilde{\eta}_t = \frac{k(\theta_{t-1}^2)}{\sqrt{E[k(\theta_{t-1}^2)]^2}} \eta_t,$$

Now the stochastic volatility model (5.1) can be rewritten as:

$$\begin{aligned} r_t &= A\theta_t^2 + B\tilde{Z}_t + C \\ \theta_t^2 &= a\theta_{t-1}^2 + b\tilde{\eta}_t + c \end{aligned} \tag{5.2}$$

where $B = \sqrt{E[K(\theta_t^2)]^2}$, $b = \sqrt{E[k(\theta_{t-1}^2)]^2}$, $E(\tilde{Z}_t) = 0$, $E(\tilde{Z}_t\tilde{Z}_{t+1}) = 0$, $E(\tilde{Z}_t^2) = \sigma_Z^2$, $E(\tilde{\eta}_t) = 0$, $E(\tilde{\eta}_t\tilde{\eta}_{t+1}) = 0$, $E(\tilde{\eta}_t^2) = \sigma_\eta^2$, and $\text{cov}(\tilde{Z}_t, \tilde{\eta}_t) = bB\rho\sigma_Z\sigma_\eta$.

Given the information of previous log returns F_t^r, the differences between r_{t-1}, θ_t^2 and their expectations are

$$\begin{aligned} r_{t-1} - E[r_{t-1}|F^r_{t-1}] &= A(\theta^2_{t-1} - \hat{\theta}^2_{t-1}) + B\tilde{Z}_{t-1}, \\ \theta^2_t - E[\theta^2_t|F^r_t] &= a(\theta^2_{t-1} - \hat{\theta}^2_{t-1}) + b\tilde{\eta}_t. \end{aligned}$$

And the variances and covariance of the two differences above are

$$\begin{aligned} \mathrm{Var}(r_{t-1} - E[r_{t-1}|F^r_{t-1}]) &= E[A(\theta^2_{t-1} - \hat{\theta}^2_{t-1}) + B\tilde{Z}_{t-1}]^2 \\ &= E[A^2(\theta^2_{t-1} - \hat{\theta}^2_{t-1})^2 + B^2\tilde{Z}^2_{t-1}] \\ &= A^2\gamma_{t-1} + B^2\sigma^2_Z, \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(\theta^2_t - E[\theta^2_t|F^r_t]) &= E[a(\theta^2_{t-1} - \hat{\theta}^2_{t-1}) + b\tilde{\eta}_t]^2 \\ &= E[a^2(\theta^2_{t-1} - \hat{\theta}^2_{t-1})^2 + b^2\tilde{\eta}^2_t] \\ &= a^2\gamma_{t-1} + b^2\sigma^2_\eta, \end{aligned}$$

$$\begin{aligned} &\mathrm{Cov}(r_{t-1} - E[r_{t-1}|F^r_{t-1}]; [\theta^2_t - E[\theta^2_t|F^r_t]]) \\ &\quad = E\{[a(\theta^2_{t-1} - \hat{\theta}^2_{t-1}) + b\tilde{\eta}_t][A(\theta^2_{t-1} - \hat{\theta}^2_{t-1}) + B\tilde{Z}_{t-1}]\} \\ &\quad = E[aA(\theta^2_{t-1} - \hat{\theta}^2_{t-1})^2 + bB\tilde{Z}_{t-1}\tilde{\eta}_t] \\ &\quad = aA\gamma_{t-1} + bB\rho\sigma_Z\sigma_\eta. \end{aligned}$$

Now using the well known result for the bivariate normal distribution, we can write,

$$\hat{\theta}^2_t = E(\theta^2_t|F^r_t) = a\hat{\theta}^2_{t-1} + c + \left(\frac{aA\gamma_{t-1} + bB\rho\sigma_Z\sigma_\eta}{A^2\gamma_{t-1} + B^2\sigma^2_Z}\right)[r_{t-1} - A\hat{\theta}^2_{t-1} - C],$$

$$\gamma_t = E[(\theta^2_t - \hat{\theta}^2_t)^2|F^r_t] = a^2\gamma_{t-1} + b^2\sigma^2_\eta - \frac{(aA\gamma_{t-1} + bB\rho\sigma_Z\sigma_\eta)^2}{A^2\gamma_{t-1} + B^2\sigma^2_Z}. \tag{5.3}$$

The parameters A, a, C, and c can be estimated by using the Moment of Estimation proposed by Dufour, J. and Valery (2005), which was introduced in the Chapter 1, based on the historical time series data.

To illustrate, consider the Stochastic Volatility model given by Shiryaev (1995)

$$\begin{aligned} r_t &= A\theta_t^2 + Z_t \\ \theta_t^2 &= a\theta_{t-1}^2 + (1+\theta_{t-1}^2)\eta_t \end{aligned}$$

where $Z_t \sim N(0, \sigma_Z^2)$ and $\eta_t \sim N(0, \sigma_\eta^2)$, and Z_t and η_t are independent. Using the transformation $\tilde{\eta}_t = \dfrac{1+\theta_{t-1}^2}{\sqrt{E(1+\theta_{t-1}^2)^2}}\eta_t$, we have

$$\begin{aligned} r_t &= A\theta_t^2 + Z_t \\ \theta_t^2 &= a\theta_{t-1}^2 + b\tilde{\eta}_t \end{aligned}$$

where, $b = \sqrt{E(1+\theta_{t-1}^2)^2} = \sqrt{\dfrac{1-a^2}{1-a^2-\sigma_\eta^2}}$. Applying (5.3), the recursive estimate is obtained as

$$\begin{aligned} \hat{\theta}_t^2 &= a\hat{\theta}_{t-1}^2 + \frac{Aa\gamma_{t-1}}{A^2\gamma_{t-1}+\sigma_Z^2}(r_{t-1} - A\hat{\theta}_{t-1}^2) \\ \gamma_t &= \frac{a^2\sigma_Z^2\gamma_{t-1}}{A^2\gamma_{t-1}+\sigma_Z^2} + \Big(\frac{1-a^2}{1-a^2-\sigma_\eta^2}\Big)\sigma_\eta^2. \end{aligned}$$

5.3 Estimating Log Returns Recursively: Mean Squared Error Approach

The recursive estimate constructed by the transformation approach above is based on two important assumptions: $Z_t \overset{iid}{\sim} N(0, \sigma_Z^2)$, $\eta_t \overset{iid}{\sim} N(0, \sigma_\eta^2)$. If the random terms Z_t and η_t are not normally distributed then we cannot treat them as bivariate normal distribution and cannot interpret the recursive estimate as a conditional mean.

Hence as an alternative we propose an optimal MSE (minimum mean square error) approach to construct a recursive estimate. Recall the system (5.2),

$$\begin{aligned} r_t &= A\theta_t^2 + B\tilde{Z}_t + C \\ \theta_t^2 &= a\theta_{t-1}^2 + b\tilde{\eta}_t + c. \end{aligned}$$

From the recursive estimate by the transformation approach, we notice that $\hat{\theta}_t^2$ is a function of $r_{t-1} - A\hat{\theta}_{t-1}^2$, so we assume that the estimate of θ_t^2 is of the form

$$\hat{\theta}_t^2 = a\hat{\theta}_{t-1}^2 + \delta(r_{t-1} - A\hat{\theta}_{t-1}^2). \tag{5.4}$$

By minimizing the mean square error $\gamma_t = E[(\theta_t^2 - \hat{\theta}_t^2)^2 | F_t^r]$ with respect to δ, an optimal estimate of δ can be obtained. Thus $\hat{\theta}_t^2$ and γ_t are achieved.

To illustrate, reconsider the Stochastic Volatility model given by Shiryaev (1995)

$$\begin{aligned} r_t &= A\theta_t^2 + Z_t & (5.5)\\ \theta_t^2 &= a\theta_{t-1}^2 + (1+\theta_{t-1}^2)\eta_t & (5.6) \end{aligned}$$

where Z_t and η_t are still independent with mean 0 and variance σ_Z^2 and σ_η^2 respectively, but they are no longer needed to be normally distributed. The difference between (5.6) and (5.4) are

$$\theta_t^2 - \hat{\theta}_t^2 = a(\theta_{t-1}^2 - \hat{\theta}_{t-1}^2) - \delta(r_{t-1} - A\hat{\theta}_{t-1}^2) + (1+\theta_{t-1}^2)\eta_t.$$

Squaring this difference, we have

$$\begin{aligned} (\theta_t^2 - \hat{\theta}_t^2)^2 &= a^2(\theta_{t-1}^2 - \hat{\theta}_{t-1}^2)^2 + \delta^2[A^2(\theta_{t-1}^2 - \hat{\theta}_{t-1}^2)^2 + 2A(\theta_{t-1}^2 - \hat{\theta}_{t-1}^2)Z_t + Z_t^2] \\ &\quad + (1 + 2\theta_{t-1}^2 + \theta_{t-1}^4)\eta_t^2 - 2a\delta[A(\theta_{t-1}^2 - \hat{\theta}_{t-1}^2)^2 + (\theta_{t-1}^2 - \hat{\theta}_{t-1}^2)Z_t] \\ &\quad + 2a(\theta_{t-1}^2 - \hat{\theta}_{t-1}^2)(1+\theta_{t-1}^2)\eta_t - 2\delta(1+\theta_{t-1}^2)\eta_t[A(\theta_{t-1}^2 - \hat{\theta}_{t-1}^2) + Z_t] \end{aligned}$$

After taking expectation to this squared difference, we obtain

$$\gamma_t = E[(\theta_t^2 - \hat{\theta}_t^2)^2 | F_t^r] = (a - A\delta)^2\gamma_{t-1} + \left(\frac{1-a^2}{1-a^2-\sigma_\eta^2}\right)\sigma_\eta^2 + \sigma_Z^2\delta^2, \tag{5.7}$$

where $E(\theta_{t-1}^2) = 0$ and $E(\theta_{t-1}^4) = \dfrac{\sigma_\eta^2}{1-a^2-\sigma_\eta^2}$. Differentiating γ_t with respect to δ and setting the derivative function to zero, that is,

$$\frac{\partial \gamma_t}{\partial \delta} = -2A(a - A\delta)\gamma_{t-1} + 2\sigma_Z^2\delta = 0,$$

we have $\hat{\delta} = \dfrac{aA\gamma_{t-1}}{A^2\gamma_{t-1}+\sigma_Z^2}$. Considering that $\dfrac{\partial^2\gamma_t}{\partial\delta^2} = 2A^2\gamma_{t-1} + 2\sigma_Z^2$ is always positive, γ_t attains its minimum value at $\hat{\delta}$. After plugging this optimal $\hat{\delta}$ into (5.4) and (5.7), the recursive estimates are obtained

$$\begin{aligned}
\hat{\theta}_t^2 &= a\hat{\theta}_{t-1}^2 + \frac{Aa\gamma_{t-1}}{A^2\gamma_{t-1}+\sigma_Z^2}(r_{t-1} - A\hat{\theta}_{t-1}^2) \\
\gamma_t &= \frac{a^2\sigma_Z^2\gamma_{t-1}}{A^2\gamma_{t-1}+\sigma_Z^2} + \Big(\frac{1-a^2}{1-a^2-\sigma_\eta^2}\Big)\sigma_\eta^2.
\end{aligned}$$

Note. It is of interest to note that the optimal linear estimates of $\hat{\theta}_t^2$ and γ_t turn out to be the exact same as the transformation approach. However, we do not make any assumptions regarding the distributions of Z_t and η_t when estimating the optimal values for the parameters.

The recursive estimate through either approach makes stochastic volatility models only depend on the random component Z_t and does not include the random component η_t in the equations. Thus the log returns and volatilities can be updated recursively through the recursive estimate.

For a specific stochastic volatility model, its parameters can be estimated by the method of moments introduced in the Chapter 1 and one specific model was introduced in Chapter 4, through the historical log returns data $r_1, \ldots, r_s$. Then if setting the last historical log return r_s and the variance of the historical log returns $\mathrm{Var}(r_t)$, that is θ_s^2, as the start point, the future log return r_{s+1} and future variance θ_{s+1} can be updated through the recursive estimate. The newly obtained r_{s+1} and θ_{s+1} can be used to update the next log return r_{s+2} and variance θ_{s+2}. Repeating the process till the expiry date T, the asset price at time T can be simulated by $S_T = S\exp(r_{s+1} + \ldots + r_T)$. With N simulations, we obtain N asset prices $S_{i,T}$ at expiry T where $1 \leq i \leq N$, such as $S_{1,T} = S\exp(r_{1,s+1} + \ldots + r_{1,T}), \ldots, S_{N,T} = S\exp(r_{N,s+1} + \ldots + r_{N,T})$. Then the call price at expiry time T can be estimated by

$$\frac{e^{-rT}}{N}\sum_{i=1}^{N}\max(S_{i,T} - K, 0).$$

CHAPTER 6

Conclusion and Future Research

6.1 Conclusion

This dissertation focused on the estimation of call option prices allowing market volatility to be random varying with time. Chapter 1 introduced the notations and the definitions of the returns from an investment at different time points. Since the returns from most investments are subject to risk factors, their statistical modeling is reviewed briefly. It is shown empirically that the conditional distributions of the returns depend on the time varying random volatility parameters. Their unconditional distribution has heavy tails due to outliers in the returns from periods of significant volatility. For incorporating the random volatility aspect of the returns in option pricing, the GARCH and Stochastic Volatility models are reviewed in Chapter 1.

Chapter 2 sketched a derivation of the Black-Scholes formula for call option pricing allowing the random volatility in a Geometric Brownian Motion to describe the returns. For pricing the options by the Black-Scholes formula, we proposed its expectation with respect to the volatility distribution. Since an exact expression of the expectation for pricing the options is intractable, an approximation is proposed. The approximate expression incorporates not only random volatility but also kurtosis information of the returns. Chapter 3 and 4 are then respectively devoted to computing the approximate expression for option prices when the random volatility is assumed to follow a GARCH model and a Stochastic Volatility models. These option prices are computed with those of the Black-Scholes formula for two finance series.

Chapter 5 outlines a Monte Carlo simulation method for estimating option prices when random volatility is assumed to have a Stochastic Volatility model. Our procedure is outlined along the lines of one in the literature when the volatility is defined by a GARCH model.

6.2 Future Research

This dissertation used GARCH and Stochastic Volatility models for incorporating random volatility in option pricing. In this dissertation, how we select one volatility model over the other one was not explained. The comparison of variances of estimated call price from each volatility model would be appropriate and promising. In future research, this aspect is worth further investigation. Moreover, since the Stochastic Volatility models don't have a generalized form, in this dissertation, the selection of Anderson Model as an illustration was arbitrary. In the future, we also intend to discover some tool or method to pick up a specific Stochastic Volatility model for obtained data.

Besides those, both these families are discreet time models. In future research we intend to explore the possibility of letting the volatility to have continuous time models. Also, we intend to pursue Monte Carlo simulation procedure as outlined in Chapter 5 for estimating option prices. Obviously, other research avenues will be pursued as we will be increasingly aware of the research in making this area.

Meanwhile, given the nature of financial time series data, we plan to extend our method to different time windows and to discover the impact from long term volatility memory to our method.

Finally, the results from this dissertation are promising and inspiring, so in the future we have the plan to apply our method to more real financial time series data and hope to create potential increase of profit.

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