STOCHASTIC HOMOGENIZATION OF NONCONVEX HAMILTON-JACOBI EQUATIONS IN ONE DIMENSION

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ABSTRACT

Hamilton-Jacobi equations are a class of partial differential equations that arise in many areas of science and engineering. Originating from classical mechanics, they are widely used in various fields such as optimal control theory, quantitative finance, and game theory.

Stochastic homogenization is a phenomenon used to study the behavior of solutions to partial differential equations in stationary ergodic media, aiming to understand how these solutions average out or ’homogenize’ over large scales. This process results in effective deterministic descriptions, called effective Hamiltonians, which capture the essential behavior of the system.

We consider nonconvex Hamilton-Jacobi equations in one space dimension. We provide a fully constructive proof of homogenization, which yields a formula for the effective Hamiltonian. Our proof employs sublinear correctors, functions extensively discussed in the literature. The proof involves strong induction: we first show homogenization for our base cases, then use gluing processes to generalize the solution for the strong induction. Finally, we extend the result to a wide class of functions. We study the properties of the resulting effective Hamiltonian and investigate the occurrence of flat pieces. Additionally, we develop a Python-based computational tool that performs the same homogenization steps in a computing environment, returning the effective Hamiltonian along with its graph and properties.

To Zeynep…

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CHAPTER 1

INTRODUCTION

Stochastic homogenization of Hamilton-Jacobi (HJ) equations is a topic of great interest in the intersection of probability and analysis. HJ equations are partial differential equations that arise in many areas of science and engineering, including classical mechanics, fluid dynamics, control theory, game theory, and quantitative finance (see below for some references). There are great expositions on HJ equations written by Bardi and Capuzzo-Dolcetta [3], Evans [8, Chapter 10], Lions [16], and Tran [21].

The general form of the HJ equation that we will consider is as follows:

$$\frac{\partial u}{\partial t} = H(Du, x), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d, \quad d \geq 1, \tag{1.1}$$

where $H : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is a function, called the Hamiltonian. The first instances of such equations appeared in Hamilton's work on geometrical optics [11] and Jacobi's work on classical mechanics [15].

Generally, these kind of equations appear in optimal control theory (see, e.g., [4, 7]) when the Hamiltonian H is convex in p. In this setting, HJ equations are called Bellman equations, and they play pivotal role in determining optimal strategies and trajectories for control systems. In order to encounter more general, nonconvex Hamiltonians, one needs to work with two-person zero-sum differential games [14]. In this setting, HJ equations are referred to as Isaacs equations. These games involve a conflict between two players, where one tries to maximize the value of a certain function, called the payoff function, while the other attempts to minimize it.

Many microscopic models lead to partial differential equations with rapidly oscillating coefficients. To capture these oscillations, we introduce a small parameter $\epsilon > 0$ to the general form (1.1):

$$\frac{\partial u^{\epsilon}}{\partial t} = H(Du^{\epsilon}, \frac{x}{\epsilon}, \omega), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d, \tag{1.2}$$

where ω is an element of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As $\epsilon \to 0$, we anticipate that the presence of the factor $\frac{x}{\epsilon}$ results in an averaging effect and the equation in (1.2) effectively simplifies, or homogenizes, to a HJ equation of the form

$$\frac{\partial \overline{u}}{\partial t} = \overline{H}(D\overline{u}), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d, \tag{1.3}$$

under certain assumptions (see Section 2.1 for precise definition). The function $\overline{H}$ is called the effective Hamiltonian.

The first homogenization result for the HJ equation in (1.2) was established by Lions, Papanicolaou, and Varadhan [17], focusing on the special case of periodic media. Later, Souganidis [20], and independently Rezakhanlou and Tarver [19], extended this result to convex Hamiltonians. These contributions represent classical advancements in the field of homogenization. We focus on exploring the more general case of nonconvex Hamiltonians without any periodicity assumption. In the following sections, we provide an in-depth overview of the literature concerning the homogenization of HJ equations (see Section 2.2).

Afterwards, our attention turns toward the specific framework of our study which is nonconvex Hamiltonians. Although there are counter-examples of homogenization in higher dimensions, see [24, 9], things are much better understood in one dimension. The first homogenization result for nonconvex Hamiltonians in the stationary and ergodic setting is due to Armstrong, Tran and Yu [1] who considered separated Hamiltonians of the form

$$H(p, x, \omega) = G(p) + V(x, \omega)$$

where $V(\cdot, \omega)$ is a bounded and uniformly continuous potential, and a special choice of $G(p) = (|p|^2 - 1)^2$. Later, in [2], they generalized this result to any continuous and coercive function G in one space dimension.

The results of Armstrong, Tran and Yu [2] do not provide much information regarding the effective Hamiltonian $\overline{H}$. Yilmaz [23] gave a new and constructive proof of homogenization which yields a formula for the effective Hamiltonian $\overline{H}$ under the extra assumption that G is a double-well function (i.e., it has precisely two local minima). They also used this formula to characterize all of the flat pieces of the graph of $\overline{H}$.

In this dissertation, our aim is to provide a constructive proof of homogenization in one space dimension without assuming that G is a double-well function. Our goal is to identify the effective Hamiltonian and characterize its properties. The proof of homogenization will consist of several stages. We will also develop a code that replicates these stages numerically and graphs the effective Hamiltonian. In Section 2.3, we will present the main finding of this dissertation and highlight any distinctions between our research and previous studies in the field.

In the subsequent section, we will present our proof strategy and introduce the subclasses and auxiliary theorems essential for establishing the main theorem. We define two subclasses of functions G: the first is a simplified version of the that we are trying to prove homogenization for, while the second is tailored specifically to align with our proof strategy, satisfying certain assumptions. We will show that these assumptions imply homogenization. Using strong induction and gluing procedures, we will prove that the first class also satisfies these assumptions, thus achieving homogenization. Finally, we will extend the solution to the most general class (see Section 2.4 for more details).

As a final step, our focus shifts towards exploring the practical applications of our dissertation's findings. Following the methodology laid out in the proof, we will employ a Python code to replicate the process including the strong induction and subclass distinctions which will generate the effective Hamiltonian. We will further enhance our explanation by providing illustrative examples to accompany our code (see Section 2.5).

CHAPTER 2

STATEMENTS AND RESULTS

2.1 Notation and Standing Assumptions

Throughout the dissertation, we will denote by $\mathrm{UC}(X), \mathrm{BUC}(X)$, $\mathrm{Lip}(X)$, $\mathrm{USC}(X)$, and $\mathrm{LSC}(X)$ the sets of functions on X that are uniformly continuous, bounded and uniformly continuous, Lipschitz continuous, upper semicontinuous, and lower semicontinuous functions on the metric space X, respectively.

Fix $d \geq 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a group of measure-preserving transformations $\tau_x : \Omega \to \Omega, x \in \mathbb{R}^d$, such that $\tau_0 = \mathrm{id}$ and $\tau_x \circ \tau_y = \tau_{x+y}$ for every $x, y \in \mathbb{R}^d$. Assume that $\mathbb{P}$ is ergodic under this group, i.e., if $A \in \mathcal{F}$ satisfies $\cap_{x \in \mathbb{R}^d} \tau_x A = A$, then $\mathbb{P}(A) \in \{0, 1\}$. We will use $\mathbb{E}$ to denote expectation with respect to $\mathbb{P}$. For every $\epsilon > 0$ and $\omega \in \Omega$, consider the HJ equation

$$\partial_t u^\epsilon(t, x, \omega) + H(\nabla_x u^\epsilon(t, x, \omega), x/\epsilon, \omega) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d, \tag{2.1}$$

where the function $H : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to [0, +\infty)$ is measurable and it satisfies the following properties:

$$(p, x) \mapsto H(p, x, \omega) \text{ is in BUC}(K \times \mathbb{R}^d), \tag{2.2}$$

for every bounded $K \subset \mathbb{R}^d$ and $\omega \in \Omega$,

$$\lim_{|p| \to \pm+\infty} H(p, x, \omega) = +\infty \tag{2.3}$$

uniformly in $(x, \omega) \in \mathbb{R}^d \times \Omega$, and

$$H(p, x, \omega) = H(p, 0, \tau_x \omega) \tag{2.4}$$

for every $(p, x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega$.

Since we are in a stationary and ergodic setting, as $\epsilon \to 0$, we anticipate that the presence of the factor $\frac{x}{\epsilon}$ results in an averaging effect and the equation in (2.1) effectively simplifies to the following form:

$$\frac{\partial \overline{u}}{\partial t} + \overline{H}(D\overline{u}) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d. \tag{2.5}$$

If this phenomenon indeed holds, it is called homogenization. Precisely, the HJ equation in (2.1) is said to homogenize to the one in (2.5) if, for $\mathbb{P}$-a.e. ω and for all $g \in \text{UC}(\mathbb{R}^d)$, the unique viscosity solution (will be defined in Chapter 3) of (2.1) converges locally uniformly to the unique viscosity solution of (2.5) as $\epsilon \to 0$. The function $\overline{H} : \mathbb{R}^d \to \mathbb{R}$ is referred to as the effective Hamiltonian.

2.2 Previous results

Homogenization of HJ equations has received much attention in the last three decades. The first results were obtained by Lions, Papanicolaou and Varadhan

[17]. They focused on the case where $x = (x_1, x_2, \dots, x_d) \mapsto H(p, x)$ is 1-periodic in x_i for every $i \in \{1, 2, \dots, d\}$ and used the compactness of $\mathbb{T}^d$ to solve the following auxiliary problem which is called the cell problem: for every $\theta \in \mathbb{R}^d$, find a $\lambda = \lambda(\theta) \in \mathbb{R}$ and a periodic function $h = h_\theta : \mathbb{T}^d \to \mathbb{R}$ such that

$$H(\theta + Dh(x), x) = \lambda, \quad x \in \mathbb{T}^d, \tag{2.6}$$

in the viscosity sense. The solutions of this cell problem are called correctors. Their existence implies that the HJ equation in (2.1) homogenizes to the one in (2.5) with $\overline{H}(\theta) = \lambda(\theta)$ for every $\theta \in \mathbb{R}^d$.

Souganidis [20] and independently Rezakhanlou and Tarver [19] established homogenization in the stationary and ergodic setting (i.e., generalizing the periodicity assumption) when H is convex in p. In both papers, the main idea is to apply the subadditive ergodic theorem to a control representation for the viscosity solutions of the HJ equation in (2.1) that involves the Legendre transform of $p \mapsto H(p, x)$.

If we remove the convexity assumption, the homogenization literature is relatively sparse. There are some special classes of nonconvex Hamiltonians for which homogenization has been established. For instance, Armstrong, Tran and Yu [1] considered separated Hamiltonians of the form

$$H(p, x, \omega) = G(p) + V(x, \omega).$$

They assume that $V(x,\omega)$ is in BUC($\mathbb{R}^d$) and $G(p) = (|p|^2 - 1)^2$. This special choice of G lets them construct a family of static HJ equations that involves a free parameter $\sigma \in [-1,1]$ and generalizes (2.6). They establish the desired homogenization result by applying the subadditive ergodic theorem to the maximal subsolutions of these equations and using comparison arguments. However, there is no general homogenization result when $d \geq 1$. In fact, there are counterexamples to homogenization which have been constructed using two-person zero-sum differential games ([10, 25]). In [10], Ziliotto provides an example in the two-dimensional case such that $\mathbb{P}$-almost surely, $u_\epsilon(0,1,\omega)$ does not converge when ϵ goes to 0. In this example, the Hamiltonian H satisfies all the standard assumptions of the literature, except the convexity with respect to p. He uses weight function and two segments to create this counterexample.

Things are much better understood in one dimension. The first homogenization result for a general class of nonconvex Hamiltonians in the stationary and ergodic setting in one space dimension is due to Armstrong, Tran and Yu [2], they generalize their previous homogenization result in [1] to essentially all separated Hamiltonians with a coercive G in one space dimension. By a stability argument and a gluing procedure, they showed that the desired homogenization result for the HJ equation

$$\partial_t u^\epsilon + G(\partial_x u^\epsilon(t,x,\omega)) + \beta V(x/\epsilon,\omega) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{2.7}$$

with $\beta, \epsilon > 0$ under these assumptions:

G has a unique absolute minimum at 0 and $G(0) = 0$,

G has L local minima (and hence L local maxima) in $(-\infty, 0)$ for some L ≥ 0,

G has no local minima in $(0, +\infty)$, and

$x \mapsto V(x, \omega)$ is a smooth function whose level sets have no cluster points.

Whenever $L \geq 1$, denote the local minimum and local maximum values of G in $(-\infty, 0)$ by $m_1, m_2, \ldots, m_L$ and $M_1, M_2, \ldots, M_L$, respectively and let

$$m_{\min} = \min\{m_1, m_2, \ldots, m_L\} \quad \text{and} \quad M_{\max} = \max\{M_1, M_2, \ldots, M_L\}.$$

If $\beta < M_{\max} - m_{\min}$, then they employ two gluing procedures to show that the desired result follows from that for Hamiltonians with strictly less number of local minima. This is a strong induction argument with two base cases.

The first base case is when $L = 0$, i.e., there is no critical points other than $G(0) = 0$. In this case, the effective Hamiltonian $\overline{H}$ has three pieces, namely, strictly decreasing piece, flat piece at height β, and strictly increasing piece.

The second base case for the induction argument is when $L \geq 1$ and $\beta \geq M_{\max} - m_{\min}$. In this case, they show that the effective Hamiltonian $\overline{H}$ can be described by five pieces, namely, strictly decreasing piece, non-increasing piece, strictly decreasing piece on a possibly empty interval, flat piece at height β, and strictly increasing piece.

In each of these cases, whenever the effective Hamiltonian $\overline{H}$ does not have a flat piece, there is a sublinear corrector. In the first base case, it is easy to derive a formula for the effective Hamiltonian. However, in the second base case, the proof of existence of a sublinear corrector is not constructive and it does not yield a formula for $\overline{H}$.

Yilmaz [23] gave a new and constructive proof of homogenization which yields a formula for the effective Hamiltonian $\overline{H}$ under the extra assumption that G is a double-well function (i.e., it has precisely two local minima). He also used this formula to characterize all of the flat pieces of the graph of $\overline{H}$.

The double-well assumption that they employed is as follows: There exist $m, M, p_m, p_M \in \mathbb{R}$ such that

$$p_m < p_M < 0, \quad G(0) = 0 \leq G(p_m) = m < G(p_M) = M,$$

and G is of the form

$$G(p) = \begin{cases} G_1(p) \text{ if } p \in (-\infty, p_m], \\ G_2(p) \text{ if } p \in (p_m, p_M], \\ G_3(p) \text{ if } p \in (p_m, 0], \\ G_4(p) \text{ if } p \in (0, +\infty), \end{cases}$$

where G_1 and G_3 are strictly decreasing, and G_2 and G_4 are strictly increasing.

He defines

$$\theta_1 : [m+\beta, +\infty) \to (-\infty, p_m),$$

$$\theta_2 : [m+\beta, M] \to (p_m, p_M),$$

$$\theta_3 : [\beta, M] \to (p_M, 0),$$

$$\theta_4 : [\beta, +\infty) \to (0, +\infty)$$

by

$$\theta_i(\lambda) = \mathbb{E}[G_i^{-1}(\lambda - \beta V(0,\omega))], \quad i \in \{1,2,3,4\}$$

whenever their domains are nonempty. Then, θ_1 and θ_3 are strictly decreasing and continuous, and θ_2 and θ_4 are strictly increasing and continuous.

Yilmaz shows that if (2.2)-(2.4) hold and G is a double-well function, then the homogenization of HJ equation in (2.7) is achieved. They also give the description of the effective Hamiltonian $\overline{H}$ as follows:

- If $\beta \leq m+\beta < M$, then

$$\overline{H}(\theta) = \begin{cases} \theta_1^{-1}(\theta) & \text{on } (-\infty, \theta_1(m+\beta)], \\ m+\beta & \text{on } (\theta_1(m+\beta), \theta_2(m+\beta)), \\ \theta_2^{-1}(\theta) & \text{on } [\theta_2(m+\beta), \theta_2(M)], \\ M & \text{on } (\theta_2(M), \theta_3(M)), \\ \theta_3^{-1}(\theta) & \text{on } [\theta_3(M), \theta_3(\beta)], \\ \beta & \text{on } (\theta_3(\beta), \theta_4(\beta)), \\ \theta_4^{-1} & \text{on } [\theta_4(\beta), +\infty). \end{cases}$$

- If $\beta < M \leq m+\beta$, then

$$\overline{H}(\theta) = \begin{cases} \theta_1^{-1}(\theta) & \text{on } (-\infty, \theta_1(m+\beta)], \\ m+\beta & \text{on } (\theta_1(m+\beta), \theta_{1,3}(m+\beta)], \\ \Lambda(\theta) & \text{on } (\theta_{1,3}(m+\beta), \theta_{1,3}(M)), \\ M & \text{on } [\theta_{1,3}(M), \theta_3(M)), \\ \theta_3^{-1}(\theta) & \text{on } [\theta_3(M), \theta_3(\beta)], \\ \beta & \text{on } (\theta_3(\beta), \theta_4(\beta)), \\ \theta_4^{-1} & \text{on } [\theta_4(\beta), +\infty), \end{cases}$$

  where $\theta_{1,3}(\theta)$ is defined piecewise, combining the expectation of $G_1^{-1}(\lambda - \beta V(0,\omega))$ and $G_3^{-1}(\lambda - \beta V(0,\omega))$ at a well-chosen point, and $\Lambda(\theta)$ is a nonincreasing function that will be defined in (4.19). Note that if $M = m+\beta$, then $(\theta_1(m+\beta), \theta_{1,3}(m+\beta)]$ and $(\theta_{1,3}(m+\beta), \theta_{1,3}(M))$ are empty intervals and we have one flat piece at height M on $[\theta_{1,3}(M), \theta_3(M))$.

- If $M \leq \beta \leq m+\beta$, then

$$\overline{H}(\theta) = \begin{cases} \theta_1^{-1}(\theta) & \text{on } (-\infty, \theta_1(m+\beta)], \\ m+\beta & \text{on } (\theta_1(m+\beta), \theta_{1,3}(m+\beta)], \\ \Lambda(\theta) & \text{on } (\theta_{1,3}(m+\beta), \theta_{1,3}(\beta)), \\ \beta & \text{on } [\theta_{1,3}(\beta), \theta_4(\beta)), \\ \theta_4^{-1} & \text{on } [\theta_4(\beta), +\infty), \end{cases}$$

where $\theta_{1,3}(\theta)$ is defined piecewise, combining the expectation of $G_1^{-1}(\lambda - \beta V(0,\omega))$ and $G_3^{-1}(\lambda - \beta V(0,\omega))$ at a well-chosen point, and $\Lambda(\theta)$ is a nonincreasing function that will be defined in (4.19). Note that if $m = 0$, then $(\theta_1(m+\beta), \theta_{1,3}(m+\beta)]$ and $(\theta_{1,3}(m+\beta), \theta_{1,3}(\beta))$ are empty intervals and we have one flat piece at height β on $[\theta_1(\beta), \theta_4(\beta))$.

2.3 Our results

In this dissertation, our objective is to build upon the ideas and techniques introduced by Yilmaz in [23] to provide a constructive proof of the complete homogenization result established by Armstrong, Tran, and Yu in [1]. Specifically, we extend their result to encompass essentially all continuous functions

G exhibiting superlinear growth. We also offer a comprehensive analysis of the effective Hamiltonian $\overline{H}$. This includes providing a formula for $\overline{H}$, studying its various properties, and characterizing the flat pieces of its graph. We also illustrate our results by numerical studies and look for further properties of the effective Hamiltonian, which we will then try to justify analytically. Through these endeavors, we seek to advance the understanding of the state of the art in stochastic homogenization of HJ equations.

Consider a Hamilton-Jacobi (HJ) equation of the form

$$\partial_t u^\epsilon(t,x,\omega) + G(\partial_x u^\epsilon(t,x,\omega)) + \beta V(x/\epsilon,\omega) = 0, \quad (t,x) \in (0,+\infty)\times\mathbb{R}, \tag{2.8}$$

with the following ingredients: (i) $\beta,\epsilon > 0$; (ii) $\omega \in \Omega$; (iii) the function $G : \mathbb{R} \to [0,+\infty)$ satisfies

$$G \in C(\mathbb{R}) \quad \text{and} \quad \lim_{p\longrightarrow\pm+\infty} G(p) = +\infty, \tag{2.9}$$

i.e., it is continuous and coercive on $\mathbb{R}$; (iv) the so-called potential $V : \mathbb{R}\times\Omega \to \mathbb{R}$ is nonconstant and measurable,

$$V(x,\omega) = V(0,\tau_x\omega) \quad \text{for every } \omega \in \Omega, \tag{2.10}$$

and the map

$$x \mapsto V(x,\omega) \text{ is in BUC}(\mathbb{R}) \text{ for every } \omega \in \Omega, \tag{2.11}$$

i.e., it is bounded and uniformly continuous on $\mathbb{R}$.

The parameter ϵ adjusts the length scale of the potential while the parameter β adjusts the amplitude of the potential. Since adding a constant to V corresponds to adding a linear (in time) term to any solution of (2.8), we will assume without further loss of generality that

$$\mathbb{P}(\inf\{V(x,\omega) : x \in \mathbb{R}\} = 0 < 1 = \sup\{V(x,\omega) : x \in \mathbb{R}\}) = 1. \tag{2.12}$$

The main objective of this thesis, is to show that as $\epsilon \to 0$, the HJ equation in (2.8) homogenizes to a deterministic HJ equation of the form

$$\partial_t \overline{u}(t,x) + \overline{H}(\partial_x \overline{u}(t,x)) = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{2.13}$$

with a coercive $\overline{H} \in C(\mathbb{R})$, referred to as the effective Hamiltonian for continuous and coercive function G.

Definition 2.1. *A function $G : \mathbb{R} \to \mathbb{R}$ is said to be in the class $\mathcal{G}$ if it satisfies (2.9), i.e., continuity and coercivity.*

Theorem 2.2. *Suppose V satisfies (2.10)-(2.12), $\beta > 0$, and $G \in \mathcal{G}$. Then, the HJ equation in (2.8) homogenizes to the HJ equation in (2.13) with a coercive $\overline{H} \in C(\mathbb{R})$ that will defined in the upcoming sections in detail.*

2.4 Proof Strategy and Outline

In this section, we outline our strategy for establishing homogenization result presented in Theorem 2.2. We initiate by defining two subclasses, denoted as $\mathcal{G}_0$ and $\mathcal{G}_1$, within the broader class $\mathcal{G}$. The former comprises functions characterized by having a finite number of local extrema, while the latter encompasses functions that fulfill specific criteria conducive to homogenization. Subsequently, we demonstrate that $\mathcal{G}_0$ is a subset of $\mathcal{G}_1$ through a strong induction argument. This entails addressing two base cases initially and then constructing three cohesive gluing arguments. These combined efforts showcase how functions belonging to $\mathcal{G}_1$ satisfy the conditions outlined for $\mathcal{G}_0$. It is worth noting that the effective Hamiltonian $\overline{H}$ will be contingent upon the gluing techniques employed during the homogenization process.

For every $\lambda \in [\beta, +\infty)$ and $\omega \in \Omega$, consider the static HJ equation

$$G(f'(x,\omega)) + \beta V(x,\omega) = \lambda, \quad x \in \mathbb{R}. \tag{2.14}$$

There is an elementary connection between the static HJ equation in (2.14) and the evolutionary HJ equation in (2.8). Namely, if $f(\cdot,\omega) \in C(\mathbb{R})$ is a subsolution (respectively supersolution) of (2.14), then $u(\cdot,\cdot,\omega) \in C([0,+\infty)\times\mathbb{R})$, defined by

$$u(t,x,\omega) = -\lambda t + f(x,\omega), \tag{2.15}$$

is a subsolution (respectively supersolution) of (2.8).

Definition 2.3. *A function $G \in \mathcal{G}$ is said to be in the class $\mathcal{G}_0$ if $G(0) = 0$ and G has finitely many local extrema.*

We are interested in the solutions of equation (2.14) that satisfy the conditions stated in the next definition.

Definition 2.4. *A function $G \in \mathcal{G}$ is said to be in the class $\mathcal{G}_1$ if, $G(0) = 0$ and for every $\theta \in \mathbb{R}$, there exists a constant $\lambda = \lambda(\theta) \geq \beta$ and stationary functions $\underline{f}, \overline{f} : \mathbb{R} \times \Omega \to \mathbb{R}$ (depending on θ and not necessarily distinct) that satisfy the following conditions:*

$$\underline{f}(\cdot, \omega), \overline{f}(\cdot, \omega) \text{ are } Lip(\mathbb{R}) \text{ solutions of } (2.14) \forall \omega \in \Omega_0 \text{ with } \mathbb{P}(\Omega_o) = 1; \tag{2.16}$$

$$\forall \epsilon > 0, \exists x_0 \in \mathbb{R} : \forall p \in [\underline{f}'(x_0, \omega), \overline{f}'(x_0, \omega)] : G(p) + \beta V(x_0, \omega) \geq \lambda - \epsilon; \tag{2.17}$$

$$\forall \epsilon > 0, \exists y_0 \in \mathbb{R} : \forall p \in [\overline{f}'(y_0, \omega), \underline{f}'(y_0, \omega)] : G(p) + \beta V(y_0, \omega) \leq \lambda + \epsilon; \tag{2.18}$$

$$\mathbb{E}[\underline{f}'(0, \omega)] \leq \theta \leq \mathbb{E}[\overline{f}'(0, \omega)]. \tag{2.19}$$

Next, we claim that for functions G that satisfy the above four conditions, i.e., for $G \in \mathcal{G}_1$, homogenization is achieved.

Theorem 2.5. *Suppose V satisfies (2.10)-(2.12), $\beta > 0$, and $G \in \mathcal{G}_1$. Then, (2.8) homogenizes.*

Before generalizing homogenization for a broad class of functions $\mathcal{G}$, we will show that $\mathcal{G}_0$ is a subset of $\mathcal{G}_1$ and hence any $G \in \mathcal{G}_0$, satisfies the above four conditions and therefore, homogenization is also achieved for $G \in \mathcal{G}_0$.

Theorem 2.6. *Suppose that V satisfies (2.10)-(2.12) and $\beta > 0$. Then, $\mathcal{G}_0 \subset \mathcal{G}_1$.*

In Chapter 3, we revisit well-known results in the theory of viscosity solutions, provide background on homogenization, and list examples of widely used potential functions.

Then, in Chapter 4, we construct correctors, which will be one of the main components of our discussion on homogenization.

Chapter 5 marks the beginning of preparations for our main proof. We address two base cases for the strong induction argument. Base case 1 focuses on functions $\mathcal{G}$ with one critical point, while base case 2 handles functions $\mathcal{G}$ with multiple critical points.

Chapter 6 is about the derivation of gluing procedures for the strong induction argument. The first gluing procedure deals with gluing at the origin, while the second addresses more complex gluing procedures with small β under two different scenarios.

In Chapter 7, we consolidate all the components and prove our main result. We first establish homogenization for the class $\mathcal{G}_1$. Then, we demonstrate that the class $\mathcal{G}_0$ is a subset of the class $\mathcal{G}_1$, implying homogenization for $\mathcal{G}_0$. Finally, we generalize the homogenization result from $\mathcal{G}_0$ to our broadest class, $\mathcal{G}_0$.

2.5 Python Implementation

Lastly, we will focus on the practical implementation of the theoretical concepts discussed in previous sections. The goal is to develop an effective Hamiltonian calculator using Python. The chapter follows a logical progression, mirroring the theoretical discussions, particularly regarding the proof of homogenization.

The implementation starts with outlining the steps to translate homogenization theory into a Python environment, utilizing libraries such as pandas, numpy, matplotlib, and sympy. The code development involves creating a function that evaluates the effective Hamiltonian and plots its graph. Users input the function G and constant β for calculations while potential function V is chosen from one of the pre-defined widely-used potential examples (See Section 3.3 for examples of potentials).

The code discretizes $G(p)$ over a range of p values, stores the values as an array, and identifies critical points. Based on the critical points, it defines G_i for i in $1, 2, ..., 2N$, as outlined in a specific condition. Inverse arrays of G_i are then created, followed by the definition of the potential function V.

Subsequently, an Ergodic approximation function is defined to evaluate functions that will be used in the construction of the effective Hamiltonian. Base case functions are established to handle various scenarios based on the critical points of G. For functions not falling under base cases, gluing processes are employed. These processes include gluing at the origin, splitting G into two parts, and checking for small β cases.

The working logic of Python implementation is elucidated through a flowchart, providing a visual understanding of the process (See Chapter 8 for more detail).

CHAPTER 3

PRELIMINARIES AND EXAMPLES

In this section, we review the theory of viscosity solutions for HJ equations, emphasizing key aspects essential for the upcoming discussions. Additionally, we utilize significant theorems from prior homogenization studies, adapting them to our context, and provide examples of potentials that will be used in our numerical implementation.

3.1 Viscosity Solutions for HJ Equations

Consider the Cauchy problem for the HJ equation

$$\begin{cases} u_t + H(x, Du) & = 0 \text{ in } (0, +\infty) \times \mathbb{R}^d, \\ u(0, x) & = u_0(x) \text{ on } \mathbb{R}^d, \end{cases} \tag{3.1}$$

where $H : \mathbb{R}^d \to \mathbb{R}$ is the Hamiltonian and u_0 is the initial data.

Definition 3.1. *For each* $T > 0$, *a function* $u \in BUC([0, T] \times \mathbb{R}^d)$ *is called*

- *a viscosity subsolution of* (3.1) *if for all* $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$,

$$\varphi_t(t_0, x_0) + H(x_0, D\varphi(t_0, x_0)) \leq 0,$$

and $\varphi(0,\cdot) \le u_0$ *for any* $\varphi \in C^1((0,T)\times\mathbb{R}^d)$ *such that* $u(t_0,x_0) = \varphi(t_0,x_0)$ *and* $u-\varphi$ *has a strict maximum at* $(t_0,x_0) \in (0,T)\times\mathbb{R}^d$*;*

- *a viscosity supersolution* (3.1) *if for all* $(t_0,x_0) \in (0,T)\times\mathbb{R}^d$*,*

  $$\psi_t(t_0,x_0) + H(x_0, D\psi(t_0,x_0)) \ge 0,$$

  and $\psi(0,\cdot) \ge u_0$ *for any* $\psi \in C^1((0,T)\times\mathbb{R}^d)$ *such that* $u(t_0,x_0) = \psi(t_0,x_0)$ *and* $u-\psi$ *has a strict minimum at* $(t_0,x_0) \in (0,T)\times\mathbb{R}^d$*;*

- *a viscosity solution of* (3.1) *if it is both a viscosity subsolution and a viscosity supersolution.*

Next, we have the existence result for viscosity solutions to (3.1).

Theorem 3.2. *Suppose that* $H \in UC(\mathbb{R}^d)$ *and there exists two continuous, coercive, and nondecreasing continuous* $\alpha_1, \alpha_2 : \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$\alpha_1(|p|) \le H(x,p) \le \alpha_2(|p|) \text{ for every } x, p \in \mathbb{R}^d.$$

Then, for every $g \in UC(\mathbb{R}^d)$*, there exists a continuous function* u *which solves* (3.1) *and satisfies the initial condition* $u(0,\cdot) = g$ *on* $\mathbb{R}^d$*.*

The proof is a special case of [5, Theorem 3.1]. It uses Arzela-Ascoli theorem to find a subsequence $\{\epsilon_j\} \searrow 0$ such that $u^{\epsilon_j} \to u$ locally uniformly for some function u.

Next, we will derive a comparison principle, which will be used several times throughout the dissertation.

Theorem 3.3. *Suppose $H \in UC(\mathbb{R}^d)$. Let $v \in USC([0, +\infty) \times \mathbb{R}^d)$ and $w \in LSC([0, +\infty) \times \mathbb{R}^d)$ be, respectively, a sub and a supersolution of (3.1). Then,*

$$v(t, x) - w(t, x) \leq \sup_{\mathbb{R}^d} \big(v(0, \cdot) - w(0, \cdot) \big) \text{ for every } (t, x) \in [0, +\infty) \times \mathbb{R}^d.$$

The proof is derived from [5, Proposition 2.7], tailored to the context of one-dimensional space, employing the doubling variable technique.

Now, uniqueness of the solution is just a corollary of the comparison principle.

Corollary 3.4. *Suppose that $H \in UC(\mathbb{R}^d)$. If $v, w \in BUC(\mathbb{R}^d)$ are the viscosity solutions of (3.1), then $v \equiv w$ in $\mathbb{R}^d$.*

Proof. If $v(t, \cdot), w(t, \cdot) \in BUC(\mathbb{R}^d)$ are the viscosity solution of (3.1), then $v \in \mathrm{USC}([0, +\infty) \times \mathbb{R}^d)$ and $w \in \mathrm{LSC}([0, +\infty) \times \mathbb{R}^d)$ are sub and supersolution of (3.1). Therefore, by Theorem 3.3, we have

$$v(t, x) - w(t, x) \leq \sup_{\mathbb{R}^d} \big(v(0, \cdot) - w(0, \cdot) \big) \text{ for every } (t, x) \in [0, +\infty) \times \mathbb{R}^d.$$

However, we also know that $w \in \mathrm{USC}([0, +\infty) \times \mathbb{R}^d)$ and $v \in \mathrm{LSC}([0, +\infty) \times \mathbb{R}^d)$ are sub and supersolution of (3.1). Hence,

$$w(t, x) - v(t, x) \leq \sup_{\mathbb{R}^d} \big(w(0, \cdot) - v(0, \cdot) \big) \text{ for every } (t, x) \in [0, +\infty) \times \mathbb{R}^d.$$

Together, they imply that $\sup_{\mathbb{R}^d}(v(0, \cdot) - w(0, \cdot)) = \sup_{\mathbb{R}^d}(w(0, \cdot) - v(0, \cdot)) = 0$, and we have $v \equiv w$ in $[0, +\infty) \times \mathbb{R}^d$. $\square$

Henceforth, the term "solution" shall denote a solution in the viscosity sense.

3.2 Homogenization Background

In this section, we will compile some fundamental results on which we will build our arguments. We will start by showing that in order to establish homogenization, it suffices to work with only linear initial data.

Fix $\theta \in \mathbb{R}$ and $\epsilon > 0$. We denote by u_θ^ϵ and $\overline{u}_\theta$ the unique continuous functions that solve (2.8) and (2.13), respectively, subject to the initial conditions

$$u_\theta^\epsilon(0, x, \omega) = \overline{u}_\theta(0, x) = \theta x.$$

When $\epsilon = 1$, (2.8) becomes

$$\partial_t u(t, x, \omega) + G(\partial_x u(t, x, \omega)) + \beta V(x, \omega) = 0, \qquad (t, x) \in (0, +\infty) \times \mathbb{R}, \quad (3.2)$$

and we drop the superscript of u_θ^ϵ and simply write u_θ.

If the HJ equation in (2.8) homogenizes to the HJ equation in (2.13) (as claimed in Theorem 2.2 with general $g \in \mathrm{UC}(\mathbb{R})$), then we can choose to restrict our attention to linear initial conditions and deduce that, for every $\theta \in \mathbb{R}$, and $\mathbb{P}$-a.e. $\omega, u_\theta^\epsilon(\cdot, \cdot, \omega)$ converges locally uniformly on $[0, +\infty) \times \mathbb{R}$ as $\epsilon \to 0$ to $\overline{u}_\theta$. In particular, we have pointwise convergence at $(t, x) = (1, 0)$, which is equivalent to

$$\mathbb{P}\big(\lim_{\epsilon \to 0} \epsilon u_\theta(1/\epsilon, 0, \omega) = -\overline{H}(\theta)\big) = 1.$$

Define

$$\overline{H}^L(\theta) = \liminf_{t\to+\infty} \frac{-u_\theta(t,0,\omega)}{t} \quad \text{and} \quad \overline{H}^U(\theta) = \limsup_{t\to+\infty} \frac{-u_\theta(t,0,\omega)}{t}. \tag{3.3}$$

for a.e. ω. Observe that if we take $\epsilon = 1/t$, we see that homogenization implies the existence of almost sure limit above, i.e.,

$$\overline{H}(\theta) = \overline{H}^L(\theta) = \overline{H}^U(\theta), \tag{3.4}$$

Now, we will show that the converse is also true.

Theorem 3.5. *Assume (2.9)–(2.11) hold. If $\overline{H}^L(\theta) = \overline{H}^U(\theta)$ for every $\theta \in \mathbb{R}$, then the HJ equation in (2.8) homogenizes to a HJ equation of the form in (2.13) as the statement of the Theorem 2.2. Moreover, the effective Hamiltonian $\overline{H}$ is given by (3.4).*

Proof. It follows from $u_\theta(t,x,\tau_y\omega) = u_\theta(t,x+y,\omega) - \theta y$ and our ergodicity assumption that the uniform Lipschitz constant of $u_\theta(\cdot,\cdot,\omega)$ is $\mathbb{P}$-essentially constant. Hence, the desired result is a special case of Theorem [5, Lemma 4.1]. □

Now, we will derive two propositions that will be used in the base cases of the strong induction proof of Theorem 2.6.

Proposition 3.6. *Assume (2.9)-(2.11). If there exist a constant $\theta \in \mathbb{R}$, a* *function* $f : \mathbb{R} \times \Omega \to \mathbb{R}$, *and an event* $\Omega_0 \in \mathcal{F}$ *with* $\mathbb{P}(\Omega_0) = 1$ *such that*

$$f'(\cdot, \omega) \text{ exists and is uniformly bounded on } \mathbb{R}, \tag{3.5}$$

$$\lim_{x \to \pm+\infty} \frac{f(x, \omega)}{x} = \theta, \text{ and} \tag{3.6}$$

$$f(\cdot, \omega) \text{ is a classical solution of (2.14) for some } \lambda \in [\beta, +\infty), \tag{3.7}$$

for every $\omega \in \Omega_0$, *then*

$$\overline{H}(\theta) = \overline{H}^L(\theta) = \overline{H}^U(\theta) = \lambda.$$

Proof. Let $\varphi(x) = \sqrt{1 + x^2}$. Observe that $|\varphi'(x)| < 1$ for any $x \in \mathbb{R}$. For every $\epsilon > 0$ and $\omega \in \Omega_0$, define $v = v(\cdot, \cdot, \omega) \in \mathrm{Lip}([0, +\infty) \times \mathbb{R})$ by

$$v(t, x, \omega) = -(\lambda + \epsilon)t + f(x, \omega) - \delta\varphi(x) - K,$$

where $\delta, K > 0$ are to be determined. We claim that v is a subsolution of (3.2). Fix $p \in [\partial_x^+ v(t, x, \omega), \partial_x^- v(t, x, \omega)]$. Note that if this interval is empty, then there is nothing to do by the definition of viscosity subsolution. If it is nonempty, then we note that $\partial_x^+ v(t, x, \omega) = f'_+(x, \omega) - \delta\varphi'(x)$ and $\partial_x^- v(t, x, \omega) = f'_-(x, \omega) - \delta\varphi'(x)$. Therefore,

$$p + \delta\varphi'(x) \in [f'_+(x, \omega), f'_-(x, \omega)].$$

Then, using the modulus of continuity of G, we have

$$\begin{aligned} G(p) + \beta V(x,\omega) &= G(p + \delta\varphi'(x) - \delta\varphi'(x)) + \beta V(x,\omega) \\ &\leq G(p + \delta\varphi'(x)) + \epsilon + \beta V(x,\omega) \\ &\leq \lambda + \epsilon \\ &= \partial_t v(t,x,\omega) \end{aligned}$$

when $\delta = \delta(\epsilon,\omega) > 0$ is sufficiently small. Therefore, v is indeed a subsolution of (3.2). Moreover, for $K = K(\delta,\omega) > 0$ sufficiently large,

$$v(0,x,\omega) = f(x,\omega) - \delta\varphi(x) - K = \theta x + o(|x|) - \delta\varphi(x) - K \leq \theta x = u_\theta(0,x,\omega)$$

for every $x \in \mathbb{R}$. By the comparison principle in Theorem 3.3, we get

$$\overline{H}^U(\theta) = \limsup_{t\to+\infty} \frac{-u_\theta(t,0,\omega)}{t} \leq \lim_{t\to+\infty} \frac{-v(t,0,\omega)}{t} = \lambda + \epsilon.$$

Similarly, for $\epsilon > 0$ and $\omega \in \Omega_0$, there exist $\delta, K > 0$ such that

$$w(t,x,\omega) = -(\lambda - \epsilon)t + f(x,\omega) + \delta\varphi(x) + K$$

is a supersolution of (3.2). Moreover,

$$w(0,x,\omega) \geq \theta x = u_\theta(0,x,\omega)$$

for every $x \in \mathbb{R}$. By the comparison principle in Theorem 3.3, we have

$$\overline{H}^L(\theta) = \liminf_{t\to+\infty} \frac{-u_\theta(t,0,\omega)}{t} \geq \lim_{t\to+\infty} \frac{-w(t,0,\omega)}{t} = \lambda - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that we have the desired result. $\square$

The second proposition is similar to first one but it will be tailored to use it in our second base case.

Proposition 3.7. *Assume (2.9)-(2.11). Suppose there exist constants $\theta_-, \theta_+ \in \mathbb{R}$, a function $f : \mathbb{R} \times \Omega \to \mathbb{R}$ and an event $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that (3.6) hold,*

$$\lim_{x \to -\infty} \frac{f(x,\omega)}{x} = \theta_- \quad \text{and} \quad \lim_{x \to +\infty} \frac{f(x,\omega)}{x} = \theta_+$$

for every $\omega \in \Omega_0$.

(a) If $\theta_+ < \theta_-$ and $f(\cdot,\omega)$ is a subsolution of (2.14) with some $\lambda \in [\beta, +\infty)$ for every $\omega \in \Omega_0$, then

$$\overline{H}^U(\theta) \leq \lambda \text{ for every } \theta \in (\theta_+, \theta_-).$$

(b) If $\theta_- < \theta_+$ and $f(\cdot,\omega)$ is a supersolution of (2.14) with some $\lambda \in [\beta, +\infty)$ for every $\omega \in \Omega_0$, then

$$\overline{H}^L(\theta) \geq \lambda \text{ for every } \theta \in (\theta_-, \theta_+).$$

Proof. The proof is similar to the proof of Proposition 3.6 albeit easier. Assume $\theta_+ < \theta_-$ and define $v(t,x,\omega) = -\lambda t + f(x,\omega) - K$. Observe that for any $\theta \in (\theta_+, \theta_-)$, v is a subsolution of (3.2) and $v(0,x,\omega) \leq \theta x = u_\theta(0,x,\omega)$ when $K = K(\theta,\omega)$ is sufficiently large. By the comparison principle in Theorem 3.3, we have

$$\overline{H}^U(\theta) = \limsup_{t \to +\infty} \frac{-u_\theta(t,0,\omega)}{t} \leq \lim_{t \to +\infty} \frac{-v(t,0,\omega)}{t} = \lambda.$$

The proof of part (b) is almost identical to the proof of part (a). ☐

3.3 Examples of Potentials

In this part of the dissertation, we will give several examples of potential. Recall that $V : \mathbb{R} \times \Omega \to \mathbb{R}$ is a nonconstant and measurable function with

$$V(x,\omega) = V(0,\tau_x\omega) \quad \text{for every } \omega \in \Omega,$$

and the map

$$x \mapsto V(x,\omega) \text{ is in BUC}(\mathbb{R}) \text{ for every } \omega \in \Omega,$$

i.e., it is bounded and uniformly continuous on $\mathbb{R}$. Moreover,

$$\mathbb{P}(\inf\{V(x,\omega) : x \in \mathbb{R}\} = 0 < 1 = \sup\{V(x,\omega) : x \in \mathbb{R}\}) = 1.$$

First, we will consider the following lemma.

Lemma 3.8. *Let $V : \mathbb{R} \to \mathbb{R}$ be 1-periodic and $\hat{V}(x,\omega) := V(x+U(\omega))$ where $U \sim$ Unif$[0,1]$. Then, $\hat{V}$ is stationary.*

Proof. Let $A = \{\omega \in \Omega : (\hat{V}(x_1,\omega), \hat{V}(x_2,\omega), \ldots, \hat{V}(x_k,\omega)) \in B\}$ where $B \in \mathcal{B}(\mathbb{R}^k)$, $x_1 < x_2 < \cdots < x_k$. Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is our probability space equipped with a group of measure-preserving transformations $\tau_x : \Omega \to \Omega, x \in \mathbb{R}$, i.e., $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$. Then,

$$\begin{aligned}
\tau_x A &= \{\tau_x\omega \in \Omega : (\hat{V}(x_1,\omega), \hat{V}(x_2,\omega), \ldots, \hat{V}(x_k,\omega)) \in B\} \\
&= \{\omega \in \Omega : (\hat{V}(x_1,\tau_{-x}\omega), \hat{V}(x_2,\tau_{-x}\omega), \ldots, \hat{V}(x_k,\tau_{-x}\omega)) \in B\} \\
&= \{\omega \in \Omega : (\hat{V}(x_1 - x,\omega), \hat{V}(x_2 - x,\omega), \ldots, \hat{V}(x_k - x,\omega)) \in B\}.
\end{aligned}$$

Since $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$, $V(\cdot, \omega)$ and $V(\cdot - x, \omega)$ have the same finite dimensional distributions. We know that $V \in \mathrm{BUC}(\mathbb{R})$. Therefore, $V(\cdot, \omega)$ and $V(\cdot - x, \omega)$ have the same distributions. $\square$

We can also explore periodic functions having periods different from 1 by slightly altering the above lemma.

Example 3.9. *Let* $V_1 : \mathbb{R} \to \mathbb{R}$ *be defined as follows:*

$$V_1(x) = \begin{cases} x - \lfloor x \rfloor & \textit{if } x \in [2n, 2n+1), \\ 1 - (x - \lfloor x \rfloor) & \textit{if } x \in [2n+1, 2n+2), \end{cases}$$

for $n \in \mathbb{Z}$*. Clearly,* V_1 *is 2-periodic.*

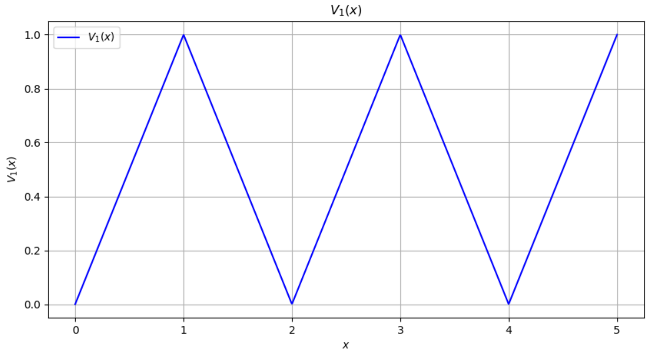

Figure 3.1: Triangle wave function

Note that V_1 *is not stationary. We use Lemma 3.8 to define*

$$\hat{V}_1(x, \omega) := V_1(x + U(\omega))$$

where $U \sim$ *Unif*$[0, 1]$*. Then,* $\hat{V}_1$ *is stationary and satisfies all the required assumptions of a potential.*

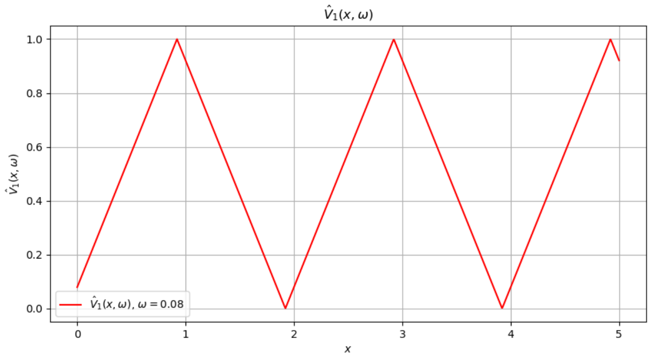

Figure 3.2: Stationary triangle wave function

Example 3.10. *Let* $V_2 : \mathbb{R} \to \mathbb{R}$ *and* $\tilde{V}_2 : \mathbb{R} \to \mathbb{R}$ *be defined as follows:*

$$V_2(x) = \frac{1}{2}(\sin(x) + 1) \quad \textit{and} \quad \overline{V}_2(x) = -\frac{1}{2}(\sin(x) - 1).$$

Clearly, V_2 *and* $\overline{V}_2$ *are* 2π*-periodic.*

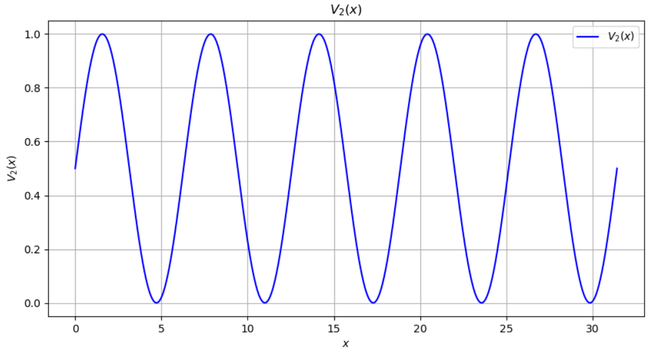

Figure 3.3: Half-wave rectified sine function

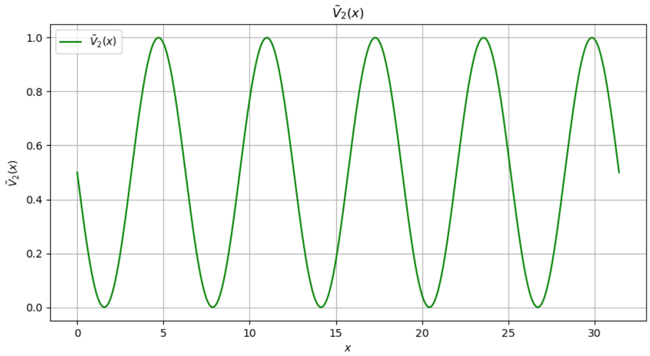

Figure 3.4: Half-wave rectified cosine function

Each time $x = 2\pi n$ for $n \in \mathbb{Z}$, we flip a coin. If coin lands on heads, we will let $\hat{V}_2(x,\omega) = V_2(x + U(\omega))$, and if coin lands on tails, we will let $\hat{V}_2(x,\omega) = \overline{V_2}(x + U(\omega))$, where $U \sim$ Unif$[0, 2\pi]$.

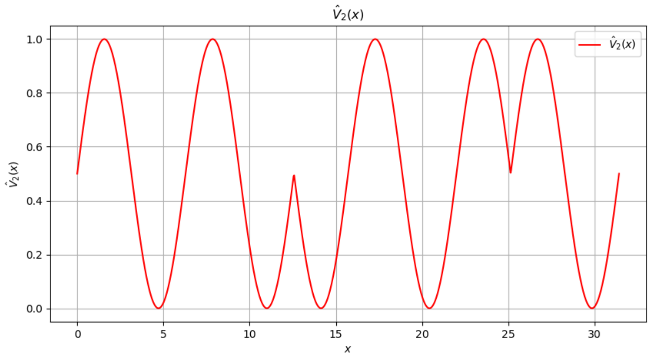

Figure 3.5: Random combination of half-wave rectified sine and cosine functions

Now, $\hat{V}_2$ is stationary and satisfies all the required assumptions of a potential.

Example 3.11. *Pick $\omega_i \sim$ Unif$[0, 1]$ for $i \in \mathbb{Z}$ and uniformly interpolate ω_i's.*

Let $V_3 : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$V_3(x) = \begin{cases} \omega_{\lfloor x \rfloor} & \textit{if } x \textit{ is an integer,} \\ \\ \omega_{\lfloor x \rfloor} + (x - \lfloor x \rfloor)(\omega_{\lceil x \rceil} - \omega_{\lfloor x \rfloor}) & \textit{if } x \textit{ is not an integer.} \end{cases}$$

Then, V_3 *satisfies all the required assumptions of a potential but stationarity.*

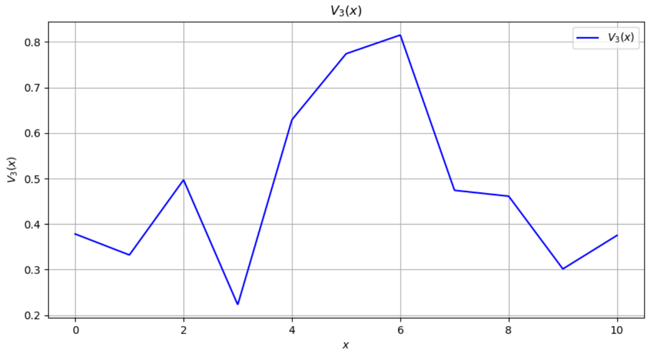

Figure 3.6: Linear interpolation of uniformly selected points in $[0, 1]$

We again use Lemma 3.8 to define

$$\hat{V}_3(x, \omega) := V_3(x + U(\omega))$$

where $U \sim$ *Unif*$[0, 1]$. *Then* $\hat{V}_3$ *is stationary and satisfies all the required assumptions of a potential.*

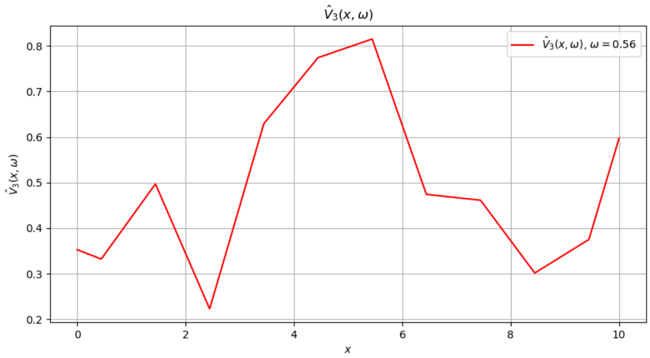

Figure 3.7: Stationary linear interpolation of uniformly selected points in $[0, 1]$

Example 3.12. *Let*

$$B_t = W_t + L_t^0 - L_t^1,$$

where W_t *is a standard Brownian motion,* L_t^0 *is the local time of* B_t *at* 0 *up to time* t*, and* L_t^1 *is the local time of* B_t *at* 1 *up to time* t*. Essentially,* L_t^0 *and* L_t^1 *push* B_t *away from* 0 *and* 1 *respectively whenever* W_t *tries to go below* 0 *or above* 1.

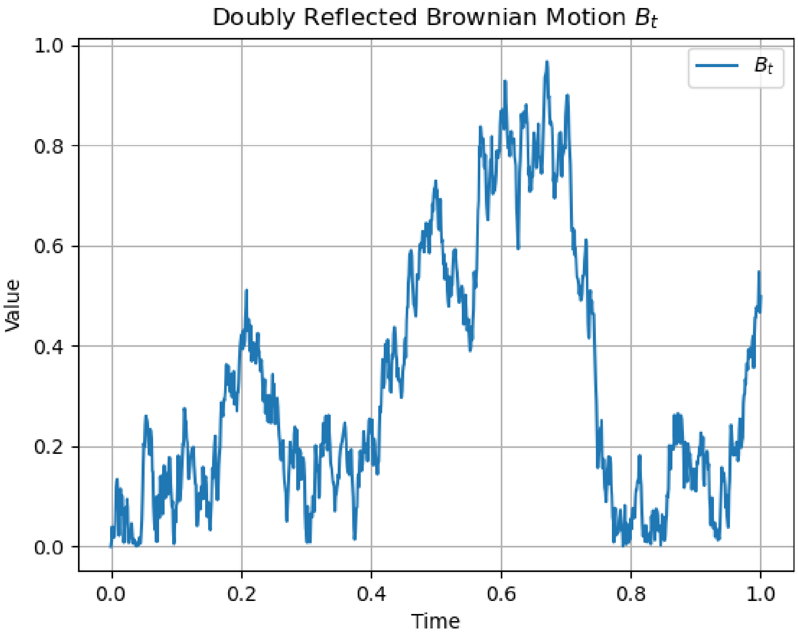

Figure 3.8: Doubly-reflected Brownian motion B_t

Let $\varphi \in C_c^{+\infty}$ *and consider*

$$B_t^\epsilon = (B_t * \varphi_\epsilon)(t) = \int_{-\infty}^{+\infty} B_s \varphi_\epsilon(t-s)ds$$

where

$$\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi(\frac{x}{\epsilon}).$$

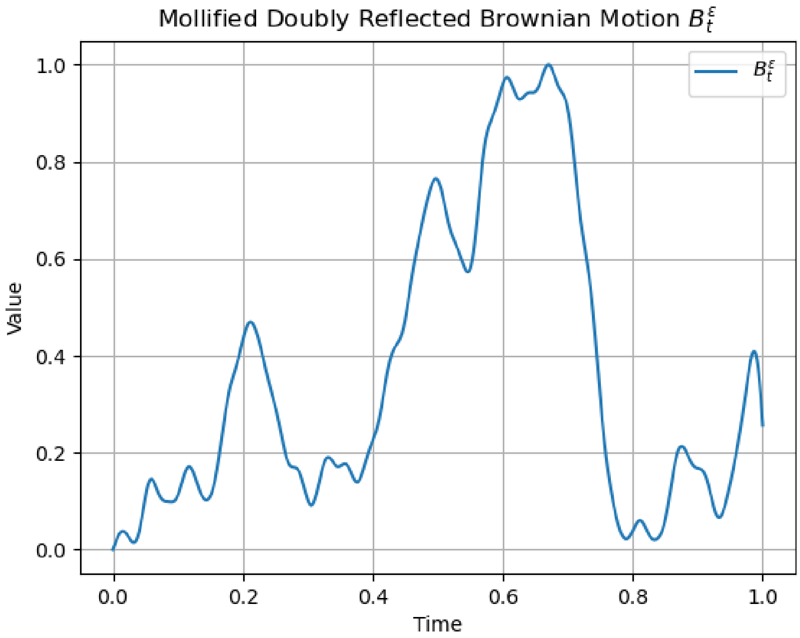

Figure 3.9: Mollified doubly-reflected Brownian motion B_t^ϵ

Now, thanks to the mollification, doubly reflected Brownian motion B_t^ϵ will not touch 0 *or* 1 *and it will satisfy all the assumptions of a potential.*

Example 3.13. *Consider the following SDE:*

$$dX_t = b(X_t)dt + dB_t,$$

where B_t is the standard Brownian motion. If we take $b(x) = \log(\frac{1-x}{x})$, then

$$b(1/2) = 0, \quad \lim_{x \to 0^+} b(x) = +\infty, \text{ and } \lim_{x \to 1^-} b(x) = -\infty.$$

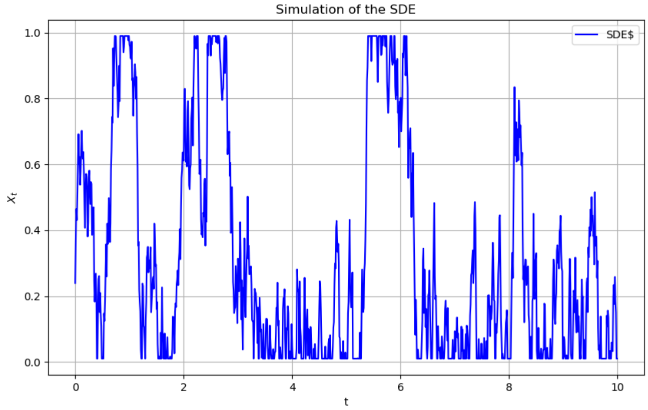

Figure 3.10: Simulation of the SDE

This is another example of a potential.

CHAPTER 4

CONSTRUCTING CORRECTORS

In this chapter, we establish the foundational framework necessary for subsequent discussions. We begin by exploring the fundamental definitions and concepts essential to our study.

For every $\lambda \in [\beta, +\infty)$ and $\omega \in \Omega$, recall the static HJ equation in (2.14):

$$G(f'(x,\omega)) + \beta V(x,\omega) = \lambda, \quad x \in \mathbb{R}.$$

We will look for a function f of the form $f(x,\omega) = \theta x$. This kind of function commonly referred to in the literature as a sublinear corrector. In the upcoming sections, we will construct these functions in detail and explore their properties. Due to the nature of the Hamiltonian, i.e., the fact that G is not one-to-one, derivative of the corresponding correctors will exhibit jumps and therefore will not be continuous.

4.1 Setting the Stage

For the sake of simplicity, we will state and prove our result in this section under the assumption that the function G satisfies the below condition, i.e., its graph has N wells all in the second quadrant. Later, we will generalize this to basically any continuous and coercive G.

Condition 4.1. *For $N \geq 2$, we say that a function G satisfies the N-well condition if there exist $p_1, p_2, \dots, p_{2N-1} \in \mathbb{R}$ such that*

$$p_1 < p_2 < \cdots < p_{2N-1} = 0, \text{ and } 0 = G(0) \leq m < M,$$

with $m = \min\{G(p_{2i-1}) : i = 1, 2, \dots, N-1\}$, $M = \max\{G(p_{2i}) : i = 1, 2, \dots, N-1\}$, and G is of the form

$$G(p) = \begin{cases} G_1(p) & \text{if } p \in (-\infty, p_1), \\ G_{i+1}(p) & \text{if } p \in [p_i, p_{i+1}), \\ G_{2N}(p) & \text{if } p \in [0, +\infty), \end{cases}$$

for $i = 1, 2, \dots, 2N-2$, where $G_1, G_3, \dots, G_{2N-1}$ are strictly decreasing and $G_2, G_4, \dots, G_{2N}$ are strictly increasing.

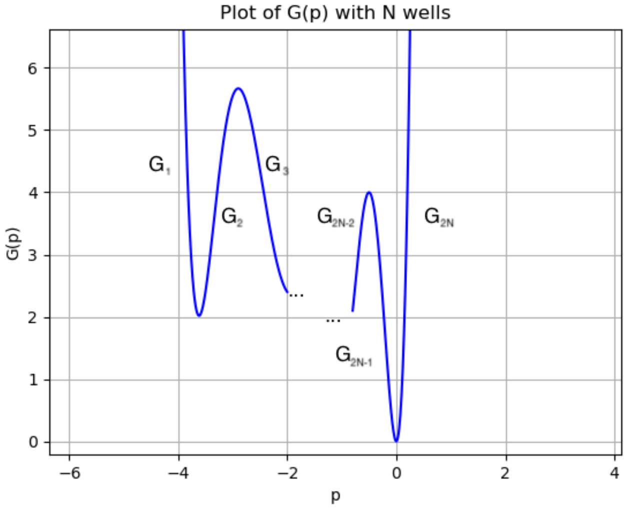

Figure 4.1: The graph of a function $G(p)$ with N wells

Before introducing the fundamental definitions, we will start with a lemma that is required for our discussion.

Lemma 4.2. *For every* $\lambda > M$,

$$\inf\{x \in \mathbb{R} : \lambda - \beta V(x,\omega) > M\} = -\infty.$$

Proof. Let $A_k = \{0 \le V(y,\omega) < \frac{\lambda - M}{\beta}$ for some $y \in [k, k+1]\}$ for $k \in \mathbb{Z}$ and $\alpha = \mathbb{P}(A_0)$. Then, by ergodicity,

$$\mathbb{P}(A_k) = \mathbb{P}(\tau_k A_0) = \mathbb{P}(A_0) = \alpha$$

and

$$\begin{aligned}\mathbb{P}(0 \leq V(y,\omega) < \frac{\lambda - M}{\beta} \text{ for some } y \in \mathbb{R}) &= \mathbb{P}(\cup_{k\in\mathbb{Z}} A_k)\\ &\leq \sum_{k\in\mathbb{Z}} \mathbb{P}(A_k)\\ &= \sum_{k\in\mathbb{Z}} \alpha.\end{aligned}$$

Since $\lambda > M$, by (2.12), we know that $\mathbb{P}(\cup_{k\in\mathbb{Z}} A_k) > 0$, so $\alpha > 0$. Then, by the Birkhoff ergodic theorem,

$$\begin{aligned}\lim_{K\to+\infty} \sum_{k=-K}^{-1} \mathbb{1}_{A_k}(\omega) &= \int \mathbb{1}_{A_0}(\omega) d\mathbb{P}\\ &= \mathbb{E}[\mathbb{1}_{A_0}]\\ &= \mathbb{P}(A_0)\\ &= \alpha > 0,\end{aligned}$$

and $\mathbb{P}(\lim_{K\to+\infty} \frac{1}{K} \sum_{k=-K}^{-1} \mathbb{1}_{A_k}(\omega) = \alpha) = 1$. So, A_k occurs for infinitely many k with probability 1, or

$$\mathbb{P}(\exists \{z_k\}_{k\geq 0} : z_k \to -\infty \text{ as } k \to +\infty, \quad 0 \leq V(z_k,\omega) < \frac{\lambda - M}{\beta}) = 1.$$

Hence, for any $B > 0$, we can find K such that for every $k \geq K$, we have $z_k \leq -B$ and $\inf\{x \in \mathbb{R} : \lambda - \beta V(x,\omega) > M\} = -\infty$. □

We start by giving fundamental definitions and introducing concepts that will serve as the groundwork for our subsequent discussions. Now that we have separated G into branches and each branch of G is invertible, we will construct f' by selectively taking the inverse of certain branches of G. Specifically, we begin by constructing two right-continuous odd-integer-valued functions L^λ and R^λ that will give us the branch number but first, we need to define jumping time sequences inductively for the selection process. We will define two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ that will give us the jumping times when we extend them to bi-infinite sequences later and two more sequences $(L^\lambda(x_{n+1},\omega))_{n\geq 0}$ and $(R^\lambda(y_{n+1},\omega))_{n\geq 0}$ that are going to define two right-continuous integer-valued branch-indicating functions.

Definition 4.3. *Take* $\lambda \geq \beta$. *Let* $x_0 = \inf\{x \geq 0 : \lambda - \beta V(x,\omega) \geq M\}$ *and* $L^\lambda(x_0,\omega) = 1$. *Define* $(x_n)_{n\geq 0} = (x_n(\lambda,\omega))_{n\geq 0}$ *and* $(L^\lambda(x_{n+1},\omega))_{n\geq 0}$ *inductively as follows:*

$$\begin{aligned} x_{n+1} = \inf\{x \geq x_n : \lambda - \beta V(x,\omega) \geq G(p_{L^\lambda(x_n,\omega)-1}) \textit{ or } \\ \lambda - \beta V(x,\omega) < G(p_{L^\lambda(x_n,\omega)})\}, \end{aligned} \tag{4.1}$$

where

$$L^\lambda(x_{n+1},\omega) = \begin{cases} \max\{2j+1 \in (-\infty, L^\lambda(x_n,\omega)) : G(p_{2j}) > G(p_{L^\lambda(x_n,\omega)-1})\} \\ \qquad\qquad \textit{if } \lambda - \beta V(x_{n+1},\omega) \geq G(p_{L^\lambda(x_n,\omega)-1}), \\ \min\{2j+1 \in (L^\lambda(x_n,\omega), +\infty) : G(p_{2j+1}) < G(p_{L^\lambda(x_n,\omega)})\} \\ \qquad\qquad \textit{if } \lambda - \beta V(x_{n+1},\omega) < G(p_{L^\lambda(x_n,\omega)}), \end{cases} \tag{4.2}$$

and $p_0 = \inf\{p \le 0 : G(p) = \lambda + 1\}$.

Similarly, let $y_0 = \inf\{x \ge 0 : \lambda - \beta V(y,\omega) > M\}$ *and* $R^\lambda(y_0,\omega) = 1$. *Define* $(y_n)_{n\ge 0} = (y_n(\lambda,\omega))_{n\ge 0}$ *and* $(R^\lambda(y_{n+1},\omega))_{n\ge 0}$ *inductively as follows:*

$$\begin{aligned} y_{n+1} = \inf\{y \ge x_n : \lambda - \beta V(y,\omega) > G(p_{R^\lambda(y_n,\omega)-1}) \ \textit{or} \\ \lambda - \beta V(y,\omega) \le G(p_{R^\lambda(y_n,\omega)})\}, \end{aligned} \tag{4.3}$$

where

$$R^\lambda(y_{n+1},\omega) = \begin{cases} \max\{2j+1 \in (-\infty, R^\lambda(y_n,\omega)) : G(p_{2j}) > G(p_{R^\lambda(y_n,\omega)-1})\} \\ \qquad\qquad \textit{if } \lambda - \beta V(y_{n+1},\omega) > G(p_{R^\lambda(y_n,\omega)-1}), \\ \min\{2j+1 \in (R^\lambda(y_n,\omega), +\infty) : G(p_{2j+1}) < G(p_{R^\lambda(y_n,\omega)})\} \\ \qquad\qquad \textit{if } \lambda - \beta V(y_{n+1},\omega) \le G(p_{R^\lambda(y_n,\omega)}), \end{cases} \tag{4.4}$$

and $p_0 = \inf\{p \le 0 : G(p) = \lambda + 1\}$.

Next, we will take right continuous extension of L^λ and R^λ as follows:

$$\begin{aligned} L^\lambda(x,\omega) &= L^\lambda(x_n,\omega) \quad \text{if } x \in [x_n, x_{n+1}) \quad , \text{ and,} \\ R^\lambda(y,\omega) &= R^\lambda(y_n,\omega) \quad \text{if } y \in [y_n, y_{n+1}) \end{aligned} \tag{4.5}$$

for any $n \ge 0$. Throughout the dissertation, $L^\lambda(x,\omega)$ and $R^\lambda(x,\omega)$ will be called left-leaning branch indicating function and right-leaning branch-indicating function, respectively.

In the Definition 4.3, the starting point of the sequence, x_0, was the first nonnegative real number that satisfied $\lambda - \beta V(x_0, \omega) \geq M$. The choice of x_0 is not unique, and we can, in fact, start with any point $\tilde{x}_0$ satisfying $\lambda - \beta V(\tilde{x}_0, \omega) \geq M$. Moreover, regardless of the choice of the initial point, the subsequent steps in the sequence will be identical to those starting from x_0.

Lemma 4.4. *The jumping time sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ stay the same regardless of the choice of initial values, x_0 and y_0, provided that the conditions $\lambda - \beta V(x_0, \omega) \geq M$ and $\lambda - \beta V(y_0, \omega) > M$ are satisfied. In other words, if x_0 and $\tilde{x}_0$ are two distinct initial values satisfying $\lambda - \beta V(x, \omega) \geq M$, then*

$$\{x_n : n \geq 0\} \cap [\tilde{x}_0, +\infty) = \{\tilde{x}_n : n \geq 0\} \cap [x_0, +\infty).$$

Proof. Let $\{\tilde{x}_n\}_{n\geq 0}$ be a sequence that is defined in the same manner as in Definition 4.3 but replacing x_0 with $\tilde{x}_0 < x_0$ with $\lambda - \beta V(\tilde{x}_0, \omega) \geq M$. Since $\lambda - \beta V(x_0, \omega) \geq M$, we know that there exists $m \geq 0$ such that

$$\tilde{x}_m = x_0 \quad \text{and} \quad L^\lambda(\tilde{x}_m, \omega) = 1.$$

Since $\{x_n\}_{n\geq 0}$ and $\{\tilde{x}_n\}_{n\geq 0}$ are constructed in the same way, observe that for any $n \geq 0$, we have

$$x_n = \tilde{x}_{m+n-1}, \quad L^\lambda(x_n, \omega) = L^\lambda(\tilde{x}_{m+n-1}, \omega).$$

The consistency of the sequence $\{y_n\}_{n\geq 0}$ follows the same way. $\square$

Using Lemma 4.2 and Lemma 4.4, we can extend $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ to bi-infinite sequences, $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$, with the following anchoring conditions:

$$x_{-1} < 0 \leq x_0, \text{ and,}$$

$$y_{-1} \leq 0 < y_0.$$

Lastly, we have continuity results for $\lambda \mapsto x_{n+1}(\lambda,\omega)$ and $\lambda \mapsto y_{n+1}(\lambda,\omega)$.

Lemma 4.5. *The functions $\lambda \mapsto x_{n+1}(\lambda,\omega)$ and $\lambda \mapsto y_{n+1}(\lambda,\omega)$ are right-continuous and left-continuous, respectively, for any $\omega \in \Omega$.*

Proof. First, we have a few definitions and a claim. Let $W \in C(\mathbb{R})$ and $K \in \mathbb{R}$. Introduce the following notations:

$$\overline{x}(\lambda) := \inf\{x \geq 0 : W(x) > \lambda\},$$

$$\underline{x}(\lambda) := \inf\{0 \leq x \leq K : W(x) \leq \lambda\},$$

$$\overline{y}(\lambda) := \inf\{0 \leq x \leq K : W(x) \geq \lambda\},$$

$$\underline{y}(\lambda) := \inf\{x \geq 0 : W(x) < \lambda\}.$$

We claim that $\lambda \mapsto \overline{x}(\lambda)$ and $\lambda \mapsto \underline{x}(\lambda)$ are right-continuous while $\lambda \mapsto \overline{y}(\lambda)$ and $\lambda \mapsto \underline{y}(\lambda)$ are left-continuous. Since proofs are very similar, we will only prove the first one of the above claims.

Let $\lambda_n \searrow \lambda$ and define $A_n = \{x \geq 0 : W(x) > \lambda_n\}$ and $A = \{x \geq 0 : f(x) > \lambda\}$. Observe that $A_n \nearrow A$. So, we have $\lim_{n\to+\infty} \inf A_n = \inf A$ which implies that

$$\lim_{n\to+\infty} \overline{x}(\lambda_n) = \overline{x}(\lambda),$$

i.e., $\overline{x}(\lambda)$ is right-continuous.

Now, to show that $\lambda \mapsto x_{n+1}(\lambda,\omega)$ is right-continuous, recall the definition of x_{n+1} : For each $n \in \mathbb{Z}$ and $\omega \in \Omega$, we have

$$\begin{aligned} x_{n+1}(\lambda,\omega) &= \inf\{x \geq x_n : \lambda - \beta V(x,\omega) < G(p_{L^\lambda(x_n,\omega)}) \text{ or} \\ &\qquad\qquad \lambda - \beta V(x,\omega) \geq G(p_{L^\lambda(x_n,\omega)-1})\} \\ &= \inf\{x_n \leq x \leq x_{n+2} : \beta V(x,\omega) > \lambda - G(p_{L^\lambda(x_n,\omega)}) \text{ or} \\ &\qquad\qquad \beta V(x,\omega) \leq \lambda - G(p_{L^\lambda(x_n,\omega)-1})\} \\ &= \min\{A(\lambda,\omega), B(\lambda,\omega)\}, \end{aligned}$$

where $A(\lambda,\omega) = \{\inf\{x_n \leq x \leq x_{n+2} : \beta V(x,\omega) > G(p_{L^\lambda(x_n,\omega)})\}$ and $B(\lambda,\omega) = \inf\{x_n \leq x \leq x_{n+2} : \beta V(x,\omega) \leq G(p_{L^\lambda(x_n,\omega)-1})\}$. Since $\beta V \in \mathrm{C}(R)$, by the first two of the above claims, we know that $A(\lambda,\omega)$ and $B(\lambda,\omega)$ are both right-continuous. Observe that

$$x_{n+1}(\lambda,\omega) = \min\{A(\lambda,\omega), B(\lambda,\omega)\} = \frac{A(\lambda,\omega) + B(\lambda,\omega) - |A(\lambda,\omega) - B(\lambda,\omega)|}{2}.$$

Therefore, $\lambda \mapsto x_{n+1}(\lambda,\omega)$ is right-continuous for any $\omega \in \Omega$.

Similarly, to show that $\lambda \mapsto y_{n+1}(\lambda,\omega)$ is left-continuous, recall the definition of y_{n+1} : For each $n \in \mathbb{Z}$ and $\omega \in \Omega$, we have

$$\begin{aligned} y_{n+1}(\lambda,\omega) &= \inf\{x \geq y_n : \lambda - \beta V(x,\omega) \leq G(p_{R^\lambda(x_n,\omega)}) \text{ or} \\ &\qquad\qquad \lambda - \beta V(x,\omega) > G(p_{R^\lambda(x_n,\omega)-1})\} \\ &= \inf\{y_n \leq x \leq y_{n+2} : \beta V(x,\omega) \geq \lambda - G(p_{R^\lambda(y_n,\omega)}) \text{ or} \\ &\qquad\qquad \beta V(x,\omega) < \lambda - G(p_{R^\lambda(y_n,\omega)-1})\} \\ &= \min\{C(\lambda,\omega), D(\lambda,\omega)\}, \end{aligned}$$

where $C(\lambda,\omega) = \inf\{y_n \leq x \leq y_{n+2} : \beta V(x,\omega) \geq G(p_{R^\lambda(y_n,\omega)})\}$ and $D(\lambda,\omega) = \inf\{y_n \leq x \leq y_{n+2} : \beta V(x,\omega) < G(p_{R^\lambda(y_n,\omega)-1})\}$. Since $\beta V \in \mathrm{C}(R)$, by the last two of the above claims, we know that $C(\lambda,\omega)$ and $D(\lambda,\omega)$ are both left-continuous. Observe that

$$y_{n+1}(\lambda,\omega) = \min\{C(\lambda,\omega), D(\lambda,\omega)\} = \frac{C(\lambda,\omega) + D(\lambda,\omega) - |C(\lambda,\omega) - D(\lambda,\omega)|}{2}.$$

Therefore, $\lambda \mapsto y_{n+1}(\lambda,\omega)$ is left-continuous for any $\omega \in \Omega$. □

4.2 Properties of Branch-indicating Functions

In this section, we will explore properties of branch-indicating functions introduced in Definition 4.3. We start by showing that left-leaning branch-indicating function is always less than or equal to right-leaning branch-indicating function.

Lemma 4.6. *For any $x \in \mathbb{R}$, $\omega \in \Omega$, and $\lambda \geq \beta$, we have $L^\lambda(x,\omega) \leq R^\lambda(x,\omega)$.*

Proof. Let

$$\overline{x} = \sup\{x \in \mathbb{R} : L^\lambda(y,\omega) \leq R^\lambda(y,\omega) \text{ for all } y \in (-\infty, x)\}. \tag{4.6}$$

Suppose that $\overline{x} < +\infty$. We claim that $R^\lambda(\overline{x},\omega) < L^\lambda(\overline{x},\omega)$. Assume, for contradiction, that $R^\lambda(\overline{x},\omega) \geq L^\lambda(\overline{x},\omega)$. Then, since L^λ and R^λ are right-continuous and piece-wise constant functions, for some $n, m \in \mathbb{Z}$, we have

$$\overline{x} \in [x_n, x_{n+1}) \text{ and } R^\lambda(\overline{x},\omega) = R^\lambda(x_n,\omega), \quad \text{and,}$$
$$\overline{x} \in [x_m, x_{m+1}) \text{ and } L^\lambda(\overline{x},\omega) = L^\lambda(x_m,\omega).$$

By the definition of $\overline{x}$, we know that $R^\lambda(\overline{x}^+,\omega) < L^\lambda(\overline{x}^+,\omega)$. This means that either $x_{n+1} = \overline{x}$, or $x_{m+1} = \overline{x}$. In both cases, we have a contradiction since either $\overline{x} \notin [x_n, x_{n+1})$ or $\overline{x} \notin [x_m, x_{m+1})$. Let $\xi = \min\{x_{n+1}, x_{m+1}\}$. Then $\overline{x} < \xi$ and for any $x \in [\overline{x}, \xi)$, we have

$$L^\lambda(x,\omega) = L^\lambda(\overline{x},\omega) \leq R^\lambda(\overline{x},\omega) = R^\lambda(x,\omega).$$

This contradicts (4.6). Thus, $R^\lambda(\overline{x},\omega) < L^\lambda(\overline{x},\omega)$.

By (4.6), we have $L^\lambda(x,\omega) \leq R^\lambda(x,\omega)$ for all $x \leq \overline{x}$, and $L^\lambda(\overline{x},\omega) > R^\lambda(\overline{x},\omega)$. In order this to hold, either L^λ has to increase at $\overline{x}$, or R^λ has to decrease at $\overline{x}$. Note that, by Definition 4.3, we know that L^λ and R^λ cannot jump at the same time to different directions. Then, there are two possible cases.

1. If L^λ increases at $\overline{x}$, then there are two subcases.

- Suppose that $L^\lambda(\overline{x}^-,\omega) = R^\lambda(\overline{x}^-,\omega)$. This equality implies that $G(p_{L^\lambda(\overline{x}^-,\omega)}) = G(p_{R^\lambda(\overline{x}^-,\omega)})$ and we have

$$\begin{aligned} L^\lambda(\overline{x},\omega) &= \min\{2j+1 \in (L^\lambda(\overline{x}^-,\omega),+\infty) : G(p_{2j+1}) < G(p_{L^\lambda(\overline{x}^-,\omega)})\} \\ &= \min\{2j+1 \in (R^\lambda(\overline{x}^-,\omega),+\infty) : G(p_{2j+1}) < G(p_{R^\lambda(\overline{x}^-,\omega)})\} \\ &= R^\lambda(\overline{x},\omega). \end{aligned}$$

- Suppose now $L^\lambda(\overline{x}^-,\omega) < R^\lambda(\overline{x}^-,\omega)$. Then, this implies that $(R^\lambda(\overline{x}^-,\omega),+\infty) \subset (L^\lambda(\overline{x}^-,\omega),+\infty)$ and by (2.14), we have

$$\begin{aligned} G(p_{L^\lambda(\overline{x}^-,\omega)}) &= G(f_L^\lambda(\overline{x}^-,\omega)) \\ &= \lambda - \beta V(\overline{x}^-,\omega) \\ &= G(f_R^\lambda(\overline{x}^-,\omega)) \\ &\geq G(p_{R^\lambda(\overline{x}^-,\omega)}) \\ &\geq G(p_{R^\lambda(\overline{x},\omega)}). \end{aligned}$$

So, it follows that

$$R^\lambda(\overline{x},\omega) \in \{2j+1 \in (L^\lambda(\overline{x}^-,\omega),+\infty) : G(p_{2j+1}) < G(p_{L^\lambda(\overline{x}^-,\omega)})\}.$$

Therefore,

$$\begin{aligned} L^\lambda(\overline{x},\omega) &= \min\{2j+1 \in (L^\lambda(\overline{x}^-,\omega),+\infty) : G(p_{2j+1}) < G(p_{L^\lambda(\overline{x}^-,\omega)})\} \\ &\leq R^\lambda(\overline{x},\omega). \end{aligned}$$

2. If R^λ decreases at $\overline{x}$, then there are two more subcases.

- Suppose that $L^\lambda(\overline{x}^-,\omega) = R^\lambda(\overline{x}^-,\omega)$. Then $G(p_{L^\lambda(\overline{x}^-,\omega)}) = G(p_{R^\lambda(\overline{x}^-,\omega)})$ and we have

$$\begin{aligned} R^\lambda(\overline{x},\omega) &= \max\{2j+1 \in (-\infty, R^\lambda(\overline{x}^-,\omega)) : G(p_{2j}) > G(p_{R^\lambda(\overline{x}^-,\omega)-1})\} \\ &= \max\{2j+1 \in (-\infty, L^\lambda(\overline{x}^-,\omega)) : G(p_{2j}) > G(p_{L^\lambda(\overline{x}^-,\omega)-1})\} \\ &= L^\lambda(\overline{x},\omega). \end{aligned}$$

- Suppose now $L^\lambda(\overline{x}^-,\omega) < R^\lambda(\overline{x}^-,\omega)$. Then, $(-\infty, L^\lambda(\overline{x}^-,\omega)) \subset (-\infty, R^\lambda(\overline{x}^-,\omega))$ and by (2.14), we have

$$\begin{aligned} G(p_{L^\lambda(\overline{x}^-,\omega)}) &= G(f_L^\lambda(\overline{x}^-,\omega)) \\ &= \lambda - \beta V(\overline{x}^-,\omega) \\ &= G(f_R^\lambda(\overline{x}^-,\omega)) \\ &\geq G(p_{R^\lambda(\overline{x}^-,\omega)}) \\ &\geq G(p_{R^\lambda(\overline{x},\omega)}). \end{aligned}$$

So, it follows that

$$L^\lambda(\overline{x},\omega) \in \{2j+1 \in (-\infty, R^\lambda(\overline{x}^-,\omega)) : G(p_{2j}) > G(p_{R^\lambda(\overline{x}^-,\omega)-1})\}.$$

Therefore,

$$\begin{aligned} L^\lambda(\overline{x},\omega) &= \max\{2j+1 \in (L^\lambda(\overline{x}^-,\omega), +\infty) : G(p_{2j+1}) < G(p_{L^\lambda(\overline{x}^-,\omega)})\} \\ &\leq R^\lambda(\overline{x},\omega). \end{aligned}$$

□

Our next observation is that branch-indicating functions are decreasing in λ.

Lemma 4.7. *The functions* $\lambda \mapsto L^{\lambda}(x,\omega)$ *and* $\lambda \mapsto R^{\lambda}(x,\omega)$ *are decreasing.*

Proof. Take $\lambda_1 < \lambda_2$. Let

$$\overline{x} = \sup\{x \in \mathbb{R} : L^{\lambda_1}(y,\omega) \geq L^{\lambda_2}(y,\omega) \text{ for all } y \in (-\infty, x)\}. \tag{4.7}$$

Suppose that $\overline{x} < +\infty$. We claim that $L^{\lambda_1}(\overline{x},\omega) < L^{\lambda_2}(\overline{x},\omega)$. Assume, for contradiction, that $L^{\lambda_1}(\overline{x},\omega) \geq L^{\lambda_2}(\overline{x},\omega)$. Then, since L^{λ} is a right-continuous function, we know that for some $n, m \in \mathbb{Z}$, we have

$$\overline{x} \in [x_n, x_{n+1}) \text{ and } L^{\lambda_1}(\overline{x},\omega) = L^{\lambda_1}(x_n,\omega), \quad \text{and,}$$
$$\overline{x} \in [x_m, x_{m+1}) \text{ and } L^{\lambda_2}(\overline{x},\omega) = L^{\lambda_2}(x_m,\omega).$$

Let $\xi = \min\{x_{n+1}, x_{m+1}\}$. Then $\overline{x} < \xi$ and for any $x \in [\overline{x}, \xi)$, we have

$$L^{\lambda_1}(x,\omega) = L^{\lambda_1}(x_n,\omega) \geq L^{\lambda_2}(x_m,\omega) = L^{\lambda_2}(x,\omega).$$

This contradicts (4.7). Thus, $L^{\lambda_1}(\overline{x},\omega) < L^{\lambda_2}(\overline{x},\omega)$.

First, we claim that L^{λ_1} and L^{λ_2} cannot move at the same time to different directions. Indeed, assume for the sake of contradiction that L^{λ_1} decreases and L^{λ_2} increases at $\overline{x}$. Then,

$$\lambda_1 - \beta V(\overline{x},\omega) = G(p_{L^{\lambda_1}(\overline{x}^-,\omega)-1}), \quad \text{and,}$$
$$\lambda_2 - \beta V(\overline{x},\omega) = G(p_{L^{\lambda_2}(\overline{x}^-,\omega)}).$$

Let $\alpha = \lambda_2 - \lambda_1$. Then,

$$\begin{aligned}\alpha &= \lambda_2 - \lambda_1 \\ &= (\lambda_2 - \beta V(\overline{x}, \omega)) - (\lambda_1 - \beta V(\overline{x}, \omega)) \\ &= G(p_{L^{\lambda_2}(\overline{x}^-, \omega)}) - G(p_{L^{\lambda_1}(\overline{x}^-, \omega) - 1}).\end{aligned}$$

We can find a small $\epsilon > 0$ such that $L^{\lambda_1}(\overline{x} - \epsilon, \omega) = L^{\lambda_1}(\overline{x}^-, \omega)$ and $L^{\lambda_2}(\overline{x} - \epsilon, \omega) = L^{\lambda_2}(\overline{x}^-, \omega)$. Then,

$$\begin{aligned}&\lambda_1 - \beta V(\overline{x} - \epsilon, \omega) < G(p_{L^{\lambda_1}(\overline{x}^-, \omega) - 1}), \qquad \text{and,} \\ &\lambda_2 - \beta V(\overline{x} - \epsilon, \omega) \geq G(p_{L^{\lambda_2}(\overline{x}^-, \omega)}),\end{aligned}$$

since we know that if $\lambda_1 - \beta V(\overline{x} - \epsilon, \omega) = G(p_{L^{\lambda_1}(\overline{x}^-, \omega) - 1})$, it would have to jump to left and we would not have $L^{\lambda_1}(\overline{x} - \epsilon, \omega) = L^{\lambda_1}(\overline{x}^-, \omega)$. This is not the case for λ_2 because $\lambda_2 - \beta V(\overline{x} - \epsilon, \omega) = G(p_{L^{\lambda_2}(\overline{x}^-, \omega)})$ is not enough to observe a right jump. This implies that

$$\alpha > G(p_{L^{\lambda_2}(\overline{x}^-, \omega)}) - G(p_{L^{\lambda_1}(\overline{x}^-, \omega) - 1})$$

which is a contradiction. The other direction will follow similarly.

In order to (4.7) to hold, we know that either L_1^{λ} has to decrease at $\overline{x}$, or L_2^{λ} has to increase at $\overline{x}$. Let us analyze each of these cases.

- Suppose that L^{λ_1} decreases at $\overline{x}$. Then $\lambda_1 - \beta V(\overline{x}, \omega) = G(p_{L^{\lambda_1}(\overline{x}^-, \omega) - 1})$ and

  $$L^{\lambda_1}(\overline{x}, \omega) = \max\{2j + 1 \in (-\infty, L^{\lambda_1}(\overline{x}^-, \omega)) : G(p_{2j}) > G(p_{L^{\lambda_1}(\overline{x}^-, \omega) - 1})\}. \tag{4.8}$$

We know that L^{λ_2} cannot increase at $\overline{x}$. So, $L^{\lambda_2}(\overline{x},\omega) \leq L^{\lambda_2}(\overline{x}^-,\omega) \leq L^{\lambda_1}(\overline{x}^-,\omega)$. Moreover,

$$p_{L^{\lambda_2}(\overline{x},\omega)-1} \leq (f_L^{\lambda_2})'(\overline{x},\omega) \leq p_{L^{\lambda_2}(\overline{x},\omega)}$$

which implies that $G(p_{L^{\lambda_2}(\overline{x},\omega)}) \leq G((f_L^{\lambda_2})'(\overline{x},\omega)) \leq G(p_{L^{\lambda_2}(\overline{x},\omega)-1})$. By (2.14), we have

$$\lambda_2 - \beta V(\overline{x},\omega) \leq G(p_{L^{\lambda_2}(\overline{x},\omega)-1}).$$

Since $\lambda_2 > \lambda_1$, we have $\lambda_1 - \beta V(\overline{x},\omega) = G(p_{L_1^{\lambda}(\overline{x}^-,\omega)-1}) < \lambda_2 - \beta V(\overline{x},\omega)$. Therefore,

$$G(p_{L^{\lambda_1}(\overline{x}^-,\omega)-1}) < G(p_{L^{\lambda_2}(\overline{x},\omega)-1}).$$

So, we have

$$L^{\lambda_2}(\overline{x},\omega) \in \{2j+1 \in (-\infty, L^{\lambda_1}(\overline{x}^-,\omega)) : G(p_{2j}) > G(p_{L^{\lambda_1}(\overline{x}^-,\omega)-1})\},$$

and $L^{\lambda_1}(\overline{x},\omega) \geq L^{\lambda_2}(\overline{x},\omega)$ by (4.8).

- Suppose that L^{λ_2} increases at $\overline{x}$. Then $\lambda_2 - \beta V(\overline{x},\omega) = G(p_{L^{\lambda_2}(\overline{x}^-,\omega)})$ and

$$L^{\lambda_2}(\overline{x},\omega) = \min\{2j+1 \in (L^{\lambda_2}(\overline{x}^-,\omega), +\infty) : G(p_{2j+1}) < G(p_{L^{\lambda_2}(\overline{x}^-,\omega)})\}. \tag{4.9}$$

We know that L^{λ_1} cannot decrease at $\overline{x}$. So, $L^{\lambda_1}(\overline{x},\omega) \geq L^{\lambda_1}(\overline{x}^-,\omega) \geq L^{\lambda_2}(\overline{x}^-,\omega)$. Moreover,

$$p_{L^{\lambda_1}(\overline{x},\omega)-1} \leq (f_L^{\lambda_1})'(\overline{x},\omega) \leq p_{L^{\lambda_1}(\overline{x},\omega)}$$

which implies that $G(p_{L^{\lambda_1}(\overline{x},\omega)}) \leq G((f_L^{\lambda_1})'(\overline{x},\omega)) \leq G(p_{L^{\lambda_1}(\overline{x},\omega)-1})$. By (2.14), we have

$$G(p_{L^{\lambda_1}(\overline{x},\omega)}) \leq \lambda_1 - \beta V(\overline{x},\omega).$$

Since $\lambda_2 > \lambda_1$, we have $\lambda_2 - \beta V(\overline{x},\omega) = G(p_{L^{\lambda_2}(\overline{x}^-,\omega)}) > \lambda_1 - \beta V(\overline{x},\omega)$. Therefore,

$$G(p_{L^{\lambda_1}(\overline{x},\omega)}) < G(p_{L^{\lambda_2}(\overline{x}^-,\omega)}).$$

So, we have

$$L^{\lambda_1}(\overline{x},\omega) \in \{2j+1 \in (L^{\lambda_2}(\overline{x}^-,\omega),+\infty) : G(p_{2j+1}) < G(p_{L^{\lambda_2}(\overline{x}^-,\omega)})\},$$

and $L^{\lambda_1}(\overline{x},\omega) \geq L^{\lambda_2}(\overline{x},\omega)$ by (4.9).

The proof of the second part is similar. □

Next, we demonstrate the stationarity of branch-indicating functions.

Lemma 4.8. *For any $x \in \mathbb{R}$, $\omega \in \Omega$, and $\lambda \geq \beta$ we have $L^\lambda(x,\omega) = L^\lambda(0,\tau_x\omega)$ and $R^\lambda(x,\omega) = R^\lambda(0,\tau_x\omega)$.*

Proof. Let $\lambda \geq \beta$. Fix $\tilde{x} \in \mathbb{R}$ such that $\lambda - \beta V(\tilde{x},\omega) > M$. Then, $L^\lambda(\tilde{x},\omega) = 1$ By (2.10), we know that

$$\lambda - \beta V(\tilde{x},\omega) = \lambda - \beta V(0,\tau_{\tilde{x}}\omega) > M$$

and $L^\lambda(0,\tau_{\tilde{x}}\omega) = 1$. We start running sequences from $(\tilde{x},\omega)$ and $(0,\tau_{\tilde{x}}\omega)$. By Lemma 4.4, we know that shifting the index of the jumping times does not affect its behavior. Since L^λ depends only on $\lambda - \beta V(x,\omega)$ and the jumping times, by (2.10) and the fact that the jumping times are the same, we have

$$L^\lambda(x,\omega) = L^\lambda(0,\tau_x\omega) \text{ for any } x \geq \tilde{x}.$$

Since we know that $\inf\{x \in \mathbb{R} : \lambda - \beta V(x,\omega) > M\} = -\infty$ by Lemma 4.2, we conclude that

$$L^\lambda(x,\omega) = L^\lambda(0,\tau_x\omega) \text{ for all } x \in \mathbb{R} \text{ and } \omega \in \Omega.$$

We can do the same for R^λ and have

$$R^\lambda(x,\omega) = R^\lambda(0,\tau_x\omega) \text{ for all } x \in \mathbb{R} \text{ and } \omega \in \Omega. \qquad \square$$

Now, let us compare the left-leaning branch-indicating function and the right-leaning branch-indicating function for different λ values.

Lemma 4.9. *Let* $\lambda_1 < \lambda_2$ *be such that* $\beta \leq \lambda_1$ *and* $\lambda_2 - \lambda_1 < \delta$ *where* $\delta < \min_{1\leq i,j\leq N} |G(p_{2i+1}) - G(p_{2j+1})|$*. Then, for any* $x \in \mathbb{R}$ *and* $\omega \in \Omega$*, we have* $L^{\lambda_1}(x,\omega) \geq R^{\lambda_2}(x,\omega)$*.*

Proof. Let

$$\overline{x} = \sup\{x \in \mathbb{R} : L^{\lambda_1}(y,\omega) \geq R^{\lambda_2}(y,\omega) \text{ for all } y \in (-\infty, x)\}. \tag{4.10}$$

Assume, for the sake of contradiction, that $\overline{x} < +\infty$. We claim that $L^{\lambda_1}(\overline{x},\omega) < R^{\lambda_2}(\overline{x},\omega)$. Indeed, if we had $L^{\lambda_1}(\overline{x},\omega) \geq R^{\lambda_2}(\overline{x},\omega)$, since L^{λ_1} and R^{λ_2} are right-continuous and piecewise-constant functions, for some $n, m \in \mathbb{Z}$, we would have

$$\overline{x} \in [x_n, x_{n+1}) \text{ and } L^{\lambda_1}(\overline{x}, \omega) = L^{\lambda_1}(x_n, \omega), \text{ and,}$$
$$\overline{x} \in [x_m, x_{m+1}) \text{ and } R^{\lambda_2}(\overline{x}, \omega) = R^{\lambda_2}(x_m, \omega).$$

Let $\xi = \min\{x_{n+1}, x_{m+1}\}$. Then $\overline{x} < \xi$ and for any $x \in [\overline{x}, \xi)$, we would have

$$L^{\lambda_1}(x, \omega) = L^{\lambda_1}(\overline{x}, \omega) \geq R^{\lambda_2}(\overline{x}, \omega) = R^{\lambda_2}(x, \omega).$$

This contradicts with (4.10). Thus, $L^{\lambda_1}(\overline{x}, \omega) < R^{\lambda_2}(\overline{x}, \omega)$.

Then, there are four possible cases.

1. If L^{λ_1} increases at $\overline{x}$, then in order to $L^{\lambda_1}(\overline{x}, \omega) < R^{\lambda_2}(\overline{x}, \omega)$ to hold, R^{λ_2} has to increase even more at $\overline{x}$. Due to the distinctness of $G(p_i)'$s and the fact that $\lambda_2 - \lambda_1 < \delta$ where $\delta < \min_{1 \leq i,j \leq N} |G(p_{2i+1}) - G(p_{2j+1})|$, we know that L^{λ_1} and R^{λ_2} can jump at the same time if $L^{\lambda_1}(\overline{x}^-, \omega) = R^{\lambda_2}(\overline{x}^-, \omega)$. Recall that

   $$L^{\lambda_1}(\overline{x}, \omega) = \min\{2j+1 \in (L^{\lambda_1}(\overline{x}^-, \omega), +\infty) : G(p_{2j+1}) < G(p_{L^{\lambda_1}(\overline{x}^-, \omega)})\}$$
   $$G^{\lambda_w}(\overline{x}, \omega) = \min\{2j+1 \in (R^{\lambda_2}(\overline{x}^-, \omega), +\infty) : G(p_{2j+1}) < G(p_{R^{\lambda_2}(\overline{x}^-, \omega)})\}.$$

   Thus, $L^{\lambda_1}(\overline{x}, \omega) = R^{\lambda_2}(\overline{x}, \omega)$.

2. If L^{λ_1} decreases at $\overline{x}$, then there are two subcases.

   - Suppose that $L^{\lambda_1}(\overline{x}^-, \omega) = R^{\lambda_2}(\overline{x}^-, \omega)$. Then we have $G(p_{L^{\lambda_1}(\overline{x}^-, \omega)}) =$

$G(p_{R^{\lambda_2}(\overline{x}^-,\omega)})$ and

$$\begin{aligned} L^{\lambda_1}(\overline{x},\omega) &= \max\{2j+1 \in (-\infty, L^{\lambda_1}(\overline{x}^-,\omega)) : G(p_{2j}) > G(p_{L^{\lambda_1}(\overline{x}^-,\omega)-1})\} \\ &= \max\{2j+1 \in (-\infty, R^{\lambda_2}(\overline{x}^-,\omega)) : G(p_{2j}) > G(p_{R^{\lambda_2}(\overline{x}^-,\omega)-1})\} \\ &= R^{\lambda_2}(\overline{x},\omega). \end{aligned}$$

- Suppose now $L^{\lambda_1}(\overline{x}^-,\omega) > R^{\lambda_2}(\overline{x}^-,\omega)$. Then, since $\lambda_2 - \lambda_1 < \delta$ where $\delta < \min_{1\leq i,j\leq N} |G(p_{2i+1}) - G(p_{2j+1})|$, and $G(p_i)'$s are distinct, when L^{λ_1} decreases at $\overline{x}$, R^{λ_2} has to stay the same at $\overline{x}$. Hence

  $$R^{\lambda_2}(\overline{x},\omega) = R^{\lambda_2}(\overline{x}^-,\omega) \leq L^{\lambda_1}(\overline{x}^-,\omega).$$

  Observe that $R^{\lambda_2}(\overline{x},\omega) = 2j+1$ for some $j \in \mathbb{Z}^+$ and $R^{\lambda_2}(\overline{x},\omega) \in (-\infty, L^{\lambda_1}(\overline{x}^-,\omega))$. Moreover,

  $$\begin{aligned} G(p_{R^{\lambda_2}(\overline{x},\omega)-1}) &= G(p_{R^{\lambda_2}(\overline{x}^-,\omega)-1}) \\ &= G(f_{R^{\lambda_2}(\overline{x}^-,\omega)-1}) \\ &= \lambda_2 - \beta V(\overline{x}^-,\omega) \\ &= (\lambda_2 - \lambda_1) + \lambda_1 - \beta V(\overline{x}^-,\omega) \\ &> \lambda_1 - \beta V(\overline{x}^-,\omega) \\ &= G(f_{L^{\lambda_1}(\overline{x}^-,\omega)}) \\ &= G(p_{L^{\lambda_1}(\overline{x}^-,\omega)-1}). \end{aligned}$$

  Therefore,

  $$R^{\lambda_2}(\overline{x},\omega) \in \{2j+1 \in (-\infty, L^{\lambda_1}(\overline{x}^-,\omega)) : G(p_{2j}) > G(p_{L^{\lambda_1}(\overline{x}^-,\omega)-1})\}.$$

Since

$$L^{\lambda_1}(\overline{x},\omega) = \max\{2j{+}1 \in (-\infty, L^{\lambda_1}(\overline{x}^-,\omega)) : G(p_{2j}) > G(p_{L^{\lambda_1}(\overline{x}^-,\omega)-1})\},$$

we conclude that $L^{\lambda_1}(\overline{x},\omega) \geq R^{\lambda_2}(\overline{x},\omega)$.

3. If R^{λ_2} increases at $\overline{x}$, then there are two subcases.

   - Suppose that $L^{\lambda_1}(\overline{x}^-,\omega) = R^{\lambda_2}(\overline{x}^-,\omega)$. Then $G(p_{L^{\lambda_1}(\overline{x}^-,\omega)}) = G(p_{R^{\lambda_2}(\overline{x}^-,\omega)})$ and we have

     $$\begin{aligned} R^{\lambda_2}(\overline{x},\omega) &= \min\{2j+1 \in (R^{\lambda_2}(\overline{x}^-,\omega), +\infty) : G(p_{2j+1}) < G(p_{R^{\lambda_2}(\overline{x}^-,\omega)})\} \\ &= \min\{2j+1 \in (L^{\lambda_1}(\overline{x}^-,\omega)) : G(p_{2j+1}) > G(p_{L^{\lambda_1}(\overline{x}^-,\omega)})\} \\ &= L^{\lambda_1}(\overline{x},\omega). \end{aligned}$$

   - Suppose now $L^{\lambda_1}(\overline{x}^-,\omega) > R^{\lambda_2}(\overline{x}^-,\omega)$. Then, since $\lambda_2 - \lambda_1 < \delta$ where $\delta < \min_{1\leq i,j\leq N} |G(p_{2i+1}) - G(p_{2j+1})|$, and $G(p_i)'$s are distinct, when R^{λ_2} increases at $\overline{x}$, L^{λ_1} has to stay the same at $\overline{x}$. Hence

     $$L^{\lambda_1}(\overline{x},\omega) = L^{\lambda_1}(\overline{x}^-,\omega) > R^{\lambda_2}(\overline{x}^-,\omega).$$

     Observe that $L^{\lambda_1}(\overline{x},\omega) = 2j+1$ for some $j \in \mathbb{Z}^+$ and $L^{\lambda_1}(\overline{x},\omega) \in$

$(R^{\lambda_2}(\overline{x}^-, \omega), +\infty)$. Moreover,

$$\begin{aligned} G(p_{L^{\lambda_1}(\overline{x},\omega)}) &= G(p_{L^{\lambda_1}(\overline{x}^-,\omega)}) \\ &= G(f_{L^{\lambda_1}(\overline{x}^-,\omega)}) \\ &= \lambda_1 - \beta V(\overline{x}^-, \omega) \\ &= (\lambda_1 - \lambda_2) + \lambda_2 - \beta V(\overline{x}^-, \omega) \\ &< \lambda_2 - \beta V(\overline{x}^-, \omega) \\ &= G(f_{R^{\lambda_2}(\overline{x}^-,\omega)}) \\ &= G(p_{R^{\lambda_2}(\overline{x}^-,\omega)}). \end{aligned}$$

Therefore,

$$L^{\lambda_1}(\overline{x}, \omega) \in \{2j{+}1 \in (R^{\lambda_2}(\overline{x}^-, \omega), +\infty) : G(p_{2j+1}) < G(p_{R^{\lambda_2}(\overline{x}^-,\omega)})\}.$$

Since

$$R^{\lambda_2}(\overline{x}, \omega) = \min\{2j{+}1 \in (R^{\lambda_2}(\overline{x}^-, \omega), +\infty) : G(p_{2j+1}) < G(p_{R^{\lambda_2}(\overline{x}^-,\omega)})\},$$

we conclude that $R^{\lambda_2}(\overline{x}, \omega) \leq L^{\lambda_1}(\overline{x}, \omega)$.

4. If R^{λ_2} decreases at $\overline{x}$, then in order to $L^{\lambda_1}(\overline{x}, \omega) < R^{\lambda_2}(\overline{x}, \omega)$ to hold, L^{λ_1} has to decrease even more at $\overline{x}$. Due to the distinctness of $G(p_i)'$s and the fact that $\lambda_2 - \lambda_1 < \delta$ where $\delta < \min_{1 \leq i,j \leq N} |G(p_{2i+1}) - G(p_{2j+1})|$, we know that L^{λ_1} and R^{λ_2} can jump at the same time if $L^{\lambda_1}(\overline{x}^-, \omega) = R^{\lambda_2}(\overline{x}^-, \omega)$.

Recall that

$$L^{\lambda_1}(\overline{x},\omega) = \max\{2j+1 \in (-\infty, L^{\lambda_1}(\overline{x}^-,\omega)-1) : G(p_{2j}) > G(p_{L^{\lambda_1}(\overline{x}^-,\omega)-1})\}$$

$$G^{\lambda_w}(\overline{x},\omega) = \max\{2j+1 \in (-\infty, R^{\lambda_2}(\overline{x}^-,\omega)-1) : G(p_{2j}) > G(p_{R^{\lambda_2}(\overline{x}^-,\omega)-1})\}.$$

Thus, $L^{\lambda_1}(\overline{x},\omega) = R^{\lambda_2}(\overline{x},\omega)$.

In each case, we reach a contradiction. Therefore $\overline{x} = +\infty$ and we have $L^{\lambda_1}(x,\omega) \geq R^{\lambda_2}(x,\omega)$. □

4.3 Properties of Correctors

Using the left-leaning and right-leaning branch indicating functions and their properties that we derived in the previous section, we are ready to define our correctors. For every $\lambda \in [\beta, +\infty)$ and $\omega \in \Omega$, consider the static HJ equation in (2.14). The functions $f_L^\lambda, f_R^\lambda \in \mathrm{C}(\mathbb{R})$ defined by $f_L^\lambda(0,\omega) = 0, f_R^\lambda(0,\omega) = 0$, and

$$\begin{aligned} (f_L^\lambda)'(x,\omega) &= G^{-1}_{L^\lambda(x,\omega)}(\lambda - \beta V(x,\omega)), \quad x \in \mathbb{R}, \\ (f_R^\lambda)'(x,\omega) &= G^{-1}_{R^\lambda(x,\omega)}(\lambda - \beta V(x,\omega)), \quad x \in \mathbb{R}, \end{aligned} \tag{4.11}$$

are solutions of (2.14), respectively. In this section, we obtain properties of correctors and their derivatives.

Lemma 4.10. *For any $x \in \mathbb{R}$, $\omega \in \Omega$, and $\lambda \geq \beta$, we have $(f_L^\lambda)'(x,\omega) \leq (f_R^\lambda)'(x,\omega)$.*

Proof. Recall, from Lemma 4.6 that $L^\lambda(x,\omega) \leq R^\lambda(x,\omega)$. First, observe that if $L^\lambda(x,\omega) = R^\lambda(x,\omega)$, then,

$$\begin{aligned}(f_L^\lambda)'(x,\omega) &= G^{-1}_{L^\lambda(x,\omega)}(\lambda - \beta V(x,\omega)) \\ &= G^{-1}_{R^\lambda(x,\omega)}(\lambda - \beta V(x,\omega)) \\ &= (f_R^\lambda)'(x,\omega).\end{aligned}$$

Assume that $L^\lambda(x,\omega) < R^\lambda(x,\omega)$. Then, we know that

$$p_{L^\lambda(x,\omega)-1} \leq G^{-1}_{L^\lambda(x,\omega)}(\lambda - \beta V(x,\omega)) \leq p_{L^\lambda(x,\omega)} \tag{4.12}$$

$$p_{R^\lambda(x,\omega)-1} \leq G^{-1}_{R^\lambda(x,\omega)}(\lambda - \beta V(x,\omega)) \leq p_{R^\lambda(x,\omega)} \tag{4.13}$$

and since $L^\lambda(x,\omega)$ and $R^\lambda(x,\omega)$ are odd-integer-valued functions, we have

$$p_{L^\lambda(x,\omega)} < p_{R^\lambda(x,\omega)}. \tag{4.14}$$

So,

$$\begin{aligned}(f_L^\lambda)'(x,\omega) &= G^{-1}_{L^\lambda(x,\omega)}(\lambda - \beta V(x,\omega)) && \text{by (4.11)} \\ &\leq p_{L^\lambda(x,\omega)} && \text{by (4.12)} \\ &< p_{R^\lambda(x,\omega)-1} && \text{by (4.14)} \\ &\leq G_{R^\lambda(x,\omega)}(\lambda - \beta V(x,\omega)) && \text{by (4.13)} \\ &= (f_R^\lambda)'(x,\omega) && \text{by (4.11)}\end{aligned}$$

∎

By combining (2.10), (4.11), and Lemma 4.8, we get

$$\begin{aligned}(f_L^\lambda)'(x,\omega) &= G^{-1}_{L^\lambda(x,\omega)}(\lambda-\beta V(x,\omega))\\ &= G^{-1}_{L^\lambda(0,\tau_x\omega)}(\lambda-\beta V(0,\tau_x\omega))\\ &= (f_L^\lambda)'(0,\tau_x\omega).\end{aligned}$$

Using this, we define

$$\begin{aligned}\theta_L(\lambda) &:= \lim_{x\to\pm+\infty}\frac{f_L^\lambda(x,\omega)}{x} = \lim_{x\to\pm+\infty}\frac{1}{x}\int_0^x (f_L^\lambda)'(y,\omega)dy = \mathbb{E}[(f_L^\lambda)'(0,\omega)],\\ \theta_R(\lambda) &:= \lim_{x\to\pm+\infty}\frac{f_R^\lambda(x,\omega)}{x} = \lim_{x\to\pm+\infty}\frac{1}{x}\int_0^x (f_R^\lambda)'(y,\omega)dy = \mathbb{E}[(f_R^\lambda)'(0,\omega)],\end{aligned} \tag{4.15}$$

where the last equality follows from the ergodic theorem.

Lemma 4.11. *The functions* $\lambda\mapsto(f_L^\lambda)'(0,\omega)$ *and* $\lambda\mapsto(f_R^\lambda)'(0,\omega)$ *are decreasing.*

Proof. Let $\lambda_1<\lambda_2$. Then since we know that $L^{\lambda_1}(0,\omega)\geq L^{\lambda_2}(0,\omega)$ by Lemma 4.7, by using a similar argument to the one in the proof of Lemma 4.10 and the fact that G^{-1} is decreasing, we see that

$$\begin{aligned}(f_L^{\lambda_1})'(0,\omega) &= G^{-1}_{L^{\lambda_1}(0,\omega)}(\lambda_1-\beta V(0,\omega))\\ &\geq G^{-1}_{L^{\lambda_2}(0,\omega)}(\lambda_1-\beta V(0,\omega))\\ &\geq G^{-1}_{L^{\lambda_2}(0,\omega)}(\lambda_2-\beta V(0,\omega))\\ &= (f_L^{\lambda_2})'(0,\omega),\end{aligned}$$

and the function $\lambda \mapsto (f_L^\lambda)'(0,\omega)$ is decreasing. Similarly, $\lambda \mapsto (f_R^\lambda)'(0,\omega)$ is decreasing. $\square$

Corollary 4.12. *The functions $\lambda \mapsto \theta_L(\lambda)$ and $\lambda \mapsto \theta_R(\lambda)$ are decreasing.*

Proof. Since $\theta_L(\lambda) = \mathbb{E}[(f_L^\lambda)'(0,\omega)]$ and $\theta_R(\lambda) = \mathbb{E}[(f_R^\lambda)'(0,\omega)]$, using the monotonicity of expectation, we conclude that the function $\lambda \mapsto \theta_L(\lambda)$ and $\lambda \mapsto \theta_R(\lambda)$ are decreasing as well. $\square$

Next, we want to show that the functions $\lambda \mapsto \theta_L(\lambda)$ and $\lambda \mapsto \theta_R(\lambda)$ are right-continuous and left-continuous, respectively, using the continuity result that derived in Lemma 4.5 for x_n and y_n.

Lemma 4.13. *The functions $\lambda \mapsto \theta_L(\lambda)$ and $\lambda \mapsto \theta_R(\lambda)$ are right-continuous and left-continuous, respectively.*

Proof. Recall that

$$\theta_L(\lambda) := \lim_{x\to\pm\infty} \frac{f_L^\lambda(x,\omega)}{x} = \lim_{x\to\pm\infty} \frac{1}{x}\int_0^x (f_L^\lambda)'(y,\omega)dy = \mathbb{E}[(f_L^\lambda)'(0,\omega)]$$
$$\theta_R(\lambda) := \lim_{x\to\pm\infty} \frac{f_R^\lambda(x,\omega)}{x} = \lim_{x\to\pm\infty} \frac{1}{x}\int_0^x (f_R^\lambda)'(y,\omega)dy = \mathbb{E}[(f_R^\lambda)'(0,\omega)].$$

Let $\lambda_n \searrow \lambda$ and consider

$$\begin{aligned}\lim_{n\to+\infty} \theta_L(\lambda_n) &= \lim_{n\to+\infty} \mathbb{E}[(f_L^{\lambda_n})'(0,\omega)],\\ &= \lim_{n\to+\infty} \mathbb{E}[G^{-1}_{L^{\lambda_n}(0,\omega)}(\lambda_n - \beta V(0,\omega))].\end{aligned}$$

By (4.2), we know that $\lim_{n\to+\infty} L^{\lambda_n}(0,\omega) = L^{\lambda}(0,\omega)$ since $\lambda \mapsto y_{n+1}(\lambda,\omega)$ is right-continuous by Lemma 4.5. Thus $\lim_{n\to+\infty} \theta_L(\lambda_n) = \theta_L(\lambda)$ and the function $\lambda \mapsto \theta_L(\lambda)$ is right-continuous.

Similarly, let $\lambda_n \nearrow \lambda$ and consider

$$\begin{aligned}\lim_{n\to+\infty} \theta_R(\lambda_n) &= \lim_{n\to+\infty} \mathbb{E}[(f_R^{\lambda_n})'(0,\omega)] \\ &= \lim_{n\to+\infty} \mathbb{E}[G^{-1}_{R^{\lambda_n}(0,\omega)}(\lambda_n - \beta V(0,\omega))].\end{aligned}$$

By (4.4), we know that $\lim_{n\to+\infty} R^{\lambda_n}(0,\omega) = R^{\lambda}(0,\omega)$ since $\lambda \mapsto x_{n+1}(\lambda,\omega)$ is left-continuous by Lemma 4.5. Thus $\lim_{n\to+\infty} \theta_R(\lambda_n) = \theta_R(\lambda)$ and the function $\lambda \mapsto \theta_R(\lambda)$ is left-continuous. □

Lemma 4.14. *Let $\beta \le \lambda_1 < \lambda_2$. Then, for any $x \in \mathbb{R}$ and $\omega \in \Omega$, we have $\theta_L(\lambda_1) > \theta_R(\lambda_2)$.*

Proof. First, suppose that $\beta \le \lambda_1$ and $\lambda_2 - \lambda_1 < \delta$ where $\delta < \min_{1\le i,j\le N} |G(p_{2i+1}) - G(p_{2j+1})|$. Then, by Lemma 4.9, we know that $L^{\lambda_1}(x,\omega) \ge R^{\lambda_2}(x,\omega)$ for any $x \in \mathbb{R}$ and $\omega \in \Omega$.

If $L^{\lambda_1}(x,\omega) = R^{\lambda_2}(x,\omega)$, then

$$\begin{aligned}
\theta_L(\lambda_1) &= \mathbb{E}[(f_L^{\lambda_1})'(0,\omega)] \\
&= \mathbb{E}[G^{-1}_{L^{\lambda_1}(0,\omega)}(\lambda_1 - \beta V(0,\omega))] \\
&= \mathbb{E}[G^{-1}_{R^{\lambda_2}(0,\omega)}(\lambda_1 - \beta V(0,\omega))] \\
&> \mathbb{E}[G^{-1}_{R^{\lambda_2}(0,\omega)}(\lambda_2 - \beta V(0,\omega))] \\
&= \mathbb{E}[(f_R^{\lambda_2})'(0,\omega)] \\
&= \theta_R(\lambda_2)
\end{aligned}$$

since $G^{-1}_{R^{\lambda_2}(0,\omega)}$ is decreasing.

If $L^{\lambda_1}(0,\omega) > R^{\lambda_2}(0,\omega)$, then, we know that

$$p_{L^{\lambda_1}(0,\omega)-1} \leq G^{-1}_{L^{\lambda_1}(0,\omega)}(\lambda_1 - \beta V(0,\omega)) \leq p_{L^{\lambda_1}(0,\omega)}, \tag{4.16}$$

and

$$p_{R^{\lambda_2}(0,\omega)-1} \leq G^{-1}_{R^{\lambda_2}(0,\omega)}(\lambda_2 - \beta V(0,\omega)) \leq p_{R^{\lambda_2}(0,\omega)}. \tag{4.17}$$

Since $L^{\lambda_1}(0,\omega)$ and $R^{\lambda_2}(0,\omega)$ are odd-integer-valued functions, we have

$$p_{L^{\lambda_1}(0,\omega)-1} > p_{R^{\lambda_2}(0,\omega)-1}.$$

Therefore,

$$\begin{aligned}
(f_L^{\lambda_1})'(0,\omega) &= G^{-1}_{L^{\lambda_1}(0,\omega)}(\lambda_1 - \beta V(0,\omega)) \\
&\geq p_{L^{\lambda_1}(0,\omega)-1} \\
&> p_{R^{\lambda_2}(0,\omega)-1} \\
&\geq G^{-1}_{R^{\lambda_2}(0,\omega)}(\lambda_2 - \beta V(0,\omega)) \\
&= (f_R^{\lambda_2})'(0,\omega).
\end{aligned}$$

By the monotonicity of expectation, this implies that $\theta_L(\lambda_1) > \theta_R(\lambda_2)$.

Now, let $\beta \leq \lambda_3 < \lambda_4$ be arbitrary. Then, we can find $\beta \leq \lambda_1$ and $\lambda_2 - \lambda_1 < \delta$ where $\delta < \min_{1\leq i,j\leq N} |G(p_{2i+1}) - G(p_{2j+1})|$ such that

$$\lambda_3 < \lambda_1 < \lambda_2 < \lambda_4$$

and by Corollary 4.12, we have

$$\theta_L(\lambda_3) \geq \theta_L(\lambda_1) > \theta_R(\lambda_2) \geq \theta_R(\lambda_4). \quad \square$$

Now, we are finally ready to state and prove the main result of this chapter that will be used in the proof of Theorem 2.6. First, using Lemma 4.10 and Lemma 4.14, we define the quantity

$$\theta(\beta) := \theta_R(\beta+). \tag{4.18}$$

Theorem 4.15. *The collection* $\mathcal{C} = \{[\theta_L(\lambda), \theta_R(\lambda)] : \lambda \geq \beta\}$ *is a partition of* $(-\infty, \theta(\beta))$*. In other words, for every* θ *in this open interval, there is a unique* $\Lambda(\theta) \geq \beta$ *such that* $\theta \in [\theta_L(\Lambda(\theta)), \theta_R(\Lambda(\theta))]$.

Proof. The intervals of $\mathcal{C}$ are disjoint by Corollary 4.12. It remains to show that their union is $(-\infty, \theta(\beta))$. Fix an arbitrary θ in this open interval and let

$$\Lambda(\theta) = \inf\{\lambda \in (-\infty, \beta] : \theta_L(\lambda) \leq \theta\}. \tag{4.19}$$

Then,

$$\begin{aligned}
\theta_L(\Lambda(\theta)) &= \theta_L(\Lambda(\theta)+) && \text{by Lemma 4.13} \\
&\leq \theta && \text{by (4.19)} \\
&\leq \theta_L(\Lambda(\theta)-) && \text{by (4.19)} \\
&\leq \theta_R(\Lambda(\theta)-) && \text{by Lemma 4.14} \\
&= \theta_R(\Lambda(\theta)) && \text{by Lemma 4.13.}
\end{aligned}$$

□

CHAPTER 5

BASE CASES FOR THE STRONG INDUCTION ARGUMENT

In this chapter, we consider the base cases for the strong induction argument that we will employ in the proof of Theorem 2.6. Our first focus will be on functions G that have exactly one strictly decreasing and one strictly increasing part. Then, we treat a more general case in which functions can have infinitely many extrema but the parameter β has to be suitably large.

5.1 Base Case 1

Suppose that $G \in \mathcal{G}$ is defined as follows:

$$G(p) = \begin{cases} G_1(p) & \text{if } p < 0, \\ \\ G_2(p) & \text{if } p \geq 0, \end{cases}$$

where G_1 is strictly decreasing and G_2 is strictly increasing. Note that this function is quasiconvex, i.e.,

$$G(cp + (1-c)q) \leq \max\{G(p), G(q)\} \text{ for any } p \neq q \text{ and } c \in [0, 1].$$

We want to show that G is in the class $\mathcal{G}_1$ (see Definition 2.4) and therefore applies to Theorem 2.5.

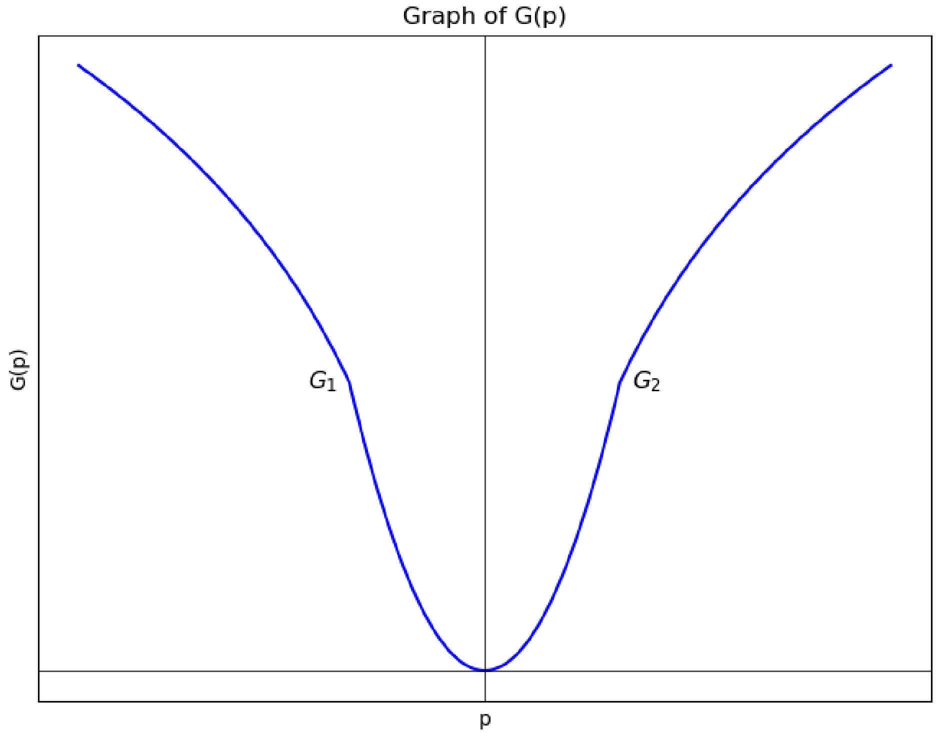

Figure 5.1: Single-well $G(p)$

Recall that in Definition 4.5, we have defined left-leaning and right-leaning branch indicating functions $L^\lambda(x,\omega)$ and $R^\lambda(x,\omega)$. In this case, $L^\lambda(x,\omega) = R^\lambda(x,\omega) = 1$ for any $x \in \mathbb{R}, \omega \in \Omega$, and $\lambda \geq \beta$ since there is only one branch in the second quadrant. Since both G_1 and G_2 are invertible, for each $\lambda \geq \beta$, we can define $f_1^\lambda, f_2^\lambda \in C_b^1(\mathbb{R})$ by setting

$$f_i^\lambda(x,\omega) = \int_0^x (f_i^\lambda)'(y,\omega)dy \tag{5.1}$$

where

$$(f_i^\lambda)'(x,\omega) = G_i^{-1}(\lambda - \beta V(x,\omega)) \text{ for every } x \in \mathbb{R}, i \in \{1,2\}. \tag{5.2}$$

Note that whenever $\underline{f} = f_i^\lambda$ or $\overline{f} = f_i^\lambda$ for some $\lambda \geq \beta$ and $i \in \{1,2\}$, the condition (2.16) in the Definition 2.4 is satisfied since $f_i^\lambda \in \mathrm{Lip}(\mathbb{R})$ is a solution of the equation in (2.14).

Let $\theta_i(\lambda) = \mathbb{E}[(f_i^\lambda)'(0,\omega)]$ for $i \in \{1,2\}$. Then, by (5.2), $\lambda \mapsto \theta_1(\lambda)$ is strictly decreasing and $\lambda \mapsto \theta_2(\lambda)$ is strictly increasing. Moreover, $\lim_{\lambda\to+\infty} \theta_1(\lambda) = -\infty$ and $\lim_{\lambda\to+\infty} \theta_2(\lambda) = +\infty$. Hence,

$$\lambda \mapsto \theta_1(\lambda) \text{ is invertible from } [\beta,+\infty) \text{ to } (-\infty,\theta_1(\beta)], \text{ and,}$$

$$\lambda \mapsto \theta_2(\lambda) \text{ is invertible from } [\beta,+\infty) \text{ to } [\theta_2(\beta),+\infty).$$

We will partition $\mathbb{R}$ into three pieces, namely $(-\infty,\theta_1(\beta)], (\theta_1(\beta),\theta_2(\beta))$, and $[\theta_2(\beta),+\infty)$, and consider these intervals separately.

- If $\theta \in (-\infty,\theta_1(\beta)]$, we take $\lambda = \theta_1^{-1}(\theta)$ and $\underline{f} = \overline{f} = f_1^\lambda$. Let $\epsilon > 0$ be given. Since $\underline{f} = \overline{f}$, for any $x_0, y_0 \in \mathbb{R}$, intervals $[\underline{f}'(x_0,\omega), \overline{f}'(x_0,\omega)] = \{(f_1^\lambda)'(x_0,\omega)\}$ and $[\underline{f}'(y_0,\omega), \overline{f}'(y_0,\omega)] = \{(f_1^\lambda)'(y_0,\omega)\}$ are just singletons and

$$G((f_1^\lambda)'(x_0,\omega)) + \beta V(x_0,\omega) = \lambda \geq \lambda - \epsilon,$$

$$G((f_1^\lambda)'(y_0,\omega)) + \beta V(y_0,\omega) = \lambda \leq \lambda + \epsilon.$$

Similarly, we have

$$\mathbb{E}[\underline{f}'(0,\omega)] = \mathbb{E}[\overline{f}'(0,\omega)] = \mathbb{E}[(f_1^\lambda)'(0,\omega)] = \theta$$

and condition (2.19) trivially holds.

- If $\theta \in [\theta_2(\beta), +\infty)$, we take $\lambda = \theta_2^{-1}$ and $\underline{f} = \overline{f} = f_2^\lambda$ and follow the same steps as in the first case.

- If $\theta \in (\theta_1(\beta), \theta_2(\beta))$, then we take $\lambda = \beta$, $\underline{f} = f_1^\beta$, and $\overline{f} = f_2^\beta$. Let $\epsilon > 0$ be given. Using (2.12) and the Intermediate Value Theorem, we can find $x_0 \in \mathbb{R}$ such that $\beta - \beta V(x_0,\omega) = \epsilon$. Then

$$\underline{f}'(x_0,\omega) = (f_1^\beta)'(x_0,\omega) = G_1^{-1}(\beta - \beta V(x_0,\omega)) = G_1^{-1}(\epsilon) < 0,$$

$$\overline{f}'(x_0,\omega) = (f_2^\beta)'(x_0,\omega) = G_2^{-1}(\beta - \beta V(x_0,\omega)) = G_2^{-1}(\epsilon) > 0.$$

For any $p \in [\underline{f}'(x_0,\omega), \overline{f}'(x_0,\omega)] = [G_1^{-1}(\epsilon), G_2^{-1}(\epsilon)]$, we have

$$G(p) + \beta V(x_0,\omega) = G(p) + \beta - \epsilon \geq \beta - \epsilon.$$

So, condition (2.17) holds.

To show (2.18), we will use the quasiconvexity of $G(p)$. Observe that for any $y_0 \in \mathbb{R}$, we have

$$\underline{f}'(y_0,\omega) = (f_1^{\beta})'(y_0,\omega) = G_1^{-1}(\beta - \beta V(y_0,\omega)) < 0,$$

$$\overline{f}'(y_0,\omega) = (f_2^{\beta})'(y_0,\omega) = G_2^{-1}(\beta - \beta V(y_0,\omega)) > 0.$$

For any $p \in [G_1^{-1}(\beta - \beta V(y_0,\omega)), G_2^{-1}(\beta - \beta V(y_0,\omega))]$, we have

$$G(p) + \beta V(y_0,\omega) \leq G(\beta - \beta V(y_0,\omega)) + \beta V(y_0,\omega) \leq \beta + \epsilon.$$

Lastly, since $\mathbb{E}[\underline{f}'(0,\omega)] = \theta_1(\beta)$ and $\mathbb{E}[\overline{f}'(0,\omega)] = \theta_2(\beta)$, condition (2.19) is also true since $\theta \in (\theta_1(\beta), \theta_2(\beta))$.

Thus, $G \in \mathcal{G}_1$ with the following choices:

- $\lambda = \theta_1^{-1}(\theta)$ and $\underline{f} = \overline{f} = f_1^{\lambda}$ for $\theta \in (-\infty, \theta_1(\beta)]$;
- $\lambda = \beta$, $\underline{f} = f_1^{\beta}$, and $\overline{f} = f_2^{\beta}$ for $\theta \in (\theta_1(\beta), \theta_2(\beta))$;
- $\lambda = \theta_2^{-1}(\theta)$ and $\underline{f} = \overline{f} = f_2^{\lambda}$ for $\theta \in [\theta_2(\beta), +\infty)$.

Hence, the effective Hamiltonian is given by

$$\overline{H}(\theta) = \begin{cases} \theta_1^{-1}(\theta) & \text{if } \theta \in (-\infty, \theta_1(\beta)], \\ \beta & \text{if } \theta \in (\theta_1(\beta), \theta_2(\beta)), \\ \theta_2^{-1}(\theta) & \text{if } \theta \in [\theta_2(\beta), +\infty). \end{cases}$$

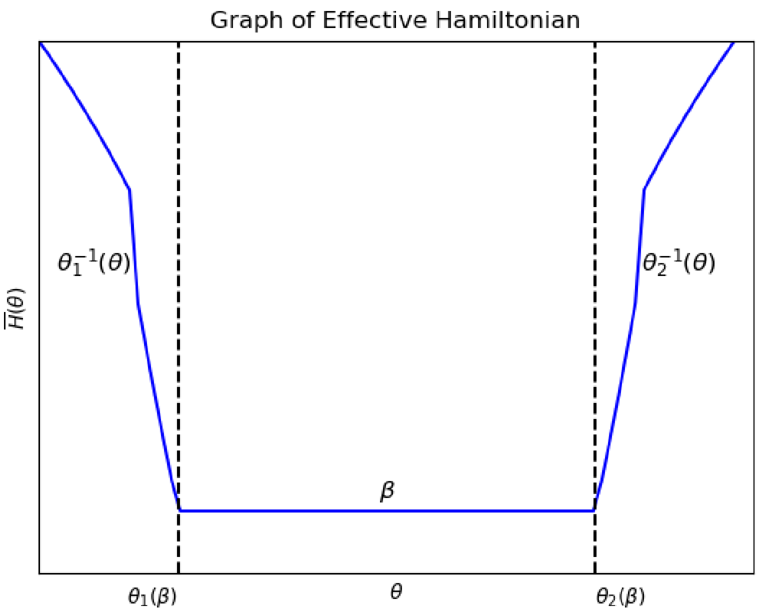

Figure 5.2: Effective Hamiltonian for single-well $G(p)$ in Figure 5.1

5.2 Base Case 2

Suppose that G is defined as in Condition 4.1 and that $\beta > M - m$. In this case, in addition to what we have seen in the base case 1, namely, a nonincreasing piece, a flat piece at height β, and a nondecreasing piece, we have an additional choice that accounts for possible jumps.

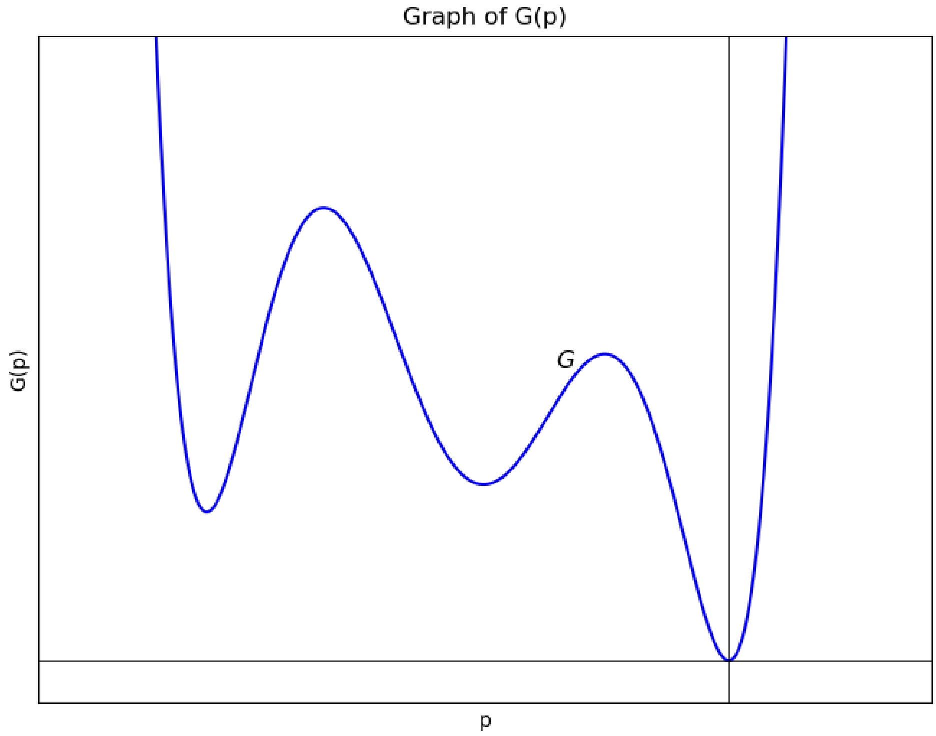

Figure 5.3: Multiple-well $G(p)$

In order to deal with the case with possible jumps, we will use Theorem 4.15. Let $\theta \in (-\infty, \theta(\beta))$ where $\theta(\beta) = \theta_{2N-1}(\beta)$. We know that there is a unique $\Lambda(\theta) \geq \beta$ such that $\theta \in [\theta_L(\Lambda(\theta)), \theta_R(\Lambda(\theta))]$. There are two subcases: one covering cases where there is no jump, e.g., $\theta > \theta_1(p_1 + \beta)$, and the other covering cases where there might be a jump.

- If $\theta_L(\Lambda(\theta)) = \theta_R(\Lambda(\theta))$, then we take $\lambda = \Lambda(\theta)$, and $\underline{f} = \overline{f} = f_L^\lambda = f_R^\lambda$ where f_L^λ and f_R^λ are defined as in (4.11). Then, the procedure is almost identical to the nonincreasing part in base case 1.

- If $\theta_L(\Lambda(\theta)) \neq \theta_R(\Lambda(\theta))$, then we note that (4.15) implies

$$(f_L^\lambda)'(x,\omega) \neq (f_R^\lambda)'(x,\omega) \text{ and } L^\lambda(x,\omega) \neq R^\lambda(x,\omega) \tag{5.3}$$

for some $x \in \mathbb{R}$ and $\omega \in \Omega$. We take the following choices:

$$\lambda = \Lambda(\theta), \underline{f} = f_L^\lambda, \text{ and } \overline{f} = f_R^\lambda.$$

First, observe that since f_L^λ and f_R^λ satisfy (2.14), we know that condition (2.16) in Definition 2.4 holds.

Let $\epsilon > 0$ be given. Since $\beta > M - m$ and (2.12), we can find $x_0 \in \mathbb{R}$ such that

$$\lambda - \beta V(x_0,\omega) = G(p_{2i-1})$$

for some $i \in \{1,2,\ldots,N\}$ and $\omega \in \Omega$ since otherwise, we would have a contradiction with (5.3). Then,

$$\underline{f}'(x_0,\omega) = (f_L^\lambda)'(x_0,\omega) = G_L^{-1}(\lambda - \beta V(x_0,\omega)) = G_L^{-1}(G(p_{2i-1})) = p_{2i-1},$$

$$\overline{f}'(x_0,\omega) = (f_R^\lambda)'(x_0,\omega) = G_R^{-1}(\lambda - \beta V(x_0,\omega)) = G_R^{-1}(G(p_{2i-1})) = \tilde{p}_{2i-1},$$

for some $\tilde{p}_{2i-1} \in \mathbb{R}$ with $\tilde{p}_{2i-1} > p_{2i-1}$ since we assumed that $\theta_L(\Lambda(\lambda)) \neq \theta_R(\Lambda(\lambda))$ which means that a jump has occurred to the right at x_0 and $L^\lambda(x_0,\omega) < R^\lambda(x_0,\omega)$ by Lemma 4.6. Then, for any $p \in [p_{2i-1}, \tilde{p}_{2i-1}]$, we know that $G(p) \geq G(p_{2i-1})$. Hence, for any $p \in [\underline{f}'(x_0,\omega), \overline{f}'(x_0,\omega)]$, we have

$$G(p) + \beta V(x_0, \omega) = \lambda + G(p) - G(p_{2i-1}) \geq \lambda - \epsilon,$$

and condition (2.17) is satisfied.

Similarly, using $\beta > M - m$ and (2.12), we can find $y_0 \in \mathbb{R}$ such that $\lambda - \beta V(y_0, \omega) = G(p_{2j})$ for some $j \in \{1, 2, \ldots, N\}$. Then

$$\underline{f}'(y_0, \omega) = (f_L^\lambda)'(y_0, \omega) = G_L^{-1}(\lambda - \beta V(y_0, \omega)) = G_L^{-1}(G(p_{2j})) = \tilde{p}_{2j},$$

$$\overline{f}'(y_0, \omega) = (f_R^\lambda)'(y_0, \omega) = G_R^{-1}(\lambda - \beta V(y_0, \omega)) = G_R^{-1}(G(p_{2j})) = \tilde{p}_{2j},$$

for some $\tilde{p}_{2j} \in \mathbb{R}$ with $\tilde{p}_{2j} < p_{2j}$ since we assumed that $\theta_L(\lambda) \neq \theta_R(\lambda)$ which implies that a jump has occurred to the left at y_0 and $L^\lambda(y_0, \omega) < R^\lambda(y_0, \omega)$ by Lemma 4.6. Then, for any $p \in [\tilde{p}_{2j}, p_{2j}]$, we know that $G(p) \leq G(p_{2j})$. Hence, for any $p \in [\underline{f}'(y_0, \omega), \overline{f}'(y_0, \omega)]$, we have

$$G(p) + \beta V(y_0, \omega) = \lambda + G(p) - G(p_{2j}) \leq \lambda + \epsilon,$$

and (2.18) is satisfied.

Lastly, note that we have

$$\mathbb{E}[\underline{f}'(0, \omega)] = \mathbb{E}[(f_L^\lambda)'(0, \omega)] \leq \mathbb{E}[(f_R^\lambda)'(0, \omega)] = \mathbb{E}[\overline{f}'(0, \omega)]$$

by Lemma 4.10 and the linearity of expectation.

Thus, $G \in \mathcal{G}_1$ with the following choices:

- If $\theta \in (-\infty, \theta_{2N-1}(\beta)]$, then $\lambda = \Lambda(\theta)$, $\underline{f} = f_L^\lambda$ and $\overline{f} = f_R^\lambda$;

- If $\theta \in (\theta_{2N-1}(\beta), \theta_{2N}(\beta))$, then $\lambda = \beta$, $\underline{f} = f^{\beta}_{2N-1}$, and $\overline{f} = f^{\beta}_{2N}$;
- If $\theta \in [\theta_{2N}(\beta), +\infty)$, then $\lambda = \theta^{-1}_{2N}(\theta)$ and $\underline{f} = \overline{f} = f^{\lambda}_{2N}$,

where $\Lambda(\theta)$ is given in (4.19), or equivalently, the effective Hamiltonian is

$$\overline{H}(\theta) = \begin{cases} \Lambda(\theta) & \text{if } \theta \in (-\infty, \theta_{2N-1}(\beta)], \\ \beta & \text{if } \theta \in (\theta_{2N-1}(\beta), \theta_{2N}(\beta)), \\ \theta^{-1}_{2N}(\theta) & \text{if } \theta \in [\theta_{2N}(\beta), +\infty). \end{cases}$$

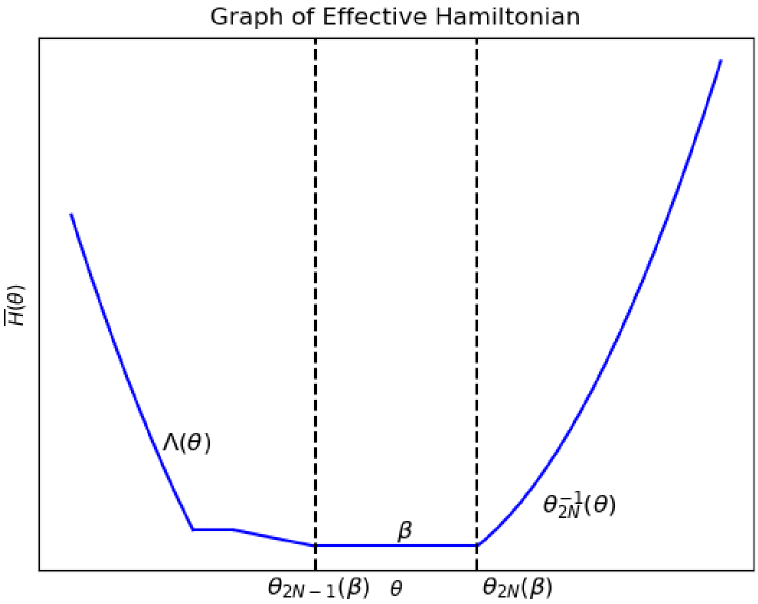

Figure 5.4: Effective Hamiltonian for multiple-well $G(p)$ in Figure 5.3

CHAPTER 6

GLUING PROCEDURES FOR THE STRONG INDUCTION ARGUMENT

In this chapter, we will explore one of the most significant part of the proof of Theorem 2.6: the gluing procedures. We will consider two types of gluing. First, we will discuss gluing at the origin, which will help us prove homogenization for functions with extrema not only in the second quadrant but also in other quadrants. Next, we will examine gluing when β is small, which involves two scenarios based on the behavior of G. This will be essential for the strong induction part of the proof.

6.1 Gluing at the Origin

Take any $G \in \mathcal{G}_0$ and define

$$G_1(p) = \begin{cases} G(p) & \text{if } p \leq 0, \\ I(p) & \text{if } p > 0, \end{cases} \quad \text{and} \quad G_2(p) = \begin{cases} I(p) & \text{if } p \leq 0, \\ G(p) & \text{if } p > 0, \end{cases}$$

where $I : \mathbb{R} \to [0, +\infty)$ is chosen such that $I_1 := I|_{[0,+\infty)}$ and $I_2 := I|_{(-\infty,0]}$ are strictly increasing and decreasing, respectively, and $G = \min\{G_1, G_2\}$. We want to show that if $G_1, G_2 \in \mathcal{G}_1$, then $G \in \mathcal{G}_1$.

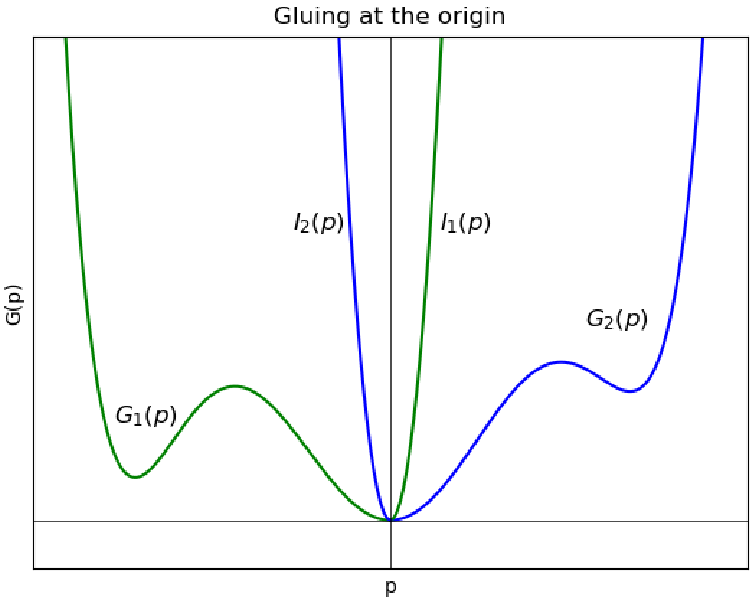

Figure 6.1: Gluing at the origin

Let $q_1 := \max\{p \leq 0 : G(p) = \beta\}$ and $q_2 := \min\{p \geq 0 : G(p) = \beta\}$. Observe that in $(-\infty, q_1]$, G is equal to G_1 and in $[q_2, +\infty)$, G is equal to G_2. Since we know that $G_1, G_2 \in \mathcal{G}_1$, we know that for any $\theta \in (-\infty, q_1] \cup [q_2, +\infty)$, there exist $\lambda = \lambda(\theta)$ and stationary functions $\underline{f}(\cdot, \omega), \overline{f}(\cdot, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying (2.16)–(2.19). We can use those choices for G as well. Hence, we only need to focus $\theta \in (q_1, q_2)$.

For $\theta \in (q_1, q_2)$, we claim that there is a flat piece at height β, i.e., $\lambda = \beta$, $\underline{f}(x, \omega) = f^{\beta}_{1,L}(x, \omega)$, and $\overline{f} = f^{\beta}_{2,R}(x, \omega)$, where $f^{\beta}_{1,L}(x, \omega)$ and $f^{\beta}_{2,R}(x, \omega)$ are defined as in (4.11), the former one is the corrector corresponding to G_1 and the latter one is the corrector corresponding to G_2.

First, observe that

$$\begin{aligned} G(\underline{f}'(x,\omega)) + \beta V(x,\omega) &= G(((f_{1,L}^{\beta})'(x,\omega)) + \beta V(x,\omega) \\ &= G(G_{1,L}^{-1}(\beta - \beta V(x,\omega)) + \beta V(x,\omega)) \\ &= \beta - \beta V(x,\omega) + \beta V(x,\omega) \\ &= \beta, \end{aligned}$$

since $G = G_1$ on $(-\infty, 0]$. Similarly,

$$\begin{aligned} G(\overline{f}'(x,\omega)) + \beta V(x,\omega) &= G(((f_{2,R}^{\beta})'(x,\omega)) + \beta V(x,\omega) \\ &= G(G_{2,R}^{-1}(\beta - \beta V(x,\omega)) + \beta V(x,\omega)) \\ &= \beta - \beta V(x,\omega) + \beta V(x,\omega) \\ &= \beta, \end{aligned}$$

since $G = G_2$ on $[0, +\infty)$.

Let $\epsilon > 0$ be given. Using (2.12) and the Intermediate Value Theorem, we can find $x_0 \in \mathbb{R}$ such that $\beta - \beta V(x_0,\omega) = \epsilon$. Then,

$$\begin{aligned} \underline{f}'(x_0,\omega) &= (f_{1,L}^{\beta})'(x_0,\omega) = G_{1,L}^{-1}(\beta - \beta V(x_0,\omega)) = G_{1,L}^{-1}(\epsilon) < 0, \\ \overline{f}'(x_0,\omega) &= (f_{2,R}^{\beta})'(x_0,\omega) = G_{2,R}^{-1}(\beta - \beta V(x_0,\omega)) = G_{2,R}^{-1}(\epsilon) > 0. \end{aligned}$$

Observe that for any $p \in [\underline{f}'(x_0,\omega), \overline{f}'(x_0,\omega)]$, we have $G(p) \le \epsilon$ and

$$G(p) + \beta V(x_0,\omega) = G(p) + \beta - \epsilon \ge \beta - \epsilon.$$

Now, take any $y_0 \in \mathbb{R}$. Then,

$$\underline{f}'(y_0, \omega) = (f_{1,L}^{\beta})'(y_0, \omega) = G_{1,L}^{-1}(\beta - \beta V(y_0, \omega)) < 0,$$

$$\overline{f}'(y_0, \omega) = (f_{2,R}^{\beta})'(y_0, \omega) = G_{2,R}^{-1}(\beta - \beta V(y_0, \omega)) > 0.$$

Observe that for any $p \in [\underline{f}'(y_0, \omega), \overline{f}'(y_0, \omega)]$, we have $G(p) \leq \beta - \beta V(y_0, \omega)$ and

$$G(p) + \beta V(x_0, \omega) \leq \beta - \beta V(y_0, \omega) + \beta V(y_0, \omega) \leq \beta + \epsilon.$$

Lastly, observe that

$$\begin{aligned}\mathbb{E}[\underline{f}'(0, \omega)] &= \mathbb{E}[(f_{1,L}^{\beta})'(0, \omega)] \\ &= \mathbb{E}[G_{1,L}^{-1}(\beta - \beta V(0, \omega))] \\ &\geq \mathbb{E}[G_{1,L}^{-1}(\beta)] \\ &= q_1,\end{aligned}$$

since $G_{1,L}^{-1}$ is decreasing, and

$$\begin{aligned}\mathbb{E}[\overline{f}'(0, \omega)] &= \mathbb{E}[(f_{2,R}^{\beta})'(0, \omega)] \\ &= \mathbb{E}[G_{2,R}^{-1}(\beta - \beta V(0, \omega))] \\ &\leq \mathbb{E}[G_{2,R}^{-1}(\beta)] \\ &= q_2,\end{aligned}$$

since $G_{2,R}^{-1}$ is increasing. Since $\theta \in (q_1, q_2)$, we have $\mathbb{E}[\underline{f}'(0, \omega)] \leq \theta \leq \mathbb{E}[\overline{f}'(0, \omega)]$. Hence, G satisfies (2.16) $-$ (2.19) and we conclude that $G \in \mathcal{G}_1$ with these choices:

- If $\theta \in (-\infty, q_1]$, then $\lambda = \Lambda_1(\theta)$, $\underline{f} = f^{\lambda}_{1,L}$, and $\overline{f} = f^{\lambda}_{1,R}$;

- If $\theta \in (q_1, q_2)$, then $\lambda = \beta$, $\underline{f} = \underline{f}_1 = f^{\beta}_{1,L}$, and $\overline{f} = \overline{f}_2 = f^{\beta}_{2,R}$;

- If $\theta \in [q_2, +\infty)$, then $\lambda = \Lambda_2(\theta)$, $\underline{f} = f^{\lambda}_{2,L}$, and $\overline{f} = f^{\lambda}_{2,R}$,

where Λ_1 and Λ_2 are defined as in (4.19), respectively, for G_1 and G_2. Therefore, the effective Hamiltonian is

$$\overline{H}(\theta) = \begin{cases} \Lambda_1(\theta) & \text{if } \theta \in (-\infty, q_1], \\ \beta & \text{if } \theta \in (q_1, q_2), \\ \Lambda_2(\theta) & \text{if } \theta \in [q_2, +\infty). \end{cases}$$

6.2 Gluing when β is Small

Take any $G \in \mathcal{G}_0$ satisfying Condition 4.1. Recall that $M = G(p_{2i})$ and $m = G(p_{2j-1})$ for some $i, j \in \{1, 2, \cdots, N-1\}$. Assume that $\beta \leq M - m$. We have two subcases.

6.2.1 $M - \min\{G(p_1), G(p_3), \ldots, G(p_{2i-1})\} < \beta$.

Define

$$G_1(p) = \begin{cases} G(p) & \text{if } p \leq p_{2j-1} \\ I(p) & \text{if } p > p_{2j-1}, \end{cases} \quad \text{and} \quad G_2(p) = \begin{cases} I(p) & \text{if } p \leq p_{2i} \\ G(p) & \text{if } p > p_{2i}, \end{cases}$$

where $I : \mathbb{R} \to \mathbb{R}$ is chosen such that $I_1 := I|_{[p_{2j-1}, +\infty)}$ and $I_2 := I|_{(-\infty, p_{2i}]}$ are strictly increasing and decreasing, respectively and $G = \min\{G_1, G_2\}$. We want to show that if $G_1, G_2 \in \mathcal{G}_1$, then $G \in \mathcal{G}_1$. Note that $G_1(0) \neq 0$ but we can always fix this by a simple change of variable (see Section 7.3).

Observe that on $(-\infty, p_{2i}]$, G is equivalent to G_1, and on $[p_{2j-1}, +\infty)$, G is equivalent to G_2. Since we know that $G_1, G_2 \in \mathcal{G}_1$, we know that for any $\theta \in (-\infty, p_{2i}] \cup [p_{2j-1}, +\infty)$, there exist $\lambda = \lambda(\theta)$ and stationary functions $\underline{f}(\cdot, \omega), \overline{f}(\cdot, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying (2.16)-(2.19). We can use those choices for G as well. Hence, we only need to focus the case $\theta \in (p_{2i}, p_{2j-1})$. For this case, we will employ Theorem 4.15.

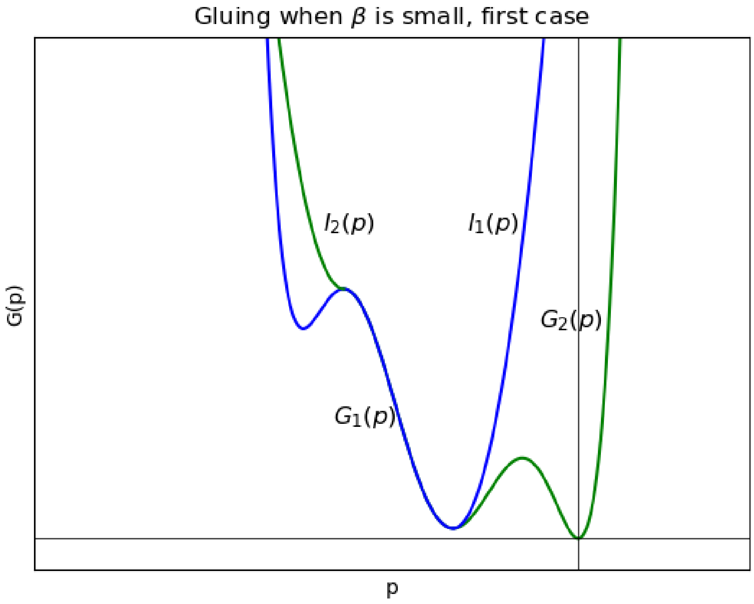

Figure 6.2: Gluing when β is small, first case

For $\theta \in (p_{2i}, p_{2j-1})$, we know that there exists a unique $\Lambda(\theta) \geq \beta$ such that $\theta \in [\theta_L(\Lambda(\theta)), \theta_R(\Lambda(\theta))]$ by Theorem 4.15. So, we take $\lambda = \Lambda(\theta)$, $\underline{f} = f_L^\lambda$, and $\overline{f} = f_R^\lambda$ where f_L^λ and f_R^λ are defined as in (4.11). Note that these stationary functions are not necessarily distinct. We have two subcases.

- If $\theta_L(\Lambda(\theta)) = \theta_R(\Lambda(\theta))$, then $\theta = \theta_L(\Lambda(\theta)) = \theta_R(\Lambda(\theta))$ and $L^\lambda(x, \omega) = R^\lambda(x, \omega)$, which implies that $\underline{f} = \overline{f}$. So, we take

  $$\lambda = \lambda(\theta) = \theta_L^{-1}(\theta) = \theta_R^{-1}(\theta) \text{ and } \underline{f} = \overline{f} = f_L^\lambda = f_R^\lambda.$$

  Then, we have

  $$\begin{aligned} G(\overline{f}'(x,\omega)) + \beta V(x,\omega) &= G(\underline{f}'(x,\omega)) + \beta V(x,\omega) \\ &= G((f_L^\lambda)'(x,\omega)) + \beta V(x,\omega) \\ &= G(G_L^{-1}(\lambda - \beta V(x,\omega))) + \beta V(x,\omega) \\ &= \lambda - \beta V(x,\omega) + \beta V(x,\omega) \\ &= \lambda. \end{aligned}$$

  Next, let $\epsilon > 0$ be given. Since for any $x_0, y_0 \in \mathbb{R}$, both $[\underline{f}'(x_0,\omega), \overline{f}'(x_0,\omega)]$ and $[\underline{f}'(y_0,\omega), \overline{f}'(y_0,\omega)]$ are singletons, we clearly have

  $$G(\underline{f}'(x_0,\omega)) + \beta V(x_0,\omega) = \lambda \geq \lambda - \epsilon$$

  $$G(\overline{f}'(y_0,\omega)) + \beta V(y_0,\omega) = \lambda \leq \lambda + \epsilon.$$

  Lastly, we note that $\theta = \mathbb{E}[\underline{f}'(0,\omega)] = \mathbb{E}[\overline{f}'(0,\omega)]$. Hence, these choices satisfy (2.16)-(2.19).

- If $\theta_L(\Lambda(\theta)) \neq \theta_R(\Lambda(\theta))$, we employ Theorem 4.15 and take

$$\lambda = \Lambda(\theta), \quad \underline{f} = f_L^\lambda, \quad \overline{f} = f_R^\lambda.$$

Note that in this case there exist $x \in \mathbb{R}$ and $\omega \in \Omega$ such that $L^\lambda(x,\omega) \neq R^\lambda(x,\omega)$ and $(f_L^\lambda)'(x,\omega)$ and $(f_R^\lambda)(x,\omega)$. Then, observe that

$$\begin{aligned} G(\underline{f}'(x,\omega)) + \beta V(x,\omega) &= G((f_L^\lambda)'(x,\omega)) + \beta V(x,\omega) \\ &= G(G_L^{-1}(\lambda - \beta V(x,\omega))) + \beta V(x,\omega) \\ &= \lambda - \beta V(x,\omega) + \beta V(x,\omega) \\ &= \lambda, \end{aligned}$$

and

$$\begin{aligned} G(\overline{f}'(x,\omega)) + \beta V(x,\omega) &= G((f_R^\lambda)'(x,\omega)) + \beta V(x,\omega) \\ &= G(G_R^{-1}(\lambda - \beta V(x,\omega))) + \beta V(x,\omega) \\ &= \lambda - \beta V(x,\omega) + \beta V(x,\omega) \\ &= \lambda. \end{aligned}$$

We can find $x_0 \in \mathbb{R}$ such that $\lambda - \beta V(x_0,\omega) = G(p_L)$. Then, observe that

$$\begin{aligned} \underline{f}'(x_0,\omega) = f_L^\lambda(x_0,\omega) = G_L^{-1}(\lambda - \beta V(x_0,\omega)) = G_L^{-1}(G(p_L)) = p_L, \\ \overline{f}'(x_0,\omega) = f_R^\lambda(x_0,\omega) = G_R^{-1}(\lambda - \beta V(x_0,\omega)) = G_R^{-1}(G(p_L)) = \tilde{p}_L, \end{aligned}$$

for some $\tilde{p}_L$ with $\tilde{p}_L > p_L$ by Lemma 4.6. Note that for any $p \in [p_L, \tilde{p}_L]$, we have $G(p) \geq G(p_L)$. Hence, for any $\epsilon > 0$,

$$G(p) + \beta V(x_0, \omega) = G(p) + \lambda - G(p_L) \geq G(p_L) + \lambda - G(p_L) \geq \lambda - \epsilon.$$

Similarly, we take $y_0 \in \mathbb{R}$ such that $\lambda - \beta V(y_0, \omega) = G(p_{R-1})$. Then,

$$\underline{f}'(y_0, \omega) = f_L^\lambda(y_0, \omega) = G_L^{-1}(\lambda - \beta V(y_0, \omega)) = G_L^{-1}(G(p_{R-1})) = \tilde{p}_{R-1},$$

$$\overline{f}'(y_0, \omega) = f_R^\lambda(y_0, \omega) = G_R^{-1}(\lambda - \beta V(y_0, \omega)) = G_R^{-1}(G(p_{R-1})) = p_{R-1},$$

for some $\tilde{p}_{R-1}$ with $\tilde{p}_{R-1} < p_{R-1}$.Note that for any $p \in [\tilde{p}_{R-1}, p_{R-1}]$, we have $G(p) \leq G(p_{R-1})$. Then, for any $\epsilon > 0$, we have

$$G(p) + \beta V(y_0, \omega) = G(p) + \lambda - G(p_{R-1}) \leq G(p_{R-1}) + \lambda - G(p_{R-1}) \leq \lambda + \epsilon.$$

Lastly, we know that $\theta_L(\lambda) = \mathbb{E}[(f_L)'(0, \omega)] = \mathbb{E}[\underline{f}'(0, \omega)]$ and $\theta_R(\lambda) = \mathbb{E}[(f_R)'(0, \omega)] = \mathbb{E}[\overline{f}'(0, \omega)]$. Since $\theta \in [\theta_L(\omega), \theta_R(\omega)]$, we conclude that these choices satisfy (2.16)-(2.19).

Therefore, we conclude that $G_1, G_2 \in \mathcal{G}_1$ implies that $G \in \mathcal{G}_1$.

6.2.2 $M - \min\{G(p_1), G(p_3), \ldots, G(p_{2i-1})\} \geq \beta$.

Define

$$G_1(p) = \begin{cases} G(p) & \text{if } p \leq p_{2i} \\ I(p) & \text{if } p > p_{2i}, \end{cases} \quad \text{and} \quad G_2(p) = \begin{cases} I(p) & \text{if } p \leq p_{2i} \\ G(p) & \text{if } p > p_{2i}, \end{cases}$$

where $I : \mathbb{R} \to [M, +\infty)$ is chosen such that $I_1 := I|_{[p_{2i}, +\infty)}$ and $I_2 := I|_{(-\infty, p_{2i}]}$ are strictly increasing and decreasing, respectively and $G = \min\{G_1, G_2\}$. We want to show that if $G_1, G_2 \in \mathcal{G}_1$, then $G \in \mathcal{G}_1$. Note that $G_1(0) \neq 0$ but we can always fix this by a simple change of variable (see Section 7.3).

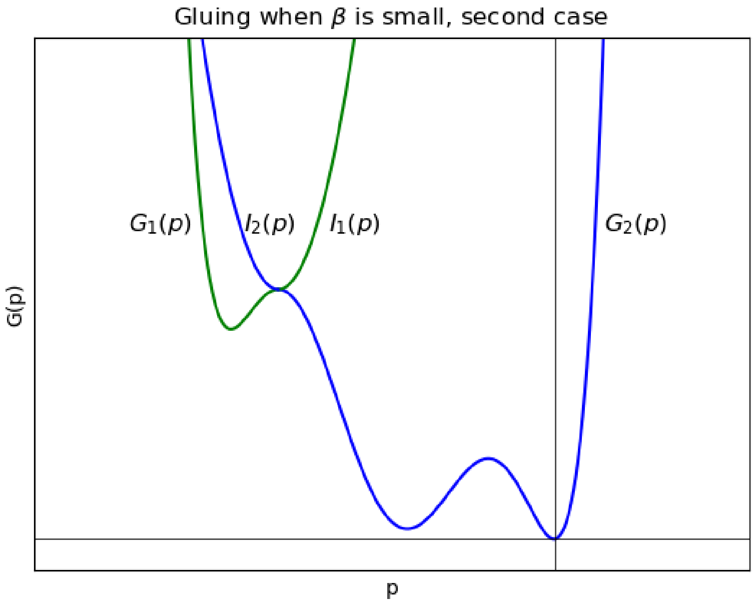

Figure 6.3: Gluing when β is small, second case

Let $q_1 := \max\{p \leq p_{2i} : G(p) = M - \beta\}$ and $q_2 := \min\{p > p_{2i} : G(p) = M - \beta\}$. Observe that in $(-\infty, q_1]$, G is equivalent to G_1, and in $[q_2, +\infty)$, G is equivalent to G_2. Since we know that $G_1, G_2 \in \mathcal{G}_1$, we know that for any $\theta \in (-\infty, q_1] \cup [q_2, +\infty)$, there exist $\lambda = \lambda(\theta)$ and stationary functions $\underline{f}(\cdot, \omega), \overline{f}(\cdot, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying (2.16)-(2.19). We can use those choices for G as well. Hence, we only need to focus the case $\theta \in (q_1, q_2)$. For this case, we will follow a similar approach like we did in subsection 6.2.1.

Assume that $\theta \in (q_1, q_2)$. We claim that the effective Hamiltonian has a flat piece at height M in this interval, i.e., if we take $\lambda = M, \underline{f} = f_{1,L}^M(x,\omega)$, and $\overline{f} = f_{2,R}^M(x,\omega)$, then these choices will satisfy (2.16) – (2.19).

First, notice that we have

$$\begin{aligned} G(\underline{f}'(x,\omega)) &= G((f_{1,L}^M)'(x,\omega)) + \beta V(x,\omega) \\ &= G(G_1^{-1}(M - \beta V(x,\omega))) + \beta V(x,\omega) \\ &= M - \beta V(x,\omega) + \beta V(x,\omega) \\ &= M, \end{aligned}$$

and

$$\begin{aligned} G(\overline{f}'(x,\omega)) &= G((f_{2,R}^M)'(x,\omega)) + \beta V(x,\omega) \\ &= G(G_2^{-1}(M - \beta V(x,\omega))) + \beta V(x,\omega) \\ &= M - \beta V(x,\omega) + \beta V(x,\omega) \\ &= M. \end{aligned}$$

Next, let $\epsilon > 0$ be given. Take $x_0 \in \mathbb{R}$ such that $M - \beta V(x_0,\omega) = M - \beta V(x_0,\omega)$. Then, observe that

$$\begin{aligned} \underline{f}'(x_0,\omega) &= (f_{1,L}^M)'(x_0,\omega) = G_1^{-1}(M - \beta V(x_0,\omega)) = \overline{p} \in (q_1, p_{2i}), \\ \overline{f}'(x_0,\omega) &= (f_{2,R}^M)'(x_0,\omega) = G_2^{-1}(M - \beta V(x_0,\omega)) = \tilde{p} \in (p_{2i}, q_2). \end{aligned}$$

Note that for any $p \in (\overline{p}, \tilde{p})$, we have $G(p) \geq M - \beta V(x_0, \omega)$. Therefore,

$$G(p) + \beta V(x_0, \omega) \geq M - \beta V(x_0, \epsilon) + \beta V(x_0, \epsilon) \geq M - \epsilon.$$

Similarly, we take $y_0 \in \mathbb{R}$ such that $M - \beta V(y_0, \omega) = M - \epsilon$. Then,

$$\underline{f}'(y_0, \omega) = (f_{1,L}^M)'(y_0, \omega) = G_1^{-1}(M - \beta V(y_0, \omega)) = G_1^{-1}(M - \epsilon) = \overline{q} \in (q_1, p_{2i})$$

$$\overline{f}'(y_0, \omega) = (f_{2,R}^M)'(y_0, \omega) = G_2^{-1}(M - \beta V(y_0, \omega)) = G_2^{-1}(M - \epsilon) = \tilde{q} \in (p_{2i}, q_2).$$

Note that for any $p \in (\overline{q}, \tilde{q})$, we have $G(p) \leq M - \epsilon$. Therefore

$$G(p) + \beta V(y_0, \omega) \leq G(p) + \epsilon \leq M + \epsilon.$$

Lastly, we have

$$\begin{aligned}
\mathbb{E}[\underline{f}'(0, \omega)] &= \mathbb{E}[(f_{1,L}^M)'(0, \omega)] \\
&= \mathbb{E}[G_1^{-1}(M - \beta V(0, \omega))] \\
&\geq \mathbb{E}[G_1^{-1}(M - \beta)] \\
&= q_1,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\overline{f}'(0, \omega)] &= \mathbb{E}[(f_{2,R}^M)'(0, \omega)] \\
&= \mathbb{E}[G_2^{-1}(M - \beta V(0, \omega))] \\
&\leq \mathbb{E}[G_2^{-1}(M - \beta)] \\
&= q_2,
\end{aligned}$$

since $G_{1,L}^M$ and $G_{2,R}^M$ are increasing and decreasing functions, respectively. Lastly, we know that $\theta \in (q_1, q_2)$, which implies

$$\mathbb{E}[\underline{f}'(0,\omega)] \leq \theta \leq \mathbb{E}[\overline{f}'(0,\omega)].$$

Hence, $G_1, G_2 \in \mathcal{G}_1$ implies that $G \in \mathcal{G}_1$.

CHAPTER 7

HOMOGENIZATION

We are finally ready to prove homogenization. We will first show that the HJ equation of the form as in (2.8)

$$\partial_t u^\epsilon(t,x,\omega) + G(\partial_x u^\epsilon(t,x,\omega)) + \beta V(x/\epsilon,\omega) = 0, \quad (t,x) \in (0,+\infty)\times\mathbb{R},$$

homogenizes to the HJ equation of the form as in (2.13)

$$\partial_t \overline{u}(t,x) + \overline{H}(\partial_x \overline{u}(t,x)) = 0, \quad (t,x) \in (0,+\infty)\times\mathbb{R},$$

for functions $G \in \mathcal{G}_1$. Then, we will prove that $\mathcal{G}_0$ is a subset of $\mathcal{G}_1$ implying homogenization for functions $G \in \mathcal{G}_0$. Lastly, we will generalize this result for the broadest class in the dissertation which is $\mathcal{G}$, i.e., the class of continuous and coercive functions.

7.1 Homogenization for the Class $\mathcal{G}_1$

Let us first recall the Theorem 2.5. We want to show homogenization for potential function V that satisfies (2.10)-(2.12), $\beta > 0$, and $G \in \mathcal{G}_1$, i.e., G

that satisfies the following four conditions:

$$\underline{f}(\cdot,\omega), \overline{f}(\cdot,\omega) \text{ are Lip}(\mathbb{R}) \text{ solutions of (2.14) on } \Omega_0 \text{ with } \mathbb{P}(\Omega_o) = 1;$$

$$\forall \epsilon > 0, \exists x_0 \in \mathbb{R} : \forall p \in [\underline{f}'(x_0,\omega), \overline{f}'(x_0,\omega)] : G(p) + \beta V(x_0,\omega) \geq \lambda - \epsilon;$$

$$\forall \epsilon > 0, \exists y_0 \in \mathbb{R} : \forall p \in [\overline{f}'(y_0,\omega), \underline{f}'(y_0,\omega)] : G(p) + \beta V(y_0,\omega) \leq \lambda + \epsilon;$$

$$\mathbb{E}[\underline{f}'(0,\omega)] \leq \theta \leq \mathbb{E}[\overline{f}'(0,\omega)].$$

Proof of Theorem 2.5. Let $\theta \in \mathbb{R}$. Since $G \in \mathcal{G}_1$, we know that there exist a constant $\lambda = \lambda(\theta) \geq \beta$, $x_0, y_0 \in \mathbb{R}$, and stationary functions $\underline{f}^\lambda, \overline{f}^\lambda : \mathbb{R}\times\Omega \to \mathbb{R}$ that satisfies (2.16), (2.17), (2.18), and (2.19).

First, observe that if $\theta = \mathbb{E}[(\underline{f}^\lambda)'(0,\omega)]$ or $\theta = \mathbb{E}[(\overline{f}^\lambda)'(0,\omega)]$, then, by Proposition 3.6, we are done since $\underline{f}^\lambda(\cdot,\omega), \overline{f}^\lambda(\cdot,\omega)$ are C_b^1 solutions of (2.14).

Without loss of generality, suppose that $\theta \in (\mathbb{E}[(\underline{f}^\lambda)'(0,\omega)], \mathbb{E}[(\overline{f}^\lambda)'(0,\omega)])$. Construct $f_1, f_2 \in \text{Lip}(\mathbb{R})$ by setting $f_1(0,\omega) = f_2(0,\omega) = 0$,

$$(f_1)'(x,\omega) = \begin{cases} (\underline{f}^\lambda)'(x,\omega) & \text{if } x < x_0 \\ (\overline{f}^\lambda)'(x,\omega) & \text{if } x > x_0, \end{cases} \tag{7.1}$$

and

$$(f_2)'(x,\omega) = \begin{cases} (\underline{f}^\lambda)'(x,\omega) & \text{if } x > y_0 \\ (\overline{f}^\lambda)'(x,\omega) & \text{if } x < y_0. \end{cases} \tag{7.2}$$

Observe that, at every $x \in \mathbb{R} \setminus \{x_0\}$, f_1 satisfies both

$$G(p) + \beta V(x,\omega) \leq \lambda \quad \text{for every } p \in D^+ f_1(x,\omega) \tag{7.3}$$

and

$$G(p) + \beta V(x,\omega) \geq \lambda \quad \text{for every } p \in D^- f_1(x,\omega), \tag{7.4}$$

since f_1 is differentiable and $D^+ f_1(x,\omega) = D^- f_1(x,\omega) = \{Df_1(x,\omega)\}$, i.e., f_1 is a solution of (2.14). Similarly, at every $x \in \mathbb{R} \setminus \{y_0\}$, f_2 satisfies both (7.3) and (7.4).

Observe that

$$\begin{aligned}
\lim_{x\to+\infty} \frac{f_1(x,\omega)}{x} &= \lim_{x\to+\infty} \frac{\overline{f}(x,\omega)}{x} \\
&= \theta_R(\lambda) \\
&> \theta_L(\lambda) \\
&= \lim_{x\to-\infty} \frac{\underline{f}(x,\omega)}{x} \\
&= \lim_{x\to-\infty} \frac{f_1(x,\omega)}{x}.
\end{aligned}$$

So, for $x = x_0$, we have $D^- f_1(x_0,\omega) = [\underline{f}(x_0,\omega), \overline{f}(x_0,\omega)]$ and by (2.17), we know that

$$G(p) + \beta V(x_0,\omega) \geq \lambda - \epsilon \text{ for every } p \in D^- f_1(x_0,\omega) = [\underline{f}(x_0,\omega), \overline{f}(x_0,\omega)].$$

Therefore, by Proposition 3.7, we have

$$\overline{H}^L(\theta) \geq \lambda - \epsilon \text{ for every } \theta \in (\mathbb{E}[(\underline{f}^\lambda)'(0,\omega)], \mathbb{E}[(\overline{f}^\lambda)'(0,\omega)]).$$

In a similar way, using $x = y_0$, we can show that $\overline{H}^U(\theta) \leq \lambda + \epsilon$ for every $\theta \in (\mathbb{E}[(\underline{f}^\lambda)'(0,\omega)], \mathbb{E}[(\overline{f}^\lambda)'(0,\omega)])$.

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\overline{H}(\theta) = \overline{H}^L(\theta) = \overline{H}^U(\theta) = \lambda(\theta). \quad \square$$

7.2 Proof of the Inclusion $\mathcal{G}_0 \subset \mathcal{G}_1$

In this section, our aim is to prove Theorem 2.6. From the previous function, we know that the homogenization result holds for $G \in \mathcal{G}_1$. If we show that $\mathcal{G}_0 \subset \mathcal{G}_1$, this will imply that the homogenization result holds for $G \in \mathcal{G}_0$ as well. To prove Theorem 2.6, we will use a strong induction argument.

Proof of Theorem 2.6. For every $n \geq 0$, let

$$\mathcal{G}_{0,n} := \{G \in \mathcal{G}_0 : G \text{ has exactly } 2n+1 \text{ local extrema}\}.$$

Note that $\mathcal{G}_0 = \cup_{n=0}^{+\infty} \mathcal{G}_{0,n}$. We will prove by strong induction that $\mathcal{G}_{0,n} \subset \mathcal{G}_1$ for every $n \geq 0$.

If $G \in \mathcal{G}_{0,0}$, i.e., G is quasiconvex, then $G \in \mathcal{G}_1$ by our result in Subsection 5.1.

Fix an $N \geq 1$ and suppose that $\mathcal{G}_{0,n} \subset \mathcal{G}_1$ for all $n < N$. Take any $G \in \mathcal{G}_{0,N}$. Since we know that we can glue two functions in $\mathcal{G}_1$ with extremas in the first quadrant and in the second quadrant at the origin by Subsection 6.1, it is enough to consider a function with extrema on one quadrant.

Case 1: G is strictly increasing on $[0, +\infty)$. Then, we have two cases depending on β :

- If $\beta > M - m$, then $G \in \mathcal{G}_1$ by large β case in Subsection 5.2.

- If $\beta \leq M - m$, then we have two more subcases:

  1. If $\beta > M - \min\{G(p_1), G(p_3), \ldots, G(p_{2i-1})\}$ where $M = G(p_{2i})$, then we use the gluing method in Subsection 6.2.1 by expressing $G = \min\{G_1, G_2\}$. Note that $G_1 \in \mathcal{G}_{0,n_1}$ and $G_2 \in \mathcal{G}_{0,n_2}$ for some $n_1, n_2 < N$. So, by our induction hypothesis, $G_1(\cdot), G_2(\cdot + p_{2j-1}) - m \in \mathcal{G}_1$. Since $G_2(\cdot + p_{2j-1}) - m \in \mathcal{G}_1$, we know that for every $\theta \in \mathbb{R}$, there exist $\lambda_0 \geq \beta - m$, and stationary functions $\underline{f}, \overline{f}$ satisfying (2.16)-(2.19). Let $\tilde{G}(p) := G_2(p + p_{2j-1})$. We claim that $\tilde{G} \in \mathcal{G}_1$ as well. Take $\lambda = \lambda_0 + m \geq \beta$, $\tilde{\underline{f}} = \underline{f}$, and $\tilde{\overline{f}} = \overline{f}$. Then

     $$\begin{aligned}\tilde{G}(\tilde{\underline{f}}'(x,\omega)) + \beta V(x,\omega) &= \tilde{G}(\underline{f}'(x,\omega)) + \beta V(x,\omega)\\ &= G_2(\underline{f}'(x,\omega) + p_{2j-1}) + \beta V(x,\omega)\\ &= \lambda_0 + \beta.\end{aligned}$$

     Similarly, we also have

     $$\tilde{G}(\tilde{\overline{f}}'(x,\omega)) + \beta V(x,\omega) = \lambda_0 + \beta.$$

     Since $\tilde{\underline{f}} = \underline{f}$ and $\tilde{\overline{f}} = \overline{f}$, (2.17)-(2.19) will follow directly. Therefore, $\tilde{G} \in \mathcal{G}_1$. We conclude that $G \in \mathcal{G}_1$ by our result in Subsection 6.2.1.

  2. If $\beta \leq M - \min\{G(p_1), G(p_3), \ldots, G(p_{2i-1})\}$, where $M = G(p_{2i})$, then we use the gluing method in Subsection 6.2.2 by letting $G =$

min$\{G_1, G_2\}$. Note that $G_1 \in \mathcal{G}_{0,n_1}$ and $G_2 \in \mathcal{G}_{0,n_2}$ for some $n_1, n_2 < N$. So, by our induction hypothesis, $G_1(\cdot, \omega), G_2(\cdot + p_{2i}) - M \in \mathcal{G}_1$. Next, we use change of variable on $G_2(\cdot + p_{2i}) - M$, similar to the one that we use above and see that $G_2(\cdot) \in \mathcal{G}_1$. Therefore, we conclude that $G \in \mathcal{G}_1$ by our result in Subsection 6.2.2.

Case 2: G is strictly decreasing on $(-\infty, 0]$.

Set $\tilde{G}(p) := G(-p)$ for all $p \in \mathbb{R}$. Then, by the first case, we have $\tilde{G} \in \mathcal{G}_1$. Notice that if we let $\tilde{f}(x, \omega) := f(-x, \omega)$ and $\tilde{V}(x, \omega) = V(-x, \omega)$, we realize that for any $x \in \mathbb{R}$ and $\omega \in \Omega$,

- $\tilde{f}$ is a stationary function with a new group of measure-preserving $\tilde{\tau}_x$ defined by $\tilde{\tau}_x := \tau_{-x}$ since

$$\tilde{f}(x, \omega) = f(-x, \omega) = f(0, \tau_{-x}\omega) = \tilde{f}(0, \tilde{\tau}_x, \omega);$$

- The relationship between f and $\tilde{f}$'s derivatives is as follows:

$$\tilde{f}'(x, \omega) = \frac{d}{dx}\tilde{f}(x, \omega) = \frac{d}{dx} f(-x, \omega) = -f'(-x, \omega);$$

- $\tilde{f}$ satisfies the equation in 2.14 with $\tilde{G}$ since

$$\begin{aligned}
\tilde{G}(\tilde{f}'(x, \omega)) + \beta V(x, \omega) &= \tilde{G}(\tilde{f}'(-x, \omega)) + \beta \tilde{V}(-x, \omega) \\
&= G(-\tilde{f}'(-x, \omega)) + \beta \tilde{V}(-x, \omega) \\
&= G(f'(x, \omega)) + \beta V(x, \omega) \\
&= \lambda.
\end{aligned}$$

Hence, we have $G \in \mathcal{G}_1$.

Therefore, we conclude that $\mathcal{G}_{0,N} \in \mathcal{G}_1$ and $\mathcal{G}_0 \subset \mathcal{G}_1$. $\square$

7.3 Generalizing from $\mathcal{G}_0$ to $\mathcal{G}$

We have the homogenization result for the class of $\mathcal{G}_0$. We now need to generalize this result for $\mathcal{G}$. Recall that a function $G \in \mathcal{G}$ is in the class of $\mathcal{G}_0$ if $G(0) = 0$ and G has finitely many extrema whereas $G \in \mathcal{G}$ means that G is continuous and coercive.

First, we will show that while functions within $\mathcal{G}$ do not necessarily pass through the origin, we can circumvent this requirement through translation and change of variables.

Proof of Theorem 2.2. Let $G \in \mathcal{G}$. Since G is continuous and coercive, there exist $\alpha, p_\alpha \in \mathbb{R}$ such that

$$G(p_\alpha) = \alpha := \inf_{p \in \mathbb{R}} G(p).$$

Assume that $\alpha \neq 0$, or $p_\alpha \neq 0$. We define

$$G_0(p) := G(p + p_\alpha) - \alpha$$

so that $G_0 \in \mathcal{G}$.

Recall the differential equation

$$\begin{cases} \partial_t u + G(\partial_x u(t,x,\omega)) + \beta V(x,\omega) = 0, \\ \\ u_\theta(0,x) = \theta x. \end{cases}$$

For

$$\tilde{u}_\theta(t,x) = u_\theta(t,x) - p_\alpha x + \alpha t,$$

we have

$$\partial_t \tilde{u} = \partial_t u + \alpha, \quad \text{and} \quad \partial_x \tilde{u} = \partial_x u - p_\alpha.$$

So,

$$\begin{cases} \partial_t u(t,x,\omega) + G_0(\partial_x u(t,x,\omega) - p_\alpha) + \alpha + \beta V(x,\omega) = 0, \\ \\ u_\theta(0,x) = \theta x, \end{cases}$$

or, equivalently,

$$\begin{cases} \partial_t \tilde{u} + G_0(\partial_x \tilde{u}) + \beta V(x,\omega) = 0, \\ \\ \tilde{u}_\theta(0,x) = (\theta - p_\alpha)x. \end{cases}$$

Then, using (3.3), we have

$$\begin{aligned} \overline{H}_0(\theta - p_\alpha) &= \lim_{t\to+\infty} \frac{-1}{t}\tilde{u}_\theta(t,0) \\ &= \lim_{t\to+\infty} \frac{-1}{t}(u_\theta(t,0) + \alpha t) \\ &= \overline{H}(\theta) - \alpha. \end{aligned}$$

Hence

$$\overline{H}(\theta) = \overline{H}_0(\theta - p_\alpha) + \alpha, \tag{7.5}$$

or, equivalently,

$$\overline{H}_0(\theta) = \overline{H}(\theta + p_\alpha) - \alpha. \tag{7.6}$$

Therefore, the homogenization holds for G if and only if it holds for G_0.

Next, we will discuss that for any $G \in \mathcal{G}$, there exists $G_n \in \mathcal{G}_0$ such that $G_n \to G$ locally uniformly, using a stability argument. For every fixed $n \in \mathcal{N}$, we can choose $G_n \in \mathcal{G} \subset \mathcal{G}_0$ such that $||G - G_n||_{L^\infty([-n,n])} < 1/n$, G_n is strictly decreasing on $(-\infty, -n]$, and G_n is strictly increasing on $[n, +\infty)$.

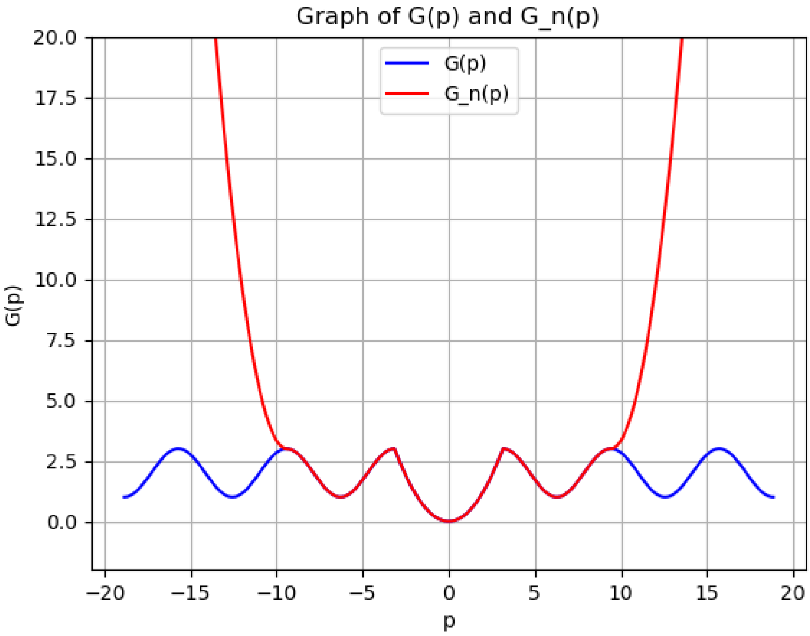

Figure 7.1: Graphical representation of local uniform convergence

By assumption, (2.8) homogenizes for each G_n. By using [6, Theorem B.4], we conclude that (2.8) homogenizes for G as well. $\square$

CHAPTER 8

NUMERICAL IMPLEMENTATION

In this chapter, we transition from theoretical derivations to practical implementation, using the methods developed in the previous chapters. Our aim is to develop an effective Hamiltonian calculator with Python, which will allow us to perform computations efficiently. This chapter follows closely to the logical progression of our earlier discussions, particularly the proof of homogenization. Through this implementation, we aim to demonstrate the applicability of our theoretical framework in computational settings. We begin by outlining the steps involved in translating the homogenization theory into a Python environment using the following libraries pandas [22], numpy [12], matplotlib [13], and sympy [18]. Then, we will consider some examples to illustrate the practical application of these methods. The details of the numerical implementation can be found in the following link: https://github.com/ademirelli/Homogenizator.

8.1 Outline of the Code in Python

In this section, we will develop a code that evaluates the effective Hamiltonian and plots its graph. The code will allow users to input the function G and the constant β, essential components for the calculations. However, due to the complexity of potential function V, we adopt a different approach. Instead of directly inputting V, we utilize pre-defined functions, as demonstrated in Chapter 3.3.

After receiving the function G and β as inputs from the user, we discretize $G(p)$ over the range $[-100, 100]$, store the values as an array, and find its critical points. Based on the critical points, we define G_i for $i \in \{1, 2, \ldots, 2N\}$ as specified in Condition 4.1. Then, for each G_i, we create inverse arrays of G_i, namely G_i^{-1}. Finally, we define our potential function V.

Once we have all the ingredients that we need, we start by defining an Ergodic approximation function. This function takes G_i^{-1} values as an array, λ values, β, V values as an array, and the number of critical points as an input. The aim of this function is to evaluate $\theta_i(\lambda) = \mathbb{E}[G_i^{-1}(\lambda - \beta V(0, \omega)]$.

The next step is to define a function called BaseCase1. This function deals with functions G with only one critical point, $G(0) = 0$. If there is only one critical point, we know that from Section 5.1, the effective Hamiltonian is

$$H = \begin{cases} \theta_1^{-1}(\theta) & \text{if } \theta \in (-\infty, \theta_1(\beta)), \\ \beta & \text{if } \theta \in [\theta_1(\beta), \theta_2(\beta)], \\ \theta_2^{-1}(\theta) & \text{if } \theta \in (\theta_1(\beta), +\infty). \end{cases}$$

We will find $\theta_1^{-1}(\lambda)$, and $\theta_2^{-1}(\lambda)$ by using Ergodic approximation function that we defined earlier and considering their inverse arrays. Similarly, we will evaluate $\theta_1(\beta)$ and $\theta_2(\beta)$ values. Then, we can define the effective Hamiltonian $\overline{H}$. The base case 1 function will return linearly spaced θ values from the minimum value of $\theta_1(\lambda)$ and maximum value of $\theta_2(\lambda)$, and $\overline{H}(\theta)$ values evaluated at these θ values.

Then, we move on to the second function, BaseCase2. If the function G has multiple critical points, all located in the second quadrant, and if β is large enough, i.e., $\beta > M - m$, we will use this function to evaluate effective Hamiltonian. In addition to steps of base case 1, in this case, we need to deal with possible jumps. We know that if there is a jump from one of the critical points p_i to somewhere, first, we need to find the second jump point. To do that, we find $\tilde{p} = \min\{p > p_i : G(p) - G(p_i) = 0\}$. Next, we find the branch number of this point. We store the jump point and the branch number of the jump point in two lists. The rest is similar to the previous case. We find the inverses of θ_i's and define the effective Hamiltonian $\overline{H}$. The base case 2 function returns linearly spaced θ values from the minimum value of $\theta_1(\lambda)$ and maximum value of $\theta_{2N}(\lambda)$, and $\overline{H}(\theta)$ values evaluated at these θ values.

For the functions that are neither in base case 1 nor base case 2, we employ gluing processes. First, we will focus on gluing at the origin case, i.e., the function G has multiple critical points but not all of them are in the second quadrant. Here, we split the function G into to parts, say $\overline{G}_1$ and $\overline{G}_2$ so that $G = \min\{\overline{G}_1, \overline{G}_2\}$. To achieve this, we assume that $100p^2 > G(p)$ for any $p \in \mathbb{R}$ and use

$$\overline{G}_1(p) = \begin{cases} G(p) & \text{if } p \leq 0, \\ 100p^2 & \text{if } p > 0, \end{cases}$$

and

$$\overline{G}_2(p) = \begin{cases} 100p^2 & \text{if } p \leq 0, \\ G(p) & \text{if } p > 0. \end{cases}$$

Then, we define gluing at the origin function. This function takes G as an input, defines $\overline{G}_1(p)$ and $\overline{G}_2(p)$, takes the reflection of $\overline{G}_2(p)$ over the y-axis, and returns $\overline{G}_1(p)$, critical points of $\overline{G}_1(p)$, reflected $\overline{G}_2(p)$, and critical points of reflected $\overline{G}_2(p)$. We are taking the reflection of $\overline{G}_2(p)$ because after gluing at the origin, we will revisit base case 1 and base case 2 for homogenization and these base cases are triggered when the only critical point is at the origin or Condition 4.1 is satisfied, respectively, so we need to have all of the possible critical values in the second quadrant other than the one at the origin. After the homogenization, we will take the reflection again to have the exact effective Hamiltonian.

The second type of gluing function, Gluing2, takes G and critical points of G as input. It replicates Section 6.2.1. It first locates points p_{2j-1} and p_{2i}. Then, it defines

$$G_1(p) = \begin{cases} G(p) & \text{if } p \leq p_{2j-1} \\ \\ I(p) & \text{if } p > p_{2j-1}, \end{cases} \quad \text{and} \quad G_2(p) = \begin{cases} I(p) & \text{if } p \leq p_{2i} \\ \\ G(p) & \text{if } p > p_{2i}, \end{cases}$$

where $I : \mathbb{R} \to \mathbb{R}$ is chosen such that $I_1 := I|_{[p_{2j-1},+\infty)}$ and $I_2 := I|_{(-\infty,p_{2i}]}$ are strictly increasing and decreasing, respectively and $G = \min\{G_1, G_2\}$. Specifically,

$$I_1(p) = 10(p - p_{2j-1})^2 + G(p_{2j-1})$$

and

$$I_2(p) = 10(p - p_{2i})^2 + G(p_{2i}).$$

Note that $G_2(p)$ does not pass through the origin right now. In order to move on to the next steps, we need to shift this function so that it passes through the origin. So, it defines

$$\tilde{G}_2(p) = G_2(p + p_{2i}) - G(p_{2i}).$$

After the homogenization, we will shift this function back to its original place.

Similar to the above step, we define the third type of gluing function, Gluing3, which takes G and critical points of G as input. This function replicates

Section 6.2.2. It locates p_{2i} and defines

$$G_1(p) = \begin{cases} G(p) & \text{if } p \leq p_{2i} \\ I(p) & \text{if } p > p_{2i}, \end{cases} \quad \text{and} \quad G_2(p) = \begin{cases} I(p) & \text{if } p \leq p_{2i} \\ G(p) & \text{if } p > p_{2i}, \end{cases}$$

where $I : \mathbb{R} \to \mathbb{R}$ is chosen such that $I_1 := I|_{[p_{2i},+\infty)}$ and $I_2 := I|_{(-\infty,p_{2i}]}$ are strictly increasing and decreasing, respectively and $G = \min\{G_1, G_2\}$. Specifically,

$$I_1(p) = I_2(p) = 10(p - p_{2i})^2 + G(p_{2j-1}).$$

This time, $G_1(p)$ does not pass through the origin, so we consider the reflection of $\tilde{G}_1(p) = G_1(p + p_{2i}) - G(p_{2i})$ over the y-axis.

We lastly define the main function, Homogenizator. This function takes G, critical points of G, and β as inputs. First, it checks that whether the function G has its minimum value at the origin or not. If not, using a change of variable, we make sure that its minimum value, 0, is attained when $p = 0$. Then, it checks the number of critical points. If there is only one critical point, it calls the BaseCase1 function. If the number of critical points is greater than 1 and $\beta > M - m$, it employs BaseCase2 function. If neither BaseCase1 or BaseCase2 is triggered, then it chooses appropriate gluing procedure so that BaseCase1 or BaseCase2 is applicable. Finally, it returns the effective Hamiltonian.

The working logic of the Python implementation can be visually understood through the flowchart provided below.

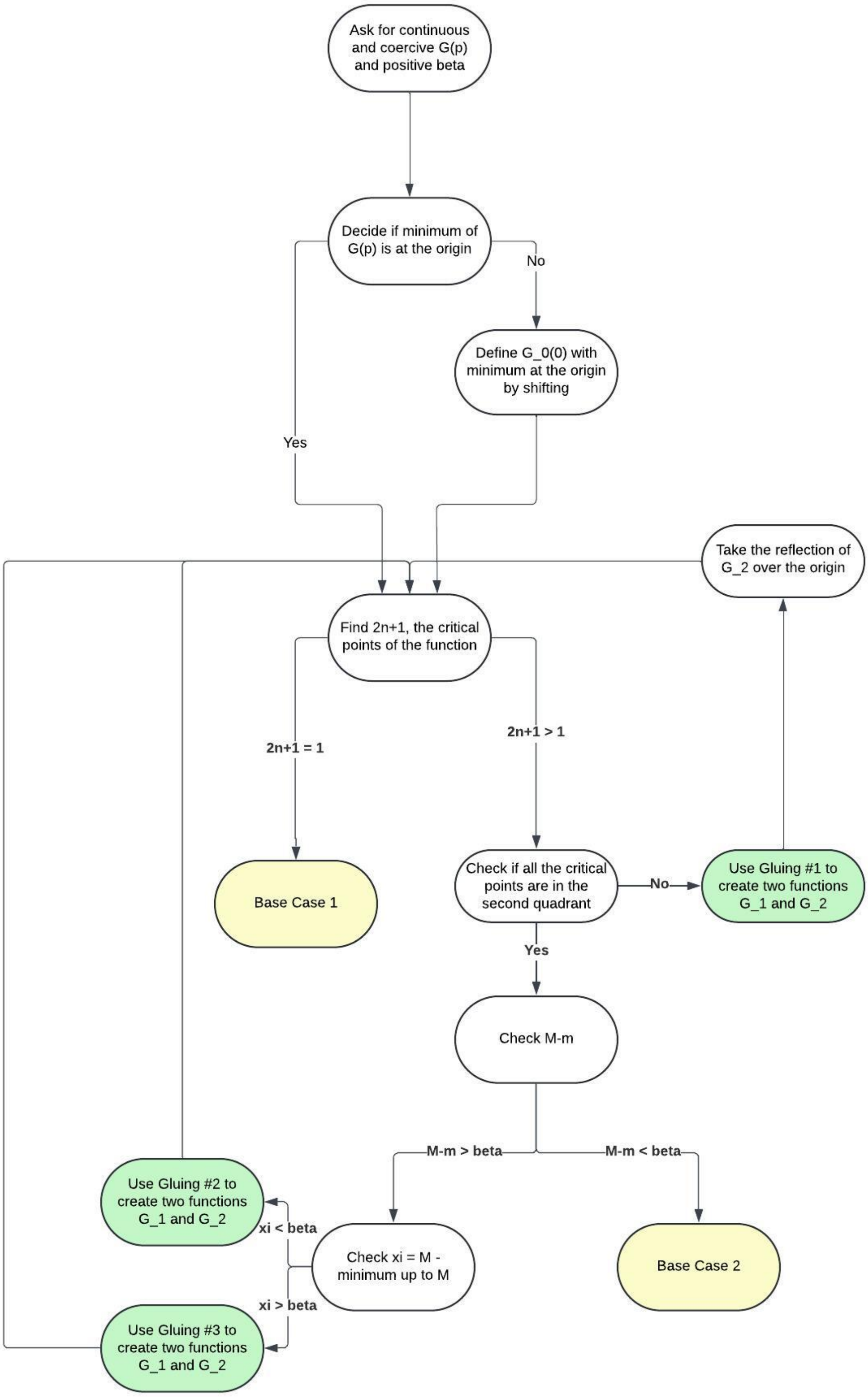

Figure 8.1: Flowchart of the Python implementation

8.2 Examples

In this section, we will consider some examples. We will find the effective Hamiltonian $\overline{H}$ and plot its graph using the code that we developed in the previous section.

Example 8.1. *The most natural example comes to mind when we think of single-well function is* $G(p) = p^2$.

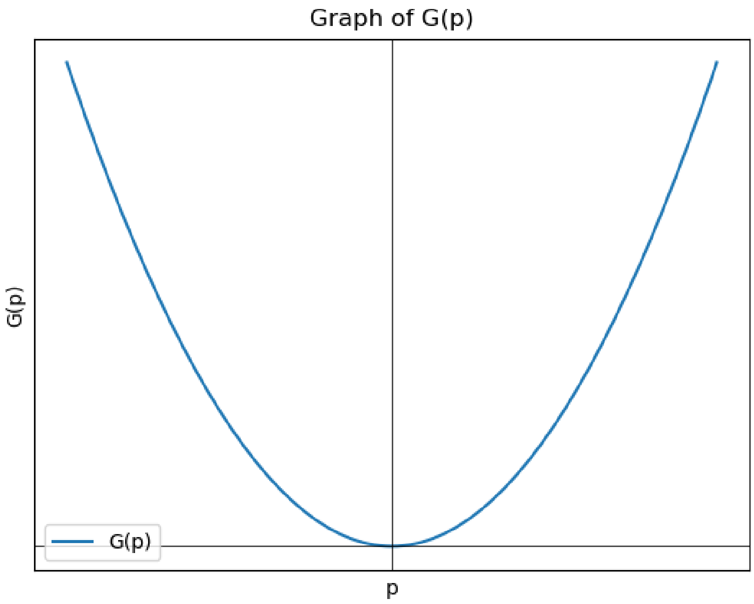

Figure 8.2: Single-well $G(p)$

Take $\beta = 2$ *and let* V *be defined as in Example 3.11, i.e.,* $V(x, \omega) = V(x +$

$U(\omega))$, *where* $U \sim Unif[0,1]$, *and*

$$V_3(x) = \begin{cases} \omega_{\lfloor x \rfloor} & \textit{if } x \textit{ is an integer,} \\ \\ \omega_{\lfloor x \rfloor} + (x - \lfloor x \rfloor)(\omega_{\lceil x \rceil} - \omega_{\lfloor x \rfloor}) & \textit{if } x \textit{ is not an integer.} \end{cases}$$

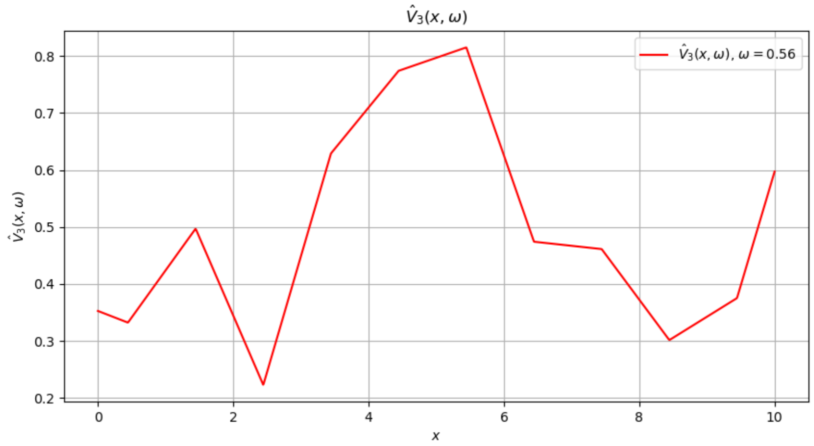

Figure 8.3: Stationary linear interpolation of uniformly selected points in $[0,1]$

When we run the code, it checks the number of critical points of $G(p)$*. Since there is only one critical point, located at the origin, we employ base case 1 function which evaluates* $\theta_1(\beta)$ *and* $\theta_2(\beta)$ *values and finds inverses of* $\theta_1(\lambda)$ *and* $\theta_2(\lambda)$ *functions using Ergodic approximation. Then, it defines the effective Hamiltonian* $\overline{H}(\theta)$ *and plots its graph with a flat piece at height* $\beta = 2$ *for* $\theta \in [\theta_1(\beta), \theta_2(\beta)]$.

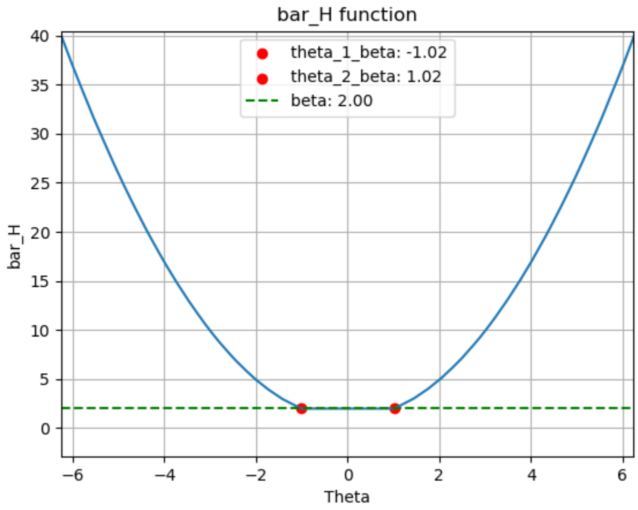

Figure 8.4: The effective Hamiltonian $\overline{H}(\theta)$ for single-well $G(p)$

Example 8.2. *Let us do the same using a different potential function. Take* $G(p) = p^2$ *and* $\beta = 2$ *as in the previous example. This time, we will take* V *to be defined as in Example 3.10 to see what happens in the periodic setting, i.e.,*

$$\hat{V}_2(x,\omega) = \begin{cases} V_2(x + U(\omega)) & \textit{if the coin lands on heads,} \\ \overline{V}_2(x + U(\omega)) & \textit{if the coin lands on tails,} \end{cases}$$

where $V_2 : \mathbb{R} \to \mathbb{R}$ *and* $\overline{V}_2 : \mathbb{R} \to \mathbb{R}$ *be defined as follows:*

$$V_2(x) = \frac{1}{2}(\sin(x) + 1) \quad \textit{and} \quad \overline{V}_2(x) = -\frac{1}{2}(\sin(x) - 1),$$

we are flipping a coin each time $x = 2\pi n$ *for* $n \in \mathbb{Z}$, *and* $U \sim \mathit{Unif}[0, 2\pi]$. *Clearly,* $\tilde{V}_2$ *is* 2π*-periodic.*

When we run the same code with the new potential function, we see that *the overall shape of the effective Hamiltonian stays the same but the length of the flat piece decreases.*

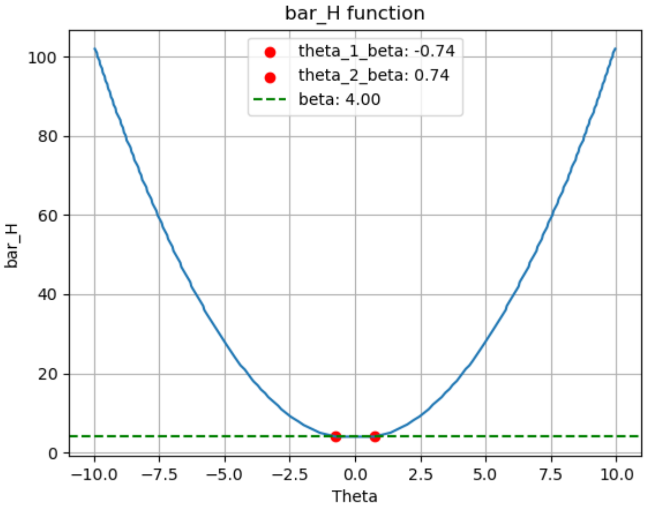

Figure 8.5: The effective Hamiltonian $\overline{H}(\theta)$ for single-well $G(p)$ with periodic potential

Example 8.3. *Consider the function* $G(p) = 4.5p^4 + 17p^3 + 16.5p^2$. *This function has three critical points at* $p_1 = \frac{-11}{6}$, $p_2 = -1$, *and* $p_3 = 0$.

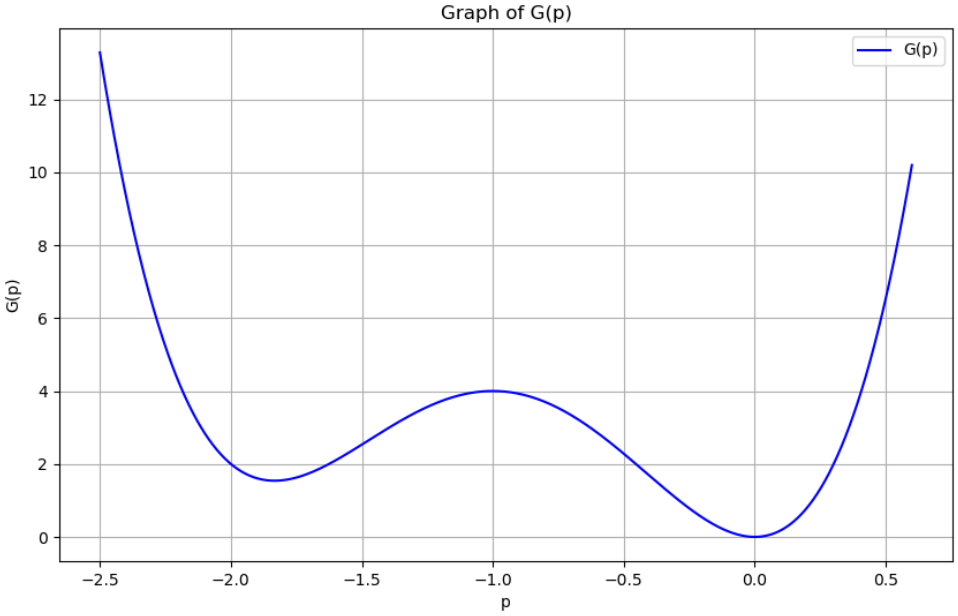

Figure 8.6: Double-well $G(p)$

Take $\beta = 4$ *and let* V *be defined as in Example 3.11, i.e.,* $V(x,\omega) = V(x + U(\omega))$, *where* $U \sim Unif[0,1]$, *and*

$$V_3(x) = \begin{cases} \omega_{\lfloor x \rfloor} & \textit{if } x \textit{ is an integer,} \\ \omega_{\lfloor x \rfloor} + (x - \lfloor x \rfloor)(\omega_{\lceil x \rceil} - \omega_{\lfloor x \rfloor}) & \textit{if } x \textit{ is not an integer.} \end{cases}$$

When we run the code, it checks the number of critical points of $G(p)$. *There are three critical points, one at the origin and the rest are in the second quadrant, so we employ base case 2 function which finds the cutoff points for the effective Hamiltonian, uses Ergodic approximation to find inverse of necessary* $\theta_i(\lambda)'s$, *and flat pieces. Then, after defining the* $\overline{H}$, *it plots the graph of the* *effective Hamiltonian.*

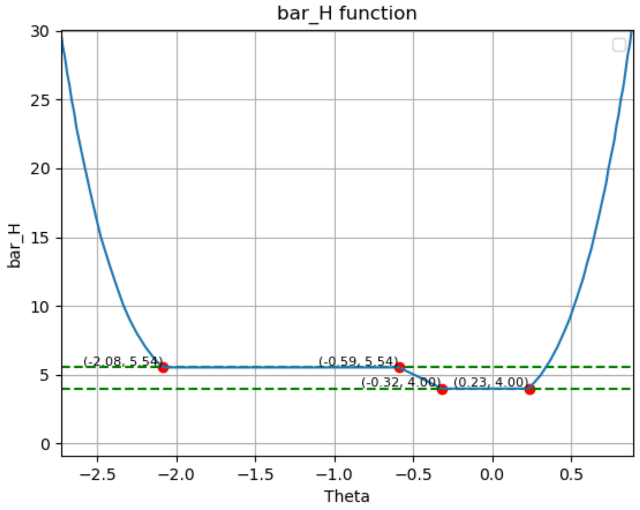

Figure 8.7: The effective Hamiltonian $\overline{H}(\theta)$ for double-well $G(p)$

Example 8.4. *Consider the function* $G(p) = p^6 - 5p^4 + \frac{1}{4}p^3 + 7p^2$. *This function has five critical points, approximately located at* $p_1 = -1.57, p_2 = -0.96, p_3 = 0, p_4 = 1.05,$ *and* $p_5 = 1.48$.

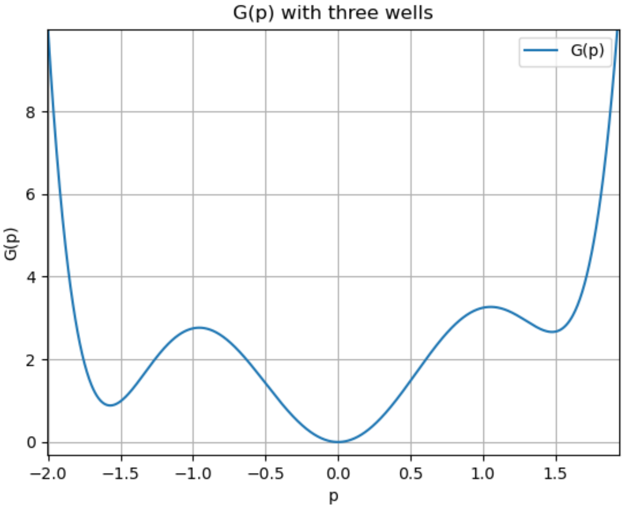

Figure 8.8: Triple-well $G(p)$

We take $\beta = 4$ *and let* V *be defined as in Example 3.11, as above. Then, since not all the critical points are located in the second quadrant, we use gluing at the origin (see 6.1). We split* $G(p)$ *into two parts as follows:*

$$\tilde{G}_1(p) = \begin{cases} G(p) & \textit{if } p \leq 0, \\ 10p^2 & \textit{if } p > 0, \end{cases}$$

and

$$\tilde{G}_2(p) = \begin{cases} 10p^2 & \textit{if } p < 0, \\ G(p) & \textit{if } p \geq 0. \end{cases}$$

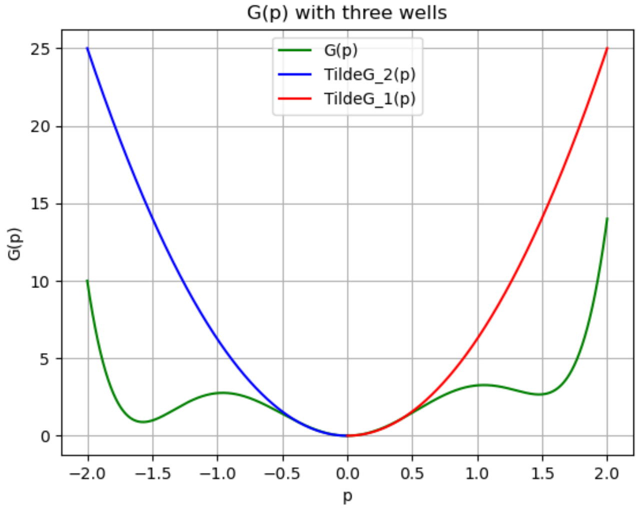

Figure 8.9: Gluing triple-well $G(p)$ at the origin

Now, $\tilde{G}_1(p)$ *has all of its critical points in the second quadrant, but* $\tilde{G}_2(p)$ *has not. We take the reflecton of* $\tilde{G}_2(p)$ *as follows:*

$$\overline{\tilde{G}}_2(p) = \tilde{G}_2(-p).$$

Now, we run the algorithm for $\tilde{G}_1(p)$ *and* $\overline{\tilde{G}}_2(p)$*. They are both in base case 2. After finding their effective Hamiltonians as in above question, we take the reflection* $\overline{\tilde{G}}_2(p)'s$ *effective Hamiltonian above the* $y-$*axis and merge two effective Hamiltonians over the flat piece at height* $\beta = 4$.

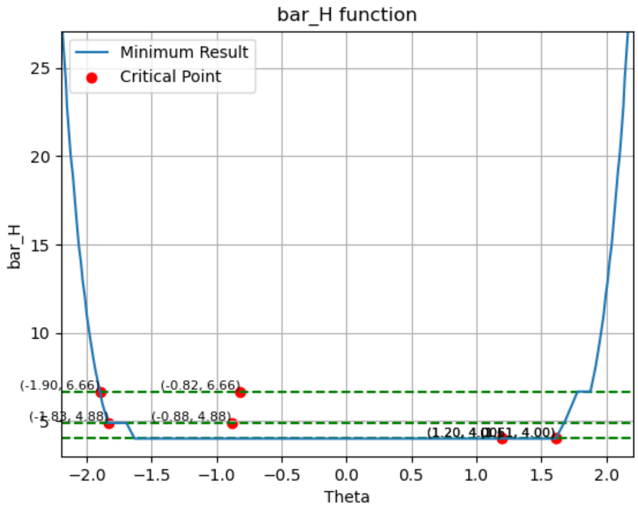

Figure 8.10: The effective Hamiltonian $\overline{H}(\theta)$ for triple-well $G(p)$ using the first gluing proce

Example 8.5. *Let* $G(p) = 1.2p^6 - 6p^4 + 0.25p^3 + 7p^2 - 1.2p$, $\beta = 4$, *and and let* V *be defined as in Example 3.11. We note that although* $G(0) = 0$, *the minimum value of* $G(p)$ *is not attained at* $p = 0 : \min_{p \in \mathbb{R}} G(p) = G(1.595) \approx -2.166$. *So, the code first shifts this function so that the minimum value is attained at the origin. Then, since the number of wells is greater than one and* β *is not larger than* $M - m$, *it cannot use base case 1 and base case 2 without appropriate gluing. So, it checks* $M - \min\{G(p_1), G(p_3), \ldots, G(p_{2i-1})\}$ *where* $M = G(p_{2i})$. *Since* $\beta = 4 > M - \min\{G(p_1), G(p_3), \ldots, G(p_{2i-1})\}$, *it uses the second gluing process, i.e., we define*

$$\tilde{G}_1(p) = \begin{cases} G(p) & \text{if } p \leq p_{2j-1}, \\ 10(p - p_{2j-1})^2 + G(p_{2j-1}), & \text{if } p > p_{2j-1}, \end{cases}$$

and

$$\tilde{G}_2(p) = \begin{cases} 10(p - p_{2i})^2 + G(p_{2i}) & \text{if } p < p_{2i}, \\ G(p) & \text{if } p \geq p_{2i}. \end{cases}$$

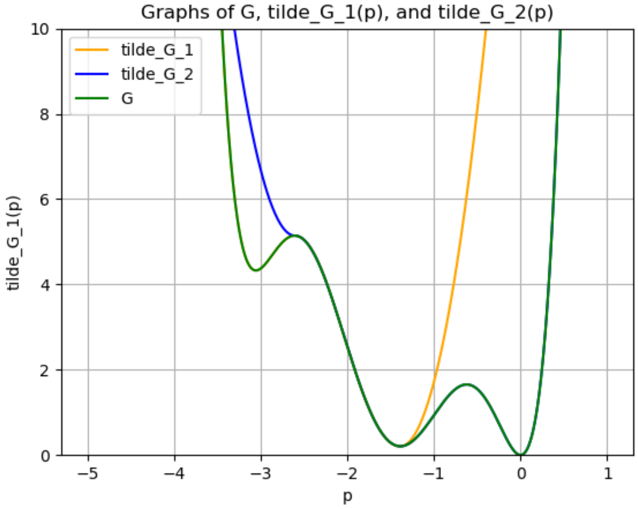

Figure 8.11: Gluing triple-well $G(p)$ using second gluing process

Now, observe that $\tilde{G}_1(0) \neq 0$*. So, we need to shift* $\tilde{G}_1(0)$ *as well before continuing the process. After shifting the minimum point of* $\tilde{G}_1$ *to the origin, the code uses base case 2 since the number of critical points of* $\tilde{G}_1$ *is greater than* 1*. There is no need to shift* $\tilde{G}_2$ *since it attains its minimum value at the origin. It uses base case 2 again since the number of critical points of* $\tilde{G}_2$ *is also greater than* 1*. Then, after shifting the effective Hamiltonian of* $\tilde{G}_1$ *back and gluing two effective Hamiltonians and taking their minimum, we have the graph of the effective Hamiltonian as follows:*

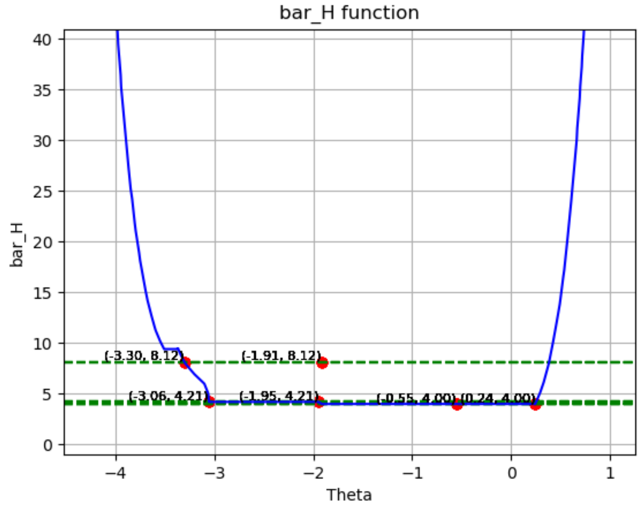

Figure 8.12: The effective Hamiltonian $\overline{H}(\theta)$ for triple-well $G(p)$ and large β

Example 8.6. *Consider the same example with* $\beta = 0.2$. *This time,* $\beta < M - \min\{G(p_1), G(p_3), \dots, G(p_{2i-1})\} = 13$, *we will use Section 6.2.2. The rest of the process is almost the same and we have the following effective Hamiltonian:*

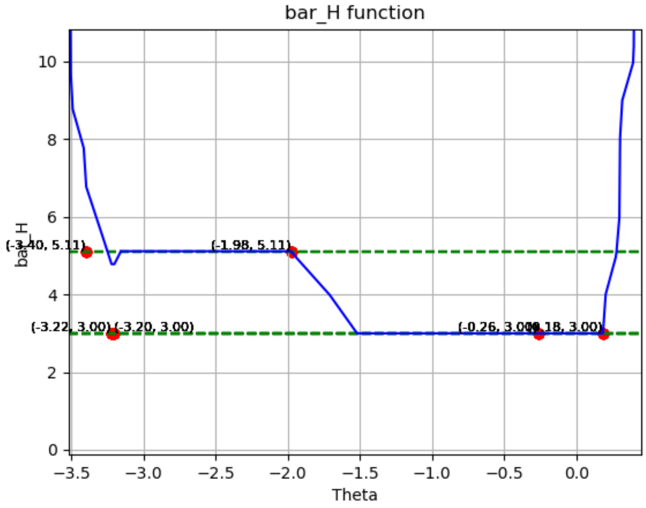

Figure 8.13: The effective Hamiltonian $\overline{H}(\theta)$ for triple-well $G(p)$ and small β

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