ELECTROMAGNETIC FORM FACTORS AND THEIR INTERPRETATION

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by
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ABSTRACT

The electromagnetic form factors in elastic electron-proton scattering are used to determine the finite size of the proton. Through the use of Feynman Diagrams and Fermi's "golden rule", several key results for cross sections of elastic electron scattering will be re-derived. This will ultimately lead to the calculation for the Rosenbluth formula, that describes in detail the process of electron-proton scattering. Furthermore, the process used for determining the size of the proton from the form factors will be shown. In addition, a recent paper by R. Jaffe, which argues the validity of this process, will be discussed in detail.

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CHAPTER 1

INTRODUCTION

Although the proton was discovered over one-hundred years ago, there remains debate on its intrinsic size, which is typically defined by its electric charge radius. Through electron scattering experiments, the electric and magnetic form factors can be obtained, from which the proton charge radius can be determined. Using this method, developed by Hofstader, McAllister and co-workers over a ten year period [4–8], they arrived at the (root-mean-square) charge radius $\sqrt{\langle r^2 \rangle} = (0.74 \pm 0.24)\,\text{fm}$ in 1956 [9] for which Hofstader received the Nobel Prize in Physics in 1961. Many similar experiments have since followed, reducing the error significantly.

Over the last two decades new experimental results have renewed the interest in this process. A significant discrepancy between unpolarized and double-polarized measurements [10] of the electric form factor were observed. It is now generally believed that radiative corrections can largely account for this discrepancy — see, for instance, Ref. [11]. In addition, while both hydrogen spectroscopy and electron-proton scattering agree on a value of 0.875 fm [12], results from muonic hydrogen spectroscopy are 4% less at 0.84 fm [13]. Although this issue is not completely resolved, some more recent papers suggest that a careful analysis of the electron scattering data actually provides smaller numbers for the proton radius which agree with the result from muonic hydrogen spectroscopy — see, e.g., Ref. [14]. Finally, the circumstances in which a 3D charge density can be obtained from the electric form factor [3, 15, 16], remains under debate. It is on this final point that this paper will focus.

Starting with Fermi's "golden rule" for scattering [17, 18], which is a (frame-independent) master formula for the cross section of the collision of two particles, both the center-of-mass (cm) frame and the laboratory frame (lab) formulas are derived for elastic electron scattering. Then, to the leading-order in quantum electrodynamics (QED), cross sections for electron scattering off a spin-0 target and a spin-1/2 target are calculated. The finite extension of the hadrons is taken into account through the form factors. In particular, the classic Rosenbluth formula for electron-proton scattering [19] is re-derived. The last part of the thesis is devoted to a recent work by R. Jaffe [3], which addresses the challenges of defining a 3D charge density through the electric form factor. The work concentrates on a spin-0 particle and illustrates important results by means of a toy model calculation. The analytical part of this calculation have been repeated and analyzed. Although several discrepancies were observed, we are in agreement with the overall finding that, relativistic corrections seem to prevent one from a rigorous definition of a 3D charge distribution.

CHAPTER 2

SCATTERING CROSS SECTION

We first consider the scattering of two particles, a and b, with an arbitrary number of particles in the final state,

$$a + b \to c + d + \ldots \tag{2.1}$$

and discuss the master formula for the corresponding scattering cross section. Generally, the cross section of a given process is a measure for the strength of the interaction between the particles.

The differential cross section for the process in Eq. (2.1) can be written as

$$d\sigma = \frac{|\mathcal{M}|^2}{F} dQ\,, \tag{2.2}$$

where $\mathcal{M}$ is the scattering amplitude which contains the information about the dynamics of the process. The quantity F represents the incident flux, which for a general collision between two particles is,

$$F = 4((p_a \cdot p_b)^2 - {m_a}^2 {m_b}^2)^{\frac{1}{2}}, \tag{2.3}$$

with $p_a(m_a)$ and $p_b(m_b)$ the energy-momentum 4-vectors (masses) of the particles a and b, respectively. This means, $p_a = (E_a, \vec{p}_a)$, $E_a = \sqrt{{m_i}^2 + {\vec{p}_i}^{\,2}}$, and likewise for the particle b. All equations are done using "natural units", such that $c = \hbar = 1$.

The phase space factor, dQ, is represented as,

$$dQ = (2\pi)^4 \delta^4 \Big(p_a + p_b - \sum_f p_f \Big) \prod_f \frac{d^3\vec{p}_f}{(2\pi)^3 2E_f} \tag{2.4}$$

where each final-state particle provides a factor of

$$\frac{d^3\vec{p}}{(2\pi)^3 2E}$$

The delta function is used to ensure that both energy and momentum are conserved.

Combining Eq. (2.3) and Eq. (2.4) into the equation for the differential cross section Eq. (2.2) provides:

$$\boxed{d\sigma = \frac{|\mathcal{M}|^2}{4((p_a \cdot p_b)^2 - {m_a}^2 {m_b}^2)^{\frac{1}{2}}} (2\pi)^4 \delta^4 \Big(p_a + p_b - \sum_f p_f \Big) \prod_f \frac{d^3\vec{p}_f}{(2\pi)^3 2E_f}} \tag{2.5}$$

This equation, which can be considered the master formula for the calculation of the scattering cross section, is known as "Fermi's golden rule" for scattering [17, 18].

In the following, Fermi's golden rule will be used for the specific case of two final-state particles. Furthermore, we will concentrate on elastic scattering, where both particle a and particle b remain intact in the collision process. For this case, we will further evaluate the master formula for the cross section by considering the center of mass frame as well as the laboratory frame.

2.1 Center of Mass Frame

We will first examine the process of two-body elastic scattering in the center of mass (cm) frame as shown in Fig. 2.1, although not notated individually, all variables in this section should be considered in the CM frame. In this process the incident particle a (mass, m) will be scattered off the target particle b (mass, M), while the outgoing particles will be annotated as c and d, such that

$$a(k) + b(p) \to c(k') + d(p'), \tag{2.6}$$

where, k and p represent the 4-momenta of the particles.

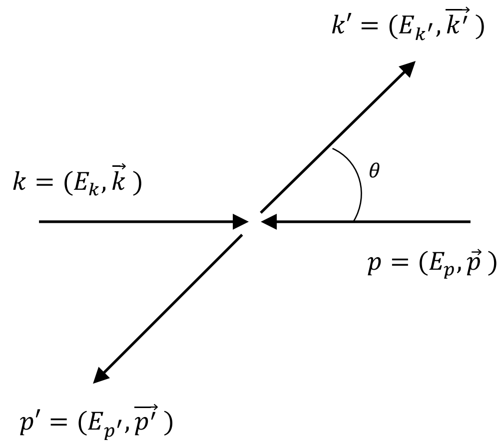

Figure 2.1: Two-body elastic scattering in the CM frame

In the CM frame, the incoming and target particle have the same momentum, $\vec{p} = -\vec{k}$, meaning the total momentum is always zero, which simplifies the equations.

$$\begin{aligned}(k \cdot p)^2 - m^2M^2 &= (E_kE_p + \vec{k}^2)^2 - m^2M^2 \\ &= E_k^2E_p^2 + (E_kE_p)\vec{k}^2 + \vec{k}^4 - (E_k^2 - \vec{k}^2)(E_p^2 - \vec{k}^2) \\ &= \vec{k}^2(E_k^2 + E_p^2 + 2E_kE_p) \\ &= \vec{k}^2(E_k + E_p)^2\end{aligned} \tag{2.7}$$

therefore,

$$\sqrt{(k \cdot p)^2 - m^2M^2} = |\vec{k}|(E_k + E_p). \tag{2.8}$$

The "golden rule", Eq. (2.5), then becomes:

$$d\sigma = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{(E_k + E_p)|\vec{k}|}\ \delta^4(k + p - k' - p')\ \frac{d^3\vec{k'}\ d^3\vec{p'}}{E_{k'}\ E_{p'}}. \tag{2.9}$$

Next, to evaluate the δ-function, using $p_i = (E_i, \vec{p_i})$, and $\vec{p} = -\vec{k}$,

$$\delta^4(k + p - k' - p') = \delta(E_k + E_p - E_{k'} - E_{p'})\delta^3(-\vec{k'} - \vec{p'}). \tag{2.10}$$

For $E_i = \sqrt{m_i^2 + \vec{p_i}^{\,2}}$,

$$\frac{d\sigma}{d^3\vec{k'}} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{(E_k + E_p)|\vec{k}|} \frac{\delta\left((E_k + E_p) - \sqrt{m^2 + \vec{k'}^2} - \sqrt{M^2 + \vec{p'}^2}\right)}{\sqrt{m^2 + \vec{k'}^2}\,\sqrt{M^2 + \vec{p'}^2}}\, \delta^3(-\vec{k'} - \vec{p'})\, d^3\vec{p'} \tag{2.11}$$

Performing the $\vec{p}'$ integration yields $\vec{p}' = -\vec{k}\,'$ from the delta function, allowing the removal of $\vec{p}'$ from the equation.

$$\frac{d\sigma}{d^3\vec{k'}} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{(E_k + E_p)|\vec{k}|} \frac{\delta\left((E_k + E_p) - \sqrt{m^2 + \vec{k'}^2} - \sqrt{M^2 + \vec{k'}^2}\right)}{\sqrt{m^2 + \vec{k'}^2}\,\sqrt{M^2 + \vec{k'}^2}} \tag{2.12}$$

Rewriting

$$d^3\vec{k'} = |\vec{k'}|^2\; d\rho\; d\Omega \tag{2.13}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{(cm)} = \left(\frac{1}{8\pi}\right)^2 \frac{1}{(E_k + E_p)|\vec{k}|} \times \int_0^\infty |\mathcal{M}|^2 \frac{\delta\left((E_k + E_p) - \sqrt{m^2 + |\vec{k'}|^2} - \sqrt{M^2 + |\vec{k'}|^2}\right)}{\sqrt{m^2 + |\vec{k'}|^2}\,\sqrt{M^2 + |\vec{k'}|^2}}\, |\vec{k'}|^2 d|\vec{k'}| \tag{2.14}$$

Performing the integration over $|\vec{k'}|$ is simplified by the change of variable:

$$E \equiv \left(\sqrt{m^2 + |\vec{k'}|^2} + \sqrt{M^2 + |\vec{k'}|^2}\right), \tag{2.15}$$

where E represents the total energy of the outgoing particles. then,

$$dE = \frac{E\;|\vec{k'}|}{\sqrt{m^2 + |\vec{k'}|^2}\,\sqrt{M^2 + |\vec{k'}|^2}}\; d|\vec{k'}| \tag{2.16}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{(cm)} = \left(\frac{1}{8\pi}\right)^2 \frac{1}{(E_k + E_p)|\vec{k}|} \int_{m+M}^\infty |\mathcal{M}|^2\, \frac{|\vec{k'}|}{E}\, \delta((E_k + E_p) - E) dE \tag{2.17}$$

Where the delta function confirms conservation of energy in that the total energy of the outgoing particles, E is equal to the energy of the incoming particles$(E_k + E_p)$. Replacing $E \to (E_k + E_p)$,

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(cm)} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{(E_k + E_p)^2}} \tag{2.18}$$

Giving us the equation for the scattering angle in the center of mass frame in terms of the scattering amplitude $\mathcal{M}$, the initial energies E_k and E_p, as well as the ratio of the outgoing vs incoming momentum of particle, k.

2.2 Laboratory Frame

The center of mass equation, although simpler, is not always a direct representation of experimental scattering. To model the behavior as seen in many experiments, the laboratory frame (lab) is used. In this frame, the center point is the point of collision and remains stationary, as the target particle, b, starts at rest.

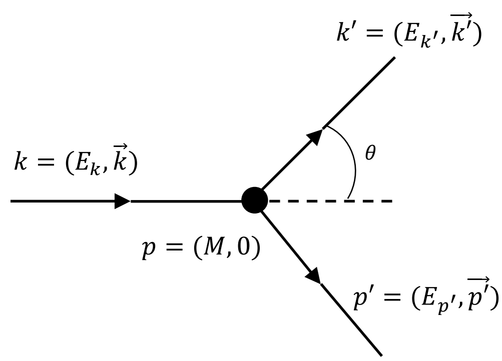

Figure 2.2: Two-Body Scattering in the Lab Frame

To derive the lab frame equation, we will start with the same "golden rule" equation, Eq. (2.5), as before, but with different initial conditions. All remaining equations in the paper will be carried out in the lab frame, therefore all variables should be considered as such unless otherwise noted. As seen in Fig. 2.2, the incident particle, a, approaches the stationary target particle, b, and is deflected at an angle θ. In the lab frame, the momentum of the incident and target particles are $k = (E_k, \vec{k})$ and

$p = (M, 0)$, respectively. Then,

$$
\begin{aligned}
(k \cdot p)^2 - m^2 M^2 &= E_k^2 M^2 - m^2 M^2 \\
&= M^2 (E_k^2 - m^2) \\
&= M^2 \vec{k}^2
\end{aligned}
$$

therefore,

$$\sqrt{(k \cdot p)^2 - m^2 M^2} = M|\vec{k}| \tag{2.19}$$

The "golden rule" equation then becomes:

$$d\sigma = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{M|\vec{k}|} \, \delta^4(k + p - k' - p') \, \frac{d^3\vec{k'} \; d^3\vec{p'}}{E_{k'} \; E_{p'}} \tag{2.20}$$

where $\vec{p} = 0$, and $E_p = M$

$$
\begin{aligned}
\frac{d\sigma}{d^3\vec{k'}} = \left(\frac{1}{8\pi}\right)^2 \frac{1}{M|\vec{k}|} \int |\mathcal{M}|^2 \delta \left(E_k + M - \sqrt{\vec{k'}^2 + m^2} - \sqrt{\vec{p'}^2 + M^2} \right) \\
\times \, \delta^3(\vec{k} - \vec{k'} - \vec{p'}) \, \frac{d^3\vec{p'}}{\sqrt{\vec{k'}^2 + m^2} \, \sqrt{\vec{p'}^2 + M^2}}
\end{aligned}
\tag{2.21}
$$

Performing the $\vec{p'}$ integration yields $\vec{p'} = (\vec{k} - \vec{k'})$ which reflects the conservation of momentum for the target particle $\vec{p} = 0$.

$$\frac{d\sigma}{d^3\vec{k'}} = \left(\frac{1}{8\pi}\right)^2 \frac{1}{M|\vec{k}|} \int |\mathcal{M}|^2 \frac{\delta \left(E_k + M - \sqrt{\vec{k'}^2 + m^2} - \sqrt{(\vec{k} - \vec{k'})^2 + M^2} \right)}{\sqrt{\vec{k'}^2 + m^2} \, \sqrt{(\vec{k} - \vec{k'})^2 + M^2}} \tag{2.22}$$

where,

$$(\vec{k} - \vec{k'})^2 = \vec{k}^2 + \vec{k'}^2 - 2(\vec{k} \cdot \vec{k'}) = \vec{k}^2 + \vec{k'}^2 - 2|\vec{k}||\vec{k'}| \cos\theta \tag{2.23}$$

and making the substitution, $d^3\vec{k'} = |\vec{k'}|^2 d|\vec{k'}| d\Omega$.

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{1}{8\pi}\right)^2 \frac{1}{M|\vec{k}|} \int |\mathcal{M}|^2 \times \frac{\delta\left(E_k + M - \sqrt{|\vec{k'}|^2 + m^2} - \sqrt{\vec{k}^2 + |\vec{k'}|^2 - 2|\vec{k}||\vec{k'}|| \cos\theta + M^2}\right)}{\sqrt{\rho^2 + m^2}\,\sqrt{\vec{k}^2 + |\vec{k'}|^2 - 2|\vec{k}||\vec{k'}| \cos\theta + M^2}} |\vec{k'}|^2 \, d^3|\vec{k'}| \tag{2.24}$$

where, θ is the scattering angle of the incident particle.

Let,

$$z \equiv \sqrt{|\vec{k'}|^2 + m^2} + \sqrt{|\vec{k'}|^2 - 2|\vec{k'}||\vec{k}| \cos\theta + \vec{k}^2 + M^2} \tag{2.25}$$

then,

$$\frac{dz}{d|\vec{k'}|} = \frac{|\vec{k'}|z - |\vec{k}| \cos\theta \sqrt{|\vec{k'}|^2 + m^2}}{\sqrt{|\vec{k'}|^2 + M^2}\sqrt{|\vec{k'}|^2 - 2|\vec{k'}||\vec{k}| \cos\theta + \vec{k}^2 + m^2}} \tag{2.26}$$

therefore, the scattering angle becomes,

$$\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \left(\frac{1}{8\pi}\right)^2 \frac{1}{M|\vec{k}|} \int |\mathcal{M}|^2 \frac{\delta(E_k + M - z)}{|\vec{k'}|z - |\vec{k}| \cos\theta \sqrt{|\vec{k'}|^2 + m^2|}} |\vec{k'}|^2 dz \tag{2.27}$$

From the integration of z, we see $z = (E_k + M)$

$$\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{M|\vec{k}|} \frac{|\vec{k'}|^2}{||\vec{k'}|(E_k + M) - |\vec{k}| \cos\theta \sqrt{\rho^2 + m^2|}} \tag{2.28}$$

then, $\sqrt{\vec{k'}^2 + m^2} = E_3$, and the equation for the scattering angle becomes:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{M|\vec{k}|} \frac{|\vec{k'}|^2}{(E_k + M)|\vec{k'}| - E_{k'}|\vec{k}| \cos\theta}} \tag{2.29}$$

In the lab frame, the scattering angle becomes dependant on both the incoming and outgoing particle momentum and energy, and the angle between them.

For this paper we will be focusing on the scattering of electrons from a stationary proton. In this case, the mass of the electron is much smaller than the electron's energy, allowing us to simplify the equation by assuming the mass of the electron is negligible. Therefore, for the mass of the incident particle, $m = 0$, then, $\vec{k} = E_k$ and $\vec{k'} = E_{k'}$. Starting with the denominator in Eq. (2.29),

$$\begin{aligned}(E_k + M)|\vec{k'}| - E_{k'}|\vec{k}|\cos\theta &= E_k E_{k'} + E_{k'} M - E_k E_{k'} \cos\theta \\ &= E_k E_{k'}(1 - \cos\theta) + E_{k'} M = (E_k - E_{k'})M + E_{k'} M \\ &= E_k M - E_{k'} M + E_{k'} M = E_k M\end{aligned}$$

Therefore,

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{M^2} \frac{E_{k'}^2}{E_k^2}} \qquad \text{for } m = 0 \tag{2.30}$$

CHAPTER 3

ELASTIC ELECTRON SCATTERING IN QED

The scattering amplitude, $\mathcal{M}$, describes the interaction between the two particles and can be derived using the Feynman rules [20]. The Feynman diagram, as shown below in Fig. 3.1, shows the incoming particle with momentum, k, and spin, λ, while the target particle momentum and spin is denoted as p, and s, respectively. The outgoing momenta and spins are denoted with a $'$ for each respective variable. The line, connecting vertices μ and ν, represent the photon, with momentum q.

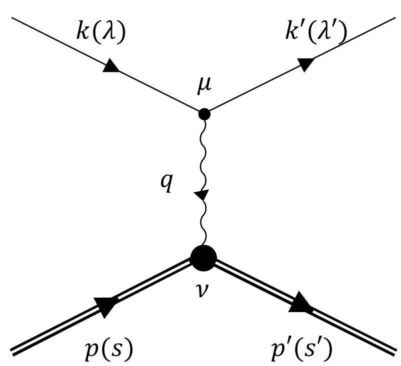

Figure 3.1: Feynman Diagram For Two-Body Scattering

The Feynman rules are dependant on the type of particle. For quantum electrodynamics (QED), the spin of the particle determines the rules that are used. This paper will explore both spin-0 and spin-1/2 particles. For these equations, Dirac γ-matrices will be utilized and are defined in Appendix A.

Lab Frame Kinematics: Looking at the kinematics of this process, the momentum is transferred from the incident particle to the target particle by a photon. The photon's momentum q is transferred to the target particle, giving it a non-zero momentum p' after collision. We can further relate these momenta to their energies and the scattering angle. It can be shown, due to conservation of momentum that:

$$k - k' = q = p' - p \tag{3.1}$$

where the momentum lost from the incident particle is given to the target particle. And similarly, for conservation of energy:

$$E_k - E_k' = \nu = E_p' - E_p \tag{3.2}$$

where, in the lab frame, $E_p = M$,

$$\nu = (E_{p'} - M) \tag{3.3}$$

Squaring the photon momentum, leads to:

$$\begin{aligned} q^2 &= -\nu^2 + \vec{q}^{\,2} = -(E_{p'} - M)^2 + |\vec{p}\,' - \vec{p}\,|^2 \\ &= -(E_{p'} - M)^2 + |\vec{p}\,'|^2 \end{aligned} \tag{3.4}$$

for $\vec{p} = 0$.

Factoring out the first term yields:

$$q^2 = -E_p'^2 - M^2 + 2E_{p'}M + |\vec{p}\,'|^2 \tag{3.5}$$

where, $M^2 = E_{p'}^2 - \vec{p}\,'^2$,

$$\begin{aligned} q^2 &= -2M^2 + 2E_{p'}M \\ &= 2M(E_p' - M) = 2\nu M \end{aligned} \tag{3.6}$$

Equating q^2 to the incident particle results in a another valuable relationship.

$$
\begin{aligned}
q^2 &= k^2 + k'^2 - 2k \cdot k' \approx -2k \cdot k' \\
&= 2E_k E_{k'} - 2|\vec{k}||\vec{k}\,'| \cos\theta \\
&\approx -2E_k E_{k'}(1 - \cos\theta) = -4E_k E_{k'} \sin^2 \tfrac{\theta}{2}
\end{aligned}
\tag{3.7}
$$

for the mass of the incident particle, $m \approx 0$.

3.1 Spin-0 Particle

We will start with the simplified process of a spin-1/2 electron scattering off of a spin-0 stationary particle (for example, a pion) as the target particle. Using the Feynman rules, as listed in Appendix D, we obtain from the electron fermion line:

$$
[\overline{u}(k', \lambda')(ie\gamma^\mu)u(k, \lambda)] \tag{3.8}
$$

where $(ie\gamma^\mu)$ comes from the μ vertex for an electron with $Q = -e$.

From the internal photon line we obtain:

$$
\frac{-ig_{\mu\nu}}{q^2} \tag{3.9}
$$

The pion, because of being a spin-0 particle, does not have an external line equation, only a vertex factor:

$$
-iQ(p + p')^\nu \tag{3.10}
$$

where Q is the charge of the pion.

Combining these equations give us the starting equation for the scattering amplitude.

$$
\begin{aligned}
i\mathcal{M} &= [\overline{u}(k', \lambda')(ie\gamma^\mu)u(k, \lambda)] \left(\frac{ig_{\mu\nu}}{q^2} \right) (iQ(p + p')^\nu) \\
&= \frac{eQ}{q^2} [\overline{u}(k', \lambda')(\gamma^\mu)u(k, \lambda)](p + p')_\mu
\end{aligned}
\tag{3.11}
$$

Our equation for the scattering angle from chapter 2, requires the scattering amplitude to be squared, therefore,

$$
|\mathcal{M}|^2 = \frac{e^2Q^2}{q^4} [\overline{u}(k', \lambda')(\gamma^\mu)u(k, \lambda)](p + p')_\mu [\overline{u}(k', \lambda')(\gamma^\mu)u(k, \lambda)]^* (p + p')^*_\mu \tag{3.12}
$$

where, $*$ denotes the complex conjugate, which, because the value being squared is a 1x1 matrix, the complex conjugate is equal to the Hermitian conjugate, then:

$$[\overline{u}(k')(\gamma^{\mu})u(k)]^{*} = [\overline{u}(k)(\gamma^{\nu})u(k')] \tag{3.13}$$

$$(p+p')^{*}_{\mu} = (p+p')_{\nu} \tag{3.14}$$

Therefore,

$$|\mathcal{M}|^2 = \frac{e^2Q^2}{q^4}[\overline{u}(k',\lambda')(\gamma^{\mu})u(k,\lambda)][\overline{u}(k,\lambda)(\gamma^{\nu})u(k',\lambda')](p+p')_{\mu}(p+p')_{\nu} \tag{3.15}$$

Now that we have a working equation for the scattering amplitude, it will need to be simplified into an equation that can be integrated into the equation for the "golden rule". We will first break the equation into two parts, and simplify individually. The amplitude can be written as a function of two tensors, one for each particle,

$$\langle|\mathcal{M}|^2\rangle = \frac{e^2Q^2}{q^4}L^{\mu\nu}_{(e)}Y^{(\pi)}_{\mu\nu} \tag{3.16}$$

Working first with the pion tensor, $Y^{(\pi)}_{\mu\nu}$,

$$\begin{aligned} Y^{(\pi)}_{\mu\nu} &= (p+p')_{\mu}(p+p')_{\nu} \\ &= [p_{\mu}p_{\nu} + p_{\mu}p'_{\nu} + p'_{\mu}p_{\nu} + p'_{\mu}p'_{\nu}] \end{aligned} \tag{3.17}$$

The electron tensor, $L^{\mu\nu}_{(e)}$, is more complicated and requires the use of "Casimir's Trick"[1] which is shown in Appendix B.

$$\begin{aligned} L^{\mu\nu}_{(e)} &= \frac{1}{2}\sum_{\lambda,\lambda'}[\overline{u}(k',\lambda')(\gamma^{\mu})u(k,\lambda)][\overline{u}(k,\lambda)(\gamma^{\nu})u(k',\lambda')] \\ &= 1/2\ Tr[\gamma^{\nu}(\not{k}'+m)\gamma^{\mu}(\not{k}+m)] \\ &= 1/2\ Tr[\gamma^{\nu}\not{k}'\gamma^{\mu}\not{k} + \gamma^{\nu}\not{k}'\gamma^{\mu}m + \gamma^{\nu}m\gamma^{\mu}\not{k} + \gamma^{\nu}m\gamma^{\mu}m] \end{aligned} \tag{3.18}$$

To calculate the electron tensor, a series of trace identities (listed in Appendix A) are utilized. Because $\not{k} = k_{\lambda}\gamma^{\lambda}$, the second and third term in the final line of Eq. (3.18) have an odd number of γ-matrices, where the $\mathrm{Tr}[\gamma_{(odd)}] = 0$. The remaining two terms are,

$$
\begin{aligned}
Tr[\gamma^\nu \not{k}' \gamma^\mu \not{k}] &= k'_\sigma k_\lambda Tr[\gamma^\nu \gamma^\sigma \gamma^\mu \gamma^\lambda] \\
&= k'_\sigma k_\lambda \; 4(g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu}) \\
&= 4(k^\mu k'^\nu - g^{\mu\nu}(k \cdot k') + k'^\mu k^\nu)
\end{aligned} \tag{3.19}
$$

$$
Tr[\gamma^\nu m \gamma^\mu m] = m^2 Tr[\gamma^\nu \gamma^\mu] = 4m^2 g^{\mu\nu} \tag{3.20}
$$

Therefore,

$$
L^{\mu\nu}_{(e)} = 2(k^\mu k'^\nu - (k \cdot k') g^{\mu\nu} + k'^\mu k^\nu + m^2 g^{\mu\nu}) \tag{3.21}
$$

Then, putting Eq. (3.17) and Eq. (3.21) into Eq. (3.16), the scattering amplitude becomes:

$$
\langle |\mathcal{M}|^2 \rangle = \frac{2e^2 Q^2}{q^4} [k^\mu k'^\nu + k'^\mu k^\nu - (k \cdot k') g^{\mu\nu} + m^2 g^{\mu\nu}][p_\mu p_\nu + p_\mu p'_\nu + p'_\mu p_\nu + p'_\mu p'_\nu] \tag{3.22}
$$

Combining each term from the electron tensor with the tensor for the pion yields the following:

$$
\begin{aligned}
&(k^\mu k'^\nu)(p_\mu p_\nu + p_\mu p'_\nu + p'_\mu p_\nu + p'_\mu p'_\nu) \\
&\qquad = [(k' \cdot p)(k \cdot p) + (k' \cdot p)(k \cdot p') + (k' \cdot p')(k \cdot p) + (k' \cdot p')(k \cdot p')]
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
&(k'^\mu k^\nu)(p_\mu p_\nu + p_\mu p'_\nu + p'_\mu p_\nu + p'_\mu p'_\nu) \\
&\qquad = [(k' \cdot p)(k \cdot p) + (k' \cdot p)(k \cdot p') + (k' \cdot p')(k \cdot p) + (k' \cdot p')(k \cdot p')]
\end{aligned} \tag{3.24}
$$

$$
-(k \cdot k') g_{\mu\nu} (p_\mu p_\nu + p_\mu p'_\nu + p'_\mu p_\nu + p'_\mu p'_\nu) = -(k \cdot k')[(p \cdot p) + (p' \cdot p) + (p \cdot p') + (p' \cdot p')] \tag{3.25}
$$

$$
m^2 g_{\mu\nu} (p_\mu p_\nu + p_\mu p'_\nu + p'_\mu p_\nu + p'_\mu p'_\nu) = m^2[(p \cdot p) + (p' \cdot p) + (p \cdot p') + (p' \cdot p')] \tag{3.26}
$$

Eq. (3.23) and Eq. (3.24) yield the same result and can be combined, while Eq. (3.25) and Eq. (3.26) contain a common factor, therefore,

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle = \frac{4e^2 Q^2}{q^4} \Big[&(k' \cdot p)(k \cdot p) + (k' \cdot p)(k \cdot p') + (k' \cdot p')(k \cdot p) + (k' \cdot p')(k \cdot p') \\
&+ \frac{1}{2}(m^2 - k \cdot k')((p \cdot p) + 2(p \cdot p') + (p' \cdot p')) \Big]
\end{aligned} \tag{3.27}
$$

Eq. (3.27) is a general equation for the scattering amplitude, $\langle|\mathcal{M}|^2\rangle$, and holds for any spin-1/2 to spin-0 scattering. At this point it is frame-independent and no approximations have been made.

In order to calculate the scalar products from Eq. (3.27), the initial and final conditions must be declared. Using conservation of energy and momentum, the outgoing momentum of the target pion can be rewritten as, $p' = k - k' + p$.

The scattering amplitude becomes:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle = \frac{4e^2Q^2}{q^4}\Big[&(k'\cdot p)(k\cdot p) + (k'\cdot p)(k\cdot(k-k'+p)) \\ &+ (k'\cdot(k-k'+p))(k\cdot p) + (k'\cdot(k-k'+p))(k\cdot(k-k'+p)) \\ &- \frac{1}{2}(k\cdot k')\Big((p\cdot p) + 2(p\cdot(k-k'+p)) + ((k-k'+p)\cdot(k-k'+p))\Big)\Big]\end{aligned} \tag{3.28}$$

For the pion, $Q = \pm e$, but regardless of the sign of the charge, $Q^2 = e^2$. In addition, for an incident electron, with $m \approx 0$ we can eliminate all terms containing the electron mass.

For the target particle, initially at rest, $p = (M, 0)$, $k = (m, \vec{k})$, $k' = (m, \vec{k'})$, and $q = (k - k')$. With $E_p = M$, and removing the p' terms, the remaining energy terms belong to the incoming and outgoing electron, k, k'. Therefore, to simplify notation, the k subscript will be removed from future energy terms, such that $E_k = E$, and $E_{k'} = E'$. The necessary scalar products are as follows:

$$\begin{aligned}(k\cdot k) &= E^2 - \vec{k}^2 \approx 0 & (p\cdot p) &= M^2 \\ (k'\cdot k') &= E'^2 - \vec{k'}^2 \approx 0 & (k\cdot p) &= ME \\ (k\cdot k') &= EE' - \vec{k}\vec{k'}\cos\frac{\theta}{2} \approx \frac{-q^2}{2} & (k'\cdot p) &= ME'\end{aligned} \tag{3.29}$$

Inserting these values into Eq. (3.28),

$$\langle|\mathcal{M}|^2\rangle = \frac{4e^4}{q^4}\Big[4M^2EE' + M^2q^2\Big] \tag{3.30}$$

where, $q^2 \approx -4EE'\sin^2\frac{\theta}{2}$.

$$\langle|\mathcal{M}|^2\rangle = \frac{4e^4}{q^4}\left[4M^2EE'(\cos^2\frac{\theta}{2})\right] \tag{3.31}$$

$$\boxed{\langle|\mathcal{M}|^2\rangle = \frac{16M^2e^4EE'}{q^4}\left(\cos^2\frac{\theta}{2}\right)} \tag{3.32}$$

Then, using equation (2.33), the scattering cross section for a point-like spin-0 pion in the Lab frame becomes:

$$\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{e^4E'^2}{4\pi^2q^4}\left(\frac{E'}{E}\right)\cos^2\frac{\theta}{2} \tag{3.33}$$

for $\alpha = \frac{e^2}{4\pi}$, and $q^2 = -4EE'\sin^2\frac{\theta}{2}$,

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left(\frac{E'}{E}\right)\cos^2\frac{\theta}{2}} \tag{3.34}$$

For a target particle without recoil, meaning that the target particle does not gain any momentum from the incident particle. This also insinuates that there is no energy transfer such that $E = E'$. This is commonly referred to as the "brick wall" scenario, and although it is not realistic, it can be used to approximate scattering of large mass targets. Then,

$$\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\cos^2\frac{\theta}{2} \tag{3.35}$$

Showing that the term $\left(\frac{E'}{E}\right)$ creates the recoil for the scattering particles. This equation is known as the Mott [21] formula for $m \approx 0$. Evaluating the $\cos^2\frac{\theta}{2}$ term,

$$\begin{aligned}\cos^2\frac{\theta}{2} &= 1 - \sin^2\frac{\theta}{2}\\ &= 1 + \frac{q^2}{4EE'}\\ &= 1 - \frac{2M(E-E')}{4EE'}\end{aligned} \tag{3.36}$$

Where, for $E = E'$, $cos^2\frac{\theta}{2} = 1$. Then,

$$\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}} \tag{3.37}$$

which is defined as the Rutherford equation [22].

Extended Spin-0 Particle: For a particle with physical size, known as an extended particle, an additional term is required. Adding the scalar quantity $F(q^2)$ to the vertex factor, Eq. (3.10), the scattering amplitude, Eq. (3.16) becomes:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^2Q^2}{q^4}L^{\mu\nu}_{(e)}Y^{(\pi)}_{\mu\nu}\left[F(q^2)\right]^2 \tag{3.38}$$

Such that,

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}}\left[F(q^2)\right]^2 \tag{3.39}$$

Therefore,

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{e^4E'^2}{4\pi^2q^4}\left(\frac{E'}{E}\right)\cos^2\frac{\theta}{2}\;\left[F(q^2)\right]^2} \tag{3.40}$$

for $\alpha = \frac{e^2}{4\pi}$, and $q^2 = -4EE'\sin^2\frac{\theta}{2}$,

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left(\frac{E'}{E}\right)\cos^2\frac{\theta}{2}\;\left[F(q^2)\right]^2} \tag{3.41}$$

where, $F(q^2)$ is called the form factor, and contains information about the particle's charge distribution. The form factor will be examined in closer detail in Chapter 4.

3.2 Point-like Spin-1/2 Particle

For a spin-$\frac{1}{2}$ particle, a new scattering amplitude will need to be calculated using Feynman's rules. Both the electron fermion line and the internal photon line will be the same as with the spin-0 case. The addition of a spin-$\frac{1}{2}$ particle as the target adds an additional fermion line, with $Q = e$ for the proton, which replaces the spin-0 vertex term:

$$[\overline{u}(p',s')(-ie\gamma^\nu)u(p,s)] \tag{3.42}$$

Combining the equations for the fermion line of the electron, Eq. (3.8), the internal photon line, Eq. (3.9) and the proton line, Eq. (3.42), we obtain the equation for the scattering amplitude.

$$\begin{aligned} i\mathcal{M} &= [\overline{u}(k',\lambda')(ie\gamma^\mu)u(k,\lambda)]\left(\frac{ig_{\mu\nu}}{q^2}\right)[\overline{u}(p',s')(-ie\gamma^\nu)u(p,s)] \\ &= \frac{ie^2}{q^2}[\overline{u}(k',\lambda')(\gamma^\mu)u(k,\lambda)][\overline{u}(p',s')(\gamma_\mu)u(p,s)] \end{aligned} \tag{3.43}$$

therefore,

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{e^4}{q^4}[\overline{u}(k',\lambda')(\gamma^\mu)u(k,\lambda)][\overline{u}(p',s')(\gamma_\mu)u(p,s)][\overline{u}(k',\lambda')(\gamma^\mu)u(k,\lambda)]^*[\overline{u}(p',s')(\gamma_\mu)u(p,s)]^* \\ &= \frac{e^4}{(k-k')^4}[\overline{u}(k',\lambda')(\gamma^\mu)u(k,\lambda)][\overline{u}(p',s')(\gamma_\mu)u(p,s)] \\ &\qquad \times [\overline{u}(k,\lambda)(\gamma^\mu)u(k',\lambda')][\overline{u}(p,s)(\gamma_\mu)u(p',s')] \end{aligned} \tag{3.44}$$

where, $q = (k - k')$

Same as in the previous section, the amplitude can be separated into two tensors,

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{q^4}L^{\mu\nu}_{(e)}K^{(p)}_{\mu\nu} \tag{3.45}$$

where the electron tensor remains unchanged.

$$L^{\mu\nu}_{(e)} = 2(k^\mu k'^\nu - (k\cdot k')g^{\mu\nu} + k'^\mu k^\nu + m^2 g^{\mu\nu}) \tag{3.46}$$

For the proton tensor we have,

$$K^{(p)}_{\mu\nu} = \frac{1}{2}\sum_{s,s'}[\overline{u}(p',s')(\gamma_\mu)u(p,s)][\overline{u}(p,s)(\gamma_\mu)u(p',s')] \tag{3.47}$$

Again, "Casimir's Trick" from Appendix B, is used to simplify the expression.

$$K^{(p)}_{\mu\nu} = 1/2\ Tr[\gamma_\nu(\not{p}' + M)\gamma_\mu(\not{p} + M)]$$
$$= 1/2\ Tr[\gamma_\nu\not{p}'\gamma_\mu\not{p} + \gamma_\nu\not{p}'\gamma_\mu M + \gamma_\nu M\gamma_\mu\not{p} + \gamma_\nu M\gamma_\mu M] \quad (3.48)$$
$$= 1/2\ Tr[\gamma_\nu\not{p}'\gamma_\mu\not{p} + \gamma_\nu M\gamma_\mu M]$$

Using the same trace identities from Appendix A, the two remaining terms become,

$$Tr[\gamma_\nu\not{p}'\gamma_\mu\not{p}] = p'^\lambda p^\sigma Tr[\gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\sigma]$$
$$= p'^\lambda p^\sigma\ 4(g_{\nu\lambda}g_{\mu\sigma} - g_{\nu\mu}g_{\lambda\sigma} + g_{\nu\sigma}g_{\lambda\mu}) \quad (3.49)$$
$$= 4(p_\nu p'_\mu - g_{\nu\mu}(p'\cdot p) + p'_\mu p_\nu)$$

$$Tr[\gamma_\nu M\gamma_\mu M] = M^2 Tr[\gamma_\nu\gamma_\mu] = 4M^2 g_{\mu\nu} \quad (3.50)$$

Therefore,

$$K^{(p)}_{\mu\nu} = 2(p_\nu p'_\mu - g_{\nu\mu}(p'\cdot p) + p'_\nu p_\mu + M^2 g_{\nu\mu}) \quad (3.51)$$

Then, putting Eq. (3.46) and Eq. (3.51) into Eq. (3.45), the scattering amplitude becomes:

$$\langle|\mathcal{M}|^2\rangle = \frac{2e^2Q^2}{q^4}[k^\mu k'^\nu + k'^\mu k^\nu - (k\cdot k')g^{\mu\nu} + m^2 g^{\mu\nu}][p_\nu p'_\mu + p'_\nu p_\mu - (p'\cdot p)g_{\nu\mu} + M^2 g_{\mu\nu}] \quad (3.52)$$

Combining the tensors gives the following:

$$(k^\mu k'^\nu)(p_\nu p'_\mu + p'_\nu p_\mu - (p'\cdot p)g_{\nu\mu} + M^2 g_{\nu\mu})$$
$$= [(k\cdot p')(k'\cdot p) + (k\cdot p)(k'\cdot p') - (p'\cdot p)(k\cdot k') + M^2(k\cdot k')] \quad (3.53)$$

$$(k'^\mu k^\nu)(p_\nu p'_\mu + p'_\nu p_\mu - (p'\cdot p)g_{\nu\mu} + M^2 g_{\nu\mu})$$
$$= [(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - (p'\cdot p)(k'\cdot k) + M^2(k'\cdot k)] \quad (3.54)$$

$$-(k\cdot k')g^{\mu\nu}(p_\nu p'_\mu + p'_\nu p_\mu - (p'\cdot p)g_{\nu\mu} + M^2 g_{\nu\mu})$$
$$= 2(k\cdot k')(p\cdot p') - 4M^2(k\cdot k') = 2(k\cdot k')[(p\cdot p') - 2M^2] \quad (3.55)$$

$$m^2 g^{\mu\nu}(p_\nu p'_\mu + p'_\nu p_\mu - (p'\cdot p)g_{\nu\mu} + M^2 g_{\nu\mu})$$
$$= -2m^2(p\cdot p') + 4m^2M^2 = -2m^2[(p\cdot p') - 2M^2] \quad (3.56)$$

Combining these equations into Eq. (3.52) yields,

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{q^4}\Big[(k\cdot p)(k'\cdot p') + (k'\cdot p)(k\cdot p') - M^2(k\cdot k') - m^2(p\cdot p') + 2m^2M^2\Big] \quad (3.57)$$

removing terms for $m \approx 0$,

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{q^4}\Big[(k\cdot p)(k'\cdot p') + (k'\cdot p)(k\cdot p') - M^2(k\cdot k')\Big] \quad (3.58)$$

Using the scalar products from Eq. (3.29),

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{8e^4}{q^4}\Big[(ME(-\frac{q^2}{2} + ME') + (ME')(\frac{q^2}{2} + ME) + \frac{M^2q^2}{2})\Big] \\ &= \frac{8e^4}{q^4}\Big[\frac{q^2}{2}M(E'-E) + 2M^2EE' + \frac{M^2q^2}{2}\Big] \\ &= \frac{8e^4}{q^4}(2M^2EE')\Big[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\Big]\end{aligned} \quad (3.59)$$

Therefore,

$$\boxed{\langle|\mathcal{M}|^2\rangle = \frac{16e^4M^2EE'}{q^4}\Big(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\Big)} \quad (3.60)$$

Combining this equation with Eq. (2.30) we obtain the equation for the scattering cross section for a spin-$\frac{1}{2}$ point-like proton with recoil.

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{e^4E'^2}{4\pi^2q^4}\left(\frac{E'}{E}\right)\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)} \quad \text{for } m = 0 \quad (3.61)$$

for $q^2 \approx -4EE'\sin^2\frac{\theta}{2}$, and $\alpha = \frac{e^2}{4\pi}$:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{\alpha^2}{16E^2\sin^4\frac{\theta}{2}}\left(\frac{E'}{E}\right)\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)} \quad (3.62)$$

Comparing Eq. (3.62) for the spin-$\frac{1}{2}$ proton with Eq. (3.34) for the spin-0 pion, it is clear that the $\left(\cos^2\frac{\theta}{2}\right)$ term expresses the charge interaction between the two particles, while, the $\left(\sin^2\frac{\theta}{2}\right)$ term expresses the spin interaction.

3.3 Extended Spin-1/2 Particle

Unlike the simple addition of the form factor $F(q)^2$ to the point-like pion, calculating the scattering cross section for the extended proton is more complicated. In addition to the proton's charge, its magnetic moment is also involved in the scattering process.

In place of a single form factor, a general equation will be created that incorporates two form factors with the two independent terms, γ^μ and $i\sigma^{\mu\nu}q_\nu$.

$$\Gamma^\mu = \left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\right] \tag{3.63}$$

Using the Gordon Decomposition Identity[23], derived in Appendix C, the $i\sigma^{\mu\nu}$ term can be simplified, such that,

$$\Gamma^\mu = \left[\gamma^\mu F_1(q^2) + \left(\gamma^\mu - \frac{(p+p')^\mu}{2M}\right) F_2(q^2)\right] \tag{3.64}$$

Starting with Eq. (3.45) for the scattering amplitude for the electron-proton scattering, with $W^{(p)}_{\mu\nu} \to K^{(p)}_{\mu\nu}$,

$$\langle|\mathcal{M}|^2\rangle = \frac{e^2Q^2}{q^4}L^{\mu\nu}_{(e)}W^{(p)}_{\mu\nu} \tag{3.65}$$

where,

$$W^{(p)}_{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left[\overline{u}(p',s')(\Gamma_\mu)u(p,s)\right]\left[\overline{u}(p,s)(\Gamma_\nu)u(p',s')\right] \tag{3.66}$$

Again, making use of "Casimir's Trick"[1] this can be rewritten as:

$$W^{(p)}_{\mu\nu} = \frac{1}{2}Tr\left[\Gamma_\mu(\not{p}+M)\Gamma_\nu(\not{p}'+M)\right] \tag{3.67}$$

After inserting Γ_μ and Γ_ν into Eq. (3.67) and expanding the terms into four separate traces,

$$
\begin{aligned}
W^{(p)}_{\mu\nu} = {} & \tfrac{1}{2}Tr\Big[(F_1+F_2)^2\gamma_\mu(\not{p}+M)\gamma_\nu(\not{p}'+M)\Big] \\
& - \tfrac{1}{2}Tr\Big[(F_1+F_2)F_2\frac{(p+p')_\nu}{2M}\gamma_\mu(\not{p}+M)(\not{p}'+M)\Big] \\
& - \tfrac{1}{2}Tr\Big[(F_1+F_2)F_2\frac{(p+p')_\mu}{2M}\gamma_\nu(\not{p}+M)(\not{p}'+M)\Big] \\
& + \tfrac{1}{2}Tr\Big[F_2^2\frac{(p+p')_\mu(p+p')_\nu}{4M^2}(\not{p}+M)(\not{p}'+M)\Big]
\end{aligned}
\tag{3.68}
$$

then, using the trace identities listed in Appendix A,

$$
\begin{aligned}
W^{(p)}_{\mu\nu} = {} & 2(F_1+F_2)^2\Big[p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}(p\cdot p') + g_{\mu\nu}M^2\Big] \\
& - 2(F_1+F_2)F_2\Big[(p_\mu+p'_\mu)(p_\nu+p'_\nu\Big] \\
& + \frac{1}{2}\frac{F_2^2}{M^2}\Big[(p+p')_\mu(p+p')_\nu((p\cdot p')+M^2))\Big]
\end{aligned}
\tag{3.69}
$$

This equation is then inserted into Eq. (3.65), where it is multiplied by the electron tensor from the previous section.

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle = {} & \frac{8e^4}{q^4}\Bigg\{ \\
& (F_1+F_2)^2\Big[(k'\cdot p)(k\cdot p') + (k'\cdot p')(k\cdot p) - (k'\cdot k)M^2 + 2m^2M^2 - (p'\cdot p)m^2\Big] \\
& -(F_1+F_2)F_2\Big[(k'\cdot p)(k\cdot p) + (k'\cdot p')(k\cdot p') + (k'\cdot p)(k\cdot p') + (k'\cdot p')(k\cdot p)\Big] \\
& +\frac{F_2^2}{4M^2}\Big[(p\cdot p') + M^2\Big]\Big[(k'\cdot p)(k\cdot p) + (k'\cdot p')(k\cdot p') + (k'\cdot p)(k\cdot p') + (k'\cdot p')(k\cdot p) \\
& \qquad + \frac{1}{2}\Big((p+p')^2(m^2-(k\cdot k'))\Big)\Big]\Bigg\}
\end{aligned}
\tag{3.70}
$$

Again, setting $m = 0$ and using the results in Eq. (3.29) for the scalar products this is then further simplified.

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{q^4}\Bigg\{\Big(F_1+F_2\Big)^2\Big[2M^2EE' + \frac{q^4}{4} + \frac{q^2M^2}{2} - 2M^2EE' + \frac{q^4}{8} - \frac{q^4}{4} - \frac{q^2M^2}{2} + \frac{q^4}{8}\Big]$$
$$+\Big(F_1^2 - \frac{q^2}{4M^2}F_2^2\Big)\Big[-2M^2EE' + \frac{q^4}{8} - \frac{q^4}{4} - \frac{q^2M^2}{2} + \frac{q^4}{8}\Big]\Bigg\}$$

$$= \frac{8e^4}{q^4}\Bigg\{\Big(F_1+F_2\Big)^2\Big[\frac{q^4}{8} + \frac{q^4}{8}\Big]$$
$$+\Big(F_1^2 - \frac{q^2}{4M^2}F_2^2\Big)\Big[-2M^2EE' - \frac{q^2M^2}{2}\Big]\Bigg\}$$

$$= \frac{8e^4}{q^4}\Bigg\{\Big(F_1+F_2\Big)^2\Big[\frac{q^4}{4}\Big] + \Big(F_1^2 - \frac{q^2}{4M^2}F_2^2\Big)\Big[-2M^2EE' - \frac{q^2M^2}{2}\Big]\Bigg\} \tag{3.71}$$

Therefore,

$$\boxed{\langle|\mathcal{M}|^2\rangle = \frac{16M^2EE'e^4}{q^4}\left[\Big(F_1^2 - \frac{q^2}{4M^2}F_2^2\Big)\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\Big(F_1+F_2\Big)^2\sin^2\frac{\theta}{2}\right]} \tag{3.72}$$

Inserting the scattering amplitude into Eq. (2.30) for the scattering cross-section in the lab frame, we obtain:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{(lab)} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left(\frac{E'}{E}\right)\left[\Big(F_1^2 - \frac{q^2}{4M^2}F_2^2\Big)\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\Big(F_1+F_2\Big)^2\sin^2\frac{\theta}{2}\right]} \tag{3.73}$$

Typically, this equation is written using linear combinations of the form factors, F_1 and F_2, to avoid the interference term, F_1F_2.

$$\begin{aligned} G_E &= F_1 + \frac{q^2}{4M^2}F_2 \\ G_M &= F_1 + F_2 \end{aligned} \tag{3.74}$$

for $q^2 \approx -4EE' \sin^2 \frac{\theta}{2}$ and using the fine-structure constant $\alpha = \frac{e^2}{4\pi}$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \left(\frac{E'}{E}\right) \left[\frac{G_E^2 + \tau G_M^2}{1+\tau}(\cos^2 \frac{\theta}{2}) + 2\tau G_M^2 \sin^2 \frac{\theta}{2}\right]} \tag{3.75}$$

where, $\tau = -\frac{q^2}{4M^2}$

This equation is known as the Rosenbluth formula [19], and describes the scattering interaction between an electron and an extended proton. The form factors $G_E(q^2)$ and $G_M(q^2)$ are closely related to the proton charge and magnetic moment, respectively.

Comparing Eq. (3.73) to Eq. (3.61) for the point-like proton, it can be seen how adding the linear combination of two form factors creates the extended proton. In the next chapter, we will analyze how these form factors can be used to determine the size of the target particle.

3.4 Summary of Elastic Electron Scattering

As shown through Chapters 2 and 3, the process for determining the cross section for elastic scattering can be broken down into two main parts. First, to develop a reference frame appropriate differential cross section equation using Fermi's "golden rule" for scattering. Then determining the scattering amplitude using the spin appropriate Feynman's rules, and simplified using the frame dependent scalar products.

Although the main goal though the first three chapters was to derive the Rosenbluth formula for electron-proton scattering, several more elementary processes were derived. By deriving these additional results, the final equation is able to be broken down and better understood.

As seen in the equation below, the first term represents the Rutherford formula[22], derived in Eq. (3.37), which represents elastic Coulomb scattering off a point-like target without recoil. This can be seen as the most basic approximation of the electron-proton scattering process. Each term thereafter adds additional information for a more precise model of the interaction. The second term takes into account the recoil of the target particle, while the form factors, G_E, and G_M, give information about the target particle's size. The final two terms represent the electric and magnetic interaction due to the Coulomb potential as well as the spin interaction between the two spin=$\frac{1}{2}$ particles.

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \overbrace{\frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}}^{\text{Rutherford Formula}} \underbrace{\left(\frac{E'}{E}\right)}_{\text{Recoil}} \left[\overbrace{\frac{G_E^2+\tau G_M^2}{1+\tau}(\cos^2\frac{\theta}{2})}^{\text{Electric/Magnetic Interaction}} + \underbrace{2\tau G_M^2\sin^2\frac{\theta}{2}}_{\text{Spin Interaction}}\right] \tag{3.76}$$

Once the scattering cross section is determined, the charge distribution of the target particle as well as the mean square radius can be calculated. The following chapter will analyze the form factors shown above to better understand how they can be utilized to extract information about the scattering process.

CHAPTER 4

ANALYSIS OF FORM FACTORS

As seen in the previous chapters, the form factor, $F(q^2)$, in general, takes a known cross section for a point-like structure and adds a spacial element to the equation.

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \left[F(q^2)\right]^2 \tag{4.1}$$

Experimentally, an object can be "mapped" by focusing an electron beam towards the object and measuring the angular distribution of the scattered electrons. These results are then compared to the equation above to determine the charge distribution. For a static,point-like, spin-0 target, it is found that the form factor is the Fourier transform of the charge distribution.

$$F(q^2) = \int \rho(\vec{x})\ e^{i\vec{q}\cdot\vec{x}} d^3x \tag{4.2}$$

where, for normalization,

$$\int \rho(\vec{x})\ d^3x = 1 \tag{4.3}$$

If we assume, for example, that the charge distribution has an exponential form, i.e. $\rho(r) = Ae^{-mr}$, then,

$$\begin{aligned} F(q^2) &= \int \frac{m^3}{8\pi} e^{-mr}\ e^{i\vec{q}\cdot\vec{x}} d^3x \\ &= \frac{m^3}{4} \int e^{-mr} \left(\frac{e^{i\vec{q}r} - e^{-i\vec{q}r}}{i\vec{q}r}\right) r^2\ d^3r \\ &= \frac{m^4}{(m^2 + \vec{q}^2)^2} \end{aligned} \tag{4.4}$$

where $A = \frac{m^3}{8\pi}$ for the normalization. The form factor becomes:

$$\boxed{F(q^2) = \left(1 - \frac{\vec{q}^2}{m^2}\right)^{-2}} \tag{4.5}$$

with $q^2 = -|\,\vec{q}\,|^2$

In the laboratory setting, the reverse equation is required. Scattering experiments measure the scattering cross section, from which the form factor can calculated. Then, typically, from these form factors, the charge distribution $\rho(r)$ and the mean-square radius $\langle r^2 \rangle$ is calculated.

Starting with Eq. (4.5) in the breit frame, and applying an inverse fourier transform yields the charge distribution,

$$\rho(\vec{r}\,) = \frac{1}{(2\pi)^3} \int d^3\vec{q}\; F(q^2 = -\vec{q}^{\,2}) e^{-i\vec{q}\cdot\vec{r}} \tag{4.6}$$

then, the radius can be calculated as the expectation value from the charge distribution,

$$\begin{aligned} \langle r^2 \rangle &= \int d^3r\; r^2\; \rho(r) \\ &= 4\pi \int_0^\infty dr\; r^4 \rho(r) \end{aligned} \tag{4.7}$$

The form factors for the proton, $G_E(q^2)$ and $G_M(q^2)$, from Eq. (3.74) can be regarded as generalizations of the form factor $F(q^2)$ in Eq. (4.1). Therefore, the Fourier transform in Eq. (4.2) can still be used.

Looking at Eq. (3.75) for the scattering cross-section for the proton, the values of the electric form factor, G_E, and magnetic form factor, G_M, can be determined by analyzing data on the angular dependence of electron-proton at varying values of q^2.

The data in Fig. 4.1, obtained from a textbook from Halzen and Martin [2], shows the result for G_E,

$$G_E(q^2) \approx \left(1 - \frac{q^2}{0.71}\right)^{-2} \qquad \text{(in units of GeV}^2\text{)}. \tag{4.8}$$

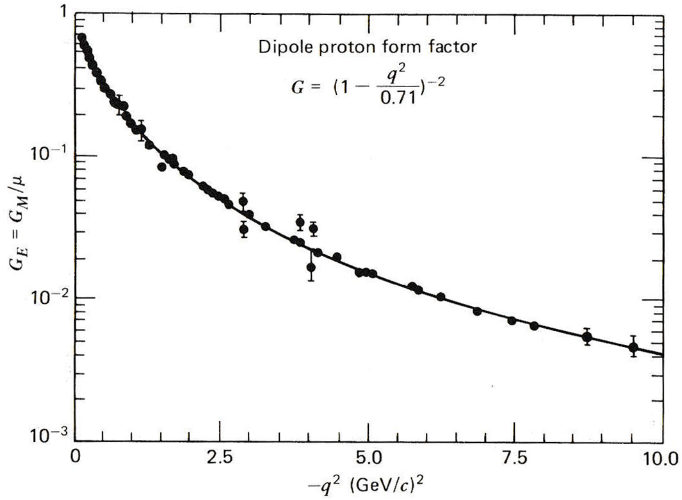

Figure 4.1: Proton form factors as a function of q^2. [2]

This equation, agrees with the theoretical expression in Eq. (4.5), which was derived using an exponential charge distribution. This form for the charge distribution corresponds to a form factor with a dipole representation, that is commonly used to determine the radius of the proton.

In the non-relativistic limit, we can write

$$\begin{aligned} F(-\vec{q}^2) &= \int d^3\vec{r}\,\rho(r)\,e^{i\vec{q}\cdot\vec{r}} \\ &= \int d^3\vec{r}\,\rho(r)\left[1 + i\,\vec{q}\cdot\vec{r} - \frac{1}{2}\,\vec{q}^2\,\vec{r}^2\cos^2\vartheta + \dots\right] \\ &= 1 - \frac{1}{6}\,\vec{q}^2\,\langle r^2\rangle + \dots \end{aligned} \tag{4.9}$$

Here we used that in the non-relativistic limit $q^2 = -\vec{q}^2$, which implies that the form factor is a function of $|\vec{q}|$. Furthermore, the $\vec{q}\cdot\vec{r}$ term vanishes due to spherical symmetry, while the averaging of $\cos^2\vartheta$ provides the factor $\frac{1}{3}$. This leads to the model-independent result

$$\langle r^2\rangle = -6\left(\frac{\partial F}{\partial \vec{q}^2}\right)_{\vec{q}^2=0} = 6\left(\frac{\partial F}{\partial q^2}\right)_{q^2=0} \tag{4.10}$$

Using this equation for the data from Fig. 4.1 gives the radius of the proton as,

$$\langle r^2 \rangle = 6 \, \frac{\partial}{\partial \vec{q}^{\,2}} \left(1 - \frac{q^2}{0.71}\right)^{-2} \Bigg|_{q^2=0} = \left(8.14 \times 10^{-}14 \text{ cm}\right)^2 \tag{4.11}$$

which is close to the expected results.

4.1 Arguments to the Common Practice

Although this method of determining the charge radius, $\langle r^2 \rangle$, from the electric form factor, $G_E(q^2)$, is common practice, there still remains debate about its validity. Specifically, a paper written by R.L. Jaffe [3] suggests that this process, that he credits to Hofstadter [6] and Sachs [24, 25], is not valid when considering a system whose Compton wavelength is comparable to its size. This would most effect nucleons, with a Compton wavelength of 0.21 fm [3], and size of 0.85 fm [3].

Jaffe argues that this becomes problematic when trying to localize the wave packet. As the precision in the radius is increased, so too is the momentum and the relativistic effects. Using a spin-0 system as an example to make his argument, he starts by defining the charge density operator, $\hat{\rho}(\vec{r}, 0)$ at t=0 as,

$$\hat{Q} = \int d^3r \; \hat{\rho}(\vec{r}) \tag{4.12}$$

and,

$$\langle p' | p \rangle = 2E(2\pi)^3 \delta^3(\vec{p}\,' - \vec{p}\,) \tag{4.13}$$

where $p = (E, \vec{p})$ and $E = \sqrt{m^2 + \vec{p}^{\,2}}$
then,

$$\langle p' | \hat{\rho}(\vec{r}, 0) | p \rangle = e^{i(\vec{p}\,' - \vec{p}) \cdot \vec{r}} \langle p' | \hat{\rho}(0) | p \rangle \tag{4.14}$$

where, for a spin-0 system a single form factor, $F(q^2)$, is used.

$$\langle p' | \hat{\rho}(0) | p \rangle = (E + E')F(q^2) \tag{4.15}$$

where,

$$q^2 = (p' - p)^2 = (E' - E)^2 - (\vec{p}\,' - \vec{p}\,)^2 \tag{4.16}$$

Combining Eqs. (4.14)-(4.16) we have,

$$\langle p'|\hat{\rho}(\vec{r},0)|p\rangle = e^{i(\vec{p}\,'-\vec{p})\cdot\vec{r}}(E + E')F(q^2) \tag{4.17}$$

Next, he constructs a wave packet state by superimposing the energy-momentum eigenstates.

$$|\Psi, x\rangle = \int \frac{d^3p}{\sqrt{2E(2\pi)^3}}\phi(\vec{p})e^{i\vec{p}\cdot\vec{x}}\;|p\rangle \tag{4.18}$$

which is normalized by,

$$\int d^3p\;|\phi(\vec{p})|^2 = 1 \tag{4.19}$$

where the wave packet is defined as a spherically symmetric gaussian,

$$\phi(\vec{p}) = \phi(p) = \left(\frac{2R^2}{3\pi}\right)^{\frac{3}{4}} e^{-\vec{p}^2R^2/3} \tag{4.20}$$

Then, the charge density distribution in the localized state,

$$\rho(r) = \langle\Psi|\rho(\vec{r},0)|\Psi\rangle \tag{4.21}$$

Inserting Eqs. (4.18)-(4.20) into Eq. (4.21),

$$\rho(r) = \int \frac{d^3p\;d^3p'}{(2\pi)^3\sqrt{4EE'}}(E + E')F(q^2)\phi(p)\phi(p')e^{i\vec{q}\cdot\vec{r}} \tag{4.22}$$

where, $\vec{q} = \vec{p}\,' - \vec{p}$ and $q^2 = (E' - E)^2 - \vec{q}^2$

To simplify the calculations, a gaussian is used to express the form factor.

$$F(q^2) = e^{\frac{1}{6}q^2\Delta^2} \tag{4.23}$$

Here, Δ^2, is defined as the "naive" mean-square radius, typically obtained by taking the derivitive of the electric form factor at zero momentum transfer, as done in Eq. (4.10).

In addition, he converts to center-of-momentum variables using,

$$\begin{aligned} \vec{p} &= \vec{P} + \frac{\vec{q}}{2} \\ \vec{p}\,' &= \vec{P} - \frac{\vec{q}}{2} \end{aligned} \tag{4.24}$$

leading to the complete equation for the charge density distribution.

$$\begin{aligned} \rho(r) = \left(\frac{2R^2}{3\pi}\right)^{\frac{3}{2}} \int \frac{d^3p\ d^3p'}{(2\pi)^3\sqrt{4EE'}}(E+E') \\ \times\, exp\left[\tfrac{1}{6}q^2\Delta^2 - \tfrac{2}{3}\vec{P}^2R^2 - \tfrac{1}{6}\vec{q}^{\,2}R^2 + i\vec{q}\cdot\vec{r}\right] \end{aligned} \tag{4.25}$$

This is the Jaffe's master equation for the charge distribution for a spin-0 system, obtained through quantum mechanics. The author states that this equation must be evaluated numerically for systems like the nucleon, in which the following equation in not satisfied.

$$\Delta >> R >> \frac{1}{m} \tag{4.26}$$

Where, Δ is the charge radius, R, is the radius of the wave packet, and $\frac{1}{m}$ is the Compton wavelength.

For systems in which $\Delta >>>> \frac{1}{m}$, then several of the terms in the integral for ρ can be expanded so that the integral can be calculated analytically. The derivations for these expansions can be found in Appendix E.

The first expansion is for the Energy terms,

$$\frac{E'+E}{\sqrt{4EE'}} = 1 + \frac{1}{2m^4}(\vec{P}\cdot\vec{q})^2 + O\left[\frac{1}{m}\right]^6 \tag{4.27}$$

As shown in Appendix E, a slightly different result for the expansion was obtained, where the $\frac{1}{m^4}$ term has a $\frac{1}{8}$ pre-factor in contrast to the $\frac{1}{2}$ as shown above from the Jaffe paper.

The second expansion is for the momentum exchange term, q^2,

$$\begin{aligned} q^2 &= (E' - E)^2 - \vec{q}^{\,2} \\ &= \left(\sqrt{m^2 + (\vec{P} - \tfrac{\vec{q}}{2})^2} - \sqrt{m^2 + (\vec{P} + \tfrac{\vec{q}}{2})^2} - \vec{q}^{\,2}\right) \\ &= -\vec{q}^{\,2} + \left(\frac{\vec{P}\cdot\vec{q}}{m}\right)^2 + O\left[\frac{1}{m}\right]^4 \end{aligned} \tag{4.28}$$

And the third and final expansion required to simplify Eq. (4.25) is,

$$F(q^2) = e^{-\frac{1}{6}\vec{q}^{\,2}\Delta^2}\left(1 + \frac{\Delta^2(\vec{P}\cdot\vec{q})^2}{6m^2} + O\left[\frac{1}{m}\right]^4\right) \tag{4.29}$$

In this expansion, the author chose not to include the $\left[\frac{1}{m}\right]^4$ term, presumably to avoid a more complicated integral for $\rho(r)$. This is problematic though in that his expansion in Eq. (4.27) does include the $\frac{1}{m^4}$ term, and is used for his integration of $\rho(r)$. This results in a final expression that is incomplete.

Substituting these expansions into Eq. (4.25), the expression for $\rho(r)$ is simplified into an integral that can be calculated analytically.

$$\begin{aligned} \rho(r) \approx \left(\frac{2R^2}{3\pi}\right)^{\frac{3}{2}} \int \frac{d^3p\; d^3p'}{(2\pi)^3}\left(1 + \left(\frac{1}{2m^4} + \frac{\Delta^2}{6m^2}\right)(\vec{P}\cdot\vec{q})^2\right) \\ \times\, exp\left[-\tfrac{1}{6}\vec{q}^{\,2}\Delta^2 - \tfrac{2}{3}\vec{P}^2R^2 - \tfrac{1}{6}\vec{q}^2R^2 + i\vec{q}\cdot\vec{r}\right] \end{aligned} \tag{4.30}$$

This integral is three-dimensional in $\vec{P}$, and three-dimensional in $\vec{q}$. In addition, the integral needs to converted into spherical coordinates. Furthermore, the dot product of $(\vec{P}\cdot\vec{q})$, adds additional complications.

$$\begin{aligned} (\vec{P}\cdot\vec{q}) &= \left|\vec{P}\right|\left|\vec{q}\right|\cos\theta_{(Pq)} \\ &= \left(\sqrt{P_x^2 + P_y^2 + P_z^2}\,\right)\left(\sqrt{q_x^2 + q_y^2 + q_z^2}\,\right)\cos\theta_{(Pq)} \\ &= P_r q_r \cos\theta_{(Pq)} \end{aligned} \tag{4.31}$$

for $i\vec{q}\cdot\vec{r}$;

$$
\begin{aligned}
i\vec{q}\cdot\vec{r} &= \left|\vec{q}\right|\left|\vec{r}\right|\cos\theta_{(qr)} \\
&= \left(\sqrt{q_x^2+q_y^2+q_z^2}\,\right)\left(\sqrt{x^2+y^2+z^2}\,\right)\cos\theta_{(qr)} \\
&= q_r r\cos\theta_{(qr)}
\end{aligned} \tag{4.32}
$$

For $\vec{P}^{\,2}$ and $\vec{q}^{\,2}$:

$$
\begin{aligned}
\vec{A}^{\,2} &= \vec{A}\cdot\vec{A} \\
&= (A_xA_x + A_yA_y + A_zA_z) = A_r^2
\end{aligned} \tag{4.33}
$$

And for the conversion to spherical coordinates:

$$
\begin{aligned}
&\int P(x,y,z)q(x,y,z)dP_xdP_ydP_zdq_xdq_ydq_z \\
&= \int P(r,\theta,\phi)q(r,\theta,\phi)P_r^2\sin(P_\phi)q_r^2\sin(q_\phi)dP_rdP_\theta dP_\phi dq_rdq_\theta dq_\phi
\end{aligned} \tag{4.34}
$$

Making these changes, the integral becomes:

$$
\begin{aligned}
\rho(r) \approx \left(\frac{2R^2}{3\pi}\right)^{\frac{3}{2}}\int\frac{1}{(2\pi)^3}\left(1+\left(\frac{1}{2m^4}+\frac{\Delta^2}{6m^2}\right)P_r^2q_r^2\cos\theta_{(Pq)}\right)P_r^2\sin(P_\phi)q_r^2\sin(q_\phi) \\
\times\, exp\left[-\tfrac{1}{6}q_r^2(\Delta^2+R^2)-\tfrac{2}{3}P_r^2R^2+iq_rr\cos\theta_{(qr)}\right]dP_rdP_\theta dP_\phi dq_rdq_\theta dq_\phi
\end{aligned} \tag{4.35}
$$

The angles from the two dot products $\cos\theta_{(Pq)}$, and $\cos\theta_{(qr)}$ add two additional integrals, where x substitution can be used. The q_r integral which is a function of e^{ix} can be done by taking:

$$
\int_0^\infty F(q_r)dq_r = \frac{1}{2}\int_{-\infty}^{\infty} q_rdq_r \tag{4.36}
$$

as the Real part of the function being integrated is an even function.

$$
e^{ix} = \cos x + i\sin x \tag{4.37}
$$

The final result becomes:

$$
\rho(r) \approx \left(1+\frac{27}{8}\left(\frac{1}{m^4}+\frac{\Delta^2}{3m^2}\right)\left(\frac{R^2+\Delta^2-r^2}{R^2(R^2+\Delta^2)^2}\right)\right)\rho_0(r) \tag{4.38}
$$

where,

$$\rho_0(r) = \left(\frac{3}{2\pi(R^2+\Delta^2)}\right)^{\frac{3}{2}} e^{-3r^2/2(R^2+\Delta^2)} \tag{4.39}$$

From this equation, the author shows that one can derive the "naive" form of the charge density by first taking the mass, m, to infinity in Eq. (4.38), which would eliminate all of the $\frac{1}{m}$ terms and be left with Eq. (4.39). Then, take the wave packet radius R to zero. Therefore,

$$\rho_{\text{naive}}(r) = \left(\frac{3}{2\pi(\Delta^2)}\right)^{\frac{3}{2}} e^{-3r^2/2\Delta^2} \tag{4.40}$$

which can also be obtained by taking the fourier transform of the form factor, $F(q^2)$, Eq. (4.23).

Once the charge distribution is calculated, the radius can be determined using Eq. (4.7)

$$\langle r^2 \rangle = 4\pi \int_0^\infty dr\ r^4 \rho(r) \tag{4.41}$$

This integral can be broken into three separate integrals.

$$4\pi \int_0^\infty r^4 \left(\frac{3}{2\pi(R^2+\Delta^2)}\right)^{\frac{3}{2}} exp\left[\frac{-3r^2}{2(R^2+\Delta^2)}\right] dr = R^2 + \Delta^2$$

$$4\pi \int_0^\infty r^4\ \frac{27}{8}\frac{1}{m^4}\left(\frac{R^2+\Delta^2-r^2}{R^2(R^2+\Delta^2)^2}\right)\left(\frac{3}{2\pi(R^2+\Delta^2)}\right)^{\frac{3}{2}} exp\left[\frac{-3r^2}{2(R^2+\Delta^2)}\right] dr = -\frac{9}{4m^4R^2}$$

$$4\pi \int_0^\infty r^4\ \frac{27}{8}\frac{\Delta^2}{3m^2}\left(\frac{R^2+\Delta^2-r^2}{R^2(R^2+\Delta^2)^2}\right)\left(\frac{3}{2\pi(R^2+\Delta^2)}\right)^{\frac{3}{2}} exp\left[\frac{-3r^2}{2(R^2+\Delta^2)}\right] dr = -\frac{3\Delta^2}{4m^2R^2}$$

Combining these three equations yields the same result for the mean charge radius.

$$\begin{aligned} \langle r^2 \rangle &= R^2 + \Delta^2 - \frac{9}{4m^4R^2} - \frac{3\Delta^2}{4m^2R^2} \\ &= \Delta^2\left(1 - \frac{3}{4m^2R^2}\right) + R^2\left(1 - \frac{9}{4m^4R^4}\right) \end{aligned} \tag{4.42}$$

It is from this equation, where his argument is based. In order to obtain the desired result of $\langle r^2 \rangle = \Delta^2$, the Compton wavelength, $\frac{1}{m^2}$, must be much smaller than the wave packet radius, R^2, to minimize the relativistic corrections in both terms. Furthermore, R^2 must be much smaller than Δ^2 to minimize the R^2 second term in Eq. (4.42). These requirements lead to the condition set in Eq. (4.26).

The author evaluates the difference between his localized equation for the charge distribution, Eq. (4.25) and what he labels as the "naive" charge distribution Eq. (4.40).

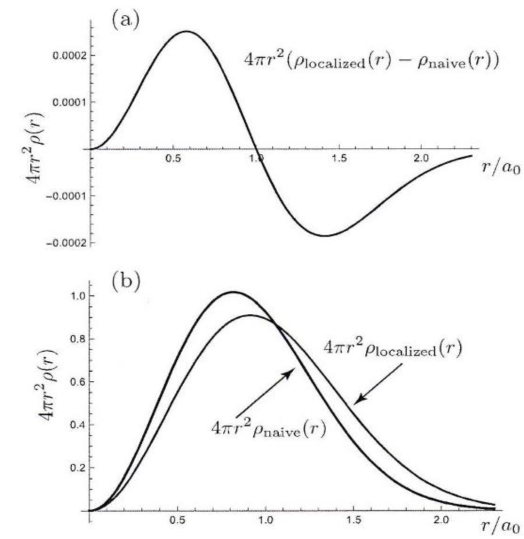

Figure 4.2: Radial charge distribution, $4\pi r^2\rho(r)$, for the hydrogen atom. Difference between $\rho_{\text{localized}}(r)$ and $\rho_{\text{naive}}(r)$. [3]

These graphs, show the charge distribution for the hydrogen atom Fig. 4.2 and the proton Fig. 4.3. The hydrogen atom has a radius of about $a_0 \approx 5 \times 10^{-}11$ m, and it's Compton wavelength, $\frac{1}{m} = 2.1{\times}10^{-}16$ m. With a radius of about $2.5{\times}10^5$ times larger than its Compton wavelength, the hydrogen atom falls within the authors condition in Eq. (4.26). The proton, with a radius ≈ 0.85 fm, and Compton wavelength of ≈ 0.2 fm, does not meet the author's requirements to use the expansions to simplify the equation for the charge density in Eq. (4.25).

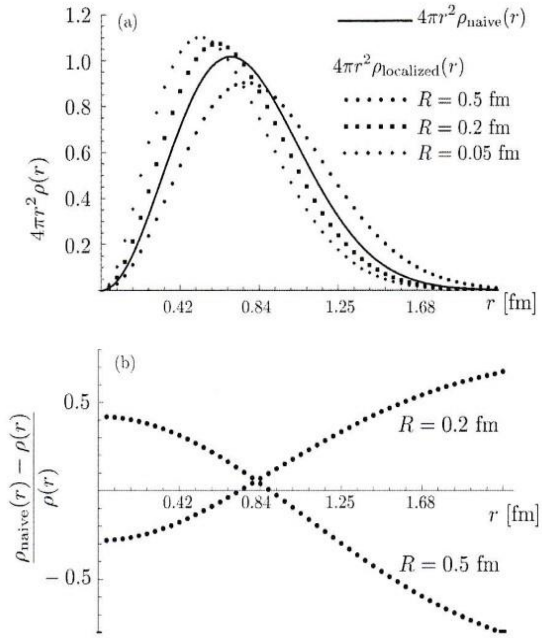

Figure 4.3: Radial charge distribution, $4\pi r^2\rho(r)$, for the proton. Difference between $\rho_{\text{localized}}(r)$ and $\rho_{\text{naive}}(r)$. [3]

4.2 Analysis of the Jaffe Paper

In summary, Jaffe uses a quantum mechanical approach to define a localized wave packet, Eq. (4.20), and charge density distribution Eq. (4.25). He then uses several series expansions to simplify $\rho(r)$ in order to calculate the integral analytically. Because of their size and mass, the author states that these expansions are not valid for the nucleon. This also limits the validity of the final result for the charge distribution, Eq. (4.38), for the nucleon.

The author uses a qaussian form factor as seen in Eq. (4.23). Using the common practice method for determining the radius yields:

$$\begin{aligned} \langle r^2 \rangle &= 6\ \frac{dF}{dq^2}\Big|_{q^2=0} \\ &= 6\frac{d}{dq^2}\left(e^{\frac{1}{6}q^2\Delta^2}\right)\Big|_{q^2=0} \\ &= \left(\Delta^2\ e^{\frac{1}{6}q^2\Delta^2}\right)\Big|_{q^2=0} = \Delta^2 \end{aligned} \tag{4.43}$$

It can also be shown that the fourier transform of this form factor results in the charge distribution.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{6}\vec{q}~^2\Delta^2} e^{i\vec{r}\vec{q}} d\vec{q} = \left(\frac{3}{2\pi\Delta^2}\right)^{\frac{3}{2}} e^{-(3r^2)/2\Delta^2} \tag{4.44}$$

Which, when used in Eq. (4.42) yields the same result as above.

$$\begin{aligned} \langle r^2\rangle &= 4\pi\int_0^\infty dr\ r^4\rho(r) \\ &= 4\pi\int_0^\infty dr\ r^4\left(\frac{3}{2\pi\Delta^2}\right)^{\frac{3}{2}} e^{-(3r^2)/2\Delta^2} = \Delta^2 \end{aligned} \tag{4.45}$$

This shows the direct relationship between the form factor, $F(q^2)$, the charge distribution, $\rho(r)$, and the mean squared charge radius, $\langle r^2\rangle$. It also stands as an example to the common practice treatment in electron scattering.

There are several issues with the expansions that the author utilizes to simplify the integral of $\rho(r)$, as mentioned earlier. In Jaffe's expansion of the energy term in Eq. (4.27), there is a difference in the results from those obtained here. Where the $\frac{1}{m^4}$ term differs by a factor by $\frac{1}{4}$, as detailed in Appendix E. This difference, leads to a charge distribution of:

$$\rho(r)_{\text{corrected}} \approx \left(1+\frac{27}{32}\left(\frac{1}{m^4}+\frac{4\Delta^2}{3m^2}\right)\left(\frac{R^2+\Delta^2-r^2}{R^2(R^2+\Delta^2)^2}\right)\right)\rho_0(r) \tag{4.46}$$

which in turn effects the radius.

$$\langle r^2\rangle_{\text{corrected}} = \Delta^2\left(1-\frac{3}{4m^2R^2}\right)+R^2\left(1-\frac{9}{16m^4R^4}\right) \tag{4.47}$$

This discrepancy is minor, and overall does not effect the claim that Jaffe makes through out the paper.

A more concerning issue is found in the expansion for the form factor $F(q^2)$ in Eq. (4.29), where the expansion does not include the term of the order of $\left[\frac{1}{m}\right]^4$. This is problematic in that there are other terms of the order $\left[\frac{1}{m}\right]^4$ in the subsequent equation for the charge density. This results in missing information in both the charge density, Eq. (4.40), and the derived equation for $\langle r^2 \rangle$ in Eq. (4.42). This equation forms the basis for the author's argument, making this a concerning discrepancy. This equation is further complicated in that the process used to obtain Eq. (4.40) is put into question by Epelbaum, et al. [16], stating that taking the mass to infinity then the wave packet radius to zero does not commute. Suggesting that this invalidates the equation.

This approach to derive Eq. (4.40) can be further examined by looking at the earlier expansions used by Jaffe, to better understand the difference between his "naive" and "localized" charge distributions. In the expansion of the energy term, Eq. (4.27), as shown in Appendix E, the leading term of 1, exists when $E' = E$, which would apply to a system in which the target particle does not recoil. This is further confirmed in that the higher order terms vanish as $M \to \infty$. In the second expansion, taking only the leading term, puts the system in the Breit frame, $q = -\vec{q}$, which again removes terms associated with energy transfer. Using only the leading terms in the three expansions to simplify Eq. (4.25), leads to $\rho_0(r)$ in Eq. (4.39). Then by minimizing the wave packet radius, yields the "naive" representation of $\rho(r)$, Eq. (4.40). This agrees with Jaffe's assessment that this "naive" charge density is incomplete.

Despite these issues, a closer look at the data provided in the paper shows a distinguishable difference between the treatment of the hydrogen atom and the proton. In Fig. 4.2(a), the difference between the localized and naive charge distribution is of the order of 0.02%, indicating a strong agreement between the two approaches for $R = 10$ fm. In contrast, the graph in Fig. 4.2(b) where the wave packet radius is much larger, $R = 13,250$ fm, the difference between the two equations is significant, showing that even for a larger atom, the "naive" representation fails for large R.

Examining the graphs in Fig. 4.3, the top graph, (a), shows the localized charge distribution for several values of R, as well as the naive $\rho(r)$. It can be seen that small changes in R, results in significant variations in the "localized" charge density. This suggests the inability to localize the proton over a range of R. In the lower graph (b), two data sets of the localized charge density, at different values of R, are shown. This graph represents the variation between the naive and localized values of ρ at both 0.2 fm and 0.5 fm, further expressing the extreme difference in the charge density for two similar values of R.

4.3 Further Analysis

Additional arguments to the common practice of deriving the charge distribution, $\rho(r)$, and charge radius, $\langle r^2 \rangle$, from the form factor, $G_E(q^2)$, are presented by Miller [26, 27]. Miller presents a very detailed argument to the use of using the equation derived by Sachs [25], Eq. (4.6), and therefore the subsequent for the charge radius Eq. (4.7). The problem with making this connection is that it forces the wave function of the proton to be the same before and after collision. The author states that this is incorrect and any attempt to force the wave functions to be equal results in a non-relativistic charge density. This approach agrees with Jaffe's representation of the "naive" charge distribution, by removing the recoil of the proton, and moving to the Breit frame. Miller acknowledges that Sachs, states that for this relationship to work that the charge density, $\rho(r)$, "applies in the non-relativistic limit in which $\rho(r)$ is the static density distribution".

In place of using the three-dimensional charge density, Miller uses Wigner[28] distributions to create a two-dimensional charge distribution, which allows spacial and momentum aspects of the wave function and preserves the relativistic effects. Furthermore, he shows that using the derivative of the form factor at $q = 0$, Eq. (4.10), is a valid method for determining the charge radius, and suggest that it in fact should be the definition.

CHAPTER 5

SUMMARY

The theoretical model for elastic electron scattering is generated using a combination of Feynman's rules [20] for scattering and Fermi's "golden-rule"[17, 18]. These two processes combine the kinematics and dynamics of the scattering process for specific particles to develop an expression for the scattering cross section, $\frac{d\sigma}{d\Omega}$. The scattering cross section, combined with experimental data, can be used to find both the charge distribution, $\rho(r)$, and mean charge radius, $\langle r^2 \rangle$.

The electric form factor in the scattering cross section formula is the key to understanding the finite structure of the proton. Two methods for determining the radius from the form factor are commonly used. In theory both methods for calculating the radius should yield the same result, although as shown by Miller [27], determining the radius through the charge density Eq. (4.10) only holds in the non-relativistic limit. Jaffe argues that this method is invalid in the specific case when the radius of the particle and its Compton wavelength are in the same order, which is most problematic for nucleons. The paper by Jaffe, does not provide an adequate alternative to calculate the radius of the proton. It suggests the use of numerical integration of the charge density in Eq. (4.25), before large M expansions have been applied. This equation though, has a strong dependence on the wave packet radius R, which is an undefined quantity, and therefore is not a viable method.

From the analysis of these papers, there is convincing evidence that neither commonly used method can be considered an exact result for relativistic particles, but instead, should be considered a good approximation. Specifically, these equations can be considered in the breit frame such that $q = -\vec{q}$, and for $M >> m$, when the recoil is negligible. It is shown that for smaller particles including the nucleon, that these methods begin to fail. This is due in large part to the assumption that the particle's mass is large enough to minimize the recoil induced from the incident particle. But, as shown for the proton, this is not the case and therefore, large mass expansions are not valid. In addition, using the expectation value to define the radius would also fail due to the difference in the target's initial and final wave function. Allowing only an approximation for the radius at large M, when the charge density distribution becomes static. This suggest that none of the current methods of determining the charge radius are valid and additional methods should be explored.

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APPENDIX A

γ-MATRICES AND IDENTITIES

Dirac-Pauli Matrices $(\alpha_i, \beta)_{4\text{x}4}$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac $\gamma-$matrices

$$\gamma^\mu \equiv (\beta, \beta\alpha)$$

$$\text{such that, } \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

because $\gamma^0 = \beta$,

$$\gamma^{0\dagger} = \gamma^0, \quad (\gamma^0)^2 = I$$

$$\left.\begin{array}{r} \gamma^{k\dagger} = (\beta\alpha^k)^\dagger = \alpha^k\beta = -\gamma^k \\ (\gamma^k)^2 = \beta\alpha^k\beta\alpha^k = -I \end{array}\right\} \quad k = 1, 2, 3$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

Commutator and Anticommutator Relationships

$$[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\mathrm{g}^{\mu\nu}$$

where,

$$\mathrm{g}^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Product Rules:

$$\mathrm{g}_{\mu\nu}\mathrm{g}^{\mu\nu} = 4$$
$$\gamma_\mu\gamma^\mu = 4$$
$$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$$
$$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4\mathrm{g}^{\nu\lambda}$$
$$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu$$

$$\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b$$
$$\gamma_\mu\not{a}\,\gamma^\mu = -2\not{a}$$
$$\gamma_\mu\not{a}\not{b}\,\gamma^\mu = 4a \cdot b$$
$$\gamma_\mu\not{a}\not{b}\not{c}\,\gamma^\mu = -2\not{c}\not{b}\not{a}$$

where $\not{a} = a_\mu\gamma^\mu$ and a_μ is any 4-vector

Trace Identities:

$$Tr(A + B) = Tr(A) + Tr(B)$$
$$Tr(\alpha A) = \alpha Tr(A)$$
$$Tr(AB) = Tr(BA)$$
$$Tr(I) = 4$$

$$Tr(\gamma^\mu\gamma^\nu) = 4\mathrm{g}^{\mu\nu}$$
$$Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(\mathrm{g}^{\mu\nu}\mathrm{g}^{\lambda\sigma} - \mathrm{g}^{\mu\lambda}\mathrm{g}^{\nu\sigma} + \mathrm{g}^{\mu\sigma}\mathrm{g}^{\nu\lambda})$$
$$Tr(\not{a}\not{b}) = 4a \cdot b$$
$$Tr(\not{a}\not{b}\not{c}\not{d}) = 4(a \cdot b\; c \cdot d - a \cdot c\; b \cdot d + a \cdot d\; b \cdot c)$$

The trace of an odd number of γ-matrices is equal *zero*.

APPENDIX B

DIRAC EQUATION

Starting with the relativistic energy-momentum relation:

$$(p^\mu p_\mu - m^2c^2) = 0$$

which can be factored into two terms

$$(\gamma^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = 0$$

where, by convention, the second term (after substitution of $p_\mu \to i\partial_\mu$) is used to create the covariant form of the Dirac equation

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

using a plane-wave solution for $\psi(\vec{r}, t)$

$$\psi(\vec{r}, t) = ae^{-i(Et - \vec{p}\cdot\vec{r})}u(E, \vec{p})$$

or simply

$$\psi(x) = ae^{-i(x\cdot p)}u(p)$$

because $p \equiv (E, \vec{p})$

$$\partial_\mu \psi = -ip_\mu a e^{-i(x^\mu \cdot p_\mu)}u$$

inserting this into the Dirac equation, we get

$$\gamma^{\mu} p_{\mu} a e^{-i(x \cdot p)} u - m a e^{-i(x \cdot p)} u = 0$$

or

$$\boxed{(\gamma^{\mu} p_{\mu} - m) u = 0}$$

This is known as the "momentum space Dirac equation", and if u satisfies this equation then ψ (as defined above) will satisfy the Dirac equation.

Completeness Relation:

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix}, \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x + i p_y}{E+m} \\ \frac{p_z}{E+m} \end{pmatrix}$$

where N is the normalization constant

$$u^{(r)\dagger} u^{(s)} = 2E\delta_{rs}$$

$$u^{(1)\dagger} u^{(1)} = N * \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x + i p_y}{E+m} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix} N$$

$$= |N|^2 \left[1 + 0 + \frac{p_z^2}{(E+m)^2} + \frac{(p_x + i p_y)(p_x - i p_y)}{(E+m)^2} \right]$$

$$= \frac{|N|^2}{(E+m)^2} \left[(E+m)^2 + p_x^2 + p_y^2 + p_z^2 \right] = \frac{|N|^2}{(E+m)^2} \left[(E+m)^2 + \vec{p}^{\,2} \right]$$

where $\vec{p}^{\,2} = E^2 - m^2 = (E-m)(E+m)$ therefore,

$$u^{(1)\dagger}u^{(1)} = \frac{|N|^2}{(E+m)^2}\left[(E+m)^2 + (E-m)(E+m)\right] = \frac{2E\;|N|^2}{(E+m)}$$

and,

$$\boxed{N = \sqrt{E+m}}$$

$$\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) =$$

$$= |N|^2 \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x+ip_y}{E+m} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{-p_z}{E+m} \\ \frac{-p_x+ip_y}{E+m} \end{pmatrix} + \begin{pmatrix} 0 & 1 & \frac{p_x-ip_y}{E+m} & \frac{-p_z}{E+m} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{-p_x-ip_y}{E+m} \\ \frac{p_z}{E+m} \end{pmatrix}$$

$$= \begin{pmatrix} E+m & 0 & -p_z & -(p_x-ip_y) \\ 0 & E+m & -(p_x+ip_y) & p_z \\ p_z & (p_x-ip_y) & -\frac{\vec{p}^{\,2}}{E+m} & 0 \\ (p_x+ip_y) & -p_z & 0 & -\frac{\vec{p}^{\,2}}{E+m} \end{pmatrix}$$

where $\frac{\vec{p}^{\,2}}{E+m} = \frac{E^2-m^2}{E+m} = E-m$ and $\vec{p}\cdot\sigma = \begin{pmatrix} p_z & (p_x-ip_y) \\ (p_x+ip_y) & -p_z \end{pmatrix}$

then,

$$\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \begin{pmatrix} E+m & -(\vec{p}\cdot\sigma) \\ (\vec{p}\cdot\sigma) & -E+m \end{pmatrix} = E\,\gamma^0 - \vec{p}\cdot\begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} + m$$

therefore,

$$\boxed{\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p} + m}$$

Casimir's Trick:

This process is an application of the Completeness Relation and simplifies the equations required for the Scattering Amplitude $\mathcal{M}$. When calculating the square of the scattering amplitude $\mathcal{M}^2$ each equation has two spinors, i.e. ($\overline{u}(a)$ and $u(b)$) and a vertex factor, Γ.

$$G \equiv [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^*$$

where,

$$[\overline{u}(a)\Gamma_1 u(b)]^* = \overline{u}(b)\overline{\Gamma}_2 u(a)$$

where, $\overline{\Gamma}_2 = \gamma^0 {\Gamma_2}^\dagger \gamma^0$ Summing over the spin orientations and using the completeness relation:

$$\sum_{a\ spins} \sum_{b\ spins} G = \overline{u}(a)\Gamma_1 \boxed{u(b)\ \overline{u}(b)}\ \overline{\Gamma}_2 u(a)$$

where the terms in the box correspond to the completeness relation

$$\sum_{b\ spins} u(b)\overline{u}(b) = \not{p}_b + m$$

moving the spinor, $u(a)$ from the end of the line to the beginning we can repeat this process for the *a spins*

$$\Gamma_1(\not{p}_b + m)_{ij}\overline{\Gamma}_2 \sum_{a\ spins} \boxed{u(a)\ \overline{u}(a)} = \Gamma_1(\not{p}_b + m)_{ij}\overline{\Gamma}_2(\not{p}_a + m)_{ji}$$

therefore,

$$\boxed{\sum_{all\ spins} [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(a)\Gamma_2 u(b)]^* = Tr[\Gamma_1(\not{p}_b + m)\overline{\Gamma}_2(\not{p}_a + m)}$$

APPENDIX C

GORDON DECOMPOSITION

$$[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = \gamma^\mu\gamma^\nu - (2\mathrm{g}^{\mu\nu} - \gamma^\mu\gamma^\nu) = 2\gamma^\mu\gamma^\nu - 2\mathrm{g}^{\mu\nu}$$

therefore

$$i\sigma^{\mu\nu} = -(\gamma^\mu\gamma^\nu - \mathrm{g}^{\mu\nu}) = \mathrm{g}^{\mu\nu} - \gamma^\mu\gamma^\nu$$

also

$$i\sigma^{\mu\nu} = \mathrm{g}^{\mu\nu} - (2\mathrm{g}^{\mu\nu} - \gamma^\nu\gamma^\mu) = \gamma^\nu\gamma^\mu - \mathrm{g}^{\mu\nu}$$

$$\overline{u}(p')i\sigma^{\mu\nu}(p'_\nu - p_\nu)u(p) = \overline{u}(p')[(\gamma^\nu\gamma^\mu - \mathrm{g}^{\mu\nu})p'_\nu - (\mathrm{g}^{\mu\nu} - \gamma^\mu\gamma^{nu})p_\nu]u(p)$$

$$= \overline{u}(p')\left[\gamma^\nu p'_\nu\gamma^\mu - p'^\mu - p^\mu + \gamma^\mu\gamma^\nu p_\nu\right]u(p)$$

using the Dirac equation and it's conjugate

$$(\gamma^\mu p_\mu - M) = 0 \quad \Longrightarrow \quad \gamma^\mu p_\mu \; u(p) = M \; u(p)$$

$$\overline{u}(p')(\gamma^\mu p'_\mu - M) = 0 \quad \Longrightarrow \quad \overline{u}(p') \; \gamma^\mu p'_\mu = \overline{u}(p') \; M$$

then

$$\overline{u}(p')i\sigma^{\mu\nu}(p'_\nu - p_\nu)u(p) = \overline{u}(p')[M\gamma^\mu - (p' + p)^\mu + \gamma^\mu M]u(p)$$

$$= \overline{u}(p')[2M\gamma^\mu - (p' + p)^\mu]u(p)$$

therefore

$$\boxed{\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\left[\frac{(p' + p)^\mu}{2M} + \frac{i\sigma^{\mu\nu}(p' - p)_\nu)}{2M}\right]u(p)}$$

APPENDIX D

FEYNMAN RULES

External Lines

$$\text{Spin 0: (nothing)}$$

$$\text{Spin } \frac{1}{2}\text{:} \begin{cases} \text{Incoming particle: } u \\ \text{Incoming antiparticle: } \overline{v} \\ \text{Outgoing particle: } \overline{u} \\ \text{Outgoing antiparticle: } v \end{cases}$$

$$\text{Spin 1:} \begin{cases} \text{Incoming: } \epsilon^{\mu} \\ \text{Outgoing: } \epsilon^{\mu *} \end{cases}$$

Propagators

$$\text{Spin 0: } \frac{i}{q^2 - (mc)^2}$$

$$\text{Spin } \tfrac{1}{2}\text{: } \frac{i(\not{q} + mc)}{q^2 - (mc)^2}$$

$$\text{Spin 1:} \begin{cases} \text{Massless: } \dfrac{-i\mathrm{g}_{\mu\nu}}{q^2} \\ \\ \text{Massive: } \dfrac{-i[\mathrm{g}_{\mu\nu} - q_\mu q_\nu/(mc)^2}{q^2 - (mc)^2} \end{cases}$$

Vertex Factors

$$\text{Spin 0: } iQ(p + p')^{\mu}$$

$$\text{Spin } \frac{1}{2}\text{: } - iQ\gamma^{\mu}$$

APPENDIX E

SERIES EXPANSIONS

Series expansion of Eq. (4.27):

$$\frac{(E' + E)}{\sqrt{4EE'}} = \frac{\sqrt{m^2 + \vec{p'}^2} + \sqrt{m^2 + \vec{p}^2}}{\sqrt{4\sqrt{m^2 + \vec{p'}^2}\sqrt{m^2 + \vec{p}^2}}} \tag{E.1}$$

$$= \frac{\sqrt{1 + \left(\frac{\vec{P} - \frac{\vec{q}}{2}}{m}\right)^2} + \sqrt{1 + \left(\frac{\vec{P} + \frac{\vec{q}}{2}}{m}\right)^2}}{\sqrt{4\sqrt{1 + \left(\frac{\vec{P} - \frac{\vec{q}}{2}}{m}\right)^2}\sqrt{1 + \left(\frac{\vec{P} + \frac{\vec{q}}{2}}{m}\right)^2}}} \tag{E.2}$$

using center of momentum variables from equation (4.27). This equation can be broken down into multiple expressions of the form:

$$\sqrt{1 + x^2} \approx \left\{1 + \frac{x^2}{2} - \frac{x^4}{8} + O[x]^6\right\} \tag{E.3}$$

Therefore,

$$\sqrt{1 + \left(\frac{\vec{P} \pm \frac{\vec{q}}{2}}{m}\right)^2} \approx \left\{1 + \frac{\left(\vec{P} \pm \frac{\vec{q}}{2}\right)^2}{2m^2} - \frac{\left(\vec{P} \pm \frac{\vec{q}}{4}\right)^4}{8m^4} + O\left[\frac{1}{m}\right]^6\right\} \tag{E.4}$$

For the denominator in (E.2), The expansion can be used for both $\sqrt{1 + x^2}$ term, then, once combined, used a third time for the complete square root term before combining it with the numerator.

Therefore,

$$\boxed{\frac{E' + E}{\sqrt{4EE'}} = 1 + \frac{(\vec{P}\cdot\vec{q}\,)^2}{8m^4} + O\left[\frac{1}{m}\right]^6} \tag{E.5}$$

It is noted that the $\left[\frac{1}{m}\right]^2$ terms cancel when the expansion expressions are combined.

<u>Series expansion of Eq. (4.28):</u>

Using the same expansion from equation (E.4), the two square roots can be eliminated to simplify the expression.

$$q^2 = \left[m\left(1 + \frac{\left(\vec{P} - \frac{\vec{q}}{2}\right)^2}{2m^2} - \frac{\left(\vec{P} - \frac{\vec{q}}{4}\right)^4}{8m^4}\right) - m\left(1 + \frac{\left(\vec{P} + \frac{\vec{q}}{2}\right)^2}{2m^2} - \frac{\left(\vec{P} + \frac{\vec{q}}{4}\right)^4}{8m^4}\right)\right]^2 - \vec{q}^{\,2} \tag{E.6}$$

$$\boxed{q^2 = -\vec{q}^{\,2} + \left(\frac{\vec{P}\cdot\vec{q}}{m}\right)^2 - \frac{\left(4\vec{P}^{\,2} + \vec{q}^{\,2}\right)\left(\vec{P}\cdot\vec{q}\right)^2}{4m^4} + O\left[\frac{1}{m}\right]^6} \tag{E.7}$$

<u>Series expansion of Eq. (4.29):</u>

Using the expansion from (E.7),

$$\boxed{e^{\frac{1}{6}q^2\Delta^2} = e^{-\frac{1}{6}\vec{q}^{\,2}\Delta^2}\left(1 + \frac{\Delta^2\left(\vec{P}\cdot\vec{q}\right)^2}{6m^2} + \frac{\Delta^2\left(\vec{P}\cdot\vec{q}\right)^2\left(-12\vec{P}^{\,2} - 3\vec{q}^{\,2} + \Delta^2\left(\vec{P}\cdot\vec{q}\right)^2\right)}{72m^4} + O\left[\frac{1}{m}\right]^6\right)} \tag{E.8}$$