AN EXPLORATION OF COMPATIBILITY

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ABSTRACT

Let $G = \mathrm{SO}^\circ(3,1)$, $H = \mathrm{SO}^\circ(4,1)$, $\Gamma < G$ be a lattice, and $\rho : \Gamma \to H$ be a Zariski-dense representation. This thesis considers conditions required for ρ to extend to a continuous homomorphism $\widetilde{\rho} : G \to H$, formulated as a classic superrigidity theorem. These conditions require Γ to contain an infinite family $\{\Gamma_i\}$ of nonconjugate maximal Fuchsian subgroups associated with closed $\mathrm{SO}^\circ(2,1)$-orbits on $\Gamma\backslash G$ such that $\rho(\Gamma_i)$ fails to be Zariski-dense in H. This extends work of Bader–Fisher–Miller–Stover, who used similar techniques find a geometric characterization of arithmeticity. Their work relies on G and H satisfying a technical compatibility condition that does not hold for the pair G, H considered here.

In particular, the situation considered here falls outside the scope covered by their work. This is a interesting case, because G and H are the isometry groups of hyperbolic 3- and 4-space respectively.

This thesis applies a construction of Bader–Furman to the map that ρ will extend to, called ϕ, and upgrades ϕ from a map to a homomorphism, while utilizing a subgroup $J < H$, related to the failure of compatibility. The first goal of this thesis is to understand the failure of compatibility in terms of the triples (G, H, J) with $J < H$ a real algebraic subgroup. A triple is *compatible* or *incompatible* depending on whether the methods of Bader–Fisher–Miller–Stover apply to prove superrigidity via the techniques of Bader–Furman.

We then split into two cases: when (G, H, J) is a compatibility triple and when (G, H, J) is an incompatibility triple. In the former case, we use the methods of Bader–Fisher–Miller–Stover mentioned above to prove superrigidity. In the latter case, we prove there is only one such J that makes an incompatibility triple. In the incompatible case, we use works of Kim–Oh to understand the boundary map $\partial\mathbb{H}^3 \to \partial\mathbb{H}^4$ classically associated with the representation ρ, and this ultimately allows us to conclude that no such ρ could ever exist.

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CHAPTER 1

INTRODUCTION

Bader–Fisher–Miller–Stover proved the following seminal result in their paper [2], first posed as an informal question by Alan Reid in the mid-2000s and independently by Curtis McMullen in the case of hyperbolic 3-manifolds.

Theorem 1.1 ([2] Theorem 1.1)**.** *Let Γ be a lattice in $\mathrm{SO}^\circ(n,1)$. If the associated locally symmetric space contains infinitely many maximal totally geodesic subspaces of dimension at least 2, then Γ is arithmetic.*

To better understand the above theorem, we begin with defining the objects in it.

Definition 1.2. *The **associated locally symmetric space** of Γ in $\mathrm{SO}^\circ(n,1)$ is the Riemannian manifold $\Gamma\backslash\mathbb{H}^n$. A space is **locally symmetric** if for every point p and every geodesic g going through p, the reflection of g about p is an isometry.*

Definition 1.3. *A subspace $S \subset \Gamma\backslash\mathbb{H}^n$ is **totally geodesic** if for every two points $p, q \in S$, the geodesic $[p,q] \subset \Gamma\backslash\mathrm{SO}^\circ(n,1)$ is fully contained in S. This subspace is **maximal** if there does not exist some $S' \supset S$ so that S' is also a totally geodesic subspace.*

Note that for dimension 3, every totally geodesic subspace of dimension at least 2 is maximal, since the only totally geodesic subspaces are surfaces and they are codimension 1.

The above theorem is the reformulation of a *superrigidity theorem*, meaning it extends the work of Margulis' original superrigidity theorem.

Theorem 1.4 (Margulis Superrigidity [11])**.** *Suppose we have the following:*

•G *is a semisimple Lie group of real rank at least 2*

•Γ *is an irreducible lattice in* G

•H *is a connected, noncompact, simple Lie group with trivial center*

•$\rho : \Gamma \to H$ *is an irreducible, Zariski-dense representation*

Then ρ *extends to a continuous homomorphism* $\widetilde{\rho} : G \to H$.

The following puts Theorem 1.1 in the perspective of a superrigidity theorem:

Theorem 1.5 ([2] Theorem 1.6)**.** *Let* $G = \mathrm{SO}^\circ(n, 1)$ *for* $n \geq 3$*,* $W < G$ *be a noncompact simple subgroup, and* $\Gamma < G$ *a lattice. Suppose* k *is a local field and* H *is a connected* k*-algebraic group such that the pair consisting of* k *and* H *is compatible with* G*. Let* $\rho : \Gamma \to H$ *be a homomorphism with unbounded Zariski dense image. If there exists a* k*-rational faithful irreducible representation* $H \to \mathrm{SL}(V)$ *on an* k*-vector space* V *and a* W*-invariant measure* ν *on* $(G \times \mathbb{P}(V))/\Gamma$ *that projects to Haar measure on* G/Γ*, then* ρ *extends to a continuous homomorphism* $\widetilde{\rho} : G \to H$.

The above covers the case where the fundamental groups of infinitely many maximal totally geodesic surfaces do **not** have Zariski-dense images under ρ. Notice this relies on H and G satisfying a *compatibility condition*, which we will address in Definition 1.10. In their later work [3], Bader–Fisher–Miller–Stover prove the following superrigidity theorem that does **not** rely on a compatibility condition, for a case that is critical for extending Theorem 1.1 above to the complex hyperbolic setting.

Theorem 1.6 ([3] Theorem 1.3)**.** *Let $G = \mathrm{SU}(n,1)$ for $n \geq 2$, $W < G$ be a noncompact connected almost simple subgroup, and $\Gamma < G$ a lattice. Suppose k is a local field, H is a connected adjoint k-algebraic group, and $\rho : \Gamma \to H$ is a representation with unbounded Zariski-dense image. Moreover, suppose there is a faithful irreducible representation of H on a k-vector space V of dimension at least 2 and a W-invariant, ergodic measure ν on $(G \times \mathbb{P}(V))/\Gamma$ that projects to Haar measure on G/Γ. If either*

(1) the pair (H, k) is compatible with G, or

(2) $k = \mathbb{R}$ and $H \cong \mathrm{PU}(n,1)$,

then ρ extends to a continuous homomorphism $\widetilde{\rho} : G \to H$.

While $G = \mathrm{SU}(n,1)$ and $H = \mathrm{PU}(n,1)$ are not compatible, Bader–Fisher–Miller–Stover were still able to prove a superrigidity theorem about them.

This thesis analyzes an interesting case that is not handled in the above theorems: when $G = \mathrm{SO}^\circ(3,1)$ and $H = \mathrm{SO}^\circ(4,1)$. This case is particularly important because $\mathrm{SO}^\circ(n,1)$ is the isometry group of hyperbolic n-space. While $\mathrm{SO}^\circ(3,1)$ and $\mathrm{SO}^\circ(4,1)$ are not compatible, we are still able to prove a superrigidity theorem.

1.1 Statement of theorem and sketch of proof

Let $G := \mathrm{SO}^\circ(3,1)$, $H := \mathrm{SO}^\circ(4,1)$, and $\Gamma < G$ be a lattice.

Definition 1.7. *A representation $\rho : \Gamma \to H$ is **convex cocompact** if $\rho(\Gamma)$ acts convex cocompactly on H, meaning it acts cocompactly on some nonempty closed convex subset of the associated locally symmetric space of H.*

Theorem 1.8. *Let*

$$\rho : \Gamma \to H$$

be a convex cocompact Zariski-dense representation and let $\{\Gamma_i\}$ *be an infinite family of distinct conjugacy classes of maximal finite covolume totally geodesic surface subgroups so that* $\Gamma_i < \Gamma$. *Then all but finitely many images* $\rho(\Gamma_i)$ *are Zariski-dense in* H.

Similarly to Bader–Fisher–Miller–Stover in [2] and [3], we are able to reformulate this result as a superrigidity theorem.

Theorem 1.9. *Let* $G := \mathrm{SO}^\circ(3,1)$, $H := \mathrm{SO}^\circ(4,1)$, $\Gamma < G$ *be a lattice, and* $W < G$ *be a noncompact simple subgroup. Let*

$$\rho : \Gamma \to H$$

be a convex cocompact representation so that the Zariski closure $\overline{\rho(\Gamma)}$ *of* $\rho(\Gamma)$ *in* H *contains a conjugate of the standard embedding of* $\mathrm{SO}^\circ(3,1)$. *Moreover, suppose there is a faithful irreducible representation of* H *on a vector space* V *of dimension at least 2 and a* W*-invariant, ergodic measure* ν *on* $(G \times \mathbb{P}(V))/\Gamma$ *that projects to Haar measure on* G/Γ. *Then* ρ *extends to a continuous homomorphism*

$$\widetilde{\rho} : G \to H.$$

We prove this theorem by contradiction. When there is an infinite family of lattices $\{\Gamma_i\}$ in conjugates of $\mathrm{O}^\circ(2,1)$ in G with $\Gamma_i < \Gamma$ so that the $\rho(\Gamma_i)$ are not Zariski-dense in H, we obtain a W-invariant map $\psi : G \to H/L$ (see Proposition 3.7, where we define L). We use this map to show $\rho(\Gamma)$ must act on H/L with a fixed point, which contradicts the assumed Zariski-density of ρ. This allows us to use the Bader–Furman construction described in Chapter 3 to build the map $G \to H$ that will extend to ρ. The existence of this continuous extension depends on whether G, H, and a subgroup $J < H$ satisfy the aforementioned *compatibility condition*, defined below.

Definition 1.10. *Let $G = \mathrm{SO}^\circ(n,1)$, let H be a connected semisimple Lie group, and let $\Gamma < G$ be a lattice. Let*

$$\rho : \Gamma \to H$$

be a representation. Let $P < G$ be a maximal parabolic subgroup and $U < P$ its unipotent radical.

Let $J < H$ and let $N_H(J)$ be the normalizer of J in H. We call the triple (G, H, J) a ***compatibility triple*** *if for every continuous homomorphism*

$$\tau : P \to N_H(J)/J$$

and for every nontrivial subgroup $U' < U$, we have equality $\overline{\tau(U')} = \overline{\tau(U)}$ of Zariski closures.

- *If (G, H, J) do not form a compatibility triple, we call it an* ***incompatibility triple****.*

• *If there does not exist some $J < H$ so that (G, H, J) is an incompatibility triple, we call G and H* ***compatible****.*

In Section 3 we consider when (G, H, J) is a compatibility triple following the roadmap from [2]. In this case, we use the Bader–Furman construction to show that $\rho : \Gamma \to H$ must extend to a representation $\widetilde{\rho} : G \to H$.

Consider the assumptions for Theorem 1.8. Since ρ is Zariski-dense, we know $\overline{\rho(\Gamma)} = H$. However $\widetilde{\rho}(G)$ is contained in a copy of $\mathrm{SO}^\circ(3,1) < H$, which is Zariski-closed. Then $\overline{\rho(\Gamma)}$ must also be in that copy of $\mathrm{SO}^\circ(3,1)$. This means ρ cannot be a Zariski-dense map, garnering a contradiction.

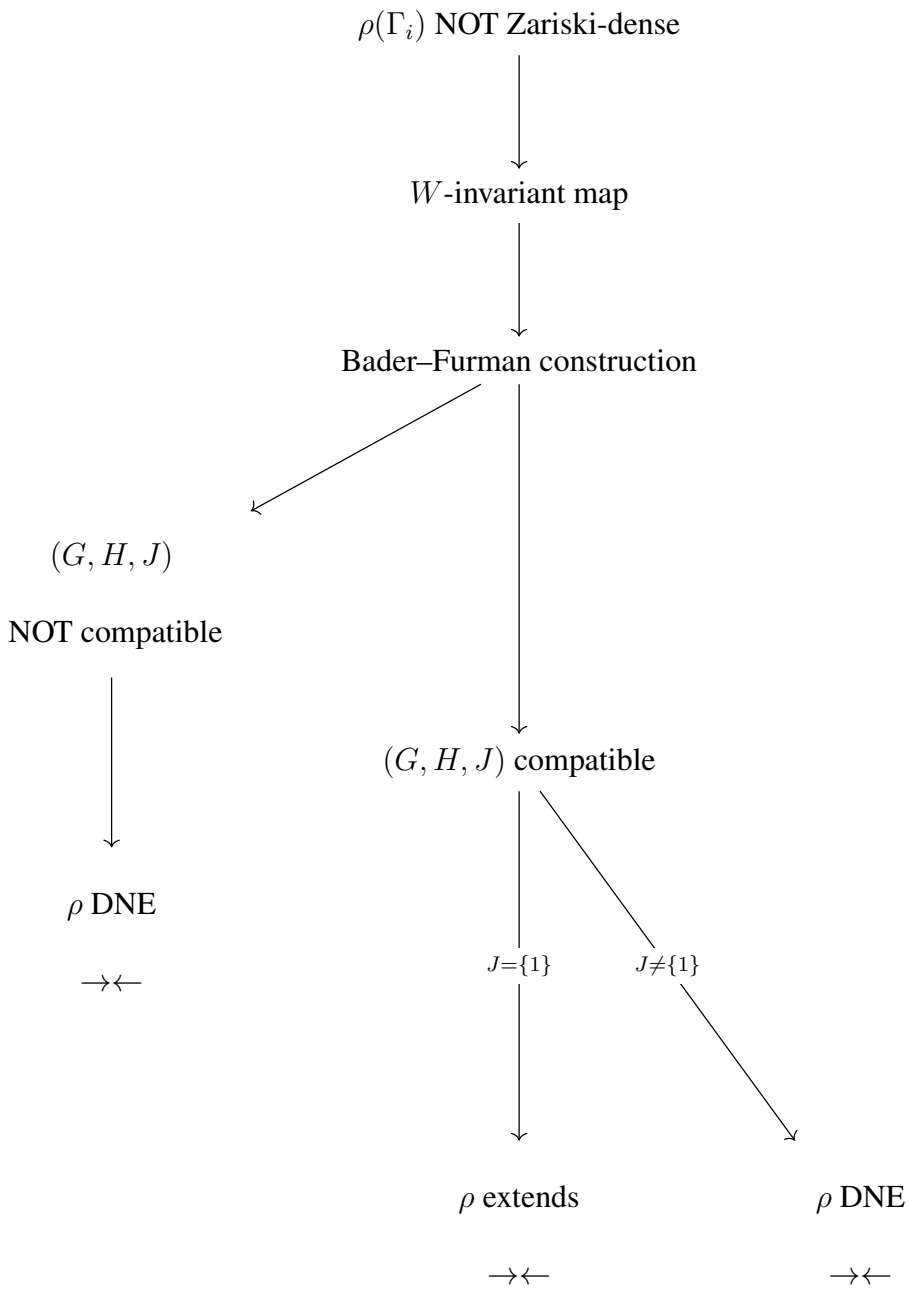
Chapter 4 considers the case when (G, H, J) is an incompatibility triple. Now following the basic outline used to prove Theorem 1.5, we begin by proving the following theorem.

Theorem 1.11. *Let $G = \mathrm{SO}^\circ(3,1)$ and $H = \mathrm{SO}^\circ(4,1)$. Then there exists a unique $J < H$ so that (G, H, J) is an incompatibility triple. In this case,*

$$J = \left\{ \begin{pmatrix} 1 & -t\boldsymbol{e}_1^\top & \frac{-t^2}{2} \\ & \mathrm{Id} & t\boldsymbol{e}_1 \\ & & 1 \end{pmatrix} : \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

The above is the only case not covered by Section 3. We consider the action of W and P on $\partial\mathbb{H}^3$ to analyze the ρ-boundary map $\overline{\phi} : \partial\mathbb{H}^3 \to \partial\mathbb{H}^4$. This allows us to use work of Kim–Oh in [9] and [10] to determine what ρ must be. We ultimately can conclude that in this case, the representation $\rho : \Gamma \to G$ cannot exist.

1.1.1 *Proof flow chart for Theorem*



CHAPTER 2

BACKGROUND

Let G act by orientation-preserving isometries on the hyperboloid model of hyperbolic n-space with quadratic form

$$q(x) = 2x_0x_n + \sum_{i=1}^{n-1} x_i^2$$

represented by the matrix

$$q := \begin{pmatrix} & & 1 \\ & \mathrm{Id} & \\ 1 & & \end{pmatrix}. \tag{2.1}$$

Throughout this paper, blank spaces in matrices represent zero entries.

We define the hyperboloid model of $\mathbb{H}^n$ as follows:

$$\mathbb{H}^n = \{w = (w_0, \ldots, w_n) \in \mathbb{R}^{n+1} : w_0 = 1, q(w) < 0\}$$

and the boundary of $\mathbb{H}^n$ is the space

$$\partial\mathbb{H}^n = \{w \in \mathbb{R}^n_{\neq 0} : q(1, w) = 0\}/\mathbb{R}^*.$$

As shown via the definition above, the boundary of $\mathbb{H}^n$ consists of null lines of q. We can conventionally write the point at ∞ as the null line

$$\lambda \begin{pmatrix} 1 \\ \vec{0} \\ 0 \end{pmatrix},$$

where $\vec{0} \in \mathbb{R}^{n-2}$. This is why our definition of $\partial\mathbb{H}^n$ quotients by $\mathbb{R}^*$— this identifies the entire null *line* with the *point* at ∞. Other than the point at ∞, we can normalize the null lines so that each point in $\partial\mathbb{H}^n$ has a unique representative as

$$\begin{pmatrix} w_0 \\ \vec{v} \\ 1 \end{pmatrix}, \tag{2.2}$$

where $w_0 = -|\vec{v}|^2$ and $\vec{v} \in \mathbb{R}^{n-2}$. This has a clear identification with $\mathbb{R}^{n-1}$, since the above vectors are normalized to have the last entry be 1, and the point at ∞ provices the one-point compactification giving us $\partial\mathbb{H}^n \cong \mathbb{S}^{n-1}$.

With this interpretation, we can identify the oriented orthonormal frame bundle of $\mathbb{H}^n$ with its isometry group G:

$$G = \mathrm{SO}^\circ(n,1) = \{g \in \mathrm{SL}_{n+1}(\mathbb{R}) : g^T q g = q \text{ and } g \text{ preserves the upper hyperboloid}\}.$$

2.1 Maximal parabolic subgroups of $\mathrm{SO}^\circ(n,1)$

Let P be the proper maximal parabolic subgroup of G, meaning G/P is a projective algebraic variety and there is no $Q > P$ such that G/Q is also a projective algebraic variety [7]. Then by [12] we know $P = \mathrm{stab}_G(\infty)$, which provides a G-equivariant homeomorphism from G/P to the boundary of $\partial\mathbb{H}^n$.

Lemma 2.1. *The maximal parabolic subgroup $P < G$ is the following matrix group:*

$$P = \left\{ \begin{pmatrix} \varepsilon\lambda & -u^\top M_0 & \frac{\varepsilon\lambda^{-1}|u|^2}{2} \\ & M_0 & \varepsilon\lambda^{-1}u \\ & & \varepsilon\lambda^{-1} \end{pmatrix} : \begin{array}{c} \varepsilon \in \{\pm 1\}, \\ \lambda \in A := \mathbb{R}^+, \\ M_0 \in M := \mathrm{SO}(n-1), \\ u \in U := \mathbb{R}^{n-1} \end{array} \right\}. \tag{2.3}$$

Proof. Let

$$P_1 := \begin{pmatrix} \varepsilon\lambda & -u^\top M_0 & \frac{\varepsilon\lambda^{-1}|u|^2}{2} \\ & M_0 & \varepsilon\lambda^{-1}u \\ & & \varepsilon\lambda^{-1} \end{pmatrix}.$$

Then

$$P_1 \cdot \infty = \begin{pmatrix} \varepsilon\lambda & -u^\top M_0 & \frac{\varepsilon\lambda^{-1}|u|^2}{2} \\ & M_0 & \varepsilon\lambda^{-1}u \\ & & \varepsilon\lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ \vec{0} \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon\lambda \\ \vec{0} \\ 0 \end{pmatrix},$$

meaning P_1 does indeed stabilize infinity, so $P_1 \in P$ if it is contained in G.

Consider $N \in G$. Any matrix that stabilizes infinity must have the first column equal to

$$\begin{pmatrix} \varepsilon\lambda \\ \vec{0} \\ 0 \end{pmatrix}. \tag{2.4}$$

Then

$$N = \begin{pmatrix} \varepsilon\lambda & a & b \\ & C_0 & d \\ & e & f \end{pmatrix},$$

where a, e are row vectors, d is a column vector, C_0 is a square matrix, and b, f are real numbers. This matrix must also preserve the quadratic form q, meaning that $N^\top qN = q$ with q defined in Equation 2.1. Equivalently, we need

$$N^\top qN = \begin{pmatrix} \varepsilon\lambda & & \\ a^\top & C_0^\top & e^\top \\ b & d^\top & f \end{pmatrix} \begin{pmatrix} & & 1 \\ & \mathrm{Id} & \\ 1 & & \end{pmatrix} \begin{pmatrix} \varepsilon\lambda & a & b \\ & C_0 & d \\ & e & f \end{pmatrix} = \begin{pmatrix} & & 1 \\ & \mathrm{Id} & \\ 1 & & \end{pmatrix}.$$

By matrix multiplication, we see that the left hand side is

$$\begin{pmatrix} 0 & \varepsilon\lambda e & \varepsilon\lambda f \\ \varepsilon\lambda e^T & ea^\top + e^\top a + C_0^\top C_0 & be^\top + C_0^\top d + fa^\top \\ \varepsilon\lambda f & fa + d^\top C_0 + be & 2bf + |d|^2 \end{pmatrix}.$$

From looking at the $(2,1)$ and $(3,1)$ entries, we see that $e = \vec{0}$ and $f = \varepsilon\lambda^{-1}$. Knowing that, the $(2,2)$ entry tells us that $C_0 \in \mathrm{O}(n-1)$; following from $e = \vec{0}$, $f = \varepsilon\lambda^{-1}$, and the overall matrix having $\det = 1$, we know that $\det(C_0) = 1$, so $C_0 \in \mathrm{SO}(n-1)$. This then tells us that $a = -\varepsilon\lambda(C_0^\top d)^\top$, and that $b = \frac{-\varepsilon\lambda|d|^2}{2}$. Thus

$$N = \begin{pmatrix} \varepsilon\lambda & -\varepsilon\lambda(C_0^\top d)^\top & \frac{-\varepsilon\lambda|d|^2}{2} \\ & C_0 & d \\ & & \varepsilon\lambda^{-1} \end{pmatrix}$$

By reparameterizing so that $u = \varepsilon\lambda d$, we get

$$N = \begin{pmatrix} \varepsilon\lambda & -u^\top C_0 & \frac{-\varepsilon\lambda^{-1}|u|^2}{2} \\ & C_0 & \varepsilon\lambda^{-1}u \\ & & \varepsilon\lambda^{-1} \end{pmatrix}$$

Therefore $N \in P$, and moreover this proves that $P < G$. □

Let $A := \mathbb{R}^+$, $M := \mathrm{SO}(n-1)$, and $U = \mathbb{R}^{n-1}$. We canonically identify these groups with the following subgroups of P:

$$A = \left\{ \begin{pmatrix} \lambda & & \\ & \mathrm{Id} & \\ & & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^* \right\}$$

$$M = \left\{ \begin{pmatrix} 1 & & \\ & M & \\ & & 1 \end{pmatrix} : M \in \mathrm{SO}(n-1) \right\}$$

$$U = \left\{ \begin{pmatrix} 1 & -u^\top & \\ & \mathrm{Id} & u \\ & & 1 \end{pmatrix} : u \in \mathbb{R}^{n-1} \right\}.$$

Then A acts on U by scaling under the identification with $\mathbb{R}^{n-1}$, M acts by rotation, and U acts on itself by translation. Since these are all subgroups of P, then they all fix the point at infinity, meaning they similarly act on $\partial\mathbb{H}^n \cong \mathbb{S}^{n-1}$.

2.2 J and its normalizer in H

Let $H = \mathrm{SO}^\circ(n+1,1)$, and let J be defined as follows:

$$J = \left\{ \begin{pmatrix} 1 & -t\mathbf{e}_1^\top & \frac{-t^2}{2} \\ & \mathrm{Id} & t\mathbf{e}_1 \\ & & 1 \end{pmatrix} : \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

Claim 2.2. *The normalizer of J in H, denoted $N_H(J)$, is the following matrix*

$$N_H(J) = \left\{ \begin{pmatrix} \varepsilon\lambda & -r^\top M & \frac{\varepsilon\lambda^{-1}|r|^2}{2} \\ & \begin{pmatrix} \det(M_0) & \\ & M_0 \end{pmatrix} & \varepsilon\lambda^{-1} r \\ & & \varepsilon\lambda^{-1} \end{pmatrix} : \begin{array}{c} \varepsilon \in \{\pm 1\} \\ t \in \mathbb{R} \\ \lambda \in \mathbb{R}^+ \\ M_0 \in \mathrm{O}(n-1) \\ r = t\boldsymbol{e}_1 + u, u \in \mathbb{R}^{n-1} \end{array} \right\}.$$

Proof. Let

$$N = \begin{pmatrix} \varepsilon\lambda & -r^\top M & \frac{\varepsilon\lambda^{-1}|r|^2}{2} \\ & M & \varepsilon\lambda^{-1} r \\ & & \varepsilon\lambda^{-1} \end{pmatrix}$$

as described in the statement of the claim, where

$$M = \begin{pmatrix} \det(M_0) & \\ & M_0 \end{pmatrix}.$$

Then

$$N^{-1} = \begin{pmatrix} \varepsilon\lambda^{-1} & \varepsilon\lambda^{-1} r^\top & \frac{\varepsilon\lambda^{-1}|r|^2}{2} \\ & M^\top & -M^\top r \\ & & \varepsilon\lambda \end{pmatrix}.$$

Let $J_0 \in J$. Then for some $t \in \mathbb{R}$ we have

$$J_0 = \begin{pmatrix} 1 & -t\mathbf{e}_1^\top & \frac{-t^2}{2} \\ & \mathrm{Id} & t\mathbf{e}_1 \\ & & 1 \end{pmatrix}.$$

By matrix multiplication, we can see that

$$N^{-1}J_0N = \begin{pmatrix} 1 & -\varepsilon\lambda^{-1}(M^\top t\mathbf{e}_1)^\top & \lambda^{-2}(2|u|^2 + 4tu_1 + \frac{5t^2}{2}) \\ & \mathrm{Id} & \varepsilon\lambda^{-1}M^\top t\mathbf{e}_1 \\ & & 1 \end{pmatrix}.$$

By reparameterizing so we have $s \in \mathbb{R}$ such that $s\mathbf{e}_1 = \varepsilon\lambda^{-1}M^\top t\mathbf{e}_1$, we get

$$N^{-1}J_0N = \begin{pmatrix} 1 & -s\mathbf{e}_1^\top & \frac{-s^2}{2} \\ & \mathrm{Id} & s\mathbf{e}_1 \\ & & 1 \end{pmatrix}.$$

This tells us that

$$\left\{ \begin{pmatrix} \varepsilon\lambda & -r^\top M & \frac{\varepsilon\lambda^{-1}|r|^2}{2} \\ & \begin{pmatrix} \det(M_0) & \\ & M_0 \end{pmatrix} & \varepsilon\lambda^{-1} r \\ & & \varepsilon\lambda^{-1} \end{pmatrix} : \begin{array}{c} \varepsilon \in \{\pm 1\} \\ t \in \mathbb{R} \\ \lambda \in \mathbb{R}^+ \\ M_0 \in \mathrm{O}(n-1) \\ r = t\mathbf{e}_1 + u, u \in \mathbb{R}^{n-1} \end{array} \right\} \leq N_H(J).$$

Suppose N normalizes J_0, and let Q be the maximal parabolic subgroup of $\mathrm{SO}^\circ(n+1,1)$. Since $J < Q$ and the normalizer of a unipotent subgroup also stabilizes ∞, we know $N \in Q$. Then N must have a first column of

$$\begin{pmatrix} \varepsilon\lambda \\ \vec{0} \\ 0 \end{pmatrix}.$$

That means that N must look like

$$N = \begin{pmatrix} \varepsilon\lambda & a & b \\ & C_0 & d \\ & & f \end{pmatrix}$$

where a is a row-vector, C_0 is a square matrix, d is a column vector, and b, f are values. Since N must satisfy the quadratic form, meaning $N^\top q N = q$, we can see that $f = \varepsilon\lambda^{-1}$, $C_0 \in \mathrm{SO}(n-1)$, and $a = -\varepsilon\lambda(C_0^\top d)^\top$. This means that

$$N = \begin{pmatrix} \varepsilon\lambda & -\varepsilon\lambda(C_0^\top d)^\top & b \\ & C_0 & d \\ & & \varepsilon\lambda^{-1} \end{pmatrix}.$$

By definition we have

$$N^{-1} = \begin{pmatrix} \varepsilon\lambda^{-1} & d^\top & -\varepsilon\lambda|d|^2 - b \\ & C_0^\top & -\varepsilon\lambda C_0^\top d \\ & & \varepsilon\lambda \end{pmatrix}.$$

Again, let $J_0 \in J$. Then by matrix multiplication we see that

$$N^{-1} J_0 N = \begin{pmatrix} 1 & -\varepsilon\lambda^{-1} t \mathbf{e}_1^\top C_0 & 2\varepsilon\lambda^{-1} t d_1 + \lambda^{-2}\frac{t^2}{2} \\ & \mathrm{Id} & \varepsilon\lambda^{-1} C_0^\top t \mathbf{e}_1 \\ & & 1 \end{pmatrix}.$$

Since $N \in N_H(J)$, we know that

$$\begin{pmatrix} 1 & -\varepsilon\lambda^{-1}t\mathbf{e}_1^\top C_0 & 2\varepsilon\lambda^{-1}td_1 + \lambda^{-2}\frac{t^2}{2} \\ & \mathrm{Id} & \varepsilon\lambda^{-1}C_0^\top t\mathbf{e}_1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & -s\mathbf{e}_1^\top & \frac{-s^2}{2} \\ & \mathrm{Id} & s\mathbf{e}_1 \\ & & 1 \end{pmatrix}$$

for some $s \in \mathbb{R}$. By reparameterizing so that $-\varepsilon\lambda^{-1}t = s$, we get that

$$N^{-1}J_0N = \begin{pmatrix} 1 & -s\mathbf{e}_1^\top C_0 & sd_1 + \frac{s^2}{2} \\ & \mathrm{Id} & C_0^\top s\mathbf{e}_1 \\ & & 1 \end{pmatrix}.$$

Thus we must have

$$C_0 = \begin{pmatrix} \det(M_0) & \\ & M_0 \end{pmatrix}$$

for $M_0 \in \mathrm{O}(n-2)$. We also know that $d_1 = -s$, which in conjunction with $t = -\varepsilon\lambda s$ tells us that $d = t\mathbf{e}_1 + u$ for some $u \in \mathbb{R}^2$. Thus

$$N \in \left\{ \begin{pmatrix} \varepsilon\lambda & -r^\top M & \frac{\varepsilon\lambda^{-1}|r|^2}{2} \\ & \begin{pmatrix} \det(M_0) & \\ & M_0 \end{pmatrix} & \varepsilon\lambda^{-1}r \\ & & \varepsilon\lambda^{-1} \end{pmatrix} : \begin{array}{c} \varepsilon \in \{\pm 1\} \\ t \in \mathbb{R} \\ \lambda \in \mathbb{R}^+ \\ M_0 \in \mathrm{O}(n-1) \\ r = t\mathbf{e}_1 + u, u \in \mathbb{R}^{n-1} \end{array} \right\}.$$

□

Corollary 2.3. *The group $N_H(J)$ is contained in the maximal parabolic subgroup $Q < H$.*

CHAPTER 3

WHEN (G, H, J) IS NOT AN INCOMPATIBILITY TRIPLE, THE MAP ρ EXTENDS.

Let $G = \mathrm{SO}^\circ(n, 1)$, let H be a connected semisimple Lie group, and let $\Gamma < G$ be a lattice. Let

$$\rho : \Gamma \to H$$

be a representation.

3.1 Bader–Furman construction.

In their paper that generalizes Theorem 1.4 [5], Bader–Furman prove the following superrigidity theorem:

Theorem 3.1. *Let k be a local field, G be the locally compact group formed by the k-points of a connected, semi-simple, algebraic group defined over k. Assume the k-rank of G is at least two. Let $\Gamma < G$ be a lattice and assume the projection of Γ to G/N is nondiscrete whenever $N \lhd G$ is the k-points of a proper normal k-isotropic subgroup.*

Let l be a field with an absolute value so that as a metric space l is complete. Let H be the l-points of a connected, adjoint, l-simple algebraic group H defined over l. Let $\rho : \Gamma \to H$ be a homomorphism and assume $\rho(\Gamma)$ is Zariski-dense and unbounded in H. Then there exists a unique continuous homomorphism $\widetilde{\rho} : G \to H$ so that $\widetilde{\rho}$ is an extension of ρ.

The proof of this theorem relies on Bader–Furman building an R-algebraic representation of G, as defined below.

Definition 3.2 ([5] Definition 4.1)**.** *Let $R < G$ be a closed subgroup, and let k be a separable and complete (as a metric space) field with a non-trivial absolute value so that G is a k-algebraic group. An R-**algebraic representation of** G consists of the following:*

- *a k-algebraic group N*
- *a $k - (H \times N)$-algebraic variety V, regarded as a left H, right N space, on which the N-action is faithful*
- *a homomorphism $\tau : R \to N$ with a Zariski-dense image, and*
- *an associated algebraic representation of the $\Gamma \times R$-space G on V, where Γ acts on the left and R acts on the right of G. In other words, there must be a Haar almost everywhere defined measurable map $\phi : G \to V$ so that for almost every $g \in G$ and every $\gamma \in \Gamma$ and $r \in R$ we have*

$$\phi(\gamma g r) = \rho(\gamma)\phi(g)\tau(r).$$

The R-algebraic representation of G is a general definition, but we will apply the following *coset R-algebraic representation* to this thesis's specific setting.

Definition 3.3 ([5] Defintion 4.2)**.** *An R-algebraic representation of G is said to be a **coset R-algebraic representation** if it is isomorphic as a R-algebraic representation to H/J for some algebraic subgroup $J < H$, and N corresponds to a subgroup of $N_H(J)/J$ which acts on H/J as described above. In other words, a coset R algebraic representation consists of the following:*

• *the k-algebraic group $N_H(J)/J$*

• *a $k-(H \times N_H(J)/J)$-algebraic variety H/J, regarded as a left H, right $N_H(J)/J$ space, on which the $N_H(J)/J$-action is faithful*

• *a homomorphism $\tau : R \to N_H(J)/J$ with a Zariski-dense image, and*

• *an associated algebraic representation of the $\Gamma \times R$-space G on H/J, where Γ acts on the left and R acts on the right of G. In other words, there must be a Haar almost everywhere defined measurable map $\phi : G \to H/J$ so that for almost every $g \in G$ and every $\gamma \in \Gamma$ and $r \in R$ we have*

$$\phi(\gamma g r) = \rho(\gamma)\phi(g)\tau(r).$$

This collection

$$H/J, \qquad \tau : P \to N_H(J)/J, \qquad \phi : G \to H/J$$

where $P < G$ is the maximal parabolic subgroup of G is precisely the Bader–Furman construction we will use to prove Theorem 1.8. Coset R-algebraic representations are well understood, and we have a number of helpful theorems that will be used in our proofs. We begin with a theorem that shows these coset R-algebraic representations do actually exist in our setting.

Theorem 3.4 ([5] Theorem 4.3)**.** *Suppose the R-action on G/Γ is weakly mixing. Then the category of R-algebraic representations of G has an initial object and this initial object is a coset R-algebraic representation.*

The following theorem allows us to extend from a coset R-algebraic representation to a coset T-algebraic representation, where $T = N_G(R)$ is the normalizer of R in G.

Theorem 3.5 ([5] Theorem 4.6)**.** *Suppose the action of R on G/Γ is weakly mixing and let H/J, $\tau : R \to N < N_H(J)/J$, and $\phi : G \to H/J$ be an initial object in the category of R-algebraic representations of G, which we know exists by Theorem 3.4. Then the map $\tau : R \to N_H(J)/J$ extends to the normalizer $T = N_G(R)$ of R in G, and the map ϕ can be seen as T-algebraic representations of R. More precisely, there is a continuous homomorphism $\widetilde{\tau} : T \to N_H(J)/J$ satisfying $\widetilde{\tau}|_R = \tau$ so that the data*

$$H/J, \qquad \widetilde{\tau} : T \to \overline{N}, \qquad \phi : G \to H/J$$

forms a T-algebraic coset representation, where $\overline{N}$ represents the Zariski closure of $\tau(T)$ in $N_H(J)/J$. Moreover, this T-algebraic coset representation is an initial object in the category of T-algebraic representations.

3.2 Proof that ρ extends when (G, H, J) is not an incompatibility triple.

For the rest of this section, let $G = \mathrm{SO}^\circ(3,1)$, $H = \mathrm{SO}^\circ(4,1)$, $P < G$ be the maximal parabolic subgroup, and $W < G$ be $\mathrm{SO}^\circ(2,1)$. Let $U < P$ be the unipotent radical, and $U' = U \cap W$.

Proposition 3.6 ([2] Proposition 3.4, [6] Theorem 2.1)**.** *Suppose there exists an infinite family of distinct conjugacy classes of maximal finite covolume totally geodesic surface subgroups* $\{\Gamma_i\}$ *so that* $\Gamma_i < \Gamma$ *and* $\rho(\Gamma_i)$ *is not Zariski dense for infinitely many* i*. Then there is a* k*-rational faithful irreducible representation* $H \to \mathrm{SL}(V)$ *on a* k*-vector space* V *and a* W*-invariant measure on* $(G \times \mathbb{P}(V))/\Gamma$ *that projects to Haar measure on* G/Γ*.*

The rest of this section relies on having the W-invariant measure ν on the bundle $(G \times \mathbb{P}(V))/\Gamma$ that projects to Haar measure on G/Γ, as described in Proposition 3.6.

Proposition 3.7 ([2] Prop 4.1)**.** *There exists a proper* k*-algebraic subgroup* $L < H$ *and a measurable* W*-invariant,* Γ*-equivariant map* $\psi : G \to H/L$*. We can also view* ψ *as a measurable* Γ*-map from* $W\backslash G$ *to* H/L*.*

The above proposition is critical— we will show that the gate ϕ in our coset U-algebraic representation must factor through this ψ. This eventually allows us to conclude that ψ is essentially constant, as we will show it is both U- and W-invariant.

Lemma 3.8 ([2] Lemma 4.4)**.** *Suppose the action of* R *on* G/Γ *is weakly mixing. Then the initial objects in the category of* R*-algebraic representations and the category of* $N_G(R)$*-algebraic representations have the same map* $\phi : G \to H/J$*. By iterating, we have that the initial object in the category of* R*-algebraic representations and the initial object in the category of* $N_G(N_G(\dots(N_G(R))\dots))$*-algebraic representations have the same map* $\phi : G \to H/J$*.*

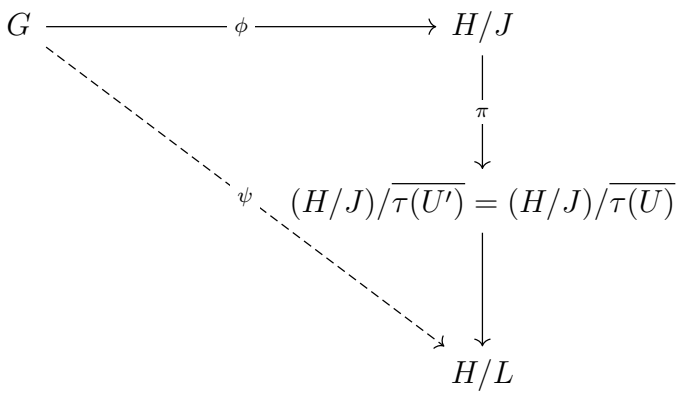
Lemma 3.8 is an extension of Bader–Furman's Theorem 3.5. This will allow us to transition from looking at coset U-algebraic representations to coset P-algebraic representations. This move gives us the map $\tau : P \to N_H(J)/J$, which we will use to build the extension of ρ from $G \to H$.

Lemma 3.9. *Suppose, under the hypotheses of Theorem 1.8, (G, H, J) is **not** an incompatibility triple. Then the map $\rho : \Gamma \to H$ extends to a continuous homomorphism $\widetilde{\rho} : G \to H$.*

Proof. The following proof is derived from Section 4.3 in [2]. Notice the action of G on G/Γ is mixing by the Howe–Moore theorem. Specifically, we know the action of every noncompact subgroup of G is weakly mixing on G/Γ.

Consider $R = U'$, which is noncompact. Given an initial object in the category of U'-algebraic representations on G, by Theorem 3.4 there is a subgroup $J < H$ so that $\phi : G \to H/J$ is that initial object, meaning it is $(U' \times \Gamma)$-equivariant for a continuous homomorphism $\tau : U' \to N_H(J)/J$. Since $U' \triangleleft U \triangleleft P$, then Theorem 3.5 implies τ extends to a continuous homomorphism $\widetilde{\tau} : P \to N_H(J)/J$, making ϕ an initial object in the category of P-algebraic representations of G.

Suppose J is nontrivial. Since (G, H, J) is a compatibility triple, we know that the Zariski closures satisfy $\overline{\widetilde{\tau}(U')} = \overline{\widetilde{\tau}(U)}$. Let $L < H$ and $\psi : G \to H/L$ be as described in Proposition 3.7. Since $U' < W$, then the W-invariant map ψ is also U'-invariant. This tells us that ψ factors not only through ϕ, but also through the following maps



where $\overline{\tau(U')}$ and $\overline{\tau(U)}$ represent the Zariski closures of $\tau(U')$ and $\tau(U)$ respectively.

Compatibility tells us $\overline{\tau(U')} = \overline{\tau(U)}$, which means π is not only U'-invariant but also U-invariant. Since ϕ is U-equivariant, we get that ψ must be U-invariant. Since ψ is also W-invariant, and $G = \langle U, W \rangle$, we get that $\psi : G \to H/L$ is an essentially constant Γ-equivariant map. This tells us that $\rho(\Gamma)$ has a fixed point on H/L, meaning $\rho(\Gamma)$ is contained in a conjugate of L. Since $\rho(\Gamma)$ is Zariski dense in H, this is a contradiction. Thus J must be trivial.

Since J is trivial, then the map

$$\tau : P \to H$$

now has codomain H, since $H/J \cong H$. Let Ψ be the $(P \times \Gamma)$-map

$$\Psi : G \to H,$$

as introduced in Definition 3.2. Since $A < P$, then Ψ must be A-equivariant via the homomorphism $\tau|_A$, and thus must be an initial object in the category of A-algebraic representations by 3.8. Then $\tau|_A$ extends to the homomorphism

$$\tau' : N_G(A) \to H,$$

where $N_G(A)$ represents the normalizer of A in G.

Again by [2], we know that $\langle P, N_G(A)\rangle = G$. Since Ψ is equivariant for both P and $N_G(A)$, Theorem 1.3 and Proposition 1.5 from [5] tells us that $\rho : \Gamma \to H$ must extend to $\widetilde{\rho} : G \to H$. □

Theorem 3.10. *Suppose $\rho : \Gamma \to H$ extends to $\widetilde{\rho} : G \to H$. Then ρ cannot be a Zariski-dense representation.*

Proof. If ρ is a Zariski-dense representation, then $\overline{\rho(\Gamma)} = H$. Since $G = \mathrm{SO}^\circ(3,1)$, then $\widetilde{\rho}(G)$ is contained in some copy of $\mathrm{SO}^\circ(3,1) \lneq H$. Since $\rho(\Gamma) < \widetilde{\rho}(G)$, then $\overline{\rho(\Gamma)}$ must also be in that copy of $\mathrm{SO}^\circ(3,1)$. Since $\mathrm{SO}(3,1)$ is almost simple and ρ is a continuous homomorphism, this means $\overline{\rho(\Gamma)}$ cannot be Zariski dense in H, contradicting Zariski-density of ρ. □

CHAPTER 4

WHEN (G, H, J) IS AN INCOMPATIBILITY TRIPLE, THE MAP ρ DOES NOT EXIST.

For the rest of this paper, let $G := \mathrm{SO}^\circ(3,1)$, $\Gamma < G$ be a lattice, and $H := \mathrm{SO}^\circ(4,1)$.

4.1 There is only one J that gives an incompatibility triple (G, H, J)

Theorem 4.1. *Up to finite index, there is only one J that gives an incompatibility triple (G, H, J), and that J has connected component of the identity*

$$J_0 = \left\{ \begin{pmatrix} 1 & -t\boldsymbol{e}_1^\top & \frac{-t^2}{2} \\ & \mathrm{Id} & t\boldsymbol{e}_1 \\ & & 1 \end{pmatrix} : \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}. \tag{4.1}$$

First we fix some notation used in the proof of Theorem 4.1. Let (G, H, J) be an incompatibility triple, $U < G$ be the unipotent radical, and $\tau : P \to N_H(J)$. Then there exists some $U' < U$ so that the Zariski closures $\overline{\tau(U')} \neq \overline{\tau(U)}$. This tells us that τ is nontrivial, lest $\overline{\tau(U')} = \overline{\tau(U)} = 0$.

Lemma 4.2. *The map τ is almost injective on P.*

Proof. This proof recapitulates the discussion in Section 3.4 of [2].

Note that U is the derived subgroup of the solvable subgroup AU of P, meaning the Zariski closure $\overline{\tau(U)}$ is contained in the Zariski closure $\overline{\tau(AU)}$, and $\overline{\tau(AU)}$ is also solvable. This tells us that $\overline{\tau(U)}$ is a nontrivial unipotent group.

Suppose there is some subgroup one parameter $U'' < U \cong \mathbb{R}^2$ so that $\tau(U'') = \mathrm{Id}$. Then for any other one-parameter unipotent subgroup $U''' \triangleleft U$ there is some $m \in M$ so that $mU''m^{-1} = U'''$, which tells us that $\tau(U''') = \mathrm{Id}$ as well. Since $m \in \mathrm{SO}(2)$, which acts on $\mathbb{R}^2$ transitively by rotations, we get that $\tau(U) = \mathrm{Id}$, which contradicts $\overline{\tau(U)}$ being nontrivial. Hence τ is injective on U.

Note that MA acts on $U \cong \mathbb{R}^2$ by homotheties and is transitive on lines $\mathbb{R}^2 \setminus \{0\}$. This tells us that if $MA \leq \ker(\tau)$, then τ would be trivial on U, which is a contradiction because we assumed τ is injective on U. Since $M = \mathrm{SO}(2)$ is abelian and τ is a continuous homomorphism, no subgroups of M can be in $\ker(\tau)$. Suppose $A \subset \ker(\tau)$. Then for all $u \in U$, we have $u = \lambda u$ for all $\lambda \in \mathbb{R}^*$, which tells us that $u = 0$. Then $U \subset \ker(\tau)$ which is a contradiction. Hence τ is injective on AMU, meaning it is almost injective on P. It is possible for τ to be trivial on the $\pm$ part of A, but that is the finite group $\mathbb{Z}/2\mathbb{Z}$. □

4.1.1 *Dimension Count Arguments*

From this point on, when talking about dimension it will always be the real dimension.

Lemma 4.3. *The real dimension of J is in the set $\{0, 1, 2, 3\}$.*

Proof. By Lemma 2.1, we know

$$\dim(P) = \dim(\mathbb{R}^*) + \dim(\mathrm{SO}(2)) + \dim(\mathbb{R}^2) = 1 + 1 + 2 = 4,$$

and

$$\dim(Q) = \dim(\mathbb{R}^*) + \dim(\mathrm{SO}(3)) + \dim(\mathbb{R}^3) = 1 + 3 + 3 = 7.$$

Since τ has finite kernel by Lemma 4.2, then $\dim(P) = \dim(\tau(P)) = 4$. Let $\widetilde{P} < H$ be the preimage of $\tau(P)$ before quotienting out by J:

$$\begin{array}{ccc} & & N_H(J) > \widetilde{P} \triangleright J \\ & & \Big\downarrow \\ P & \xrightarrow[\tau]{} & N_H(J)/J > \tau(P) \end{array} .$$

Due to the above, we have $7 \geq \dim(\widetilde{P}) = \dim(\tau(P)) + \dim(J) = 4 + \dim(J)$. Ergo, $\dim(J) \in \{0, 1, 2, 3\}$. □

Corollary 4.4. *The following are subgroups of Q:*

$$\widetilde{P} \leq N_H(J) \leq Q$$

meaning

$$\dim(\widetilde{P}) \leq \dim(N_H(J)) \leq 7.$$

Recall from Chapter 2 that we identify A, M, and U with their obvious embeddings into P, as well as A, L, and V with their obvious embeddings into Q.

Proposition 4.5. $\dim(J) \neq 0$.

Proof. Suppose $J < Q = (\pm A)LV$ is finite. Since there are no finite subgroups of A or V, we know that $J < \pm L$. By Theorem 19.2 in [1], the following are the finite subgroups of $L = \mathrm{SO}(3)$:

- cyclic group of order n

- dihedral group of order $2n$
- alternating group A_4 (order 12)
- symmetric group S_4 (order 24)
- alternating group A_5 (order 60).

This tells us that J can be any of the above groups, or any of the above groups cross $\mathbb{Z}/2\mathbb{Z}$, since we are considering finite subgroups of $\pm L$— note that by definition, $\pm$ and L commute.

Suppose J is cyclic. Then up to conjugation, J acts on $\mathbb{R}^3$ by rotation around the z-axis. Then the normalizer of J will be the homotheties that preserve this J-action, which are are rotation, vertical translation, and scaling. Since those actions all commute with each other, we get that

$$N_H(J) = \mathbb{R} \times \mathbb{R}^* \times \mathrm{SO}(2),$$

where the $\mathbb{R}$ represents vertical translation, $\mathbb{R}^*$ represents scaling, and the $\mathrm{SO}(2)$ factor gives rotations. That tells us that $\dim(N_H(J)) = 1 + 1 + 1 = 3 < 4 = \dim(P)$, which is a contradiction.

Similarly, when J is dihedral then it can be conjugated so that it acts on $\mathbb{R}^3$ by rotation about the z-axis and by reflections. The only homotheties that conjugate with this action are rotation and scaling — vertical translation does not preserve the reflections. This indicates that the normalizer for this J is contained in the normalizer for the case when J is cyclic, so again we get $\dim(N_H(J)) < \dim(P)$.

Suppose J is A_4, S_4, or A_5. Then $N_H(J)$ acts on $\mathbb{R}^3$ with quotient a platonic solid that preserves lines through vertices, allowing scaling along those lines. This means that $N_H(J)$ acts on $\mathbb{R}^3$ just by scaling, so again $\dim(N_H(J)) < \dim(P)$.

Suppose J is any of the above groups cross $\mathbb{Z}/2\mathbb{Z}$. Then $N_H(J)$ is even smaller than the above normalizers, so we will get the same contradiction. Hence $\dim(J) \neq 0$. □

Claim 4.6. *If* $\dim(J) \geq 1$*, then* J *contains a unipotent element.*

Proof. In this proof we will consider the upper triangular matrices of Q. Since the only upper triangular matrices in $\mathrm{SO}(3)$ are $\{\pm \mathrm{Id}\}$, then by Lemma 2.1 the upper triangular subgroup of Q is:

$$Q^{\mathrm{UT}} := \left\{ \begin{pmatrix} \varepsilon\lambda & -v^T & \frac{\varepsilon\lambda^{-1}|v|^2}{2} \\ 0 & \pm \mathrm{Id} & \varepsilon\lambda^{-1} v \\ 0 & 0 & \varepsilon\lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^*, v \in \mathbb{R}^3 \right\}.$$

Note that $\dim(Q^{\mathrm{UT}}) = 1 + 3 = 4$. By Corollary 2.3, we know $N_H(J), Q^{\mathrm{UT}} \leq Q$ and $\dim(Q) = 7$, so $\dim(N_H(J) \cap Q^{\mathrm{UT}}) \geq (5 + 4) - 7 = 2$.

Let $m \in N_H(J) \cap Q^{\mathrm{UT}}$. Then by matrix multiplication, we see that $m^2 =: n \in N_H(J) \cap Q^{\mathrm{UT}}$ must look like

$$n := \left\{ \begin{pmatrix} \mu & -w^T & -\mu\frac{|w|^2}{2} \\ 0 & \mathrm{Id} & \mu^{-1} w \\ 0 & 0 & \mu^{-1} \end{pmatrix} : \mu \in \mathbb{R}^*_{>0}, w \in \mathbb{R}^3 \right\}.$$

Let $g \in J < Q$. Then g must look like

$$g = \left\{ \begin{pmatrix} \delta\lambda & -v^\top B & -\delta\lambda\frac{|v|^2}{2} \\ 0 & B & \delta\lambda^{-1}v \\ 0 & 0 & \delta\lambda^{-1} \end{pmatrix} : \delta \in \{\pm 1\}, \lambda \in \mathbb{R}^*, B \in \mathrm{SO}(3), v \in \mathbb{R}^3 \right\}.$$

Since $g \in J$, we know $g^2 \in J$, and the $(1,1)$ and $(3,3)$ entries of g^2 are $\delta^2\lambda^2 = \lambda^2$ and $\delta^2\lambda^{-2} = \lambda^{-2}$ respectively, so without loss off generality assume $\delta = 1$.

Commutator Calculation 4:

$$
\begin{aligned}
[n,g] := &\begin{pmatrix} \mu & -w^T & -\mu\frac{|w|^2}{2} \\ 0 & \mathrm{Id} & \mu^{-1}w \\ 0 & 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \lambda & -v^\top B & -\lambda\frac{|v|^2}{2} \\ 0 & B & \lambda^{-1}v \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \\
&\cdot \begin{pmatrix} \mu^{-1} & \mu^{-1}w^\top & * \\ 0 & \mathrm{Id} & -w \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \lambda^{-1} & \lambda^{-1}v^\top & * \\ 0 & B^{-1} & -B^{-1}v \\ 0 & 0 & \lambda \end{pmatrix} \\
= &\begin{pmatrix} \mu\lambda & -(v^\top + w^\top)B & * \\ & B & \lambda^{-1}(v+\mu^{-1}w) \\ & & \mu^{-1}\lambda^{-1} \end{pmatrix} \\
&\cdot \begin{pmatrix} \mu^{-1}\lambda^{-1} & \mu^{-1}(\lambda^{-1}v^\top + w^\top B^{-1}) & * \\ & B^{-1} & -B^{-1}v - \lambda w \\ & & \mu\lambda \end{pmatrix} \\
= &\begin{pmatrix} \mu\lambda\mu^{-1}\lambda^{-1} & (\lambda B^{-1} - \mathrm{Id})w^\top & * \\ 0 & \mathrm{Id}\cdot B\cdot \mathrm{Id}\cdot B^{-1} & (\mu-1)v + (\mathrm{Id} - \lambda B)w \\ 0 & 0 & \mu^{-1}\lambda^{-1}\mu\lambda \end{pmatrix} \\
= &\begin{pmatrix} 1 & (\lambda B^{-1} - \mathrm{Id})w^\top & * \\ 0 & \mathrm{Id} & (\mu-1)v + (\mathrm{Id} - \lambda B)w \\ 0 & 0 & 1 \end{pmatrix} \in J.
\end{aligned}
$$

Therefore J contains a nontrivial unipotent element. □

Claim 4.7. *There is only one 6-dimensional subgroup Y of Q, and it is the following:*

$$Y = \left\{ \begin{pmatrix} \pm 1 & -v^\top L & \frac{-|v|^2}{2} \\ 0 & L & v \\ 0 & 0 & \pm 1 \end{pmatrix} : L \in \mathrm{SO}(3), v \in \mathbb{R}^3 \right\} = \pm LV,$$

or Y is the index 2 subgroup

$$Y = \left\{ \begin{pmatrix} 1 & -v^\top L & \frac{-|v|^2}{2} \\ 0 & L & v \\ 0 & 0 & 1 \end{pmatrix} : L \in \mathrm{SO}(3), v \in \mathbb{R}^3 \right\} = LV.$$

Proof. By Lemma 2.1, we know

$$Q = \left\{ \begin{pmatrix} \varepsilon\mu & -v^\top L_0 & -\frac{|v|^2}{2} \\ & L_0 & v \\ & & \varepsilon\mu^{-1} \end{pmatrix} : \begin{array}{c} \varepsilon \in \{\pm 1\}, \\ \mu \in A := \mathbb{R}^+, \\ L_0 \in L := \mathrm{SO}(3), \\ v \in V := \mathbb{R}^3 \end{array} \right\} = (\pm A)LV,$$

so $\dim(Q) = 7$. By [1] and [12], we know L only has 0-, 1-, or 3-dimensional subgroups, since two rotations, each around distinct axes, will give a rotation around the third axis, excluding the case of a 2-dimensional subgroup. This means L itself must be contained in any 6-dimensional subgroup of Q, as only including its 0- or 1-dimensional subgroups will result in a group of dimension less than 6. Since V is 3-dimensional, we know $V \cap Y \neq \emptyset$. Since L acts transitively on V, then the entirety of V must be in Y. This tells us that Y is one of the matrices in Claim 4.7. □

Proposition 4.8. $\dim(J) \neq 2$.

Proof. We know that $J \leq \widetilde{P} \leq N_H(J) \leq Q$, and that $\dim(\widetilde{P}) = \dim(J) + \dim(\tau(P)) = 2 + 4 = 6$. Since $\widetilde{P} < Q$ is 6-dimensional, by Claim 4.7 we know that

$$\widetilde{P} = \left\{ \begin{pmatrix} \pm 1 & -v^\top L & \frac{-|v|^2}{2} \\ 0 & L & v \\ 0 & 0 & \pm 1 \end{pmatrix} : L \in \mathrm{SO}(3), v \in \mathbb{R}^3 \right\} = \pm LV.$$

By definition $\widetilde{P} \leq N_H(J) \leq Q$. We know that Q has no normal subgroup of dimension 2 because any normal subgroup of Q is unipotent; since $J \triangleleft N_H(J)$ and $\dim(J) = 2$, we know that $N_H(J) \lneq Q$, so $\dim(N_H(J)) = 6$. That means $N_H(J)$ virtually equals $\widetilde{P}$.

By Claim 4.6, we know that any normal subgroup of Q contains at least one nontrivial unipotent element, and by transitivity of the L-action on V, we know a normal subgroup of Q must contain all of V. Since $\dim(V) = 3$, this tells us that any normal subgroup of Q must be at least 3-dimensional, so importantly Q does not have any 2-dimensional normal subgroup.

However, by definition $J \triangleleft N_H(J)$ and $N_H(J)$ is virtually $\widetilde{P}$. Since $L = \mathrm{SO}(3)$ also has no 2-dimensional subgroup, then $J \cap V \neq \emptyset$. That means that $V < J$ by the transitivity of the L-action mentioned above, which is a contradiction because $\dim(V) = 3 > 2 = \dim(J)$. □

Proposition 4.9. $\dim(J) \neq 3$.

Proof. Similarly to the above cases, if J is 3-dimensional, then $\dim(\widetilde{P}) = \dim(J) + \dim(\tau(P)) = 3 + 4 = 7$. Since $J \leq \widetilde{P} \leq N_H(J) \leq Q$ and $\dim(\widetilde{P}) = \dim(Q) = 7$, we know that virtually $\widetilde{P} = N_H(J) = Q$.

Again, by the commutator computation 4 we know that J must contain a unipotent element. Since M still acts on V transitively, we know that virtually $J \cong V$, so virtually

$$\tau(J) \cong \widetilde{P}/J = (AM)/(AM \cap J).$$

However, the group AM has a compact component and $AM \cap J$ is finite, which means $(AM)/(AM \cap J) \cong \mathbb{R} \times \{\text{something compact}\}$. Note that $\mathbb{R} \times \{\text{something compact}\}$ contains no noncompact, nonabelian solvable subgroup. However, $\tau(AU) < \tau(P)$ is a noncompact, nonabelian solvable group which yields a contradiction. □

Proposition 4.10. *If (G, H, J) is an incompatibility triple, then the connected component of the identity of J, denoted J_0, is the 1-parameter unipotent subgroup*

$$J_0 = \left\{ \begin{pmatrix} 1 & -t\boldsymbol{e}_1^\top & \frac{-t^2}{2} \\ & \mathrm{Id} & t\boldsymbol{e}_1 \\ & & 1 \end{pmatrix} : \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

Since $\dim(J_0) = 1$ and J_0 contains a unipotent element, then it must be conjugate to the one-parameter unipotent subgroup:

$$J_0 = \begin{pmatrix} 1 & (-x, 0, 0) & \frac{-x^2}{2} \\ 0 & \mathrm{Id} & \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

4.2 If (G, H, J) is an incompatibility triple, then ρ does not exist.

Let $J < H$ be such that (G, H, J) is an incompatibility triple. Then as shown in Equation 4.2, the connected component of the identity of J is defined as follows:

$$J_0 = \left\{ \begin{pmatrix} 1 & -t^T\mathbf{e}_1 & \frac{-t^2}{2} \\ & \mathrm{Id} & t\mathbf{e}_1 \\ & & 1 \end{pmatrix} : \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

Let P be the maximal parabolic subgroup of G and Q be the maximal parabolic subgroup of H. Recall from Lemma 2.1 that

$$P = \left\{ \begin{pmatrix} \varepsilon\lambda & -u^\top M_0^\top & \frac{\varepsilon\lambda^{-1}|u|^2}{2} \\ & M_0 & \varepsilon\lambda^{-1}u \\ & & \varepsilon\lambda^{-1} \end{pmatrix} : \begin{array}{c} \varepsilon \in \{\pm 1\}, \\ \lambda \in A := \mathbb{R}^+, \\ M_0 \in M := \mathrm{SO}(2), \\ u \in U := \mathbb{R}^2 \end{array} \right\}$$

$$= \pm AMU,$$

and

$$Q = \left\{ \begin{pmatrix} \varepsilon\lambda & -v^\top L_0^\top & \frac{\varepsilon\lambda^{-1}|v|^2}{2} \\ & L_0 & \varepsilon\lambda^{-1}v \\ & & \varepsilon\lambda^{-1} \end{pmatrix} : \begin{array}{c} \varepsilon \in \{\pm 1\}, \\ \lambda \in A := \mathbb{R}^+, \\ M_0 \in M := \mathrm{SO}(3), \\ v \in V := \mathbb{R}^3 \end{array} \right\}$$

$$= \pm ALV.$$

4.2.1 *Identifying important objects with circles on* $\mathbb{S}^2$

Let E_i represent the i^{th} coordinate axis of $\mathbb{R}^n$, and $\mathbf{e}_i$ represent the basis vector for the i^{th} coordinate.

Claim 4.11. *Let W be the stabilizer of the standard $\mathbb{H}^2$ in $\mathbb{H}^3$, defined by deleting the x_2 coordinate from $q(x)$. Then $W \cong \mathrm{O}^+(2,1)$.*

Proof. Since W stabilizes a copy of $\mathbb{H}^2 \subset \mathbb{H}^3$, we know that W must act on that hyperbolic plane by isometries of $\mathbb{H}^2$. The isometry group of $\mathbb{H}^2$ is $\mathrm{O}^+(2,1)$, so W must be isomorphic to $\mathrm{O}^+(2,1)$. Additionally, every isometry can be extend to an orientation-preserving isometry of $\mathbb{H}^3$. □

Note that $\mathrm{O}^+(2,1)$ has two connected components. Should we want to consider the collection of oriented circles on $\mathbb{S}^2$, we can look at the connected component of the identity of $\mathrm{O}^+(2,1)$. The element of $\mathrm{O}^+(2,1)$ that swaps the connected components is precisely the element that flips orientation on the stabilized $\mathbb{H}^2 \subset \mathbb{H}^3$.

Claim 4.12. *The homogeneous space G/W can be identified with the space of circles on $\mathbb{S}^2 \cong \partial\mathbb{H}^3$.*

Proof. Since W stabilizes $\mathbb{H}^2 \subset \mathbb{H}^3$, then W stabilizes $\partial\mathbb{H}^2 \subset \partial\mathbb{H}^3$. Since $\partial\mathbb{H}^2$ is a circle, then W stabilizes a circle. Because G acts transitively on circles in $\mathbb{S}^2 = \partial\mathbb{H}^3$, the orbit of $\partial\mathbb{H}^2$ is all circles in $\mathbb{S}^2$. Then by the orbit-stabilizer theorem, we know that G/W is the collection of all circles in $\mathbb{S}^2$. □

Define $u_1(t)$ to be the following one-parameter unipotent subgroup of $\mathrm{SO}^\circ(n,1)$:

$$u_1(t) := \left\{ \begin{pmatrix} 1 & -t\,\mathbf{e}_1^\top & \frac{-t^2}{2} \\ & \mathrm{Id} & t\,\mathbf{e}_1 \\ & & 1 \end{pmatrix} : t \in \mathbb{R} \right\} < J < P.$$

Note that

$$u_1(t) \cdot \infty = \infty$$

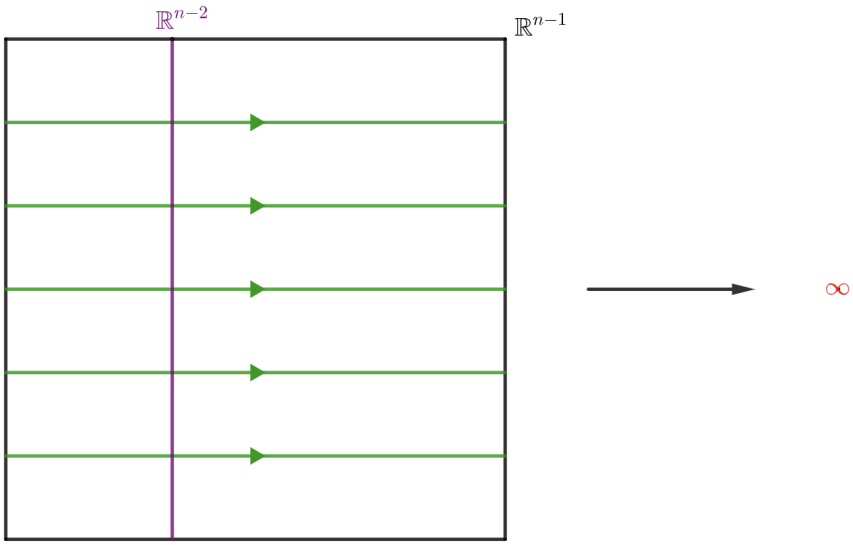


Figure 4.1: Foliation fixed by $u_1(t)$

and

$$u_1(t) \cdot \begin{pmatrix} w_0 \\ \vec{v} \\ 1 \end{pmatrix} = \begin{pmatrix} w_0 - tv_1 - \frac{t^2}{2} \\ t\,\mathbf{e}_1 + \vec{v} \\ 1 \end{pmatrix}.$$

We can see from the middle entry $t\,\mathbf{e}_1 + \vec{v}$ that $u_1(t)$ translates $\vec{v} \in \mathbb{R}^{n-2}$ towards the point at ∞, as shown in Figure 4.1, where the purple line represents the vector $\vec{v}$ and the green lines are a foliation of $\mathbb{R}^{n-1}$ that is stabilized by $u_1(t)$.

When we perform the one-point compactification which adds in the point at ∞, we get Figure 4.2, where the green foliation lines from Figure 4.1 turn into circles going through the point at ∞.

Each of these circles are the ideal boundary of a copy of $\mathbb{H}^2 \subset \mathbb{H}^3$, so by Claim 4.11 we know that each circle is stabilized by a conjugate of W in G.

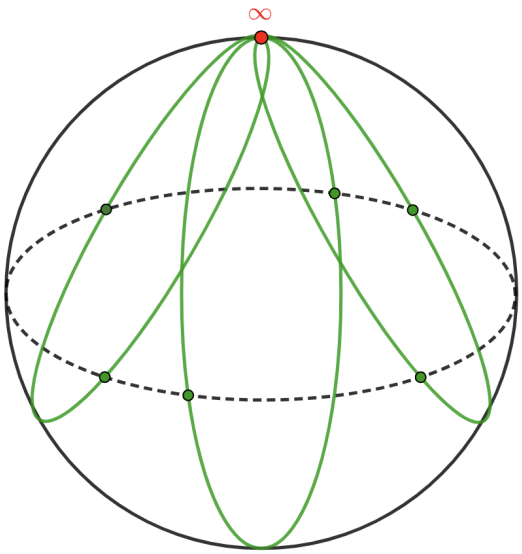


Figure 4.2: Foliation with the point at infinity added in

Claim 4.13. *Any element of G that stabilizes the pair (C, p), where C is a circle on $\mathbb{S}^2 \cong \partial\mathbb{H}^3$ and $p \in C$ is a point, acts on $\mathbb{R}^3$ by some combination of scaling, translating along E_1, or flipping the orientation of E_1.*

Proof. Suppose $g \in G$ fixes the pair (C, p), where C is a circle on $\mathbb{S}^2 \cong \partial\mathbb{H}^3$ and $p \in C$ is a point. Then we can conjugate g so that it fixes infinity, so without loss of generality assume $g \cdot \infty = \infty$. Then $g \in P$. Consider how g acts on some other point in $\partial\mathbb{H}^3$; by matrix multiplication, we know

$$g \cdot \begin{pmatrix} w_0 \\ \vec{v}_t \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ M_o\vec{v}_t + u \\ \lambda^{-1} \end{pmatrix}. \tag{4.3}$$

Recall from Equation 2.2 that we have standardized the points on $\partial\mathbb{H}^3$ to have a final entry of 1, so without loss of generality we can rescale the vector in 4.3 to be

$$\begin{pmatrix} *_2 \\ \lambda\left(M_0\vec{v}_t + u\right) \\ 1 \end{pmatrix}. \tag{4.4}$$

Suppose

$$\begin{pmatrix} * \\ \vec{v}_t \\ 1 \end{pmatrix} \in C.$$

Since g fixes C, then for all t there must be some s so that

$$g \cdot \begin{pmatrix} * \\ \vec{v}_t \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ \vec{v}_s \\ 1 \end{pmatrix}.$$

By Equation 4.4, it must be the case that

$$\lambda\left(M_0\vec{v}_t + u\right) = \vec{v}_s.$$

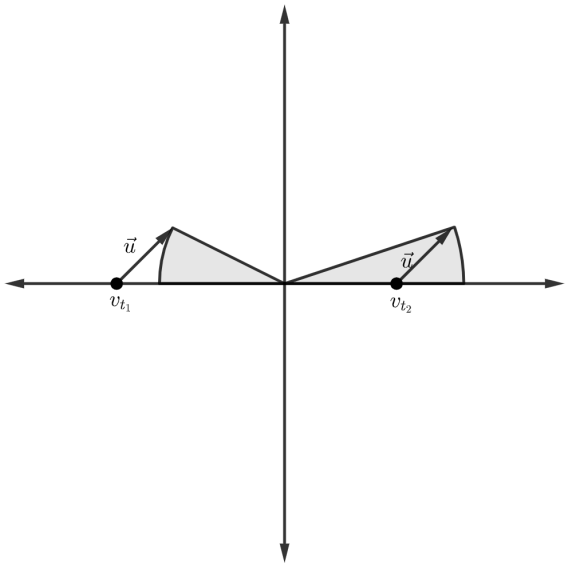


Figure 4.3: Translations at different points require different rotations to fix the x-axis

Suppose u translates $\vec{v}_t$ off of E_1. Then there exists some M_0 and some s so that $M_0\vec{v}_t + u = v_s$. However for $t_1 \neq t$, the subsequent $M_0\vec{v}_{t_1} + u$ will not make its way back to E_1; the vector $\vec{v}_{t_1} + u$ would require a different angle of rotation, as shown in Figure 4.3. Hence u cannot translate $\vec{v}_t$ off of E_1.

Since u can only translate along E_1, the only possible rotation matrices M_0 that will not move $\vec{v}_t$ off of E_1 are $\pm \mathrm{Id}$. □

Corollary 4.14. *The group*

$$\mathrm{stab}_G((C, p)) = \pm A(u_1(t))$$

where C is a circle in $\mathbb{S}^2$ and $p \in C$ is a point. Recall from Section 3 that $U' = U \cap W$.

Proof. By Claim 4.13, we know that $\mathrm{stab}_G((C,p))$ must act on $\mathbb{R}^3$ by scaling, translating along E_1, or flipping orientation of E_1. The $\pm$ flips the orientation of E_1, the group A acts by scaling, and the group $(u_1(t))$ acts by translation along E_1. For a fixed $t \in \mathbb{R}$, the group $(u_1(t))$ is generated by the matrix

$$\begin{pmatrix} 1 & -t\,\mathbf{e}_1^\top & \frac{-t^2}{2} \\ & \mathrm{Id} & t\,\mathbf{e}_1 \\ & & 1 \end{pmatrix}.$$

□

Definition 4.15. *A collection of n lines in $\mathbb{R}^n$ is called **extra-parallel** if it includes E_1 and a collection of $(n-1)$ lines, each parallel to E_1, intersecting a distinct $\{E_i\}_{i\neq 1}$. A collection of n circles are called **extra-parallel** if they are derived from the one-point compactification of a collection of extra-parallel lines. Examples of each are shown in Figure 4.4 and Figure 4.5 respectively.*

Lemma 4.16. *The following are equivalent*

$$\mathrm{SO}(n,1)/(\pm u_1(t)) \cong \{(p \in \partial\mathbb{H}^n, \{B_n\})\}, \tag{4.5}$$

where $p \in \partial\mathbb{H}^n$ is a point and $\{B_n\}$ is a collection of n extra-parallel circles going through $p = \infty$.

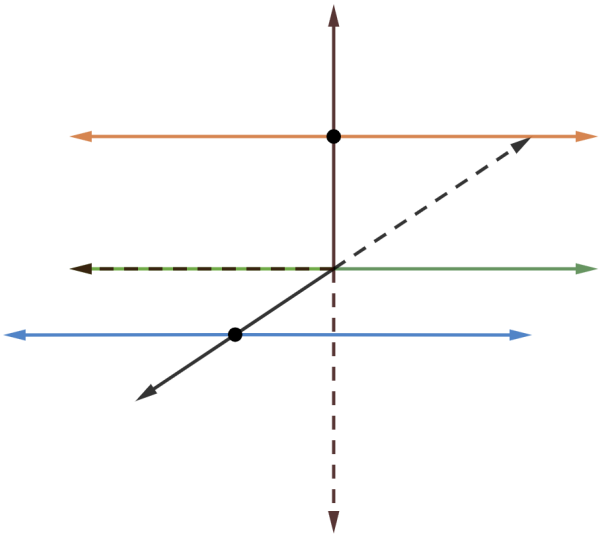


Figure 4.4: A collection of extra-parallel lines

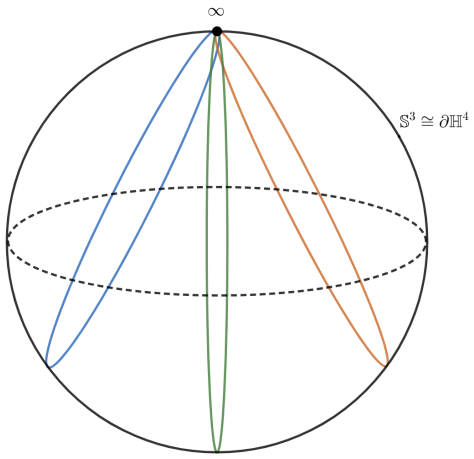


Figure 4.5: Extra-parallel lines with the point at infinity added in

Proof. By Corollary 4.14, we know that $\{u'(t)\} < \mathrm{stab}((C,p))$. Let $\{L_i\}$ be a collection of extra-parallel lines in $\mathbb{R}^{n-1}$. Then $\mathrm{stab}_G((\{L_i\},\infty))$ must act on $\mathbb{R}^{n-1}$ by either translation along E_1 or my permuting the L_i. Since there is no homothety of $\mathbb{R}^{n-1}$ that permutes a collection of extra-parallel lines, we know $\mathrm{stab}_G((\{L_i\},\infty)) = \pm u_1(t)$. Let $\{B_i\}$ be derived from the one point compactification of $\{L_i\}$, meaning that $\{B_i\}$ is a collection of extra-parallel circles in $\partial\mathbb{H}^n$– note that we still have $\pm u_1(t) = \mathrm{stab}_G(\{B_i\},p)$ where p is the point at ∞ added in with the one-point compactification. Since G acts transitively on circles and points in $\partial\mathbb{H}^n$, then G acts transitively on $\mathrm{stab}_G((\{B_i\},p))$. Thus by the orbit-stabilizer theorem we get the lemma. □

Claim 4.17. *Let P be a maximal parabolic subgroup of $G = \mathrm{SO}^\circ(n,1)$ and $W < G$ be the stabilizer of $\mathbb{H}^{n-1} \subset \mathbb{H}^n$ for a fixed isometric embedding of $\mathbb{H}^{n-1}$ into $\mathbb{H}^n$. Then we can canonically identify*

$$G/(P \cap W) \cong \{(Y,p) : Y \subset \partial\mathbb{H}^n \cong \mathbb{S}^{n-1} \textit{ an } (n-2)\textit{-sphere}, p \in Y \textit{ a point}\}.$$

Proof. Recall that P fixes the point $\infty \in \partial\mathbb{H}^n$ and W stabilizes the standard $\partial\mathbb{H}^{n-1} \subset \partial\mathbb{H}^n$, meaning $P \cap W$ must do both simultaneously. Hence on $\partial\mathbb{H}^n$, we know that $P \cap W$ stabilizes an $(n-2)$-sphere Y and the point $p \in Y$. Since G acts transitively on both points and $(n-2)$-spheres, by the orbit-stabilizer theorem we get the claim. □

Let $X := G/\pm A$ and $X' := H/J\widetilde{\tau}(\pm A)$ where

$$J\widetilde{\tau}(\pm A) = \begin{pmatrix} \varepsilon\lambda & * & * \\ & \mathrm{Id} & \varepsilon\lambda^{-1}t\mathbf{e}_1 \\ & & \varepsilon\lambda^{-1} \end{pmatrix}.$$

Claim 4.18. *The space $X = \{(C, x, y) : C$ is a circle, $x, y \in C$ distinct$\}$.*

Proof. Without loss of generality, suppose the two points on $\partial\mathbb{H}^3$ we are stabilizing are 0 and ∞. Since U acts on $\mathbb{R}^3$ by translation, it does not stabilize 0. Hence $\pm MA < P$ is the subgroup that stabilizes both 0 and ∞. By definition, we know W stabilizes a circle in $\partial\mathbb{H}^3$. Ergo the stabilizer of a circle and two points is $W \cap \pm MA = \pm A$. Since G acts transitively on circles and pairs of points, by the orbit-stabilizer theorem we get the claim. □

4.2.2 *Fiber product measures*

Definition 4.19. *Let X, Y, and Z be Lebesgue spaces with probability measures μ_X, μ_Y, and μ_Z respectively. Let $\phi : X \to Z$ and $\psi : Y \to Z$ be maps so that the pushforward measures $\phi_*\mu_X$ and $\psi_*\mu_Y$ are in the same measure class as μ_Z. Then the* ***fiber product*** *is*

$$X \times_Z Y = \{(x, y) \in X \times Y : \phi(x) = \psi(y) \subset X \times Y\}.$$

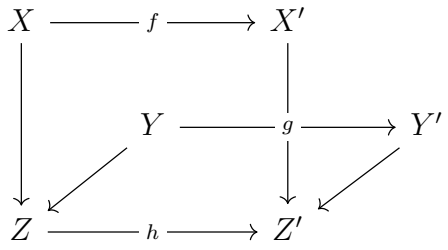
Definition 4.20. *For μ_Z-almost every $z \in Z$, disintegration gives measures $\nu_{X,z}$ and $\nu_{Y,z}$ on almost every fiber $\phi^{-1}(z) \subset X$ and $\psi^{-1}(z) \subset Y$ so that*

$$\mu_X = \int_Z \mu_{X,z} d\mu_Z \qquad\qquad \mu_Y = \int_Z \nu_{Y,z} d\mu_Z.$$

*This defines the **fiber product measure***

$$\mu_X \times_Z \mu_Y = \int_Z (\nu_{X,z} \times \nu_{Y,z}) d\mu_Z.$$

Lemma 4.21 ([3])**.** *Consider Lebesgue spaces X, Y, and Z endowed with morphisms $X \to Z$ and $Y \to Z$. Assume X', Y', and Z' are standard Borel spaces endowed with Borel maps $X' \to Z'$ and $Y' \to Z'$, and moreover that there are almost everywhere defined measurable maps $f : X \to X'$, $g : Y \to Y'$, and $h : Z \to Z'$ such that the following 3-dimensional diagram commutes:*



Consider the product map $f \times g : X \times Y \to X' \times Y'$ and the fiber product measure class $\mu_x \times_Z \mu_Y$ on $X \times_Z Y \subset X \times Y$. Then for $\mu_X \times_Z \mu_Y$ almost every (x, y), one has that $(f \times g)(x, y) \in X' \times_{Z'} Y'$.

4.2.3 *Using limit sets to identify ρ*

Definition 4.22. *A representation $\rho : \Gamma \to H$ is called **convex cocompact** if it has finite kernel and its image acts convex cocompactly on a properly convex domain in real projective space.*

Definition 4.23. *Let $\Gamma < G$ be a finitely generated torsion-free Kleinian group. Then $\Lambda < \mathbb{S}^2$ is the **limit set** of Γ, meaning Λ is the set of accumulations of an orbit of Γ on $\partial\mathbb{H}^3$. In other words,*

$$\Lambda = \overline{\Gamma \cdot o} \cap \partial\mathbb{H}^3,$$

for $o \in \mathbb{H}^3$.

Let $\rho : \Gamma \to H$ be a Zariski-dense convex cocompact representation and $\Lambda \subset \mathbb{S}^2$ be the limit set of Γ. By [13] we know there is a unique ρ-equivariant continuous embedding $\overline{\phi} : \Lambda \to \mathbb{S}^3$ called the **ρ-boundary map**.

Definition 4.24. *A circle $C \subset \mathbb{S}^n$ is **Λ-doubly stable** if for any sequence of circles C_k converging to C, we have*

$$\#\limsup(C_k \cap \Lambda) \geq 2.$$

Lemma 4.25 ([10] Lemma 4.2)**.** *For $\Gamma < G = \mathrm{SO}^\circ(3,1)$ a lattice, there exists a Λ-doubly stable circle.*

Definition 4.26. *Let $\Gamma < \mathrm{SO}^\circ(n,1)$ be a lattice and $\rho : \Gamma \to \mathrm{SO}^\circ(m,1)$ be a Zariski-dense discrete faithful representation. Let $f : \Lambda \to \mathbb{S}^{m-1}$ be the ρ-boundary map.*

$$\Lambda_f := \bigcup \left\{ C \cap \Lambda : \begin{array}{c} C \subset \mathbb{S}^2 = G/P \text{ a circle and} \\ f(C \cap \Lambda) \subset \text{ a 2-sphere in } \mathbb{S}^{m-1} \end{array} \right\}.$$

Definition 4.27. *Let $\mathcal{H}^\delta$ denote the δ-**dimensional Hausdorff measure** on $\mathbb{S}^n$.*

Definition 4.28. *Let $\Lambda \subset \mathbb{S}^n$. Then the **Hausdorff dimension** of Λ is defined as follows:*

$$\dim_{\textit{Haus}}(\Lambda) = \begin{cases} 0 \text{ if } H^\delta(\Lambda) = 0 \text{ for all } \delta \\ \\ \sup\{\delta : H^\delta(\Lambda) = \infty\} \text{ else.} \end{cases}$$

Claim 4.29. *The Hausdorff dimension $\dim_{\textit{Haus}}(\Gamma)$ of a lattice $\Gamma < G$ is 2.*

Theorem 4.30 (Theorem 5.1 in [9])**.** *Suppose there exists a Λ-doubly stable circle. Then either*

$$\Lambda_f = \Lambda \qquad \textit{or} \qquad \mathcal{H}^\delta(\Lambda_f) = 0.$$

In the former case, we have $n = m$ and f extends to some $g \in$ Möb$(\mathbb{S}^n)$ and ρ is conjugation by g.

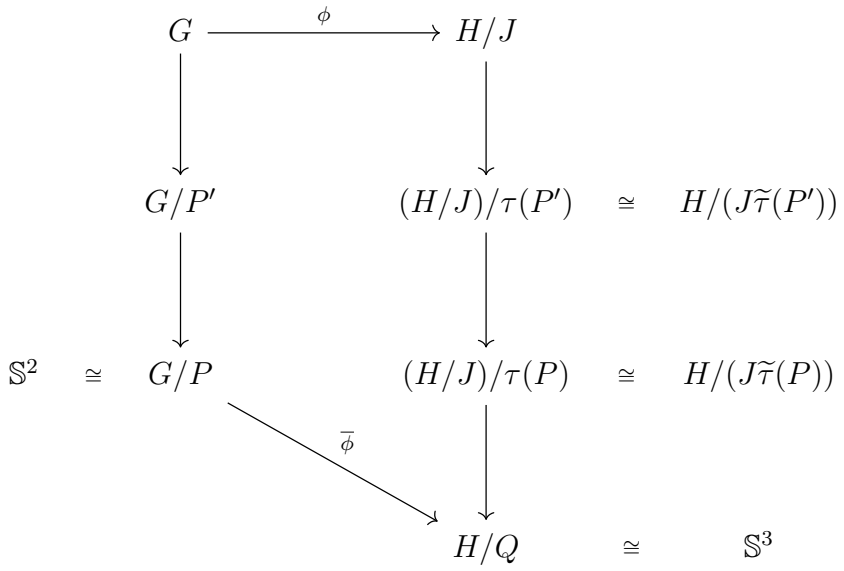
By Claim 4.29, the above theorem is claiming that $\mathcal{H}^2(\Lambda_f) = 0$. We will show that Λ_f has full measure, meaning $\Lambda_f = \Lambda$. This contradiction is precisely what will give us Theorem 1.8.

Theorem 4.31. *Let ρ have the same conditions as described in Theorem 1.8. Then the ρ-boundary map $\overline{\phi} : \mathbb{S}^2 \to \mathbb{S}^3$ maps almost every circle in $\mathbb{S}^2$ into a 2-sphere in $\mathbb{S}^3$.*

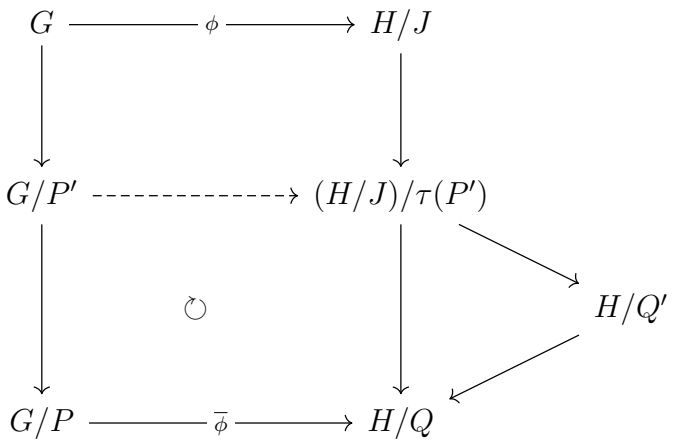
Proof of Theorem 4.31. Let $P' := P \cap W_1$ and $Q := Q \cap W_2$, where W_1 is the stabilizer of the standard $\mathbb{H}^2$ in $\mathbb{H}^3$ and W_2 is the stabilizer of the standard $\mathbb{H}^3$ in $\mathbb{H}^4$. Let J be as defined in Proposition 4.10. Let $\phi : G \to H/J$ be an initial object in the category of P-algebraic coset representations, which induces the ρ-equivariant boundary map

$$\overline{\phi} : G/P \to H/Q$$

as shown by factoring through the following commutative diagram.



Recall from Claim 4.17 that $P' \cong \{(C, p)\}$ where C is a circle and $p \in C$ is a point, and $Q' \cong \{(S, q)\}$ where S is a 2-sphere and $q \in S$ is a point. Since $J\widetilde{\tau}(P') < Q'$, we additionally get the following commutative diagram.



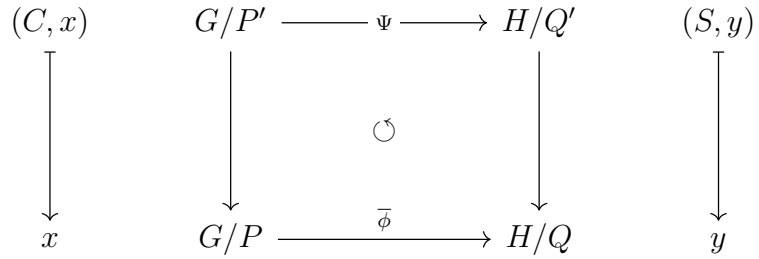
We now get a map

$$\Psi : G/P' \to H/Q',$$

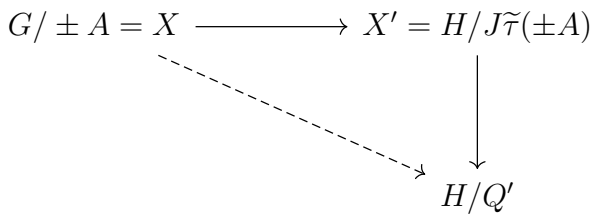
defined by

$$(C, x) \longmapsto \big(\Psi_x(C), \overline{\phi}(x)\big)$$

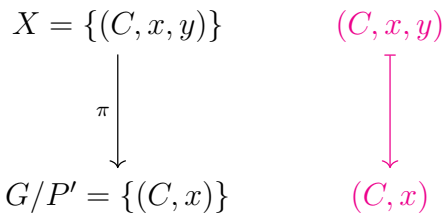
where $\Psi_x(C)$ is a sphere containing the point $\overline{\phi}(x)$. We can see that x necessarily maps to $\overline{\phi}(x)$ by chasing through the following section of the above commutative diagram.



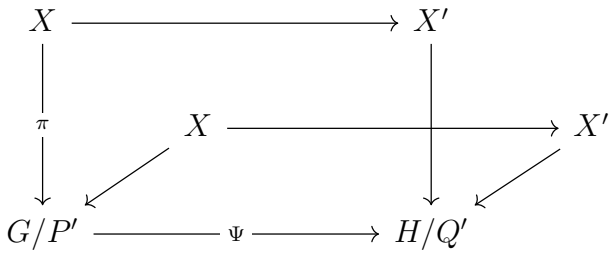
Since $J\widetilde{\tau}(\pm A) < Q'$, we get a map from $\hat{\Psi} : X \to H/Q'$ by factoring through X'.



Also consider the following map $\pi : X \to G/P'$.



Lemma 4.21 and the above map π gives the following 3-dimensional commutative diagram.



Following the above commutative diagram, we get a map $X \to H/Q'$ defined as follows

$$(C, x, \cdot) \longmapsto \left(\hat{\Psi}(C), x\right).$$

Since the fiber product measure is supported on the subset

$$\{((C, x, \cdot), (C, x, \cdot))\} < X \times_{G/P'} X,$$

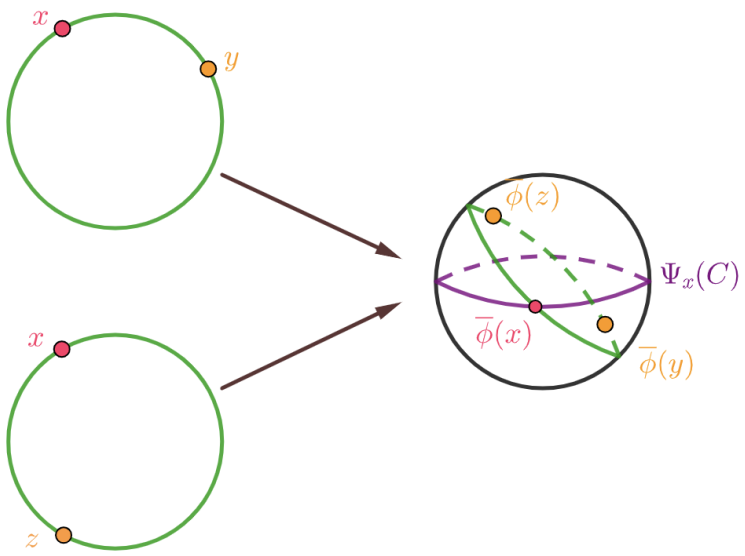


Figure 4.6: Each circle and pair of points $(C, x, \cdot) \in X$ and gets mapped into one 2-sphere in H/Q'.

we can apply Lemma 4.21 to the map

$$X \times_{G/P'} X \to X' \times_{H/Q'} X'$$

where

$$((C, x, y), (C, x, z)) \mapsto ((\hat{\Psi}_{x,y}(C), \overline{\phi}(x), \overline{\phi}(y)), (\hat{\Psi}_{x,z}(C), \overline{\phi}(x), \overline{\phi}(z)))$$

with $\hat{\Psi}_{x,t}(C)$ being a sphere containing $\overline{\phi}(x)$ and $\overline{\phi}(t)$ for $t \in \{y, z\}$.

As shown in Figure 4, we can see that since $\overline{\phi}(x)$, $\overline{\phi}(y)$, and $\overline{\phi}(z)$ are all contained in the same 2-sphere, that $\hat{\Psi}_{x,y}(C) = \hat{\Psi}_{x,z}(C)$. □

CHAPTER 5

FINAL PROOFS OF RESULTS

Proof of Theorem 1.9. Let $J < H$. Suppose $J = \{1\}$. Then by Lemma 3.9, we know that $\rho : \Gamma \to H$ extends to a continuous homomorphism $\widetilde{\rho} : G \to H$.

Suppose (G, H, J) is a compatibility triple. Then from the proof of Lemma 3.9, we know $J = \{1\}$, and thus is covered by the above case.

Suppose (G, H, J) is an incompatibility triple. Then by Theorem 4.31, we know that the ρ-boundary map $\overline{\phi} : \mathbb{S}^2 \to \mathbb{S}^3$ maps almost every circle in $\mathbb{S}^2$ into a 2-sphere in $\mathbb{S}^3$. Then the Hausdorff measure $\mathcal{H}^2(\Lambda_{\overline{\phi}}) \neq 0$. Then by Theorem 4.30, we know $\Lambda_{\overline{\phi}} = \Lambda$ and $\rho : \Gamma \to \mathrm{SO}^\circ(3, 1)$. However, we assumed that $\rho : \Gamma \to \mathrm{SO}^\circ(4, 1)$, yielding a contradiction. Thus no such ρ exists when (G, H, J) is an incompatibility triple, so every representation $\rho : \Gamma \to H$ extends to a continuous homomorphism $\widetilde{\rho} : G \to H$. □

Proof of Theorem 1.8. Let $\{\Gamma_i\}$ be an infinite family of distinct conjugacy classes of maximal finite covolume totally geodesic surface subgroups so that $\Gamma_i < \Gamma$ and $\rho(\Gamma_i)$ is not Zariski-dense in H. Then by Proposition 3.6, we satisfy the conditions of Theorem 1.9. Thus ρ extends to a continuous homomorphism $\widetilde{\rho} : G \to H$. Then by Theorem 3.10, we know ρ cannot be a Zariski-dense representation, yielding a contradiction. □

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