

# 3-Coloring in Time $\mathcal{O}(1.3289^n)$

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## Abstract

We consider worst case time bounds for NP-complete problems including 3-SAT, 3-coloring, 3-edge-coloring, and 3-list-coloring. Our algorithms are based on a constraint satisfaction (CSP) formulation of these problems. 3-SAT is equivalent to  $(2, 3)$ -CSP while the other problems above are special cases of  $(3, 2)$ -CSP; there is also a natural duality transformation from  $(a, b)$ -CSP to  $(b, a)$ -CSP. We give a fast algorithm for  $(3, 2)$ -CSP and use it to improve the time bounds for solving the other problems listed above. Our techniques involve a mixture of Davis-Putnam-style backtracking with more sophisticated matching and network flow based ideas.

## 1 Introduction

There are many known NP-complete problems including such important graph theoretic problems as coloring and independent sets. Unless  $P=NP$ , we know that no polynomial time algorithm for these problems can exist, but that does not obviate the need to solve them as efficiently as possible, indeed the fact that these problems are hard makes efficient algorithms for them especially important.

We are interested in this paper in worst case analysis of algorithms for 3-coloring, a basic NP-complete problem. We will also discuss other related problems including 3-SAT, 3-edge-coloring and 3-list-coloring.

Our algorithms for these problems are based on the following simple idea: to find a solution to a 3-coloring problem, it is not necessary to choose a color for each vertex (giving something like  $\mathcal{O}(3^n)$  time). Instead, it suffices to only partially solve the problem by restricting each vertex to two of the three colors. We can then test whether the partial solution can be extended to a complete coloring in polynomial time (e.g. as a 2-SAT instance). This idea applied naively already gives a simple  $\mathcal{O}(1.5^n)$  time randomized algorithm; we improve this by taking advantage of local structure (if we choose a color for one vertex, this restricts the colors of several neighbors at once). It seems likely that our idea of only searching for a partial solution can be applied to many other combinatorial search problems.

If we perform local reductions as above in a 3-coloring problem, we eventually reach a situation in which some uncolored vertices are surrounded by partially colored neighbors, and we run out of good local configurations to use. To avoid this problem, we translate our 3-coloring problem to one

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that also generalizes the other problems listed above: *constraint satisfaction* (CSP). In an  $(a, b)$ -CSP instance, we are given a collection of  $n$  variables, each of which can be given one of  $a$  different colors. However certain color combinations are disallowed: we also have input a collection of  $m$  *constraints*, each of which forbids one coloring of some  $b$ -tuple of variables. Thus 3-satisfiability is exactly  $(2, 3)$ -CSP, and 3-coloring is a special case of  $(3, 2)$ -CSP in which the constraints disallow adjacent vertices from having the same color.

As we show,  $(a, b)$ -CSP instances can be transformed in certain interesting and useful ways: in particular, one can transform  $(a, b)$ -CSP into  $(b, a)$ -CSP and vice versa, one can transform  $(a, b)$ -CSP into  $(\max(a, b), 2)$ -CSP, and in any  $(a, 2)$ -CSP instance one can eliminate variables for which only two colors are allowed, reducing the problem to a smaller one of the same form. Because of this ability to eliminate partially colored variables immediately rather than saving them for a later 2-SAT instance, we can solve a  $(3, 2)$ -CSP instance without running out of good local configurations.

Our actual algorithm solves  $(3, 2)$ -CSP by applying such reductions only until we reach instances with a certain simplified structure, which can then be solved in polynomial time as an instance of graph matching. We further improve our time bound for graph 3-vertex-coloring by using methods involving network flow to find a large set of good local reductions which we apply before treating the remaining problem as a  $(3, 2)$ -CSP instance. And similarly, we solve 3-edge-coloring by using graph matching methods to find a large set of good local reductions which we apply before treating the remaining problem as a 3-vertex-coloring instance.

## 1.1 New Results

We show the following:

- A  $(3, 2)$ -CSP instance with  $n$  variables can be solved in worst case time  $\mathcal{O}(1.3645^n)$ , independent of the number of constraints. We also give a very simple randomized algorithm for solving this problem in expected time  $\mathcal{O}(n^{\mathcal{O}(1)} 2^{n/2}) \approx \mathcal{O}(1.4142^n)$ .
- A  $(d, 2)$ -CSP instance with  $n$  variables and  $d > 3$  can be solved by a randomized algorithm in expected time  $\mathcal{O}((0.4518d)^n)$ .
- 3-coloring in a graph of  $n$  vertices can be solved in time  $\mathcal{O}(1.3289^n)$ , independent of the number of edges in the graph.
- 3-list-coloring (graph coloring given a list at each vertex of three possible colors chosen from some larger set) can be solved in time  $\mathcal{O}(1.3645^n)$ , independent of the number of edges.
- 3-edge-coloring in an  $n$ -vertex graph can be solved in time  $\mathcal{O}(2^{n/2})$ , again independent of the number of edges.
- 3-satisfiability of a formula with  $t$  3-clauses can be solved in time  $\mathcal{O}(n^{\mathcal{O}(1)} + 1.3645^t)$ , independent of the number of variables or 2-clauses in the formula.

Except where otherwise specified,  $n$  denotes the number of vertices in a graph or variables in a SAT or CSP instance, while  $m$  denotes the number of edges in a graph, constraints in an CSP instance, or clauses in a SAT problem.

## 1.2 Related Work

There is a growing body of papers on worst case analysis of algorithms for NP-hard problems. Several authors have described algorithms for maximum independent sets [2, 5, 12, 21, 24, 29, 30]; the best of these is Robson’s [24], which takes time  $\mathcal{O}(1.2108^n)$ . Others have described algorithms for Boolean formula satisfiability [6–10, 14, 15, 18, 19, 22, 25, 26, 28]; the best of these satisfiability algorithms are Schöning’s, which solves 3-SAT in expected time  $\mathcal{O}((4/3)^n)$  [28], and Hirsch’s, which solves SAT in time  $\mathcal{O}(1.239^n)$  [10].

For three-coloring, we know of several relevant references. Lawler [17] is primarily concerned with the general chromatic number, but he also gives the following very simple algorithm for 3-coloring: for each maximal independent set, test whether the complement is bipartite. The maximal independent sets can be listed with polynomial delay [13], and there are at most  $3^{n/3}$  such sets [20], so this algorithm takes time  $\mathcal{O}(1.4422^n)$ . Schiermeyer [27] gives a complicated algorithm for solving 3-colorability in time  $\mathcal{O}(1.415^n)$ , based on the following idea: if there is one vertex  $v$  of degree  $n - 1$  then the graph is 3-colorable iff  $G - v$  is bipartite, and the problem is easily solved. Otherwise, Schiermeyer performs certain reductions involving maximal independent sets that attempt to increase the degree of  $G$  while partitioning the problem into subproblems, at least one of which will remain solvable. Our  $\mathcal{O}(1.3289^n)$  bound significantly improves both of these results.

There has also been some related work on approximate or heuristic 3-coloring algorithms. Blum and Karger [4] show that any 3-chromatic graph can be colored with  $\tilde{\mathcal{O}}(n^{3/14})$  colors in polynomial time. Alon and Kahale [1] describe a technique for coloring random 3-chromatic graphs in expected polynomial time, and Petford and Welsh [23] present a randomized algorithm for 3-coloring graphs which also works well empirically on random graphs although they prove no bounds on its running time. Finally, Vlasie [31] has described a class of instances which are (unlike random 3-chromatic graphs) difficult to color.

Very recently, Schöning [28] has described a simple and powerful randomized algorithm for  $k$ -SAT and more general constraint satisfaction problems, including the CSP instances that we use in our solution of 3-coloring. However, for  $(d, 2)$ -CSP, Schöning notes that his method is not as good as a randomized approach based on an idea from our previous conference paper [3]: simply choose a random pair of values for each variable and solve the resulting 2-SAT instance in polynomial time. The table below compares the resulting  $(d/2)^n$  bound with our new results; an entry with value  $x$  in column  $d$  indicates a time bound of  $\mathcal{O}(x^n)$  for  $(d, 2)$ -CSP.

	$d = 3$	$d = 4$	$d = 5$	$d = 6$
Schöning [28]	1.5	2	2.5	3
New results	1.3645	1.8072	2.2590	2.7108

We were unable to locate prior work on worst case edge coloring. Since any 3-edge-chromatic graph has at most  $3n/2$  edges, one can transform the problem to 3-vertex-coloring at the expense of increasing  $n$  by a factor of  $3/2$ . If we applied our vertex coloring algorithm we would then get time  $\mathcal{O}(1.5319^n)$  which is significantly improved by the bound stated above.

It is interesting that, historically, until the work of Schöning [28], the time bounds for 3-coloring have been smaller than those for 3-satisfiability (in terms of the number of vertices or variables respectively). Schöning’s  $\mathcal{O}((4/3)^n)$  bound for 3-SAT reversed this pattern by being smaller than the previous  $\mathcal{O}(1.3443^n)$  bound for 3-coloring from our 1995 conference paper [3]. The present work restores 3-coloring to a smaller time bound than 3-SAT.

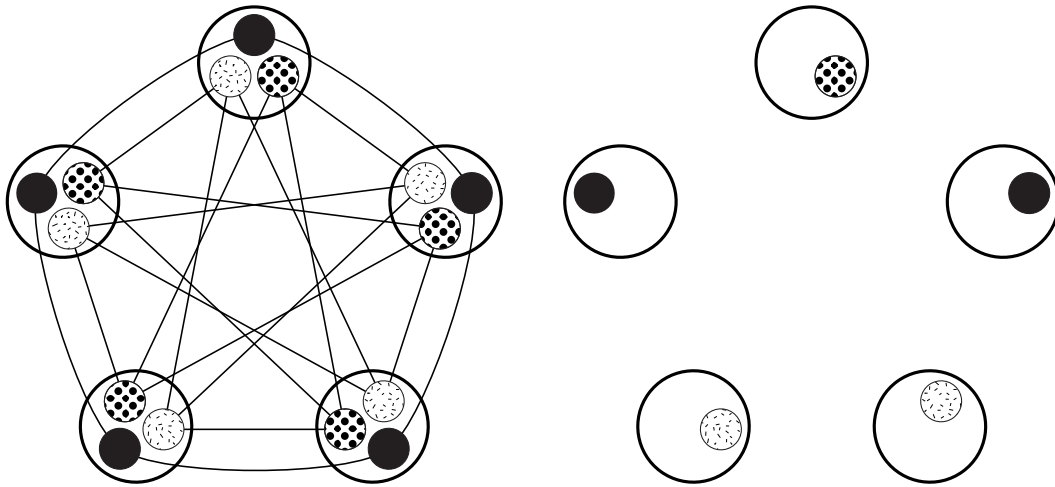


Figure 1: Example  $(3, 2)$ -CSP instance with five variables and twenty constraints (left), and a solution of the instance (right).

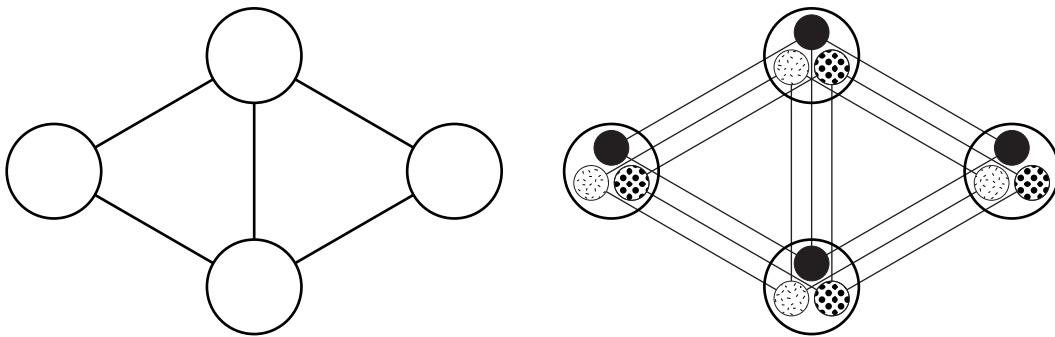


Figure 2: Example 3-coloring instance (left) and translation into a  $(3, 2)$ -CSP instance (right).

## 2 Constraint Satisfaction Problems

We now describe a common generalization of satisfiability and graph coloring as a *constraint satisfaction problem* (CSP) [16, 28]. We are given a collection of  $n$  variables, each of which has a list of possible colors allowed. We are also given a collection of  $m$  constraints, consisting of a tuple of variables and a color for each variable. A constraint is *satisfied* by a coloring if not every variable in the tuple is colored in the way specified by the constraint. We would like to choose one color from the allowed list of each variable, in a way not conflicting with any constraints.

For instance, 3-satisfiability can easily be expressed in this form. Each variable of the satisfiability problem may be colored (assigned the value) either *true* ( $T$ ) or *false* ( $F$ ). For each clause like  $(x_1 \vee x_2 \vee \neg x_3)$ , we make a constraint  $((v_1, F), (v_2, F), (v_3, T))$ . Such a constraint is satisfied if and only if at least one of the corresponding clause's terms is true.

In the  $(a, b)$ -CSP problem, we restrict our attention to instances in which each variable has at

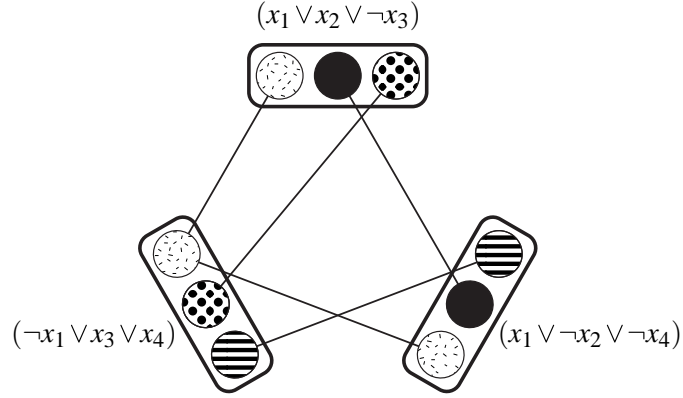


Figure 3: Translation from 3-SAT (or  $(2, 3)$ -CSP) to  $(3, 2)$ -CSP.

most  $a$  possible colors and each constraint involves at most  $b$  variables. The CSP instance constructed above from a 3-SAT instance is then a  $(2, 3)$ -CSP instance, and in fact 3-SAT is easily seen to be equivalent to  $(2, 3)$ -CSP.

In this paper, we will concentrate our attention instead on  $(3, 2)$ -CSP and  $(4, 2)$ -CSP. We can represent a  $(d, 2)$ -CSP instance graphically, by interpreting each variable as a vertex containing up to  $d$  possible colors, and by drawing edges connecting incompatible pairs of vertex colors (Figure 1). Note that this graphical structure is not actually a graph, as the edges connect colors within a vertex rather than the vertices themselves. However, graph 3-colorability and graph 3-list-colorability can be translated directly to a form of  $(3, 2)$ -CSP: we keep the original vertices of the graph and their possible colors, and add up to three constraints for each edge of the graph to enforce the condition that the edge's endpoints have different colors (Figure 2).

Of course, since these problems are all NP-complete, the theory of NP-completeness provides translations from one problem to the other, but the translations above are size-preserving and very simple. We will later describe more complicated translations from 3-coloring and 3-edge-coloring to  $(3, 2)$ -CSP in which the input graph is partially colored before treating the remaining graph as an CSP instance, leading to improved time bounds over our pure CSP algorithm.

As we now show,  $(a, b)$ -CSP instances can be transformed in certain interesting and useful ways. We first describe a form of duality that transforms  $(a, b)$ -CSP instances into  $(b, a)$ -CSP instances, exchanging constraints for variables and vice versa.

**Lemma 1** *If we are given an  $(a, b)$ -CSP instance, we can find an equivalent  $(b, a)$ -CSP instance in which each constraint of the  $(a, b)$ -CSP instance corresponds to a single variable of the transformed problem, and each constraint of the transformed problem corresponds to a single variable of the original problem.*

**Proof:** An assignment of colors to the original  $(a, b)$ -CSP instance's variables solves the problem if and only if, for each constraint, there is at least one pair  $(V, C)$  in the constraint that does not appear in the coloring. In our transformed problem, we choose one variable per original constraint, with the colors available to the new variable being these pairs  $(V, C)$  in the corresponding constraint in the original problem. Choosing such a pair in a coloring of the transformed problem is interpreted as ruling out  $C$  as a possible color for  $V$  in the original problem. We then add constraints to our transformed problem to ensure that for each  $V$  there remains at least one color that is not ruled out:

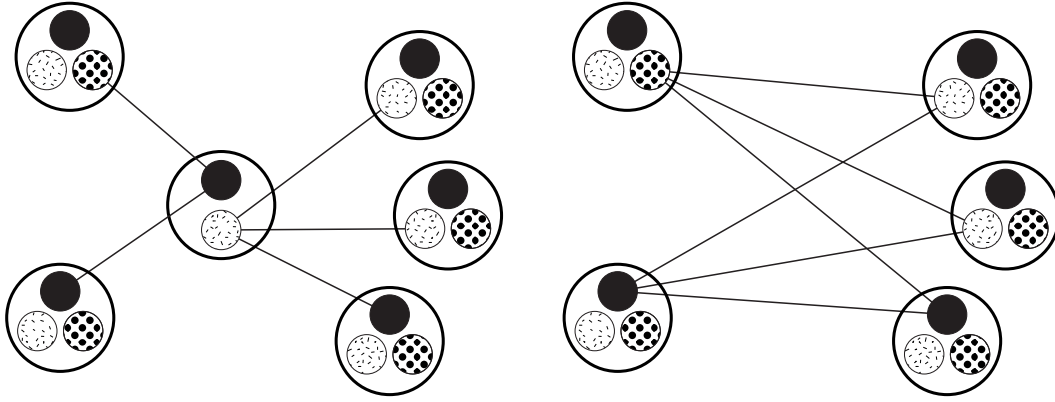


Figure 4:  $(3, 2)$ -CSP instance with a two-color variable (left) and reduced instance after application of Lemma 2 (right).

we add one constraint for each  $a$ -tuple of colors of new variables—recall that each such color is a pair  $(V, C)$ —such that all colors in the  $a$ -tuple involve the same original variable  $V$  and exhaust all the choices of colors for  $V$ .  $\square$

This duality may be easier to understand with a small example. As discussed above, 3-SAT is essentially the same as  $(2, 3)$ -CSP, so Lemma 1 can be used to translate 3-SAT to  $(3, 2)$ -CSP. Suppose we start with the 3-SAT instance  $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_4)$ . Then we make a  $(3, 2)$ -CSP instance (Figure 3) with three variables  $v_i$ , one for each 3-SAT clause. Each variable has three possible colors:  $(1, 2, 3)$  for  $v_1$ ,  $(1, 3, 4)$  for  $v_2$ , and  $(1, 2, 4)$  for  $v_3$ . The requirement that value  $T$  or  $F$  be available to  $x_1$  corresponds to the constraints  $((v_1, 1), (v_2, 1))$  and  $((v_2, 1), (v_3, 1))$ ; we similarly get constraints  $((v_1, 2), (v_3, 2))$ ,  $((v_1, 3), (v_2, 3))$ , and  $((v_2, 4), (v_3, 4))$ . One possible coloring of this  $(3, 2)$ -CSP instance would be to color  $v_1$  1,  $v_2$  3, and  $v_3$  4; this would give satisfying assignments in which  $x_1$  and  $x_3$  are  $T$ ,  $x_4$  is  $F$ , and  $x_2$  can be either  $T$  or  $F$ .

We can similarly translate an  $(a, a)$ -CSP instance into an  $(a, 2)$ -CSP instance in which each variable corresponds to either a constraint or a variable, and each constraint forces the variable colorings to match up with the dual constraint colorings; we omit the details as we do not use this construction in our algorithms.

### 3 Simplification of CSP Instances

Before we describe our CSP algorithms, we describe some situations in which the number of variables in an CSP instance may be reduced with little computational effort.

**Lemma 2** *Let  $v$  be a variable in an  $(a, 2)$ -CSP instance, such that only two of the  $a$  colors are allowed at  $v$ . Then we can find an equivalent  $(a, 2)$ -CSP instance with one fewer variable.*

**Proof:** Let the two colors allowed at  $v$  be  $R$  and  $G$ . Define  $\text{conflict}(C)$  to be the set of pairs  $\{(u, A) : ((u, A), (v, C)) \text{ is a constraint}\}$ . We then include  $\text{conflict}(R) \times \text{conflict}(G)$  to our set of constraints.

Any pair  $((u, A), (w, B)) \in \text{conflict}(R) \times \text{conflict}(G)$  does not reduce the space of solutions to the original problem since if both  $(u, A)$  and  $(w, B)$  were present in a coloring there would be no possible color left for  $v$ . Conversely if all such constraints are satisfied, one of the two colors for  $v$  must be available. Therefore we can now find a smaller equivalent problem by removing  $v$ , as shown in Figure 4.  $\square$

When we apply this variable elimination scheme, the number of constraints can increase, but there can exist only  $(an)^2$  distinct constraints, which in our applications will be a small polynomial.

**Lemma 3** *Let  $(v, X)$  and  $(w, Y)$  be (variable,color) pairs in an  $(a, 2)$ -CSP instance, such that  $v \neq w$  the only constraints involving these pairs are either of the form  $((v, X), (w, Z))$  with  $Y \neq Z$ , or  $((v, Z), (w, Y))$  with  $X \neq Z$ . Then we can find an equivalent  $(a, 2)$ -CSP instance with two fewer variables.*

**Proof:** It is safe to choose the colors  $(v, X)$  and  $(w, Y)$ , since these two choices do not conflict with each other nor with anything else in the CSP instance.  $\square$

**Lemma 4** *Let  $(v, R)$  and  $(v, B)$  be (variable,color) pairs in an  $(a, 2)$ -CSP instance, such that whenever the instance contains a constraint  $((v, R), (w, X))$  it also contains a constraint  $((v, B), (w, X))$ . Then we can find an equivalent  $(a, 2)$ -CSP instance with one fewer variable.*

**Proof:** Any solution involving  $(v, B)$  can be changed to one involving  $(v, R)$  without violating any additional constraints, so it is safe to remove the option of coloring  $v$  with color  $B$ . Once we remove this option,  $v$  is restricted to two colors, and we can apply Lemma 2.  $\square$

**Lemma 5** *Let  $(v, R)$  be a (variable,color) pair in an  $(a, b)$ -CSP instance that is not involved in any constraints. Then we can find an equivalent  $(a, b)$ -CSP instance with one fewer variable.*

**Proof:** We may safely assign color  $R$  to  $v$  and remove it from the instance.  $\square$

**Lemma 6** *Let  $(v, R)$  be a (variable,color) pair in an  $(a, 2)$ -CSP instance that is involved in constraints with all three color options of another variable  $w$ . Then we can find an equivalent  $(a, b)$ -CSP instance with one fewer variable.*

**Proof:** No coloring of the instance can use  $(v, R)$ , so we can restrict  $v$  to the remaining two colors and apply Lemma 2.  $\square$

We say that a CSP instance in which none of Lemmas 2–6 applies is *reduced*.

## 4 Simple Randomized CSP Algorithm

We first demonstrate the usefulness of Lemma 2 by describing a very simple randomized algorithm for solving  $(3, 2)$ -CSP instances in expected time  $\mathcal{O}(2^{n/2}n^{\mathcal{O}(1)})$ .

**Lemma 7** *If we are given a  $(3, 2)$ -CSP instance  $I$ , then in random polynomial time we can find an instance  $I'$  with two fewer variables, such that if  $I'$  is solvable then so is  $I$ , and if  $I$  is solvable then with probability at least  $\frac{1}{2}$  so is  $I'$ .*

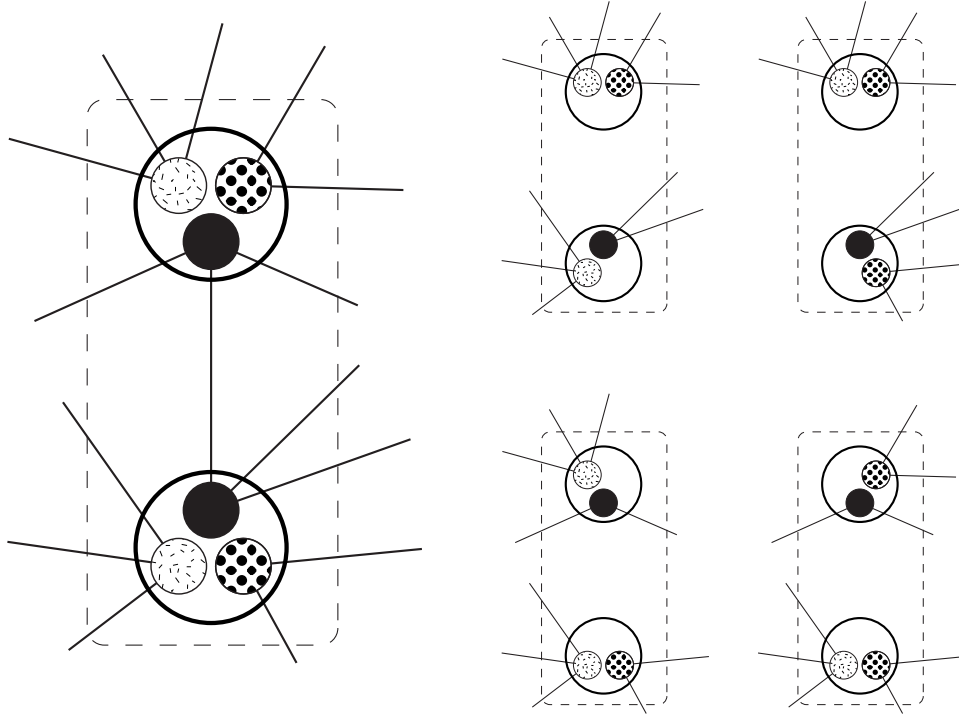


Figure 5: Randomized  $(3, 2)$ -CSP algorithm: constraint  $((v, R), (w, R))$  (left) and four equally-likely two-color restrictions in which exactly one of the variables may still use color  $R$  (right).

**Proof:** If no constraint exists, we can solve the problem immediately. Otherwise choose some constraint  $((v, X), (w, Y))$ . Rename the colors if necessary so that both  $v$  and  $w$  have available the same three colors  $R, G$ , and  $B$ , and so that  $X = Y = R$ . Restrict the colorings of  $v$  and  $w$  to two colors each in one of four ways, chosen uniformly at random from the four possible such restrictions in which exactly one of  $v$  and  $w$  is restricted to colors  $G$  and  $B$  (Figure 5). Then it can be verified by examination of cases that any valid coloring of the problem remains valid for exactly two of these four restrictions, so with probability  $\frac{1}{2}$  it continues to be a solution to the restricted problem. Now apply Lemma 2 and eliminate both  $v$  and  $w$  from the problem.  $\square$

**Corollary 1** *In expected time  $\mathcal{O}(2^{n/2}n^{\mathcal{O}(1)})$  we can find a solution to a  $(3, 2)$ -CSP instance if one exists.*

**Proof:** We perform the reduction above  $n/2$  times, taking polynomial time and giving probability at least  $2^{-n/2}$  of finding a correct solution. If we repeat this method until a solution is found, the expected number of repetitions is  $2^{n/2}$ .  $\square$

## 5 Faster CSP Algorithm

We now describe a more complicated method of solving  $(3, 2)$ -CSP instances deterministically with the somewhat better time bound of  $\mathcal{O}(1.36443^n)$ . More generally, our algorithm can actually handle  $(4, 2)$ -CSP instances. Any  $(4, 2)$ -CSP instance can be transformed into a  $(3, 2)$ -CSP instance



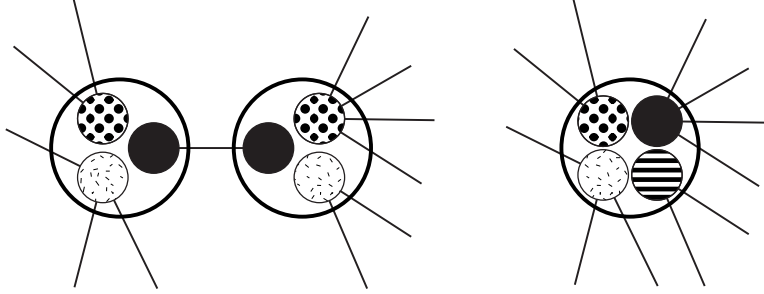


Figure 6: Isolated constraint between two three-color variables (left) can be replaced by a single four-color variable (right).

by expanding each of its four-color variables to two three-color variables, each having two of the original four colors, with a constraint connecting the third color of each new variable (Figure 6). Therefore, the natural definition of the “size” of a  $(4, 2)$ -CSP instance is  $n = n_3 + 2n_4$ , where  $n_i$  denotes the number of variables with  $i$  colors. However, we instead define the size to be  $n = n_3 + (2 - \epsilon)n_4$ , where  $\epsilon \approx 0.095543$  is a constant to be determined more precisely later. In any case, the size of a  $(3, 2)$ -CSP instance remains equal to its number of variables, so any bound on the running time of our algorithm in terms of  $n$  applies directly to  $(3, 2)$ -CSP.

The basic idea of our algorithm is to find a set of local configurations that must occur within any  $(4, 2)$ -CSP instance  $I$ , such that any instance containing such a configuration can be replaced by a small number of smaller instances.

In more detail, for each configuration we describe a set of smaller instances  $I_i$  of size  $|I| - r_i$  such that  $I$  is solvable if and only if at least one of the instances  $I_i$  is solvable. If one particular configuration occurred at each step of the algorithm, this would lead to a recurrence of the form

$$T(n) = \sum T(n - r_i) + \text{poly}(n) = \mathcal{O}(\lambda(r_1, r_2, \dots)^n)$$

for the worst-case running time of our algorithm, where the base  $\lambda(r_1, r_2, \dots)$  of the exponent in the running time is the largest zero of the function  $f(x) = 1 - \sum x^{-r_i}$  (such a function is not necessarily a polynomial because the  $r_i$  will not necessarily be integers). We call this value  $\lambda(r_1, r_2, \dots)$  the *work factor* of the given local configuration. The overall time bound will be  $\lambda^n$  where  $\lambda$  is the largest work factor among the configurations we have identified. This value  $\lambda$  will depend on our previous choice of  $\epsilon$ ; we will choose  $\epsilon$  in such a way as to minimize  $\lambda$ .

## 5.1 Single Constraints and Multiple Adjacencies

We first consider local configurations in which some (variable,color) pair is incident on only one constraint, or has multiple constraints to the same variable. First, suppose that (variable,color) pair  $(v, R)$  is involved in only a single constraint  $((v, R), (w, R))$ . If this is also the only constraint involving  $(w, R)$ , we call it an *isolated constraint*. Otherwise, we call it a *dangling constraint*.

**Lemma 8** *Let  $((v, R), (w, R))$  be an isolated constraint in a  $(4, 2)$ -CSP instance, and let  $\epsilon \leq 0.545$ . Then the instance can be replaced by smaller instances with work factor at most  $\lambda(2 - \epsilon, 3 - \epsilon)$ .*

**Proof:** If  $v$  and  $w$  are both three-color variables, then the instance can be colored if and only if we can color the instance formed by replacing them with a single four-color variable, in which the

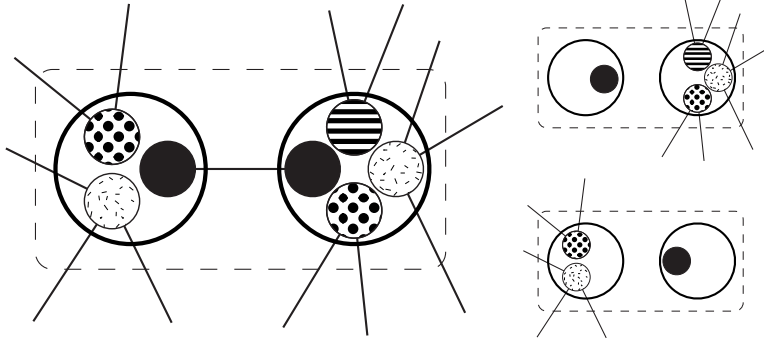


Figure 7: Isolated constraint between three-color and four-color variable (left) can be replaced by two instances with size smaller by  $2 - \epsilon$  (right top) and  $3 - \epsilon$  (right bottom).

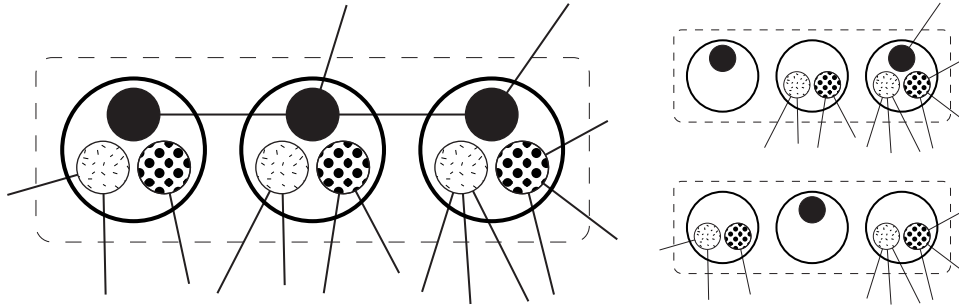


Figure 8: Dangling constraint: a choice with one constraint, where the second choice in the constraint is also constrained by a third variable. We choose either to use or not use that second color.

four colors are the remaining choices for  $v$  and  $w$  other than  $R$  (Figure 6). Thus in this case we can reduce the problem size by  $\epsilon$ , with no additional work.

Otherwise, if there exists a coloring of the given instance, there exists one in which exactly one of  $v$  and  $w$  is given color  $R$ . Suppose first that  $v$  has four colors while  $w$  has only three. Thus we can reduce the problem to two instances, in one of which  $(v, R)$  is used (so  $v$  is removed from the problem, and  $(w, R)$  is removed as a choice for variable  $w$ , allowing us to remove the variable by Lemma 2) and in the other of which  $(w, R)$  is used (Figure 7). The first subproblem has its size reduced by  $3 - \epsilon$  since both variables are removed, while the second's size is reduced by  $2 - \epsilon$  since  $w$  is removed while  $v$  loses one of its colors but is not removed. Thus the work factor is  $\lambda(2 - \epsilon, 3 - \epsilon)$ . Similarly, if both are four-color variables, the work factor is  $\lambda(3 - 2\epsilon, 3 - 2\epsilon)$ . For the given range of  $\epsilon$ , this second work factor is smaller than the first.  $\square$

**Lemma 9** *Let  $((v, R), (w, R))$  be a dangling constraint in a reduced  $(4, 2)$ -CSP instance. Then the instance can be replaced by smaller instances with work factor at most  $\lambda(2 - \epsilon, 3 - \epsilon)$ .*

**Proof:** The second constraint for  $(w, R)$  can not involve  $v$ , or we would be able to apply Lemma 4. We choose either to use color  $(w, R)$  or to restrict  $w$  to avoid that color (Figure 8). If we use color  $(w, R)$ , we eliminate choice  $(v, R)$  and another choice on the other neighbor of  $w$ . If we avoid color  $(w, R)$ , we may safely use color  $(v, R)$ .

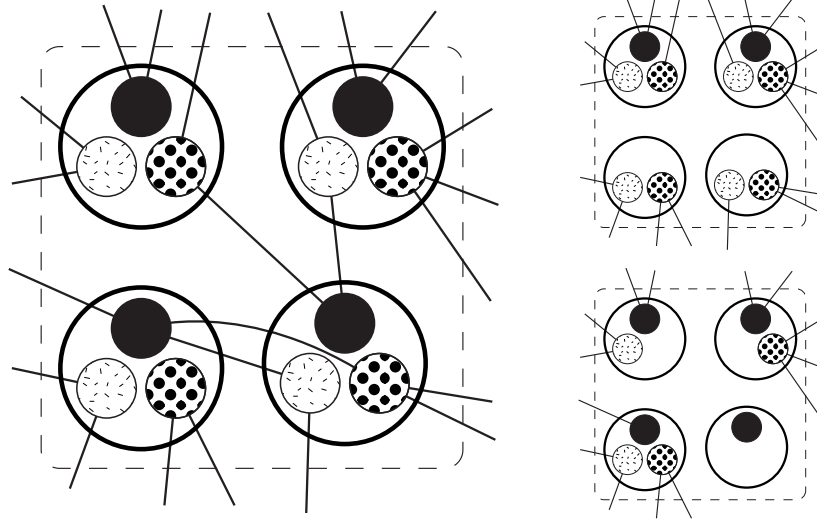


Figure 9: Implication from  $(v, R)$  to  $(w, R)$ , such that  $(w, R)$  has two distinct neighbors. Restricting  $w$  eliminates  $v$  and  $w$  (top right) while assigning  $w$  color  $R$  eliminates three variables (bottom right).

In the worst case, the other neighbor of  $(w, R)$  has four colors, so removing one only reduces the problem size by  $1 - \epsilon$ . There are four cases depending on the number of colors of  $v$  and  $w$ : If both have three colors, the work factor is  $\lambda(2, 3 - \epsilon)$ . If only  $v$  has four colors, the work factor is  $\lambda(3 - \epsilon, 3 - 2\epsilon)$ . If only  $w$  has four colors, the work factor is  $\lambda(2 - \epsilon, 4 - 2\epsilon)$ . If both have four colors, the work factor is  $\lambda(3 - 2\epsilon, 4 - 3\epsilon)$ . These factors are all dominated by the one in the statement of the lemma.  $\square$

**Lemma 10** *Suppose a reduced  $(4, 2)$ -CSP instance includes two constraints such as  $((v, R), (w, B))$  and  $((v, R), (w, G))$  that connect one color of variable  $v$  with two colors of variable  $w$ , and let  $\epsilon \leq 0.4$ . Then the instance can be replaced by smaller instances with work factor at most  $\lambda(2 - \epsilon, 3 - 2\epsilon)$ .*

**Proof:** We assume that the instance has no color choice with only a single constraint, or we could apply one of Lemmas 8 and 9 to achieve the given work factor.

We say that  $(v, R)$  implies  $(w, R)$  if there are constraints from  $(v, R)$  to every other color choice of  $w$ . If the target  $(w, R)$  of an implication is not the source of another implication, then using  $(w, R)$  eliminates  $w$  and at least two other colors, while avoiding  $(w, R)$  forces us to also avoid  $(v, R)$  (Figure 9). Thus, in this case we achieve work factor either  $\lambda(2 - \epsilon, 3 - 2\epsilon)$  if  $w$  has three color choices, or  $\lambda(2 - 2\epsilon, 4 - 3\epsilon)$  if it has four.

If the target of every implication is the source of another, then we can find a cycle of colors each of which implies the next in the cycle (Figure 10). If no other constraints involve colors in the cycle (as is true in the figure), we can use them all, reducing the problem by the length of the cycle for free. Otherwise, let  $(v, R)$  be a color in the cycle that has an outside constraint. If we use  $(v, R)$ , we must use the colors in the rest of the cycle, and eliminate the  $(\text{variable}, \text{color})$  pair outside the cycle constrained by  $(v, R)$ . If we avoid  $(v, R)$ , we must also avoid the colors in the rest of the cycle. The maximum work factor for this case is  $\lambda(2, 3 - \epsilon)$ , and arises when the cycle consists of only two variables, both of which have only three allowed colors.

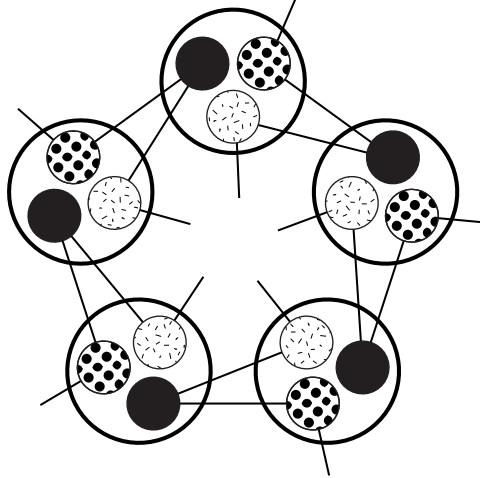


Figure 10: Cycle of implications.

Finally, if the situation described in the lemma exists without forming any implication, then  $w$  must have four color choices, exactly two of which are constrained by  $(v, R)$ . In this case restricting  $w$  to those two choices reduces the size by at least  $3 - 2\epsilon$ , while restricting it to the remaining two choices reduces the size by  $2 - \epsilon$ , again giving work factor  $\lambda(2 - \epsilon, 3 - 2\epsilon)$ .  $\square$

## 5.2 Highly Constrained Colors

We next consider cases in which choosing one color for a variable eliminates many other choices, or in which adjacent (variable,color) pairs have different numbers of constraints.

**Lemma 11** *Suppose a reduced  $(4, 2)$ -CSP instance includes a color pair  $(v, R)$  involved in three or more constraints, where  $v$  has four color choices, or a pair  $(v, R)$  involved in four or more constraints, where  $v$  has three color choices. Then the instance can be replaced by smaller instances with work factor at most  $\lambda(1 - \epsilon, 5 - 4\epsilon)$ .*

**Proof:** We can assume from Lemma 10 that each constraint connects  $(v, R)$  to a different variable. Then if we choose to use color  $(v, R)$ , we eliminate  $v$  and remove a choice from each of its neighbors, either eliminating them or reducing their number of choices from four to three. If we don't use  $(v, R)$ , we eliminate that color only. So if  $v$  has four choices, the work factor is at most  $\lambda(1 - \epsilon, 5 - 4\epsilon)$ , and if it has three choices and four or more constraints, the work factor is at most  $\lambda(1, 5 - 4\epsilon)$ .  $\square$

**Lemma 12** *Suppose a reduced  $(4, 2)$ -CSP instance includes a (variable,color) pair  $(v, R)$  with three constraints, one of which connects it to a variable with four color choices, and let  $\epsilon \leq 0.3576$ . Suppose also that none of the previous lemmas applies. Then the instance can be replaced by smaller instances with work factor at most  $\lambda(3 - \epsilon, 4 - \epsilon, 4 - \epsilon)$ .*

**Proof:** For convenience suppose that the four-color neighbor is  $(w, R)$ . We can assume  $(w, R)$  has only two constraints, else it would be covered by a previous lemma.

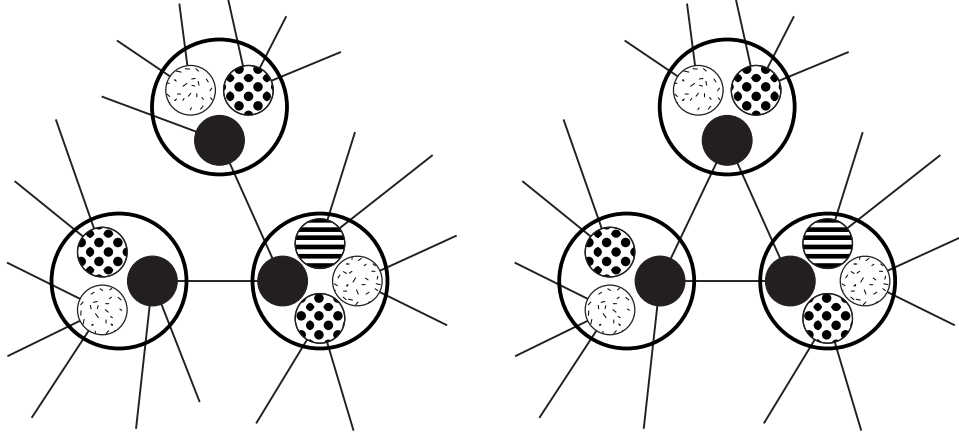


Figure 11: Cases for Lemma 12:  $(v, R)$  and  $(w, R)$  either have disjoint neighbors (left) or form a triangle (right).

Then, if  $(v, R)$  and  $(w, R)$  do not form a triangle with a third (variable,color) pair (Figure 11, left), we choose either to use or avoid color  $(v, R)$ . If we use  $(v, R)$ , we eliminate  $v$  and the three adjacent color choices. If we avoid  $(v, R)$ , we create a dangling constraint at  $(w, R)$ , which we have seen in Lemma 9 allows us to further subdivide the instance with work factor  $\lambda(3 - \epsilon, 3 - 2\epsilon)$  in addition to the elimination of  $v$ . Thus, the overall work factor in this case is  $\lambda(4 - \epsilon, 4 - 2\epsilon, 4 - 3\epsilon)$ .

On the other hand, suppose we have a triangle of constraints formed by  $(v, R)$ ,  $(w, R)$ , and a third (variable,color) pair  $(x, R)$ , as shown in Figure 11, right. Then  $(v, R)$  and  $(x, R)$  are the only choices constraining  $(w, R)$ , so if  $(v, R)$  and  $(x, R)$  are both not chosen, we can safely choose to use color  $(w, R)$ . Therefore, we make three smaller instances, in each of which we choose to use one of the three choices in the triangle. We can assume from the previous cases that  $(v, R)$  has only three choices, and further its third neighbor (other than  $(w, R)$  and  $(x, R)$ ) must also have only three choices or we could apply the previous case of the lemma. In the worst case,  $(x, R)$  has only two constraints and  $x$  has only three color choices. Therefore, the size of the subproblems formed by choosing  $(v, R)$ ,  $(w, R)$ , and  $(x, R)$  is reduced by at least  $4 - \epsilon$ ,  $4 - \epsilon$ , and  $3 - \epsilon$  respectively, leading to a work factor of  $\lambda(3 - \epsilon, 4 - \epsilon, 4 - \epsilon)$ . If instead  $x$  has four color choices, we get the better work factor  $\lambda(4 - 2\epsilon, 4 - 2\epsilon, 4 - 2\epsilon)$ .

For the given range of  $\epsilon$ , the largest of these work factors is  $\lambda(3 - \epsilon, 4 - \epsilon, 4 - \epsilon)$ .  $\square$

**Lemma 13** *Suppose a reduced  $(4, 2)$ -CSP instance includes a (variable,color) pair  $(v, R)$  with three constraints, one of which connects it to a variable with two constraints. Suppose also that none of the previous lemmas applies. Then the instance can be replaced by smaller instances with work factor at most  $\max\{\lambda(1 + \epsilon, 4), \lambda(3, 4 - \epsilon, 4)\}$ .*

**Proof:** Let  $(w, R)$  be the neighbor with two constraints. Note that (since the previous lemma is assumed not to apply) all neighbors of  $(v, R)$  have only three color choices.

First, suppose  $(v, R)$  and  $(w, R)$  are not part of a triangle of constraints (Figure 12, top). Then, if we choose to use color  $(v, R)$  we eliminate four variables, while if we avoid using it we create a dangling constraint on  $(w, R)$  which we further subdivide into two more instances according to Lemma 9. Thus, the work factor in this case is  $\lambda(3, 4 - \epsilon, 4)$ .

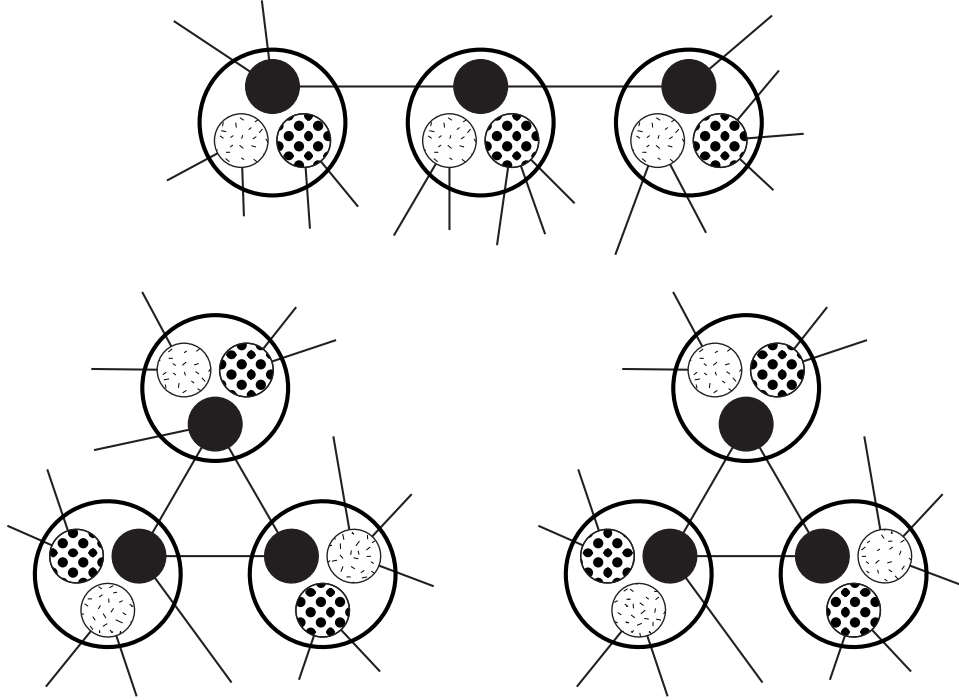


Figure 12: Cases for Lemma 13. Top:  $(v, R)$  and  $(w, R)$  do not form a triangle; avoiding  $(v, R)$  produces a dangling constraint. Bottom left:  $(v, R)$  and  $(w, R)$  are part of a triangle with two triply-constrained colors; choosing each triangle vertex gives work factor  $\lambda(3, 4, 4)$ . Bottom right:  $(v, R)$  and  $(w, R)$  are part of a triangle with two doubly-constrained colors; avoiding  $(v, R)$  produces an isolated constraint.

Second, suppose that  $(v, R)$  and  $(w, R)$  are part of a triangle with a third (variable,color) pair  $(x, R)$ , and that  $(x, R)$  has three constraints (Figure 12, bottom left). Then (as in the previous lemma) we may choose to use one of the three choices in the triangle, resulting in work factor  $\lambda(3, 4, 4)$ .

Finally, suppose that  $(v, R)$ ,  $(w, R)$ , and  $(x, R)$  form a triangle as above, but that  $(x, R)$  has only two constraints (Figure 12, bottom right). Then if we choose to use  $(v, R)$  we eliminate four variables, while if we avoid using it we create an isolated constraint between  $(w, R)$  and  $(x, R)$ . Thus in this case the work factor is  $\lambda(1 + \epsilon, 4)$ .  $\square$

If none of the above lemmas applies to an instance, then each color choice in the instance must have either two or three constraints, and each neighbor of that choice must have the same number of constraints.

### 5.3 Triply-Constrained Colors

Within this section we assume that we have a  $(4, 2)$ -CSP instance in which none of the previous reduction lemmas applies, so any (variable,color) pair must be involved in exactly as many constraints as each of its neighbors.

We now consider the remaining (variable,color) pairs that have three constraints each. Define a *three-component* to be a subset of such pairs such that any pair in the subset is connected to any other by a path of constraints. We distinguish two such types of components: a *small three-component* is

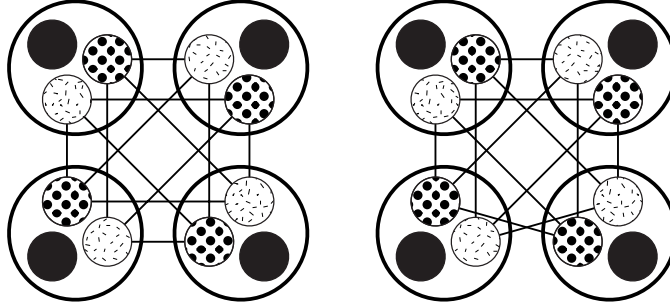


Figure 13: The two possible small three-components with  $k = 8$ .

one that involves only four distinct variables, while a *large three-component* involves five or more variables. Note that we can assume by the previous lemmas that each variable in a component has only three color choices.

**Lemma 14** *Let  $C$  be a small three-component involving  $k$  (variable,color) pairs. Then  $k$  must be a multiple of four, and each variable involved in the component has exactly  $k/4$  pairs in  $C$ .*

**Proof:** Let  $v$  and  $w$  be variables in a small component  $C$ . Then each (variable,color) pair in  $C$  from variable  $v$  has exactly one constraint to a distinct (variable,color) pair from variable  $w$ , so the numbers of pairs from  $v$  equals the number of pairs from  $w$ . The assertions that each variable has the same number of pairs, and that the total number of pairs is a multiple of four, then follow.  $\square$

We say that a small three-component is *good* if  $k = 4$  in the lemma above.

**Lemma 15** *Let  $C$  be a small three-component that is not good. Then the instance can be replaced by smaller instances with work factor at most  $\lambda(4, 4, 4)$ .*

**Proof:** A component with  $k = 12$  uses up all color choices for all four variables. Thus we may consider these variables in isolation from the rest of the instance, and either color them all (if possible) or determine that the instance is unsolvable.

The remaining small components have  $k = 8$ . Such a component may be drawn with the four variables at the corners of a square, and the top, left, and right pairs of edges uncrossed (Figure 13). If only the center two pairs were crossed, we would actually have two  $k = 4$  components, and if any other two or three of the remaining pairs were crossed, we could reduce the number of crossings in the drawing by swapping the colors at one of the variables. Thus, the only possible small components with  $k = 8$  are the one with all six pairs uncrossed, and the one with only one pair crossed.

The first of these allows all four variables to be colored and removed, while in the other case there exist only three maximal subsets of variables that can be colored. (In the figure, these three sets are formed by the bottom two vertices, and the two sets formed by removing one bottom vertex). We split into instances by choosing to color each of these maximal subsets, eliminating all four variables in the component and giving work factor  $\lambda(4, 4, 4)$ .  $\square$

Define a *witness* to a large three-component to be a set of five (variable,color) pairs with five distinct variables, such that there exist constraints from one pair to three others, and from at least



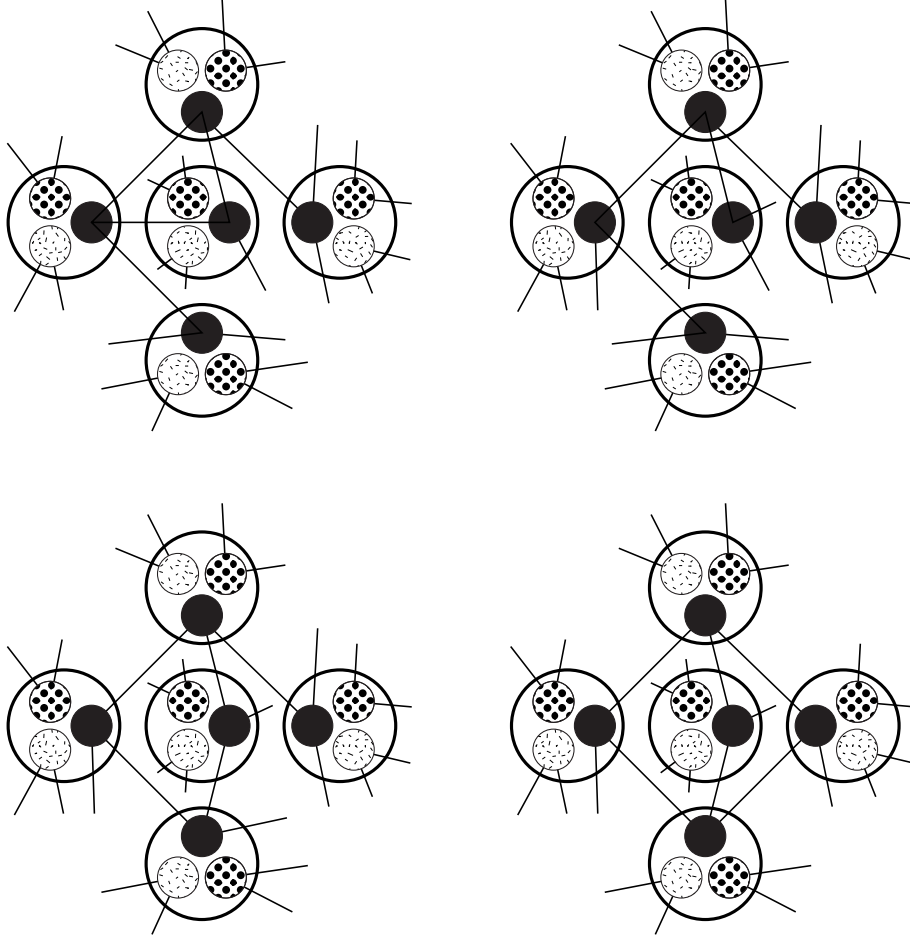


Figure 14: Cases for Lemma 17:  $(z, R)$  is constrained by one neighbor of  $(v, R)$  forming a triangle with a second neighbor (top left), one neighbor not part of a triangle (top right), two neighbors (bottom left), or three neighbors (bottom right).

one of those three to the fifth. By convention we use  $(v, R)$  to denote the first pair,  $(w, R)$ ,  $(x, R)$ , and  $(y, R)$  to denote the pairs connected by constraints to  $(v, R)$ , and  $(z, R)$  to be the fifth pair in the witness.

**Lemma 16** *Every large three-component has a witness.*

**Proof:** Choose some arbitrary pair  $(u, R)$  as a starting point, and perform a breadth first search in the graph formed by the pairs and constraints in the component. Let  $(z, R)$  be the first pair reached by this search where  $z$  is not one of the variables adjacent to  $(u, R)$ , let  $(v, R)$  be the grandparent of  $(z, R)$  in the breadth first search tree, and let the other three pairs be the neighbors of  $(v, R)$ . Then it is easy to see that  $(v, R)$  and its neighbors must use the same four variables as  $(u, R)$  and its neighbors, while  $z$  by definition uses a different variable.  $\square$

**Lemma 17** *Suppose that a  $(4, 2)$ -CSP instance contains a large three-component. Then the instance can be replaced by smaller instances with work factor at most  $\lambda(4, 4, 5, 5)$ .*



**Proof:** Let  $(v, R)$ ,  $(w, R)$ ,  $(x, R)$ ,  $(y, R)$ , and  $(z, R)$  be a witness for the component. Then we distinguish subcases according to how many of the neighbors of  $(z, R)$  are pairs in the witness.

1. If  $(z, R)$  has a constraint with only one pair in the witness, say  $(w, R)$ , then we choose either to use color  $(z, R)$  or to avoid it. If we use it, we eliminate some four variables. If we avoid it, then we cause  $(w, R)$  to have only two constraints. If  $(w, R)$  is also constrained by one of  $(x, R)$  or  $(y, R)$ , we then have a triangle of constraints (Figure 14, top left). We can assume without loss of generality that the remaining constraint from this triangle does not connect to a different color of variable  $z$ , for if it did we could instead use the same five variables in a different order to get a witness of this form. We then further subdivide into three more instances, in each of which we choose to use one of the pairs in the triangle, as in the second case of Lemma 13. This gives overall work factor  $\lambda(4, 4, 5, 5)$ .

On the other hand, if  $(v, R)$  and  $(w, R)$  are not part of a triangle (Figure 14, top right), then (after avoiding  $(z, R)$ ) we can apply the first case of Lemma 13 again achieving the same work factor.

2. If  $(z, R)$  has constraints with two pairs in the witness (Figure 14, bottom left), then choosing to use  $(z, R)$  eliminates four variables and causes  $(v, R)$  to dangle, while avoiding  $(z, R)$  eliminates a single variable. The work factor is thus  $\lambda(1, 6, 7)$ .
3. If  $(z, R)$  has constraints with all three of  $(w, R)$ ,  $(y, R)$ , and  $(z, R)$  (Figure 14, bottom right), then choosing to use  $(z, R)$  also allows us to use  $(v, R)$ , eliminating five variables. The work factor is  $\lambda(1, 5)$ .

The largest of the three work factors arising in these cases is the first one,  $\lambda(4, 4, 5, 5)$ .  $\square$

## 5.4 Doubly-Constrained Colors

As in the previous section, we define a *two-component* to be a subset of (variable,color) pairs such that each has two constraints, and any pair in the subset is connected to any other by a path of constraints. A two-component must have the form of a cycle of pairs, but it is possible for more than one pair in the cycle to involve the same variable. We distinguish two such types of components: a *small two-component* is one that involves only three pairs, while a *large two-component* involves four or more pairs.

**Lemma 18** *Suppose a reduced  $(4, 2)$ -CSP instance includes a large two-component, and let  $\epsilon \leq 0.287$ . Then the instance can be replaced by smaller instances with work factor at most  $\lambda(3, 3, 5)$ .*

**Proof:** We split into subcases:

1. Suppose the cycle passes through five consecutive distinct variables, say  $(v, R)$ ,  $(w, R)$ ,  $(x, R)$ ,  $(y, R)$ , and  $(z, R)$ . We can assume that, if any of these five variables has four color choices, then this is true of one of the first four variables. Any coloring that does not use both  $(v, R)$  and  $(y, R)$  can be made to use at least one of the two colors  $(w, R)$  or  $(x, R)$  without violating any of the constraints. Therefore, we can divide into three subproblems: one in which we use  $(w, R)$ , eliminating three variables, one in which we use  $(x, R)$ , again eliminating three variables, and one in which we use both  $(v, R)$  and  $(y, R)$ , eliminating all five variables. If all

five variables have only three color choices, The work factor resulting from this subdivision is  $\lambda(3, 3, 5)$ . If some of the variables have four color choices, the work factor is at most  $\lambda(3 - \epsilon, 4 - \epsilon, 5 - 2\epsilon)$ , which is smaller for the given range of  $\epsilon$ .

2. Suppose two colors three constraints apart on a cycle belong to the same variable; for instance, the sequence of colors may be  $(v, R), (w, R), (x, R), (v, G)$ . Then any coloring can be made to use one of  $(w, R)$  or  $(x, R)$  without violating any constraints. If we form one subproblem in which we use  $(w, R)$  and one in which we use  $(x, R)$ , we get work factor at most  $\lambda(3 - \epsilon, 3 - \epsilon)$  (the worst case occurring when only  $v$  has four color choices).
3. Any long cycle which does not contain one of the previous two subcases must pass through the same four variables in the same order one, two, or three times. If it passes through two or three times, all four variables may be safely colored using colors from the cycle, reducing the problem with work factor one. And if the cycle has length exactly four, we may choose one of two ways to use two diagonally opposite colors from the cycle, giving work factor at most  $\lambda(4, 4)$ .

For the given range of  $\epsilon$ , the largest of these work factors is  $\lambda(3, 3, 5)$ .  $\square$

## 5.5 Matching

Suppose we have a  $(4, 2)$ -CSP instance to which none of the preceding reduction lemmas applies. Then, every constraint must be part of a good three-component or a small two-component. As we now show, this simple structure enables us to solve the remaining problem quickly.

**Lemma 19** *If we are given a  $(4, 2)$ -CSP instance in which every constraint must be part of a good three-component or a small two-component, then we can solve it or determine that it is not solvable in polynomial time.*

**Proof:** We form a bipartite graph, in which the vertices correspond to the variables and components of the instance. We connect a variable to a component by an edge if there is a  $(\text{variable}, \text{color})$  pair using that variable and belonging to that component.

Since each pair in a good three-component or small two-component is connected by a constraint to every other pair in the component, any solution to the instance can use at most one  $(\text{variable}, \text{color})$  pair per component. Thus, a solution consists of a set of  $(\text{variable}, \text{color})$  pairs, covering each variable once, and covering each component at most once. In terms of the bipartite graph constructed above, this is simply a matching. So, we can solve the problem by using a graph maximum matching algorithm to determine the existence of a matching that covers all the variables.  $\square$

## 5.6 Overall CSP Algorithm

This completes the case analysis needed for our result.

**Theorem 1** *We can solve any  $(3, 2)$ -CSP instance in time  $\mathcal{O}(\lambda(4, 4, 5, 5)^n) \approx \mathcal{O}(1.36443^n)$ .*

**Proof:** We employ a backtracking (depth first) search in a state space consisting of  $(3, 2)$ -CSP instances. At each point in the search, we examine the current state, and attempt to find a set of

smaller instances to replace it with, using one of the reduction lemmas above. Such a replacement can always be found in polynomial time by searching for various simple local configurations in the instance. We then recursively search each smaller instance in succession. If we ever reach an instance in which Lemma 19 applies, we perform a matching algorithm to test whether it is solvable. If so, we find a solution and terminate the search. If not, we backtrack to the most recent branching point of the search and continue with the next alternative at that point.

A bound of  $\lambda^n$  on the number of recursive calls in this search algorithm, where  $\lambda$  is the maximum work factor occurring in our reduction lemmas, can be proven by induction on the size of an instance. The work within each call is polynomial and does not add appreciably to the overall time bound.

To determine the maximum work factor, we need to set a value for the parameter  $\epsilon$ . We used *Mathematica* to find a numerical value of  $\epsilon$  minimizing the maximum of the work factors involving  $\epsilon$ , and found that for  $\epsilon \approx 0.095543$  the work factor is  $\approx 1.36443 \approx \lambda(4, 4, 5, 5)$ . For  $\epsilon$  near this value, the two largest work factors are  $\lambda(3 - \epsilon, 4 - \epsilon, 4 - \epsilon)$  (from Lemma 12) and  $\lambda(1 + \epsilon, 4)$  (from Lemma 13); the remaining work factors are below 1.36. The true optimum value of  $\epsilon$  is thus the one for which  $\lambda(3 - \epsilon, 4 - \epsilon, 4 - \epsilon) = \lambda(1 + \epsilon, 4)$ .

As we now show, for this optimum  $\epsilon$ ,  $\lambda(3 - \epsilon, 4 - \epsilon, 4 - \epsilon) = \lambda(1 + \epsilon, 4) = \lambda(4, 4, 5, 5)$ , which also arises as a work factor in Lemma 17. Consider subdividing an instance of size  $n$  into one of size  $n - (1 + \epsilon)$  and another of size  $n - 4$ , and then further subdividing the first instance into subinstances of size  $n - (1 + \epsilon) - (3 - \epsilon)$ ,  $n - (1 + \epsilon) - (4 - \epsilon)$ , and  $n - (1 + \epsilon) - (4 - \epsilon)$ . This four-way subdivision combines subdivisions of type  $\lambda(1 + \epsilon, 4)$  and  $\lambda(3 - \epsilon, 4 - \epsilon, 4 - \epsilon)$ , so it must have a work factor between those two values. But by assumption those two values equal each other, so they also equal the work factor of the four-way subdivision, which is just  $\lambda(4, 4, 5, 5)$ .  $\square$

We use the quantity  $\lambda(4, 4, 5, 5)$  frequently in the remainder of the paper, so we use  $\Lambda$  to denote this value. Theorem 1 immediately gives algorithms for some more well known problems, some of which we improve later. Of these, the least familiar is likely to be *list  $k$ -coloring*: given at each vertex of a graph a list of  $k$  colors chosen from some larger set, find a coloring of the whole graph in which each vertex color is chosen from the corresponding list [11].

**Corollary 2** *We can solve the 3-coloring and 3-list coloring problems in time  $\mathcal{O}(\Lambda^n)$ , the 3-edge-coloring problem in time  $\mathcal{O}(\Lambda^m)$ , and the 3-SAT problem in time  $\mathcal{O}(\Lambda^t)$ ,*

**Corollary 3** *There is a randomized algorithm which finds the solution to any solvable  $(d, 2)$ -CSP instance (with  $d > 3$ ) in expected time  $\mathcal{O}((0.4518d)^n)$ .*

**Proof:** Randomly choose a subset of four values for each variable and apply our algorithm to the resulting  $(4, 2)$ -CSP problem. Repeat with a new random choice until finding a solvable  $(4, 2)$ -CSP instance. The random restriction of a variable has probability  $4/d$  of preserving solvability so the expected number of trials is  $(d/4)^n$ . Each trial takes time  $\mathcal{O}(\Lambda^{(2-\epsilon)n}) \approx \mathcal{O}(1.8072^n)$ . The total expected time is therefore  $\mathcal{O}((d/4)^n 1.8072^n)$ .  $\square$

## 6 Vertex Coloring

Simply by translating a 3-coloring problem into a  $(3, 2)$ -CSP instance, as described above, we can test 3-colorability in time  $\mathcal{O}(\Lambda^n)$ . We now describe some methods to reduce this time bound even

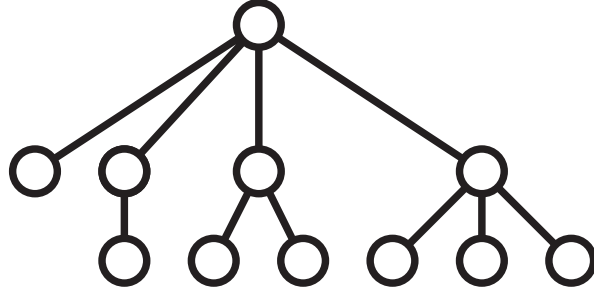


Figure 15: Types of branch in a height-two tree (left-right): club, stick, fork, broom.

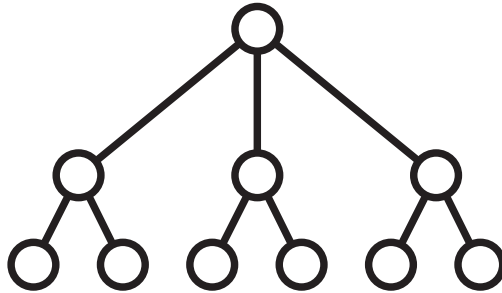


Figure 16: Worst-case tree for 3-coloring algorithm: three forks.

further.

The basic idea is as follows: we find a small set of vertices  $S \subset V(G)$  with a large set  $N$  of neighbors, and choose one of the  $3^{|S|}$  colorings for all vertices in  $S$ . For each such coloring, we translate the remaining problem to a  $(3, 2)$ -CSP instance. The vertices in  $S$  are already colored and need not be included in the  $(3, 2)$ -CSP instance. The vertices in  $N$  now have a colored neighbor, so for each such vertex at most two possible colors remain; therefore we can eliminate them from the  $(3, 2)$ -CSP instance using Lemma 2. The remaining instance has  $k = |V(G) - S - N|$  vertices, and can be solved in time  $\mathcal{O}(\Lambda^k)$  by Theorem 1. The total time is thus  $\mathcal{O}(3^{|S|}\Lambda^k)$ . By choosing  $S$  appropriately we can make this quantity smaller than  $\mathcal{O}(\Lambda^n)$ .

We can assume without loss of generality that all vertices in  $G$  have degree three or more, since smaller degree vertices can be removed without changing 3-colorability.

As a first cut at our algorithm, choose  $X$  to be any set of vertices, no two adjacent or sharing a neighbor, and maximal with this property. Let  $Y$  be the set of neighbors of  $X$ . We define a rooted forest  $F$  covering  $G$  as follows: let the roots of  $F$  be the vertices in  $X$ , let each vertex in  $Y$  be connected to its unique neighbor in  $X$ , and let each remaining vertex  $v$  in  $G$  be connected to some neighbor of  $v$  in  $Y$ . (Such a neighbor must exist or  $v$  could have been added to  $X$ ). We let the set  $S$  of vertices to be colored consist of all of  $X$ , together with each vertex in  $Y$  having three or more children in  $F$ .

We classify the subtrees of  $F$  rooted at vertices in  $Y$  as follows (Figure 15). If a vertex  $v$  in  $Y$  has no children, we call the subtree rooted at  $v$  a *club*. If  $v$  has one child, we call its subtree a *stick*. If it has two children, we call its subtree a *fork*. And if it has three or more children, we call its subtree a *broom*.

We can now compute the total time of our algorithm by multiplying together a factor of 3 for

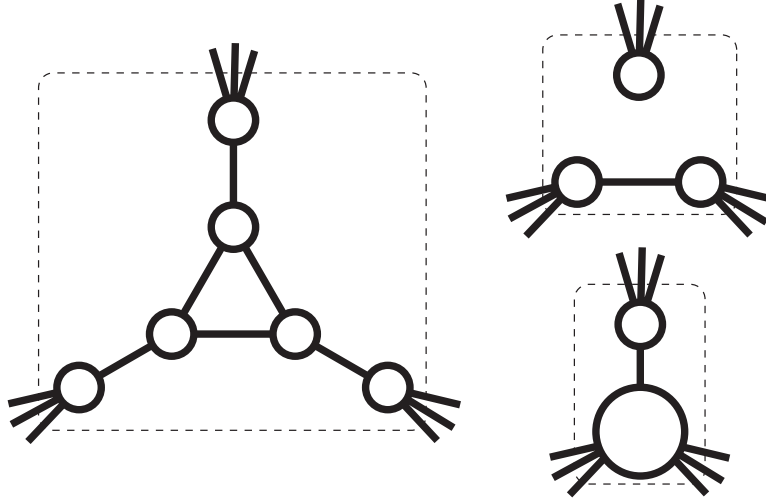


Figure 17: Cases for elimination of a triangle of degree-three vertices (left): add edge between two neighbors and eliminate triangle (top right), or merge two neighbors and third triangle vertex into a single supervertex (bottom right).

each vertex in  $S$  (that is, the roots of the trees of  $F$  and of broom subtrees) and a factor of  $\Lambda$  for each leaf in a stick or fork. We define the *cost* of a vertex in a tree  $T$  to be the product  $p$  of such factors involving vertices of  $T$ , spread evenly among the vertices—if  $T$  contains  $k$  vertices the cost is  $p^{1/k}$ . The total time of the algorithm will then be  $\mathcal{O}(c^n)$  where  $c$  is the maximum cost of any vertex. It is not hard to show that this maximum is achieved in trees consisting of three forks (Figure 16), for which the cost is  $(3(\Lambda)^6)^{1/10} \approx 1.34488$ . Therefore we can three-color any graph in time  $\mathcal{O}(1.34488^n)$ .

We can improve this somewhat with some more work.

## 6.1 Cycles of Degree-Three Vertices

We begin by showing that we can assume that our graph has a special structure: the degree-three vertices do not form any cycles. For if they do form a cycle, we can remove it cheaply as follows.

**Lemma 20** *Let  $G$  be a 3-coloring instance in which some cycle consists only of degree-three vertices. Then we can replace  $G$  by smaller instances with work factor at most  $\lambda(5, 6, 7, 8) \approx 1.2433$ .*

**Proof:** Let the cycle  $C$  consist of vertices  $v_1, v_2, \dots, v_k$ . We can assume without loss of generality that it has no chords, since otherwise we could find a shorter cycle in  $G$ ; therefore each  $v_i$  has a unique neighbor  $w_i$  outside the cycle, although the  $w_i$  need not be distinct from each other.

Note that, if any  $w_i$  and  $w_{i+1}$  are adjacent, then  $G$  is 3-colorable iff  $G \setminus C$  is; for, if we have a coloring of  $G \setminus C$ , then we can color  $C$  by giving  $v_{i+1}$  the same color as  $w_i$ , and then proceeding to color the remaining cycle vertices in order  $v_{i+2}, v_{i+3}, \dots, v_k, v_1, v_2, \dots, v_i$ . Each successive vertex has only two previously-colored neighbors, so there remains at least one free color to use, until we return to  $v_i$ . When we color  $v_i$ , all three of its neighbors are colored, but two of them have the same color, so again there is a free color.

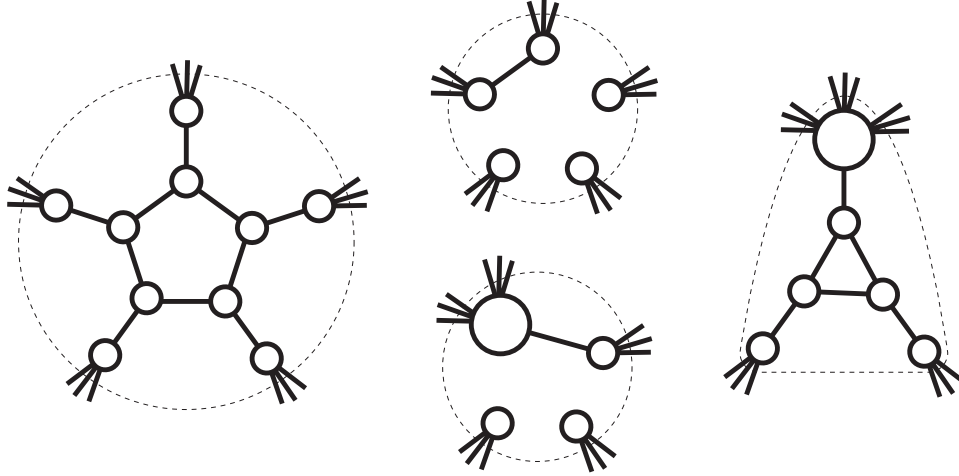


Figure 18: Cases for elimination of a long odd cycle of degree-three vertices (left): add edge between two neighbors and eliminate cycle (top middle), merge two neighbors, add edge to third neighbor, and eliminate cycle (bottom middle), or merge three neighbors and reduce cycle length by two (right).

As a consequence, if  $C$  has even length, then  $G$  is 3-colorable iff  $G \setminus C$  is; for if some  $w_i$  and  $w_{i+1}$  are given different colors, then the above argument colors  $C$ , while if all  $w_i$  have the same color, then the other two colors can be used in alternation around  $C$ .

The first remaining case is that  $k = 3$  (Figure 17, left). Then we divide the problem into two smaller instances, by forcing  $w_1$  and  $w_2$  to have different colors in one instance (by adding an edge between them, Figure 17 top right) while forcing them to have the same color in the other instance (by collapsing the two vertices into a single supervertex, Figure 17 bottom right). If we add an edge between  $w_1$  and  $w_2$ , we may remove  $C$ , reducing the problem size by three. If we give them the same color as each other, the instance is only colorable if  $v_3$  is also given the same color, so we can collapse  $v_3$  into the supervertex and remove the other two cycle vertices, reducing the problem size by four. Thus the work factor in this case is  $\lambda(3, 4) \approx 1.2207$ .

If  $k$  is odd and larger than three, we form three smaller instances, as shown in Figure 18. In the first, we add an edge between  $w_1$  and  $w_2$ , and remove  $C$ , reducing the problem size by  $k$ . In the second, we collapse  $w_1$  and  $w_2$ , add an edge between the new supervertex and  $w_3$ , and again remove  $C$ , reducing the problem size by  $k + 1$ . In the third instance, we collapse  $w_1$ ,  $w_2$ , and  $w_3$ . This forces  $v_1$  and  $v_3$  to have the same color as each other, so we also collapse those two vertices into another supervertex and remove  $v_2$ , reducing the problem size by four. For  $k \geq 7$  this gives work factor at most  $\lambda(4, 7, 8) \approx 1.1987$ . For  $k = 5$  the subproblem with  $n - 4$  vertices contains a triangle of degree-three vertices, and can be further subdivided into two subproblems of  $n - 7$  and  $n - 8$  vertices, giving the claimed work factor.  $\square$

Any degree-three vertices remaining after the application of this lemma must form components that are trees. As we now show, we can also limit the size of these trees.

**Lemma 21** *Let  $G$  be a 3-coloring instance containing a connected subset of eight or more degree-three vertices. Then we can replace  $G$  by smaller instances with work factor at most  $\lambda(2, 5, 6) \approx 1.3247$ .*

**Proof:** Suppose the subset forms a  $k$ -vertex tree, and let  $v$  be a vertex in this tree such that each subtree formed by removing  $v$  has at most  $k/2$  vertices. Then, if  $G$  is 3-colored, some two of the three neighbors of  $v$  must be given the same color, so we can split the instance into three smaller instances, each of which collapses two of the three neighbors into a single supervertex. This collapse reduces the number of vertices by one, and allows the removal of  $v$  (since after the collapse  $v$  has degree two) and the subtree connected to the third vertex. Thus we achieve work factor  $\lambda(a, b, c)$  where  $a + b + c = k + 3$  and  $\max\{a, b, c\} \leq k/2$ . The worst case is  $\lambda(2, 5, 6)$ , achieved when  $k = 8$  and the tree is a path.  $\square$

## 6.2 Planting Good Trees

We define a *bushy forest* to be an unrooted forest within a given instance graph, such that each internal node has degree four or more (for an example, see the top three levels of Figure 21). A bushy forest is *maximal* if no internal node is adjacent to a vertex outside the forest, no leaf has three or more neighbors outside the forest, and no vertex outside the forest has four or more neighbors outside the forest. If a leaf  $v$  does have three or more neighbors outside the forest, we could add those neighbors to the tree containing  $v$ , producing a bushy forest with more vertices. Similarly, if a vertex outside the forest has four or more neighbors outside the forest, we could extend the forest by adding another tree consisting of that vertex and its neighbors.

As we now show, a maximal bushy forest must cover at least a constant fraction of a 3-coloring instance graph.

**Lemma 22** *Let  $G$  be a graph in which all vertex degrees are three or more, and in which there is no cycle of degree-three vertices, let  $F$  be a maximal bushy forest in  $G$ , and let  $r$  denote the number of leaves in  $F$ . Then  $|G \setminus F| \leq 20r/3$ .*

**Proof:** Divide  $G \setminus F$  into two subsets  $X$  and  $Y$ , where  $X$  consists of the vertices of degree four or more and  $Y$  consists of the degree-three vertices.

Let  $m_{A,B}$  denote the number of edges connecting sets  $A$  and  $B$ . Then each vertex in  $X$  must have at least one edge connecting it to  $F$ , and at most three edges connecting it to  $Y$ , so  $m_{X,F} \geq |X|$  and  $m_{X,Y} \leq 3|X|$ . Further, to avoid cycles, each connected component in  $Y$  must form a tree, and if such a component has  $k$  vertices, it must have  $k + 2$  edges leaving it, and  $k \leq 7$  else we could apply Lemma 21. So,  $m_{Y,X \cup F} \geq 9|Y|/7$ . If  $3|X| \leq 9|Y|/7$ ,  $m_{F,X \cup Y} = m_{F,X} + m_{F,X \cup Y} - m_{X,Y} \geq 9|Y|/7 - 2|X| = 3|X \cup Y|/10 + (69|Y|/70 - 23|X|/10) \geq 3|X \cup Y|/10$ . And if  $3|X| \geq 9|Y|/7$ , then again  $m_{F,X \cup Y} \geq m_{F,X} \geq |X| \geq 3|X \cup Y|/10$ .

However, each leaf in  $F$  has at most two edges outside  $F$ , or  $F$  would not be maximal, so  $|X \cup Y| \leq 10m_{F,X \cup Y}/3 \leq 20r/3$ .  $\square$

## 6.3 Pruning Bad Trees

After finding a maximal bushy forest  $F$ , we find a second forest  $H$  in the remaining graph  $G \setminus F$ , as follows. Note that, due to the maximality of  $F$ , each vertex in  $G \setminus F$  has at most three neighbors in  $G \setminus F$ . We first choose a maximal set  $T$  of disjoint  $K_{1,3}$  subgraphs in  $G \setminus F$ . Then, we increase the size of  $T$  as much as possible by operations in which we remove one  $K_{1,3}$  from  $T$  and form two  $K_{1,3}$  subgraphs from the remaining vertices.

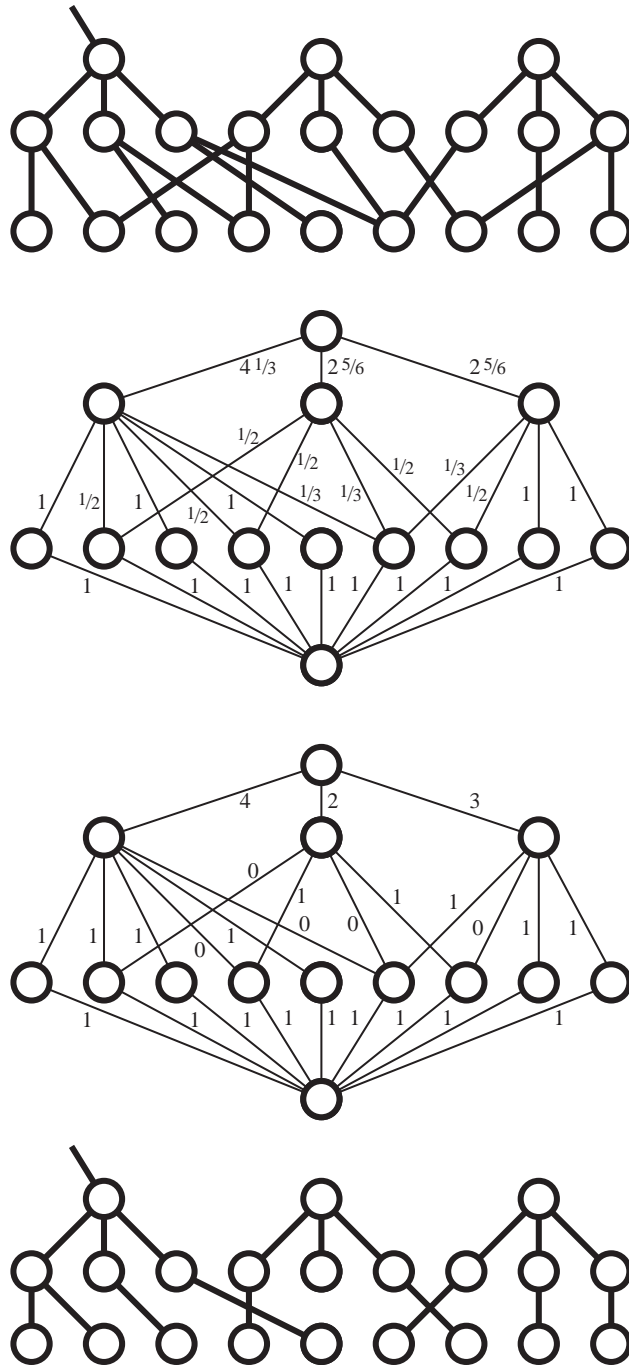


Figure 19: Use of maximum flow to find a good height-two forest. Top: forest  $T$  of  $K_{1,3}$  subgraphs and adjacent vertices in  $Y$ . Top middle: flow graph and fractional flow formed by dividing flow equally at each vertex in  $Y$ . The edge capacities are all one, except for the top three which are respectively 5, 3, and 3. Bottom middle: maximum integer flow for the same flow graph. Bottom: height-two forest corresponding to the given integer flow.



Let  $X$  denote the set of vertices in  $G \setminus (T \cup F)$  that are adjacent to vertices in  $F$ . By the maximality of  $F$ , each vertex in  $F$  is adjacent to at most two vertices in  $X$ . Let  $Y = G \setminus (X \cup T \cup F)$  denote the remaining vertices. By the maximality of  $T$ , each vertex in  $Y$  is adjacent to at most two vertices in  $X \cup Y$ , and so must have a neighbor in  $T$ . Since  $G \setminus F$  contains no degree-four vertices, each vertex in  $T$  must have at most two neighbors in  $Y$ . As we now show, we can assign vertices in  $Y$  to trees in  $T$ , extending each tree in  $T$  to a tree of height at most two, in such a way that we do not form any tree with three forks, which would otherwise be the worst case for our algorithm.

**Lemma 23** *Let  $F$ ,  $T$ ,  $X$ , and  $Y$  be as above. Then there exists a forest  $H$  of height two trees with three branches each, such that the vertices of  $H$  are exactly those of  $S \cup Y$ , such that each tree in  $H$  has at most five grandchildren, and such that any tree with four or more grandchildren contains at least one vertex with degree four or more in  $G$ .*

**Proof:** We first show how to form a set  $H'$  of non-disjoint trees in  $T \cup Y$ , and a set of weights on the grandchildren of these trees, such that each tree's grandchildren have weight at most five.

To do this, let each tree in  $H'$  be formed by one of the  $K_{1,3}$  trees in  $T$ , together with all possible grandchildren in  $Y$  that are adjacent to the  $K_{1,3}$  leaves. We assign each vertex in  $Y$  unit weight, which we divide equally among the trees it belongs to.

Then, suppose for a contradiction that some tree  $h$  in  $H'$  has grandchildren with total weight more than five. Then, its grandchildren must form three forks, and at least five of its six grandchildren must have unit weight; i.e., they belong only to tree  $h$ . Note that each vertex in  $Y$  must have degree three, or we could have added it to the bushy forest, and all its neighbors must be in  $S \cup Y$ , or we could have added it to  $X$ . The unit weight grandchildren each have one neighbor in  $h$  and two other neighbors in  $Y$ . These two other neighbors must be one each from the two other forks in  $h$ , for, if to the contrary some unit-weight grandchild  $v$  does not have neighbors in both forks, we could have increased the number of trees in  $T$  by removing  $h$  and adding new trees rooted at  $v$  and at the missed fork.

Thus, these five grandchildren each connect to two other grandchildren, and (since no grandchild connects to three grandchildren) the six grandchildren together form a degree-two graph, that is, a union of cycles of degree-three vertices. But after applying Lemma 20 to  $G$ , it contains no such cycles. This contradiction implies that the weight of  $h$  must be at most five.

Similarly, if the weight of  $h$  is more than three, it must have at least one fork, at least one unit-weight grandchild outside that fork, and at least one edge connecting that grandchild to a grandchild within the fork. This edge together with a path in  $h$  forms a cycle, which must contain a high degree vertex.

We are not quite done, because the assignment of grandchildren to trees in  $H'$  is fractional and non-disjoint. To form the desired forest  $H$ , construct a network flow problem in which the flow source is connected to a node representing each tree  $t \in T$  by an edge with capacity  $w(t) = 5$  if  $t$  contains a high degree vertex and capacity  $w(t) = 3$  otherwise. The node corresponding to tree  $t$  is connected by unit-capacity edges to nodes corresponding to the vertices in  $Y$  that are adjacent to  $t$ , and each of these nodes is connected by a unit-capacity edge to a flow sink. Then the fractional weight system above defines a flow that saturates all edges into the flow sink and is therefore maximum (Figure 19, middle top). But any maximum flow problem with integer edge capacities has an integer solution (Figure 19, middle bottom). This solution must continue to saturate the sink edges, so each vertex in  $Y$  will have one unit of flow to some tree  $t$ , and no flow to the other adjacent trees. Thus, the flow corresponds to an assignment of vertices in  $Y$  to adjacent trees in  $T$  such that

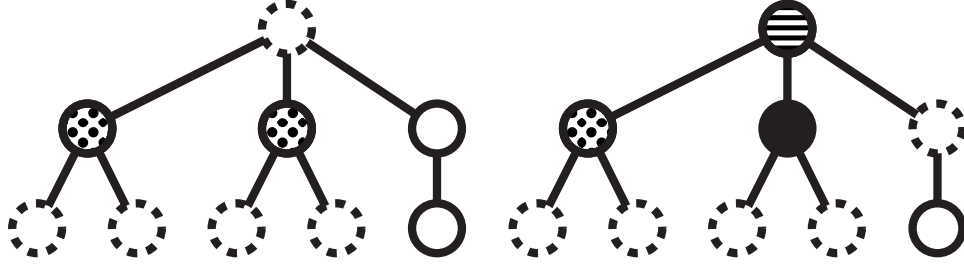


Figure 20: Coloring a tree with two forks and one stick. If the two fork vertices are colored the same (left), five neighbors (dashed) are restricted to two colors, leaving the two stick vertices for the  $(3, 2)$ -CSP instance. If the two forks are colored differently (right), they force the tree root to have the third color, leaving only one vertex for the  $(3, 2)$ -CSP instance.

each tree is assigned at most  $w(t)$  vertices. We then simply let each tree in  $H$  consist of a tree in  $T$  together with its assigned vertices in  $Y$  (Figure 19, bottom).  $\square$

## 6.4 Improved Tree Coloring

We now discuss how to color the trees in the height-two forest  $H$  constructed in the previous subsection. As in the discussion at the start of this section, we color some vertices (typically just the root) of each tree in  $H$ , leave some vertices (typically the grandchildren) to be part of a later  $(3, 2)$ -CSP instance, and average the costs over all the vertices in the tree. However, we average the costs in the following strange way: a cost of  $\Lambda$  is assigned to any vertex with degree four or higher in  $G$ , as if it was handled as part of the  $(3, 2)$ -CSP instance. The remaining costs are then divided equally among the remaining vertices.

**Lemma 24** *Let  $T$  be a tree with three children and at most five grandchildren. Then  $T$  can be colored with cost per degree-three vertex at most  $(3\Lambda^3)^{1/7} \approx 1.3366$ .*

**Proof:** First, suppose that  $T$  has exactly five grandchildren. At least one vertex of  $T$  has high degree. Two of the children  $x$  and  $y$  must be the roots of forks, while the third child  $z$  is the root of a stick. We test each of the nine possible colorings of  $x$  and  $y$ . In six of the cases,  $x$  and  $y$  are different, forcing the root to have one particular color (Figure 20, right). In these cases the only remaining vertex after translation to a  $(3, 2)$ -CSP instance and application of Lemma 2 will be the child of  $z$ , so in each such case  $T$  accumulates a further cost of  $\Lambda$ . In the three cases in which  $x$  and  $y$  are colored the same (Figure 20, left), we must also take an additional factor of  $\Lambda$  for  $z$  itself. One of these  $\Lambda$  factors goes to a high degree vertex, while the remaining work is split among the remaining eight vertices. The cost per vertex in this case is then at most  $(6 + 3\Lambda)^{1/8} \approx 1.3351$ .

If  $T$  has fewer than five grandchildren, we choose a color for the root of the tree as described at the start of the section. The worst case occurs when the number of grandchildren is either three or four, and is  $(3\Lambda^3)^{1/7} \approx 1.3366$ .  $\square$

## 6.5 The Vertex Coloring Algorithm

**Theorem 2** *We can solve the 3-coloring problem in time  $\mathcal{O}((2^{3/49} 3^{4/49} \Lambda^{24/49})^n) \approx 1.3289^n$ .*

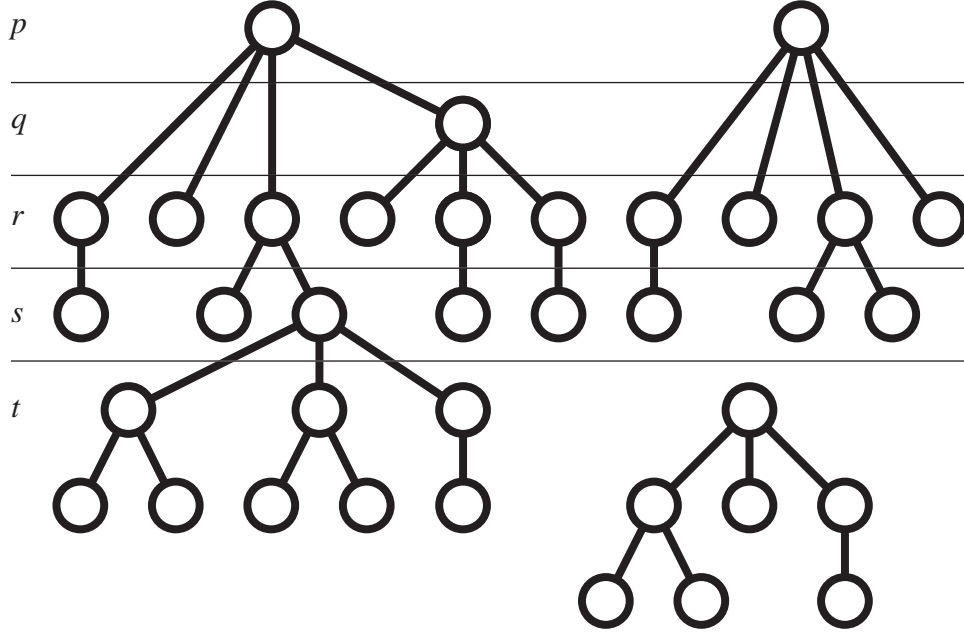


Figure 21: Partition of vertices into five groups:  $p$  bushy forest roots,  $q$  other bushy forest internal nodes,  $r$  bushy forest leaves,  $s$  vertices adjacent to bushy forest leaves, and  $t$  degree-three vertices in height-two forest.

**Proof:** As described in the preceding sections, we find a maximal bushy forest, then cover the remaining vertices by height-two trees. We choose colors for each internal vertex in the bushy forest, and for certain vertices in the height-two trees as described in Lemma 24. Vertices adjacent to these colored vertices are restricted to two colors, while the remaining vertices form a  $(3, 2)$ -CSP instance and can be colored using our general  $(3, 2)$ -CSP algorithm. Let  $p$  denote the number of vertices that are roots in the bushy forest;  $q$  denote the number of non-root internal vertices;  $r$  denote the number of bushy forest leaves;  $s$  denote the number of vertices adjacent to bushy forest leaves; and  $t$  denote the number of remaining vertices, which must all be degree-three vertices in the height-two forest (Figure 21). Then the total time for the algorithm is at most  $3^p 2^q \Lambda^s (3\Lambda^3)^{t/7}$ .

We now consider which values of these parameters give the worst case for this time bound, subject to the constraints  $p, q, r, s, t \geq 0$ ,  $p + q + r + s + t = n$ ,  $4p + 2q \leq r$  (from the definition of a bushy forest),  $2r \geq s$  (from the maximality of the forest), and  $20r/3 \geq s + t$  (Lemma 22). We ignore the slightly tighter constraint  $p \geq 1$  since it only complicates the overall solution.

Since the work per vertex in  $s$  and  $t$  is larger than that in the bushy forests, the time bound is maximized when  $s$  and  $t$  are as large as possible; that is, when  $s + t = 20r/3$ . Further since the work per vertex in  $s$  is larger than that in  $t$ ,  $s$  should be as large as possible; that is,  $s = 2r$  and  $t = 14r/3$ . Increasing  $p$  or  $q$  and correspondingly decreasing  $r$ ,  $s$ , and  $t$  only increases the time bound, since we pay a factor of 2 or more per vertex in  $p$  and  $q$  and at most  $\Lambda$  for the remaining vertices, so in the worst case the constraint  $4p + 2q \leq r$  becomes an equality.

It remains only to set the balance between parameters  $p$  and  $q$ . There are two candidate solutions: one in which  $q = 0$ , so  $r = 4p$ , and one in which  $p = 0$ , so  $r = 2q$ . In the former case  $n = p + 4p + 8p + 56p/3 = 95p/3$  and the time bound is  $3^p \Lambda^{8p} (3\Lambda^3)^{8p/3} = 3^{11p/3} \Lambda^{16p} \approx 1.3287^n$ . In the latter case  $n = q + 2q + 4q + 28q/3 = 49q/3$  and the time bound is  $2^q \Lambda^{4q} (3\Lambda^3)^{4q/3} =$

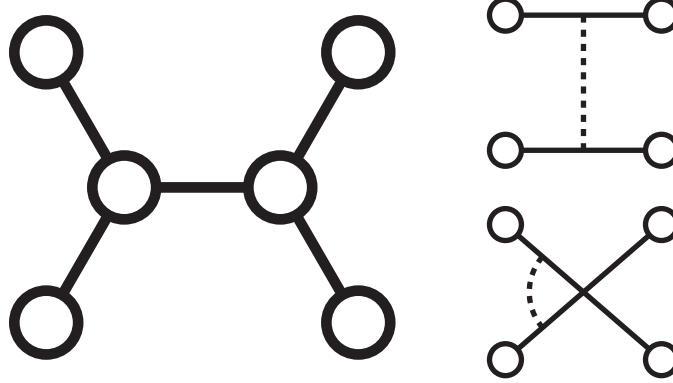


Figure 22: Replacement of five edges (left) by two constrained edges (right).

$$2^q 3^{4q/3} \Lambda^{8q} \approx 1.3289^n. \quad \square$$

## 7 Edge Coloring

We now describe an algorithm for finding edge colorings of undirected graphs, using at most three colors, if such colorings exist. We can assume without loss of generality that the graph has vertex degree at most three. Then  $m \leq 3n/2$ , so by applying our vertex coloring algorithm to the line graph of  $G$  we could achieve time bound  $1.3289^{3n/2} \approx 1.5319^n$ . Just as we improved our vertex coloring algorithm by performing some reductions in the vertex coloring model before treating the problem as a  $(3, 2)$ -CSP instance, we improve this edge coloring bound by performing some reductions in the edge coloring model before treating the problem as a vertex coloring instance.

The main idea is to solve a problem intermediate in generality between 3-edge-coloring and 3-vertex-coloring: 3-edge-coloring with some added constraints that certain pairs of edges should not be the same color.

**Lemma 25** *Suppose a constrained 3-edge-coloring instance contains an unconstrained edge connecting two degree-three vertices. Then the instance can be replaced by two smaller instances with three fewer edges and two fewer vertices each.*

**Proof:** Let the given edge be  $(w, x)$ , and let its four neighbors be  $(u, w)$ ,  $(v, w)$ ,  $(x, y)$ , and  $(x, z)$ . Then  $(w, x)$  can be colored only if its four neighbors together use two of the three colors, which forces these neighbors to be matched into equally colored pairs in one of two ways. Thus, we can replace the instance by two smaller instances: one in which we replace the five edges by the two edges  $(u, y)$  and  $(v, z)$ , and one in which we replace the five edges by the two edges  $(u, z)$  and  $(v, y)$ ; in each case we add a constraint between the two new edges.  $\square$

The reduction operation described in Lemma 25 is depicted in Figure 22.

We let  $m_3$  denote the number of edges with three neighbors in an unconstrained 3-edge-coloring instance, and  $m_4$  denote the number of edges with four neighbors. Edges with fewer neighbors can be removed at no cost, so we can assume without loss of generality that  $m = m_3 + m_4$ .

**Lemma 26** *In an unconstrained 3-edge-coloring instance, we can find in polynomial time a set  $S$  of  $m_4/3$  edges such that Lemma 25 can be applied independently to each edge in  $S$ .*

**Proof:** Use a maximum matching algorithm in the graph induced by the edges with four neighbors. If the graph is 3-colorable, the resulting matching must contain at least  $m_4/3$  edges. Applying Lemma 25 to an edge in a matching neither constrains any other edge in the matching, nor causes the remaining edges to stop being a matching.  $\square$

**Lemma 27**  $m_3 = \frac{6}{5}n - \frac{4}{5}m_4$ .

**Proof:** Assign a charge of  $6/5$  to each vertex of the graph, and redistribute this charge equally to each incident edge. Further assign an additional  $1/5$  charge to each four-neighbor edge. Then each edge receives a unit charge, so  $m_3 + m_4 = m = (6/5)n + (1/5)m_4$ . Subtracting  $m_4$  from both sides yields the result.  $\square$

**Theorem 3** *We can 3-edge-color any 3-edge-colorable graph, in time  $\mathcal{O}(2^{n/2})$ .*

**Proof:** We apply Lemma 26, resulting in a set of  $2^{m_4/3}$  constrained 3-edge-coloring problems each having only  $m_3$  edges. We then treat these remaining problems as 3-vertex-coloring problems on the corresponding line graphs, augmented by additional edges representing the constraints added by Lemma 25. The time for this algorithm is thus at most  $\mathcal{O}(1.3289^{m_3} 2^{m_4/3})$ . By Lemma 27, we can rewrite this bound as  $\mathcal{O}(1.3289^{6n/5} (2^{1/3} 1.3289^{-4/5})^{m_4})$ . Since  $2^{1/3} 1.3289^{-4/5} > 1$ , this time bound is maximized when  $m_4$  is maximized, which occurs when  $m_4 = 3n/2$  and  $m_3 = 0$ . For this value, all the work occurs within Lemma 26, and gives the stated time bound.  $\square$

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