ASYNCHRONOUS OPTIMIZED SCHWARZ METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS IN RECTANGULAR DOMAINS

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ABSTRACT

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Asynchronous iterative algorithms are parallel iterative algorithms in which communications and iterations are not synchronized among processors. Thus, as soon as a processing unit finishes its own calculations, it starts the next cycle with the latest data received during a previous cycle, without waiting for any other processing unit to complete its own calculation. These algorithms increase the number of updates in some processors (as compared to the synchronous case) but suppress most idle times. This usually results in a reduction of the (execution) time to achieve convergence.

Optimized Schwarz methods (OSM) are domain decomposition methods in which the transmission conditions between subdomains contain operators of the form $\partial/\partial\nu + \Lambda$, where $\partial/\partial\nu$ is the outward normal derivative and Λ is an optimized local approximation of the global Steklov-Poincaré operator. There is more than one family of transmission conditions that can be used for a given partial differential equation (e.g., the $OO0$ and $OO2$ families), each of these families containing a particular approximation of the Steklov-Poincaré operator. These transmission conditions have some parameters that are tuned to obtain a fast convergence rate. Optimized Schwarz methods are fast in terms of iteration count and can be implemented asynchronously.

In this thesis we analyze the convergence behavior of the synchronous and asynchronous implementation of OSM applied to solve partial differential equations with a shifted Laplacian operator in bounded rectangular domains. We analyze two cases. In the first case we have a shift that can be either positive, negative or zero, a one-way domain decomposition and transmission conditions of the $OO2$ family. In the second case we have Poisson's equation, a domain decomposition with cross-points and $OO0$ transmission conditions. In both cases we reformulate the equations defining the problem into a fixed point iteration that is suitable for our analysis, then derive convergence proofs and analyze how the convergence rate varies with the number of subdomains, the amount of overlap, and the values of the parameters introduced in the transmission conditions. Additionally, we find the optimal values of the parameters and present some numerical experiments for the second case illustrating our theoretical results. To our knowledge this is the first time that a convergence analysis of optimized Schwarz is presented for bounded subdomains with multiple subdomains and arbitrary overlap. The analysis presented in this thesis also applies to problems with more general domains which can be decomposed as a union of rectangles.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Since approximately 2005, the clock speed of processors has stagnated [5]. Consequently, the time it takes to perform a floating point operation has stagnated as well. Since then, the computational power of the computers has been improved by packing more cores into a single processor, and with this increase in the number of floating point operations (flops) per second that a computer can do. This implies that to take advantage of the computational power of current computers, parallelization of the problems to solve is important. A problem is fully parallelizable if it can be divided into independent parts, i.e., parts that do not need to communicate information among themselves. In many situations, however, we run into problems that are not fully parallelizable, i.e., the tasks performed by different processing units need some information from the tasks performed at other processors. Therefore, in these cases communication of information between different processing units in necessary. Unfortunately, there has not been a substantial improvement in the speed of communication of data among processors and between processors and memory devices [4]. Thus, although computers are able to perform a large number of flops per second, in practice the number of flops performed per second is less than the maximum possible given that some (or all) processing units are idle, waiting to receive the information needed to start performing their next assigned task. Given the gap between the number of flops that a computer can be done per second and the time required to communicate data among processors and memory, the communication part is creating a bottleneck in current computations, and thus producing a performance inefficiency. Therefore, it is desirable to formulate the problem in a way that solving it requires the less communication as possible.

In the context of simulations requiring extreme scale computing, parallel computations are necessary, and they are usually performed through parallel iterative algorithms that require the communication of information among processors and among processors and memory.

Synchronous iterative algorithms are parallel iterative algorithms in which iterations and communications are synchronized among processors. In this synchronous paradigm, in addition to the waiting time due to communication speed, any load imbalance or non-uniformity in hardware performance also causes processing units to idle at the synchronization point, waiting for the slowest unit, and deteriorates performance even more.

Given the heterogeneous and distributed architecture of the anticipated exascale computers [15], idle times in processing units will be an issue in terms of efficiency, and this problem will be in particular exacerbated due to the synchronization. Therefore, it is desirable to minimize the amount of communication and/or the time that processors remain idle.

Asynchronous iterative algorithms are parallel iterative algorithms in which communications and iterations are not synchronized among processors [8]. Thus, as soon as a processing unit finishes its own calculations, it starts the next cycle with the latest data received during a previous cycle, without waiting for any other processing unit. These algorithms increase the number of updates in some processors (with respect to the synchronous case) but suppress most idle times. This usually results in a reduction of the (execution) time to achieve convergence. In other words, in certain cases we might be able to reach the finish line faster by walking a longer path but without stopping after each step, than a shorter path with a certain waiting time between steps. Thus, in certain situations, asynchronous iterative algorithms can reduce the bottleneck problem due to communications and therefore reduce the execution time of a solver.

1.2 Background

Domain Decomposition (DD) methods are iterative in which the original problem is divided into coupled smaller subproblems that are solved to produce local approximations of the global true solution until the approximations are close enough to the true global solution [17]. In the context of the solution of partial differential equations, the original domain is divided into smaller subdomains, possibly with overlap. Some of the boundaries of these subdomains are artificial, i.e., they were created artificially after the decomposition of the original domain, and some other are physical boundaries, i.e, those who coincide with the boundaries of the original domain. Then, subproblems (local problems) are defined in each subdomain. Each of these subproblems consists of the partial differential equation (PDE) that we want to solve in the original domain but restricted to the subdomain, and the corresponding boundary conditions. There are two possible types of boundary conditions, namely, artificial boundary conditions, also known as transmission conditions, which are defined on the artificial boundaries, and the physical boundary conditions related to the physical boundaries. Once the subproblems are defined, each subproblem is solved using boundary data coming from other subdomains. This process is repeated until each local approximation is close enough to the true global solution, i.e., the true solution of the original problem.

Domain decomposition methods are naturally parallel in the sense that the only information needed to produce a new approximation in each local problem is the boundary data at the beginning of the computation of this new approximation. Besides this, each local process is completely independent from the others. Two major

DD categories are the non-overlapping methods, also known as iterative substructuring methods, and the overlapping methods known as Schwarz methods. In this thesis we focus on Schwarz methods.

There are different types of Schwarz domain decomposition methods, each of them defined by a particular choice of artificial boundary condition. Classical Schwarz methods are those in which the artificial boundary conditions are Dirichlet conditions. Optimized Schwarz methods are those in which the artificial boundary conditions contain an operator of the form $\frac{\partial}{\partial \nu} + \Lambda$ where ν is the normal derivative pointing outwards and Λ is an approximation of the Steklov-Poincaré operator by using local differential operators. There is more than one family of transmission conditions that can be used for a given PDE , each of these families consisting of a particular approximation of the optimal transmission conditions [9]. For example, we have the $OO0$ and $OO2$ families. In the $OO0$ family Λ is the zero$-th$ order approximation of the Steklov-Poincaré operator, i.e., $\Lambda = \alpha$, where α is a constant. As for the $OO2$ family, we take $\Lambda = \alpha + \beta\Delta$, where α and β are constants and Δ is the Laplace-Beltrami operator. The convergence factor (usually the spectral radius of the iteration operator) of the method depends on the value of the boundary conditions parameters (i.e., α and β). Thus, in optimized Schwarz we tune the values of these boundary parameters to improve convergence with respect to the classical Schwarz case ($\alpha = \infty, \beta = 0$).

Schwarz methods can be used as solvers or as preconditioners for a Krylov subspace method. Usually, a Krylov subspace method preconditioned with an optimized Schwarz method will be somewhat faster than the same optimized Schwarz method applied as a solver, when both solvers are implemented synchronously. However, the difference in terms of iteration count is not always very large; see e.g., [1].

Preconditioned Krylov subspace methods are very fast in terms of iteration count. Unfortunately, these type of methods are inherently synchronous containing inner products that require global synchronization. In fact, the inner products involved in the orthogonalization of the basis of the Krylov subspace are operations that necessarily require global synchronization. Schwarz methods, however, when used as solvers, do not require global synchronization. Therefore they can be implemented asynchronously.

With the goal of reducing the bottleneck problem in extreme-scale computations, Magoulès proposed in [13] a method composed of two ingredients:

1. The use of optimized Schwarz methods as outer solvers.

2. Asynchronous iterations.

Optimized Schwarz methods are fast in terms of iteration count and implementing it in an asynchronous fashion may in principle reduce substantially the execution time of the solver. The numerical results presented by Magoulés in [13] provided evidence of this idea. However, a proof of convergence was needed. Magoulés, Szyld and Venet in [14] presented a convergence proof of Asynchronous Optimized Schwarz (AOS) for the problem defined by Poisson's equation in an infinite plane using a one way decomposition; see also [10]. In this thesis we present a convergence proof of Asynchronous Optimized Schwarz for cases with bounded rectangular domains. We study the problem with the shifted Laplacian operator. The case with zero shift corresponds to Poisson's equation, the negative shift case to the Screened Poisson's equation and the positive shift case to the Helmholtz equation. Poisson's equation has applications in problems modelling the diffusion of substances or the steady-state heat distribution in materials. The Screened Poisson's equation arises for example in electric field screening in plasmas. The Helmholtz equation has applications in seismology and acoustics.

1.3 Outline

In Chapter 2, we study the convergence of AOS applied to solve a PDE with the shifted Laplacian operator, with Dirichlet physical boundary conditions, and artificial boundary conditions of the $OO2$ family. We use a one-way domain decomposition, and the analysis applies to the cases where the shift is positive, negative or zero. In Chapter 3 we study the convergence of AOS applied to the problem involving Poisson's Equation, with Dirichlet physical boundary conditions, $OO0$ artificial boudnary conditions and a partition of the domain containing cross-points. Cross-points are points where the boundaries of more than two subdomains meet.

In each of the chapters, we first recast the equations defining the corresponding problem into a fixed point iteration that is suitable for our analysis. Then we present convergence proofs of the Asynchronous Optimized Schwarz method. Next, we analyze the behavior of the convergence rate for different values of the parameters. From this analysis we obtain the evidence that confirms that the hypotheses of our convergence theorems hold, and we also determine the optimal values of the boundary parameters that optimize the convergence rate of the method.

1.4 Contribution of this thesis

Our main contribution can be divided into five parts.

1. The reformulation of the original iteration formulas (from which no conclusion can be made about convergence) into an equivalent fixed point iteration from which convergence can be analyzed and proved.

2. The proof of convergence of the synchronous and asynchronous iterations of Optimized Schwarz for the case of the shifted-Laplacian equation when we have

    - a shift that can be positive, negative or zero (this encompasses both Poisson's equation and Helmholtz equation),

- a one-way domain decomposition,
- a bounded domain with multiple subdomains,
- $OO2$ artificial boundary conditions,
- arbitrary overlap between adjacent subdomains, and
- any number of subdomains.

3. The proof of convergence of the synchronous and asynchronous iterations of Optimized Schwarz for the case of Poisson's equation when we have

   - a bounded domain,
   - a domain decompostion with cross-points,
   - $OO0$ boundary conditions, and
   - arbitrary overlap

4. The study of the convergence behavior of the Optimized Schwarz method when we vary the values of parameters such as the number of subdomains, the amount of overlap between subdomains, the shift and the artificial boundary conditions parameters.

5. The provision of empirical formulas for the optimal boundary parameters for certain cases.

CHAPTER 2

SHIFTED LAPLACIAN: ONE WAY DECOMPOSITION OF 2D DOMAIN

2.1 Preliminaries

In this chapter we consider the solution of a PDE with the shifted Laplacian operator, in a rectangular domain with Dirichlet boundary conditions using optimized Schwarz methods, where the division of the domain is carried out in a one-way decomposition. We consider that the transmission conditions on the artificial boundaries are of the $OO2$ family. The shift can be positive, negative or zero. The zero shift case corresponds to Poisson's equation, which has applications in problems involving diffusion of substances or the steady state heat distribution in a material. The PDE resulting from a negative shift is known as the Screened Poisson's equation and arises, for example, in electric field screening in plasmas. The positive shift case corresponds to the Helmholtz equation, which models, for example, the propagation of acoustic waves.

2.2 Formulation of the Problem

We want to solve the problem defined by

$$\begin{cases} \Delta u + \omega u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega = [0, W] \times [0, H]$ and $\omega \in \mathbb{R}$. We consider that ω and H are such that $\omega H^2/\pi^2 \notin \mathbb{N}$.

We divide the physical domain into p overlapping rectangular subdomains. Each of these subdomains is represented by an index s with $s \in \{1, ..., p\}$. Let h be the length of the side of each subdomain as if it were a partition with no overlap. Let us now displace (outwards) the boundaries artificial boundaries of the nonoverlapping subdomains by an amount γ so as to have overlapping rectangular subdomains. Thus, we have interior subdomains (i.e., subdomains that have artificial boundaries on their left and the right sides) with width $L := h + 2\gamma$ and height H, and the left-most and a right-most subdomains with width $h + \gamma$ and height H. The union of all subdomains is the original domain; see Figure 2.1. The overlap between two adjacent subdomains is 2γ, thus we can use γ as a parameter to quantify the amount of overlap between subdomains. The subdomain s, denoted by $\Omega^{(s)}$, is given by

$$\Omega^{(s)} = [(s-1)h - \gamma\chi_{\{1\}^c}(s), sh + \gamma\chi_{\{p\}^c}(s)]$$

where

$$\chi_{\{j\}^c}(s) := \begin{cases} 1 & \text{if } s \neq j \\ 0 & \text{if } s = j\ . \end{cases}$$

Thus, the interior subdomains are those corresponding to $1 < s < p$, the left-most subdomain corresponds to $s = 1$ and the right-most subdomain to $s = p$.

We define each local problem using local coordinates. The local coordinates associated to the subdomain s are those in the coordinate system whose origin is on the bottom-left corner of the subdomain s. The relation between the local coordinates, denoted by (x, y), and the global coordinates, denoted by (x_g, y_g), is given by the

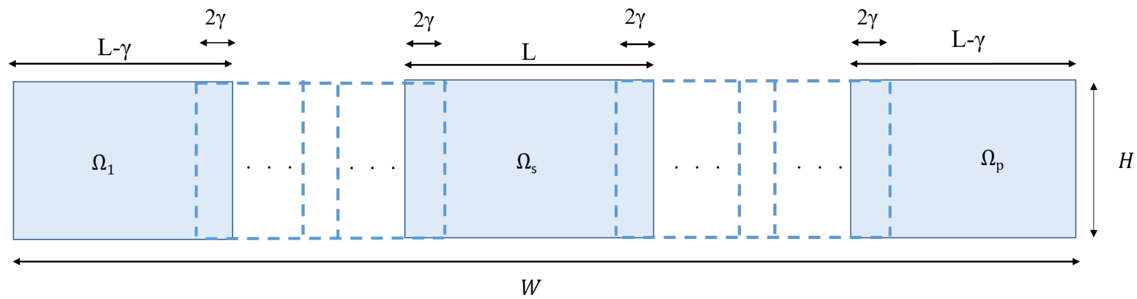

Figure 2.1: One-way decomposition of the domain

following formulas

$$x = x_g - (s-1)h - \gamma\chi_{\{1\}^c}(s), \tag{2.2}$$

$$y = y_g. \tag{2.3}$$

Thus, the optimized Schwarz (OS) iteration process associated with problem (2.1) and with $OO2$ transmission conditions, using local coordinates is defined, for an interior subdomain (i.e., for $1 < s < p$), by

$$\begin{cases} \Delta u_{n+1}^s + \omega u_{n+1}^s = f & \text{in } (0,L)\times(0,H) \\ -\frac{\partial u_{n+1}^s}{\partial x} + \alpha u_{n+1}^s + \beta\frac{\partial^2 u_{n+1}^s}{\partial y^2} = -\frac{\partial u_n^{s-1}}{\partial x} + \alpha u_n^{s-1} + \beta\frac{\partial^2 u_n^{s-1}}{\partial y^2} & \text{for } x = 0 \\ \frac{\partial u_{n+1}^s}{\partial x} + \alpha u_{n+1}^s + \beta\frac{\partial^2 u_{n+1}^s}{\partial y^2} = \frac{\partial u_n^{s+1}}{\partial x} + \alpha u_n^{s+1} + \beta\frac{\partial^2 u_n^{s+1}}{\partial y^2} & \text{for } x = L \\ u_{n+1}^s = g & \text{for } y = 0 \\ u_{n+1}^s = g & \text{for } y = H, \end{cases} \tag{2.4}$$

where $\frac{\partial}{\partial x}$ is, in this instance, a normal derivative and $\frac{\partial^2}{\partial y^2}$ a tangential second derivative corresponding to the Laplace-Beltrami operator. The equations describing the

Optimized Schwarz (OS) iteration process for the left-most subdomain are given by

$$\begin{cases} \Delta u_{n+1}^s + \omega u_{n+1}^s = f & \text{in } (0, L-\gamma) \times (0, H) \\ \alpha u_{n+1}^s = g & \text{for } x = 0 \\ \frac{\partial u_{n+1}^s}{\partial x} + \alpha u_{n+1}^s + \beta \frac{\partial^2 u_{n+1}^s}{\partial y^2} = \frac{\partial u_n^{s+1}}{\partial x} + \alpha u_n^{s+1} + \beta \frac{\partial^2 u_n^{s+1}}{\partial y^2} & \text{for } x = L-\gamma \\ u_{n+1}^s = g & \text{for } y = 0 \\ u_{n+1}^s = g & \text{for } y = H, \end{cases} \tag{2.5}$$

and for the right-most subdomain

$$\begin{cases} \Delta u_{n+1}^s + \omega u_{n+1}^s = f & \text{in } (0, L-\gamma) \times (0, H) \\ -\frac{\partial u_{n+1}^s}{\partial x} + \alpha u_{n+1}^s + \beta \frac{\partial^2 u_{n+1}^s}{\partial y^2} = -\frac{\partial u_n^{s-1}}{\partial x} + \alpha u_n^{s-1} + \beta \frac{\partial^2 u_n^{s-1}}{\partial y^2} & \text{for } x = 0 \\ u_{n+1}^s = g & \text{for } x = L-\gamma \\ u_{n+1}^s = g & \text{for } y = 0 \\ u_{n+1}^s = g & \text{for } y = H. \end{cases} \tag{2.6}$$

In all cases we consider that $\beta \leq 0$ and $\alpha > 0$.

Equations (2.4)-(2.6) describe a fixed point iteration of the form $u_{n+1} = Tu_n$, where the iteration operator T is defined implicitly by those equations. In order to analyze the convergence of the method, however, we shall obtain an equivalent fixed point iteration which is easier to analyze and for which we can obtain an explicit expression for its iteration operator. To that end, in the next section we introduce the Fourier series expansion of the local errors and then find the formulas relating the Fourier series coefficients of the local errors at iteration $n+1$ to those at iteration n. Then we will use this relation between coefficients to define the equivalent fixed point iteration.

2.3 A fixed point iteration

To determine the new fixed point iteration, we obtain first the equations defining the local errors. Hence, let us denote the local error corresponding to the subdomain s at iteration n by η_n^s. Let u_* be the solution of (2.1). Let u_*^s be the restriction of u_* to $\Omega^{(s)}$. Then, we have that $\eta_n^s = u_* - u_n^s$. Thus, plugging this expression of η_n^s in (2.4)-(2.6) we can see by linearity that η_{n+1}^s satisfies a set of equations of the form (2.4)-(2.6) but with $f = g = 0$.

We define the normalized parameters as $\bar{\alpha} := \alpha H$, $\bar{\beta} := \beta/H$, $\bar{\gamma} := \gamma/H$, $\bar{L} := L/H$, and $\bar{\omega} := \omega H^2$. As we will see later in the chapter, the advantage of working with these normalized parameters is that when we do the change of variables given by their definitions, the resulting formulas for the error coefficients and for the entries of the new iteration operator do not contain the parameters L and H. This means that when we applied optimized Schwarz to two problems that are such that one is a scaled version of the other and they have the same partitioning of the original domain (i.e., they have the same number of subdomains, and the subdomains of one of the problems are just scaled versions of the subdomains of the other problem), the iterative method resulting from both cases will have the same convergence behavior.

From the theory given in Appendix A, it follows that the solution of (2.4) can be written in terms of the following series

$$\eta_{n+1,1}^s(x,y) = \sum_{m=1}^{\infty} \left[A_{n+1,m,1}^s \psi_m^{(1)}(x) + A_{n+1,m,2}^s \psi_m^{(2)}(x) \right] \phi_m(y), \tag{2.7}$$

where

$$\phi_m(y) = \sin\left(\frac{z_m y}{H}\right), \tag{2.8}$$

with $z_m = m\pi$, and

$$\psi_m^{(1)}(x) = \begin{cases} \frac{i\sqrt{z_m^2-\bar{\omega}}+(\bar{\alpha}-\beta z_m^2)}{i\sqrt{z_m^2-\bar{\omega}}-(\bar{\alpha}-\beta z_m^2)} e^{-i\sqrt{\lambda-\omega}(x-L)} + e^{i\sqrt{\lambda-\omega}(x-L)}, & \text{if } z_m^2 < \bar{\omega}, \\ e^{-\frac{(x-L)}{H}\sqrt{z_m^2-\bar{\omega}}} - \left(\frac{\alpha-\beta z_m^2-\sqrt{z_m^2-\bar{\omega}}}{\alpha-\beta z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right) e^{\frac{(x-L)}{H}\sqrt{z_m^2-\bar{\omega}}}, & \text{if } z_m^2 > \bar{\omega}, \end{cases} \tag{2.9}$$

and

$$\psi_m^{(2)}(x) = \begin{cases} \frac{i\sqrt{z_m^2-\bar{\omega}}-(\bar{\alpha}-\beta z_m^2)}{i\sqrt{z_m^2-\bar{\omega}}+(\bar{\alpha}-\beta z_m^2)} e^{-i\sqrt{\lambda-\omega}(x)} + e^{i\sqrt{\lambda-\omega}(x)}, & \text{if } z_m^2 < \bar{\omega}, \\ e^{-\frac{x}{H}\sqrt{z_m^2-\bar{\omega}}} - \left(\frac{\alpha-\beta z_m^2+\sqrt{z_m^2-\bar{\omega}}}{\alpha-\beta z_m^2-\sqrt{z_m^2-\bar{\omega}}}\right) e^{\frac{x}{H}\sqrt{z_m^2-\bar{\omega}}}, & \text{if } z_m^2 > \bar{\omega}. \end{cases} \tag{2.10}$$

In other words, for a fixed value of x, the series in (2.7) is the Fourier sine series in the variable y of $\eta_{n+1}^s(x, y)$.

The solution of the left-most and right-most subdomain local problems can be written as

$$\eta_{n+1}^1(x, y) = \sum_{m=1}^{\infty} A_{n+1,m,2}^1 \psi_m^{(3)}(x)\phi_m(y) \tag{2.11}$$

and

$$\eta_{n+1}^p(x, y) = \sum_{m=1}^{\infty} A_{n+1,m,1}^p \psi_m^{(3)}(x - L)\phi_m(y), \tag{2.12}$$

respectively, where

$$\psi_m^{(3)}(x) = \begin{cases} 2i\sin(\sqrt{z_m^2 - \omega}x), & \text{if } z_m^2 < \bar{\omega}, \\ e^{-(\frac{x}{H}\sqrt{z_m^2-\omega})} - e^{(\frac{x}{H}\sqrt{z_m^2-\omega})}, & \text{if } z_m^2 > \bar{\omega}. \end{cases} \tag{2.13}$$

Remark 2.1. Note that the expressions on the first line of equations (2.9), (2.10), and (2.13) are qualitatively different from those on the second line of the respective equations. The error series terms containing these expressions only appear when we have $m \in \mathbb{N}$ such that $z_m^2 < \bar{\omega}$. This is only possible when $\bar{\omega} > 0$, i.e., for the

Helmholtz equation case. Given that ω is fixed and $z_m \to \infty$ as $m \to \infty$, there are only a finite number of these terms. The error series for the Helmholtz case contain two types of terms, each type having a different behavior. The error series for $\omega \leq 0$ contain only terms of one type, those containing the expressions on the second line of equations (2.9), (2.10), and (2.13). Therefore, we expect that the behavior of the error (and therefore the convergence behavior) will be somewhat different for the Helmholtz case from that of cases with $\omega \leq 0$.

We want to determine a fixed point iteration with an iteration operator $\hat{T}_k$ mapping a vector containing all error coefficients at iteration n to a vector with all the error coefficients at iteration $n+1$. To determine the expression of this operator, we derive the formulas that relate the coefficients at iteration n to those at iteration $n+1$.

2.3.1 Coefficients formulas

Before deriving the coefficients formulas, we state two lemmas which will be necessary in the derivation of these formulas. The proof of both lemmas are given in the Appendix B. These lemmas enable us to interchange the order of the derivatives, summation and integral, when integral and differential operations are applied to the series given by (2.7), (2.11), and (2.12).

Lemma 2.2. *Consider the series in (2.7), (2.11), and (2.12). If we can write their coefficients as follows*

$$A^s_{n,m,1} = \frac{B^s_{n,m,1}}{z_m^2 \left(\frac{-d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} \tag{2.14}$$

and

$$A^s_{n,m,2} = \frac{B^s_{n,m,2}}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} \tag{2.15}$$

for $1 < s < p$,

$$A^s_{n,m,2} = \frac{B^s_{n,m,2}}{z_m^2\left(\frac{d\psi_m^{(3)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta}z_m^2)\psi_m^{(3)}(L)\right)} \tag{2.16}$$

for $s = 1$, *and*

$$A^s_{n,m,1} = \frac{B^s_{n,m,1}}{z_m^2\left(\frac{-d\psi_m^{(3)}}{dx}(-L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta}z_m^2)\psi_m^{(3)}(-L)\right)} \tag{2.17}$$

for $s = p$, *where*

$$B^s_{n,m,i} \leq M_{n,s} \tag{2.18}$$

for some constants $M_{n,s} > 0$, *then the following holds.*

1. *The order of their first derivatives and summation can be interchanged in* $[0, L] \times [0, H]$

2. *The order of their second derivatives and summation can be interchanged in* $[0, L] \times [0, H]$ *if* $\beta \neq 0$ *and in* $(0, L) \times (0, H)$ *if* $\beta = 0$

3. *The order of their integral over* $y \in [0, H]$, *first derivatives and summation can be interchanged.*

Lemma 2.3. *Consider the series expansion of the local error* η^s_n *from equations (2.7), (2.11), and (2.12). Let* u_0 *be the initial approximation of the solution of* (2.1) *and such that the initial error* η_0 *is* $C^3((0, W) \times (0, H))$. *Then, for all* $n \in \mathbb{N}$, *the coefficients* $A^s_{n,m,i}$ *can be written as in (2.14)-(2.17).*

The identities (2.14)-(2.17) in Lemma 2.2 will also be useful later in the chapter to prove that the local error series converge to zero uniformly. In fact, let

$$C^s_{n,m,i} := \begin{cases} z_m^2 A^s_{n,m,i}, & \text{if } z_m^2 < \bar{\omega} \\ B^s_{n,m,i}, & \text{if } z_m^2 > \bar{\omega}, \end{cases} \tag{2.19}$$

where $B^s_{n,m,i}$ is defined in Lemma 2.3. Note that, by Lemma 2.3, we have that $|C^s_{n,m,i}| \leq \tilde{M}_{n,s}$ for all $m \in \mathbb{N}$ and some $\tilde{M}_{n,s} > 0$. We can think of the coefficients $C^s_{n,m,i}$ as scaled versions of the error coefficients. In our convergence proofs we need to show that these scaled coefficients converge to zero uniformly. Consequently, we shall determine the formulas relating the scaled error coefficients $C^s_{n+1,m,i}$ and $C^s_{n,m,i}$ instead of those relating $A^s_{n+1,m,i}$ to $A^s_{n,m,i}$.

We are ready to determine the formulas relating error coefficients at iterations $n+1$ and n. Plugging (2.7) into the non-homogenous boundary condition on the second line of (2.4) and since, by in Lemmas 2.2 and 2.3, the order of derivatives and summation interchange, we obtain for $2 < s < p$ that

$$\sum_{m=1}^{\infty}\left[A^s_{n+1,m,1}\left(-\frac{d\psi^{(1)}_m}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z^2_m)\psi^{(1)}_m(0)\right)\right.$$
$$\left.+A^s_{n+1,m,2}\left(-\frac{d\psi^{(2)}_m}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z^2_m)\psi^{(2)}_m(0)\right)\right]\phi_m(y)=$$
$$\sum_{m=1}^{\infty}\left[A^{s-1}_{n,m,1}\left(-\frac{d\psi^{(1)}_m}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z^2_m)\psi^{(1)}_m(L-2\gamma)\right)\right.$$
$$\left.+A^{s-1}_{n,m,2}\left(-\frac{d\psi^{(2)}_m}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z^2_m)\psi^{(2)}_m(L-2\gamma)\right)\right]\phi_m(y).$$

Multiplying both sides by $\phi_k(y)$, integrating over $[0,H]$, interchanging the order of the summation and integral, using the orthogonality of the set $\{\phi_m\}_{m\in\mathbb{N}}$ and noticing that $\left(-\frac{d\psi^{(2)}_m}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z^2_m)\psi^{(2)}_m(0)\right)=0$, we obtain

$$A^s_{n+1,k,1}\left(-\frac{d\psi^{(1)}_m}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z^2_m)\psi^{(1)}_m(0)\right)\int_0^H(\phi_k(y))^2dy=$$
$$\left[A^{s-1}_{n,k,1}\left(-\frac{d\psi^{(1)}_k}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z^2_k)\psi^{(1)}_k(L-2\gamma)\right)\right.$$
$$\left.+A^{s-1}_{n,k,2}\left(-\frac{d\psi^{(2)}_k}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z^2_k)\psi^{(2)}_k(L-2\gamma)\right)\right]\int_0^H(\phi_k(y))^2dy.$$

This implies that

$$A^{s}_{n+1,k,1}\left(-\frac{d\psi^{(1)}_{m}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(1)}_{m}(0)\right)= \tag{2.20}$$
$$A^{s-1}_{n,k,1}\left(-\frac{d\psi^{(1)}_{k}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi^{(1)}_{k}(L-2\gamma)\right)$$
$$+A^{s-1}_{n,k,2}\left(-\frac{d\psi^{(2)}_{k}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi^{(2)}_{k}(L-2\gamma)\right).$$

Then, solving for $A^{s}_{n+1,k,1}$ we obtain

$$A^{s}_{n+1,k,1} = A^{s-1}_{n,k,1}\left(\frac{-\frac{d\psi^{(1)}_{k}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi^{(1)}_{k}(L-2\gamma)}{-\frac{d\psi^{(1)}_{m}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(1)}_{m}(0)}\right) \tag{2.21}$$
$$+A^{s-1}_{n,k,2}\left(\frac{-\frac{d\psi^{(2)}_{k}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi^{(2)}_{k}(L-2\gamma)}{-\frac{d\psi^{(1)}_{m}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(1)}_{m}(0)}\right).$$

By a similar procedure as the one just carried out, but using the non-homogenous boundary condition on the third line of (2.4) and noticing that $\left(\frac{d\psi^{(1)}_{m}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(1)}_{m}(L)\right) = 0$, we obtain for $1 < s < p-1$,

$$A^{s}_{n+1,k,2}\left(\frac{d\psi^{(2)}_{m}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(2)}_{m}(L)\right)=$$
$$A^{s+1}_{n,k,1}\left(\frac{d\psi^{(1)}_{k}}{dx}(2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi^{(1)}_{k}(2\gamma)\right)$$
$$+\quad A^{s+1}_{n,k,2}\left(\frac{d\psi^{(2)}_{k}}{dx}(2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi^{(2)}_{k}(2\gamma)\right).$$

Consequently

$$
\begin{aligned}
A^s_{n+1,k,2} &= A^{s+1}_{n,k,1}\left(\frac{\frac{d\psi_k^{(1)}}{dx}(2\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_k^2)\psi_k^{(1)}(2\gamma)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar\alpha-\bar\beta z_m^2)\psi_m^{(2)}(L)}\right)\\
&+ A^{s+1}_{n,k,2}\left(\frac{\frac{d\psi_k^{(2)}}{dx}(2\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_k^2)\psi_k^{(2)}(2\gamma)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar\alpha-\bar\beta z_m^2)\psi_m^{(2)}(L)}\right).
\end{aligned}
$$

For $s=1$ we have

$$
\begin{aligned}
A^1_{n+1,k,2} &= A^{2}_{n,k,1}\left(\frac{\frac{d\psi_k^{(1)}}{dx}(2\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_k^2)\psi_k^{(1)}(2\gamma)}{\frac{d\psi_m^{(3)}}{dx}(L-\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_m^2)\psi_m^{(3)}(L-\gamma)}\right)\\
&+ A^{2}_{n,k,2}\left(\frac{\frac{d\psi_k^{(2)}}{dx}(2\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_k^2)\psi_k^{(2)}(2\gamma)}{\frac{d\psi_m^{(3)}}{dx}(L-\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_m^2)\psi_m^{(3)}(L-\gamma)}\right),
\end{aligned}
\tag{2.22}
$$

for $s=p$

$$
\begin{aligned}
A^p_{n+1,k,1} &= A^{p-1}_{n,k,1}\left(\frac{\frac{d\psi_k^{(1)}}{dx}(L-2\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_k^2)\psi_k^{(1)}(L-2\gamma)}{\frac{d\psi_m^{(3)}}{dx}(L-\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_m^2)\psi_m^{(3)}(L-\gamma)}\right)\\
&+ A^{p-1}_{n,k,2}\left(\frac{\frac{d\psi_k^{(2)}}{dx}(L-2\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_k^2)\psi_k^{(2)}(L-2\gamma)}{\frac{d\psi_m^{(3)}}{dx}(L-\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_m^2)\psi_m^{(3)}(L-\gamma)}\right),
\end{aligned}
\tag{2.23}
$$

for $s=2$

$$
A^2_{n+1,k,1} = A^{1}_{n,k,2}\left(\frac{-\frac{d\psi_k^{(3)}}{dx}(L-3\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_k^2)\psi_k^{(2)}(L-3\gamma)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar\alpha-\bar\beta z_m^2)\psi_m^{(1)}(0)}\right), \tag{2.24}
$$

and for $s=p-1$

$$
A^{p-1}_{n+1,k,2} = A^{p}_{n,k,1}\left(\frac{\frac{d\psi_k^{(3)}}{dx}(L-3\gamma)+\frac{1}{H}(\bar\alpha-\bar\beta z_k^2)\psi_k^{(3)}(L-3\gamma)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar\alpha-\bar\beta z_m^2)\psi_m^{(2)}(L)}\right). \tag{2.25}
$$

As for the coefficients $B^s_{n+1,k,1}$, using the identities (2.14) and (2.15) in (2.20) and solving for $B^s_{n+1,k,1}$ we obtain for $2 < s < p$ that

$$B^s_{n+1,k,1} = B^{s-1}_{n,k,1}\left(\frac{-\frac{d\psi_k^{(1)}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(1)}(L-2\gamma)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)}\right) + B^{s-1}_{n,k,2}\left(\frac{-\frac{d\psi_k^{(2)}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(2)}(L-2\gamma)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)}\right). \quad (2.26)$$

Similarly for $1 < s < p-1$ we obtain

$$B^s_{n+1,k,2} = B^{s+1}_{n,k,1}\left(\frac{\frac{d\psi_k^{(1)}}{dx}(2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(1)}(2\gamma)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(0)}\right) + B^{s+1}_{n,k,2}\left(\frac{\frac{d\psi_k^{(2)}}{dx}(2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(2)}(2\gamma)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)}\right). \quad (2.27)$$

For $s = 1$ we have,

$$B^s_{n+1,k,1} = B^{2}_{n,k,1}\left(\frac{\frac{d\psi_k^{(1)}}{dx}(2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(1)}(2\gamma)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(0)}\right) + B^{2}_{n,k,2}\left(\frac{\frac{d\psi_k^{(2)}}{dx}(2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(2)}(2\gamma)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)}\right), \quad (2.28)$$

for $s = p$

$$B^s_{n+1,k,1} = B^{s-1}_{n,k,1}\left(\frac{-\frac{d\psi_k^{(1)}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(1)}(L-2\gamma)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)}\right) + B^{s-1}_{n,k,2}\left(\frac{-\frac{d\psi_k^{(2)}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(2)}(L-2\gamma)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)}\right), \quad (2.29)$$

for $s = p-1$

$$B^{p-1}_{n+1,k,2} = B^{p}_{n,k,1}\left(\frac{\frac{d\psi_k^{(3)}}{dx}(L-3\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(3)}(L-3\gamma)}{-\frac{d\psi_m^{(3)}}{dx}(\gamma-L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(3)}(\gamma-L)}\right) \quad (2.30)$$

and for $s = 2$

$$B^2_{n+1,k,1} \quad = \quad B^1_{n,k,2}\left(\frac{-\frac{d\psi_k^{(3)}}{dx}(L-3\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(2)}(L-3\gamma)}{\frac{d\psi_m^{(3)}}{dx}(L-\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(3)}(L-\gamma)}\right). \tag{2.31}$$

Coefficients formulas for k such that $z_k^2 > \bar{\omega}$

Recall that $z_m = m\pi$ for all $m \in \mathbb{N}$ and $\bar{\omega} = \omega H^2$. From (2.19) we have for $k \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$ that $C^s_{n,k,i} = B^s_{n,k,i}$. Thus, considering $k \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$, and plugging (2.9) and (2.10) into (2.26) and (2.29) , we observe for $2 < s \leq p$ that

$$\begin{aligned} C^s_{n+1,k,1} \quad &= \quad \left(\frac{\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)^2 - e^{-4\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)^2}{\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)^2 - e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)^2}\right) e^{-(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}} C^{s-1}_{n,k,1} \\ &+ \quad \left(\frac{\left(e^{-2(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}-1\right)\left((\bar{\alpha}-\bar{\beta}z_k^2)^2-(z_k^2-\bar{\omega})\right)}{e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)^2-\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)^2}\right) e^{-2\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}} C^{s-1}_{n,k,2}. \end{aligned} \tag{2.32}$$

Similarly, plugging (2.9) and (2.10) into (2.27) and (2.28) we find for $1 \leq s < p$ that

$$\begin{aligned} C^s_{n+1,k,2} \quad &= \quad \left(\frac{\left(e^{-2(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}-1\right)\left((\bar{\alpha}-\bar{\beta}z_k^2)^2-(z_k^2-\bar{\omega})\right)}{e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)^2-\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)^2}\right) e^{-2\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}} C^{s+1}_{n,k,1} \\ &+ \left(\frac{\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)^2 - e^{-4\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)^2}{\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)^2 - e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)^2}\right) e^{-(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}} C^{s+1}_{n,k,2}\,. \end{aligned} \tag{2.33}$$

Also, using (2.13) in (2.31) and (2.30), we see that for $s = 2, p-1$ we have

$$C^2_{n+1,k,1} = \left(\frac{e^{-2(\bar{L}-3\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)-\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)}{e^{-2(\bar{L}-\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)-\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)}\right) e^{-2\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}} C^1_{n,k,2} \tag{2.34}$$

and

$$C^{p-1}_{n+1,k,2} = \left(\frac{e^{-2(\bar{L}-3\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)-\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)}{e^{-2(\bar{L}-\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_k^2-\sqrt{z_k^2-\bar{\omega}}\right)-\left(\bar{\alpha}-\bar{\beta}z_k^2+\sqrt{z_k^2-\bar{\omega}}\right)}\right) e^{-2\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}} C^p_{n,k,1}\,, \tag{2.35}$$

respectively.

Coefficients formulas for k such that $z_k^2 < \bar{\omega}$

Considering the expression of (2.9) for $k \in \mathbb{N}$ such that $z_k^2 < \bar{\omega}$ and noticing that

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

and

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

we have that

$$\begin{aligned}
&-\frac{d\psi_m^{(1)}}{dx}(x) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(x) = \\
&\frac{1}{H}\left[\frac{(i\sqrt{z_m^2 - \bar{\omega}} + (\bar{\alpha} - \beta z_m^2))^2}{i\sqrt{z_m^2 - \bar{\omega}} - (\bar{\alpha} - \beta z_m^2)} e^{-i\sqrt{\lambda - \omega}(x-L)} - \left(i\sqrt{z_m^2 - \bar{\omega}} - (\bar{\alpha} - \beta z_m^2)\right) e^{i\sqrt{\lambda-\omega}(x-L)}\right] = \\
&\frac{1}{H\left(i\sqrt{z_m^2 - \bar{\omega}} - (\bar{\alpha} - \beta z_m^2)\right)}\left[e^{-i\frac{(x-L)}{H}\sqrt{\bar{\omega} - z_m^2}}\left(2i\sqrt{\bar{\omega} - z_m^2}\,(\bar{\alpha} - \bar{\beta} z_m^2) + (\bar{\alpha} - \bar{\beta} z_m^2)^{\,2} + z_m^2 - \bar{\omega}\right) - \right. \\
&\left. e^{i\frac{(x-L)}{H}\sqrt{\bar{\omega} - z_m^2}}\left(-2i\sqrt{\bar{\omega} - z_m^2}\,(\bar{\alpha} - \bar{\beta} z_m^2) + (\bar{\alpha} - \bar{\beta} z_m^2)^{\,2} + z_m^2 - \bar{\omega}\right)\right] = \\
&\frac{1}{H\left(i\sqrt{z_m^2 - \bar{\omega}} - (\bar{\alpha} - \beta z_m^2)\right)}\left[2i\sqrt{\bar{\omega} - z_m^2}\,(\bar{\alpha} - \bar{\beta} z_m^2)\left(e^{-i\frac{(x-L)}{H}\sqrt{\bar{\omega} - z_m^2}} + e^{i\frac{(x-L)}{H}\sqrt{\bar{\omega} - z_m^2}}\right) + \right. \\
&\left. \left(e^{-i\frac{(x-L)}{H}\sqrt{\bar{\omega} - z_m^2}} - e^{i\frac{(x-L)}{H}\sqrt{\bar{\omega} - z_m^2}}\right)\left((\bar{\alpha} - \bar{\beta} z_m^2)^{\,2} + z_m^2 - \bar{\omega}\right)\right] = \\
&\frac{i\left[2\left(-(\bar{\alpha} - \bar{\beta} z_m^2)^{\,2} - z_m^2 + \bar{\omega}\right)\sin\left(\frac{(x-L)}{H}\sqrt{\bar{\omega} - z_m^2}\right) + 4\sqrt{\bar{\omega} - z_m^2}\,(\bar{\alpha} - \bar{\beta} z_m^2)\cos\left(\frac{(x-L)}{H}\sqrt{\bar{\omega} - z_m^2}\right)\right]}{H\left(i\sqrt{z_m^2 - \bar{\omega}} - (\bar{\alpha} - \beta z_m^2)\right)}.
\end{aligned} \tag{2.36}$$

Similarly, we obtain

$$-\frac{d\psi_m^{(2)}}{dx}(x) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(x) = \frac{2}{H}\left(\sqrt{z_m^2 - \bar{\omega}} + (\bar{\alpha} - \bar{\beta} z_m^2)i\right)\sin\left(\frac{x}{H}\sqrt{z_m^2 - \bar{\omega}}\right). \tag{2.37}$$

Then, plugging (2.36) and (2.37) in (2.21), multiplying both sides of the resulting equation by z_k^2 and recalling that $C_{n,k,i}^s = z_k^2 A_{n+1,k,i}^s$ for $k \in \mathbb{N}$ such that $z_k^2 < \bar{\omega}$, we have for $2 < s < p$ that

$$\begin{aligned}
C_{n+1,k,1}^s \quad &= \quad \frac{\left((\bar{\alpha} - \bar{\beta} z_k^2)^2 - (\bar{\omega} - z_k^2)\right)\sin\left(2\bar{\gamma}\sqrt{\bar{\omega} - z_k^2}\right) + 2\,(\bar{\alpha} - \bar{\beta} z_k^2)\,\sqrt{\bar{\omega} - z_k^2}\cos\left(2\bar{\gamma}\sqrt{\bar{\omega} - z_k^2}\right)}{\left((\bar{\alpha} - \bar{\beta} z_k^2)^2 - (\bar{\omega} - z_k^2)\right)\sin\left(\sqrt{(\bar{\omega} - z_k^2)}\bar{L}\right) + 2\,(\bar{\alpha} - \bar{\beta} z_k^2)\,\sqrt{(\bar{\omega} - z_k^2)}\cos\left(\sqrt{\bar{\omega} - z_k^2}\bar{L}\right)} C_{n,k,1}^{s-1} \\
&+ \quad \frac{-\left(i\sqrt{\bar{\omega} - z_k^2} - (\bar{\alpha} - \bar{\beta} z_k^2)\right)^2\sin\left(\sqrt{\bar{\omega} - z_k^2}\,(\bar{L} - 2\bar{\gamma})\right)}{\left((\bar{\alpha} - \bar{\beta} z_k^2)^2 - (\bar{\omega} - z_k^2)\right)\sin\left(\sqrt{(\bar{\omega} - z_k^2)}\bar{L}\right) + 2\,(\bar{\alpha} - \bar{\beta} z_k^2)\,\sqrt{(\bar{\omega} - z_k^2)}\cos\left(\sqrt{\bar{\omega} - z_k^2}\bar{L}\right)} C_{n,k,2}^{s-1}.
\end{aligned} \tag{2.38}$$

Similarly, we have for $1 < s < p-1$ that

$$C^s_{n+1,k,2} = \frac{-\left(i\sqrt{\bar\omega - z_k^2} + (\bar\alpha - \bar\beta z_k^2)\right)^2 \sin\left(\sqrt{\bar\omega - z_k^2}\,(\bar L - 2\bar\gamma)\right)}{\left((\bar\alpha - \bar\beta z_k^2)^2 - (\bar\omega - z_k^2)\right)\sin\left(\sqrt{(\bar\omega - z_k^2)}\bar L\right) + 2\,(\bar\alpha - \bar\beta z_k^2)\sqrt{(\bar\omega - z_k^2)}\cos\left(\sqrt{\bar\omega - z_k^2}\bar L\right)} C^s_{n,k,1}$$
$$+ \frac{\left((\bar\alpha - \bar\beta z_k^2)^2 - (\bar\omega - z_k^2)\right)\sin\left(2\bar\gamma\sqrt{\bar\omega - z_k^2}\right) + 2\,(\bar\alpha - \bar\beta z_k^2)\sqrt{\bar\omega - z_k^2}\cos\left(2\bar\gamma\sqrt{\bar\omega - z_k^2}\right)}{\left((\bar\alpha - \bar\beta z_k^2)^2 - (\bar\omega - z_k^2)\right)\sin\left(\sqrt{(\bar\omega - z_k^2)}\bar L\right) + 2\,(\bar\alpha - \bar\beta z_k^2)\sqrt{(\bar\omega - z_k^2)}\cos\left(\sqrt{\bar\omega - z_k^2}\bar L\right)} C^s_{n,k,2}. \tag{2.39}$$

For $s = 1$ we obtain

$$C^1_{n+1,k,2} = -\frac{\left(i\sqrt{\bar\omega - z_k^2} + (\bar\alpha - \bar\beta z_k^2)\right)\sin\left(\sqrt{\bar\omega - z_k^2}\,(\bar L - 2\bar\gamma)\right)}{\left((\bar\alpha - \bar\beta z_k^2)\sin\left(\sqrt{\bar\omega - z_k^2}\bar L\right) + \sqrt{\bar\omega - z_k^2}\cos\left(\sqrt{\bar\omega - z_k^2}\bar L\right)\right)} C^s_{n,k,1} \tag{2.40}$$
$$+ \frac{\left((\bar\alpha - \bar\beta z_k^2)^2 - (\bar\omega - z_k^2)\right)\sin\left(2\bar\gamma\sqrt{\bar\omega - z_k^2}\right) + 2\,(\bar\alpha - \bar\beta z_k^2)\sqrt{\bar\omega - z_k^2}\cos\left(2\bar\gamma\sqrt{\bar\omega - z_k^2}\right)}{\left(i\sqrt{\bar\omega - z_k^2} + (\bar\alpha - \bar\beta z_k^2)\right)\left((\bar\alpha - \bar\beta z_k^2)\sin\left(\sqrt{\bar\omega - z_k^2}\bar L\right) + \sqrt{\bar\omega - z_k^2}\cos\left(\sqrt{\bar\omega - z_k^2}\bar L\right)\right)} C^s_{n,k,2}\,,$$

and for $s = p$

$$C^p_{n+1,k,1} = -\frac{\left((\bar\alpha - \bar\beta z_k^2)^2 - (\bar\omega - z_k^2)\right)\sin\left(2\bar\gamma\sqrt{\bar\omega - z_k^2}\right) + 2\,(\bar\alpha - \bar\beta z_k^2)\sqrt{\bar\omega - z_k^2}\cos\left(2\bar\gamma\sqrt{\bar\omega - z_k^2}\right)}{\left(i\sqrt{\bar\omega - z_k^2} - (\bar\alpha - \bar\beta z_k^2)\right)\left((\bar\alpha - \bar\beta z_k^2)\sin\left(\sqrt{\bar\omega - z_k^2}\bar L\right) + \sqrt{\bar\omega - z_k^2}\cos\left(\sqrt{\bar\omega - z_k^2}\bar L\right)\right)} C^s_{n,k,1}$$
$$+ \frac{\left(i\sqrt{\bar\omega - z_k^2} - (\bar\alpha - \bar\beta z_k^2)\right)\sin\left(\sqrt{\bar\omega - z_k^2}\,(\bar L - 2\bar\gamma)\right)}{(\bar\alpha - \bar\beta z_k^2)\sin\left(\sqrt{\bar\omega - z_k^2}\bar L\right) + \sqrt{\bar\omega - z_k^2}\cos\left(\sqrt{\bar\omega - z_k^2}\bar L\right)} C^s_{n,k,2}\,. \tag{2.41}$$

Also, for $s = 2$ we obtain

$$C^2_{n+1,k,1} = -\frac{\left(i\sqrt{\bar\omega - z_k^2} + (\bar\alpha - \bar\beta z_k^2)\right)\left((\bar\alpha - \bar\beta z_k^2)\sin\left(\sqrt{\bar\omega - z_k^2}\,(\bar L - 2\bar\gamma)\right) - \sqrt{\bar\omega - z_k^2}\cos\left(\sqrt{\bar\omega - z_k^2}\,(\bar L - 2\bar\gamma)\right)\right)}{\left((\bar\alpha - \bar\beta z_k^2)^2 - (\bar\omega - z_k^2)\right)\sin\left(\sqrt{\bar\omega - z_k^2}\bar L\right) + 2\,(\bar\alpha - \bar\beta z_k^2)\sqrt{\bar\omega - z_k^2}\cos\left(\sqrt{\bar\omega - z_k^2}\bar L\right)} C^s_{n,k,2}\,, \tag{2.42}$$

and for $s = p-1$

$$C^{p-1}_{n+1,k,1} = \frac{\left(i\sqrt{\bar\omega - z_k^2} - (\bar\alpha - \bar\beta z_k^2)\right)\left((\bar\alpha - \bar\beta z_k^2)\sin\left(\sqrt{\bar\omega - z_k^2}\,(\bar L - 2\bar\gamma)\right) - \sqrt{\bar\omega - z_k^2}\cos\left(\sqrt{\bar\omega - z_k^2}\,(\bar L - 2\bar\gamma)\right)\right)}{\left((\bar\alpha - \bar\beta z_k^2)^2 - (\bar\omega - z_k^2)\right)\sin\left(\sqrt{\bar\omega - z_k^2}\bar L\right) + 2\,(\bar\alpha - \bar\beta z_k^2)\sqrt{\bar\omega - z_k^2}\cos\left(\sqrt{\bar\omega - z_k^2}\bar L\right)} C^s_{n,k,1}\,. \tag{2.43}$$

2.3.2 Iteration operator

Based on (2.32)- (2.35), for $k \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$ we can write

$$\begin{aligned}
C^s_{n+1,k,2} &= c_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,1} + c_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,2}, && \text{for } s=1\\
C^s_{n+1,k,1} &= c_3(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,2}, && \text{for } s=2\\
C^s_{n+1,k,1} &= c_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s-1}_{n,m,1} + c_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s-1}_{n,m,2}, && \text{for } 2 < s \le p-1\\
C^s_{n+1,k,2} &= c_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,1} + c_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,2}, && \text{for } 2 \le s < p-1\\
C^s_{n+1,k,2} &= c_3(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,2}, && \text{for } s=p-1\\
C^s_{n+1,k,1} &= c_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s-1}_{n,m,1} + c_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s-1}_{n,m,2}, && \text{for } s=p.
\end{aligned} \tag{2.44}$$

where

$$c_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) = \left(\frac{\left(\bar{\alpha} - \bar{\beta}z_k^2 + \sqrt{z_k^2-\bar{\omega}}\right)^2 - e^{-4\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha} - \bar{\beta}z_k^2 - \sqrt{z_k^2-\bar{\omega}}\right)^2}{\left(\bar{\alpha} - \bar{\beta}z_k^2 + \sqrt{z_k^2-\bar{\omega}}\right)^2 - e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha} - \bar{\beta}z_k^2 - \sqrt{z_k^2-\bar{\omega}}\right)^2} \right) e^{-(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}, \tag{2.45}$$

$$c_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) = \left(\frac{\left(e^{-2(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}} - 1\right)\left((\bar{\alpha} - \bar{\beta}z_k^2)^2 - (z_k^2-\bar{\omega})\right)}{e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha} - \bar{\beta}z_k^2 - \sqrt{z_k^2-\bar{\omega}}\right)^2 - \left(\bar{\alpha} - \bar{\beta}z_k^2 + \sqrt{z_k^2-\bar{\omega}}\right)^2} \right) e^{-2\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}}, \tag{2.46}$$

and

$$c_3(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) = \left(\frac{e^{-2(\bar{L}-3\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha} - \bar{\beta}z_k^2 + \sqrt{z_k^2-\bar{\omega}}\right) - \left(\bar{\alpha} - \bar{\beta}z_k^2 - \sqrt{z_k^2-\bar{\omega}}\right)}{e^{-2(\bar{L}-\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}}\left(\bar{\alpha} - \bar{\beta}z_k^2 - \sqrt{z_k^2-\bar{\omega}}\right) - \left(\bar{\alpha} - \bar{\beta}z_k^2 + \sqrt{z_k^2-\bar{\omega}}\right)} \right) e^{-2\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}}. \tag{2.47}$$

Let

$$C_{n,k} := \left(C^1_{n,k,2}, C^2_{n,k,1}, C^2_{n,k,2}, \dots C^{p-1}_{n,k,1}, C^{p-1}_{n,k,2}, C^p_{n,k,1}\right), \tag{2.48}$$

i.e., $C_{n,k}$ is the vector obtained by collecting the local error coefficients of frequency k from all subdomains at the iteration n. Thus, if we collect the coefficients on the left hand side of the equations in (2.44) into the vector $C_{n+1,k}$ as in (2.48), this set of equations define the operation $C_{n+1,k} = \hat{T}_k C_{n,k}$, which is the new fixed point iteration that we are seeking. The operator $\hat{T}_k$ is a pentadiagonal square matrix of order

$N = 2(p-1)$ whose non-zero entries are located in the first and second subdiagonals and first and second superdiagonals, i.e., its main diagonal is zero. We have that

$$\left(\hat{T}_k\right)_{i,i-2} = \begin{cases} 0, & \text{if } i \text{ is odd} \\ c_1, & \text{if } i \text{ is even,} \end{cases} \tag{2.49}$$

$$\left(\hat{T}_k\right)_{i,i-1} = \begin{cases} c_3, & \text{if } i = 2 \\ 0, & \text{if } i \text{ is odd} \\ c_2, & \text{if } i \text{ is even and } i \neq 2, \end{cases} \tag{2.50}$$

$$\left(\hat{T}_k\right)_{i,i+1} = \begin{cases} 0, & \text{if } i \text{ is even} \\ c_2, & \text{if } i \text{ is odd and } i \neq 2(p-1)-1 \\ c_3, & \text{if } i = 2(p-1)-1, \end{cases} \tag{2.51}$$

$$\left(\hat{T}_k\right)_{i,i+2} = \begin{cases} 0, & \text{if } i \text{ is even} \\ c_1, & \text{if } i \text{ is odd.} \end{cases} \tag{2.52}$$

Similarly, from (2.38)-(2.43) we have that, for $k \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$, the relation between the coefficients $C^s_{n+1,k,i}$ and $C^s_{n,k,i}$ can be written as

$$\begin{aligned}
C^s_{n+1,k,2} &= d_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,1} + d_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,2}, && \text{for } s = 1 \\
C^s_{n+1,k,1} &= d_3(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s-1}_{n,m,1} + d_4(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s-1}_{n,m,2}, && \text{for } 2 < s < p \\
C^s_{n+1,k,2} &= d_5(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,1} + d_4(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,2}, && \text{for } 1 < s < p-1 \\
C^s_{n+1,k,1} &= d_6(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,2}, && \text{for } s = 2 \\
C^s_{n+1,k,2} &= d_7(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s+1}_{n,m,2}, && \text{for } s = p-1 \\
C^s_{n+1,k,1} &= d_8(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s-1}_{n,m,1} + d_9(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})C^{s-1}_{n,m,2}, && \text{for } s = p,
\end{aligned}$$

where

$$
\begin{aligned}
d_1\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= -\frac{\left(i\sqrt{\bar{\omega}-z_k^2}+\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)}{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)+\sqrt{\bar{\omega}-z_k^2}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)\right)}\\
d_2\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= \frac{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(2\bar{\gamma}\sqrt{\bar{\omega}-z_k^2}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\bar{\omega}-z_k^2}\cos\left(2\bar{\gamma}\sqrt{\bar{\omega}-z_k^2}\right)}{\left(i\sqrt{\bar{\omega}-z_k^2}+\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\right)\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)+\sqrt{\bar{\omega}-z_k^2}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)\right)}\\
d_3\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= \frac{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(2\bar{\gamma}\sqrt{\bar{\omega}-z_k^2}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\bar{\omega}-z_k^2}\cos\left(2\bar{\gamma}\sqrt{\bar{\omega}-z_k^2}\right)}{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(\sqrt{\left(\bar{\omega}-z_k^2\right)}\bar{L}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\left(\bar{\omega}-z_k^2\right)}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)},\\
d_4\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= \frac{-\left(i\sqrt{\bar{\omega}-z_k^2}-\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\right)^2\sin\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)}{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(\sqrt{\left(\bar{\omega}-z_k^2\right)}\bar{L}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\left(\bar{\omega}-z_k^2\right)}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)},\\
d_5\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= \frac{-\left(i\sqrt{\bar{\omega}-z_k^2}+\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\right)^2\sin\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)}{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(\sqrt{\left(\bar{\omega}-z_k^2\right)}\bar{L}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\left(\bar{\omega}-z_k^2\right)}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)},\\
d_6\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= -\frac{\left(i\sqrt{\bar{\omega}-z_k^2}+\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\right)\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)-\sqrt{\bar{\omega}-z_k^2}\cos\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)\right)}{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\bar{\omega}-z_k^2}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)},\\
d_7\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= \frac{\left(i\sqrt{\bar{\omega}-z_k^2}-\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\right)\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)-\sqrt{\bar{\omega}-z_k^2}\cos\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)\right)}{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\bar{\omega}-z_k^2}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)},\\
d_8\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= -\frac{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(2\bar{\gamma}\sqrt{\bar{\omega}-z_k^2}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\bar{\omega}-z_k^2}\cos\left(2\bar{\gamma}\sqrt{\bar{\omega}-z_k^2}\right)}{\left(i\sqrt{\bar{\omega}-z_k^2}-\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\right)\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)+\sqrt{\bar{\omega}-z_k^2}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)\right)},\\
d_9\left(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}\right) &= \frac{\left(i\sqrt{\bar{\omega}-z_k^2}-\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)}{\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)+\sqrt{\bar{\omega}-z_k^2}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)}.
\end{aligned}
\tag{2.53}
$$

Then, with the vector $C_{n,k}$ as in (2.48), and based on (2.53) we have for $k \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$ that $C_{n+1,k} = \hat{T}_k C_{n,k}$, where $\hat{T}_k$ is also pentadiagonal and with zeros on the diagonal. We have that

$$
\left(\hat{T}_k\right)_{i,i-2} = \begin{cases} 0, & \text{if } i \text{ is odd} \\ d_3, & \text{if } i \text{ is even and } i < N \\ d_9, & \text{if } i = N, \end{cases} \tag{2.54}
$$

$$\left(\hat{T}_k\right)_{i,i-1} = \begin{cases} d_6, & \text{if } i = 2 \\ 0, & \text{if } i \text{ is odd} \\ d_4, & \text{if } i \text{ is even and } i < N \\ d_8, & \text{if } i = N, \end{cases} \tag{2.55}$$

$$\left(\hat{T}_k\right)_{i,i+1} = \begin{cases} d_1, & \text{if } i = 1 \\ 0, & \text{if } i \text{ is even} \\ d_5, & \text{if } i \text{ is odd and } i < N-1 \\ d_7, & \text{if } i = N-1, \end{cases} \tag{2.56}$$

$$\left(\hat{T}_k\right)_{i,i+2} = \begin{cases} d_2, & \text{if } i = 1 \\ 0, & \text{if } i \text{ is even} \\ d_3, & \text{if } i \text{ is odd.} \end{cases} \tag{2.57}$$

2.4 Bounds on the coefficients

2.4.1 Bound based on the max norm

The max norm of $\hat{T}_k$ is given by its maximum absolute row sum. Thus, based on (2.49)-(2.52) we have for $k \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$ that

$$||\hat{T}_k||_\infty = \max\{c_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) + c_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}), c_3(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})\} \tag{2.58}$$

and from (2.54)-(2.57) we have for $k \in \mathbb{N}$ such that $z_k^2 < \bar{\omega}$ that

$$\begin{aligned} ||\hat{T}_k||_\infty \;=\; \max\{\, & d_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) + d_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}), \\ & d_3(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) + d_4(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega}, \\ & d_4(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) + d_5(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}),\bar{L}), \\ & d_6(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}), d_7(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}), \\ & d_8(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) + d_9(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})\,\}\,. \end{aligned}$$

Note that

$$|C^s_{n+1,k,i}| \leq ||C_{k,n+1}||_\infty = ||\hat{T}_k C_{k,n}||_\infty \leq ||\hat{T}_k||_\infty ||C_{k,n}||_\infty.$$

From the above inequality we obtain the following bound for the scaled coefficients of the series expansion of the errors

$$|C^s_{n+1,k,i}| \leq ||\hat{T}_k||^n_\infty ||C_{k,0}||_\infty. \tag{2.59}$$

2.4.2 Another bound

The bound on the coefficients given in (2.59) contracts at a rate given by the contracting factor $||\hat{T}||_\infty$. We can obtain a better contraction factor and therefore a better bound for the coefficients $C^s_{n,k,i}$ by using the spectral radius of $|\hat{T}_k|$, where the absolute value is understood componentwise. In fact, the matrix $\hat{T}_k$ defined by (2.49)-(2.52) is irreducible and consequently so it is $|\hat{T}_k|$. Moreover, $|\hat{T}_k|$ is a non-negative matrix. Therefore, by the Perron-Frobenius theorem (see e.g, [18]) we have that its largest eigenvalue in modulus, ρ_k, is positive and the corresponding eigenvector v_k can be chosen to be a positive vector (i.e., a vector whose entries are all positive). Thus, we have

$$|\hat{T}_k| v_k = \rho_k v_k.$$

Then, denoting the smallest entry of v_k by $v_{k,min}$ and letting $w_{n,k} := \frac{||C_{n,k}||_\infty}{v_{k,min}} v_k$ we have

$$|C_{n+1,k}| = |\hat{T}_k C_{n,k}| \leq |\hat{T}_k||C_{n,k}| \leq |\hat{T}_k| w_{n,k} = |\hat{T}_k| \frac{||C_{n,k}||_\infty}{v_{k,min}} v_k = \rho_k \frac{||C_{n,k}||_\infty}{v_{k,min}} v_k = \rho_k w_{n,k}\ .$$

Then,

$$|C_{n,k}| \leq (\rho_k)^n w_{0,k}. \tag{2.60}$$

Note that $||\hat{T}_k||_\infty = \left\| |\hat{T}_k| \right\|_\infty$. Then,

$$\rho_k = \rho(|\hat{T}_k|) \leq \left\| |\hat{T}_k| \right\|_\infty = ||\hat{T}_k||_\infty.$$

Thus, $\rho_k \leq ||\hat{T}_k||_\infty$. Therefore, the contraction factor ρ_k of the coefficients bound given in (2.60) is not larger than the contraction factor of the bound given by (2.59). It turns out that there exists a positive vector of weights w and an associated weighted max norm such that $||\hat{T}_k||_w \leq \rho_k$. Therefore, in cases where $||\hat{T}_k||_\infty > 1$ but $\sup_{k\in\mathbb{N}} \rho_k < 1$, we could still show that there exists a bound on the error coefficients (i.e., the one whose contraction factor is the weighed max norm) that is contracting and that the local errors converge uniformly to zero even if $||\hat{T}_k||_\infty > 1$.

2.5 Asynchronous Optimized Schwarz methods

In asynchronous iterations, as soon as a processing unit finishes its own calculations, it starts the next cycle with the latest data received from the other processing units during a previous cycle, without necessarily waiting to receive new information from every other processing unit connected to it.

Before analyzing the convergence of the asynchronous implementation of the Optimized Schwarz method described in Section 2.2, we review the mathematical model of asynchronous iterations.

2.5.1 Mathematical Model of asynchronous iterations

Let $U^{1)}, \ldots, U^{(p)}$ be given sets and U be their Cartesian product, i.e., $U = U^{(1)} \times \cdots \times U^{(p)}$. Thus $u \in U$ implies $u = (u^1, \ldots, u^p)$ with $u^s \in U^{(s)}$ for $s \in \{1, \ldots, p\}$. Let $T^{(s)} : U \to U^{(s)}$ where $s \in \{1, \ldots, p\}$, and let $T : U \to U$ be a vector-valued map (the iteration map) given by $T = (T^{(1)}, \ldots, T^{(p)})$ with a fixed point u_*, i.e., $u_* = T(u_*)$. Let us define a *time stamp* as the instant of time at which at least one processor finishes its computation and produces a new update. Thus, let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of time stamps at which at least one processor updates its associated component. Let $\{\sigma(n)\}_{n\in\mathbb{N}}$ be a sequence with $\sigma(n) \subset \{1,\ldots,p\}$ $\forall n \in \mathbb{N}$. The set $\sigma(n)$ consists of labels (numbers) of the processors that update their associated component at the $n-th$ time stamp. Define for $s, q \in \{1,\ldots,p\}$, $\{\tau_q^s(n)\}_{n\in\mathbb{N}}$ a sequence of integers, representing the time-stamp index of the update of the data coming from processor q and available in processor s at the beginning of the computation of $u_n^{(s)}$ which ends at the time stamp t_n. Let $u_0 = (u_0^1,\ldots,u_0^p)$ be the initial approximation (of the fixed point u_*). Then, the new computed value updated by processor s at the $n-th$ time stamp is

$$u_n^s = \begin{cases} T^s\left(u^1_{\tau_1^s(n)},\ldots,u^p_{\tau_p^s(n)}\right), & s \in \sigma(n) \\ u_{n-1}^s, & s \notin \sigma(n). \end{cases}$$

In other words, at the time stamp t_n either u^s is updated (if $s \in \sigma(n)$) or it is not (if $s \notin \sigma(n)$). It is assumed that the three following conditions (necessary for convergence) are satisfied

$$\tau_q^{(s)}(n) < n,\ \forall s, q \in \{1,\ldots,p\}, \forall n \in \mathbb{N}, \tag{2.61}$$

$$\text{card}\,\{n \in \mathbb{N}^* | s \in \sigma(n)\} = +\infty,\ \forall s \in \{1,\ldots,p\}, \tag{2.62}$$

$$\lim_{n\to+\infty} \tau_q^{(s)}(n) = +\infty,\ \forall s, q \in \{1,\ldots,p\}. \tag{2.63}$$

Condition (2.61) indicates that data used at the time t_n must have been produced before the beginning of the computation of $u_n^{(s)}$, i.e., time does not flow backward. Condition (2.62) means that no process will ever stop updating its components. Condition (2.63) corresponds to the fact that new data will always be provided to the process. In other words, no process will have a piece of data that is never updated.

2.5.2 Asynchronous Optimized Schwarz iterations

Now that we have introduced the asynchronous iteration model, we are ready to define the asynchronous optimized Schwarz (AOS) iterations. Similarly as for the synchronous case, we shall present the equations describing the local problems for the asynchronous case in terms of local coordinates (see Section 2.1 for the definition of the local coordinates). While we understand by u_n to be the value of u at the time stamp t_n we emphasize this fact an call it u_{t_n} instead. Let $l_1 = \tau^s_{s-1}(n)$ and $l_2 = \tau^s_{s+1}(n)$, i.e., the time-stamp indexes of the updates of the data coming from the neighboring processors and available in processor s at the beginning of the computation which ends at the $n-th$ time stamp. Let

$$\begin{cases} \Delta u^s_{t_{n+1}} + \omega u^s_{t_{n+1}} = f & \text{in } (0,L)\times(0,H) \\ -\frac{\partial u^s_{t_{n+1}}}{\partial x} + \alpha u^s_{t_{n+1}} + \beta\frac{\partial^2 u^s_{t_{n+1}}}{\partial y^2} = -\frac{\partial u^{s-1}_{t_{l_1}}}{\partial x} + \alpha u^{s-1}_{t_{l_1}} + \beta\frac{\partial^2 u^{s-1}_{t_{l_1}}}{\partial y^2} & \text{for } x=0 \\ \frac{\partial u^s_{t_{n+1}}}{\partial x} + \alpha u^s_{t_{n+1}} + \beta\frac{\partial^2 u^s_{t_{n+1}}}{\partial y^2} = \frac{\partial u^{s+1}_{t_{l_2}}}{\partial x} + \alpha u^{s+1}_{t_{l_2}} + \beta\frac{\partial^2 u^{s+1}_{t_{l_2}}}{\partial y^2} & \text{for } x=L \\ u^s_{t_{n+1}} = g & \text{for } y=0 \\ u^s_{t_{n+1}} = g & \text{for } y=H, \end{cases} \tag{2.64}$$

for $1 < s < p$,

$$\begin{cases} \Delta u^s_{t_{n+1}} + \omega u^s_{t_{n+1}} = f & \text{in } (0,L-\gamma)\times(0,H) \\ u^s_{t_{n+1}} = g & \text{for } x=0 \\ \frac{\partial u^s_{t_{n+1}}}{\partial x} + \alpha u^s_{t_{n+1}} + \beta\frac{\partial^2 u^s_{t_{n+1}}}{\partial y^2} = \frac{\partial u^{s+1}_{t_{l_2}}}{\partial x} + \alpha u^{s+1}_{t_{l_2}} + \beta\frac{\partial^2 u^{s+1}_{t_{l_2}}}{\partial y^2} & \text{for } x=L-\gamma \\ u^s_{t_{n+1}} = g & \text{for } y=0 \\ u^s_{t_{n+1}} = g & \text{for } y=H, \end{cases} \tag{2.65}$$

for $s = 1$ and

$$\begin{cases} \Delta u^s_{t_{n+1}} + \omega u^s_{t_{n+1}} = f & \text{in } (0, L-\gamma) \times (0, H) \\ -\frac{\partial u^s_{t_{n+1}}}{\partial x} + \alpha u^s_{t_{n+1}} + \beta \frac{\partial^2 u^s_{t_{n+1}}}{\partial y^2} = -\frac{\partial u^{s-1}_{t_{l_1}}}{\partial x} + \alpha u^{s-1}_{t_{l_1}} + \beta \frac{\partial^2 u^{s-1}_{t_{l_1}}}{\partial y^2} & \text{for } x = 0 \\ u^s_{t_{n+1}} = g & \text{for } x = L-\gamma \\ u^s_{t_{n+1}} = g & \text{for } y = 0 \\ u^s_{t_{n+1}} = g & \text{for } y = H, \end{cases} \tag{2.66}$$

for $s = p$. Then, the local approximation of the solution at the time stamp t_n corresponding to the subdomain s is

$$u^s_{t_n} = \begin{cases} \text{solution of (2.64)}, & \text{if } 1 < s < p,\ s \in \sigma(n) \\ \text{solution of (2.65)}, & \text{if } s = 1,\ 1 \in \sigma(n) \\ \text{solution of (2.66)}, & \text{if } s = p,\ p \in \sigma(n) \\ u^s_{t_{n-1}}, & \text{if } s \notin \sigma(n) \end{cases} . \tag{2.67}$$

Again, the equations defining the local errors are given by (2.64), (2.65) and (2.66) with $f = 0$ and $g = 0$.

2.6 Convergence Proofs

We are now ready to present the convergence proof of Asynchronous Optimized Schwarz applied to the problem considered in this chapter, namely, the one having a PDE with the shifted Laplacian operator and in which a one-way domain decomposition is used. We present the proof of two convergence theorems. The second theorem requires a stronger condition than in the first theorem, but, relatively speaking, these conditions are easier to evaluate than those of the first theorem. The conditions in the first theorem are a little bit more complicated to test (it requires the computation of the spectral radius of T_k) but it allows us to guarantee convergence for a wider range of cases, i.e., for a wider range of values of $\bar{L}$, $\bar{\gamma}$, $\bar{\omega}$, $\bar{\alpha}$, $\bar{\beta}$ and number of subdomains p.

Theorem 2.4. *Let $\hat{T}_k$ be the irreducible matrix defined by (2.49)-(2.52) and (2.54)-(2.57). Let ρ_k the largest eigenvalue of $|\hat{T}_k|$ and v_k be the corresponding (positive) eigenvector. Let $v_{k,min}$ be the smallest entry of v_k. Then, the asynchronous iterations given by (2.64)-(2.67) converge provided that*

$$\sup_{k\in\mathbb{N}} \rho_k < 1$$

and

$$\sup_{k\in\mathbb{N}} \left\{ \rho_k \frac{||v_k||_\infty}{v_{k,min}} \right\} < \infty.$$

Proof. Note first that, since $|\hat{T}_k|$ is a non-negative irreducible matrix, it holds that

$$|\hat{T}_k| v_k = \rho_k v_k.$$

Let

$$w_k := \frac{||C_{t_0,k}||_\infty}{v_{k,min}} v_k.$$

Let t_j be the stamp at which the processor s produces its first new update. Let $C^s_{t_j,k}$ be the vector containing the coefficients corresponding to the subdomain s, e.g., $C^s_{t_j,k} = (C^s_{t_j,k,1}, C^s_{t_j,k,2})$ for $1 < s < p$. We have that $\hat{T}_k = (\hat{T}^1_k, \ldots, \hat{T}^p_k)$, where $\hat{T}^s_k$ is the local operator associated to the subdomain s. We partition v_k as $v_k = (v^1_k, \ldots, v^p_k)$, and if follows that

$$|\hat{T}^s_k| v_k = \rho_k v^s_k.$$

The same partition applies to w, i.e., $w_k = (w^1_k, \ldots, w^p_k)$. Then, we have

$$|C^s_{t_j,k}| = |\hat{T}^s_k C_{t_0,k}| \leq |\hat{T}^s_k||C_{t_0,k}| \leq |\hat{T}^s_k| \frac{||C_{t_0,k}||_\infty}{v_{k,min}} v_k = \rho_k \frac{||C_{t_0,k}||_\infty}{v_{k,min}} v^s_k = \rho_k \;.w^s_k < w^s_k,$$

Note that, in particular, we have

$$|C_{t_j,k}| \leq w_k.$$

Thus, for this processor it holds that

$$|C^s_{t_m,k}| \leq \rho_k w^s_k \tag{2.68}$$

for all time stamps $t_m \geq t_j$.

Then, after every processor has produced its first update, say at time stamp t_{j_1} we have that (2.68) holds for all s, and consequently $|C_{t_j,k}| \leq \rho_k w_k$ for all $t_j \geq t_{j_1}$. Dy a similar reasoning, we can see that once every processor has produced a new update after t_{j_1}, say at t_{j_2} ,we have $|C_{t_m,k}| \leq \rho_k^2 w_k$ for all $t_m \geq t_{j_2}$. Thus, we have

$$|C_{t_m,k}| \leq \rho_k^{\Phi(m)} w_k, \tag{2.69}$$

where, denoting by t_{j_ℓ} the first time stamp at which all processors have updated their values at least ℓ times, we have $\Phi(m) = \max\{\ell \in \mathbb{N} : t_{j_\ell} \leq t_m\}$, i.e., $\Phi(m)$ is the update number (at time t_m) of the processor that produced the least number of updates among all processors until the instant of time t_m.

We have that

$$|C^s_{t_m,k,i}| \leq \rho_k^{\Phi(m)} w^s_{k,i} = \rho_k^{\Phi(m)-1} \rho_k w^s_{k,i} \ .$$

Let

$$M = \sup_{k \in \mathbb{N}} \rho_k \frac{||v_k||_\infty}{v_{k,min}} \ .$$

By hypothesis, $M < \infty$. Note that

$$\rho_k w^s_{k,i} = \rho_k \frac{v^s_{k,i}}{v_{k,min}} \leq \rho_k \frac{||v_k||_\infty}{v_{k,min}} \leq \sup_{k \in \mathbb{N}} \rho_k \frac{||v_k||_\infty}{v_{k,min}} = M < \infty.$$

Then,

$$|C^s_{t_m,k,i}| \le \rho_k^{\Phi(m)-1} M \le (\sup_{k\in\mathbb{N}} \rho_k)^{\Phi(m)-1} M.$$

Consequently, letting $\rho := \sup_{k\in\mathbb{N}} \rho_k$, we have

$$|C^s_{t_m,k,i}| \le \rho^{\phi(m)-1} M. \tag{2.70}$$

Conditions 2 and 3 of the asynchronous model imply that new updates will always be produced and used by the processors (see Section 2.5), thus we have that $\Phi(m) \to \infty$ as $m \to \infty$. Consequently, from (2.69) we have that $C_{t_m,k} \to 0$ uniformly in $k \in \mathbb{N}$, $s = 1, \ldots, p$, $i = 1, 2$ as $m \to \infty$.

In order to complete the proof we need to show that $\lim_{t_m\to\infty} \eta^s_{t_m} = 0$. We present the proof of this fact for the case of an interior subdomain, i.e., for $1 < s < p$, but in a similar way it can be shown that this also holds for the left-most and right-most subdomains.

Note first that

$$
\begin{aligned}
&\frac{\psi_m^{(1)}(x)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)} = \\
&\frac{H\left(e^{-\frac{(x-L)}{H}\sqrt{z_m^2-\bar{\omega}}}-\frac{e^{\frac{(x-L)}{H}\sqrt{z_m^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{e^{L\sqrt{z_m^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)-\frac{e^{-L\sqrt{z_m^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}} = \\
&\frac{H\left(e^{\frac{(L-x)}{H}\sqrt{z_m^2-\bar{\omega}}}-\frac{e^{-\frac{(L-x)}{H}\sqrt{z_m^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)\left(e^{L\sqrt{z_m^2-\bar{\omega}}}-\frac{e^{-L\sqrt{z_m^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)^2}{\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)^2}\right)} = \\
&\frac{He^{-\frac{x}{H}\sqrt{z_m^2-\bar{\bar{\omega}}}}\left(1-\frac{e^{-2\frac{(L-x)}{H}\sqrt{z_m^2-\bar{\bar{\omega}}}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)\left(1-\frac{e^{-2L\sqrt{z_m^2-\bar{\omega}}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)^2}{\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)^2}\right)} \leq \\
&\frac{He^{-\frac{x}{H}\sqrt{z_m^2-\bar{\omega}}}\left(e^{-2\frac{(L-x)}{H}\sqrt{z_m^2-\bar{\omega}}}+1\right)}{\left(1-e^{-2\bar{L}\sqrt{z_m^2-\bar{\omega}}}\right)\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)} \leq \\
&\frac{2H}{\left(1-e^{-2\bar{L}\sqrt{z_1^2-\bar{\omega}}}\right)\left(\bar{\alpha}-\bar{\beta}z_1^2+\sqrt{z_1^2-\bar{\omega}}\right)} := K.
\end{aligned}
$$

Similarly, we have that

$$
\left|\frac{\psi_m^{(2)}(x)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)}\right| \leq \frac{2H}{\left(1-e^{-2\bar{L}\sqrt{z_1^2-\bar{\omega}}}\right)\left(\bar{\alpha}-\bar{\beta}z_1^2+\sqrt{z_1^2-\bar{\omega}}\right)} = K.
$$

Note from (2.9), (2.10), and (2.13) that for $k \in \mathbb{N}$ such that $z_k^2 < \bar{\omega}$ we have

$$
|\psi_k^{(\nu)}(x)| \leq 2
$$

for $\nu = 1, 2, 3$, $k \in \mathbb{N}$ and $x \in [0, L]$.

Let $\bar{K} = \max\{2, K\}$. Given that ω is fixed and $z_k \to \infty$ as $k \to \infty$, there is a number $N_{\bar{\omega}} \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$ for all $k \geq N_{\bar{\omega}}$ and $z_k^2 < \bar{\omega}$ for all $k < N_{\bar{\omega}}$. Then, we have

$$\begin{aligned}
|\eta_{t_n}^s(x,y)| &= \left| \sum_{m=1}^{\infty} \left[A_{t_n,m,1}^s \psi_m^{(1)}(x) + A_{t_n,m,2}^s \psi_m^{(2)}(x) \right] \phi_m(y) \right| \\
&= \left| \sum_{m=1}^{N_{\bar{\omega}}-1} \left[A_{t_n,m,1}^s \psi_m^{(1)}(x) + A_{t_n,m,2}^s \psi_m^{(2)}(x) \right] \phi_m(y) \right. \\
&+ \left. \sum_{m=N_{\bar{\omega}}}^{\infty} \left[A_{t_n,m,1}^s \psi_m^{(1)}(x) + A_{t_n,m,2}^s \psi_m^{(2)}(x) \right] \phi_m(y) \right| \\
&= \left| \sum_{m=1}^{N_{\bar{\omega}}-1} \left(\frac{C_{t_n,m,1}^s}{z_m^2} \psi_m^{(1)}(x) + \frac{C_{t_n,m,2}^s}{z_m^2} \psi_m^{(2)}(x) \right) \phi_m(y) \right. \\
&+ \sum_{m=N_{\bar{\omega}}}^{\infty} \left(\frac{C_{t_n,m,1}^s}{z_m^2 \left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(1)}(0) \right)} \psi_m^{(1)}(x) \right. \\
&+ \left. \left. \frac{C_{t_n,m,2}^s}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(2)}(L) \right)} \psi_m^{(2)}(x) \right) \phi_m(y) \right| \\
&\leq \sum_{m=1}^{N_{\bar{\omega}}-1} \left(\frac{|C_{t_n,m,1}^s|}{z_m^2} \left| \psi_m^{(1)}(x) \right| + \frac{|C_{t_n,m,2}^s|}{z_m^2} \left| \psi_m^{(2)}(x) \right| \right) |\phi_m(y)| \\
&+ \sum_{m=N_{\bar{\omega}}}^{\infty} \frac{1}{z_m^2} \left(|C_{t_n,m,1}^s| \left| \frac{\psi_m^{(1)}(x)}{-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(1)}(0)} \right| \right. \\
&+ \left. |C_{t_n,m,2}^s| \left| \frac{\psi_m^{(2)}(x)}{\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(2)}(L)} \right| \right) |\phi_m(y)| \\
&\leq \sum_{m=1}^{N_{\bar{\omega}}-1} \frac{2(2\rho^{\Phi(n)-1} M)}{z_m^2} + \sum_{m=N_{\bar{\omega}}}^{\infty} \frac{2(\rho^{\Phi(n)-1} MK)}{z_m^2} \\
&\leq \sum_{m=1}^{\infty} \frac{1}{z_m^2} 2\rho^{\Phi(n)-1} M\bar{K} = 2\rho^{\Phi(n)-1} M\bar{K} \sum_{m=1}^{\infty} \frac{1}{z_m^2}.
\end{aligned}$$

Then, since $\sum_{m=1}^{\infty} \frac{1}{z_m^2} = \sum_{m=1}^{\infty} \frac{1}{(m\pi)^2} < \infty$ and $\lim_{t_m \to \infty} \rho^{\Phi(n)} = 0$ we have

$$\lim_{t_m \to \infty} |\eta_{t_n}^s(x,y)| \leq \lim_{t_m \to \infty} 2\rho^{\Phi(n)-1} M\bar{K} \sum_{m=1}^{\infty} \frac{1}{z_m^2} = 0 \cdot$$

Thus, we just showed that for $1 < s < p$, $\eta_{t_n}^s(x,y) \to 0$ uniformly in $[0,L] \times [0,H]$

as $n \to \infty$. With a similar procedure, it can be shown that the same result holds for $s = 1, p$. Therefore,

$$\lim_{t_m \to \infty} \eta_{t_n}^s = 0$$

uniformly in $[0, L] \times [0, H]$ as $n \to \infty$ for all $s \in \{1, \dots, p\}$. Since we have a finite number of subdomains, this implies that

$$\lim_{t_m \to \infty} \eta_{t_n} = 0,$$

i.e., the global error converges to zero uniformly in $[0, W] \times [0, H]$. □

Now we present a second convergence result using a different hypothesis.

Theorem 2.5. *The asynchronous iterations given by (2.64)-(2.67) converge if the values of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\omega}$ and $\bar{L}$ are such that*

$$\sup_{k \in \mathbb{N}} ||\hat{T}_k||_\infty < 1. \tag{2.71}$$

Proof. Let $\{t_n\}$ be a monotonically increasing sequence of time stamps.

Let $\rho := \sup_{k \in \mathbb{N}} ||\hat{T}_k||_\infty$. By hypothesis we have $\sup_{k \in \mathbb{N}} ||\hat{T}_k||_\infty < 1$. Then it follows that

$$||\hat{T}_k C_{t_n,k}||_\infty \leq ||\hat{T}_k||_\infty ||C_{t_n,k}||_\infty \leq \rho ||C_{t_n,k}||_\infty.$$

Let t_j be the stamp at which the processor s produces its first new update. Let $C_{t_j,k}^s$ be the vector containing the coefficients corresponding to the subdomain s, e.g., $C_{t_j,k}^s = (C_{t_j,k,1}^s, C_{t_j,k,2}^s)$ for $1 < s < p$. We have that $\hat{T}_k = (\hat{T}_k^1, \dots, \hat{T}_k^p)$, where $\hat{T}_k^s$ is the local operator associated to the subdomain s. Then, we have

$$|C_{t_j,k,i}^s| \leq ||C_{t_j,k}^s||_\infty = ||\hat{T}_k^s C_{t_0,k}||_\infty \leq \rho ||C_{t_0,k}||_\infty.$$

In particular, we have

$$|C^s_{t_j,k,i}| < ||C_{t_0,k}||_\infty.$$

Consequently, for this processor it holds that

$$|C^s_{t_j,k,i}| \leq \rho ||C_{t_0,k}||_\infty \tag{2.72}$$

for all $t_n \geq t_j$.

In a similar way as in the Theorem 2.4, we have that after every processor has produced its first update, say at t_{j_1} we have that (2.72) holds for all s, and consequently $|C^s_{t_j,k,i}| \leq \rho ||C_{t_0,k}||_\infty$ for all $t_j \geq t_{j_1}$. Then, this implies that we have

$$|C_{t_j,k,i}| \leq \rho^{\Phi(j)} ||C_{t_0,k}||_\infty, \tag{2.73}$$

where, denoting by t_{j_ℓ} the first time stamp at which all processors have updated their values at least ℓ times, we have $\Phi(j) = \max\{\ell \in \mathbb{N} : t_{j_\ell} \leq t_j\}$, i.e., $\Phi(j)$ is the update number (at time t_j) of the processor that produced the least number of updates among all processors until the instant of time t_j.

Then, following the same procedure as in the Theorem 2.4 but using (2.73) instead of (2.70) we can see that the local error $\eta^s_{t_n} \to 0$ uniformly in $[0, L] \times [0, H]$ as $n \to \infty$. □

From (2.58), we have for $k \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$ that

$$||\hat{T}_k||_\infty = \max\left\{c_1(k, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\omega}, \bar{L}) + c_2(k, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\omega}, \bar{L}), c_3(k, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\omega}, \bar{L})\right\}.$$

Then, based on this expression and the previous theorem, we have the following corollary.

Corollary 2.6. *Let $\omega \leq 0$. Let c_1, c_2 and c_3 be defined as in (2.45)-(2.47). Then,* *the asynchronous iterations given by (2.64)-(2.67) converge if the values of* $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\omega}$ *and* $\bar{L}$ *are such that*

$$\sup_{k\in\mathbb{N}} \left\{\max\left\{c_1(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}) + c_2(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L}), c_3(k,\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\omega},\bar{L})\right\}\right\} < 1. \quad (2.74)$$

2.7 Optimal parameters

From the bound on the coefficients given in (2.60) we can see that $A^s_{n,k,i}, C^s_{n,k,i} \to 0$ as $n \to \infty$ if $\rho(|\hat{T}_k|) < 1$. In other words, if $\rho(|\hat{T}_k|) < 1$, then this quantity is a measure of how fast the coefficients of frequency k converge to zero as the number of updates increases. Then, it follows that the quantity $\rho = \sup_{k\in\mathbb{N}} \rho(|\hat{T}_k|)$ describes the convergence rate at which all the coefficients of the series expansion of the local errors decrease to zero, and therefore describes the convergence rate of the method. As we shall see, $\rho(|\hat{T}_k|)$ varies with k and it has one maximum. Therefore, we define the optimal values of the normalized artificial boundary parameters $\bar{\alpha}$ and $\bar{\beta}$ as follows.

For the $OO2$ case, i.e., for $\beta \neq 0$, the optimal $\bar{\alpha}$ and $\bar{\beta}$ are such that

$$(\bar{\alpha}_{\text{opt}}, \bar{\beta}_{\text{opt}}) = \text{argmin}_{\bar{\alpha}\in(0,\infty),\bar{\beta}\in\mathbb{R}} \left\{\max_{k\in\mathbb{N}} \rho(|\hat{T}_k|)\right\}.$$

For the $OO0$ case, i.e., for $\beta = 0$, the optimal $\bar{\alpha}$ is given by

$$(\bar{\alpha}_{\text{opt}}) = \text{argmin}_{\bar{\alpha}\in(0,\infty),\bar{\beta}=0} \left\{\max_{k\in\mathbb{N}} \rho(|\hat{T}_k|)\right\}.$$

2.7.1 Optimal parameters for the $\omega \leq 0$ case

Optimal $\bar{\alpha}$ for the OO0 case

In Figure 2.2 we can see that for any value of $\bar{\alpha}$, the maximum of $\rho(|\hat{T}_k|)$ falls into a region of low frequencies (e.g, for $k < 20$). For higher frequencies $\rho(|\hat{T}_k|)$ is very small, which implies that the coefficients $C^s_{n,k,i}$ corresponding to high frequencies are damped at a very fast rate. Thus, in order to optimize the convergence of the method it suffices to minimize the maximum value of $\rho(|\hat{T}_k|)$ over the range of low frequencies; usually considering the range $k \leq 20$ is enough to find the optimal parameters.

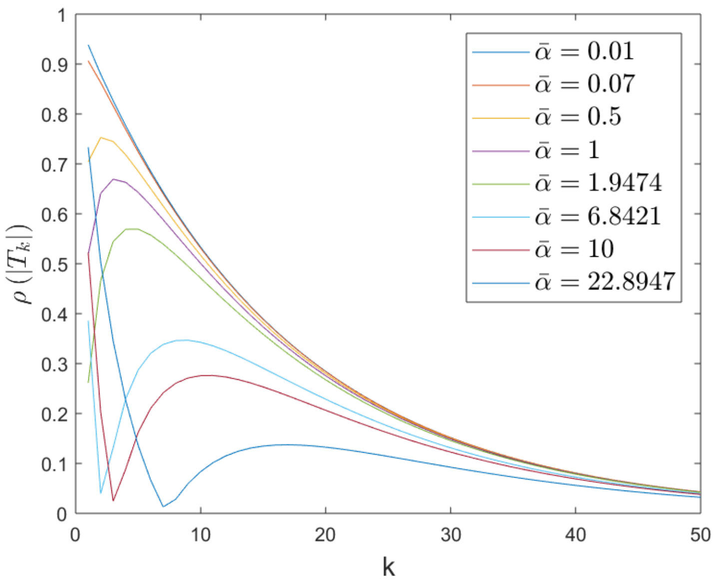

Figure 2.2: $\rho(|\hat{T}_k|)$ vs. k for $\bar{\alpha} \in [0.01, 30]$, with $\bar{\beta} = 0$, $p = 10$, $\bar{\omega} = 0$ and $\bar{\gamma} = 0.01$

In Figure 2.3 we see a typical curve of how $\sup_k \rho(|\hat{T}_k|)$ varies with $\bar{\alpha}$ for a fixed normalized overlap $\bar{\gamma}$, fixed number of subdomains p and fixed $\bar{L} = L/H$. This curve corresponds to the case $\bar{\gamma} = 0.01$, $p = 100$, $\bar{L} = 1$ and $\bar{\omega} = 0$. We can see that the curve has a minimum, which we call the optimal convergence rate factor, and the value of $\bar{\alpha}$ at which this minimum occurs is the optimal $\bar{\alpha}$ for the $OO0$ case.

In Figures 2.4-2.6 we can see blue curves showing how the optimal values of $\bar{\alpha}$ (for the case $\beta = 0$) vary with the normalized overlap $\bar{\gamma}$ for given number of subdomains p, $\bar{L} = 1$ and $\bar{\omega} = 0$. As it can be seen, these curves are essentially the same. It turns out that for small enough values of $\bar{\gamma}$ (e.g, $\bar{\gamma} \leq 0.06$, which is the case of interest in practice) these curves can be approximated well by the graph of a power function. Therefore,

$$\bar{\alpha}_{\text{opt}} \approx D(p, \bar{L}, \bar{\omega})\bar{\gamma}^{Q(p,\bar{L},\bar{\omega})}.$$

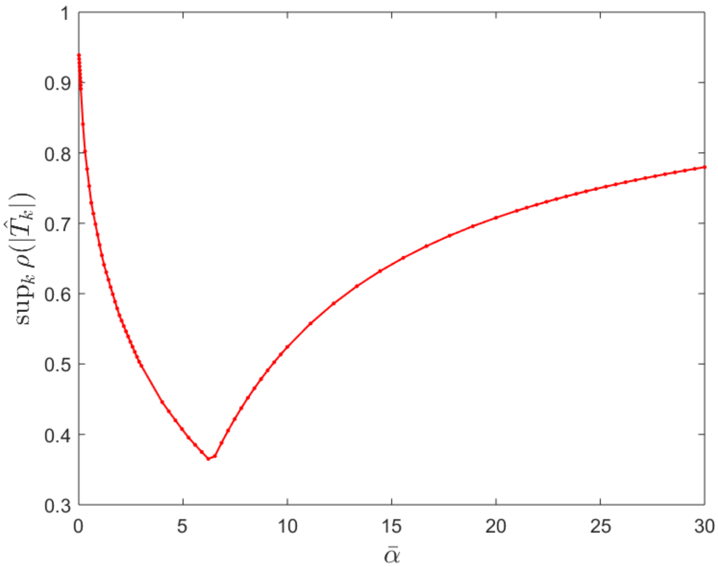

Figure 2.3: Convergence factor ρ vs. $\bar{\alpha}$ for $\bar{\gamma} = 0.01$, $p = 100$, $\bar{L} = 1$, $\bar{\beta} = 0$

The power-law approximation of these curves are shown in 2.4-2.6 by the red curves. In this case, i.e., when $\bar{L} = 1$, $\bar{\omega} = 0$ and $p \geq 10$, we have that $D = 1.7305$ and $Q = -0.2822$.

In Figures 2.7-2.9 we can see how the optimal convergence factor varies with $\bar{\gamma}$ for different values of p when $\bar{L} = 1$ and $\bar{\omega} = 0$. In all of these figures we can observe that the optimal convergence factor decreases as the normalized overlap $\bar{\gamma}$ increases.

In Figures 2.10 and 2.11 we can see how the optimal convergence rate factor and the optimal parameter $\bar{\alpha}$ vary with the number of subdomains p for $\bar{\gamma} = 0.01$, $\bar{L} = 1$ and $\bar{\omega} = 0$. In these figures we can observe that for large enough values of p, the value of $\bar{\alpha}_{\text{opt}}$ remains constant as p increases.

In Figures 2.12 and 2.13 we can see how the optimal convergence factor and $\bar{\alpha}$ vary with $\bar{L}$ for $\bar{\gamma} = 0.01\bar{L}$, $p = 10$, and $\bar{\omega} = 0$. From Figure 2.12 it follows that, for a fixed domain, as we increase the number of subdomains the optimal convergence factor deteriorates. From the same result, it can be concluded that between two domain decompositions with the same number of subdomains and same normalized overlap $\bar{\gamma}$, the decomposition with larger normalized width $\bar{L} = L/H$ will produce

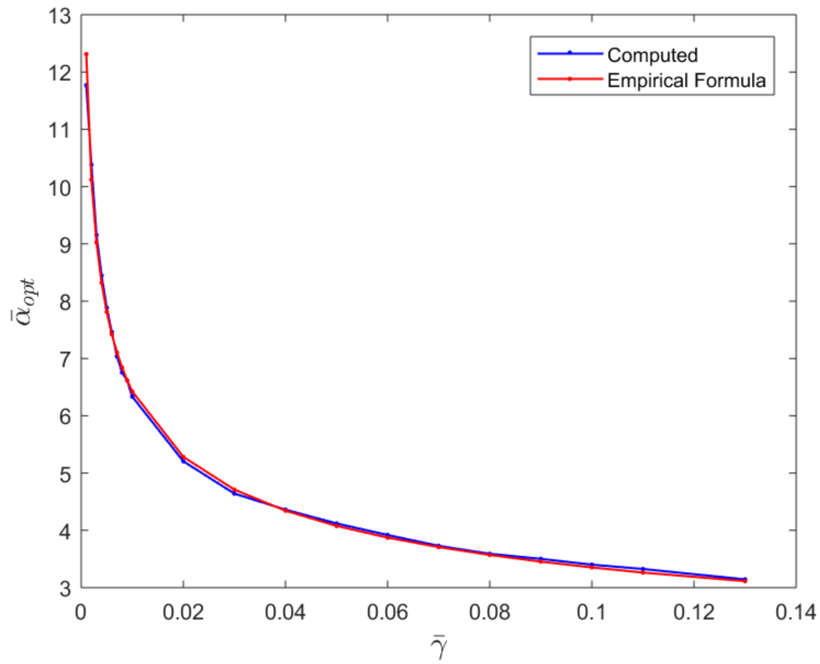

Figure 2.4: Comparison between the computed values of $\bar{\alpha}_{\text{opt}}$ and their power-law approximations for $\bar{\gamma} \in [0.001, 0.13]$, $p = 10$, $\bar{\omega} = 0$ and $\bar{L} = 1$

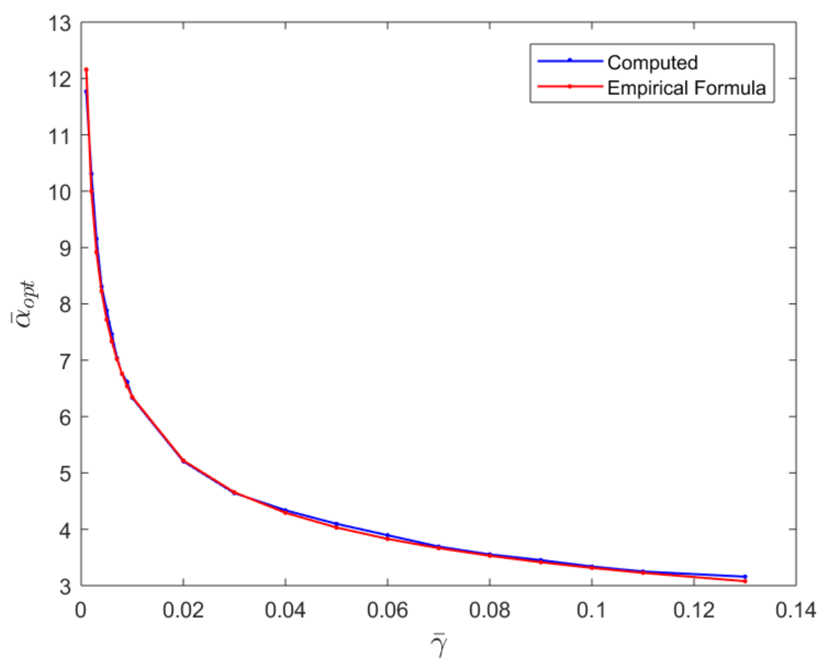

Figure 2.5: Comparison between the computed values of $\bar{\alpha}_{\text{opt}}$ and their power-law approximations for $\bar{\gamma} \in [0.001, 0.13]$, $p = 20$, $\bar{\omega} = 0$ and $\bar{L} = 1$

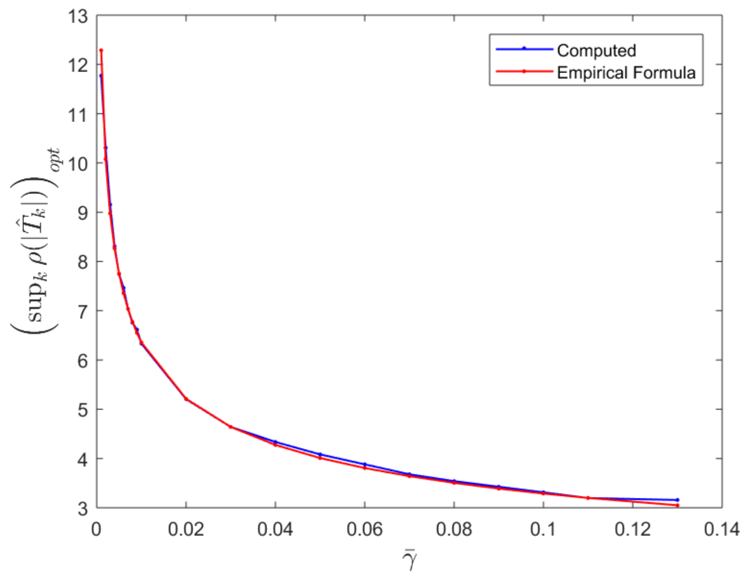

Figure 2.6: Comparison between the computed values of $\bar{\alpha}_{\text{opt}}$ and their power-law approximations for $\bar{\gamma} \in [0.001, 0.13]$, $p = 50$, $\bar{\omega} = 0$ and $\bar{L} = 1$

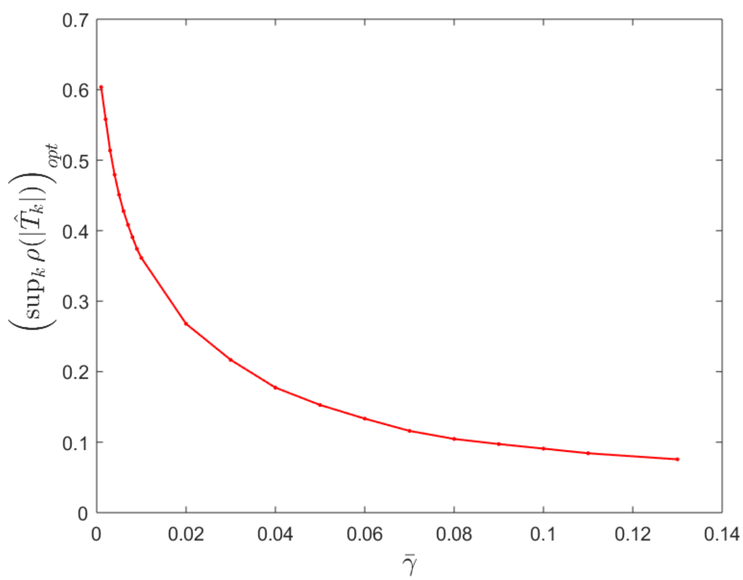

Figure 2.7: Optimal convergence factor vs. normalized overlap $\bar{\gamma}$ for $p = 10$, $\bar{L} = 1$ and $\bar{\omega} = 0$

a better optimal convergence factor. Note that the length of a physical boundary is L and the length of an artificial boundary is H. Thus the ratio $\bar{L} = L/H$ indicates what proportion of the boundaries are physical boundaries. Thus, the larger is the

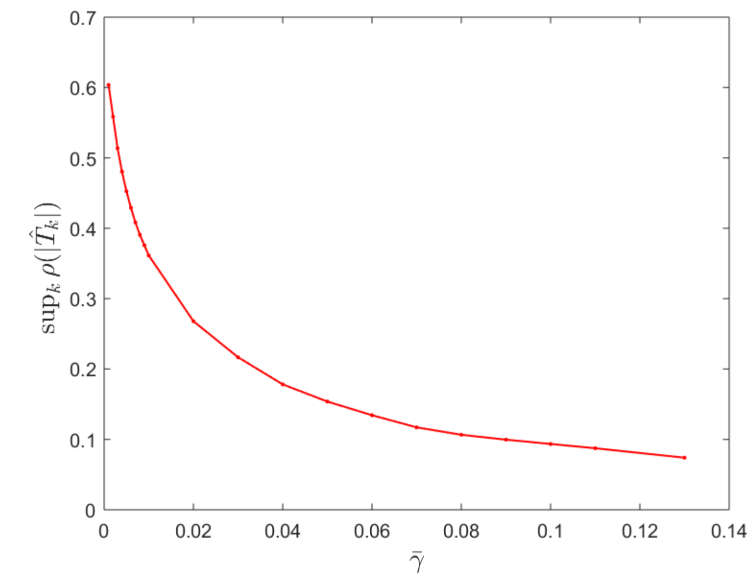

Figure 2.8: Optimal convergence factor vs. normalized overlap $\bar{\gamma}$ for $p = 20$, a$\bar{L} = 1$ and $\bar{\omega} = 0$

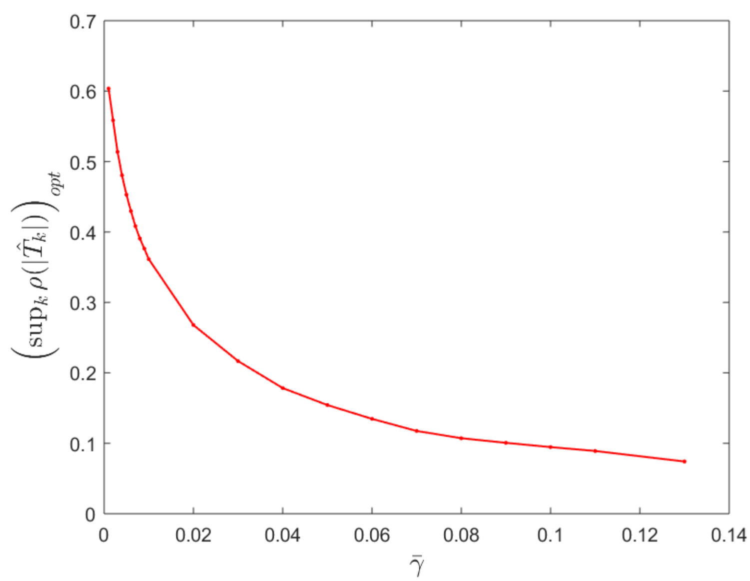

Figure 2.9: Optimal convergence factor vs. normalized overlap $\bar{\gamma}$ for $p = 50$, $\bar{L} = 1$ and $\bar{\omega} = 0$

of ratio physical boundaries vs. artificial boundaries of a subdomain, the faster is the convergence.

In Figure 2.14 we see how the optimal convergence factor varies with $\bar{\omega}$ when $\bar{\omega} \leq 0$, $p = 10$, $\bar{\gamma} = 0.01$ and $\bar{L} = 1$.

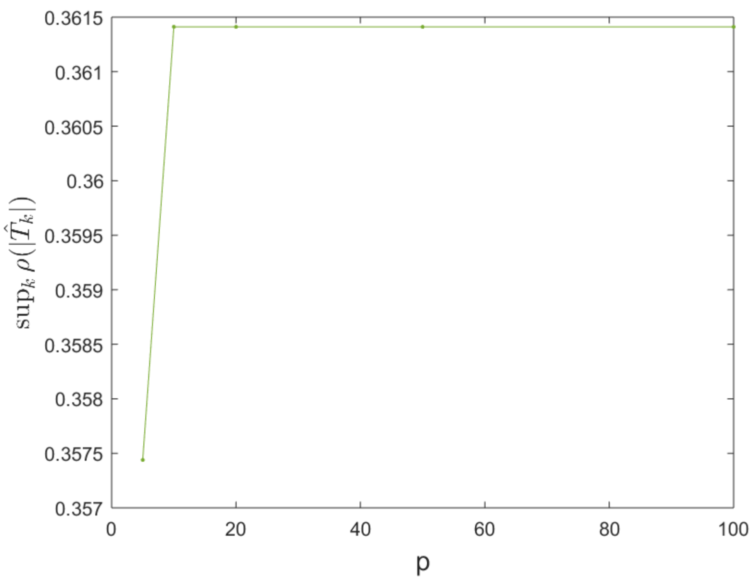

Figure 2.10: Optimal convergence factor vs. number of subdomains p for $\bar{\gamma} = 0.01$, $\bar{L} = 1$ and $\bar{\omega} = 0$

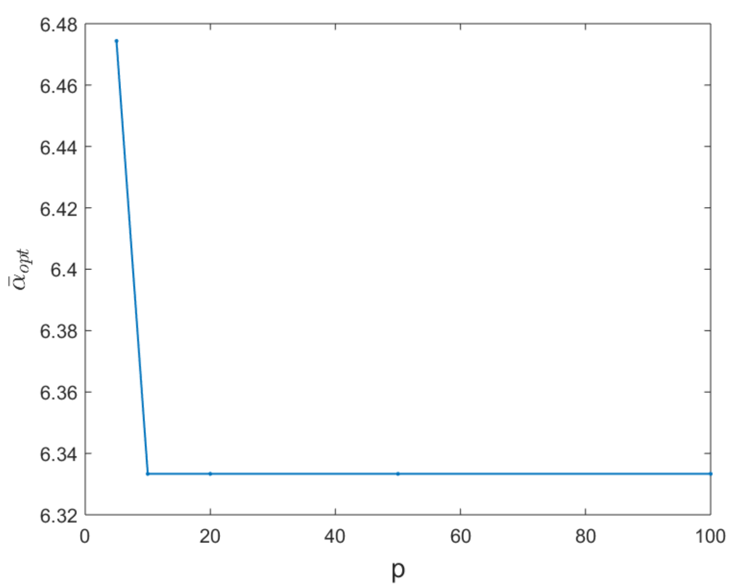

Figure 2.11: Optimal $\bar{\alpha}$ vs. number of subdomains p for $\bar{\gamma} = 0.01$, $\bar{L} = 1$ and $\bar{\omega} = 0$

As we can see, the optimal convergence factor varies monotonically with $\bar{\omega}$ for $\bar{\omega} \leq 0$.

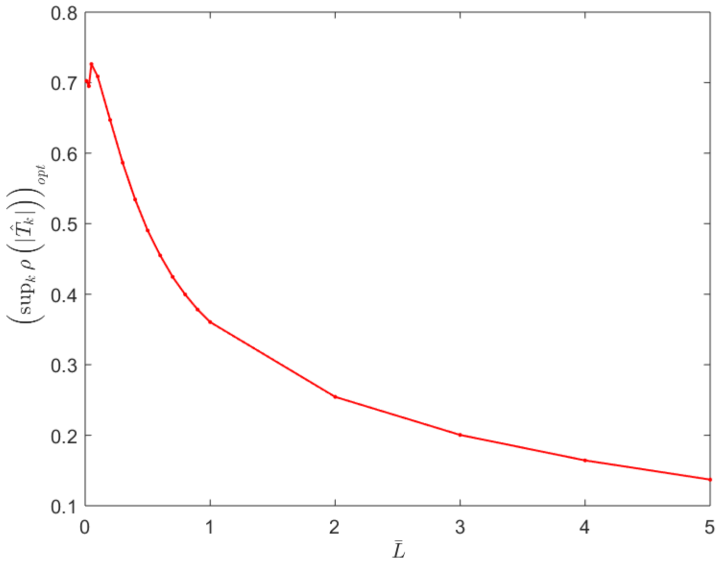

Figure 2.12: Optimal convergence factor vs. $\bar{L}$ for $\bar{\gamma} = 0.01\bar{L}$, $p = 10$ and $\bar{\omega} = 0$

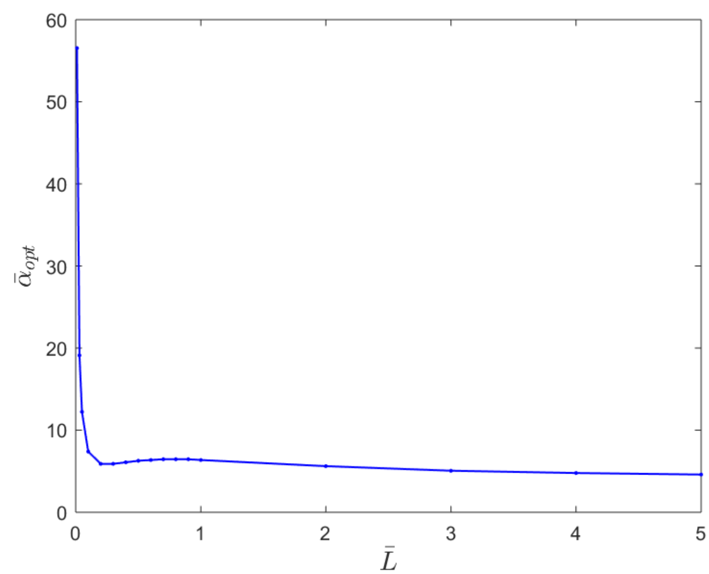

Figure 2.13: Optimal $\bar{\alpha}$ vs. $\bar{L}$ for $\bar{\gamma} = 0.01\bar{L}$, $p = 10$ and $\bar{\omega} = 0$

Optimal Parameters for the $OO2$ case

In the case where $\beta < 0$, we still have that for a fixed value of $\bar{\alpha}$, $\rho(|\hat{T}_k|)$ has a maximum over the frequencies k and it falls into a region of low frequencies (e.g, $k < 20$). In figure 2.15 we can see how $\rho(|\hat{T}_k|)$ varies with k for $\bar{\beta} = -0.044$, $\bar{\gamma} = 0.01$,

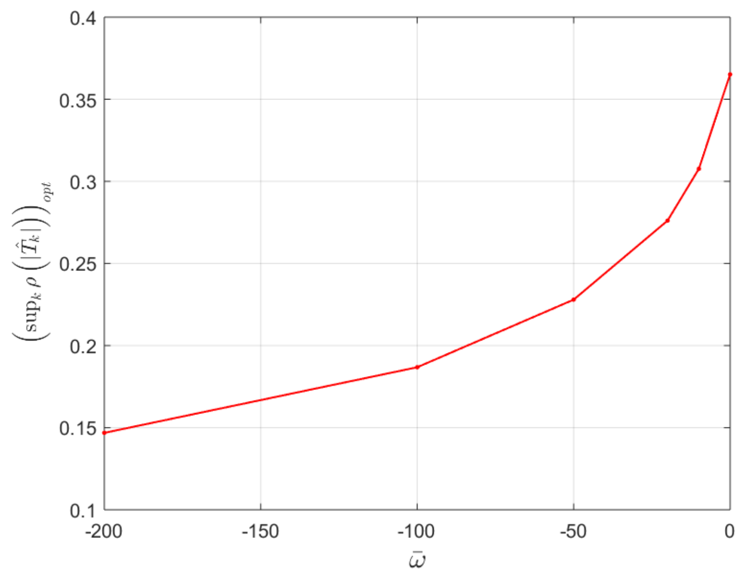

Figure 2.14: Optimal convergence factor vs. $\bar{\omega}$ for $p = 10$, $\bar{\gamma} = 0.01$ and $\bar{L} = 1$

$\bar{\omega} = 0$, $p = 10$ and $\bar{\alpha} \in [0.01, 30]$.

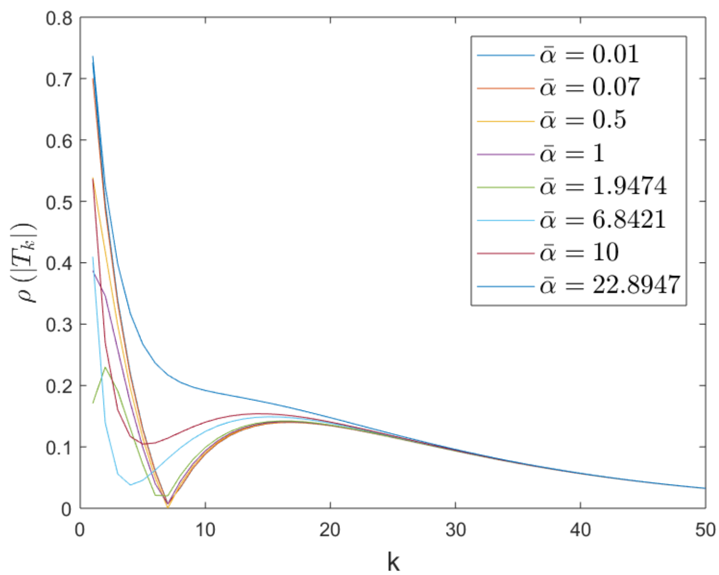

Figure 2.15: $\rho(|\hat{T}_k|)$ vs. k with $\bar{\alpha} \in [0.01, 30]$, $\bar{\beta} = -0.044$, $\bar{\gamma} = 0.01$, $p = 10$, and $\bar{\omega} = 0$

In Figure 2.16 we can see how the values of the convergence factor vary with $\bar{\alpha}$ for two values of $\bar{\beta}$, namely, $\bar{\beta} = 0$ (blue curve) and $\bar{\beta} = -0.044$ (red curve), with $\bar{\gamma} = 0.01$, $p = 100$ and $\bar{L} = 1$. From this figure we can see that there is a substantial improvement in the convergence factor when this second parameter $\bar{\beta}$ is tuned. Therefore, the $OO2$ case gives a better convergence factor with respect to the $OO0$ case, for the case of a one-way domain decomposition.

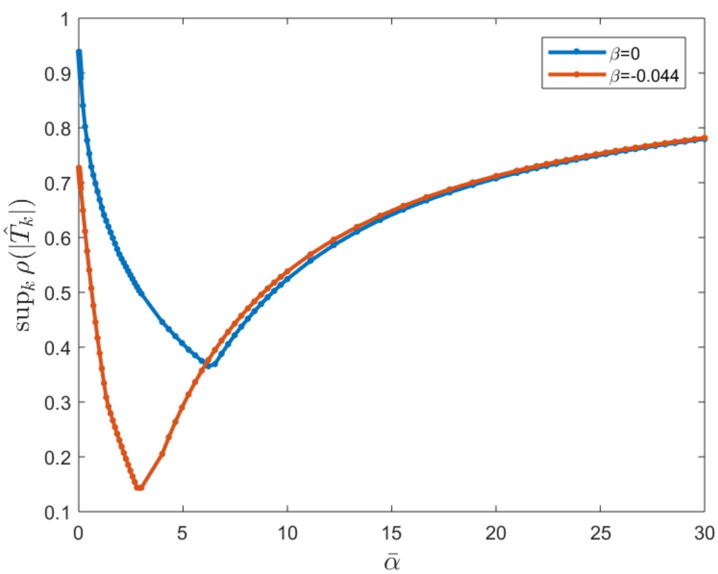

Figure 2.16: Comparison between the optimal convergence factor values for $\bar{\beta} = 0$ and $\bar{\beta} = -0.04$ when $\bar{\alpha} \in [0.01, 30]$ $\bar{\gamma} = 0.01$, $p = 100$ $\bar{L} = 1$, and $\bar{\omega} = 0$

In Figure 2.17 we can see how the convergence factor ρ varies with the values of $\bar{\alpha}$ and $\bar{\beta}$ when $\gamma = 0.01$, $\bar{\omega} = 0$ and $\bar{L} = 1$. In particular we can observe that the convergence factor has one minimum.

In Figures 2.18-2.20 we can see how the spectral radius, $\bar{\alpha}$ and $\bar{\beta}$ vary with the value of $\bar{\gamma}$ for $p = 10$, $\bar{L} = 1$ and $\bar{\omega} = 0$.

In Figures 2.21-2.23 we can see how the optimal convergence rate factor and the optimal parameters $\bar{\alpha}$ and $\bar{\beta}$ vary with the number of subdomains p for $\bar{\gamma} = 0.01$, $\bar{L} = 1$ and $\bar{\omega} = 0$. Also, in Figures 2.24-2.26 we can see how the optimal convergence factor and the optimal parameters $\bar{\alpha}$ and $\bar{\beta}$ vary for $\bar{\gamma} = 0.01\bar{L}$ and $p = 10$.

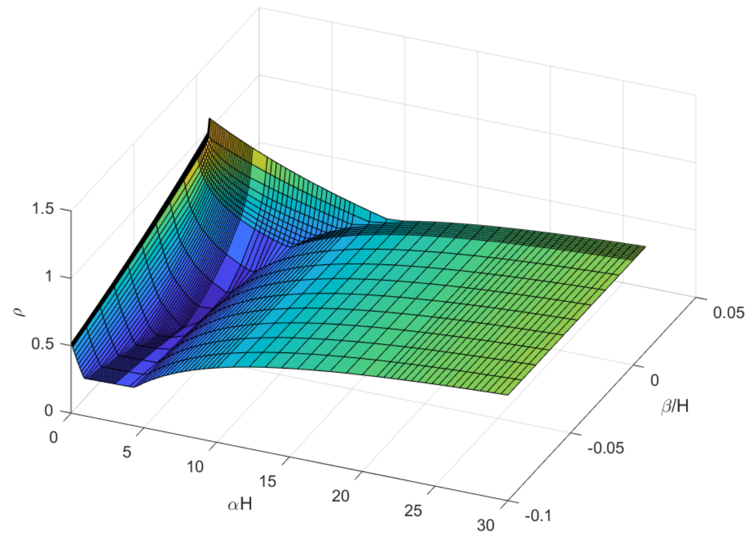

Figure 2.17: Optimal convergence factor vs. $(\bar{\alpha}, \bar{\beta})$ when $\bar{\gamma} = 0.01$, $\bar{L} = 1$, $p = 100$ and $\bar{\omega} = 0$

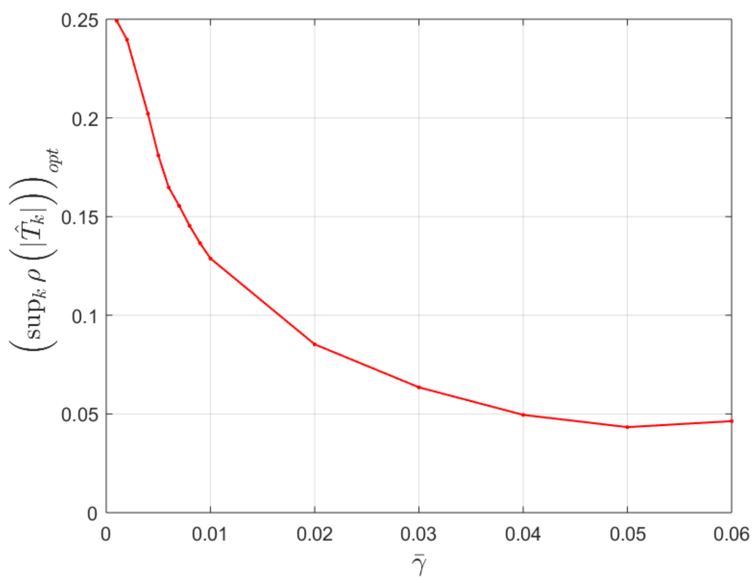

Figure 2.18: Optimal convergence factor vs. $\bar{\gamma}$ for $p = 10$, $\bar{L} = 1$ and $\bar{\omega} = 0$

2.7.2 Optimal parameters for the $\omega > 0$ case

This corresponds to the Helmholtz equations case. We analyze how the convergence factor is affected by the different parameters for the $OO0$ case.

In cases where $\bar{\omega} > 0$, in contrast with the cases with $\bar{\omega} \leq 0$, the optimal values

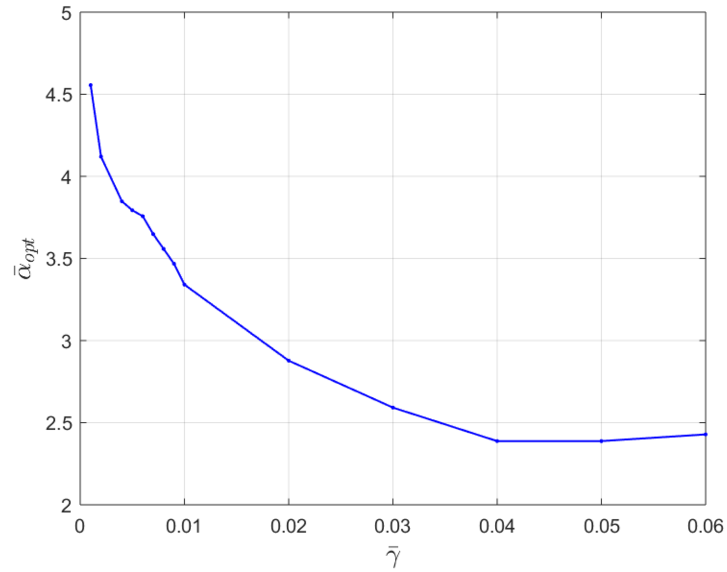

Figure 2.19: Optimal parameter $\bar{\alpha}$ vs. $\bar{\gamma}$ for $p = 10$, $\bar{L} = 1$ and $\bar{\omega} = 0$

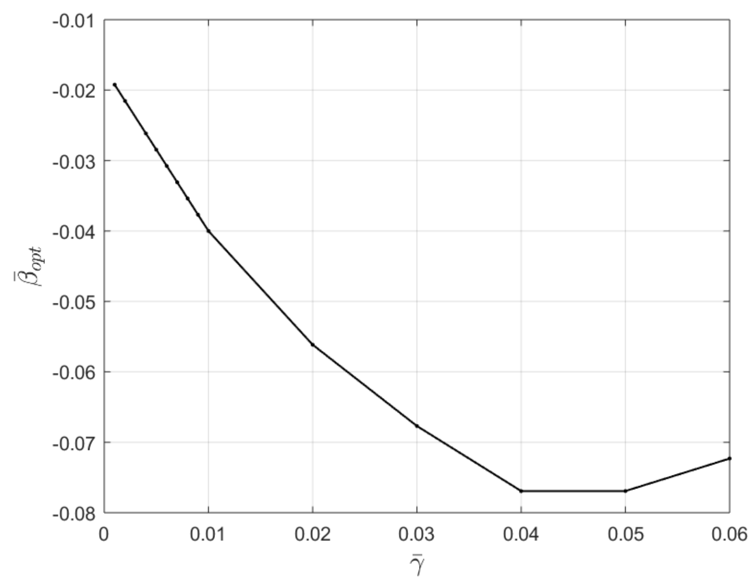

Figure 2.20: Optimal parameter $\bar{\beta}$ vs. $\bar{\gamma}$ for $p = 10$, $\bar{L} = 1$ and $\bar{\omega} = 0$

of the parameter $\bar{\alpha}$ are complex values. Also, in our computations we have obtained that $\sup_k ||\hat{T}_k||_\infty > 1$ which is not the case for $\bar{\omega} \leq 0$. However, we still observed that $\rho = \sup_k \rho(|\hat{T}_k|) < 1$ in many cases.

In Figure 2.27 we can see how ρ varies with $\bar{\alpha}$ for $\bar{\beta} = 0$, $\bar{\gamma} = 0.01$, $\bar{\omega} = 100$, $p = 10$ and $\bar{L} = 1$. In this picture the minimum of the convergence factor $\rho = \sup_k \rho(|\hat{T}_k|)$

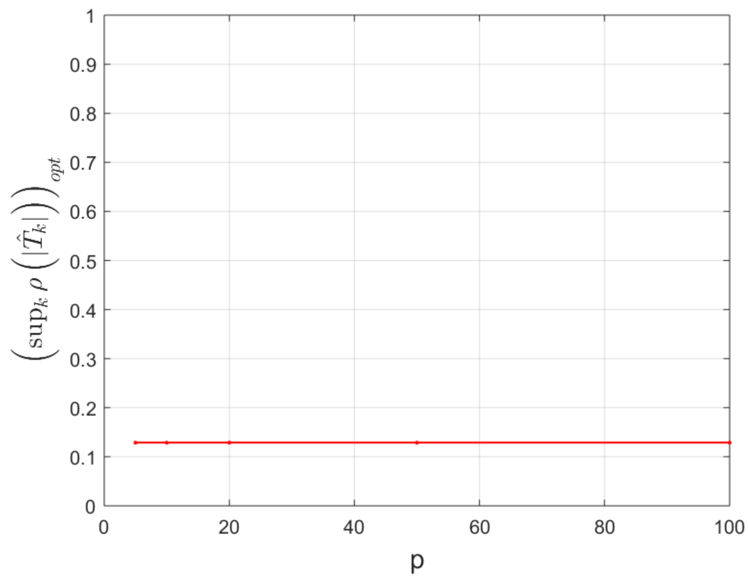

Figure 2.21: Optimal convergence factor vs. p for $\bar{\gamma} = 0.01$, $\bar{L} = 1$ and $\bar{\omega} = 0$

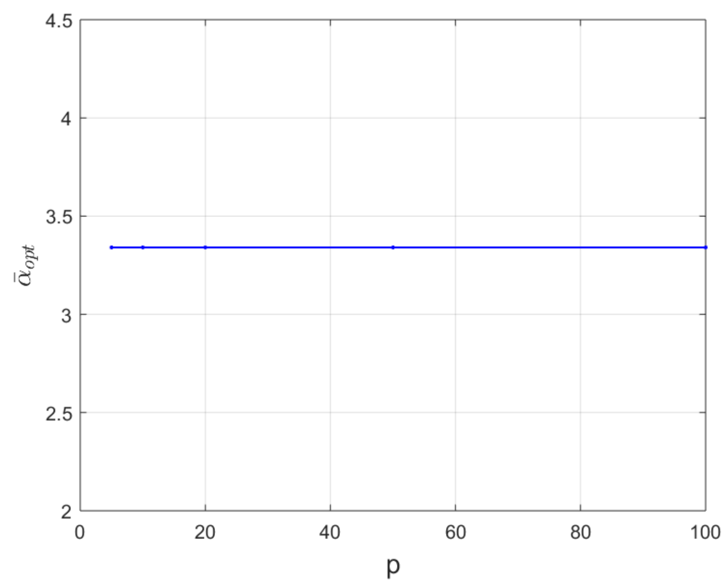

Figure 2.22: Optimal parameter $\bar{\alpha}$ vs. p for $\bar{\gamma} = 0.01$, $\bar{L} = 1$ and $\bar{\omega} = 0$

occurs when $\bar{\alpha} = \bar{\alpha}_{\text{opt}} = 0.4 + 6.755i$.

In Figure 2.28 we can observe how the optimal convergence factor vary with the normalized shift $\bar{\omega}$ when $\bar{\gamma} = 0.01$, $p = 10$ and $\bar{L} = 1$. In particular, we can observe that the convergence factor does not vary monotonically with $\bar{\gamma}$. The largest value of the convergence factor within these range of values of $\bar{\omega}$ was $\rho = 0.9994$ corresponding

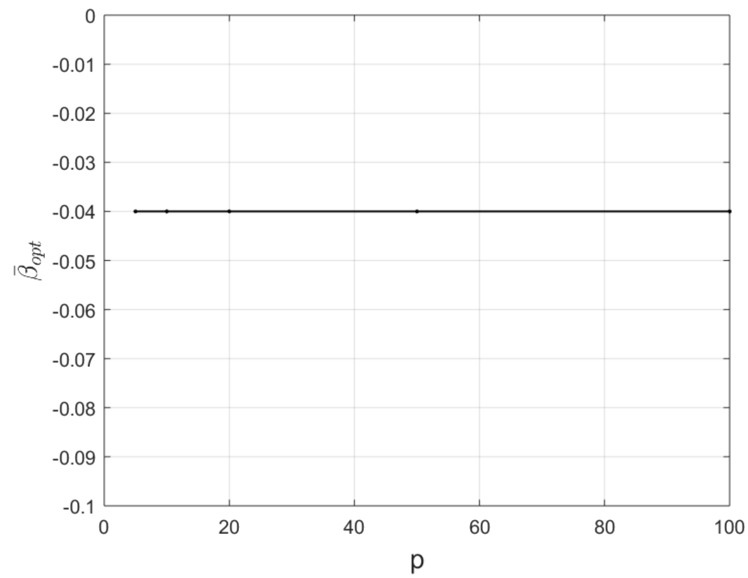

Figure 2.23: Optimal parameter $\bar{\beta}$ vs. p for $\bar{\gamma} = 0.01$, $\bar{L} = 1$ and $\bar{\omega} = 0$

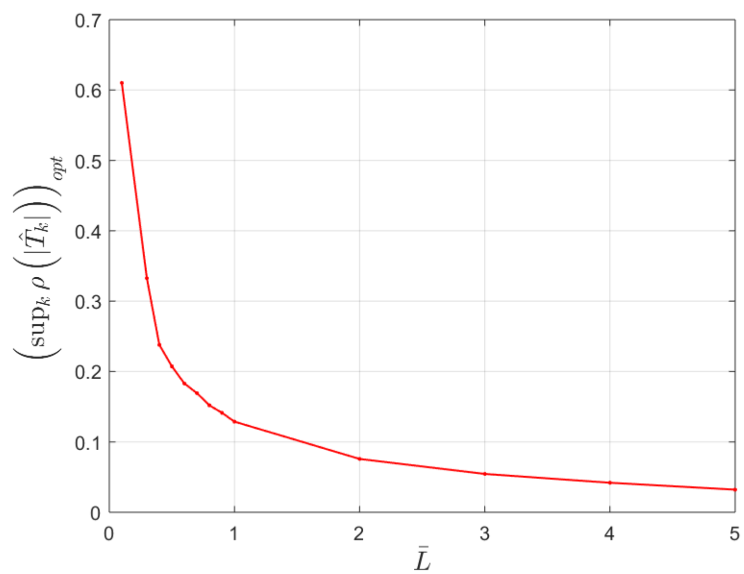

Figure 2.24: Optimal convergence factor vs. $\bar{L}$ for $\bar{\gamma} = 0.01\bar{L}$ and $p = 10$

to $\bar{\omega} = 150$. In Table 2.1 the values of $\bar{\alpha}_{\text{opt}}$ are given for different values of $\bar{\omega}$ using the same values of $\bar{\gamma}$, p and $\bar{L}$.

In Figure 2.29 we observe how the optimal convergence factor varies with the normalized overlap $\bar{\gamma}$, for $p = 10$, $\bar{\omega} = 50$ and $\bar{L} = 1$. In Table 2.2 the corresponding values of $\bar{\alpha}_{\text{opt}}$ are given. In this case we can also see that there is not a monotonic

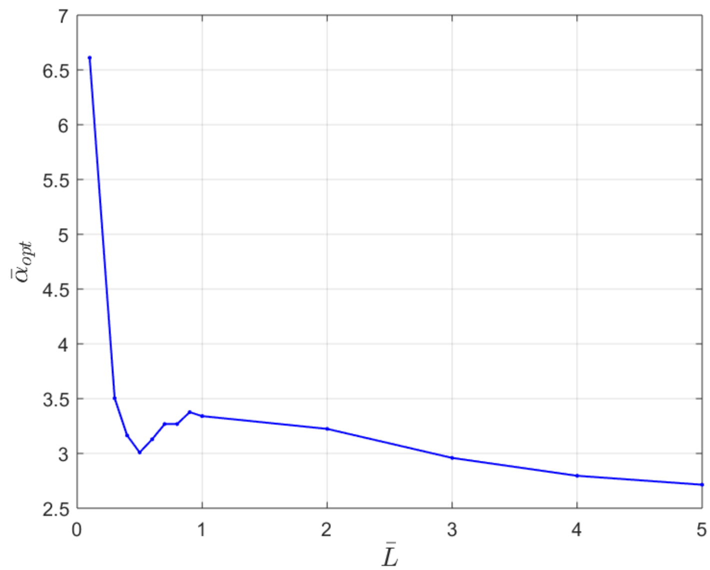

Figure 2.25: Optimal parameter $\bar{\alpha}$ vs. $\bar{L}$ for $\bar{\gamma} = 0.01\bar{L}$ and $p = 10$

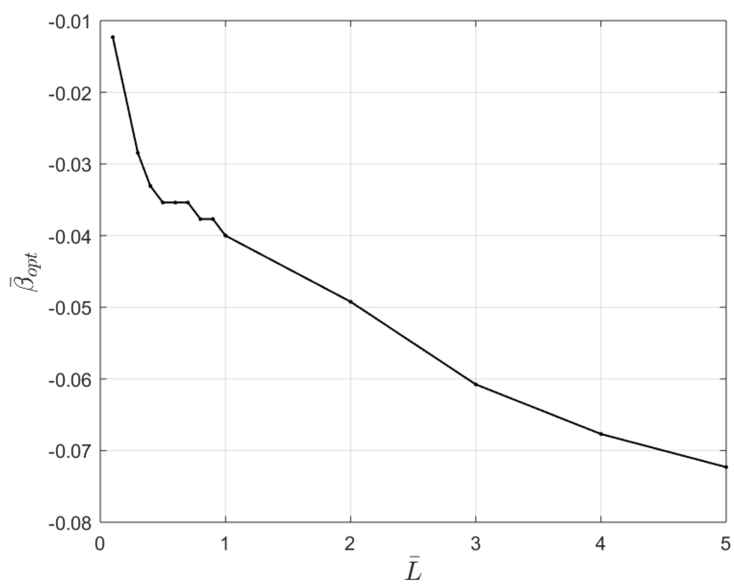

Figure 2.26: Optimal parameter $\bar{\beta}$ vs. $\bar{L}$ for $\bar{\gamma} = 0.01\bar{L}$ and $p = 10$

behavior of the convergence factor as $\bar{\gamma}$ increases. Note in this figure that for large values of $\bar{\gamma}$, the convergence factor $\rho = \sup_k \rho(|\hat{T}_k|) > 1$. Thus, our convergence theorems hold only for values of the normalized overlap $\bar{\gamma}$ smaller than 0.06, i.e., overlap of 6% of the width of the subdomain. Now, the fact that the contraction factor

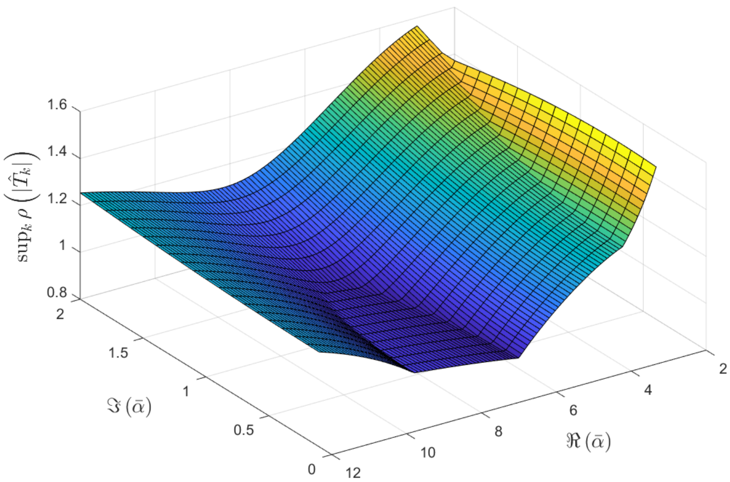

Figure 2.27: Optimal convergence factor vs. $\bar{\alpha}$ for $\bar{\gamma} = 0.01$, $p = 10$, $\bar{\omega} = 100$, $\bar{\beta} = 0$ and $\bar{L} = 1$

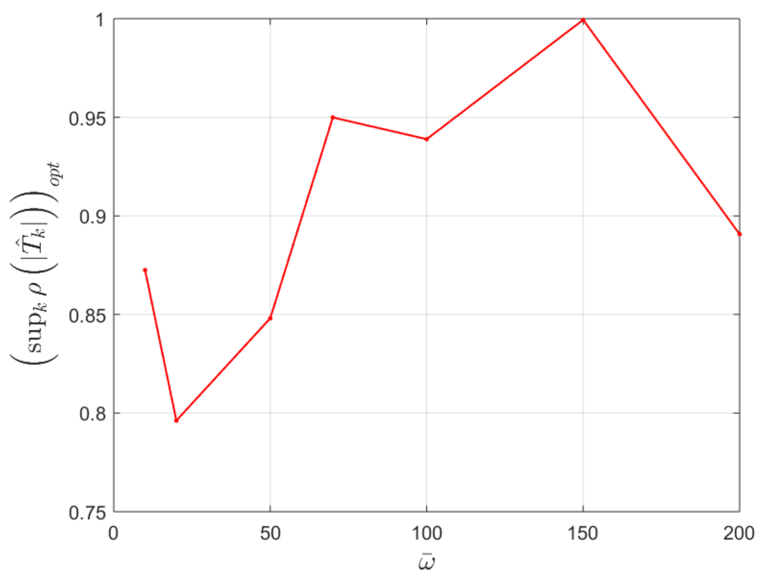

Figure 2.28: Optimal convergence factor vs. $\bar{\omega}$ for $\bar{\gamma} = 0.01$, $p = 10$ and $\bar{L} = 1$

is larger than one and that this bound does not contract does not imply necessarily that the asynchronous method will not converge. We might still get convergence in practice. Also, note that in practice we are interested in small values of overlap.

In Figure 2.30 we can see how the optimal convergence factor varies with the number of subdomains p, for $\bar{\gamma} = 0.01$, $\omega = 50$ and $\bar{L} = 1$, and in Table 2.3 we have

Table 2.1: Values of $\bar{\alpha}_{\text{opt}}$ for different values of $\bar{\omega}$, when $p = 10$, $\bar{\gamma} = 0.01$ and $\bar{L} = 1$.

$\bar{\omega}$	$\bar{\alpha}_{\text{opt}}$
10	0.0811 - 0.4000i
20	0.5222 - 2.9310i
50	0.3111 - 5.1224i
70	0.0195 - 6.5918i
100	0.5222 - 6.8367i
150	0.3111 - 9.4490i
200	0.1000 - 12.0816i

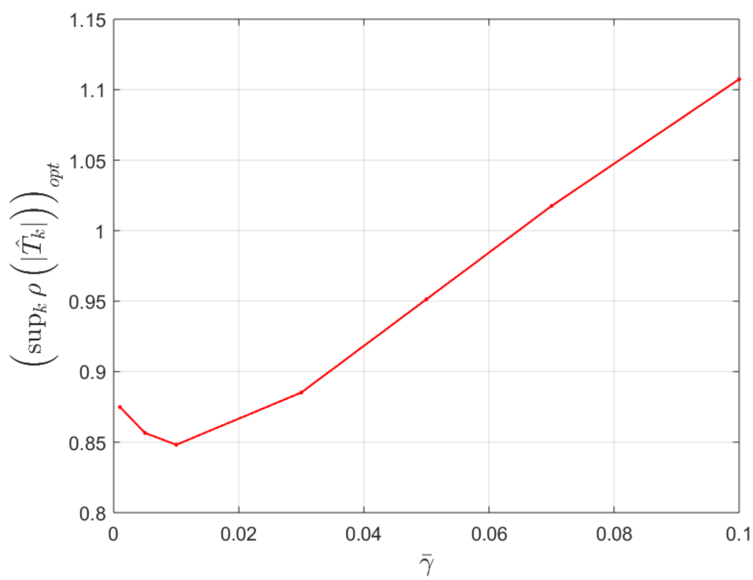

Figure 2.29: Optimal convergence factor vs. $\bar{\gamma}$ for $p = 10$, $\bar{\omega} = 50$ and $\bar{L} = 1$

Table 2.2: Values of $\bar{\alpha}_{\text{opt}}$ for different values of $\bar{\gamma}$, when $p = 10$, $\bar{\omega} = 50$ and $\bar{L} = 1$.

$\bar{\gamma}$	$\bar{\alpha}_{\text{opt}}$
0.001	1.1556 - 4.9592i
0.005	0.5222 - 4.9592i
0.01	0.3111 - 5.1224i
0.03	- 5.2041i
0.05	- 5.0408i
0.07	- 4.9592i
0.1	- 4.8776i

the corresponding values of $\bar{\alpha}_{\text{opt}}$. We can see that for $p = 100$ the convergence factor ρ goes slightly above one. Therefore, based on the previous comment, for $p = 100$ convergence cannot be guaranteed with our theory. However, in practice we might get convergence of the asynchronous iterations for this and even larger values of p.

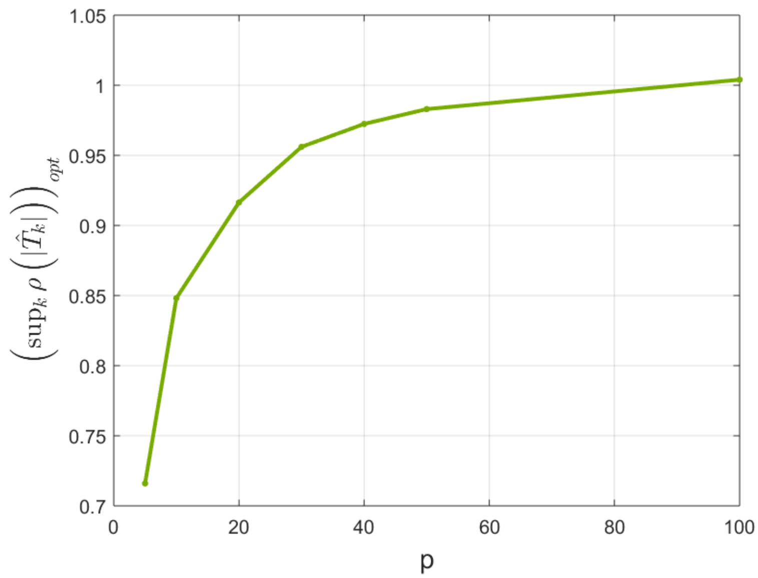

Figure 2.30: Optimal convergence factor vs. p for $\bar{\gamma} = 0.01$, $\bar{\omega} = 50$ and $\bar{L} = 1$

Table 2.3: Values of $\bar{\alpha}_{\text{opt}}$ for different values of p, when $\bar{\gamma} = 0.01$, $\bar{\omega} = 50$ and $\bar{L} = 1$

$\bar{\gamma}$	$\bar{\alpha}_{\text{opt}}$
5	1.1556 - 5.2857i
10	0.3111 - 5.1224i
20	0.0147 - 5.0408i
30	0.0905 - 5.1224i
40	0.0000 - 4.9592i
50	0.0716 - 5.0408i
100	0.1000 - 5.1224i

2.8 Conclusion

In this chapter we have studied the convergence of the asynchronous version of the Optimized Schwarz method when it is applied to solve a PDE containing the shifted Laplacian operator in a bounded rectangular domain with Dirichlet physical boundary conditions. We have considered artificial boundary conditions of the $OO2$ family, which contains two parameters α and β. We obtained a fixed point iteration mapping on the error coefficients. We showed that the error of the asynchronous implementation goes to zero uniformly provided that certain conditions hold, and then showed evidence that this conditions actually hold. We studied the dependence of the convergence factor on the parameters normalized parameters $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\omega}$, and $\bar{L}$, and obtained the optimal values for the parameters $\bar{\alpha}$ and $\bar{\beta}$. We obtained some empirical formulas for the optimal $\bar{\alpha}$ for the $OO0$ case. We compared the $OO0$ and $OO2$ cases and found that the $OO2$ case gives a better convergence rate factor, i.e., there is a substantial improvement in the convergence factor when the parameter $\bar{\beta}$ is tuned to its optimal value over the case of $\bar{\beta} = 0$. Finally, we studied how the convergence factor behaves for the positive shift case, i.e., the Helmholtz equations case, when we vary the values of the parameters $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\omega}$ and $\bar{L}$.

CHAPTER 3

POISSON'S EQUATION: DECOMPOSITION WITH CROSS-POINTS OF 2D DOMAIN

3.1 Preliminaries

In this chapter we analyze the convergence of the synchronous and asynchronous versions of an Optimized Schwarz method applied to solve Poisson's equations in a rectangular domain with Dirichlet physical boundary conditions and a domain decomposition with cross-points (i.e., points where the boundaries of more than two subdomains meet). We use boundary conditions of the $OO0$ family on the artificial boundaries, i.e, Robin boundary conditions containing a parameter α.

3.2 Formulation of the Problem

We want to solve the following problem,

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega = [0, L_1] \times [0, L_2]$.

We divide the physical domain into $p \times q$ overlapping rectangular subdomains. To simplify the presentation, we consider square subdomains where each side is of length H, i.e., all subomains have the same size. Also, we consider the same overlap on each side. Note, however that the analysis presented here is also valid for arbitrary rectangles and arbitrary overlaps. Each of these subdomains is represented by a pair of indexes, (s, r), with $s \in \{1, ..., p\}$ and $r \in \{1, ..., q\}$. We consider that the overlap on each side is of length 2γ, see Figure 3.1. There are nine types of subdomains, that can be put into three categories.

1. Interior subdomains, i.e., sudomains whose sides are all artificial boundaries.

2. Corner subdomains, which have two sides in common with the physical boundaries and the other sides are artificial boundaries. Thus, we have top-left, top-right, bottom-left and bottom-right corner subdomains.

3. Subdomains having one side in common with the physical boundaries and the other three sides being artificial boundaries. Thus we have top, bottom, left and right subdomains.

The subdomain (s, r) is denoted by $\Omega^{(s,r)}$ and given by

$$\Omega^{(s,r)} = [(s-1)(H-2\gamma), sH - 2(s-1)\gamma] \times [(r-1)(H-2\gamma), rH - 2(r-1)\gamma].$$

The local coordinates associated to the $\Omega^{(s,r)}$ are those in the coordinate system whose origin is on the bottom-left corner of the subdomain (s, r). Thus, the relation between the local coordinates, denoted by (x, y), and the global coordinates, denoted by (x_g, y_g), is given by the following formulas

$$x = x_g - (s-1)(H-2\gamma),$$

$$y = y_g - (r-1)(H-2\gamma).$$

The equations corresponding to an interior subdomain (i.e., for $1 < s < p$, $1 < r < q$) in the Optimized Schwarz (OS) iteration process associated with problem (3.1), with $OO0$ transmission conditions, using a domain decomposition with cross-points and local coordinates are given by

$$\begin{cases} -\Delta u_{n+1}^{(s,r)} = f & \text{in } (0,H)\times(0,H) \\ -\frac{\partial u_{n+1}^{(s,r)}}{\partial x} + \alpha u_{n+1}^{(s,r)} = -\frac{\partial u_n^{s-1,r}}{\partial x} + \alpha u_n^{s-1,r} & \text{for } x = 0 \\ \frac{\partial u_{n+1}^{(s,r)}}{\partial x} + \alpha u_{n+1}^{(s,r)} = \frac{\partial u_n^{s+1,r}}{\partial x} + \alpha u_n^{s+1,r} & \text{for } x = H \\ -\frac{\partial u_{n+1}^{(s,r)}}{\partial y} + \alpha u_{n+1}^{(s,r)} = -\frac{\partial u_n^{s,r-1}}{\partial y} + \alpha u_n^{s,r-1} & \text{for } y = 0 \\ \frac{\partial u_{n+1}^{(s,r)}}{\partial y} + \alpha u_{n+1}^{(s,r)} = \frac{\partial u_n^{s,r+1}}{\partial y} + \alpha u_n^{s,r+1} & \text{for } y = H. \end{cases} \tag{3.2}$$

The equations for the subdomains touching the boundaries are similar to (3.2) with the exception that one or two of the boundary conditions are Dirichlet, namely, those associated to the physical boundaries. The parameter α is the one which we want to optimize, so as to minimize the convergence rate.

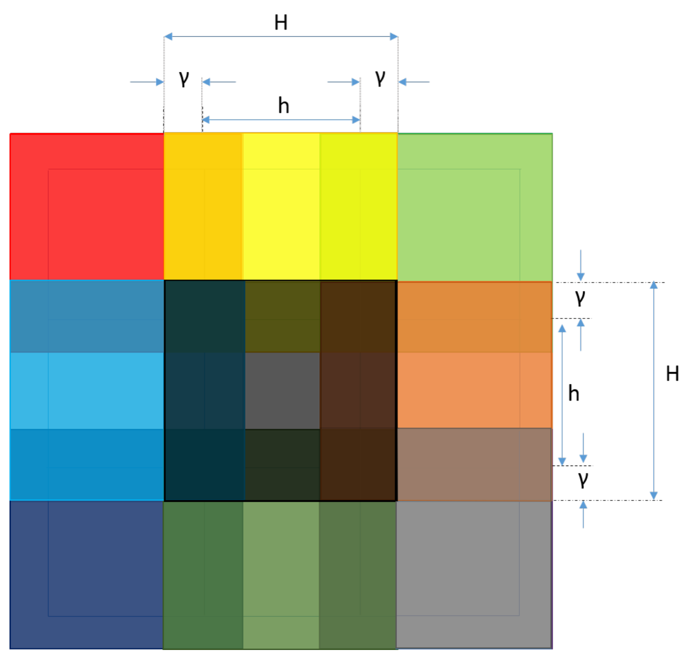

Figure 3.1: Partition of Domain

In order to study the convergence of the OS iteration described by (3.2) together with the equations associated to the subdomains touching the boundaries, and its asynchronous version, following the same approach as in Chapter 2, we obtain a series representation of the local errors and then recast the iteration into a fixed point iteration on vectors containing the error series coefficients.

3.3 Recasting equations as a fixed point iteration

To obtain the new fixed point iteration, we begin first by analyzing the local error of an interior subdomain. Let $\eta_n^{(s,r)}$ be the local error after n iterations corresponding to the subdomain (s,r). By linearity, we can see that the local error (of interior subdomains) of the iteration process is described by (3.2) with $f=0$. Furthermore, by superposition principle, we can write $\eta_n^{(s,r)} = \eta_{n,1}^{(s,r)} + \eta_{n,2}^{(s,r)} + \eta_{n,3}^{(s,r)} + \eta_{n,4}^{(s,r)}$, where $\eta_{n,i}^{(s,r)}$, $i = 1, ..., 4$, is the solution of (3.2) with $f = 0$ and with one nonhomogeneous boundary condition and the rest homogeneous (e.g., see equation (A.4) in Appendix A). We use the following convention: $i = 1$ corresponds to the case where the non-homgoneneous boundary condition is at the bottom, $i = 2$ to the case with non-homogeneous boundary condition on the right, $i = 3$ at the top, and $i = 4$ on the left. Thus, using separation of variables, superposition principle and Sturm-Liouville theory (see Appendix A), we can write each part of the local error $\eta_n^{(s,r)}$ as

$$\eta_{n,1}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} A_{n,m,1}^{(s,r)} \phi_m^{(1)}(x) \psi_m^{(1)}(H-y) \ , \tag{3.3}$$

$$\eta_{n,2}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} A_{n,m,2}^{(s,r)} \phi_m^{(1)}(y) \psi_m^{(1)}(x) \ , \tag{3.4}$$

$$\eta_{n,3}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} A_{n,m,3}^{(s,r)} \phi_m^{(1)}(x) \psi_m^{(1)}(y) \ , \tag{3.5}$$

$$\eta_{n,4}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} A_{n,m,4}^{(s,r)} \phi_m^{(1)}(y) \psi_m^{(1)}(H-x) \ , \tag{3.6}$$

where

$$\phi_m^{(1)}(x) = \frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)$$

and

$$\psi_m^{(1)}(x) = \frac{\bar{\alpha}}{z_m} \sinh\left(\frac{z_m x}{H}\right) + \cosh\left(\frac{z_m x}{H}\right),$$

with $z_m = z_m(\bar{\alpha})$ satisfying the transcendental equation

$$\tan(z) = \frac{2z\bar{\alpha}}{\bar{\alpha}^2 - z^2}\ , \tag{3.7}$$

and $\bar{\alpha} = \alpha H$ is the normalized Robin parameter.

Note that $\{\phi_m^{(1)}\}_{m\in\mathbb{N}}$ is a complete orthogonal set in $[0, H]$. Therefore, equations (3.3) and (3.5) can be seen as Generalized Fourier series in x and equations (3.4) and (3.6) as Generalized Fourier series in y.

Let $\{\tilde{z}_m\}_{m\in\mathbb{N}}$ be such that $\tilde{z}_1 < \tilde{z}_2 < \ldots$ and such that $\tilde{z}_m$ satisfies the transcendental equation

$$\tan(z) = -\frac{z}{\bar{\alpha}}.$$

Also, let

$$\psi_m^{(2)}(x) := \sinh\left(\frac{\tilde{z}_m x}{H}\right)$$

and

$$\phi_m^{(2)}(x) := \sinh\left(\frac{\tilde{z}_m x}{H}\right).$$

Note that, similarly to $\{\phi_m^{(1)}\}$, $\{\phi_m^{(2)}\}$ is a complete orthogonal set that spans the set of piecewise continuous functions. Then, the series expansion of the local errors of corner subdomains are

$$\eta^{(1,q)}(x,y) = \sum_{m=1}^{\infty} A_{n,m,1}^{(1,q)} \phi_m^{(2)}(x)\psi_m^{(2)}(y-H) + \sum_{m=1}^{\infty} A_{n,m,2}^{(1,q)} \phi_m^{(2)}(y-H)\psi_m^{(2)}(y-H),$$

$$\eta^{(p,q)}(x,y) = \sum_{m=1}^{\infty} A_{n,m,1}^{(p,q)} \phi_m^{(2)}(x-H)\psi_m^{(2)}(y-H) + \sum_{m=1}^{\infty} A_{n,m,4}^{(p,q)} \psi_m^{(2)}(x-H)\phi_m^{(2)}(y-H),$$

$$\eta^{(p,1)}(x,y) = \sum_{m=1}^{\infty} A_{n,m,3}^{(p,1)} \phi_m^{(2)}(x-H)\psi_m^{(2)}(y) + \sum_{m=1}^{\infty} A_{n,m,4}^{(p,1)} \psi_m^{(2)}(x-H)\phi_m^{(2)}(y),$$

$$\eta^{(1,1)}(x,y) = \sum_{m=1}^{\infty} A_{n,m,2}^{(1,1)} \psi_m^{(2)}(x)\phi_m^{(2)}(y) + \sum_{m=1}^{\infty} A_{n,m,3}^{(1,1)} \phi_m^{(2)}(x)\psi_m^{(2)}(y).$$

The series representation of the local errors of the subdomains touching the boundary that are not corners are

$$\begin{aligned} \eta^{(1,r)}(x,y) &= \sum_{m=1}^{\infty} A_{n,m,1}^{(1,r)} \phi_m^{(2)}(x)\psi_m^{(1)}(H-y) \\ &+ \sum_{m=1}^{\infty} A_{n,m,2}^{(1,r)} \phi_m^{(1)}(y)\psi_m^{(2)}(x) \\ &+ \sum_{m=1}^{\infty} A_{n,m,3}^{(1,r)} \phi_m^{(2)}(x)\psi_m^{(1)}(y), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \eta^{(p,r)}(x,y) &= \sum_{m=1}^{\infty} A_{n,m,1}^{(p,r)} \phi_m^{(2)}(x-H)\psi_m^{(1)}(H-y) \\ &+ \sum_{m=1}^{\infty} A_{n,m,3}^{(p,r)} \phi_m^{(2)}(x-H)\psi_m^{(1)}(y) \\ &+ \sum_{m=1}^{\infty} A_{n,m,4}^{(p,r)} \phi_m^{(1)}(y)\psi_m^{(2)}(x-H), \end{aligned} \tag{3.9}$$

$$\begin{aligned}\eta^{(s,1)}(x,y) &= \sum_{m=1}^{\infty} A_{n,m,1}^{(s,1)}\phi_m^{(2)}(y)\psi_m^{(1)}(H-x)\\ &+ \sum_{m=1}^{\infty} A_{n,m,2}^{(s,1)}\phi_m^{(1)}(x)\psi_m^{(2)}(y)\\ &+ \sum_{m=1}^{\infty} A_{n,m,3}^{(s,1)}\phi_m^{(2)}(y)\psi_m^{(1)}(x), \qquad (3.10)\end{aligned}$$

$$\begin{aligned}\eta^{(s,q)}(x,y) &= \sum_{m=1}^{\infty} A_{n,m,1}^{(s,q)}\phi_m^{(2)}(y-H)\psi_m^{(1)}(H-x)\\ &+ \sum_{m=1}^{\infty} A_{n,m,3}^{(s,q)}\phi_m^{(2)}(y-H)\psi_m^{(1)}(x)\\ &+ \sum_{m=1}^{\infty} A_{n,m,4}^{(s,q)}\phi_m^{(1)}(x)\psi_m^{(2)}(y-H). \qquad (3.11)\end{aligned}$$

For our convergence analysis, we want to show that the series in (3.3)-(3.6) and (3.8)-(3.11) converge uniformly to zero. To that end, we want to express the coefficients $A_{n,m,i}^{(s,r)}$ of the series as a quotient with a denominator having z_m^2, such as in the following theorem.

Lemma 3.1. *Consider the series expansion of* $\eta_{n,i}^{(s,r)}$ *(the* $i-th$ *part of the local error of an interior subdomain) from equations (3.3)-(3.6). Let* u_0 *be the initial approximation of the solution of (*3.1*) and such that the initial error* η_0 *is* $C^3((0,L_1)\times(0,L_2))$*. Let* $S_1 := \{(s,r) : 1 < s < p, 1 < r < q\}$*,* $S_2 := \{(\bar{s},\bar{r}) : 1 < \bar{s} < p, \bar{r} = 1\} \cup \{(\bar{s},\bar{r}) : 1 < \bar{s} < p, \bar{r} = 1\} \cup \{(\bar{s},\bar{r}) : 1 < \bar{r} < q, \bar{s} = 1\} \cup \{(\bar{s},\bar{r}) : 1 < \bar{r} < q, \bar{s} = p\}$*,* $S_3 := \{(\bar{s},\bar{r}) : \bar{s} \in 1,p, \bar{r} \in 1,q\}$*, i.e.,* S_1 *is the set of indexes for interior subdomains,* S_2 *is the set of indexes of the subdomains touching the boundaries which are not on the corners and* S_3 *are the set indexes of subdomains lying on the corners. Then, for* *all $n \in \mathbb{N}$, we have for $(s,r) \in S_1$ that*

$$A^{(s,r)}_{n,m,i} = \frac{B^{(s,r)}_{n,m,i}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}, \tag{3.12}$$

for $(s,r) \in S_2$ that

$$A^{(s,r)}_{n,m,i} = \frac{B^{(s,r)}_{n,m,i}}{z_m^2 \cosh(z_m)}, \tag{3.13}$$

and for $(s,r) \in S_3$ that

$$A^{(s,r)}_{n,m,i} = \frac{(\bar{\alpha} + 1/z_m) B^{(s,r)}_{n,m,i}}{z_m^2 \cosh(z_m)}, \tag{3.14}$$

where $B^{(s,r)}_{n,m,i} \leq \frac{M_{n,s,r}}{z_m^{1/2}}$ for all $m \in \mathbb{N}$ and some $M_{n,s,r} > 0$. Also, the series (3.3)-(3.6), (3.3)-(3.6) are uniformly convergent.

The proof of this lemma is given in Appendix B.

We want a fixed point iteration with an iteration operator mapping the vector of all the local error series coefficients at iteration n to the vector of coefficients at iteration $n+1$. Thus, the next step towards obtaining the expression of this operator, is to find the formulas that relate coefficients of the error series at iterations n and $n+1$. To that end, plugging (3.12) into (3.3)-(3.6), then plugging the resulting expressions into the nonhomogeneous boundary conditions in (3.2), multiplying both sides of the resulting equation by ϕ_k, integrating over $y \in [0, H]$ and noticing the orthogonality property of the set $\left\{\phi^{(1)}_m\right\}_{m\in\mathbb{N}}$, we obtain the expression of the error series coefficients at iteration $(n+1)$ in terms of those at iteration n. For interior subdomains that are not adjacent to subdomains touching the boundary, we have for example the following expression for $B^{(s,r)}_{n+1,k,1}$.

$$
\begin{aligned}
B^{(s,r)}_{n+1,k,1} &= \frac{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(2\bar{\gamma} z_k\right) + 2\bar{\alpha}\cosh\left(2\bar{\gamma} z_k\right)}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(z_k\right) + 2\bar{\alpha}\cosh\left(z_k\right)} B^{(s,r-1)}_{n,k,1} \qquad (3.15)\\
&+ \sum_{m=1}^{\infty}\left\{ \frac{4z_k^5\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin\left((1-2\bar{\gamma})z_m\right)}{\left[\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k)+2\bar{\alpha}\right] z_m^2\left(z_m z_k^3 + z_k z_m^3\right)} \right. \\
&\quad \left. \frac{\left\{\tanh(z_m)\left[\bar{\alpha}(z_k^2+z_m^2)\sin(z_k) - z_k(\bar{\alpha}^2 - z_m^2)\cos(z_k)\right] + z_m(\bar{\alpha}^2+z_k^2)\sin(z_k)\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2) - \bar{\alpha}^2\sin(2z_k) + 2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha}) - 2\bar{\alpha}z_k\cos(2z_k)\right]} B^{(s,r-1)}_{n,m,2} \right\} \\
&+ \frac{\left(-z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left((1-2\bar{\gamma})z_k\right)}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(z_k\right) + 2\bar{\alpha}\cosh\left(z_k\right)} B^{(s,r-1)}_{n,k,3} \\
&+ \sum_{m=1}^{\infty}\left\{ \frac{4z_k^5\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin\left((1-2\bar{\gamma})z_m\right)}{\left[\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k)+2\bar{\alpha}\right] z_m^2\left(z_m z_k^3 + z_k z_m^3\right)} \right. \\
&\quad \left. \frac{\left\{\tanh(z_m) z_k(\bar{\alpha}^2 + z_m^2) - z_m\left[-2\bar{\alpha}z_k + \frac{(\bar{\alpha}^2 - z_k^2)\sin(z_k) + 2\bar{\alpha}z_k\cos(z_k)}{\cosh(z_m)}\right]\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2) - \bar{\alpha}^2\sin(2z_k) + 2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha}) - 2\bar{\alpha}z_k\cos(2z_k)\right]} B^{(s,r-1)}_{n,m,4} \right\}.
\end{aligned}
$$

Similar expressions follows for $B^{(s,r)}_{n,m,i}$ with $i = 2, 3, 4$. The coefficients for the other types of subdomains can be obtained with the same procedure.

Note from (3.15) that, as a difference from the formulas for the coefficients of the local error series for the case in Chapter 2, in this case the coefficients of frequency k at iteration $n+1$ depend not only on the coefficients of the error series at iteration n of frequency k, but also on the coefficients of other frequencies. This means, in other words, that error modes of different frequencies are coupled.

Let B_n be the infinite vector containing all the error series coefficients at iteration n, i.e., $B_n = (b_{n_1}, b_{n_2}, ...)$ with $b_{n_j} \in \Big\{B^{(s,r)}_{n,k,i} : s \in \{1, ..., p\},\ r \in \{1, ..., q\},\ k \in \mathbb{N},$ $i \in \{1, ..., 4\}\Big\}$. Then, the relation between coefficients of the local errors can be written as $B_{n+1} = \hat{T} B_n$, where $\hat{T} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is an infinite matrix. The fact that $\hat{T}$ is an infinite matrix is a consequence of the coupling between modes of different frequencies. Note that $\hat{T} = (\hat{T}^{(1,1)}, \dots, \hat{T}^{(p,q)})$, where $\hat{T}^{(s,r)}$ is a local operator such that

$B_{n+1}^{(s,r)} = \hat{T}^{(s,r)} B_n$ with $B_{n+1}^{(s,r)}$ being a vector containing all the error coefficients of the local problem (s,r) at iteration $(n+1)$.

This new fixed point iteration means, in other words, that given the expression of the local errors $\eta_n^{(s,r)}$ in terms of the series (3.3)-(3.6),(3.8)-(3.11) but written with the coefficients $B_{n,m,i}^{(s,r)}$ as in (3.12)-(3.14), solve the local problems to obtain the new coefficients $B_{n+1,m,i}^{(s,r)}$ which determine $\eta_{n+1}^{(s,r)}$.

3.4 Convergence of Synchronous Optimized Schwarz

In this section we discuss the convergence of a synchronous implementation of the optimized Schwarz method (3.2). We have the following result. We do not call this a theorem since we use some experimental results as arguments in some parts of the proof.

Result 3.2. For any positive value of the relative overlap $\bar{\gamma}$ there exist a computable range of values of $\bar{\alpha}$ for which the OSM iteration given by (3.2) converges.

Proof. Let $(B_n)_{|k \le k_{\max}}$ denote the vector resulting after discarding all the entries of B_n corresponding to $k > k_{\max}$.

Then,we can write

$$(B_{n+1})_{|k \le k_{\max}} = \left(\hat{T}\left(B_n\right)\right)_{|k \le k_{\max}} = \hat{T}_{k_{\max}}\left((B_n)_{|k \le k_{\max}}\right) + \xi_{n+1,k_{\max}}((B_n)_{|k > k_{\max}}), \quad (3.16)$$

where $\hat{T}_{k_{\max}}$ is a finite matrix obtained by discarding the rows and columns of $\hat{T}$ related to the coefficients pertaining to $k > k_{\max}$, and $\xi_{n+1,k_{\max}}((B_n)_{|k > k_{\max}})$ is the error vector obtained by approximating $(B_{n+1})_{|k \le k_{\max}}$ by $\hat{T}_{k_{\max}}\left((B_n)_{|k \le k_{\max}}\right)$.

Note that each entry of $\xi_{n+1,k_{\max}}((B_n)_{|k > k_{\max}})$ is the truncation error that results after truncating the series in the formulas of the coefficients $B_{n+1,k,i}^{(s,r)}$, $k \le k_{\max}$, by keeping only the terms corresponding to $k \le k_{\max}$. Thus, as it can be seen in (3.15), $\xi_{n+1,k_{\max}}((B_n)_{|k > k_{\max}})$ is just a linear combination of the entries of $(B_n)_{|k > k_{\max}}$. Note also that the entries of $(B_n)_{|k>k_{\max}}$ are linear combinations of the entries of B_{n-1}.

Using equation (3.16) recursively, we obtain the following equation

$$(B_{n+1})_{|k\leq k_{\max}} = \hat{T}^{n+1}((B_0)_{|k\leq k_{\max}}) + \sum_{j=1}^{n+1} \hat{T}^{n+1-j}(\xi_{j,k_{\max}}((B_{j-1})_{|k>k_{\max}})). \tag{3.17}$$

We discuss in Section 3.7 conditions for the spectral radius $\rho(\hat{T}_{k_{\max}}) < 1$. Computing the spectral radius of $\hat{T}_{k_{\max}}$ for any (normalized) overlap $\bar{\gamma} > 0$, we can observe that $\rho(\hat{T}_{k_{\max}})$ tends to a constant less than one for large enough values of the normalized parameter $\bar{\alpha}$; see Figure 3.3. In other words, for $\bar{\gamma} > 0$ and $\bar{\alpha}$ large enough, we have

$$\lim_{k_{\max}\to\infty} \rho(\hat{T}_{k_{\max}}) = \rho_\infty < 1, \tag{3.18}$$

where $\rho_\infty = \rho_\infty(\bar{\alpha}, \bar{\gamma})$.

Let $\epsilon > 0$. From (3.15) and Result C.4 we have that the elements $B^{(s,r)}_{n+1,k,i}$ of $|B_{n+1}| = |TB_n|$ are bounded as $B^{(s,r)}_{n+1,k,i} \leq C/z_k^{1/2}$ for some $C > 0$. Thus, we can always find a large enough $k_{\max}$ such that the entries of $|B_{n+1}| = |TB_n|$ corresponding to $k > k_{\max}$ are as small as desired. From this fact and since (3.18) hold, we have for large enough n, say for $n > n_\epsilon$, that there exists a $k_{\max}(n_\epsilon)$ such that for all $k_{\max} > k_{\max}(n_\epsilon)$ we have

$$\hat{T}^{n+1}((B_0)_{|k\leq k_{\max}}) \leq \epsilon ||B_0||_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{3.19}$$

Then, from this fact and since

$$\lim_{k_{\max}\to\infty} \sum_{j=1}^{n+1} \hat{T}^{n+1-j}(\xi_{j,k_{\max}}((B_{j-1})_{|k>k_{\max}})) = 0, \tag{3.20}$$

it follows that for $n > n_\epsilon$ we have

$$|B_{n+1}| \leq \epsilon ||B_0||_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{3.21}$$

But $\epsilon > 0$ was arbitrary. Consequently, for any $\epsilon > 0$ there exists an n_ϵ such that (3.21) holds. Therefore, $|B_n| \to 0$. More over $B^{(s,r)}_{n,k,i}$ converges to zero uniformly in $s \in \{1, \ldots, p\}$, $r \in \{1, \ldots, q\}$, $k \in \mathbb{N}$ and $i \in \{1, 2, 3, 4\}$.

Using the fact that $B^{(s,r)}_{n,k,i}$ converges uniformly to zero and using the result from Lemma 3.1, we have that $\eta^{(s,r)}_{n,i}$ converges uniformly to zero in $[0, H]^2$. Consequently $\lim_{n\to\infty} \eta_n = 0$, i.e., the error goes to zero as n goes to infinity. Therefore, the synchronous implementation of OS given by (3.2) converges. □

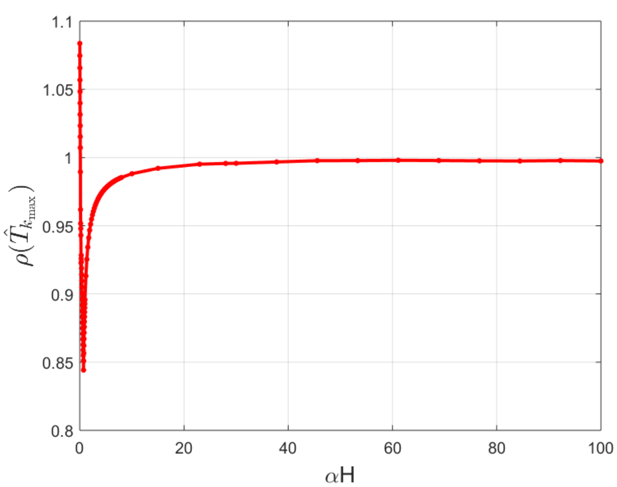

Figure 3.2: Spectral radius of $\hat{T}_{k_{\max}}$ for $p, q = 10$, $k_{\max} = 20$, normalized overlap $\bar{\gamma} = 0.01$ and normalized boundary parameter $\bar{\alpha} \in [0.01, 100]$

3.5 Asynchronous Optimized Schwarz methods

In this section we study the asynchronous optimized Schwarz iterations. We start by defining the equations for the local problems. We use the same model for the

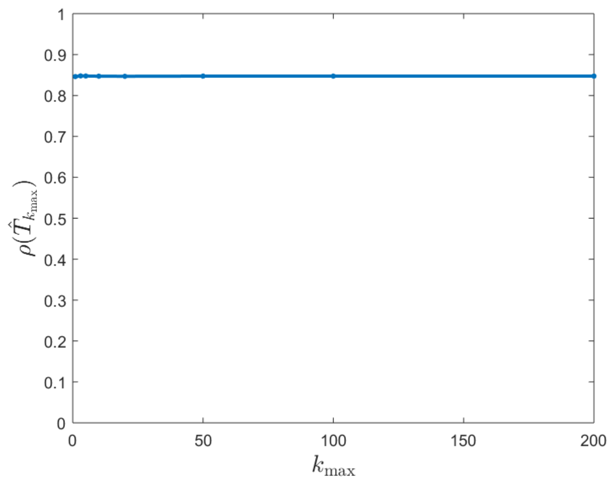

Figure 3.3: Spectral radius of $\hat{T}_{k_{\max}}$ vs. $k_{\max}$ for $p, q = 10$, normalized overlap $\bar{\gamma} = 0.01$ and normalized boundary parameter $\bar{\alpha} = 0.72$

asynchronous iterations as the one introduced in Section 2.5. Thus, let $l_1 = \tau^{(s,r)}_{s-1,r}(n)$, $l_2 = \tau^{(s,r)}_{s+1,r}(n)$, $l_3 = \tau^{(s,r)}_{s,r-1}(n)$ and $l_4 = \tau^{(s,r)}_{s,r+1}(n)$, i.e., the time-stamp indexes of the updates of the data coming from the neighboring processors and available in processor (s,r) at the beginning of the computation which will end at the $n-th$ time stamp. Let

$$\begin{cases} -\Delta u^{(s,r)}_{t_{n+1}} = f^{(s,r)} & \text{in } \Omega^{(s,r)} \\ -\frac{\partial u^{(s,r)}_{n+1}}{\partial x} + \alpha u^{(s,r)}_{t_{n+1}} = -\frac{\partial u^{s-1,r}_{t_{l_1}}}{\partial x} + \alpha u^{s-1,r}_{t_{l_1}} & \text{for } x = (s-1)h - \gamma \\ \frac{\partial u^{(s,r)}_{t_{n+1}}}{\partial x} + \alpha u^{(s,r)}_{t_{n+1}} = \frac{\partial u^{s+1,r}_{t_{l_2}}}{\partial x} + \alpha u^{s+1,r}_{t_{l_2}} & \text{for } x = sh + \gamma \\ -\frac{\partial u^{(s,r)}_{t_{n+1}}}{\partial y} + \alpha u^{(s,r)}_{t_{n+1}} = -\frac{\partial u^{s,r-1}_{t_{l_3}}}{\partial y} + \alpha u^{s,r-1}_{t_{l_3}} & \text{for } y = (r-1)h - \gamma \\ \frac{\partial u^{(s,r)}_{t_{n+1}}}{\partial y} + \alpha u^{(s,r)}_{t_{n+1}} = \frac{\partial u^{s,r+1}_{t_{l_4}}}{\partial y} + \alpha u^{s,r+1}_{t_{l_4}} & \text{for } x = rh + \gamma. \end{cases} \tag{3.22}$$

Then, the local approximation of the solution at the time stamp t_n corresponding to the (s,r) interior subdomain is

$$u^{(s,r)}_{t_n} = \begin{cases} \text{solution of (3.22)}, & \text{if } (s,r) \in \sigma(n) \\ u^{(s,r)}_{t_{n-1}}, & \text{if } (s,r) \notin \sigma(n) \end{cases}. \tag{3.23}$$

In our case, the processors are numbered with a pair of indices corresponding to the numbering of the subdomains, say (s, r). The local equations for the other type of subdomains are the same as (3.22)and (3.23) with the exception that the boundary conditions corresponding to the boundaries that coincide with the physical boundaries are Dirichlet.

Note that following the same process as in the synchronous case we can obtain local operators that relate the error coefficients at different time stamps. These local operators are the same as in the synchronous case. In the asynchronous case, the local operations are performed without synchronization, therefore the expression of the global operator is more complex than in the synchronous case. However, as it is shown in the next section, we can study the convergence of the asynchronous method by studying the spectral properties of the operator $|\hat{T}|$, where $\hat{T}$ is the global operator of the synchronous case, and the absolute value is understood componentwise.

3.6 Convergence proof of Asynchronous OS

As we shall see in Section 3.7, at least for certain subdomain configurations there are values of the normalized boundary parameter $\bar{\alpha} = \alpha H$ and the normalized overlap $\bar{\gamma} = \gamma/H$ for which $\rho(|\hat{T}_{k_{\max}}|) < 1$, where H is the side length of the subdomains. Also, for these configurations the value of $\rho(|\hat{T}_{k_{\max}}|) < 1$ remains practically constant for large enough $k_{\max}$, which implies that if $\rho(|\hat{T}_{k_{\max}}|) < 1$ for large enough $k_{\max}$, then there exists $\delta > 0$ such that $\rho(|\hat{T}_{k_{\max}}|) + \delta < 1$. We use these results to prove the convergence of Asynchronous OS (AOS) for the given Poisson's problem. We divide the convergence proof into two parts. In the first part we show that, for an arbitrary sequence of time stamps $\{t_j\}_{j \in \mathbb{N}}$, $B_{t_j} \to 0$ as $j \to \infty$ for values of $\bar{\alpha}$ and $\bar{\gamma}$ such that $\rho(|\hat{T}_{k_{\max}}|) < 1$. In the second part we show that the series expansions of the local errors converge to zero as the number of time stamps go to infinity.

The standard results one uses for the analysis of asynchronous iterations are those in [2, 3, 16] (see also [8]), where the the iteration operator is finite dimensional. In [6] the convergence conditions for the asynchronous iterations is extended to infinite dimensional operators. Our convergence proof of the asynchronous method, although inspired in part by [3] and aligned with the ideas of [6], is new.

Theorem 3.3. *Let $k_{\max} > 0$ and let $\hat{T}_{k_{\max}}$ be the operator (matrix of finite dimension) obtained by dropping the entries of $\hat{T}$, defined by (3.15), corresponding to $k > k_{\max}$. Let us then assume that there exists a $k_{\max}$ such that the absolute row sum of the rows corresponding to $k > k_{\max}$ are less than some $0 < \zeta < 1$ and for the rows corresponding to $k \leq k_{\max}$ the sum of the absolute values of the entries lying in the columns discarded are less than some $0 < \delta < 1$. Then, if $\rho(|\hat{T}_{k_{\max}}|) + \delta < 1$, the asynchronous implementation of the optimized Schwarz iteration (3.23) converges.*

We prove this theorem in two parts.

Part 1 of the Proof of Theorem 3.3. We want to show that, for an arbitrary sequence of time stamps, the infinite vector of the error series coefficients B_{t_j} goes to zero as $j \to \infty$. We begin by quoting a result in [3].

Lemma 3.4. *If A is a nonnegative matrix with spectral radius $\rho(A) < 1$, then for any $\epsilon > 0$ there exist a positive vector v_ϵ such that $Av_\epsilon \leq (\rho + \epsilon)v_\epsilon$.*

We have, by hypothesis, that $\rho(|\hat{T}_{k_{\max}}|) < 1$. Given that $|\hat{T}_{k_{\max}}|$ is a nonnegative matrix, then, for an arbitrarily small $\epsilon > 0$ there exist a positive vector v such that $|\hat{T}_{k_{\max}}|v \leq (\rho(|\hat{T}_{k_{\max}}|) + \epsilon)v$. By hypothesis, $\rho(|\hat{T}_{k_{\max}}|) + \delta < 1$. Then, since we can choose $\epsilon > 0$ as small as we want, we choose ϵ small enough so that $\rho(|\hat{T}_{k_{\max}}|) + \epsilon + \delta < 1$. Let B_{t_0} be the initial vector containing the coefficients of all local errors parts $\eta_{n,i}^{(s,r)}$, with $i \in \{1, \ldots, 4\}$, $s \in \{1, \ldots, p\}$ and $r \in \{1, \ldots, q\}$. Let $w \in \mathbb{R}^\infty$ be an infinite vector of the same size as B_{t_0} such that

$$(w)_{|k \leq k_{\max}} = \frac{||B_{t_0}||_\infty}{v_{\min}} v,$$

where $v_{\min}$ is the smallest entry of v, and

$$(w)_{|k>k_{\max}} = ||B_{t_0}||_\infty \frac{||v||_\infty}{v_{\min}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

Note that $|B_{t_0}| \leq w$ (this inequality is component wise). Let us denote by $v^{(s,r)}$ and $\xi^{(s,r)}_{t_j,k_{\max}}$ the restriction of v and $\xi_{t_j,k_{\max}}$ to the subdomain s, r, respectively, where $\xi^{(s,r)}_{t_j,k_{\max}}$ is defined by the following equation (analogous to equation (3.16))

$$(B^{(s,r)}_{t_{n+1}})_{|k\leq k_{\max}} = \left(\hat{T}^{(s,r)}_{k_{\max}}(B_*)\right)_{|k\leq k_{\max}} = \hat{T}^{(s,r)}_{k_{\max}}\left((B_*)_{|k\leq k_{\max}}\right) + \xi^{(s,r)}_{t_{n+1},k_{\max}}((B_*)_{|k>k_{\max}}). \tag{3.24}$$

where B_* is the coefficients vector whose values are the ones available at processor (s, r) when the computation of $B^{(s,r)}_{t_{n+1}}$ started.

Note that

$$\begin{aligned} \hat{T}^{(s,r)}_{k_{\max}}\left((B_{t_0})_{|k\leq k_{\max}}\right) &\leq |\hat{T}^{(s,r)}_{k_{\max}}|(|B_{t_0}|)_{|k\leq k_{\max}} \\ &\leq |\hat{T}^{(s,r)}_{k_{\max}}|\frac{||B_{t_0}||_\infty}{v_{\min}}v \leq (\rho(|\hat{T}_{k_{\max}}|) + \epsilon)\frac{||B_{t_0}||_\infty}{v_{\min}}v^{(s,r)}. \end{aligned} \tag{3.25}$$

Let t_j be the time stamp at which the processor (s, r) produces its first new update. Then, by (3.24) and (3.25) we have

$$\begin{aligned} (B^{(s,r)}_{t_j})_{|k\leq k_{\max}} &= \hat{T}^{(s,r)}_{k_{\max}}\left((B_{t_0})_{|k\leq k_{\max}}\right) + \xi^{(s,r)}_{t_j,k_{\max}}((B_{t_0})_{|k>k_{\max}}) \\ &\leq (\rho(|\hat{T}_{k_{\max}}|) + \epsilon)\frac{||B_{t_0}||_\infty}{v_{\min}}v^{(s,r)} + \xi^{(s,r)}_{t_j,k_{\max}}((B_{t_0})_{|k>k_{\max}}). \end{aligned}$$

Note that

$$\xi^{(s,r)}_{t_j,k_{\max}}((B_{t_0})_{|k>k_{\max}}) \leq \delta||(B_{t_0})_{|k>k_{\max}}||_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \leq \delta||(B_{t_0})_{|k>k_{\max}}||_\infty \frac{1}{v_{\min}} v^{(s,r)}.$$

Hence, the error series coefficients corresponding to $k \leq k_{\max}$ of the error at subdomain (s,r) are bounded as follows

$$(B^{(s,r)}_{t_j})_{|k\leq k_{\max}} \leq (\rho(|\hat{T}_{k_{\max}}|) + \epsilon + \delta) \frac{||B_{t_0}||_\infty}{v_{\min}} v^{(s,r)}. \tag{3.26}$$

As for the coefficients corresponding to $k > k_{\max}$, we have the following bound

$$(B^{(s,r)}_{t_j})_{|k>k_{\max}} \leq \zeta ||B_{t_0}||_\infty \frac{||v||_\infty}{v_{\min}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{3.27}$$

Let

$$\tilde{\rho} = \max\{(\rho(|\hat{T}_{k_{\max}}|) + \epsilon + \delta), \zeta\}. \tag{3.28}$$

Thus, from (3.26), (3.27) and (3.28) we obtain

$$B^{(s,r)}_{t_j} \leq \tilde{\rho} w. \tag{3.29}$$

Since $\tilde{\rho} < 1$, we have that $B^{(s,r)}_{t_j} \leq \tilde{\rho} w^{(s,r)} < w^{(s,r)}$. Thus, for this processor it holds that $B^{(s,r)}_{t_m} \leq (\tilde{\rho}) w^{s,r} \leq w^{s,r}$, for all $t_m \geq t_j$. Then, after every processor has produced its first update, say at t_{j_1} we have that (3.29) holds for all s,r, and consequently $B_{t_j} \leq \tilde{\rho} w$ for all $t_j \geq t_{j_1}$. By a similar reasoning, we can see that once every processor has produced a new update after t_{j_1}, say at t_{j_2} ,we have $B_{t_j} \leq \tilde{\rho}^2 w$ for all $t_j \geq t_{j_2}$. Thus, we have

$$B_{t_j} \leq \tilde{\rho}^{\Phi(j)} w, \tag{3.30}$$

where, denoting by t_{j_i} the first time stamp at which all processors have updated their values at least i times, we have $\Phi(j) = \max\{i \in \mathbb{N} : t_{j_i} \le t_j\}$, i.e., $\Phi(j)$ is the update number (at time t_j) of the processor that produced the least number of updates among all processors until the instant of time t_j. Conditions 2 and 3 of the asynchronous model imply that new updates will always be produced and used by the processors (see Section 2.5), thus we have that $\Phi(j) \to \infty$ as $j \to \infty$. Consequently, from (3.30) we have that $B_{t_j} \to 0$ as $j \to \infty$. This concludes the first part of the proof.

The weighted max norm of an operator A with a positive weight vector w is defined as

$$||A||_w = \sup_{x \neq 0} \frac{||Ax||_w}{||x||_w},$$

where

$$||x||_w = \sup_i \left| \frac{x_i}{w_i} \right|.$$

Hence, note that we just proved that the infinite dimensional operator $\hat{T}$ is a contraction in the weighted max norm corresponding to the (infinite) weight vector $w > 0$. In fact, we have

$$|\hat{T}|w \le \tilde{\rho}w.$$

Then, using the following lemma, whose proof is a simple extension to the infinite-dimensional case of that for [7, Lemma 2.1], we have that

$$||\hat{T}||_w \le \tilde{\rho} < 1.$$

Lemma 3.5. *Let $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be an infinite matrix. Let w be a positive infinite vector, and $\theta > 0$ such that*

$$|T|w \le \theta w.$$

Then, $\|T\|_w \le \theta$. In particular, $\|Tx\|_w \le \theta \|x\|_w$ for all infinite vectors x.

In other words, we have shown the following.

Theorem 3.6. *Let* $k_{\max} > 0$ *and let* $\hat{T}_{k_{\max}}$ *be the operator (matrix of finite dimension) obtained by dropping the entries of* $\hat{T}$*, defined by (3.15), corresponding to* $k > k_{\max}$*. Let us then assume that there exists a* $k_{\max}$ *such that the absolute row sum of the rows corresponding to* $k > k_{\max}$ *are less than some* $0 < \zeta < 1$ *and for the rows corresponding to* $k \leq k_{\max}$ *the sum of the absolute values of the entries lying in the columns discarded are less than some* $0 < \delta < 1$*. Then, if* $\rho(|\hat{T}_{k_{\max}}|) + \delta < 1$*, we have that* $\hat{T}$ *(infinite dimensional operator) is a contraction in a weighted max norm.*

Part 2 of the Proof of Theorem 3.3. The local error, for an interior subdomain, is given by

$$
\begin{aligned}
\eta_{t_n}^{(s,r)}(x,y) &= \sum_{i=1}^{4} \eta_{t_n,i}^{(s,r)}(x,y) \\
&= \sum_{i=1}^{4}\sum_{m=1}^{\infty} \left\{ A_{t_n,m,i}^{(s,r)} \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)\right] \times \right. \\
&\quad \left. \left[\frac{-\bar{\alpha}}{z_m} \sinh\left(\frac{z_m(y-H)}{H}\right) + \cosh\left(\frac{z_m(y-H)}{H}\right)\right]\right\} \\
&= \sum_{i=1}^{4}\sum_{m=1}^{\infty} \left\{ \frac{B_{t_n,m,i}^{(s,r)}}{z_m^2\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)\right] \times \right. \\
&\quad \left. \left[\frac{-\bar{\alpha}}{z_m} \sinh\left(\frac{z_m(y-H)}{H}\right) + \cosh\left(\frac{z_m(y-H)}{H}\right)\right]\right\}.
\end{aligned}
\tag{3.31}
$$

So far, we have shown that each of the coefficients $B_{t_n,m,i}^{(s,r)}$ goes to zero as $n \to \infty$. This implies that each term of the infinite sum in (3.31) go to zero. But this fact alone does not guarantee that the infinite sum will go to zero as $n \to \infty$. In order to insure that this series goes to zero (and consequently $\eta_{t_n}^{(s,r)}$ goes to zero), it suffices to show that it converges uniformly in $(x,y) \in [0,H] \times [0,H]$, since this implies that the order of the limit and infinite sum operations can be interchanged, and thus

$$
\begin{aligned}
&\lim_{n\to\infty} \eta_{t_n}^{(s,r)}(x,y) = \\
&\lim_{n\to\infty} \sum_{m=1}^{\infty} \left\{ \frac{B_{t_n,m,i}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)\right] \times \right. \\
&\left. \left[\frac{-\bar{\alpha}}{z_m}\sinh\left(\frac{z_m(y-H)}{H}\right) + \cosh\left(\frac{z_m(y-H)}{H}\right)\right]\right\} = \\
&\sum_{m=1}^{\infty} \lim_{n\to\infty} \left\{ \frac{B_{t_n,m,i}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)\right] \times \right. \\
&\left. \left[\frac{-\bar{\alpha}}{z_m}\sinh\left(\frac{z_m(y-H)}{H}\right) + \cosh\left(\frac{z_m(y-H)}{H}\right)\right]\right\} = 0.
\end{aligned}
$$

Hence, in order to complete the convergence proof, we just need to show that the series

$$
\begin{aligned}
\eta_{n,i}^{(s,r)}(x,y) \;=\; & \sum_{m=1}^{\infty} \left\{ \frac{B_{t_n,m,i}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)\right] \times \right. \\
& \left. \left[\frac{-\bar{\alpha}}{z_m}\sinh\left(\frac{z_m(y-H)}{H}\right) + \cosh\left(\frac{z_m(y-H)}{H}\right)\right]\right\}
\end{aligned}
\tag{3.32}
$$

converges uniformly in $[0,H]^2$.

From (3.30) we have that $B_{t_j,m,i}^{(s,r)} \leq ||w||_\infty < 1$ for all $m \in \mathbb{N}$. Then, using this fact and the result from Lemma 3.1 we have that (3.32) converges uniformly in $[0,H]^2$.

This concludes part 2 of the proof of Theorem 3.3. □

3.7 Spectral Radius of $\hat{T}_{k_{\max}}$ and $|\hat{T}_{k_{\max}}|$

Recall that $\hat{T}_{k_{\max}}$ is a finite matrix obtained by discarding the rows and columns of $\hat{T}$ related to the coefficients pertaining to $k > k_{\max}$. The subdomains form a two-dimensional array, p is the number of subdomains per row and q the number of subdomains per column. The values of the entries of the matrix $\hat{T}_{k_{\max}}$ depend on $\bar{\gamma}$, $\bar{\alpha}$ (the normalized overlap and normalized $OO0$ parameter) and $k_{\max}$. The structure of the matrix depends on $k_{\max}$, p, q and the way we order the entries of B_n, i.e., the way we order each coefficient $B_{n,k,i}^{(s,r)}$ based on the values of s, r, k and i. However, the eigenvalues (and thus the spectral radius) do not depend on the ordering of the entries, since a change in the order is a just a similarity transformation obtained through permutation matrices. For the ordering we have chosen, we computed the spectral radius of the resulting matrix $\hat{T}_{k_{\max}}$, for $\bar{\gamma} \in \{0, 0.01, 1/30, 0.04, 0.1, 0.13, 0.18, 0.2, 0.25\}$, a set of values of $\bar{\alpha}$ in the range $[0.01, 500]$, $k_{\max} \in \{1, 2, 3, 5, 10, 20, 50, 100, 200\}$, and $p, q \in \{5, 10, 20, 30, 40\}$. In these computations we have observed the following.

1. There exist values of $\bar{\alpha}$ for which the spectral radius of $\hat{T}_{k_{\max}}$ is less than one.

2. For a given $\bar{\gamma}$ and the range of $\bar{\alpha}$ considered in the experiments, $\rho(\hat{T}_{k_{\max}})$ has a minimum and it approaches a constant less than one for large values of $\bar{\alpha}$.

3. Given $\bar{\gamma}$, $\bar{\alpha}$, p and q, the value of $\rho(\hat{T}_{k_{\max}})$ remains practically constant for large enough $k_{\max}$ (see Figure 3.3).

4. For a given $\bar{\gamma}$, the optimal spectral radius of $\hat{T}_{k_{\max}}$ increases as the number of subdomains $p \times q$ increases; see Figure 3.5.

5. The optimal spectral radius of $\hat{T}_{k_{\max}}$ decreases as $\bar{\gamma}$ increases up to a certain point, see Figure 3.4.

In Figure 3.2, the result for the case $\bar{\gamma} = 0.01$, with $p, q = 10$, $k_{\max} = 20$, $\bar{\alpha} \in [0.01, 100]$, is shown.

In Figure 3.6 a plot of the values of the spectral radius of $|\hat{T}_{k_{\max}}|$ for different values of $\bar{\alpha}$ is shown for the case $\bar{\gamma} = 0.025$, $p, q = 2$ and $k_{\delta} = 40$.

With $\bar{\alpha} = 2.55$, $\gamma = 0.025$ and $k_{\delta} = 40$, we have $\rho(\hat{T}_{k_{\max}}) \leq 0.6862$ and $\delta < 0.3072$. Thus, $(\rho(\hat{T}_{k_{\max}}) + \delta) < 0.9934$. The sum of the entries of the rows discarded (i.e., the

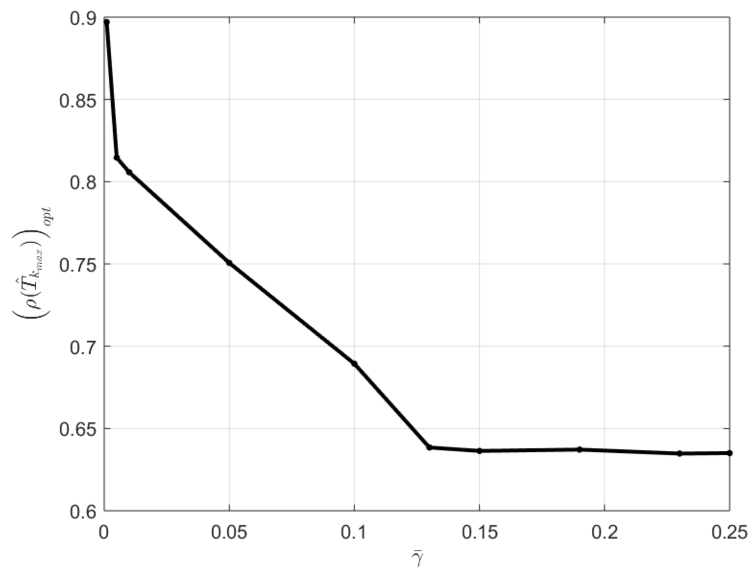

Figure 3.4: Optimal spectral radius of $\hat{T}_{k_{\max}}$ vs. the normalized overlap $\bar{\gamma} = \frac{\gamma}{H}$ for $p, q = 8$ and $k_{\max} = 10$

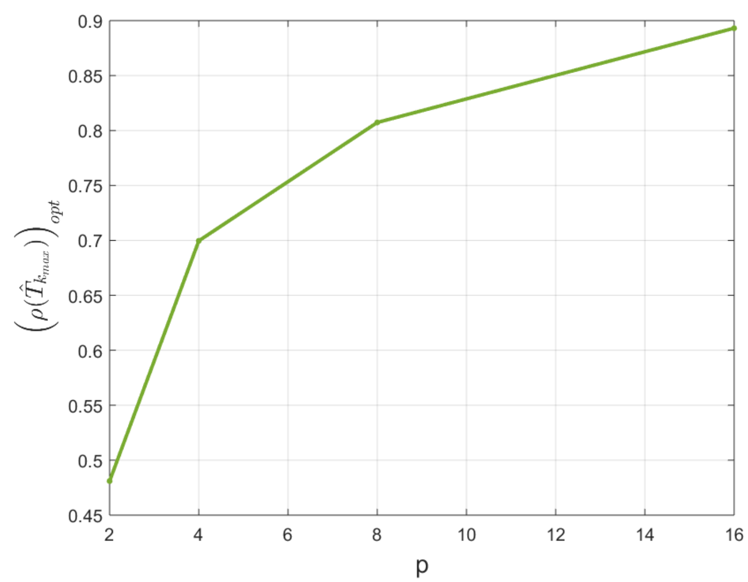

Figure 3.5: Optimal spectral radius of $\hat{T}_{k_{\max}} vs. p$ for $\bar{\gamma} = \frac{\gamma}{H} = 0.01$ and $k_{\max} = 7$, where the number of subdomains is $p \times q = p^2$ (i.e., $q = p$)

rows corresponding to $k > k_{\max}$) are less than 0.8. Consequently, we have that there exists a positive vector w whose elements are bounded and such that $|\hat{T}|w < 0.9934w$. Therefore, for the case $\bar{\alpha} = 2.55$, $\bar{\gamma} = 0.025$ and $p, q = 2$ the convergence of the asynchronous implementation of OS is guaranteed.

We remark that the discussion we just presented for $p = q = 2$ does not carry over to much larger values of p, q for the asynchronous case. This situation arises because $\rho(|\hat{T}|)$ is not a tight upper bound on the asymptotic convergence of the method. Nevertheless, as we see in the next section, convergence is achieved in the asynchronous case for higher values of p, q.

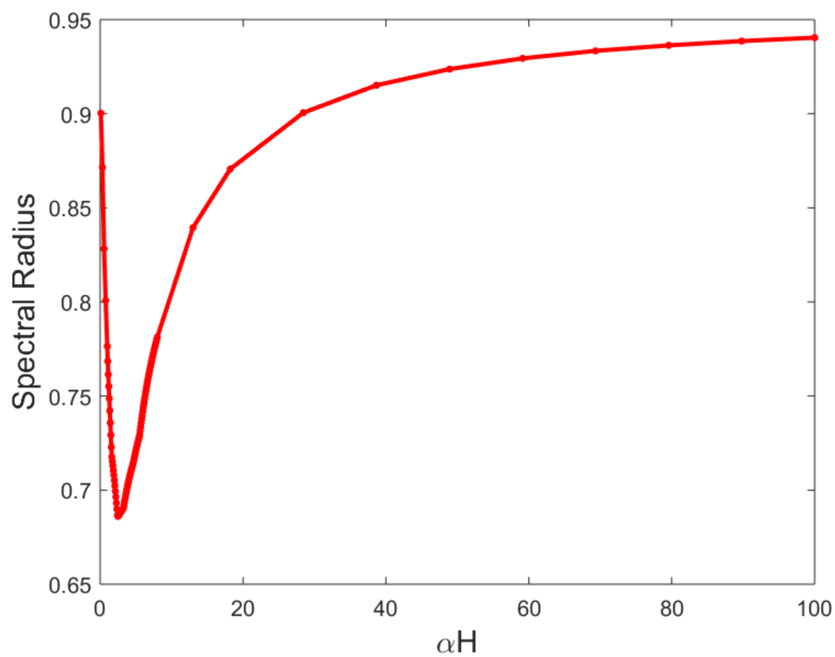

Figure 3.6: Spectral Radius of $|\hat{T}_{k_{\max}}|$ vs. normalized boundary parameter $\bar{\alpha}$ for the normalized overlap $\bar{\gamma} = 0.025$, $p, q = 2$ and $k_{\max} = 40$

3.7.1 Optimal α

For large enough $k_{\max}$, such that $\rho(\hat{T}_{k_{\max}}) < 1$, the spectral radius of $\hat{T}_{k_{\max}}$ describes the asymptotic convergence rate of the Optimized Schwarz method for the synchronous case. Thus, in the synchronous case we define the optimal $\bar{\alpha}$, for a given (normalized) overlap amount $\bar{\gamma}$, as the one which minimizes the spectral radius of $\hat{T}_{k_{\max}}$ and thus gives the optimal asymptotic convergence rate. Note that $\hat{T}_{k_{\max}}$ is a banded square matrix of dimension $N = 2k_{\max}(2pq - p - q)$. Let $\rho_\infty = \lim_{k_\delta \to \infty} \rho(\hat{T}_{k_{\max}})$. Usually, for $k_{\max} = 3$ we have that $\rho(\hat{T}_{k_{\max}})$ is a good estimation of $\rho_{\hat{T}_\infty}$. Thus, computing the spectral radius of $\hat{T}_{k_{\max}}$ is not an expensive operation. Consequently, finding the optimal $\bar{\alpha}$ is not an expensive operation in comparison with the cost of solving the discretized version of the given problem.

The spectral radius of $\hat{T}_{k_{\max}}$ is a function of the normalized Robin parameter $\bar{\alpha}$, the normalized overlap $\bar{\gamma}$, the number of subdomains in each direction p and q, and the truncation parameter $k_{\max}$. The optimal value of $\bar{\alpha}$ is a function of $\bar{\gamma}$, p, q. From the computation of the optimal $\bar{\alpha}$ (taking $k_{\max}$ large enough so that $\rho\left(\hat{T}_{k_{\max}}\right)$ remains essentially constant for larger values of $k_{\max}$) we observed that for small enough values of $\bar{\gamma}$, the values of $\bar{\alpha}_{opt}$ follow approximately a power law. Thus, for small $\bar{\gamma}$ we have that

$$\bar{\alpha}_{opt} \approx c(p,q)\bar{\gamma}^{\ell(p,q)}. \tag{3.33}$$

In Table 3.1, the values of the constant c and exponent ℓ are given for $p = q = 4, 5, 6$. In figures 3.7-3.9 the comparison between the computed values of $\bar{\alpha}_{opt}$ and their approximation obtained using (3.33) are presented for different amounts of overlap.

Table 3.1: Coefficients of empirical formula of $\bar{\alpha}_{opt}$ for different number of subdomains p, q

p, q	c	ℓ
4	0.5467	-0.2620
5	0.3722	-0.2875
6	0.2376	-0.3327

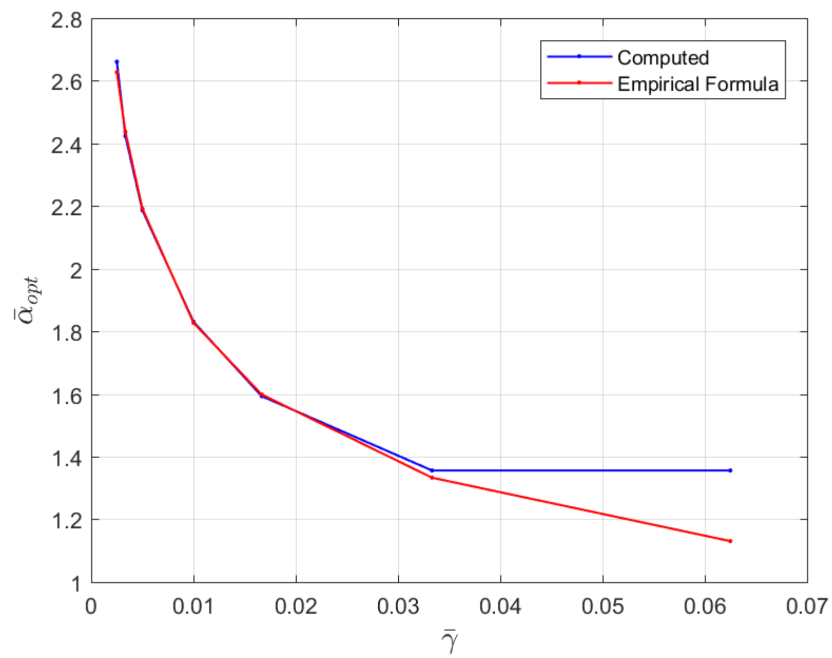

Figure 3.7: Comparison between computed values of $\bar{\alpha}_{opt}$ and the values of $\bar{\alpha}_{opt}$ using the empirical formula (3.33) for different values of overlap γ and $p, q = 4$

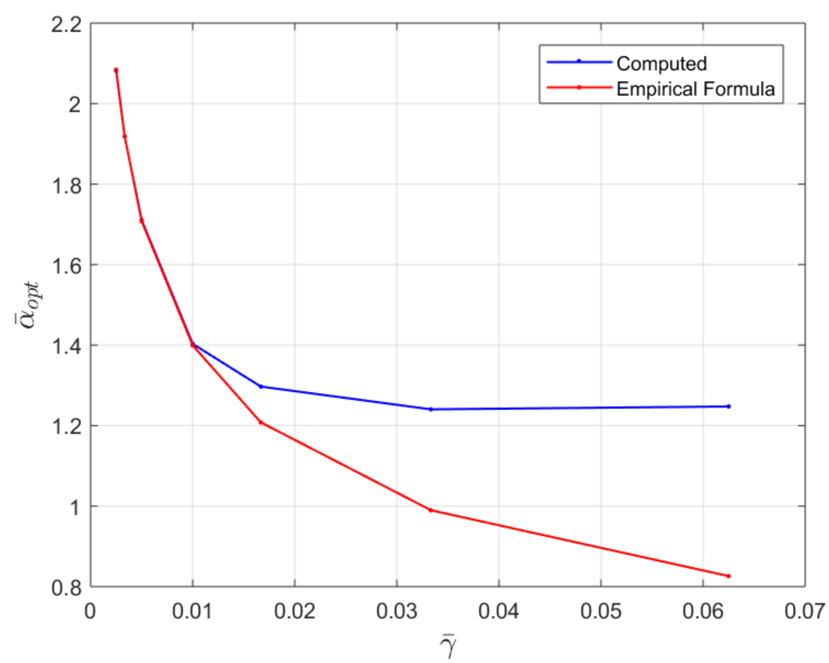

Figure 3.8: Comparison between computed values of $\bar{\alpha}_{opt}$ and the values of $\bar{\alpha}_{opt}$ using the empirical formula (3.33) for different values of overlap γ and $p, q = 5$

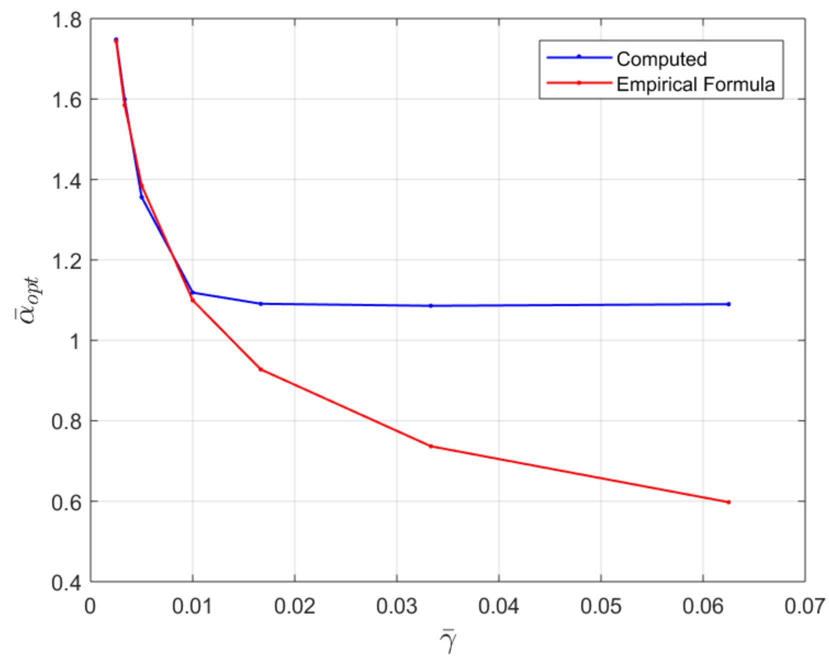

Figure 3.9: Comparison between computed values of $\bar{\alpha}_{opt}$ and the values of $\bar{\alpha}_{opt}$ using the empirical formula (3.33) for different values of overlap γ and $p, q = 6$

3.8 Numerical experiments

In this section we present some numerical experiments that validate the values of the optimal parameter α predicted by the theory developed in the previous sections. In these experiments[1] we used a finite difference MATLAB implementation of the synchronous version of the optimized Schwarz method with $OO0$ artificial boundary conditions and a domain decomposition as that from section 3.2.

In Figures 3.10 and 3.11 we can see a comparison between the optimal iteration count (blue curve), obtained by running the solver for several values of $\bar{\alpha}$ and taking the minimum iteration among these, and the iteration count obtained by running the solver with the value of $\bar{\alpha}_{opt}$ predicted by our theory (red curve), for $p = q = 4$ and $p = q = 6$, respectively, and varying amount of normalized overlap $\bar{\gamma}$.

In these numerical experiments we have used different mesh sizes. Given that the

[1]The experiments were performed using the ORAS.m MATLAB code written by Edmond Chow

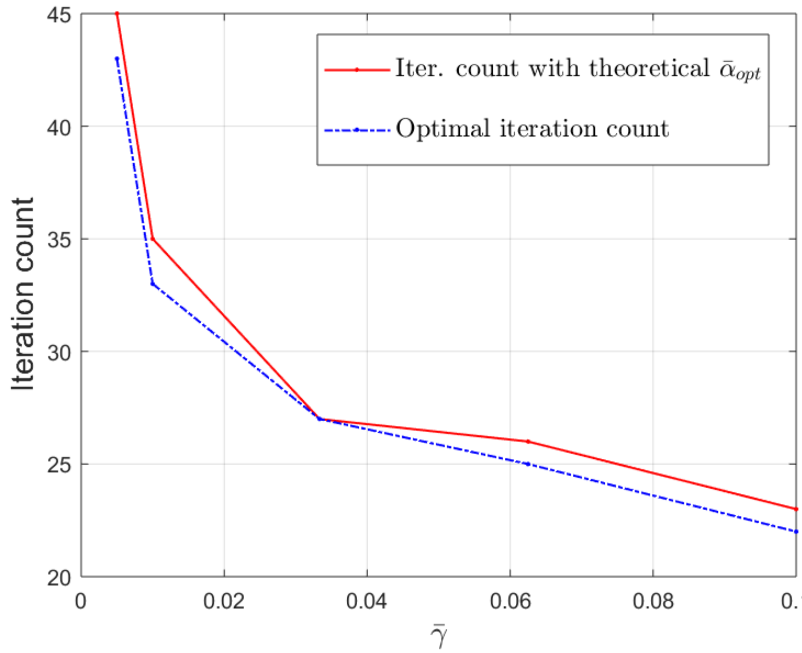

Figure 3.10: Iteration count comparison for $p = q = 6$ and $\bar{\gamma} \in [0.005, 0, 1]$

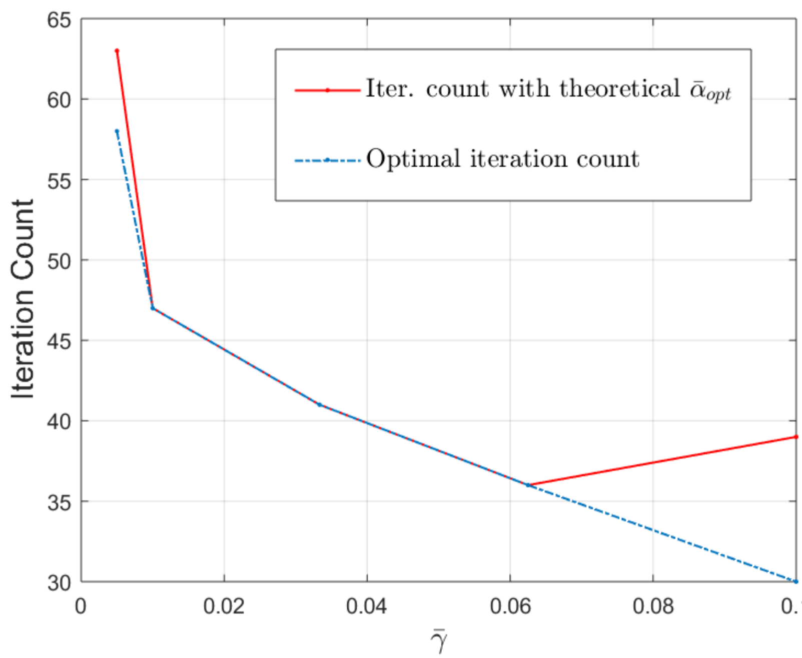

Figure 3.11: Iteration count comparison for $p = q = 6$

above curves are very close to each other, these results seems to suggest that the value of the optimal α is determined mainly by the domain decomposition and not by the discretization.

In Figure 3.12, we can see the iteration count values for different values of p, with $p = q$ and fixed $\bar{\gamma} = 0.01$, obtained by running the solver with the value of $\bar{\alpha}_{opt}$ given by our theory.

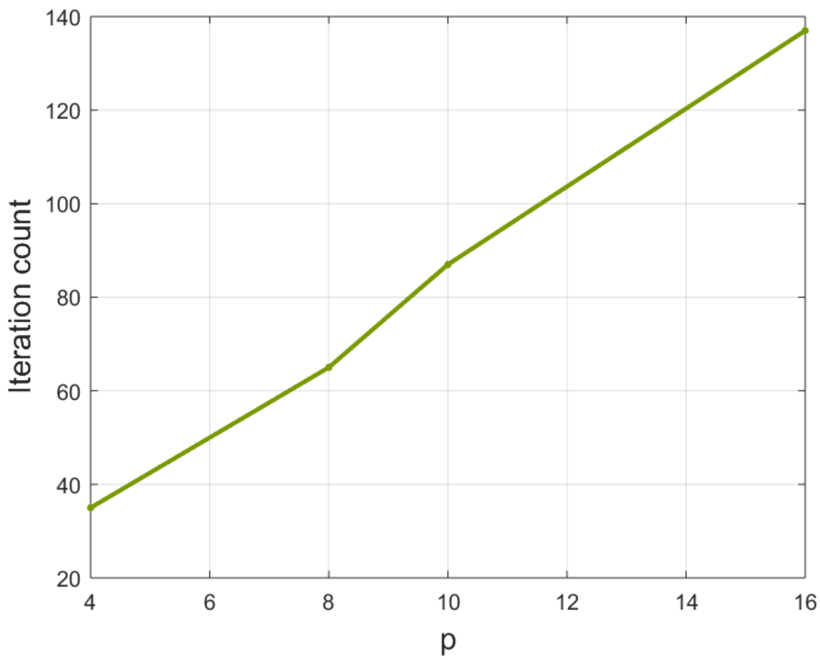

Figure 3.12: Iteration count vs. p for $\bar{\gamma} = 0.01$ and theoretical optimal $\bar{\alpha}$

In Figure 3.13 we can see the iteration count values for different values of $\bar{\alpha}$, when $p = q = 6$ and $\bar{\gamma} = 0.01$. As we can observe, when we use the optimal value of $\bar{\alpha}$, the iteration count is reduced substantially with respect to the classical Schwarz case (which corresponds to the limit of the iteration count as $\bar{\alpha} \to \infty$).

For numerical experiments from practical applications[2], see [11]. These experiments illustrate on the one hand the method indeed converges, as the theory indicates, and on the other hand that the asynchronous implementation is faster than its synchronous counterpart, in terms of execution time.

[2]These experiments were performed by Frédéric Magoulès using the optimal values of $\bar{\alpha}$ given by our theory

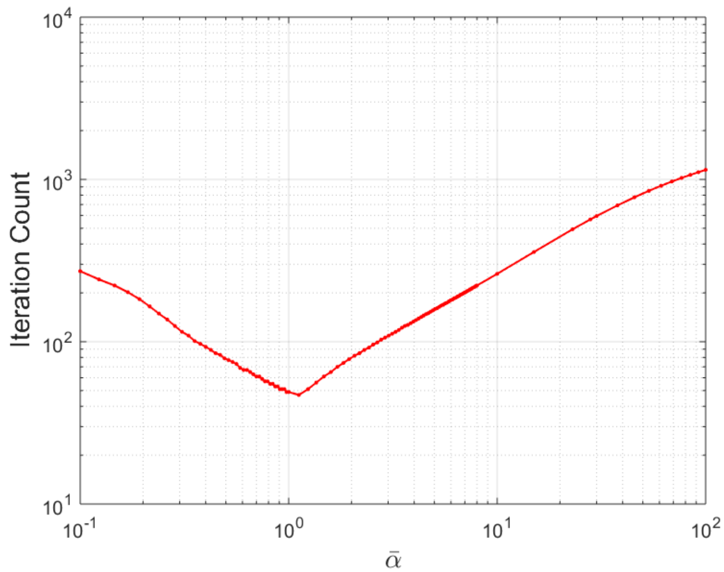

Figure 3.13: Iteration count vs. $\bar{\alpha}$ for $p = q = 6$ and $\bar{\gamma} = 0.01$

3.9 Conclusion

We have analyzed the convergence of the Optimized Schwarz method when it is used as an outer solver for the solution of Poisson's problem in a rectangular domain with Dirichlet (physical) boundary conditions and zeroth order artificial boundary conditions ($OO0$). We presented convergence proofs for the synchronous and asynchronous implementations of OS. As a key preliminary step to prove convergence we recast the problem into a fixed point iteration with an infinite matrix as the iteration operator $\hat{T}$. Then we showed that to prove convergence of the method in the synchronous case it suffices to study the spectral properties of a truncated version of this operator, $\hat{T}_{k_{\max}}$. For the convergence proof of the asynchronous case, it suffices to study the spectral properties of $|\hat{T}_{k_{\max}}|$. We defined as optimal values of the $OO0$ parameter as those whose normalized values minimize the spectral radius of $\hat{T}_{k_{\max}}$, for both the synchronous and asynchronous cases. Finally, we presented some numerical experiments that validate the choice of optimal boundary parameter values predicted by our theory.

CHAPTER 4

CONCLUSIONS

In this thesis we analyze the convergence behavior of the synchronous and asynchronous implementation of optimized Schwarz methods when they are used to solve partial differential equations with a shifted Laplacian operator and defined in bounded rectangular domains. We analyze first the case where we have a shift that can be either positive, negative or zero, a one-way domain decomposition and transmission conditions of the $OO2$ family. We analyze next the problem involving Poisson's equation, a domain decomposition with cross-points and $OO0$ transmission conditions. In both cases we recast the equations into a fixed point iteration that is suitable for our analysis, present convergence proofs of the method, and study how the convergence rate varies with the different parameter values such as the number of subdomains, the amount of overlap and the parameters introduced in the transmission conditions. We obtain the optimal values of the transmission conditions parameters. We present some numerical experiments for the second case illustrating our theoretical results. Finally, it is important to mention that although we study problems defined in rectangular domains, the analysis presented in this thesis also applies to problems with domains with more arbitrary shapes that can be decomposed as a union of rectangles.

BIBLIOGRAPHY

[1] Daniel Bennequin, Martin J. Gander, Loic Gouarin, and Laurence Halpern. Optimized schwarz waveform relaxation for advection reaction diffusion equations in two dimensions. *Numerische Mathematik*, 134:513–567, 2016.

[2] Dimitri P. Bertsekas and John N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods.* Prentice Hall, Englewood Cliffs, NJ, 1989.

[3] Daniel Chazan and Willard D. Miranker. Chaotic relaxations. *Linear Algebra and Its Applications*, 2:199–222, 1969.

[4] James Demmel, Laura Grigori, Mark Hoemmen, and Julien Langou. Communication-optimal parallel and sequential QR and LU factorizations. *SIAM Journal on Scientific Computing*, 34:206—-239, 2012.

[5] Victorita Dolean, Pierre Jolivet, and Frédéric Nataf. *An introduction to Domain Decomposition Methods: Algorithms, Theory, and Parallel Implementation.* SIAM, Philadelphia, 2015.

[6] Andreas Frommer. On asynchronous iterations in partially ordered spaces. *Numerical Functional Analysis and Optimization*, 12:315–325, 1991.

[7] Andreas Frommer, Hartmut Schwandt, and Daniel B. Szyld. Asynchronous weighted additive Schwarz methods. *Electronic Transactions on Numerical Analysis*, 5:48–61, 1997.

[8] Andreas Frommer and Daniel B. Szyld. On asynchronous iterations. *Journal of Computational and Applied Mathematics*, 123:201–216, 2000.

[9] Martin J. Gander. Optimized Schwarz methods. *SIAM Journal on Numerical Analysis*, 2:699–731, 2006.

[10] José C. Garay, Frédéric Magoulès, and Daniel B. Szyld. Convergence of asynchronous optimized Schwarz methods in the plane. Technical Report 17-05-17, Department of Mathematics, Temple University, May 2017. Revised January 2018 and March 2018. To appear in Domain Decomposition Methods in Science and Engineering XXIV, Lecture Notes in Computer Science and Engineering, Springer, Berlin and Heidelberg, 2018.

[11] José C. Garay, Frédéric Magoulès, and Daniel B. Szyld. Synchronous and asynchronous optimized Schwarz method for Poisson's equation in rectangular domains. Technical Report 17-10-18, Department of Mathematics, Temple University, October 2017. Revised April 2018.

[12] Richard Haberman. *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems.* Prentice Hall, Englewood Cliffs, NJ, 4th edition, 2003.

[13] Frédéric Magoulès. Asynchronous Schwarz methods for peta and exascale computing. In B.H.V. Topping and P. Iványi, editors, *Developments in Parallel, Distributed, Grid and Cloud Computing for Engineering*, chapter 10, pages 229–248. Saxe-Coburg, Stirlingshire, UK, 2013.

[14] Frédéric Magoulès, Daniel B. Szyld, and Cédric Venet. Asynchronous optimized Schwarz methods with and without overlap. *Numerische Mathematik*, 137:199–227, 2017.

[15] Summary Report of the Advanced Scientific Computing Advisory Committee (ASCAC) Subcommittee. The opportunities and challenges of exascale computing. Technical report, Office of Science, U.S Department of Energy, 2010. `https://science.energy.gov/~/media/ascr/ascac/pdf/reports/Exascale_subcommittee_report.pdf`.

[16] Mouhamed Nabih El Tarazi. Some convergence results for asynchronous algorithms. *Numerische Mathematik*, 39:325–340, 1982.

[17] Andrea Toselli and Olof Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Series in Computational Mathematics.* Springer, Berlin, Heidelberg, New York, 2005.

[18] Richard S. Varga. *Matrix Iterative Analysis.* Series in Computational Mathematics. Springer, Berlin and Heidelberg, 2nd edition, 2000.

APPENDIX A

Series expansion of solutions of the homogeneous Shifted-Laplacian

In Chapters 2 and 3 we deal with equations of the following form

$$\begin{cases} \Delta\eta + \omega\eta = 0 & \text{in } (0,L)\times(0,H) \\ -\frac{\partial\eta}{\partial x} + \alpha\eta + \beta\frac{\partial^2\eta}{\partial y^2} = g_1(y) & \text{for } x = 0 \\ \frac{\partial\eta}{\partial x}(x,y) + \alpha\eta(x,y) + \beta\frac{\partial^2\eta}{\partial y^2}(x,y) = g_2(y) & \text{for } x = L \\ a\frac{\partial\eta}{\partial y}(x,y) + b\eta(x,y) = 0 & \text{for } y = 0 \\ c\frac{\partial\eta}{\partial y}(x,y) + d\eta(x,y) = 0 & \text{for } y = H, \end{cases} \tag{A.1}$$

$$\begin{cases} \Delta\eta + \omega\eta = 0 & \text{in } (0,L)\times(0,H) \\ \eta = 0 & \text{for } x = 0 \\ \frac{\partial\eta}{\partial x}(x,y) + \alpha\eta(x,y) + \beta\frac{\partial^2\eta}{\partial y^2}(x,y) = g_2(y) & \text{for } x = L \\ a\frac{\partial\eta}{\partial y}(x,y) + b\eta(x,y) = 0 & \text{for } y = 0 \\ c\frac{\partial\eta}{\partial y}(x,y) + d\eta(x,y) = 0 & \text{for } y = H, \end{cases} \tag{A.2}$$

and

$$\begin{cases} \Delta\eta + \omega\eta = 0 & \text{in } (0,L)\times(0,H) \\ -\frac{\partial\eta}{\partial x} + \alpha\eta + \beta\frac{\partial^2\eta}{\partial y^2} = g_1(y) & \text{for } x = 0 \\ \eta = 0 & \text{for } x = L \\ a\frac{\partial\eta}{\partial y}(x,y) + b\eta(x,y) = 0 & \text{for } y = 0 \\ c\frac{\partial\eta}{\partial y}(x,y) + d\eta(x,y) = 0 & \text{for } y = H, \end{cases} \tag{A.3}$$

where $a, b, c, d, \omega \in \mathbb{R}$ and $b, d \neq 0$.

Since the equations in (A.1) are linear and homogenous, the solution of (A.1) can be split into two parts, i.e., $\eta = \eta_1 + \eta_2$, where one of these two parts, say η_1 is the solution of

$$\begin{cases} \Delta\eta_1 + \omega\eta_1 = 0 & \text{in } (0, L) \times (0, H) \\ -\frac{\partial\eta_1}{\partial x} + \alpha\eta_1 + \beta\frac{\partial^2\eta_1}{\partial y^2} = g_1(y) & \text{for } x = 0 \\ \frac{\partial\eta_1}{\partial x}(x, y) + \alpha\eta_1(x, y) + \beta\frac{\partial^2\eta_1}{\partial y^2}(x, y) = 0 & \text{for } x = L \\ a\frac{\partial\eta_1}{\partial y}(x, y) + b\eta_1(x, y) = 0 & \text{for } y = 0 \\ c\frac{\partial\eta_1}{\partial y}(x, y) + d\eta_1(x, y) = 0 & \text{for } y = H, \end{cases} \tag{A.4}$$

and the other is the solution of

$$\begin{cases} \Delta\eta_2 + \omega\eta_2 = 0 & \text{in } (0, L) \times (0, H) \\ -\frac{\partial\eta_2}{\partial x} + \alpha\eta_2 + \beta\frac{\partial^2\eta_2}{\partial y^2} = 0 & \text{for } x = 0 \\ \frac{\partial\eta_2}{\partial x}(x, y) + \alpha\eta_2(x, y) + \beta\frac{\partial^2\eta_2}{\partial y^2}(x, y) = g_2(y) & \text{for } x = L \\ a\frac{\partial\eta_2}{\partial y}(x, y) + b\eta_2(x, y) = 0 & \text{for } y = 0 \\ c\frac{\partial\eta_2}{\partial y}(x, y) + d\eta_2(x, y) = 0 & \text{for } y = H, \end{cases} \tag{A.5}$$

In this section we show how to obtain series expansion of the solutions of equations (A.1)-(A.5). Given that the procedure to obtain these series is the same for each case, we will show the procedure to obtain a solution series for the case given by (A.4).

Thus, let η be the solution of (A.4). To determine a series expansion of η, we find first a series expansion of the general solution of

$$\begin{cases} \Delta\eta + \omega\eta = 0 & \text{in } (0,L)\times(0,H) \\ -\frac{\partial\eta}{\partial x} + \alpha\eta + \beta\frac{\partial^2\eta}{\partial y^2} = 0 & \text{for } x = 0 \\ a\frac{\partial\eta}{\partial y} + b\eta = 0 & \text{for } y = 0 \\ c\frac{\partial\eta}{\partial y} + d\eta = 0 & \text{for } y = H. \end{cases} \tag{A.6}$$

Let us assume (ansatz) that a solution of (A.6) can be written as a product of two one-variable functions. Thus, let us assume that

$$\eta(x,y) = \psi(x)\phi(y). \tag{A.7}$$

Plugging (A.7) into the first equation in (A.6) we have

$$\phi(y)\frac{d^2\psi}{dx^2}(x) + \psi(x)\frac{d^2\phi}{dy^2}(y) + \omega\psi(x)\phi(y) = 0,$$

Dividing both sides of the above expression by $\psi(x)\phi(y)$, it follows that for all $(x,y) \in (0,L)\times(0,H)$ such that $\psi(x)\phi(y) \neq 0$ we have

$$\frac{1}{\psi(x)}\frac{d^2\psi}{dx^2} + \frac{1}{\phi(y)}\frac{d^2\phi}{dy^2} + \omega = 0,$$

which implies that

$$\frac{1}{\phi(y)}\frac{d^2\phi}{dy^2} = -\frac{1}{\psi(x)}\frac{d^2\psi}{dx^2} - \omega.$$

Note that the above equality is only possible if both sides of the equation equal a constant, i.e.,

$$\frac{1}{\phi(y)}\frac{d^2\phi}{dy^2} = -\frac{1}{\psi(x)}\frac{d^2\psi}{dx^2} - \omega = -\lambda,$$

where λ is a constant. Then, we have the following two ordinary differential equations

$$\frac{d^2\phi}{dy^2}(y) + \lambda\phi(y) = 0 \tag{A.8}$$

and

$$\frac{d^2\psi}{dx^2}(x) + (\omega - \lambda)\psi(x) = 0. \tag{A.9}$$

With $\eta(x, y) = \psi(x)\phi(y)$, from the boundary conditions $a\frac{\partial u}{\partial y}(x, 0) + bu(x, 0) = 0$ we have

$$\psi(x)\left(a\frac{d\phi}{dy}(0) + b\phi(0)\right) = 0.$$

Then since there exists at least an x such that $\psi(x) \neq 0$ it follows from the last equality that

$$a\frac{d\phi}{dy}(0) + b\phi(0) = 0. \tag{A.10}$$

Similarly, $c\frac{\partial u}{\partial y}(x, 0) + du(x, 0) = 0$ implies

$$c\frac{d\phi}{dy}(H) + d\phi(H) = 0. \tag{A.11}$$

Equations (A.8), (A.10) and (A.11) define a regular Sturm-Liouville eigenvalue problem where λ is an eigenvalue and ϕ the corresponding eigenfunction; see [12] . According to the Sturm-Liouville theory, this eigenvalue problem has infinitely many eigenvalues, and the corresponding eigenfunctions form a complete orthogonal set spanning the set of piecewise continuous functions. Hence, every piecewise continuous function can be expressed as a linear combination of the functions of this basis. This eigenvalues and eigenfunctions can be determined as follows. Let us assume (ansatz) that $\phi(y) = e^{\zeta y}$. Then, plugging this expression into (A.8) we obtain

$$\zeta^2 e^{\zeta y} + \lambda e^{\zeta y} = 0.$$

Noticing that $e^{\zeta y} \neq 0$ for all $\zeta, y \in \mathbb{R}$, we obtain

$$\zeta^2 + \lambda = 0,$$

and thus

$$\zeta = \pm\sqrt{-\lambda}.$$

As we shall see, the functional form of ϕ depends on the value of λ. For $\lambda > 0$, we have that $\zeta = \pm i\sqrt{\lambda}$, and the general solution of (A.8) has the form

$$\phi(y) = Ae^{i\sqrt{\lambda}y} + Be^{-i\sqrt{\lambda}y},$$

where A, B are arbitrary constants. In other words, for $\lambda > 0$, any solution of (A.8) can be written as a linear combination of $e^{i\sqrt{\lambda}y}$ and $e^{-i\sqrt{\lambda}y}$. Note that $\sin(\sqrt{\lambda}y) = (e^{i\sqrt{\lambda}y} - e^{-i\sqrt{\lambda}y})/(2i)$ and $\cos(\sqrt{\lambda}y) = (e^{i\sqrt{\lambda}y} + e^{-i\sqrt{\lambda}y})/2$ are solutions of (A.8). Since $\sin(\sqrt{\lambda}y)$ and $\cos(\sqrt{\lambda}y)$ are linearly independent, then any solution of (A.8) can also be written as

$$\phi(y) = A\sin\left(\sqrt{\lambda}y\right) + A\cos\left(\sqrt{\lambda}y\right), \tag{A.12}$$

where A and B are constants.

Thus, we just determined that for positive eigenvalues λ, the corresponding eigenfunctions (i.e., the solutions of the regular Sturm-Liouville eigenvalue problem) can be written as a linear combination of sines and cosines. Now, in addition to satisfying (A.8), the eigenfunctions also need to satisfy equations (A.10) and (A.11). Hence, plugging (A.12) into (A.10) we obtain

$$\left(-aB\sqrt{\lambda} + bA\right)\sin(\sqrt{\lambda}0) + \left(aA\sqrt{\lambda} + bB\right)\cos(\sqrt{\lambda}0) = 0,$$

which implies that

$$B = \frac{-aA\sqrt{\lambda}}{b}.$$

Then, we have

$$\phi(y) = \frac{-A\sqrt{\lambda}}{b}\left[-\frac{b}{\sqrt{\lambda}}\sin(\sqrt{\lambda}y) + a\cos(\sqrt{\lambda}y)\right]. \tag{A.13}$$

Now, plugging (A.13) into the remaining boundary condition (A.11) we obtain

$$-\frac{A\sqrt{\lambda}}{b}\left[\left(-ca\sqrt{\lambda} - \frac{bd}{\sqrt{\lambda}}\right)\sin(\sqrt{\lambda}H) + (-bc + ad)\cos(\sqrt{\lambda}H)\right] = 0.$$

Since the eigenfunctions are non-trivial (non-zero) solutions of (A.8), we have $A \neq 0$. Also, we assume $b \neq 0$. Then, since $\lambda > 0$, we have $\frac{A\sqrt{\lambda}}{b} \neq 0$. Therefore, the above equality implies necessarily that

$$\left(-ca\sqrt{\lambda} - \frac{bd}{\sqrt{\lambda}}\right)\sin(\sqrt{\lambda}H) + (-bc + ad)\cos(\sqrt{\lambda}H) = 0 \tag{A.14}$$

Thus, any positive eigenvalue λ of the eigenvalue problem defined by equations (A.8), (A.10) and (A.11) is a solution of the transcendental equation (A.14), and the corresponding eigenfunction is given by (A.13), where A is an arbitrary constant, and where λ in (A.13) is understood to be a solution of (A.14).

Note that if

$$(-bc + ad)\cos(\sqrt{\lambda}H) = 0, \tag{A.15}$$

given that $b, d \neq 0$ we have that the eigenvalue λ is the solution of

$$\sin(\sqrt{\lambda}H) = 0. \tag{A.16}$$

Note that equation (A.16) is satisfied for all $\lambda \in \{\lambda_m\}_{m\in\mathbb{N}}$, where $\lambda_m = (\frac{m\pi}{H})^2$. Thus, in the case that A.15 holds, there are infinitely many eigenvalues. If $(-bc +$

$ad)\cos(\sqrt{\lambda}H) \neq 0$ we have

$$\tan(\sqrt{\lambda}H) = \frac{(ad - bc)\sqrt{\lambda}}{ca\lambda + bd}, \tag{A.17}$$

which also has infinitely many solutions. Consequently, if (A.15) does not hold, there are also infinitely many eigenvalues. In both cases, the eigenfunctions are given by

$$\phi_m(y) = \frac{-A\sqrt{\lambda_m}}{b}\left[-\frac{b}{\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}y) + a\cos(\sqrt{\lambda_m}y)\right].$$

With a similar procedure we can determine the equations defining the eigenvalues and eigenvectors for the cases $\lambda < 0$ and $\lambda = 0$. For $\lambda < 0$, we have $\zeta = \pm\sqrt{-\lambda}$. This leads to have eigenfunctions of the form

$$\phi(y) = B\left[\frac{a\sqrt{-\lambda} - b}{a\sqrt{-\lambda} + b}e^{\sqrt{-\lambda}y} + e^{-\sqrt{-\lambda}y}\right],$$

and the (negative) eigenvalues, if they exist, are solutions of

$$\frac{a\sqrt{-\lambda} - b}{a\sqrt{-\lambda} + b}(c\sqrt{-\lambda} + d) + (-c\sqrt{-\lambda} + d)e^{-2\sqrt{-\lambda}L} = 0.$$

As for $\lambda = 0$, we obtain that zero is an eigenvalue only if $c + bL - da/b = 0$, and if it is, the corresponding eigenvector is given by

$$\phi(y) = A\left(y - \frac{a}{b}\right).$$

Thus, so far we determined the expressions of $\phi(y)$ for each possible value of λ. Now it remains to determine the expressions for $\psi(x)$.

Note that

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \alpha u + \beta \frac{\partial^2 u}{\partial y^2} &= \phi(y)\frac{d\psi}{dx}(x) + \alpha\psi(x)\phi(y) + \beta\psi(x)\frac{d^2\phi}{dy^2} \\
&= \phi(y)\frac{d\psi}{dx} + \alpha\psi(x)\phi(y) - \beta\lambda\psi(x)\phi(y) \\
&= \phi(y)\left[\frac{d\psi}{dx} + (\alpha - \beta\lambda)\psi\right],
\end{aligned}
$$

where in the second equality we used (A.8). Then, from the boundary condition $-\frac{\partial\eta}{\partial x}(0, y) + \alpha\eta(0, y) + \beta\frac{\partial^2\eta}{\partial y^2}(0, y) = 0$ we obtain

$$
\phi(y)\left[-\frac{d\psi}{dx}(0) + (\alpha - \beta\lambda)\psi(0)\right] = 0.
$$

But there exists at least a $y \in [0, L]$ such that $\phi(y) \neq 0$, then we have that

$$
\frac{d\psi}{dx}(L) + (\alpha - \beta\lambda)\psi(L) = 0. \tag{A.18}
$$

Similarly as for ϕ, the functional form of ψ depends on the values of $\lambda - \omega$. In fact assuming that $\psi_(x) = e^{\bar{\zeta}x}$, and plugging this expression into (A.9) we obtain that $\bar{\zeta} = \pm\sqrt{\lambda - \omega}$. Then, depending on the values of ω and λ, we have that $\bar{\zeta}$ can be positive, imaginary or zero, and each of these options give expressions of ψ with different behavior.

For $\omega - \lambda < 0$, any solution of (A.9) can be written as a linear combination of exponentials such as

$$
\psi(x) = C_1 e^{\left(x\sqrt{\lambda-\omega}\right)} + C_2 e^{-\left(x\sqrt{\lambda-\omega}\right)}, \tag{A.19}
$$

or equivalently, as a linear combination of hyperbolic sines and cosines, i.e.,

$$
\psi(x) = C_1 \sinh\left(x\sqrt{\lambda - \omega}\right) + C_2 \cosh\left(x\sqrt{\lambda - \omega}\right). \tag{A.20}
$$

Also, given that (A.9) is invariant under translations, we can also write any of its solutions as

$$\psi(x) = C_1 e^{\left(\sqrt{\lambda-\omega}(x-L)\right)} + C_2 e^{-\left(\sqrt{\lambda-\omega}(x-L)\right)}. \tag{A.21}$$

In what follows, we use the expression (A.19). Thus, plugging (A.19) into the boundary condition (A.18) we obtain

$$C_1(\sqrt{\lambda} + \alpha - \beta\lambda) + C_2(-\sqrt{\lambda} + \alpha - \beta\lambda) = 0.$$

This implies that

$$C_1 = -\left(\frac{-\sqrt{\lambda} + \alpha - \beta\lambda}{\sqrt{\lambda} + \alpha - \beta\lambda}\right) C_2,$$

and therefore

$$\psi(x) = C_2 \left[-\left(\frac{-\sqrt{\lambda} + \alpha - \beta\lambda}{\sqrt{\lambda} + \alpha - \beta\lambda}\right) e^{\left(x\sqrt{\lambda-\omega}\right)} + e^{-\left(x\sqrt{\lambda-\omega}\right)} \right], \tag{A.22}$$

For $\omega - \lambda > 0$ we have that ψ can be written as a linear combination of certain complex exponentials, as follows

$$\psi(x) = C_3 e^{-i\sqrt{\lambda-\omega}(x-L)} + C_4 e^{i\sqrt{\lambda-\omega}(x-L)}. \tag{A.23}$$

Plugging (A.23) in (A.18) we obtain

$$C_3 = \frac{i\sqrt{\lambda-\omega} + (\lambda - \beta\lambda)}{i\sqrt{\lambda-\omega} - (\lambda - \beta\lambda)} C_4.$$

Then,

$$\psi(x) = C_4 \left[\frac{i\sqrt{\lambda-\omega} + (\lambda - \beta\lambda)}{i\sqrt{\lambda-\omega} - (\lambda - \beta\lambda)} e^{-i\sqrt{\lambda-\omega}(x-L)} + e^{i\sqrt{\lambda-\omega}(x-L)} \right] \tag{A.24}$$

For $\omega - \lambda = 0$, we have

$$\psi(x) = C_5 x + C_6.$$

Plugging this equation into (A.18) we obtain

$$C_5 = (\alpha - \beta\lambda) C_6,$$

which implies that

$$\psi(x) = C_6 \left[(\alpha - \beta\lambda) x + 1 \right]. \tag{A.25}$$

Thus, we have determined the expressions for ψ for every possible value of $\lambda - \omega$.

Now, let ψ_m be the function obtained after plugging $\lambda = \lambda_m$ in equations (A.22)-(A.25). Then, we can have that any $\eta_m := \psi_m(x)\phi_m(x)$ is a solution of (A.6). Moreover, since the equations in (A.6) are linear and homogeneous, by the superposition principle we have that any finite linear combination of the η_m, say

$$\sum_{m=1}^{N} A_m \eta_m(x, y) = \sum_{m=1}^{N} A_m \psi_m(x) \phi_m(y),$$

is a solution of (A.6).

The function

$$\bar{\eta}(x, y) = \sum_{m=1}^{\infty} A_m \eta_m = \sum_{m=1}^{\infty} A_m \psi_m(x) \phi_m(y) \tag{A.26}$$

is the most general solution of (A.6) provided that the order of differential and summation operators commute when differential operators are applied to (A.26). Also, if in addition to this, the coefficients A_m are such that $\bar{\eta}$ satisfies the equation on the third line of (A.4), i.e., if

$$\frac{\partial \bar{\eta}}{\partial x}(L, y) + \alpha \bar{\eta}(L, y) + \beta \frac{\partial^2 \bar{\eta}}{\partial y^2}(L, y) = g_1(y),$$

then $\bar{\eta}$ will be a series expansion of the solution of (A.4). We show later in sections B and C Lemmas B.2, B.3, C.2 and C.3, from where it follows that such coefficients A_m exist for the case $\lambda_m > 0$ for all $m \in \mathbb{N}$, and that for these coefficients, the order of differential and summation operators commute when differential operators are applied to (A.26). This result can be extended to cases where there are also eigenvalues λ_m that are negative or zero (in this thesis we deal with problems that yield Sturm-Liouville eigenvalue problems that have only positive eigenvalues, therefore we do not present the proof for the cases containing negative or zero eigenvalues).

Thus, the solution of (A.4) indeed can be written as a series

$$\sum_{m=1}^{\infty} A_m \psi_m(x)\phi_m(y).$$

where the expression of ϕ_m depends on the value of λ_m and the expression of ψ_m depends on λ_m and ω.

A.1 Expressions for ψ_m and ϕ_m when $a = c = 0$ and $b = d = 1$

In chapter 2 we have the case where $a = b = 0$ and $c = d = 1$ and ω such that $\omega H^2/\pi^2 \notin \mathbb{N}$. These values of a, b, c, d give a Sturm-Liouville eigenvalue problem whose eigenvalues are all positive, whose eigenvalues are solutions of (A.16). Consequently, the eigenvalues are $\lambda_m = (m\pi)^2/H^2$. Taking $A = 1$, $a = 0$ and $\lambda = \lambda_m$ in (A.13), and letting $z_m := \sqrt{\lambda_m} H$, we have that the eigenfunctions are

$$\phi_m(y) = \sin\left(\frac{z_m \pi}{H}\right).$$

The condition $\omega H^2/\pi^2 \notin \mathbb{N}$ implies that $\lambda_m \neq \omega$ for all $m \in \mathbb{N}$. Then, Taking $C_2 = C_4 = 1$, and with $\bar{\alpha} := \alpha H$, $\bar{\beta} := \beta/H$ and $\bar{\omega} := \omega H^2$ in (A.22)-(A.24), we obtain

$$\psi_m(x) = \begin{cases} \frac{i\sqrt{z_m^2-\bar{\omega}}+(\bar{\alpha}-\beta z_m^2)}{i\sqrt{z_m^2-\bar{\omega}}-(\bar{\alpha}-\beta z_m^2)} e^{-i\sqrt{\lambda-\omega}(x-L)} + e^{i\sqrt{\lambda-\omega}(x-L)}, & \text{if } z_m^2 < \bar{\omega}, \\ e^{-\frac{(x-L)}{H}\sqrt{z_m^2-\bar{\omega}}} - \left(\frac{\alpha-\beta z_m^2-\sqrt{z_m^2-\bar{\omega}}}{\alpha-\beta z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right) e^{\frac{(x-L)}{H}\sqrt{z_m^2-\bar{\omega}}}, & \text{if } z_m^2 > \bar{\omega}, \end{cases} \tag{A.27}$$

A.2 Expressions for ψ_m and ϕ_m when $c = -a = 1$ and $b = d = \alpha$

In chapter 3 we have $a = -c = -1$, $b = d = \alpha$ and $\omega = 0$. These values of a, b, c, d give again a Sturm-Liouville eigenvalue problem whose eigenvalues are all positive. The eigenvalues λ_m in this case are solutions of (A.17). Taking $\lambda = \lambda_m$ and $A = b/\sqrt{\lambda_m}$ in (A.13), and again with $z_m = \sqrt{\lambda_m}H$, we have that the eigenfunctions in this case are given by

$$\phi_m(y) = \frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m y}{H}\right) + \cos\left(\frac{z_m y}{H}\right).$$

Then, from (A.19) we have that ϕ_m in this case is given by

$$\phi_m(x) = \frac{i\sqrt{z_m^2 - \bar{\omega}} + (\bar{\alpha} - \beta z_m^2)}{i\sqrt{z_m^2 - \bar{\omega}} - (\bar{\alpha} - \beta z_m^2)} e^{-i\sqrt{\lambda-\omega}(x-L)} + e^{i\sqrt{\lambda-\omega}(x-L)}.$$

Alternatively, one could use (A.20) instead of (A.19), and with we can see that another option for ψ_m is given by

$$\psi_m(x) = \frac{\bar{\alpha}}{z_m} \sinh\left(\frac{z_m x}{H}\right) + \cosh\left(\frac{z_m x}{H}\right).$$

APPENDIX B

Additional proofs for the case with a one-way decomposition

B.1 Decay of the coefficients of the series

Lemma B.1. *Let* $M > 0$ *and* $g : [0, H] \to \mathbb{R}$ *be such that* $g \in C^2((0, H))$ *and* $g(0) = g(H) = 0$*. Let* $\phi_m = \sin\left(\frac{z_m y}{H}\right)$ *and* $z_m = m\pi$*. Note that the set* $\{\phi_m\}_{m\in\mathbb{N}}$ *is a complete orthogonal set spanning the set of piecewise continuous function. Let us define* C_m *as the coefficients of the expansion of* $g(x)$ *in the basis given by* $\{\phi_m\}_{m\in\mathbb{N}}$*, i.e,*

$$g(y) = \sum_{m=1}^{\infty} C_m \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m y}{H}\right) + \cos\left(\frac{z_m y}{H}\right)\right]. \tag{B.1}$$

Then, the coefficients C_m *can be written as*

$$C_m = \frac{\bar{C}_m}{z_m^2}$$

where $\left\{\bar{C}_m\right\}_{m\in\mathbb{N}}$ *is uniformly bounded in* $m \in \mathbb{N}$*.*

Proof. The coefficients of the series expansion are given by

$$C_m = \frac{\int_0^h g\phi_m dy}{\int_0^H \phi_m^2 dy}. \tag{B.2}$$

Applying integration by parts to the integral in the numerator of the r.h.s of the previous equation we obtain

$$\begin{aligned}
\int_0^H g \sin\left(\frac{z_m y}{H}\right) &= \left[-g(y)\cos\left(\frac{z_m y}{H}\right)\frac{H}{z_m}\right]_0^H + \frac{H}{z_m}\int_0^H \cos\left(\frac{z_m y}{H}\right)\frac{dg}{dy}(y)dy \\
&= \frac{H}{z_m}\int_0^H \cos\left(\frac{z_m y}{H}\right)\frac{dg}{dy}(y)dy, \qquad \text{(B.3)}
\end{aligned}$$

since $g(0) = g(H) = 0$. Now using again integration by parts we have that

$$\begin{aligned}
\int_0^H \cos\left(\frac{z_m y}{H}\right)\frac{dg}{dy}(y)dy &= \left[\frac{H}{z_m}\frac{dg}{dy}(y)\sin\left(\frac{z_m y}{H}\right)\right]_0^H - \frac{H}{z_m}\int_0^H \sin\left(\frac{z_m y}{H}\right)\frac{d^2g}{dy^2}(y)dy \\
&= -\frac{H}{z_m}\int_0^H \sin\left(\frac{z_m y}{H}\right)\frac{d^2g}{dy^2}(y)dy, \qquad \text{(B.4)}
\end{aligned}$$

since $|\frac{dg}{dy}(0)|, |\frac{dg}{dy}(H)| < \infty$ and $\sin(z_m) = 0$.

Then, from (B.3) and (B.4) we have

$$\int_0^H g \sin\left(\frac{z_m y}{H}\right) = -\left(\frac{H}{z_m}\right)^2 \int_0^H \sin\left(\frac{z_m y}{H}\right)\frac{d^2g}{dy^2}(y)dy. \qquad \text{(B.5)}$$

We have that

$$\int_0^H \phi_m^2 dy = \int_0^H \sin^2\left(\frac{z_m y}{H}\right)dy = \frac{H}{2}\ \cdot \qquad \text{(B.6)}$$

Then plugging (B.5) and (B.6) into B.2 we obtain

$$C_m = \frac{-\left(\frac{H}{z_m}\right)^2 \int_0^H \sin\left(\frac{z_m y}{H}\right)\frac{d^2g}{dy^2}(y)dy}{\frac{H}{2}}$$

Then,

$$|C_m| \leq \frac{\left(\frac{H}{z_m}\right)^2 \left\|\frac{d^2g}{dy}\right\|_\infty H}{\frac{H}{2}} = \frac{2H^2\left\|\frac{d^2g}{dy}\right\|_\infty}{z_m^2}$$

Let $M :=:= 2H^2\left\|\frac{d^2g}{dy}\right\|_\infty$. Let

$$\bar{C}_m := z_m^2 C_m$$

Then, we have

$$|\bar{C}_m| = z_m^2 |C_m| \le z_m^2 \frac{2H^2 \left\| \frac{d^2 g}{dy} \right\|_\infty}{z_m^2} = M$$

Hence, $|\bar{C}_m| \le M$ for all $m \in \mathbb{N}$.

Therefore, we have

$$C_m = \frac{\bar{C}_m}{z_m^2} \ ,$$

where $|\bar{C}_m| < M$ for all $m \in \mathbb{N}$, i.e, $\{\bar{C}_m\}$ is uniformly bounded in $m \in \mathbb{N}$. □

B.2 Justification of order interchange between derivatives, integral and infinite summation for the one-way decompostion case

Lemma B.2. *Consider the series in (2.7), (2.11), and (2.12). If we can write their coefficients as follows*

$$A^s_{n,m,1} = \frac{B^s_{n,m,1}}{z_m^2 \left(\frac{-d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(1)}(0) \right)} \ , \tag{2.14}$$

and

$$A^s_{n,m,2} = \frac{B^s_{n,m,2}}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(2)}(L) \right)} \ , \tag{2.15}$$

for $1 < s < p$,

$$A^s_{n,m,2} = \frac{B^s_{n,m,2}}{z_m^2 \left(\frac{d\psi_m^{(3)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(3)}(L) \right)} \ , \tag{2.16}$$

for $s = 1$, *and*

$$A^s_{n,m,1} = \frac{B^s_{n,m,1}}{z_m^2 \left(\frac{-d\psi_m^{(3)}}{dx}(-L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(3)}(-L) \right)} \ , \tag{2.17}$$

for $s = p$, where

$$B^s_{n,m,i} \leq M_{n,s} \tag{B.7}$$

for some constants $M_{n,s} > 0$, then the following holds.

1. *The order of their first derivatives and summation can be interchanged in $[0, L] \times [0, H]$.*

2. *The order of their second derivatives and summation can be interchanged in $[0, L] \times [0, H]$ if $\beta \neq 0$ and in $(0, L) \times (0, H)$ if $\beta = 0$.*

3. *The order of their integral over $y \in [0, H]$, first derivatives and summation can be interchanged.*

Proof. We present the proof for the case of interior subdomains, but a similar procedure can be used to prove the above facts for the left-most and right-most subdomains cases.

We have that

$$\begin{aligned} \eta^s_n &= \sum_{m=1}^{\infty} \left(\frac{B^s_{n,m,1}}{z_m^2 \left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} \psi_m^{(1)}(x) \right. \\ &+ \left. \frac{B^s_{n,m,2}}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} \psi_m^{(2)}(x) \right) \phi_m(y) \end{aligned} \tag{B.8}$$

Let $\sigma = (\sigma_x, \sigma_y)$ be a multi-index. Let $S \subset [0, L] \times [0, H]$ be arbitrary, i.e., any set completely contained in $[0, L] \times [0, H]$. Note that the series

$$\begin{aligned} &\sum_{m=1}^{\infty} \partial^\sigma \left[\left(\frac{B^s_{n,m,1}}{z_m^2 \left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} \psi_m^{(1)}(x) \right. \right. \\ + &\left. \left. \frac{B^s_{n,m,2}}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} \psi_m^{(2)}(x) \right) \phi_m(y) \right] \end{aligned} \tag{B.9}$$

converges uniformly in S if for all $(\tilde{x},\tilde{y}) \in S$ we have

$$\left| \frac{\partial^\sigma \left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)} \right|_{|(x,y)=(\tilde{x},\tilde{y})} < M_\sigma \tag{B.10}$$

and

$$\left| \frac{\partial^\sigma \left(\psi_m^{(2)}(x)\phi_m(y)\right)}{\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L)} \right|_{|(x,y)=(\tilde{x},\tilde{y})} < M_\sigma \tag{B.11}$$

for some $M_\sigma > 0$. In fact, let us assume that (B.10) and (B.11) hold for any $(\tilde{x},\tilde{y}) \in S$. Then, we have

$$\begin{aligned}
& \sum_{m=1}^{\infty} \left\{ \partial^\sigma \left[\left(\frac{B^s_{n,m,1}}{z_m^2 \left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} \psi_m^{(1)}(x) \right. \right. \right. \\
+ & \left. \left. \left. \frac{B^s_{n,m,2}}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} \psi_m^{(2)}(x) \right) \phi_m(y) \right] \right\}_{|(x,y)=(\tilde{x},\tilde{y})} \\
= & \sum_{m=1}^{\infty} \frac{B^s_{n,m,1}}{z_m^2} \left[\frac{\left[\partial^\sigma \left(\psi_m^{(1)}(x)\phi_m(y)\right)\right]_{|(x,y)=(\tilde{x},\tilde{y})}}{\left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} \right] \\
+ & \frac{B^s_{n,m,2}}{z_m^2} \left[\frac{\left[\partial^\sigma \left(\psi_m^{(2)}(x)\phi_m(y)\right)\right]_{|(x,y)=(\tilde{x},\tilde{y})}}{\left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} \right] \\
= & \sum_{m=1}^{\infty} \frac{|B^s_{n,m,1}|}{z_m^2} \left| \frac{\partial^\sigma \left(\psi_m^{(1)}(x)\phi_m(y)\right)}{\left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} \right|_{|(x,y)=(\tilde{x},\tilde{y})} \\
+ & \frac{|B^s_{n,m,2}|}{z_m^2} \left| \frac{\partial^\sigma \left(\psi_m^{(2)}(x)\phi_m(y)\right)}{\left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} \right|_{|(x,y)=(\tilde{x},\tilde{y})} \\
\leq & \sum_{m=1}^{\infty} \frac{M_{n,s}}{z_m^2} M_\sigma + \frac{M_{n,s}}{z_m^2} M_\sigma = 2 M_{n,s} M_\sigma \sum_{m=1}^{\infty} \frac{1}{z_m^2} < \infty,
\end{aligned}$$

which implies that the series in (B.9) converges uniformly in S.

Now, note that

$$
\begin{aligned}
&\partial^\sigma \sum_{m=1}^{\infty} \left(\frac{B^s_{n,m,1}}{z_m^2 \left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(1)}(0) \right)} \psi_m^{(1)}(x) \right. \\
+ &\left. \frac{B^s_{n,m,2}}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(2)}(L) \right)} \psi_m^{(2)}(x) \right) \phi_m(y) = \\
&\sum_{m=1}^{\infty} \partial^\sigma \left[\left(\frac{B^s_{n,m,1}}{z_m^2 \left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(1)}(0) \right)} \psi_m^{(1)}(x) \right. \right. \\
+ &\left. \left. \frac{B^s_{n,m,2}}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2) \psi_m^{(2)}(L) \right)} \psi_m^{(2)}(x) \right) \phi_m(y) \right]
\end{aligned}
\tag{B.12}
$$

holds in S, i.e., the order of summation and differentiation interchanges in S when a differential operator is applied to the series in (B.8), if the series on the r.h.s of the above equation converges uniformly in S.

Thus, in order to prove parts 1. and 2. of the theorem it suffices to show that

(i) For any $(\bar{x}, \bar{y}) \in [0, L] \times [0, H]$ there exists a subset $S_{\bar{x}}$ such that $\bar{x} \in S_{\bar{x}} \subseteq [0, L] \times [0, H]$ and for any $(\tilde{x}, \tilde{y}) \in S_{\bar{x}}$ conditions (B.10) and (B.11) hold for some M_σ with $\sigma = (0, 1)$, $\sigma = (1, 0)$ and $\beta \leq 0$.

(ii) For any $(\bar{x}, \bar{y}) \in [0, L] \times [0, H]$ there exists a subset $S_{\bar{x}}$ such that $\bar{x} \in S_{\bar{x}} \subseteq [0, L] \times [0, H]$ and for any $(\tilde{x}, \tilde{y}) \in S_{\bar{x}}$ conditions (B.10) and (B.11) hold for some M_σ, with $\sigma = (0, 2)$, $\sigma = (2, 0)$ and $\beta < 0$.

(iii) For any $(\bar{x}, \bar{y}) \in (0, L) \times [0, H]$ there exists a subset $S_{\bar{x}}$ such that $\bar{x} \in S_{\bar{x}} \subset (0, L) \times [0, H]$ and for any $(\tilde{x}, \tilde{y}) \in S_{\bar{x}}$ conditions (B.10) and (B.11) hold for some M_σ, with $\sigma = (0, 2)$, $\sigma = (2, 0)$ and $\beta = 0$.

To that end, let us consider $\sigma = (0, 1)$. Then, $\partial^\sigma = \frac{\partial}{\partial y}$. Also, let $\beta \leq 0$. Then, we have that

$$\left|\frac{\frac{\partial}{\partial y}\left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)}\right|_{|(x,y)=(\tilde{x},\tilde{y})} =$$

$$\frac{z_m\left|\cos\left(\frac{z_m\tilde{y}}{H}\right)\right|\left(e^{-\frac{(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}-\frac{\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)e^{\frac{(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{e^{\frac{L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)-\frac{e^{-\frac{L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)^2}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}} =$$

$$\frac{z_m e^{-\frac{\tilde{x}\sqrt{z_m^2-\bar{\omega}}}{H}}\left|\cos\left(\frac{z_m\tilde{y}}{H}\right)\right|\left(1-\frac{\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)e^{\frac{2(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\frac{\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)^2\right)} \leq$$

$$\frac{2z_m}{\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)} =$$

$$\frac{2}{\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\sqrt{1-\frac{\bar{\omega}}{z_m^2}}+\frac{\bar{\alpha}}{z_m}-\bar{\beta}z_m\right)} \leq$$

$$\frac{2}{\sqrt{1-\frac{\bar{\omega}}{z_1^2}}\left(1-e^{-2\frac{L\sqrt{z_1^2-\bar{\omega}}}{H}}\right)} := M_1 < \infty$$

for all $(\tilde{x},\tilde{y}) \in [0,L]\times[0,H]$. Also, for $\sigma=(1,0)$, i.e., for $\partial^\sigma = \frac{\partial}{\partial x}$, we have

$$\left|\frac{\frac{\partial}{\partial x}\left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)}\right|_{|(x,y)=(\tilde{x},\tilde{y})} =$$

$$\frac{\sqrt{z_m^2-\bar{\omega}}\left|\sin\left(\frac{z_m\tilde{y}}{H}\right)\right|\left(e^{-\frac{(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}+\frac{\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)e^{\frac{(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{e^{\frac{L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)-\frac{e^{-\frac{L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)^2}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}} =$$

$$\frac{e^{-\frac{\tilde{x}\sqrt{z_m^2-\bar{\omega}}}{H}}\sqrt{z_m^2-\bar{\omega}}\left|\sin\left(\frac{z_m\tilde{y}}{H}\right)\right|\left(1+\frac{\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)e^{\frac{2(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\frac{\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)^2\right)} \leq$$

$$\frac{2\sqrt{z_m^2-\bar{\omega}}}{\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)} =$$

$$\frac{2}{\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(1+\frac{\bar{\alpha}-\bar{\beta}z_m^2}{\sqrt{z_m^2-\bar{\omega}}}\right)} \leq$$

$$\frac{2}{\left(1-e^{-2\frac{L\sqrt{z_1^2-\bar{\omega}}}{H}}\right)} := M_4 < \infty,$$

for all $(\tilde{x},\tilde{y}) \in [0,L]\times[0,H]$.

In a similar way, we have for all $(\tilde{x},\tilde{y}) \in [0,L]\times[0,H]$ that

$$\left|\frac{\frac{\partial}{\partial y}\left(\psi_m^{(2)}(x)\phi_m(y)\right)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)}\right|_{|(x,y)=(\tilde{x},\tilde{y})} \leq M_1$$

and

$$\left|\frac{\frac{\partial}{\partial x}\left(\psi_m^{(2)}(x)\phi_m(y)\right)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)}\right|_{|(x,y)=(\tilde{x},\tilde{y})} \leq M_2.$$

Thus, taking $S_{\bar{x}} = [0, L] \times [0, H]$ we see that conditions (B.10) and (B.11) hold for $\sigma = (0, 1), (1, 0)$ for every $\bar{x} \in [0, L] \times [0, H]$.

Now, considering $\sigma = (0, 2)$ we obtain

$$\left| \frac{\frac{\partial^2}{\partial y^2}\left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)} \right|_{(x,y)=(\tilde{x},\tilde{y})} = \tag{B.13}$$

$$\frac{1}{H} \frac{z_m^2 \left|-\sin\left(\frac{z_m \tilde{y}}{H}\right)\right| \left(e^{-\frac{(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}} - \frac{\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right) e^{\frac{(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}} \right)}{e^{\frac{L\sqrt{z_m^2-\bar{\omega}}}{H}} \left(\bar{\alpha} - \bar{\beta} z_m^2 + \sqrt{z_m^2 - \bar{\omega}}\right) - \frac{e^{-\frac{L\sqrt{z_m^2-\bar{\omega}}}{H}} \left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)^2}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}} = \tag{B.14}$$

$$\frac{1}{H} \frac{z_m^2 e^{-\frac{\tilde{x}\sqrt{z_m^2-\bar{\omega}}}{H}} \left|\sin\left(\frac{z_m \tilde{y}}{H}\right)\right| \left(1 - \frac{\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right) e^{\frac{2(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{\left(\bar{\alpha} - \bar{\beta} z_m^2 + \sqrt{z_m^2 - \bar{\omega}}\right) \left(1 - e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}} \left(\frac{\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)^2\right)} \leq \tag{B.15}$$

$$\frac{2 z_m^2}{H\left(1 - e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\bar{\alpha} - \bar{\beta} z_m^2 + \sqrt{z_m^2 - \bar{\omega}}\right)} e^{-\frac{\tilde{x}\sqrt{z_m^2-\bar{\omega}}}{H}} = \tag{B.16}$$

$$\frac{2}{H\left(1 - e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\sqrt{1 - \frac{\bar{\omega}}{z_m^2}} + \frac{\bar{\alpha}}{z_m^2} - \bar{\beta}\right)} e^{-\frac{\tilde{x}\sqrt{z_m^2-\bar{\omega}}}{H}} . \tag{B.17}$$

Then, if $\beta \neq 0$, it follows for all $(\tilde{x}, \tilde{y}) \in [0, L] \times [0, H]$ that

$$\left| \frac{\frac{\partial^2}{\partial y^2}\left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)} \right|_{|(x,y)=(\tilde{x},\tilde{y})} \leq$$

$$\frac{2e^{-\frac{\tilde{x}\sqrt{z_m^2 - \bar{\omega}}}{H}}}{H\left(1 - e^{-\frac{2L\sqrt{z_m^2 - \bar{\omega}}}{H}}\right)\left(\frac{1}{z_m}\sqrt{1 - \frac{\bar{\omega}}{z_m^2}} + \frac{\bar{\alpha}}{z_m^2} - \bar{\beta}\right)} \leq$$

$$\frac{2}{H\left(1 - e^{-\frac{2L\sqrt{z_m^2 - \bar{\omega}}}{H}}\right)(-\bar{\beta})} := M_2 < \infty.$$

Now, if $\beta = 0$, let $\bar{x} \in (0, L) \times [0, H]$, let $a_{\bar{x}} = \bar{x}/2$, $b_{\bar{x}} = (L - \bar{x})/2$ and $S_{\bar{x}} = (a_{\bar{x}}, b_{\bar{x}}) \times [0, L] \subset (0, L) \times [0, H]$. Note that $x \in S_{\bar{x}}$ and $0 < a_{\bar{x}} < \tilde{x}$ for all $\tilde{x} \in S_{\bar{x}}$. Then, from (B.13) and for all $\tilde{x} \in S_{\bar{x}}$ we have

$$\left| \frac{\frac{\partial^2}{\partial y^2}\left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha})\psi_m^{(1)}(0)} \right|_{|(x,y)=(\tilde{x},\tilde{y})} \leq$$

$$\frac{2z_m^2}{H\left(1 - e^{-\frac{2L\sqrt{z_m^2 - \bar{\omega}}}{H}}\right)\left(\bar{\alpha} + \sqrt{z_m^2 - \bar{\omega}}\right)} e^{-\frac{\tilde{x}\sqrt{z_m^2 - \bar{\omega}}}{H}} \leq$$

$$\frac{2z_m^2}{H\left(1 - e^{-\frac{2L\sqrt{z_m^2 - \bar{\omega}}}{H}}\right)\left(\bar{\alpha} + \sqrt{z_m^2 - \bar{\omega}}\right)} \left(\left(\frac{1}{\frac{\tilde{x}}{H}\sqrt{z_m^2 - \bar{\omega}}}\right)\right) =$$

$$\frac{2}{\left(1 - e^{-\frac{2L\sqrt{z_m^2 - \bar{\omega}}}{H}}\right)\left(\frac{\bar{\alpha}}{z_m} + \sqrt{1 - \frac{\bar{\omega}}{z_m^2}}\right)} \left(\frac{1}{\tilde{x}\sqrt{1 - \frac{\bar{\omega}}{z_m^2}}}\right) \leq$$

$$\frac{2}{\left(1 - e^{-\frac{2L\sqrt{z_m^2 - \bar{\omega}}}{H}}\right)\left(\sqrt{1 - \frac{\bar{\omega}}{z_m^2}}\right)^2 \tilde{x}} \leq$$

$$\frac{2}{\left(1 - e^{-\frac{2L\sqrt{z_1^2 - \bar{\omega}}}{H}}\right)\left(1 - \frac{\bar{\omega}}{z_1^2}\right) a_{\bar{x}}} := M_3 < \infty.$$

Similarly, for $\sigma = (2,0)$ we have

$$\left|\frac{\frac{\partial^2}{\partial y^2}\left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)}\right|_{|(x,y)=(\tilde{x},\tilde{y})} = \tag{B.18}$$

$$\frac{1}{H}\frac{(z_m^2-\bar{\omega})\left|-\sin\left(\frac{z_m\tilde{y}}{H}\right)\right|\left(e^{-\frac{(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}+\frac{\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)e^{\frac{(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{e^{\frac{L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)-\frac{e^{-\frac{L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)^2}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}} \leq \tag{B.19}$$

$$\frac{1}{H}\frac{(z_m^2-\bar{\omega})e^{-\frac{\tilde{x}\sqrt{z_m^2-\bar{\omega}}}{H}}\left|\sin\left(\frac{z_m\tilde{y}}{H}\right)\right|\left(1+\frac{\left(\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}\right)e^{\frac{2(\tilde{x}-L)\sqrt{z_m^2-\bar{\omega}}}{H}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)}{\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\left(\frac{\bar{\alpha}-\bar{\beta}z_m^2-\sqrt{z_m^2-\bar{\omega}}}{\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}}\right)^2\right)} \leq \tag{B.20}$$

$$\frac{2(z_m^2-\omega)}{H\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\bar{\alpha}-\bar{\beta}z_m^2+\sqrt{z_m^2-\bar{\omega}}\right)}e^{-\frac{\tilde{x}\sqrt{z_m^2-\bar{\omega}}}{H}}. \tag{B.21}$$

Then, if $\beta < 0$ it follows that

$$\left|\frac{\frac{\partial^2}{\partial y^2}\left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)}\right|_{|(x,y)=(\tilde{x},\tilde{y})} \leq$$

$$\frac{2\left(1-\frac{\bar{\omega}}{z_m^2}\right)e^{-\frac{x\sqrt{z_m^2-\bar{\omega}}}{H}}}{H\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\frac{1}{z_m}\sqrt{1-\frac{\bar{\omega}}{z_m^2}}+\frac{\bar{\alpha}}{z_m^2}-\bar{\beta}\right)} \leq$$

$$\frac{2}{H\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)(-\bar{\beta})} := M_4 < \infty.$$

If $\beta = 0$, let $a_{\bar{x}} = x/2$, $b_{\bar{x}} = (L-x)/2$ and $S_{\bar{x}} = (a_{\bar{x}}, b_{\bar{x}}) \times [0, L] \subset (0, L) \times [0, H]$. Note that $\bar{x} \in S_{\bar{x}}$ and $0 < a_{\bar{x}} < \tilde{x}$ for all $\tilde{x} \in S_{\bar{x}}$. Then, from (B.18) and for all $\tilde{x} \in S_{\bar{x}}$

we have

$$
\left|\frac{\frac{\partial^2}{\partial y^2}\left(\psi_m^{(1)}(x)\phi_m(y)\right)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha})\psi_m^{(1)}(0)}\right|_{|(x,y)=(\tilde{x},\tilde{y})} \leq
$$

$$
\frac{2(z_m^2-\omega)}{H\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\bar{\alpha}+\sqrt{z_m^2-\bar{\omega}}\right)}e^{-\frac{x\sqrt{z_m^2-\bar{\omega}}}{H}} \leq
$$

$$
\frac{2(z_m^2-\omega)}{H\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\bar{\alpha}+\sqrt{z_m^2-\bar{\omega}}\right)}\left(\frac{1}{\frac{\tilde{x}}{H}\sqrt{z_m^2-\bar{\omega}}}\right) =
$$

$$
\frac{2}{\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\left(\frac{\bar{\alpha}}{\sqrt{z_m^2-\bar{\omega}}}+1\right)}\left(\frac{1}{\bar{x}}\right) \leq
$$

$$
\frac{2}{\left(1-e^{-\frac{2L\sqrt{z_m^2-\bar{\omega}}}{H}}\right)\tilde{x}} \leq
$$

$$
\frac{2}{\left(1-e^{-\frac{2L\sqrt{z_1^2-\bar{\omega}}}{H}}\right)a_{\tilde{x}}} := M_5 < \infty.
$$

Note that the summation sign and the integral of a series commute if the series converges absolutely. In the proof of part 1. we have shown that (B.9) converges absolutely and uniformly in $[0,L]\times[0,H]$. From this fact, part 3. of the lemma follows. □

B.3 Coefficient identities for the one-way decomposition case

Lemma B.3. *Consider the series from equations (2.7), (2.11), and (2.12). Let* u_0 *be the initial approximation of the solution of* (2.1) *and such that the initial error* η_0 *is* $C^3((0,W)\times(0,H))$*. Then, for all* $n\in\mathbb{N}$*, the coefficients* $A^s_{n,m,i}$ *can be written as in (2.14)-(2.17).*

Proof. We present the proof for the case of interior subdomains, but a similar procedure can be used to prove the series coefficients identities for the left-most and right-most subdomains cases.

The proof is by induction. Note that $\{\phi_m\}_{m\in\mathbb{N}}$ is a complete orthogonal set spanning the set of piecewise continuous functions. Let

$$g_{n,1}(y) = \left(-\frac{\partial}{\partial x} + \alpha + \beta\frac{\partial^2}{\partial y^2}\right)\eta_n^{s-1}(L-2\gamma, y) \tag{B.22}$$

and

$$g_{n,2}(y) = \left(\frac{\partial}{\partial x} + \alpha + \beta\frac{\partial^2}{\partial y^2}\right)\eta_n^{s+1}(2\gamma, y) \tag{B.23}$$

for $n \in \mathbb{N}_0$.

By hypothesis, the initial approximation u_0 is such that the initial error η_0 is $C^3\left((0, L_1)\times(0, L_2)\right)$ (e.g., $u_0 = 0$). Hence, $g_{0,1}$ and $g_{0,2}$ satisfy the hypothesis of Lemma B.1.

The series expansion of $g_{0,1}$ and $g_{0,2}$ in the basis $\{\phi_m\}_{m\in\mathbb{N}}$ is given by

$$g_{0,1}(y) := \sum_{m=1}^{\infty} D_m\phi_m(y)$$

and

$$g_{0,2}(y) := \sum_{m=1}^{\infty} E_m\phi_m(y),$$

where

$$D_m = \frac{\int_0^H g_{0,1}\phi_m dy}{\int_0^H \phi_m^2 dy}$$

and

$$E_m = \frac{\int_0^H g_{0,2}\phi_m dy}{\int_0^H \phi_m^2 dy}\ .$$

By Lemma B.1, we have that $D_m = \bar{D}_m/z_m^2$, $E_m = \bar{E}_m/z_m^2$, and $\bar{D}_m < M_D$ and $\bar{E}_m < M_E$ for all $m \in \mathbb{N}$ and some $M_E, M_D > 0$.

Let $B^s_{1,m,1} := \bar{D}_m$ and $B^s_{1,m,2} := \bar{E}_m$ Let

$$\tilde{A}^s_{1,m,1} := \frac{B^s_{1,m,1}}{z_m^2\left(\frac{-d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)\right)}. \tag{B.24}$$

and

$$\tilde{A}_{1,m,2} := \frac{B^s_{1,m,2}}{z_m^2\left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L)\right)} \tag{B.25}$$

Now, let $\upsilon : \mathbb{R}^2 \to \mathbb{R}$ be such that

$$\upsilon(x,y) := \sum_{m=1}^{\infty} \left(\tilde{A}^s_{1,m,1}\psi_m^{(1)} + \tilde{A}^s_{1,m,2}\psi_m^{(1)}\right)\phi_m(y). \tag{B.26}$$

Then,

$$\upsilon(x,y) = \sum_{m=1}^{\infty}\left(\frac{\bar{D}_m\psi_m^{(1)}(x)}{z_m^2\left(\frac{-d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)\right)} + \frac{\bar{E}_m\psi_m^{(2)}(x)}{z_m^2\left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L)\right)}\right)\phi_m(y). \tag{B.27}$$

By Lemma B.2 we have that

- If $\beta > 0$, the first and second derivatives of the series given in (2.7), (2.11), and (2.12) are such that the order of the summation and differentiation operations can be interchanged in $[0, L] \times [0, H]$.

- If $\beta = 0$, the first and second derivatives of the series given in (2.7), (2.11), and (2.12) are such that the order of the summation and the second derivatives interchange in $(0, L)\times(0, H)$, and the summation and first derivatives interchange in $[0, L] \times [0, H]$.

Based on this fact and since $(\Delta + \omega)(\psi_m^{(1)}(x)\phi_m) = 0$ and $(\Delta + \omega)(\psi_m^{(2)}(x)\phi_m) = 0$ in

$(0, L) \times (0, H)$, we have

$$(\Delta + \omega)v(x, y) = \sum_{m=1}^{\infty} \frac{\bar{D}_m}{z_m^2 \left(\frac{-d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} (\Delta + \omega)(\psi_m^{(1)}(x)\phi_m(y))$$
$$+ \frac{\bar{E}_m}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} (\Delta + \omega)(\psi_m^{(2)}(x)\phi_m(y)) = 0$$

in $(0, L) \times (0, H)$. Also,

$$\left(-\frac{\partial}{\partial x} v + \alpha v + \beta \frac{\partial^2}{\partial y^2} v \right)(0, y) =$$
$$\sum_{m=1}^{\infty} \left\{ \left(-\frac{\partial}{\partial x} + \alpha + \beta \frac{\partial^2}{\partial y^2} \right) \left[\left(\frac{B_{1,m,1}^s}{z_m^2 \left(\frac{-d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} \psi_m^{(1)}(x) \right. \right. \right. +$$
$$\left. \left. \left. \frac{B_{1,m,2}^s}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} \psi_m^{(2)}(x) \right) \phi_m(y) \right] \right\}_{|x=0} =$$
$$\sum_{m=1}^{\infty} \left(\frac{B_{1,m,1}^s}{z_m^2 \left(\frac{-d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right)} \left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0) \right) \right.$$
$$\left. + \frac{B_{1,m,2}^s}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L) \right)} \left(-\frac{d\psi_m^{(2)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(0) \right) \right) \phi_m(y) =$$
$$\sum_{m=1}^{\infty} \frac{B_{1,m,1}^s}{z_m^2} \phi_m(y) = \sum_{m=1}^{\infty} \frac{\bar{D}_m}{z_m^2} \phi_m(y) = \sum_{m=1}^{\infty} D_m \phi_m(y) = g_{0,1}(y) = \left(-\frac{\partial}{\partial x} + \alpha + \beta \frac{\partial^2}{\partial y^2} \right) \eta_n^{s-1}(L - 2\gamma, y),$$

where on the r.h.s of the third inequality we used the fact that $\left(-\frac{d\psi_m^{(2)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(0) \right) = 0$.

Similarly, we have

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x}\upsilon+\alpha\upsilon+\beta\frac{\partial^2}{\partial y^2}\upsilon\right)(L,y)= \\
&\sum_{m=1}^{\infty}\left\{\left(\frac{\partial}{\partial x}+\alpha+\beta\frac{\partial^2}{\partial y^2}\right)\left[\left(\frac{B^s_{1,m,1}}{z_m^2\left(\frac{-d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)\right)}\psi_m^{(1)}(x)\ +\right.\right.\right. \\
&\left.\left.\left.\frac{B^s_{1,m,2}}{z_m^2\left(\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)\right)}\psi_m^{(2)}(x)\right)\phi_m(y)\right]\right\}_{|x=L}= \\
&\sum_{m=1}^{\infty}\left(\frac{B^s_{1,m,1}}{z_m^2\left(\frac{-d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)\right)}\left(\frac{d\psi_m^{(1)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(L)\right)\right. \\
&+\left.\frac{B^s_{1,m,2}}{z_m^2\left(\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)\right)}\left(\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)\right)\right)\phi_m(y)= \\
&\sum_{m=1}^{\infty}\frac{B^s_{1,m,2}}{z_m^2}\phi_m(y)=\sum_{m=1}^{\infty}\frac{\bar{E}_m}{z_m^2}\phi_m(y)=\sum_{m=1}^{\infty}E_m\phi_m(y)=g_{0,2}(y)=\left(\frac{\partial}{\partial x}+\alpha+\beta\frac{\partial^2}{\partial y^2}\right)\eta_n^{s+1}(2\gamma,y),
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\upsilon(x,0) &= \sum_{m=1}^{\infty}\left(\frac{B^s_{1,m,1}}{z_m^2\left(\frac{-d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)\right)}\psi_m^{(1)}(x)\right. \\
&+\left.\frac{B^s_{1,m,2}}{z_m^2\left(\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)\right)}\psi_m^{(2)}(x)\right)\phi_m(0)=0
\end{aligned}
$$

and

$$
\begin{aligned}
\upsilon(x,H) &= \sum_{m=1}^{\infty}\left(\frac{B^s_{1,m,1}}{z_m^2\left(\frac{-d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)\right)}\psi_m^{(1)}(x)\right. \\
&+\left.\frac{B^s_{1,m,2}}{z_m^2\left(\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)\right)}\psi_m^{(2)}(x)\right)\phi_m(H)=0,
\end{aligned}
$$

since $\phi_m(0)=\phi_m(H)=0$.

Thus, υ solves (3.2) and consequently $\eta_1^s=\upsilon$. Then, comparing (B.26) and (2.7), we see that

$$A^s_{1,m,1} = \tilde{A}^s_{1,m,1} = \frac{B^s_{1,m,1}}{z_m^2\left(\frac{-d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta}z_m^2)\psi_m^{(1)}(0)\right)},$$

and

$$A^s_{1,m,2} = \tilde{A}^s_{1,m,2} = \frac{B^s_{1,m,2}}{z_m^2\left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta}z_m^2)\psi_m^{(2)}(L)\right)}.$$

Thus, we just proved that (2.14) and (2.15) hold for $n = 1$. Now it remains to show that it also holds for $n > 1$. Note that we do not know what the regularity of $g_{n,1}$ and $g_{n,2}$ is or whether their derivatives are bounded at $y = 0$ and $y = H$. Therefore, we cannot use the integration by parts approach as we did for the case $n = 1$, and this is the reason why we have to prove the theorem by induction. Thus, given that we already proved that (2.14) and (2.15) hold for $n = 1$, it remains to show that if these coefficient identities hold for the step n then they hold for the step n+1. To that end, let us define $E^s_{n,m,1}$ and $E^s_{n,m,2}$ as the coefficients of the expansion of $g_{n,1}(y)$ and $g_{n,2}(y)$, respectively, in the basis given by $\{\phi_m\}_{m\in\mathbb{N}}$. Thus,

$$g_{n,1}(y) = \sum_{m=1}^{\infty} E^s_{n,m,1}\phi_m(y)$$

and

$$g_{n,2}(y) = \sum_{m=1}^{\infty} E^s_{n,m,2}\phi_m(y),$$

with

$$E^s_{n,m,1} = \frac{\int_0^H g_{0,1}\phi_m dy}{\int_0^H \phi_m^2 dy} \tag{B.28}$$

and

$$E^s_{n,m,2} = \frac{\int_0^H g_{0,2}\phi_m dy}{\int_0^H \phi_m^2 dy}. \tag{B.29}$$

Let us define

$$B^s_{n+1,m,1} := \frac{z_m^2 E^s_{n,m,1}}{\left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)\right)} \tag{B.30}$$

and

$$B^s_{n+1,m,2} := \frac{E^s_{n,m,2}}{\left(\frac{d\psi^{(2)m}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L)\right)} . \tag{B.31}$$

We shall see that these are indeed the coefficients we need in (2.14) and (2.15). We then write

$$E^s_{n,m,1} = \frac{B^s_{n,m,1}}{z_m^2}\left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)\right) \tag{B.32}$$

and

$$E^s_{n,m,2} = \frac{B^s_{n,m,2}}{z_m^2}\left(\frac{d\psi^{(2)m}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L)\right). \tag{B.33}$$

By the inductive hypothesis we have that η_n^s and is given by the series in (2.7)-(2.12) with the coefficients $A^s_{n,m,i}$ given by

$$A^{(s)}_{n,m,1} = \frac{B^s_{n,m,1}}{z_m^2\left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)\right)}$$

and

$$A^{(s)}_{n,m,2} = \frac{B^s_{n,m,2}}{z_m^2\left(\frac{d\psi^{(2)m}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L)\right)} .$$

Thus,

$$\begin{aligned}\eta^{s-1}_{n,1}(x,y) &= \sum_{m=1}^{\infty}\left(\frac{B^{s-1}_{n,m,1}}{z_m^2\left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)\right)}\psi_m^{(1)}(x)\right.\\ &+ \left.\frac{B^{s-1}_{n,m,2}}{z_m^2\left(\frac{d\psi^{(2)m}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L)\right)}\psi_m^{(2)}\right)\phi_m(y),\end{aligned} \tag{B.34}$$

and

$$
\begin{aligned}
\eta_{n,1}^{s+1}(x,y) &= \sum_{m=1}^{\infty}\left(\frac{B_{n,m,1}^{s+1}}{z_m^2\left(-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)\right)}\psi_m^{(1)}(x)\right. \\
&+ \left.\frac{B_{n,m,2}^{s+1}}{z_m^2\left(\frac{d\psi^{(2)}m}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)\right)}\psi_m^{(2)}\right)\phi_m(y). \qquad \text{(B.35)}
\end{aligned}
$$

Multiplying both sides of (B.22) by $\phi_m(y)$ and integrating over $[0,H]$ we obtain

$$
\begin{aligned}
&\int_0^H g_{n,1}(y)\phi_k(y)dy = \\
&\int_0^H\left(\left(-\frac{\partial}{\partial x}+\alpha+\beta\frac{\partial^2}{\partial y^2}\right)\eta_n^{s-1}(L-2\gamma,y)\right)\phi_k(y)dy = \qquad \text{(B.36)} \\
&\sum_{m=1}^{\infty}\left(B_{n,k,1}^{s-1}\left(\frac{-\frac{d\psi_k^{(1)}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(1)}(L-2\gamma)}{-\frac{d\psi_m^{(1)}}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(1)}(0)}\right)\right. \\
+\;& \left.B_{n,k,2}^{s-1}\left(\frac{-\frac{d\psi_k^{(2)}}{dx}(L-2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi_k^{(2)}(L-2\gamma)}{\frac{d\psi_m^{(2)}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi_m^{(2)}(L)}\right)\right)\int_0^H\phi_m(y)\phi_k(y)dy.
\end{aligned}
$$

Note that we have interchanged the order of the summation, derivatives and integrals in the r.h.s of the above equation. These order interchanges are justified in Lemma B.2.

Using (B.33) and (B.36) in (B.28), evaluating the integrals and then solving for $B_{n+1,k,1}^s$ we obtain for $k\in\mathbb{N}$ such that $z_k^2<\bar{\omega}$ the following expression

$$
\begin{aligned}
B_{n+1,k,1}^s &= \frac{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(2\bar{\gamma}\sqrt{\bar{\omega}-z_k^2}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\bar{\omega}-z_k^2}\cos\left(2\bar{\gamma}\sqrt{\bar{\omega}-z_k^2}\right)}{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(\sqrt{\left(\bar{\omega}-z_k^2\right)}\bar{L}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\left(\bar{\omega}-z_k^2\right)}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)}B_{n,k,1}^{s-1} \\
&+ \frac{\left(\bar{\omega}-z_k^2+\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2\right)\sin\left(\sqrt{\bar{\omega}-z_k^2}\left(\bar{L}-2\bar{\gamma}\right)\right)}{\left(\left(\bar{\alpha}-\bar{\beta}z_k^2\right)^2-\left(\bar{\omega}-z_k^2\right)\right)\sin\left(\sqrt{\left(\bar{\omega}-z_k^2\right)}\bar{L}\right)+2\left(\bar{\alpha}-\bar{\beta}z_k^2\right)\sqrt{\left(\bar{\omega}-z_k^2\right)}\cos\left(\sqrt{\bar{\omega}-z_k^2}\bar{L}\right)}B_{n,k,2}^{s-1}.
\end{aligned}
$$

and for $k \in \mathbb{N}$ such that $z_k^2 > \bar{\omega}$ we have

$$
\begin{aligned}
B^s_{n+1,k,1} &= \left(\frac{\left(\bar{\alpha} - \bar{\beta} z_k^2 + \sqrt{z_k^2 - \bar{\omega}}\right)^2 - e^{-4\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}} \left(\bar{\alpha} - \bar{\beta} z_k^2 - \sqrt{z_k^2 - \bar{\omega}}\right)^2}{\left(\bar{\alpha} - \bar{\beta} z_k^2 + \sqrt{z_k^2 - \bar{\omega}}\right)^2 - e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}} \left(\bar{\alpha} - \bar{\beta} z_k^2 - \sqrt{z_k^2 - \bar{\omega}}\right)^2} \right) e^{-(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}} B^{s-1}_{n,k,1} \\
&+ \left(\frac{\left(e^{-2(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}} - 1\right)\left((\bar{\alpha} - \bar{\beta} z_k^2)^2 - (z_k^2 - \bar{\omega})\right)}{e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}} \left(\bar{\alpha} - \bar{\beta} z_k^2 - \sqrt{z_k^2 - \bar{\omega}}\right)^2 - \left(\bar{\alpha} - \bar{\beta} z_k^2 + \sqrt{z_k^2 - \bar{\omega}}\right)^2} \right) e^{-2\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}} B^{s-1}_{n,k,2} \cdot \qquad \text{(B.37)}
\end{aligned}
$$

Note that, since ω is fixed and $z_m \to \infty$ as $m \to \infty$, there is a number $N_{\bar{\omega}} \in \mathbb{N}$ such that $z_m^2 > \bar{\omega}$ for all $m \geq N_{\bar{\omega}}$ and $z_m^2 < \bar{\omega}$ for all $m < N_{\bar{\omega}}$. We have that

$$
\begin{aligned}
M_1 &:= \max_{k < N_{\bar{\omega}}} \left| \left(\frac{\left(\bar{\alpha} - \bar{\beta} z_k^2 + \sqrt{z_k^2 - \bar{\omega}}\right)^2 - e^{-4\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}} \left(\bar{\alpha} - \bar{\beta} z_k^2 - \sqrt{z_k^2 - \bar{\omega}}\right)^2}{\left(\bar{\alpha} - \bar{\beta} z_k^2 + \sqrt{z_k^2 - \bar{\omega}}\right)^2 - e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}} \left(\bar{\alpha} - \bar{\beta} z_k^2 - \sqrt{z_k^2 - \bar{\omega}}\right)^2} \right) e^{-(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}} B^{s-1}_{n,k,1} \right. \\
&+ \left. \left(\frac{\left(e^{-2(\bar{L}-2\bar{\gamma})\sqrt{z_k^2-\bar{\omega}}} - 1\right)\left((\bar{\alpha} - \bar{\beta} z_k^2)^2 - (z_k^2 - \bar{\omega})\right)}{e^{-2\bar{L}\sqrt{z_k^2-\bar{\omega}}} \left(\bar{\alpha} - \bar{\beta} z_k^2 - \sqrt{z_k^2 - \bar{\omega}}\right)^2 - \left(\bar{\alpha} - \bar{\beta} z_k^2 + \sqrt{z_k^2 - \bar{\omega}}\right)^2} \right) e^{-2\bar{\gamma}\sqrt{z_k^2-\bar{\omega}}} B^{s-1}_{n,k,2} \right| < \infty \cdot \qquad \text{(B.38)}
\end{aligned}
$$

From (B.37), we have for $k \geq N_{\bar{\omega}}$ that

$$
\begin{aligned}
|B^s_{n+1,k,1}| &\leq \left|B^{s-1}_{n,k,1}\right| + \frac{1}{1 - e^{-2\bar{L}\sqrt{z_1^2-\bar{\omega}}}} \left|B^{s-1}_{n,k,2}\right| \\
&\leq \left(1 + \frac{1}{1 - e^{-2\bar{L}\sqrt{z_1^2-\bar{\omega}}}}\right) M_{n,s-1} := M_2 < \infty,
\end{aligned}
$$

where we used the inductive hypothesis that $|B^{s-1}_{n,k,i}| \leq M_{s-1,n}$ for some $M_{s-1,n} > 0$ and $i = 1, 2$. Then, taking $M_{n+1,s,1} := \max\{M_1, M_2\}$ we have $|B^s_{n+1,k,1}| \leq M_{n+1,s,1}$ for all $k \in \mathbb{N}$.

Similarly, multiplying both sides of (B.23) by $\phi_m(y)$, integrating over $[0, H]$, using the resulting equation together with (B.33) in (B.29) and then solving for $B^{(s,r)}_{n+1,k,2}$ we obtain the following expression

$$B^s_{n+1,k,2} = B^{s+1}_{n,k,1}\left(\frac{\frac{d\psi^{(1)}_k}{dx}(2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi^{(1)}_k(2\gamma)}{-\frac{d\psi^{(1)}_m}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(2)}_m(0)}\right) + B^{s+1}_{n,k,2}\left(\frac{\frac{d\psi^{(2)}_k}{dx}(2\gamma)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_k^2)\psi^{(2)}_k(2\gamma)}{\frac{d\psi^{(2)}_m}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(2)}_m(L)}\right). \tag{B.39}$$

From this equation if follows that

$$|B^s_{n,k,2}| \leq M_{n+1,s,2}$$

for some $M_{n+1,s,2} > 0$ and for all $k \in \mathbb{N}$.

Then, taking $M_{n+1,s} = \max M_{n+1,s,1}, M_{n+1,s,2}$, we have that $|B^s_{n,k,i}| \leq M_{n+1,s}$ for all $k \in \mathbb{N}$ and $i = 1, 2$.

Now, let

$$\tilde{A}^s_{n+1,m,1} := \frac{B^s_{n+1,m,1}}{z_m^2\left(-\frac{d\psi^{(1)}_m}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(1)}_m(0)\right)},$$

and

$$\tilde{A}^s_{n+1,m,2} := \frac{B^s_{n+1,m,1}}{z_m^2\left(\frac{d\psi^{(2)}_m}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(2)}_m(L)\right)}.$$

Also, let

$$\bar{\upsilon}(x,y) = \sum_{m=1}^{\infty}\left(\frac{B^s_{n,m,1}}{z_m^2\left(-\frac{d\psi^{(1)}_m}{dx}(0)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(1)}_m(0)\right)}\psi^{(1)}_m(x) + \frac{B^s_{n,m,2}}{z_m^2\left(\frac{d\psi^{(2)m}}{dx}(L)+\frac{1}{H}(\bar{\alpha}-\bar{\beta}z_m^2)\psi^{(2)}_m(L)\right)}\psi^{(2)}_m(x)\right)\phi_m(y). \tag{B.40}$$

By a similar reasoning as done before for υ, we can see that $\bar{\upsilon}$ solves all the equations given by (3.2) defining η^s_{n+1}. Consequently, $\eta^s_{n+1} = \bar{\upsilon}$. Therefore, defining

$A^{s,r}_{n+1,m,i}$ as

$$A^s_{n+1,m,1} = \tilde{A}^s_{n+1,m,1} = \frac{B^s_{n+1,m,1}}{z_m^2 \left(-\frac{d\psi_m^{(1)}}{dx}(0) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(1)}(0)\right)} \ ,$$

and

$$A^s_{n+1,m,2} = \tilde{A}^s_{n+1,m,2} = \frac{B^s_{n+1,m,1}}{z_m^2 \left(\frac{d\psi_m^{(2)}}{dx}(L) + \frac{1}{H}(\bar{\alpha} - \bar{\beta} z_m^2)\psi_m^{(2)}(L)\right)} \ .$$

Thus, we just showed that if (2.14) and (2.15) hold for the step n, then they also hold for step $n+1$. Then, since (2.14) and (2.15) hold for $n = 1$, it holds for every n.

Using the same procedure it can be shown that the identities (2.16) and (2.17) hold for the left-most and right-most subdomains, respectively. □

APPENDIX C

Proofs for the case with a decomposition with cross-points

C.1 Proof of convergence of series

The solutions of (3.7) are the zeros of the function $\Phi : \mathbb{R} \to \mathbb{R}$ given by $\Phi(z) = \tan(z) - \frac{2z\bar{\alpha}}{\bar{\alpha}^2 - z^2}$. Thus, z_m is a positive zero of Φ for all $m \in \mathbb{N}$. Plotting the graph of Φ we can see that $z_m > (m-1)\pi$ for all $m \in \mathbb{N}$ (see Figure C.1). Then we have

$$
\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{z_m^2} &= \frac{1}{z_1^2} + \sum_{m=2}^{\infty} \frac{1}{z_m^2} \\
&\leq \frac{1}{z_1^2} + \sum_{m=2}^{\infty} \frac{1}{((m-1)\pi)^2} \qquad \text{(C.1)} \\
&= \frac{1}{z_1^2} + \frac{1}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2}\,. \qquad \text{(C.2)}
\end{aligned}
$$

Note that $\sum_{j=1}^{\infty} \frac{1}{j^a} < \infty$ for any $a > 1$. Then, $\sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$. Consequently, since $z_1 > 0$ and $\sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$, we have $\sum_{m=1}^{\infty} \frac{1}{z_m^2} < \infty$.

C.2 Decay of the coefficients of the series

Lemma C.1. *Let $M > 0$ and $g : [0, H] \to \mathbb{R}$ be such that $g \in C^2((0, H))$, $\frac{d^\sigma g}{dx^\sigma}(x) \leq M$ for all $x \in (0, H)$ and $\sigma = 0, 1, 2,$ and $\lim_{x\to 0} \frac{d^\sigma g}{dx^\sigma}(x)$, $\lim_{x\to H} \frac{d^\sigma g}{dx^\sigma}(x) < M$ for $\sigma = 0, 1$. Let us define C_m as the coefficients of the ex-*

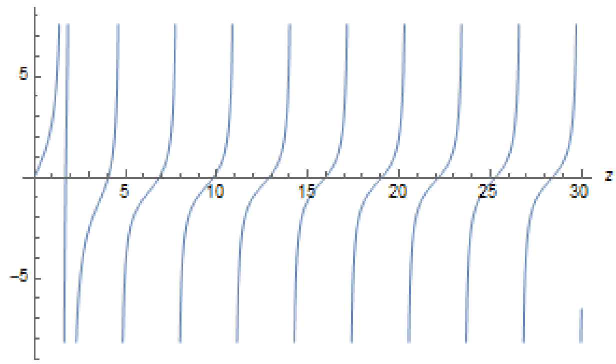

Figure C.1: Graph of Φ for $\bar{\alpha} = 2$. Note that z_m is the $m - th$ positive zero of Φ.

pansion of $g(x)$ in the basis given by $\{\phi_m\}_{m\in\mathbb{N}}$, i.e,

$$g(x) = \sum_{m=1}^{\infty} C_m \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right) \right]. \tag{C.3}$$

Then, the coefficients C_m can be written as

$$C_m = \frac{\bar{C}_m}{z_m^2}$$

where $\{\bar{C}_m\}_{m\in\mathbb{N}}$ is uniformly bounded in $m \in \mathbb{N}$.

Proof. We have that

$$C_m = \frac{\int_0^H g(x) \left(\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right) \right) dx}{\int_0^H \left(\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right) \right)^2 dx}.$$

Using integration by parts, we have

$$\int_0^H \frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) g(x) dx = \tag{C.4}$$
$$\frac{\bar{\alpha} H}{z_m^2} \left[-g(H)\cos(z_m) + g(0) + \int_0^H \frac{\bar{\alpha}}{z_m} \cos\left(\frac{z_m x}{H}\right) \frac{\partial g}{\partial x}(x) dx \right]$$

and

$$
\begin{aligned}
&\int_0^H \cos\left(\frac{z_m x}{H}\right) g(x)dx = \\
&\frac{H}{z_m}\left[g(H)\sin(z_m) - \int_0^H \sin\left(\frac{z_m x}{H}\right)\frac{\partial g}{\partial x}(x)dx\right] = \\
&\frac{H}{z_m} g(H)\sin(z_m) - \left(\frac{H}{z_m}\right)^2 \left[-\frac{\partial g}{\partial x}(H)\cos(z_m) + \frac{\partial g}{\partial x}(0)\right. \\
&\left. + \int_0^H \frac{\bar{\alpha}}{z_m}\cos\left(\frac{z_m x}{H}\right)\frac{\partial^2 g}{\partial x^2}(x)dx\right] = \\
&\frac{H}{z_m} g(H)\frac{z_m \sin(z_m)}{z_m} - \left(\frac{H}{z_m}\right)^2 \left[-\frac{\partial g}{\partial x}(H)\cos(z_m) + \frac{\partial g}{\partial x}(0)\right. \\
&\left. + \int_0^H \frac{\bar{\alpha}}{z_m}\cos\left(\frac{z_m x}{H}\right)\frac{\partial^2 g}{\partial x^2}(x)dx\right].
\end{aligned}
\tag{C.5}
$$

Using (C.4) and (C.5) we have

$$
\begin{aligned}
&\int_0^H g(x)\left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)\right] dx = \\
&\frac{\bar{\alpha} H}{z_m^2}\left[-g(H)\cos(z_m) + g(0) + \int_0^H \frac{\bar{\alpha}}{z_m}\cos\left(\frac{z_m x}{H}\right)\frac{\partial g}{\partial x}(x)dx\right] \\
&+ g(H)H\frac{z_m \sin(z_m)}{z_m^2} - \left(\frac{H}{z_m}\right)^2 \left[-\frac{\partial g}{\partial x}(H)\cos(z_m) + \frac{\partial g}{\partial x}(0)\right. \\
&\left. + \int_0^H \frac{\bar{\alpha}}{z_m}\cos\left(\frac{z_m x}{H}\right)\frac{\partial^2 g}{\partial x^2}(x)dx\right] =
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
&\frac{1}{z_m^2}\left\{\bar{\alpha} H\left[-g(H)\cos(z_m) + g(0) + \int_0^H \frac{\bar{\alpha}}{z_m}\cos\left(\frac{z_m x}{H}\right)\frac{\partial g}{\partial x}(x)dx\right]\right. \\
&+ g(H)H z_m \sin(z_m) - \left(H^2\right)\left[-\frac{\partial g}{\partial x}(H)\cos(z_m) + \frac{\partial g}{\partial x}(0)\right. \\
&\left.\left. + \int_0^H \frac{\bar{\alpha}}{z_m}\cos\left(\frac{z_m x}{H}\right)\frac{\partial^2 g}{\partial x^2}(x_l)dx\right]\right\} := \frac{1}{z_m^2} N_m.
\end{aligned}
$$

Then, using (C.6) we can write

$$C_m = \frac{\bar{C}_m}{z_m^2}, \tag{C.7}$$

where $\bar{C}_m = N_m / D_m$, with

$$D_m = \int_0^H \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right) \right]^2 dx \ .$$

We have that

$$D_m = \frac{H\left(-\bar{\alpha}^2 \sin(2z_m) + 2\bar{\alpha} z_m \left(\bar{\alpha} - \cos(2z_m) + 1\right) + z_m^2 \sin(2z_m)\right)}{4z_m^3} + \frac{H}{2} \ . \tag{C.8}$$

Note that $D_m > 0$ for all $m \in \mathbb{N}$ and that the first term in (C.8) goes to zero as m goes to infinity. Then, there exist an $\hat{m}$ such that for all $m \geq \hat{m}$ we have $D_m \geq H/4$. Let

$$\omega = \min\left\{ \min_{m \in \{1,\ldots,\hat{m}\}} D_m, \frac{H}{4} \right\} \ .$$

Then, $\omega > 0$ (since it is the minimum of a finite set of positive numbers), and for all $m \in \mathbb{N}$ we have $D_m \geq \omega$ and thus,

$$\frac{1}{D_m} \leq \frac{1}{\omega} < \infty. \tag{C.9}$$

By hypothesis we have that there exists an $M > 0$ such that $g(H^-), g(0^+)$, $\frac{dg}{dx}(H^-), \frac{dg}{dx}(0^+) < M < \infty$ and $\frac{d^2g}{dx^2} \leq M$ in $(0, H)$. Note also that $0 < z_1 < z_2 < \ldots$ Then, we have

$$|N_m| \leq \bar{\alpha}\left[2M + MH\right] H + MH + H^2 \left[2M + HM\right]. \tag{C.10}$$

Thus, from (C.9) and (C.10), we have that for all $m \in \mathbb{N}$

$$|\bar{C}_m| = \left|\frac{N_m}{D_m}\right| \leq \frac{M}{\omega}\left\{\bar{\alpha}\left(2+H\right)H + H + H^2\left(2+H\right)\right\} < \infty, \tag{C.11}$$

i.e., $\bar{C}_m$ is uniformly bounded for all $m \in \mathbb{N}$. □

C.3 Justification of order interchange between derivatives, integral and infinite summation

Lemma C.2. *The series in (3.3)-(3.6) with coefficients*

$$A^{(s,r)}_{n,m,i} = \frac{B^{(s,r)}_{n,m,i}}{z_m^2\left[\frac{\bar{\alpha}}{z_m}\sinh\left(z_m\right) + \cosh\left(z_m\right)\right]}$$

and $B^{(s,r)}_{n,m,i} \leq M_{n,s,r}/z_m^{1/2}$ *with* $M_{n,s,r} > 0$ *are such that*

1. *The order of first derivatives and summation can be interchanged in* $[0,H]^2$*,*
2. *The order of second derivatives and summation can be interchanged in* $(0,H)^2$*,*
3. *The order of the integral over* $[0,H]$*, first derivatives and summation can be interchanged.*

Proof. We present the proof for the case $i = 1$, but a similar procedure can be used for the proof in the cases $i = 2, 3, 4$.

We have that

$$\eta^{(s,r)}_{n,1}(x,y) = \sum_{m=1}^{\infty} \frac{B^{(s,r)}_{n,m,i}}{z_m^2\left[\frac{\bar{\alpha}}{z_m}\sinh\left(z_m\right) + \cosh\left(z_m\right)\right]}\phi_m(x)\psi_m(H-y).$$

Note that

$$\partial^\sigma \sum_{m=1}^{\infty} \frac{B_{n,m,i}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m} \sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y) =$$
$$\sum_{m=1}^{\infty} \partial^\sigma \left(\frac{B_{n,m,i}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m} \sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y) \right)$$

if the series in the r.h.s of the above equation converges uniformly. Thus in order to prove parts 1 and 2 of the lemma, it suffices to show that

$$\sum_{m=1}^{\infty} \partial^\sigma \left(\frac{B_{n,m,i}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m} \sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y) \right) \tag{C.12}$$

converges uniformly for $\sigma = (0,1)$, $\sigma = (1,0)$ in $[0,H]^2$ and for $\sigma = (0,1)$, $\sigma = (1,0)$ in $(0,H)^2$.

Since $B_{n,m,i}^{(s,r)} \leq M_{n,s,r}/z_m^{1/2}$, then we have that $B_{n,m,i}^{(s,r)} = C_{n,m,i}^{(s,r)}/z_m^{1/2}$ with $C_{n,m,i}^{(s,r)} \leq M_{n,s,r}$. Then

$$\eta_{n,1}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} \frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2} \left[\frac{\bar{\alpha}}{z_m} \sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y). \tag{C.13}$$

We have that

$$\sum_{m=1}^{\infty} \frac{\partial}{\partial y} \left(\frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2} \left[\frac{\bar{\alpha}}{z_m} \sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y) \right) =$$
$$\sum_{m=1}^{\infty} \frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2} \left[\frac{\bar{\alpha}}{z_m} \sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\frac{\partial \psi_m}{\partial y}(H-y).$$

Note that

$$
\begin{aligned}
\frac{\frac{\partial \psi_m}{\partial y}(H-y)}{z_m\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]} &= \frac{\frac{-\bar{\alpha}}{H}\cosh(y-H)+\frac{z_m}{H}\sinh(y-H)}{z_m\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]} \\
&= \frac{1}{H}\frac{\frac{-\bar{\alpha}}{z_m}+\tanh(y-H)}{\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1}\frac{\cosh((H-y)z_m)}{\cosh(z_m)} \\
&\leq \frac{1}{H}\left(\frac{\bar{\alpha}}{z_1}+1\right).
\end{aligned}
$$

In the last inequality we used the fact that $\cosh((H-y)z_m) \leq \cosh(z_m)$ for $y \in [0,H]$, $|\tanh(H-y)| \leq 1$ for all $y \in [0,H]$, and $\frac{1}{\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1} \leq 1$.

Then

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}\phi_m(x)\frac{\partial \psi_m}{\partial y}(H-y) \leq \\
&\sum_{m=1}^{\infty}\frac{C_{n,m,i}^{(s,r)}}{z_m^2}\frac{\frac{\partial \psi_m}{\partial y}(H-y)}{z_m\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]} \leq \\
&\frac{1}{H}\left(\frac{\bar{\alpha}}{z_1}+1\right)M_{n,s,r}\sum_{m=1}^{\infty}\frac{1}{z_m^2} < \infty.
\end{aligned}
$$

Thus the series in (C.12) converges uniformly in $[0,H]^2$ for $\sigma = (0,1)$.

Note that

$$
\frac{\partial}{\partial x}\phi_m(x) = \frac{1}{H}\left[\bar{\alpha}\cos(\frac{z_m x}{H}) - z_m\sin(\frac{z_m x}{H})\right] \leq \frac{1}{H}\left(\bar{\alpha}+z_m\right). \tag{C.14}
$$

Also $\psi_m(H-y) \leq \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]$ for y in $[0,H]$.

Then for $\sigma = (1,0)$ we have

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\frac{\partial}{\partial x}\left(\frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}\phi_m(x)\psi_m(H-y)\right) \leq \\
&\sum_{m=1}^{\infty}M_{n,s,r}\frac{1}{H}\left(\frac{\bar{\alpha}}{z_m}+1\right)\frac{1}{z_m^{\beta(m,\bar{\alpha})}} < \infty.
\end{aligned}
$$

Hence, the series in (C.12) converges uniformly in $[0, H]^2$ for $\sigma = (1, 0)$. We have that

$$\frac{\partial^2 \phi_m}{\partial x^2}(x) = \frac{1}{H^2}\left[z_m \sin(\frac{z_m x}{H}) - z_m^2 \cos(\frac{z_m x}{H})\right]. \tag{C.15}$$

Note that

$$\begin{aligned} \frac{\psi_m(H-y)}{\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} &= \frac{\frac{\bar{\alpha}}{z_m}\sinh\left(\frac{z_m(y-H)}{H}\right) + \cosh\left(\frac{z_m(y-H)}{H}\right)}{\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \\ &= \frac{\frac{\bar{\alpha}}{z_m}\tanh\left(\frac{z_m(y-H)}{H}\right) + 1}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m) + 1\right]} \frac{\cosh\left(\frac{z_m(y-H)}{H}\right)}{\cosh(z_m)}, \end{aligned} \tag{C.16}$$

and for $y \in (0, H)$

$$\begin{aligned} \frac{\cosh\left(\frac{z_m(H-y)}{H}\right)}{\cosh(z_m)} &= \frac{1 + e^{-2\left(\frac{z_m(H-y)}{H}\right)}}{1 + e^{-2z_m}} e^{z_m\frac{(H-y)}{H} - z_m} \\ &= \frac{1 + e^{-2\left(\frac{z_m(H-y)}{H}\right)}}{1 + e^{-2z_m}} e^{\frac{-z_m y}{H}} \\ &\leq 2\frac{1}{z_m^2\left(\frac{y}{H}\right)^2}. \end{aligned} \tag{C.17}$$

Then we have for $\sigma = (2, 0)$ and $0 < y < H$ that

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{\partial^2}{\partial x^2} \frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y) = \\ &\sum_{m=1}^{\infty} \frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2}} \frac{1}{H^2}\left[z_m \sin(\frac{z_m x}{H}) - z_m^2 \cos(\frac{z_m x}{H})\right] \frac{\psi_m(H-y)}{\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \leq \\ &\leq \sum_{m=1}^{\infty} \frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2}} \frac{1}{H^2}\left[z_m \sin(\frac{z_m x}{H}) - z_m^2 \cos(\frac{z_m x}{H})\right] \frac{\frac{\bar{\alpha}}{z_m}\tanh\left(\frac{z_m(y-H)}{H}\right) + 1}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m) + 1\right]} \left(2\frac{1}{z_m^2\left(\frac{y}{H}\right)^2}\right) \\ &\leq \frac{M_{n,s,r}}{y}\left(\frac{1}{z_1} + 1\right)\left(\frac{\bar{\alpha}}{z_1} + 1\right) \sum_{m=1}^{\infty} \frac{1}{z_m^{5/2}} < \infty. \end{aligned}$$

Consequently, the series in (C.12) converges uniformly in $(0, H)^2$ for $\sigma = (2, 0)$.

Finally, we analyze the case $\sigma = (0, 2)$. We have that

$$\begin{aligned}
\frac{\frac{d^2}{dy^2}(\psi_m(H-y))}{\frac{\bar{\alpha}}{H}\sinh(z_m)+\cosh(z_m)} &= \frac{z_m^2}{H^2}\frac{\psi_m(H-y)}{\frac{\bar{\alpha}}{H}\sinh(z_m)+\cosh(z_m)} \\
&\leq \frac{z_m^2}{H^2}\frac{\frac{\bar{\alpha}}{z_m}\tanh\left(\frac{z_m(y-H)}{H}\right)+1}{\left[\frac{\bar{\alpha}}{z_m}\tanh\left(z_m\right)+1\right]}\left(\frac{2}{z_m^2\left(\frac{y}{H}\right)^2}\right) \\
&\leq \frac{\frac{\bar{\alpha}}{z_m}\tanh\left(\frac{z_m(y-H)}{H}\right)+1}{\left[\frac{\bar{\alpha}}{z_m}\tanh\left(z_m\right)+1\right]}\left(\frac{2}{y^2}\right),
\end{aligned}$$

where in the second to last inequality we used (C.16) and (C.17).

Then, for $\sigma = (0, 2)$ and $0 < y < H$, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{\partial^2}{\partial y^2}\left(\frac{C_{n,m,i}^{(s,r)}\phi_m(x)\psi_m(H-y)}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh\left(z_m\right)+\cosh\left(z_m\right)\right]}\right) = \\
&\sum_{m=1}^{\infty}\frac{C_{n,m,i}^{(s,r)}}{z_m^{5/2}}\phi_m(x)\frac{\left(\frac{d^2}{dy^2}(\psi_m(H-y))\right)}{\left[\frac{\bar{\alpha}}{z_m}\sinh\left(z_m\right)+\cosh\left(z_m\right)\right]} \leq \\
&\sum_{m=1}^{\infty}\frac{M_{n,s,r}}{z_m^{5/2}}\left(\frac{\bar{\alpha}}{z_1}+1\right)\frac{\frac{\bar{\alpha}}{z_m}\tanh\left(\frac{z_m(y-H)}{H}\right)+1}{\left[\frac{\bar{\alpha}}{z_m}\tanh\left(z_m\right)+1\right]}\frac{2}{y^2} \leq \\
&M_{n,s,r}\left(\frac{\bar{\alpha}}{z_1}+1\right)^2\frac{2}{y^2}\sum_{m=1}^{\infty}z_m^{5/2} < \infty.
\end{aligned}$$

Thus, the series in (C.12) converges uniformly in $(0, H)^2$ for $\sigma = (0, 2)$, and therefore the proof is complete.

Note that the summation sign and the integral of a series commute if the series converges absolutely. In the proof of part 1. we have shown that (C.12) converges absolutely and uniformly in $[0, H]^2$. From this fact, part 3. of the lemma follows. □

C.4 Identities of the coefficients for the case of decomposition with cross-points

Lemma C.3. *Consider the series expansion of* $\eta_{n,i}^{(s,r)}$ *(the* $i-th$ *part of the local error of an interior subdomain) from equations (3.3)-(3.6),* $i = 1, 2, 3, 4$*. Let* u_0 *be the initial approximation of the solution of* (3.1) *and such that the initial error* η_0 *is* $C^3((0, L_1) \times (0, L_2))$*. Let* $S_1 := \{(s, r) : 1 < s < p, 1 < r < q\}$*,* $S_2 := \{(s, r) : 1 < s < p, r = 1\} \cup \{(s, r) : 1 < s < p, r = 1\} \cup \{(s, r) : 1 < r < q, s = 1\} \cup \{(s, r) : 1 < r < q, s = p\}$*,* $S_3 := \{(s, r) : s \in 1, p, r \in 1, q\}$*, i.e.,* S_1 *is the set of indexes for interior subdomains,* S_2 *is the set of indexes of the subdomains touching the boundaries which are not on the corners and* S_3 *are the set indexes of subdomains lying on the corners. Then, for all* $n \in \mathbb{N}$*, we have for* $(s, r) \in S_1$ *that*

$$A_{n,m,i}^{(s,r)} = \frac{B_{n,m,i}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m} \sinh(z_m) + \cosh(z_m)\right]}, \tag{3.12}$$

for $(s, r) \in S_2$ *that*

$$A_{n,m,i}^{(s,r)} = \frac{B_{n,m,i}^{(s,r)}}{z_m^2 \cosh(z_m)}, \tag{3.13}$$

and for $(s, r) \in S_3$ *that*

$$A_{n,m,i}^{(s,r)} = \frac{(\bar{\alpha} + 1/z_m) B_{n,m,i}^{(s,r)}}{z_m^2 \cosh(z_m)}, \tag{3.14}$$

where $B_{n,m,i}^{(s,r)} \leq \frac{M_{n,s,r}}{z_m^{1/2}}$ *for all* $m \in \mathbb{N}$ *and some* $M_{n,s,r} > 0$*. Also, the series (3.3)-(3.6) are uniformly convergent.*

Proof. We present the proof for the case of interior subdomains and where $i = 1$, but a similar procedure can be applied for the cases $i = 2, 3, 4$ and for subdomains touching the boundary. We prove by induction. Let us first consider the case $n = 1$.

Let

$$g_{0,1}(x) := \left(-\frac{\partial}{\partial y}\eta_0^{(s,r-1)} + \alpha\eta_0^{(s,r-1)}\right)(x, H - 2\gamma). \tag{C.18}$$

i.e., $g_{0,1}$ is the right hand side of the non-homogeneous boundary condition from the equations defining $\eta_{1,1}$. By hypothesis the initial approximation u_0 is such that the initial error η_0 is $C^3\left((0, L_1) \times (0, L_2)\right)$ (e.g., $u_0 = 0$). Then, $g_{0,1}$ satisfies the hypothesis of Lemma C.1.

Consequently, by Lemma C.1 there exist $\{C_{1,m,1}^{(s,r)}\}_{m\in\mathbb{N}}$ and $\{\bar{C}_m\}_{m\in\mathbb{N}}$ such that

$$g_{0,1}(x) = \sum_{m=1}^{\infty} C_{1,m,1}^{(s,r)} \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)\right], \tag{C.19}$$

where

$$C_{1,m,1}^{(s,r)} = \frac{\bar{C}_{1,m,1}^{(s,r)}}{z_m^2},$$

with $\left\{\bar{C}_{1,m,1}^{(s,r)}\right\}_{m\in\mathbb{N}}$ uniformly bounded in $m \in \mathbb{N}$. Let us define

$$B_{1,m,1}^{(s,r)} = \frac{\bar{C}_{1,m,1}^{(s,r)}\left[\frac{\bar{\alpha}}{z_m}\sinh\left(z_m\right) + \cosh\left(z_m\right)\right]}{\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sinh(z_m) + 2\bar{\alpha}\cosh(z_m)}.$$

Since $\bar{\alpha} > 0$, $\tanh(z_m) < 1$ for all $m \in \mathbb{N}$, $0 < z_1 < z_2 < \ldots$, and $0 < \tanh(z_1) < \tanh(z_2) < \ldots$, it follows that

$$\begin{aligned}
\frac{\left[\frac{\bar{\alpha}}{z_m}\sinh\left(z_m\right) + \cosh\left(z_m\right)\right]}{\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sinh(z_m) + 2\bar{\alpha}\cosh(z_m)} &= \frac{\frac{\bar{\alpha}}{z_m}\tanh(z_m) + 1}{z_m\left[\left(1 + \left(\frac{\bar{\alpha}}{z_m}\right)^2\right)\tanh(z_m) + \frac{2\bar{\alpha}}{z_m}\right]} \\
&\leq \frac{\frac{\bar{\alpha}}{z_m}\tanh(z_m) + 1}{z_m\tanh(z_m)} \\
&\leq \frac{1}{z_m}\left(\frac{\bar{\alpha}}{z_m} + \frac{1}{\tanh(z_m)}\right) \\
&\leq \frac{1}{z_m}\left(\frac{\bar{\alpha}}{z_1} + \frac{1}{\tanh(z_1)}\right).
\end{aligned}$$

Thus, with $\bar{M}_{1,s,r} = \bar{C}_{1,m,1}^{(s,r)}\left(\bar{\alpha}/z_1 + 1/\tanh(z_1)\right)$ and $M_{1,s,r} = \bar{M}_{1,s,r}/z_1^{1/2}$, we have

$$|B_{1,m,1}^{(s,r)}| \leq \frac{\bar{M}_{1,s,r}}{z_m} = \frac{\bar{M}_{1,s,r}}{z_m^{1/2} z_m^{1/2}} \leq \frac{\bar{M}_{1,s,r}}{z_1^{1/2} z_m^{1/2}} = \frac{M_{1,s,r}}{z_m^{1/2}}$$

for all $m \in \mathbb{N}$. Then, there exists $\{D_{1,m,1}^{(s,r)}\}_{m\in\mathbb{N}}$ such that

$$B_{1,m,1}^{(s,r)} = \frac{D_{1,m,1}^{(s,r)}}{z_m^{1/2}}$$

and $D_{1,m,1}^{(s,r)} \leq M_{1,s,r}$ for all $m \in \mathbb{N}$. Let

$$\begin{aligned} \upsilon(x,y) &= \sum_{m=1}^{\infty} \frac{B_{1,m,1}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y) \\ &= \sum_{m=1}^{\infty} \frac{D_{1,m,1}^{(s,r)}}{z_m^{5/2} \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y). \end{aligned}$$

Let $\bar{\sigma}$ be a multi-index. In Lemma C.2 it is shown that the derivative and the summation permute with each other, i.e.,

$$\partial^{\bar{\sigma}} \sum_{m=1}^{\infty} \frac{D_{1,m,1}^{(s,r)}}{z_m^{5/2} \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y) =$$
$$\sum_{m=1}^{\infty} \partial^{\bar{\sigma}} \frac{D_{1,m,1}^{(s,r)}}{z_m^{5/2} \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y)$$

in $[0,H]^2$ for $\bar{\sigma} = (1,0), \bar{\sigma} = (0,1)$ and in $(0,H)^2$ for $\bar{\sigma} = (2,0), \bar{\sigma} = (0,2)$.

Then, since $\Delta(\phi_m(x)\psi_m(H-y)) = 0$ in $(0,H)^2$ and the order of derivatives and summation interchange, we have that

$$\Delta\upsilon(x,y) = \sum_{m=1}^{\infty} \left(\frac{D_{n+1,m,i}^{(s,r)} \Delta(\phi_m(x)\psi_m(H-y))}{z_m^{5/2} \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \right) = 0.$$

Also,

$$\left(-\frac{\partial}{\partial x} + \alpha\right)\upsilon(0,y) = \sum_{m=1}^{\infty} \left(\frac{D_{n+1,m,i}^{(s,r)} \left(-\frac{d\phi_m}{dx} + \alpha\phi_m\right)(0)\psi_m(y)}{z_m^{5/2} \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \right) = 0,$$

since $\left(-\frac{d\phi_m}{dx} + \alpha\phi_m\right)(0) = 0$,

$$\left(\frac{\partial}{\partial x} + \alpha\right) \upsilon(H, y) = \sum_{m=1}^{\infty} \left(\frac{D_{n+1,m,i}^{(s,r)} \left(\frac{d\phi_m}{dx} + \alpha\phi_m\right)(H)\psi_m(y)}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \right) = 0,$$

since $\left(\frac{d\phi_m}{dx} + \alpha\phi_m\right)(H) = 0$ and

$$\left(\frac{\partial}{\partial y} + \alpha\right) \upsilon(x, H) = \sum_{m=1}^{\infty} \left(\frac{D_{n+1,m,i}^{(s,r)} \left(\frac{d\psi_m}{dy} + \alpha\phi_m\right)(0)}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}\phi_m(x) \right) = 0,$$

since $\left(\frac{d\psi_m}{dy} + \alpha\phi_m\right)(H) = 0$.

Now, note that

$$\left(-\frac{\partial}{\partial y} + \alpha\right) \upsilon(x, y) =$$
$$\sum_{m=1}^{\infty} \frac{D_{n+1,m,i}^{(s,r)} \left(\frac{d\psi_m}{dy} + \alpha\psi_m\right)(H - y)}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}\phi_m(x) =$$
$$\sum_{m=1}^{\infty} \frac{D_{n+1,m,i}^{(s,r)}\phi_m(x)\left[\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sinh\left(z_m\frac{H-y}{H}\right) + 2\bar{\alpha}\cosh\left(z_m\frac{H-y}{H}\right)\right]}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}\ .$$

Then for $y = 0$ we have

$$\begin{aligned}
\left(-\frac{\partial}{\partial y}+\alpha\right)\upsilon(x,0) &= \sum_{m=1}^{\infty}\frac{D^{(s,r)}_{n+1,m,i}\phi_m(x)\left[\left(z_m+\frac{\bar{\alpha}^2}{z_m}\right)\sinh(z_m)+2\bar{\alpha}\cosh(z_m)\right]}{z_m^{5/2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}\\
&= \sum_{m=1}^{\infty}\frac{B^{(s,r)}_{n+1,m,1}\phi_m(x)\left[\left(z_m+\frac{\bar{\alpha}^2}{z_m}\right)\sinh(z_m)+2\bar{\alpha}\cosh(z_m)\right]}{z_m^{2}\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}\\
&= \sum_{m=1}^{\infty}\frac{\bar{C}^{(s,r)}_{n+1,m,1}}{z_m^2}\phi_m(x)\\
&= \sum_{m=1}^{\infty}C^{(s,r)}_{n+1,m,1}\phi_m(x)\\
&= g(x)=\left(-\frac{\partial}{\partial y}\eta^{s,r-1}_{n-1}+\alpha\eta^{s,r-1}_{n-1}\right)(x,H-2\gamma).
\end{aligned}$$

Thus, υ solves the equations defining $\eta^{(s,r)}_{1,1}$. Consequently, $\eta^{(s,r)}_{1,1}=\upsilon$. Then, we can define $A^{(s,r)}_{1,m,1}$ as

$$A^{(s,r)}_{1,m,1}=\frac{B^{(s,r)}_{1,m,1}}{z_m^2\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}.$$

Recall that $B^{(s,r)}_{1,m,i}\le M_B/z_m^{1/2}$ for all $m\in\mathbb{N}$.

Now it remains to show that if (3.12)-(3.14) hold for the step n then they hold for the step $(n+1)$. To that end, let $g_{n,1}(x)=\left(-\frac{\partial}{\partial y}+\alpha\right)\eta^{s,r-1}_{n}(x,H-2\gamma)$, i.e., $g_{n,1}$ is the r.h.s of the non-homogeneous boundary condition from the equations defining $\eta^{s,r}_{n+1,1}$. Let us define $E^{(s,r)}_{n,m}$ as the coefficients of the expansion of $g_{n,1}(x)$ in the basis given by $\left\{\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right)+\cos\left(\frac{z_m x}{H}\right)\right\}_{m\in\mathbb{N}}$. Thus,

$$g_{n,1}(x)=\sum_{m=1}^{\infty}E^{(s,r)}_{n,m}\left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right)+\cos\left(\frac{z_m x}{H}\right)\right]$$

and

$$E^{(s,r)}_{n,m}=\frac{\int_0^H g_{n,1}(x)\left(\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right)+\cos\left(\frac{z_m x}{H}\right)\right)dx}{\int_0^H\left(\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right)+\cos\left(\frac{z_m x}{H}\right)\right)^2dx}\ . \tag{C.20}$$

Let us define

$$B_{n+1,m,1}^{(s,r)} = \frac{E_{n,m}^{(s,r)} z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}{\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sinh(z_m) + 2\bar{\alpha}\cosh(z_m)} \ . \tag{C.21}$$

We shall see that these are indeed the coefficients we need in (3.12)-(3.14). We then write

$$E_{n,m}^{(s,r)} = \frac{B_{n+1,m,1}^{(s,r)} \left[\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sinh(z_m) + 2\bar{\alpha}\cosh(z_m)\right]}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \ . \tag{C.22}$$

Following the decomposition of the error into four parts as in (3.3)-(3.6), we have that

$$g_{n,1}(x) = \sum_{i=1}^{4} \left(-\frac{\partial}{\partial y} + \alpha\right) \eta_{n,i}^{(s,r-1)}(x, H - 2\gamma). \tag{C.23}$$

By the inductive hypothesis we have that $\eta_{n,i}^{(s,r-1)}$ is given by the series in (3.3)-(3.6) with the coefficients $A_{n,m,i}^{s,r-1}$ given by

$$A_{n,m,i}^{(s,r-1)} = \frac{B_{n,m,i}^{(s,r-1)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \ ,$$

where $B_{n,m,i}^{(s,r-1)} \leq \frac{M_{n,s,r}}{z_m^{1/2}}$, i.e.,

$$\eta_{n,1}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{(s,r-1)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(H-y), \tag{C.24}$$

$$\eta_{n,2}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} \frac{B_{n,m,2}^{(s,r-1)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(y)\psi_m(x), \tag{C.25}$$

$$\eta_{n,3}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} \frac{B_{n,m,3}^{(s,r-1)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(x)\psi_m(y), \tag{C.26}$$

$$\eta_{n,4}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} \frac{B_{n,m,4}^{(s,r-1)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]} \phi_m(y)\psi_m(H-x). \tag{C.27}$$

Then, multiplying both sides of (C.23) by $\phi_m(x)$ and integrating over $[0, H]$ we obtain

$$\int_0^H g_{n,1}(x)\phi_m(\ell)dx = \tag{C.28}$$
$$\int_0^H \left(\sum_{i=1}^{4}\left(-\frac{\partial}{\partial y}+\alpha\right)\eta_{n,i}^{s,r-1}(x, H-2\gamma)\right)\phi_m(x)dx =$$
$$\sum_{m=1}^{\infty}\frac{B_{n,m,1}^{(s,r-1)}}{z_m^2\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}\int_0^H \phi_m(x)\phi_k(x)(x)dx\left(\frac{d\psi_m}{dy}+\alpha\psi_m\right)(2\gamma)$$
$$+\sum_{m=1}^{\infty}\frac{B_{n,m,1}^{(s,r-1)}}{z_m^2\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}\int_0^H \psi_m(x)\phi_k(x)dx\left(\frac{d\phi_m}{dy}+\alpha\phi_m\right)(H-2\gamma)$$
$$+\sum_{m=1}^{\infty}\frac{B_{n,m,1}^{(s,r-1)}}{z_m^2\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}\int_0^H \phi_m(x)\phi_k(x)dx\left(\frac{d\psi_m}{dy}+\alpha\psi_m\right)(H-2\gamma)$$
$$+\sum_{m=1}^{\infty}\frac{B_{n,m,1}^{(s,r-1)}}{z_m^2\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m)+\cosh(z_m)\right]}\int_0^H \psi_m(H-x)\phi_k dx\left(\frac{-d\phi_m}{dy}+\alpha\phi_m\right)(H-2\gamma).$$

Note that we have interchanged the order of the summation, derivatives and integrals in the right hand side of the above equation. These order interchanges are justified in Lemma C.2.

Let $\bar{\gamma} = \gamma/H$ be the normalized overlap. Using (C.22) and (C.28) in (C.20), evaluating the integrals and then solving for $B_{n+1,m,1}^{(s,r)}$ we obtain the following expression

$$
\begin{aligned}
B^{(s,r)}_{n+1,k,1} &= \frac{\left(z_k+\frac{\bar\alpha^2}{z_k}\right)\sinh\left(2\bar\gamma z_k\right)+2\bar\alpha\cosh\left(2\bar\gamma z_k\right)}{\left(z_k+\frac{\bar\alpha^2}{z_k}\right)\sinh\left(z_k\right)+2\bar\alpha\cosh\left(z_k\right)}B^{(s,r-1)}_{n,k,1} \qquad (C.29)\\
&+ \sum_{m=1}^{\infty}\left\{\frac{4z_k^5\left[\frac{\bar\alpha}{z_k}\tanh(z_k)+1\right]\left(z_m+\frac{\bar\alpha^2}{z_m}\right)\sin\left((1-2\bar\gamma)z_m\right)}{\left[\left(z_k+\frac{\bar\alpha^2}{z_k}\right)\tanh(z_k)+2\bar\alpha\right]z_m^2\left(z_m z_k^3+z_k z_m^3\right)}\right.\\
&\left.\frac{\left\{\tanh(z_m)\left[\bar\alpha(z_k^2+z_m^2)\sin(z_k)-z_k(\bar\alpha^2-z_m^2)\cos(z_k)\right]+z_m(\bar\alpha^2+z_k^2)\sin(z_k)\right\}}{\left[\frac{\bar\alpha}{z_m}\tanh(z_m)+1\right]\left[(z_k^2)-\bar\alpha^2\sin(2z_k)+2z_k(\bar\alpha^2+z_k^2+\bar\alpha)-2\bar\alpha z_k\cos(2z_k)\right]}B^{(s,r-1)}_{n,m,2}\right\}\\
&+ \frac{\left(-z_k+\frac{\bar\alpha^2}{z_k}\right)\sinh\left((1-2\bar\gamma)z_k\right)}{\left(z_k+\frac{\bar\alpha^2}{z_k}\right)\sinh\left(z_k\right)+2\bar\alpha\cosh\left(z_k\right)}B^{(s,r-1)}_{n,k,3}\\
&+ \sum_{m=1}^{\infty}\left\{\frac{4z_k^5\left[\frac{\bar\alpha}{z_k}\tanh(z_k)+1\right]\left(z_m+\frac{\bar\alpha^2}{z_m}\right)\sin\left((1-2\bar\gamma)z_m\right)}{\left[\left(z_k+\frac{\bar\alpha^2}{z_k}\right)\tanh(z_k)+2\bar\alpha\right]z_m^2\left(z_m z_k^3+z_k z_m^3\right)}\right.\\
&\left.\frac{\left\{\tanh(z_m)z_k(\bar\alpha^2+z_m^2)-z_m\left[-2\bar\alpha z_k+\frac{(\bar\alpha^2-z_k^2)\sin(z_k)+2\bar\alpha z_k\cos(z_k)}{\cosh(z_m)}\right]\right\}}{\left[\frac{\bar\alpha}{z_m}\tanh(z_m)+1\right]\left[(z_k^2)-\bar\alpha^2\sin(2z_k)+2z_k(\bar\alpha^2+z_k^2+\bar\alpha)-2\bar\alpha z_k\cos(2z_k)\right]}B^{(s,r-1)}_{n,m,4}\right\}.
\end{aligned}
$$

In the Result C.4 it is shown that $B^{(s,r)}_{n+1,k,1}$, defined by (C.29), satisfies the inequality $B^{(s,r)}_{n+1,k,1} \leq \frac{M_{n+1,s,r}}{z_k^{1/2}}$. Therefore, we can write $B^{s,r}_{n+1,k,1} = \frac{D^{s,r}_{n+1,k,1}}{z_k^{1/2}}$ with $D^{s,r}_{n+1,k,1} \leq M_{n+1,s,r}$. Now, let

$$
\begin{aligned}
\bar\upsilon(x,y) &= \sum_{m=1}^{\infty}\frac{B^{s,r}_{n+1,m,i}}{z_m^2\left[\frac{\bar\alpha}{z_m}\sinh\left(z_m\right)+\cosh\left(z_m\right)\right]}\phi_m(x)\psi_m(y) \qquad (C.30)\\
&= \sum_{m=1}^{\infty}\frac{D^{s,r}_{n+1,m,i}}{z_m^{5/2}\left[\frac{\bar\alpha}{z_m}\sinh\left(z_m\right)+\cosh\left(z_m\right)\right]}\phi_m(x)\psi_m(y). \qquad (C.31)
\end{aligned}
$$

By a similar reasoning as done before for υ, we can see that $\bar\upsilon$ solves all the equations defining $\eta_{n+1,1}$. Consequently, $\eta_{n+1,1} = \bar\upsilon$. Therefore, defining $A^{s,r}_{n+1,m,i}$ as

$$
A^{s,r}_{n+1,m,i} := \frac{B^{(s,r)}_{n+1,m,i}}{z_m^2\left[\frac{\bar\alpha}{z_m}\sinh\left(z_m\right)+\cosh\left(z_m\right)\right]}\ ,
$$

and using (C.30) we obtain

$$\eta_{n,1}^{(s,r)} = \bar{v}(x,y) = \sum_{m=1}^{\infty} \frac{B_{n+1,m,i}^{s,r}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}\phi_m(x)\psi_m(H-y)$$
$$= \sum_{m=1}^{\infty} A_{n,m,1}^{s,r}\phi_m(x)\psi_m(H-y).$$

Thus, we just showed for $i = 1$ that if (3.12) holds for the step n, then it also holds for step $n+1$. Then, since (3.12) holds for $n = 1$ when $i = 1$, it holds for every n. The same result can be obtained for $A_{n,m,i}^{(s,r)}$ with $i = 2, 3, 4$.

Finally, we show that each of the series of (3.3)-(3.6) with the coefficients $A_{n+1,m,i}^{(s,r)}$ given by (3.12) are uniformly convergent. We show this for $i = 1$, but the same reasoning follows for $i = 2, 3, 4$. We have that

$$\eta_{n,1}^{(s,r)}(x,y) = \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}\phi_m(x)\psi_m(H-y). \tag{C.32}$$

Since $\cosh\left(\frac{z_m(y-H)}{H}\right)$ and $\sinh\left(\frac{z_m(y-H)}{H}\right)$ are decreasing functions of y and their maximum values are attained at $y = 0$, i.e., $\cosh\left(\frac{z_m(y-H)}{H}\right) \leq \cosh(z_m)$ and $\sinh\left(\frac{z_m(y-H)}{H}\right) \leq \sinh(z_m)$, we have

$$\left|\frac{\psi_m(H-y)}{\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}\right| = \left|\frac{\left[\frac{-\bar{\alpha}}{z_m}\sinh\left(\frac{z_m(y-H)}{H}\right) + \cosh\left(\frac{z_m(y-H)}{H}\right)\right]}{\left[\frac{\bar{\alpha}}{z_m}\sinh(z_m) + \cosh(z_m)\right]}\right| \leq 1. \tag{C.33}$$

Then using (C.33), and since $|B_{n,m,1}^{(s,r)}| \leq M_{n,s,r}/z_1$ and

$$|\phi_m(x)| = |\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x}{H}\right) + \cos\left(\frac{z_m x}{H}\right)| \leq \frac{\bar{\alpha}}{z_1} + 1,$$

we have

$$\sum_{m=1}^{\infty} \frac{B_{n,m,1}^{(s,r)}}{z_m^2 \left[\frac{\bar{\alpha}}{z_m} \sinh(z_m) + \cosh(z_m)\right]} \phi_m(x) \psi_m(H-y) \leq$$
$$\frac{M_{n,s,r}}{z_1} \left(\frac{\bar{\alpha}}{z_1} + 1\right) \sum_{m=1}^{\infty} \frac{1}{z_m^2} < \infty,$$

i.e., the above series converges absolutely and uniformly. In the last inequality we used the fact that $\frac{1}{z_m^2} < \infty$, which is shown in the subsection C.1. □

C.5 Bound on the growth of the coefficients

Result C.4. The coefficients $B_{n+1,k,1}^{(s,r)}$ defined in (C.29) are such that

$$B_{n+1,k,1}^{(s,r)} \leq \frac{M_{n+1,s,r}}{z_k^{1/2}}$$

for some $M_{n+1,s,r} > 0$ and all $k \in \mathbb{N}$.

Proof. From equation (C.29) we have

$$
\begin{aligned}
B_{n+1,k,1}^{(s,r)} &= \frac{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(2\bar{\gamma}z_k\right) + 2\bar{\alpha}\cosh\left(2\bar{\gamma}z_k\right)}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(z_k\right) + 2\bar{\alpha}\cosh\left(z_k\right)} B_{n,k,1}^{(s,r-1)} \qquad \text{(C.34)} \\
&+ \sum_{m=1}^{\infty}\left\{\frac{4z_k^5\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin\left((1-2\bar{\gamma})z_m\right)}{\left[\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k) + 2\bar{\alpha}\right] z_m^2\left(z_m z_k^3 + z_k z_m^3\right)}\right. \\
&\left.\frac{\{\tanh(z_m)\left[\bar{\alpha}(z_k^2+z_m^2)\sin(z_k) - z_k(\bar{\alpha}^2 - z_m^2)\cos(z_k)\right] + z_m(\bar{\alpha}^2+z_k^2)\sin(z_k)\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2) - \bar{\alpha}^2\sin(2z_k) + 2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha}) - 2\bar{\alpha}z_k\cos(2z_k)\right]} B_{n,m,2}^{(s,r-1)}\right\} \\
&+ \frac{\left(-z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left((1-2\bar{\gamma})z_k\right)}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(z_k\right) + 2\bar{\alpha}\cosh\left(z_k\right)} B_{n,k,3}^{(s,r-1)} \\
&+ \sum_{m=1}^{\infty}\left\{\frac{4z_k^5\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin\left((1-2\bar{\gamma})z_m\right)}{\left[\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k) + 2\bar{\alpha}\right] z_m^2\left(z_m z_k^3 + z_k z_m^3\right)}\right. \\
&\left.\frac{\left\{\tanh(z_m)z_k(\bar{\alpha}^2+z_m^2) - z_m\left[-2\bar{\alpha}z_k + \frac{(\bar{\alpha}^2-z_k^2)\sin(z_k)+2\bar{\alpha}z_k\cos(z_k)}{\cosh(z_m)}\right]\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2) - \bar{\alpha}^2\sin(2z_k) + 2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha}) - 2\bar{\alpha}z_k\cos(2z_k)\right]} B_{n,m,4}^{(s,r-1)}\right\}.
\end{aligned}
$$

Let

$$
t_1 := \frac{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(2\bar{\gamma}z_k\right) + 2\bar{\alpha}\cosh\left(2\bar{\gamma}z_k\right)}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh\left(z_k\right) + 2\bar{\alpha}\cosh\left(z_k\right)}\ .
$$

Then we have

$$
t_1 = \frac{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh\left(2\bar{\gamma}z_k\right) + 2\bar{\alpha}}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh\left(z_k\right) + 2\bar{\alpha}} \cdot \frac{\cosh\left(2\bar{\gamma}z_k\right)}{\cosh\left(z_k\right)};
$$

Note that

$$
\begin{aligned}
& \frac{\cosh\left(2\bar{\gamma}z_k\right)}{\cosh\left(z_k\right)} = \frac{e^{2\bar{\gamma}z_k} + e^{-2\bar{\gamma}z_k}}{e^{z_k} + e^{-z_k}} = \frac{1+e^{-4\bar{\gamma}z_k}}{1+e^{-2z_k}}\frac{e^{2\bar{\gamma}z_k}}{e^{z_k}} = \frac{1+e^{-4\bar{\gamma}z_k}}{1+e^{-2z_k}} e^{-(1-2\bar{\gamma})z_k} \\
\leq\ & 2e^{-(1-2\bar{\gamma})z_k} \leq \frac{2}{(1-2\bar{\gamma})z_k} = \frac{2}{(1-2\bar{\gamma})z_k^{1/2}z_k^{1/2}} \\
\leq\ & \frac{2}{(1-2\bar{\gamma})z_1^{1/2}z_k^{1/2}} := \frac{C_1}{z_k^{1/2}}.
\end{aligned}
$$

Also, since $\tanh(2\bar{\gamma}z_k) < \tanh(z_k)$ for $2\bar{\gamma} < 1$, we have

$$\frac{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh(2\bar{\gamma}z_k) + 2\bar{\alpha}}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k) + 2\bar{\alpha}} \leq 1 \cdot$$

Then, $|t_1| \leq C_1/z_k^{1/2}$ for all $k \in \mathbb{N}$.

Similarly, letting

$$t_3 := \frac{\left(-z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh((1-2\bar{\gamma})z_k)}{\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\sinh(z_k) + 2\bar{\alpha}\cosh(z_k)},$$

we have that $|t_3| \leq C_3/z_k^{1/2}$ for some $C_3 > 0$ and all $k \in \mathbb{N}$.

Now, let

$$\begin{aligned} t_{2,k} \quad := \quad & \frac{4z_k^5\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin((1-2\bar{\gamma})z_m)}{\left[\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k) + 2\bar{\alpha}\right] z_m^2\left(z_m z_k^3 + z_k z_m^3\right)} \times \\ & \frac{\left\{\tanh(z_m)\left[\bar{\alpha}(z_k^2+z_m^2)\sin(z_k) - z_k(\bar{\alpha}^2 - z_m^2)\cos(z_k)\right] + z_m(\bar{\alpha}^2 + z_k^2)\sin(z_k)\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2) - \bar{\alpha}^2\sin(2z_k) + 2z_k(\bar{\alpha}^2 + z_k^2 + \bar{\alpha}) - 2\bar{\alpha}z_k\cos(2z_k)\right]} \,. \end{aligned}$$

Then,

$$\begin{aligned} & \sum_{m=1}^{\infty}\left\{\frac{4z_k^5\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin((1-2\bar{\gamma})z_m)}{\left[\left(z_k + \frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k) + 2\bar{\alpha}\right] z_m^2\left(z_m z_k^3 + z_k z_m^3\right)}\right. \\ & \left.\frac{\left\{\tanh(z_m)\left[\bar{\alpha}(z_k^2+z_m^2)\sin(z_k) - z_k(\bar{\alpha}^2 - z_m^2)\cos(z_k)\right] + z_m(\bar{\alpha}^2 + z_k^2)\sin(z_k)\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2) - \bar{\alpha}^2\sin(2z_k) + 2z_k(\bar{\alpha}^2 + z_k^2 + \bar{\alpha}) - 2\bar{\alpha}z_k\cos(2z_k)\right]} B_{n,m,2}^{(s,r-1)}\right\} \\ & \leq \sum_{m=1}^{\infty}|t_{2,k}||B_{n,m,2}^{(s,r-1)}| \leq \sum_{m=1}^{\infty}|t_{2,k}|\frac{M_{n,s,r}}{z_m^{1/2}} = M_{n,s,r}\sum_{m=1}^{\infty}|t_{2,k}|\frac{1}{z_m^{1/2}}. \end{aligned}$$

Similarly, we define

$$t_{4,k} := \frac{4z_k^5 \left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m+\frac{\bar{\alpha}^2}{z_m}\right)\sin((1-2\bar{\gamma})z_m)}{\left[\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k)+2\bar{\alpha}\right] z_m^2\left(z_m z_k^3 + z_k z_m^3\right)}$$
$$\frac{\left\{\tanh(z_m)z_k(\bar{\alpha}^2+z_m^2) - z_m\left[-2\bar{\alpha}z_k + \frac{(\bar{\alpha}^2-z_k^2)\sin(z_k)+2\bar{\alpha}z_k\cos(z_k)}{\cosh(z_m)}\right]\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2)-\bar{\alpha}^2\sin(2z_k)+2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha})-2\bar{\alpha}z_k\cos(2z_k)\right]}$$

and we have

$$\sum_{m=1}^{\infty}\left\{\frac{4z_k^5 \left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m+\frac{\bar{\alpha}^2}{z_m}\right)\sin((1-2\bar{\gamma})z_m)}{\left[\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k)+2\bar{\alpha}\right] z_m^2\left(z_m z_k^3 + z_k z_m^3\right)}\right.$$
$$\left.\frac{\left\{\tanh(z_m)z_k(\bar{\alpha}^2+z_m^2) - z_m\left[-2\bar{\alpha}z_k + \frac{(\bar{\alpha}^2-z_k^2)\sin(z_k)+2\bar{\alpha}z_k\cos(z_k)}{\cosh(z_m)}\right]\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2)-\bar{\alpha}^2\sin(2z_k)+2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha})-2\bar{\alpha}z_k\cos(2z_k)\right]} B_{n,m,4}^{(s,r-1)}\right\}$$
$$\leq \sum_{m=1}^{\infty}|t_{4,k}||B_{n,m,4}^{(s,r-1)}| \leq \sum_{m=1}^{\infty}|t_{4,k}|\frac{M_{n,s,r}}{z_m^{1/2}} = M_{n,s,r}\sum_{m=1}^{\infty}|t_{4,k}|\frac{1}{z_m^{1/2}} \ .$$

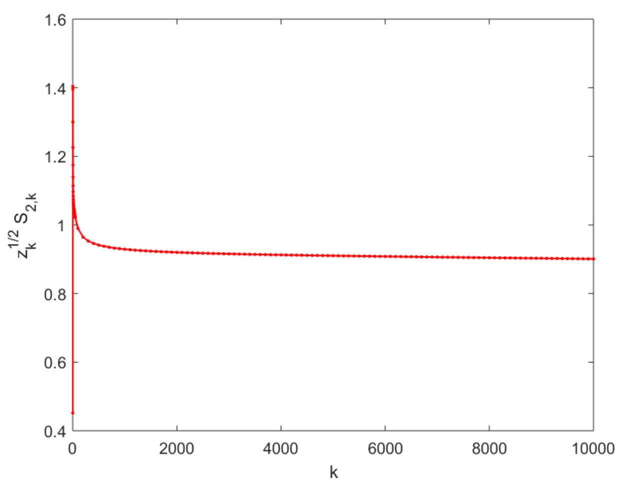

Figure C.2: $z_k^{1/2}S_{2,k}$ vs. k for $\bar{\alpha} = 0.7$ and $\bar{\gamma} = 0.01$

Let $S_{2,k} = \sum_{m=1}^{\infty}|t_{2,k}|\frac{1}{z_m^{1/2}}$ and $S_{4,k} = \sum_{m=1}^{\infty}|t_{4,k}|\frac{1}{z_m^{1/2}}$. In Figures C.2 and C.3, plots of $z_k^{1/2}S_{2,k}$ vs. k and $z_k^{1/2}S_{4,k}$ vs. k, respectively, are presented for fixed values of $\bar{\alpha}$ and $\bar{\gamma}$. For any value of $\bar{\alpha}$ and $\bar{\gamma}$ the graphs $z_k^{1/2}S_{2,k}$ vs. k and $z_k^{1/2}S_{4,k}$ vs. k look qualita-

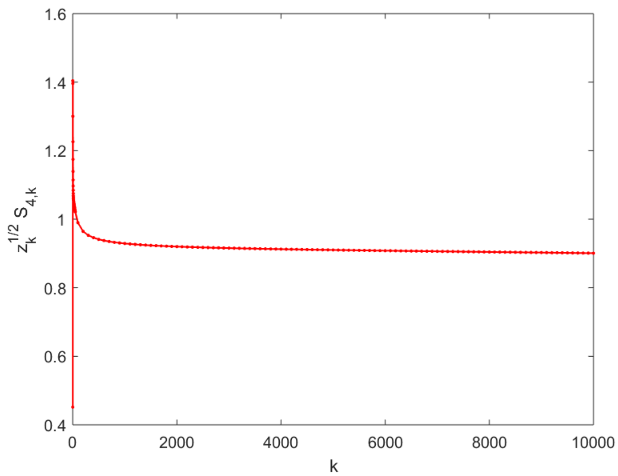

Figure C.3: $z_k^{1/2} S_{4,k}$ vs. k for $\bar{\alpha} = 0.7$ and $\bar{\gamma} = 0.01$

tively the same as in those figures. In the plots we observe that $z_k S_{2,k} = z_k S_{4,k} \le C$ for some $C = C(\bar{\alpha}, \bar{\gamma}) > 0$. This implies that $|S_{2,k}|, |S_{4,k}| \le C/z_k^{1/2}$. Consequently, we have

$$\sum_{m=1}^{\infty} \left\{ \frac{4z_k^5 \left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin\left((1-2\bar{\gamma})z_m\right)}{\left[\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k)+2\bar{\alpha}\right] z_m^2 \left(z_m z_k^3 + z_k z_m^3\right)} \right.$$
$$\left. \frac{\{\tanh(z_m)\left[\bar{\alpha}(z_k^2+z_m^2)\sin(z_k) - z_k(\bar{\alpha}^2 - z_m^2)\cos(z_k)\right] + z_m(\bar{\alpha}^2+z_k^2)\sin(z_k)\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2) - \bar{\alpha}^2\sin(2z_k) + 2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha}) - 2\bar{\alpha}z_k\cos(2z_k)\right]} B_{n,m,2}^{(s,r-1)} \right\}$$
$$\le \frac{C}{z_k^{1/2}}$$

and

$$\sum_{m=1}^{\infty} \left\{ \frac{4z_k^5 \left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m + \frac{\bar{\alpha}^2}{z_m}\right)\sin\left((1-2\bar{\gamma})z_m\right)}{\left[\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k)+2\bar{\alpha}\right] z_m^2 \left(z_m z_k^3 + z_k z_m^3\right)} \right.$$
$$\left. \frac{\left\{\tanh(z_m) z_k(\bar{\alpha}^2+z_m^2) - z_m\left[-2\bar{\alpha}z_k + \frac{(\bar{\alpha}^2-z_k^2)\sin(z_k)+2\bar{\alpha}z_k\cos(z_k)}{\cosh(z_m)}\right]\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2) - \bar{\alpha}^2\sin(2z_k) + 2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha}) - 2\bar{\alpha}z_k\cos(2z_k)\right]} B_{n,m,4}^{(s,r-1)} \right\}$$
$$\le \frac{C}{z_k^{1/2}}$$

for all $k \in \mathbb{N}$.

Using the results from above we have that

$$\begin{aligned} |B^{(s,r)}_{n+1,k,1}| &\leq |t_1||B^{(s,r-1)}_{n,k,1}| + \sum_{m=1}^{\infty} |t_{2,k}||B^{(s,r-1)}_{n,m,2}| + |t_3| + \sum_{m=1}^{\infty} |t_{4,k}||B^{(s,r-1)}_{n,m,4}| \\ &\leq \frac{(C_1 + C + C_3 + C)M_{n,s,r}}{z_k^{1/2}}. \end{aligned}$$

Therefore, with $M_{n+1,s,r} := (C_1 + C + C_3 + C)M_{n,s,r}$, we have that $|B^{(s,r)}_{n+1,k,1}| \leq \frac{M_{n+1,s,r}}{z_k^{1/2}}$. □