

ENGLISH LANGUAGE LEARNERS LEARN FROM
WORKED EXAMPLE COMPARISON
IN ALGEBRA

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ABSTRACT

This project is aimed at generating new knowledge and improving our understanding of how Modified for Language Support-Worked Example Pairs (MLS-WEPs) contribute to effective mathematics learning and teaching in an ESOL (English to Speakers of Other Languages) context. The current study investigated a novel instructional approach to help English Language Learners (ELLs) develop better understanding in mathematical reasoning, problem solving, and literacy skills (listening, reading, writing, and speaking) while they are still developing their English language proficiency. The current study followed a wait-list control design, with both the treatment and control groups receiving intervention materials. The intervention materials were administered multiple times with different topics (units) throughout the study. The lessons were audio-recorded when the selected topics were taught. Pretest and posttest were given each time when the selected topics were taught. The data analysis for this study included both qualitative and quantitative analyses. The present study revealed the following results: (1) MLS-WEPs not only enhanced ELLs' ability to solve mathematical problems, but also improved their written explanation skills and enabled them to transfer such skills to different mathematical concepts; (2) when controlling ELLs' prior knowledge, the effectiveness of the MLS-WEPs intervention did not vary by their English language proficiency; (3) the MLS-WEPs intervention materials facilitated teachers to provide ELLs with more opportunities to read, write, and speak in mathematics and enabled teachers to ask more and deeper questions. However, worked example comparisons did not appear to motivate the participant teachers to promote

equitable participation in mathematics classrooms. These findings provide direct empirical support for the need to reform mathematics teaching and learning in the ESOL context.

Keywords: worked example pair, ELL's mathematical learning, mathematics literacy, working memory (WM), cognitive load theory (CLT), classroom discourse, equitable participation, mathematical problem solving, mathematical explanation skills

To Jayden and Jaymie

It was you who gave me the motivation to follow my dreams.

I hope mama's PhD journey will inspire

and motivate you as well.

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CHAPTER 1

INTRODUCTION

According to the National Center for Education Statistics (NCES), 9.6% of the United States public school students, about 4.9 million, are English Language Learners (ELLs). An ELL is defined as any student whose native language is not English in a U.S. school setting (Kersaint, Petkova & Thompson, 2008). ELLs have been a historically underrepresented group in STEM (Science, Technology, Engineering, and Math) (The National Clearinghouse for English Language Acquisition and Language Instruction Educational Programs, 2010). Moreover, there is a gap between the performance of ELLs and native English-speaking learners that increases as the language load increases (Abedi, Leon, & Mirocha, 2005). More importantly, Abedi et al. (2005) found that a student's language proficiency and their content-based assessment performance are positively correlated. Therefore, teachers of ELLs should implement instructional and learning tools that help ELLs perform well on content-based assessments while they are still developing their language proficiency. Specifically, the current study examined whether using the supports of Modified for Language Support-Worked Example Pairs (MLS-WEPs) could help ELLs improve mathematical performance.

English language learners from diverse backgrounds join U.S. classrooms with their own cultures and strengths, but because their English language proficiency is still developing, they face the dual challenges of learning mathematics and language (Barwell, 2020). English language learners must work hard not only to master the mathematical

content, but also to overcome language limitations. However, with appropriate instructions, ELLs generally do well in mathematics, sometimes exceeding the performance of monolingual students (Barwell et al., 2017). Therefore, it is crucial to investigate instructional methods that enhance ELLs' English language proficiency and content knowledge construction.

In the current study, I examined whether MLS-WEPs, a supplemental curriculum designed to strengthen students' mathematical literacy (i.e., listening, reading, writing, and speaking) and problem-solving skills, enhance ELLs' mathematical performance and allow teachers to better ask referential questions to promote ELLs' classroom participation. The purpose of this study is therefore to compare typical discussion-based instructional practices that occur in different classroom settings with and without the use of MLS-WEPs, and whether and how these practices can enhance ELLs' mathematical performance, scaffold ELLs' mathematical discussion, and support equitable discussions of mathematics in a second language setting.

Statement of the Problem

Mathematics is considered as a “universal language” by many educators because they assume that mathematics is about numbers, not language (Brown et al., 2009). They believe as long as students have prior mathematical background, they will understand math concepts in second languages. However, in reality, mathematics requires more language proficiency than we would expect. As early as the 1990s, various researchers found that language components heavily affect mathematical learning and teaching (Ellerton & Clarkson, 1996; Clarkson & Galbraith, 1992; Jarrett, 1999). Moreover,

research has uncovered that mathematical learning becomes more difficult when mathematics is learned in a second language context (Heng & Tang, 2003). One of the explanations of such an issue is the cognitive load theory (CLT), which refers to the amount of working memory (WM) used as resources during learning and problem solving (Sweller, 1988, 1989). However, working memory can handle only a very limited number (possibly less than four) of novel interacting elements, which is far less than the number of interacting elements that requires in most human intellectual activities (Paas, Renkl, and Sweller, 2004). While ELLs engage in mathematical learning, they use their working memory not only to handle mathematical content difficulties but also to deal with language challenges. For instance, ELLs often translate their educational materials from English to their native languages (Kazima, 2007), which adds stress to their working memory system. Thus, learning mathematics in a foreign language is more likely to result in a heavier cognitive load on learners.

WM and long-term memory (LTM) work together to complete the human learning process (Paas & Ayres, 2014). WM involves the learning of novel tasks. The capacity of WM is limited to processing 4 ± 1 elements of information at any given time. When learning mathematics in a second language context, the information process is more likely to be overloaded. That is because ELLs not only struggle with registering information in a second language but also with mathematical content comprehension. Besides the constraint in capacity, WM is also restricted in the duration of storing information. It has been discovered that information can only be retained in WM for about 30s (Paas & Ayres, 2014). In sum WM is for short term and temporary memory

storage. Therefore, implementing an instructional method that reduces ELLs' cognitive load in WM is imperative.

The capacity and duration limitations of WM are removed when a learner deals with familiar tasks (Paas & Ayres, 2014). In this case, the familiar information is stored in LTM, which facilitates long term and permanent memory storage. Novel information makes its way to LTM by schema construction and automation processes, which reduces the load on WM. Schema is defined as “a structure which allows problem solvers to recognize a problem state as belonging to a particular category of problem states that normally require particular moves” (Sweller, 1988, pp. 259). Cognitive schemas can be constructed through integrating or grouping multiple elements of information into a single element with a particular function (Paas & Ayres, 2014). A schema cannot become automated until it meets two requirements: a long period of learning time and incorporation with a huge amount of related information. According to Sweller et al. (2011), a schema that contains multiple elements can be used unconsciously as an element in WM, after automation. To illustrate how cognitive schemas are constructed, I use how young children learn to comprehend the sentence “These are apples.” as an example. In this example, for novel learners, WM would register the message chunk as 14 elements (T, h, e, s, e, a, r, e, a, p, p, l, e, and s), which would overload WM. However, expert learners would process it as 3 schemas (“These”, “are”, and “apples”), which would reduce cognitive load. Hence, the limitations on WM will be removed if there are adequate schemas that are stored in LTM. As Sweller (1988) mentioned, when solving math problems, experts are able to quickly select appropriate equations leading to

the goal because they recognize each problem and each problem state from prior experience and realize which moves are appropriate. Without adequate schemas, problem solvers are not able to recognize each problem and its state and take the appropriate actions. Therefore, schema plays an important role in the problem-solving approaches and skills of experts. If students are not able to construct appropriate schemas, they will probably experience WM overload. As students establish steady schemas (i.e., background knowledge), this interference of cognitive load reduces or disappears and they are more likely to be able to select and modify the useful information. As a result, learners are more likely to achieve the desired goal.

When students learn mathematics in a language that they are not proficient in, there is an increased probability that they will experience a heavier cognitive load due to having to use more of their working memory to decipher the different parts of their non-native language. Paas et al. (2004) argued that interactive materials can be comprehended only if all of the elements and their interactions are processed simultaneously. For ELLs, in addition to using working memory to master mathematical concepts, they must sometimes use their working memory to translate or code-switch between L1 (native language) and L2 (second language). In such cases, compared to their native English-speaking peers, ELLs need to process more novel information at any given time, which takes up more space in the already limited capacity of working memory. As a result, ELLs are more likely to experience information overload, which leads to difficulty in learning mathematics. ELLs may have cognitive schemas available to solve complex math problems but language limitations may prevent them from retrieving the appropriate

cues. For example, when an ELL is asked to find the Least Common Multiple of 3 and 5, they may know that the smallest number divisible by both 3 and 5 is 15. However, they may struggle with what is meant by Least Common Multiple.

Cognitive load theory states that “effective instructional material facilitates learning by directing cognitive resources toward activities that are relevant to learning rather than toward preliminaries to learning” (Chandler & Sweller, 1991, p. 293). It aims at optimizing the learning of complex novel tasks by efficiently using the relation between the limited WM and unlimited LTM (Paas & Ayres, 2014). Sweller (1988, 1989, 1991) claimed that reducing cognitive load is one way to assist learners with task completion. Cooper and Sweller (1987) found that worked examples (i.e., a step-by-step demonstration showing all the steps needed to complete a task or solve a problem) accelerate the automation process compared to a problem-solving approach. More importantly, Sweller (2011) claimed that worked examples productively provide learners with problem-solving schemas that are stored in LTM. Once schemas are stored in LTM, cognitive load is reduced, which assists learner with problem solving.

Research over the last 30 years has shown that studying worked examples has helped to reduce students’ cognitive load (Sweller & Cooper, 1985; Cooper & Sweller, 1987; Sweller 1988; Sweller 1989). In addition, Rittle-Johnson and Star (2007) found that contrasting examples also facilitate learning. A practice guide from the US Department of Education indicates that comparison is one of the five recommendations for enhancing mathematical problem-solving skills in the middle grades (Woodward et al., 2012). However, little research has been done that shows how use of worked example

comparisons has helped ELLs master mathematical content while they are developing their English language proficiency. As a result, the current study examined if Modified for Language Support-Worked Examples Pairs (MLS-WEPs) can help English language learners effectively engage in mathematical learning.

This project investigated a research-based instructional approach that has been adapted for English language learners (ELLs) using best practices, to help ELLs overcome language limitations and develop better understanding in mathematical problem solving and literacy skills (listening, reading, writing and speaking skills). I have developed Modified for Language Support-Worked Example Pairs (MLS-WEPs) intervention materials for ELLs, which is a supplemental curriculum for ELLs that requires them to compare worked examples in order to deepen their understanding of mathematics.

Purpose Statement

This mixed method study had four purposes: (1) Establishing and testing MLS-WEPs to improve ELLs' mathematical performance; (2) Examining whether the effectiveness of the MLS-WEPs intervention is related to ELLs' English Language proficiency; (3) Exploring whether the incorporation of MLS-WEPs facilitates teachers to provide more opportunities for ELLs to engage in mathematical listening, reading, writing, and speaking; and (4) Investigating whether integration of MLS-WEPs improves teachers' attempts to ask deep questions that encourage ELLs to participate in mathematics discourse and promote equitable participation.

The ultimate goal of this study was to determine whether or not mathematical writing frameworks, reading strategies, and a supportive discussion atmosphere are effective for ELLs. Specifically, I hope to examine whether using the supports with MLS-WEPs could help ELLs improve mathematical reading, writing, speaking, listening, reasoning, and problem solving without overloading their working memory. Mathematical reading, writing, and speaking play crucial roles in mathematical learning. Such skills enable students to analyze, interpret, construct, explain, argue, and communicate mathematical knowledge. These are high level thinking skills (Bloom's taxonomy). It can be inferred that lack of mathematics literacy skills may be the main reason why ELLs do not actively participate in classroom discourse. Hence, it is imperative to investigate pedagogical strategies that enhance ELLs' mathematics literacy skills.

Broader Impact

This project is to serve ELLs, who have been historically underrepresented in the STEM areas. This project aims to enhance ELLs' mathematical education infrastructure. It is expected that the results of this research will significantly advance mathematical knowledge and understanding among ELLs because the scaffolding activities employed in this project will enhance ELLs' mathematical reading, writing, speaking, and problem-solving skills. The instructional materials developed in the study are to be easily and flexibly integrated into mathematics curricula to supplement existing lessons. The project will encourage ELLs to pursue math related fields through implementing activities and tasks that have been adapted to every student's learning ability. Through the strategies

learned during the intervention, it is my hope that ELLs will be more willing to engage in mathematical learning without fear and might even find the field interesting, thus, encouraging them to pursue math related careers.

CHAPTER 2

REVIEW OF LITERATURE

Students' mathematical performance depends heavily on their language skills (Ellerton & Clarkson, 1996; Clarkson & Galbraith, 1992; Jarrett, 1999), and mathematics acquisition becomes more difficult when learning in a second language (Heng & Tang, 2003). There may be no population group that needs more help with math literacy skills than English language learners. Therefore, it is critically important to examine teaching and learning strategies that help English language learners overcome language limitations and learn mathematics more successfully. Consequently, this project aims to highlight the impact of language on ELLs' mathematics learning.

In the body of the literature review, I describe two perspectives that guide the current research: theories on learning mathematics, and principles for creating an effective second language acquisition-rich learning environment. These two perspectives were chosen because the current study is about mathematics teaching and learning in an ESOL context. Situated in these broad perspectives, I elaborate on how instructional strategies can help English language learners continue to develop proficiency in an additional language and learn mathematics more successfully. To start, I present mathematical teaching and learning in an ESOL context. Specifically, I review empirical studies that illustrate the difficulties English language learners have in mathematics learning, including linguistic and participation challenges. I also demonstrate the challenges that teachers encounter when teaching ELLs. After discussing the challenges

of mathematical teaching and learning in an ESOL context, I then move to the potential solutions to these problems. In particular, I discuss some examples of how teachers have been successful in adding positive and meaningful discourse to their mathematics classrooms. I then present the advantages of worked example comparisons, which are designed to support discourse, and the research on how such comparisons benefit mathematics learning. Next, I evaluate research on second language acquisition that is specific to mathematics content for ELLs to show how to create an effective second language acquisition environment in order to help students develop English language proficiency and learn mathematics well. To intertwine worked example pairs with mathematics learning in a second language context, I also analyze how well the previous worked example pairs model aligns with research recommendations for ELLs.

Through analyzing the previous worked example pair studies in mathematics learning and studies that focused on ELLs learning mathematics content as they develop English language proficiency, I am able to identify gaps in the previous worked example pair model as discussed in the ESOL literature. That is, the previous worked example pair model may not have been effective in supporting English language learners' engagement with mathematical reasoning and communication prior to the development of English language skills. This is because the previous model did not employ any strategies that would help ELLs effectively learn the language as they learn math content. I will then describe how the existing worked example pair model can be adapted to allow ELLs to continue to develop English proficiency as they learn math content.

The purpose of this project is to integrate worked example pair in an ESOL context in order to improve ELLs' mathematical learning and literacy skills. All these analyses are conducted to demonstrate that worked example pair will provide opportunities for ELLs to develop literacy skills as they develop math content knowledge (i.e., mathematical problem solving, literacy, and discourse skills). However, research is lacking on the potential for comparing worked examples to improve mathematics learning in an ESOL context. The current study seeks to fill this gap by evaluating the impact of ESOL math teachers' and their ELLs' use of a supplemental algebra comparison curriculum.

Theoretical Framework

I drew on both cognitive constructivism and sociocultural theory perspectives to guide the current study because the design of the intervention materials were designed to view students as active contributors to the development of their mathematical practices (cognitive constructivist perspective). At the same, the intervention materials provided ELLs many opportunities to participate in cultural practices (sociocultural theories perspective). Therefore, I believe that ideas based on cognitive constructivist theory and sociocultural theory can be reconciled and are most appropriate to guide this study. Given the population of interest for the study, I also draw on principles for creating an effective second language acquisition-rich learning environment. All math classrooms involved in the study promote sheltered instruction (i.e., an instructional approach to teaching ELLs that integrates language development and mathematics content instruction). In addition, in an effective second language acquisition-rich mathematics classroom, second language

learners are more likely to focus explicitly on different aspects of language use in mathematics (Barwell, 2020).

Some researchers claimed it is possible and necessary to reconcile cognitive constructivist and sociocultural perspectives (Packer & Goicoechea, 2000; Cobb, 1994). Learning is a complex process that “involves becoming a member of a community, constructing knowledge at various levels of expertise as a participant, but also taking a stand on the culture of one’s community in an effort to take up and overcome the estrangement and division that are consequences of participation” (Packer & Goicoechea, 2000, p. 227). While agreeing with cognitive constructivism’s emphasis on active learners, Packer and Goicoechea (2000) argued that it is also important to recognize that knowledge is not all that is constructed. The individual human being is also a construction, as is the social world. Cognitive constructivism fails to recognize that the individual cognizer is not a natural creature, but is most likely the result of the co-creation of human culture and history. At the same time, socioculturalism cannot emphasize only context and participation. The sociocultural approach should also recognize the different ways in which people relate to any community, while valuing a socially responsive perspective.

Constructivism

Cognitive constructivists derive insight from Piaget (1968) and focus on the construction of knowledge that individuals find in their interactions with their environment (Bonk & Cunningham, 1998). Cognitive constructivists advocate using students' prior knowledge as a basis for constructing new knowledge, presenting

contradictions, resolving misconceptions, and making learning more relevant and effective (Brooks, 1990). Cognitive constructivist educational practices and orientations emphasize active, generative learning, in which the teacher serves an important learning function as a learning advisor and mentor in the curriculum (Bonk & Cunningham, 1998).

From a constructivist perspective, Cobb and colleagues (1992) asserted an approach to mathematics as an individual and collective constructive activity. They argued that each of us actively constructs relatively complex concepts, which makes it possible for us to engage in mathematical practices that are considered shared. In this account, mathematical learning is both a process of cognitive construction and acculturation. The authors argued that in mathematical learning, constructive efforts will point to a shared way of knowing mathematics. Learning should be an active, constructive process in which students attempt to solve problems that arise as they engage in mathematical practices in the classroom. This perspective emphasizes that the processes of learning and teaching are interactive and mutually negotiated. In these processes, teachers and students express what is perceived as a shared mathematical reality and form the basis for their ongoing communication. Hence, Cobb and Yackel (1992) suggested developing pedagogical approaches that allow teachers to draw on students' prior experiences and guide them in their initial interpretive negotiations.

According to Piaget (1968), learning is a dynamic process in which learners actively construct knowledge by creating and examining their own theories. Piaget (1968) argued that all cognitive development is directed toward increasing levels of complexity

and has a tendency to stabilize. The development of human cognition is a process by which new knowledge is accommodated into existing cognitive structures. When new cognition is accommodated, new cognitive structures are formed to accommodate the new information. Based on this assumption, Bonk and Cunningham (1998) suggested learning should be a personally relevant and meaningful endeavor. Teachers should connect learning to practical ideas and personal experiences and adapt content to students' responses to capitalize on personal interests and motivations. For example, when learning to solve equations, before introducing the concept of solving equations, teachers can activate the concept of equal by relating this concept to the scale (connection to existing cognitive structure). Once students understand the idea of "balance" with an equal sign (formation of new cognitive structure), they will understand why they need to "do the same thing" for both sides of the equation when solving the equation to keep the two sides balanced (accommodation of new information).

Sociocultural Theory

Vygotsky's sociocultural theory describes learning as a social process. Vygotsky's theoretical framework asserts that social interaction plays a fundamental role in the development of cognition (Cole & Wertsch, 1996). According to Vygotsky (1978), there are two levels to any learning process, the first level is social development with others through communication and the second level is individual internalization of knowledge. Thus, cognitive development is a progressive process from external to internal. Learners acquire new information through communication with the outside world, formulate new concepts internally, and apply the new knowledge in a novel context. An important

assumption of sociocultural theory is that learning is a process of knowledge transfer (Bonk & Cunningham, 1998). For example, when learning to solve equations from a sociocultural approach, teachers can have students work in small groups to discuss the meaning of the equal sign and provide examples to help each other better understand the concept. Students create their own knowledge and develop mathematical meaning as they listen carefully to the explanations of others or explain and justify their ideas to others. “As they learn to speak the mathematical language, they transform their thinking of the mathematical concepts. The mathematical language comes from society, and thought (concept) comes from the individual” (Steele, 2001, p. 405).

Another aspect of Vygotsky's theory is the view of the relationship between learning and development. Vygotsky (1978) found that children with similar levels of psychological development differed greatly in their ability to learn under the guidance of a teacher. Differential performance with and without instruction is called the zone of proximal development (ZPD). It is defined as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 33). The zone of proximal development allows us to delineate a child's approximate future and his dynamic developmental state, allowing not only for developmentally achieved results, but also for developmentally maturing results. In other words, learning awakens various internal developmental processes that can only operate when the child interacts with people in the environment and cooperates with peers. The teacher or a more capable peer may provide scaffolding to

the learner to help him or her complete a task. ZPD emphasizes that without this support the learner would not be able to complete the task. The goal of scaffolding is to actively involve the learner and provide support only as necessary, because the ultimate goal of scaffolding is that the learner will eventually be able to use the strategies independently. For example, when learning to solve equations from a ZPD approach, teachers provide scaffolding using the example of scales only when students cannot make the connection between "balance" and "equality" on their own.

From a sociocultural perspective, learning takes place through participation in cultural practices, where people co-construct knowledge (Cole & Wertsch, 1996). Whereas from a constructivist perspective, learning is characterized by a self-organizing process in which individuals reorganize their activities in order to eliminate disturbances (Cobb, 1994). The individual student is seen as an active contributor to the development of mathematical practices in the classroom. Neither the individual student's mathematical activity nor the classroom microculture can be adequately explained without considering the other factor (Cobb, 1994). In fact, the two perspectives address different problems and issues. They can therefore also be reconciled with each other.

Mathematics Literacy Development in the ESOL Context

Ellis (2005) conducted a series of “evidence-based practice” investigations and found that teachers can stimulate the language development of ELLs by creating an effective second language acquisition learning environment. Kersaint, Petkova and Thompson (2008) synthesized these research generalizations into five principles for creating an effective second language acquisition-rich learning environment. In this

article, I synthesized the five principles into three. I did not include the principle about noticing their errors and to correct their English because the current study focused on the extent to which ELLs understand mathematical concepts and their ability to discuss them, rather than on grammatical or vocabulary errors. Additionally, I synthesized the principle about giving ELLs classroom time to use their English productively and the principle about constructing activities that maximize opportunities for ELLs to interact with others in English into one principle since they are relevant and even overlap. When constructing activities, teachers should consider ways to maximize opportunities for ELLs to interact with others and use their English effectively.

Principle 1.

Give ELLs many opportunities to read, write, listen, and discuss oral and written English and mathematics text expressed in a variety of ways. Meltzer (2001) defined academic literacy as students' ability to "use reading, writing, speaking, listening and thinking to learn what they want/need to learn and communicate/demonstrate that learning to others who need/want to know" (p. 16). This definition emphasizes the importance of including all the four language domains in the preparation of lessons to help students develop academic literacy. Thus, in terms of the development of ELLs' mathematics literacy and learning, it is necessary to provide explicit instruction that enables ELLs to practice listening, reading, writing, and speaking. Boscolo and Mason (2001) suggested that teachers should provide ELLs opportunities to discuss and share mathematical concepts in order to help them develop mathematics literacy and mathematics problem solving skills. Helping ELLs engage in the discussions of

mathematical texts will help them to develop, become familiar with, and recognize mathematical content knowledge. In addition, researchers uncovered that academic literacy can be developed through enhancing reading comprehension via the following: providing explicit instruction in context; offering instruction to assist learners with reading and writing in mathematics; designing mathematics assignments that require learners to read and write; providing more time for ELLs to apply mathematical language as they process and/or develop mathematics knowledge; and facilitating the development of reasoning and critical thinking skills (Kersaint, Petkova & Thompson, 2008).

Principle 2

Draw attention to patterns in English and mathematics language structure.

This principle suggests that teachers should implement various activities that provide ELLs explicit exposure to the mathematics language structure. Based on the early research into naturalistic second language acquisition, Kersaint, Petkova, and Thompson (2008) suggested the following:

[L]earners follow a "natural" order and sequence of acquisition. What this means is that grammatical structures emerge in the communicative utterances of second language learners in a relatively fixed, regular, systematic, and universal order. The ways in which teachers can take advantage of this "built-in syllabus" are to implement an activity-centered approach that sets out to provide ELLs with language-rich instructional opportunities and offer ELLs explicit exposure and instruction related to language structures that they are trying to utter but with which they still have trouble. (p. 7)

We can't learn how to ride a bicycle if we don't ride a bicycle. Similarly, learners cannot become excellent communicators of English without using the language.

Therefore, it is critical to provide opportunities for English language learners to actively engage in practicing language output (i.e., writing and speaking).

Teachers should not only focus on providing students with subject knowledge, but should also consider the potential for language learning that may result from the same activity when preparing lessons. It is especially important for teachers who teach English language learners to be aware of this. Teachers can facilitate their students' language development by encouraging them to be aware of their mistakes, to reflect on how they use English, and to think about how English works (Kersaint, Petkova & Thompson, 2008). Based on the findings of Lyster and his colleagues (Lyster, 1998; 2001; 2004; 2007; Lyster & Ranta, 1997; Lyster & Mori, 2006), Kersaint and colleagues (2008) summarized a series of strategies that teachers can use to guide ELLs' attention to their linguistic output and help them correct their English. Those strategies are summarized in Table 2.1.

Table 2.1

Strategies to Direct ELLs' Attention to Their Linguistic Output

Name of the strategies	Definitions	Examples
Explicit correction	A clear explanation with intentional, corrective feedback.	Student: "The heart hits blood to see body..." Teacher: "The heart pumps blood to the body."
Requesting clarification	When the teacher does not understand the student's explanation or the student's response contains some kind of error, a clarification question to ask the student to repeat or rephrase the sentence.	Student: "I can experimenting with Bunsen burner." Teacher: "What? Can you say that again?"

Table 2.1

Continued

Name of the strategies	Definitions	Examples
Recast	Without directly indicating that the student's statement is incorrect, the teacher implicitly restates the English learner's error or provides correction.	Student: "After today I go to sport." Teacher: "So, tomorrow you are going to play sports?" Student; "Yes, tomorrow I am going to play sport."
Metalinguistic clue	Without providing the correct form, the teacher's use of questions or comments to provide English learners with clues to the formation of statements.	Teacher: "Is that how it is said?" or "Is that English?" or "Does that sound right to you?"
Elicitation	The teacher gets the correct form directly from the English learner, pausing to allow the student to complete the teacher's statement.	Teacher: "So, then it will be a ..." (with long stress on "a")
Repetition	The teacher repeats the English learner's mistakes and adjusts the tone of voice to get the English learner's attention.	Student: "The two boy go to town tomorrow." Teacher: "The two boyyy go to town tomorrow." (with teacher making a prolonged stress on "boy")

Principle 3

Give ELLs classroom time to use their English productively and interact with others in English while learning mathematics. This principle emphasizes the importance of providing ELLs opportunities to negotiate meaning through interaction. Mathematics language can be facilitated through engaging in discourse. During interaction, ELLs will

adjust their language to help themselves comprehend the language better. As with oral output, writing also assists ELLs with modifying their use of English language. To promote positive learning, teachers should ask a lot of guiding questions, which provides opportunities for students to contribute, as guiding questions enable students to work within their zone of proximal development to successfully solve problems (Cardimona, 2018). Furthermore, teachers should encourage students to agree and disagree with each other through evidence-based and analytical reasoning. Additionally, while monitoring collaborative activities, teachers should facilitate activities by providing instant feedback, encouraging students to participate, and guiding them to help each other. Lastly, in order for ELLs to progress in their English language development, teachers need to design instructional tasks based on the language proficiency of the ELLs, and activities that are accessible to all language learners. This prerequisite is a key to academic success since it helps students participate in class activities.

In conclusion, it is powerful to create an effective second language acquisition-rich learning environment that enables ELLs to explore collaboratively. Collaborative learning is beneficial, and it makes learning more meaningful and practical. Tavares (2018) discovered that creating a collaborative classroom atmosphere facilitates students' comprehension of concepts. An effective second language acquisition-rich learning environment enhances ELLs' mathematics literacy development (Kersaint et al., 2008). Thus, teachers should incorporate learning activities that require ELLs to work together to discuss, debate, and share knowledge using all four domains of language.

Mathematical Teaching and Learning in the ESOL Context

It is well-established that mathematical reasoning, problem solving, and literacy (listening, reading, writing, and speak) skills are all crucial components for a student to be successful in mathematics (Boscolo & Mason, 2001; Brown et al., 2009; Segerby, 2017). Mathematical reasoning and problem-solving skills allow students to extend what they have learned toward solving novel problems. Through mathematical reasoning and problem solving, students learn how to analyze different situations, determine appropriate problem-solving approaches and strategies, conclude logical outcomes, develop and write solutions, and reflect, interpret, and recognize how those solutions can be applied to novel situations (Rosenstein et al., 1996). Mathematical reasoning considerably facilitates students' meaning-making and problem-solving skills in mathematics. It has been argued that mathematical listening, reading, writing, and speaking skills play essential roles in mathematical reasoning (Segerby, 2017). Those skills enable students to evaluate, interpret, and communicate mathematical ideas effectively and efficiently (Segerby, 2017), thus indicating the need to emphasize listening, reading, writing, and speaking when learning math content.

The recognition of the important relationship between language and learning in general is not new. However, although researchers have long recognized the important role of language in mathematical achievement, they have not always acknowledged that language has as important a role in the acquisition of mathematical concepts and skills as it does in other subjects (Cuevas, 1984). Over four decades ago, researchers found that mathematics performance is highly correlated to language skills (Aiken, 1972; Cossio,

1978). In addition, Purpura and colleagues (2017) found a very high correlation between math and reading achievement, to $r = .70$, and that early math ability predicts later reading ability. The latter was found to be due to the mediating role of children's mathematical language, which again implicates the role of spoken language competence. Therefore, it is necessary to emphasize the importance of focusing on the language skills required for achievement in the mathematics curriculum (Cuevas, 1984).

In a school setting, there may be no population that needs assistance to master mathematical literacy skills more than ELLs. ELLs “not only need math support but also reading and linguistic support” (Orosco et al., 2013, p. 96). Therefore, to enhance ELLs’ mathematical learning, classroom teachers should explicitly and frequently model and teach the language of math (Mousley & Marks, 1991). Because students’ performance in mathematics is highly dependent on their level of literacy in that language, it is extremely important to investigate teaching and learning strategies to help ELLs achieve greater success in mathematical learning while developing their English language proficiency. Therefore, the current study seeks to highlight the impact of language on ELLs’ mathematical learning. If teachers are aware of these potential difficulties in mathematics learning for ELLs early on, they can adjust their instruction to help ELLs understand the information presented. Therefore, I will begin by identifying the challenges that ELLs face in learning mathematics.

Challenges of ELLs in Mathematics Learning

Through a critical review of the literature, Adoniou and Qing (2014) concluded that most of the mathematical challenges that learners encounter in school are language-

related. In fact, they found that previous studies have shown that ELLs are at a disadvantage of up to 15% in math due to language issues. Similarly, Cuevas (1984) claimed that ELLs face challenges in learning mathematics due to the complexity of the way language is used in mathematics and its impact on learning. Thus, this section will explore the challenges students may face when learning mathematics in a second language context.

ELLs require substantial support to develop mathematical literacy skills (Adoniou & Qing, 2014). The language of mathematics discourses is specific to mathematical context; therefore, mathematics teachers cannot assume that the language necessary for mathematics learning has been developed and nurtured in other language classrooms. Successful reading in the English classroom does not guarantee understanding of the text in the mathematics classroom (Adoniou & Qing, 2014). Thus, teachers must provide students with sufficient linguistic support to discuss their thinking and thereby deepen their understanding of the content. Many teachers provide scaffolding through visual representations, manipulatives, videos, and graphs (Brown et al., 2009). However, these strategies do not provide students with sufficient linguistic support. Therefore, the current study aimed to explore pedagogical approaches to improve ELLs' English language proficiency and mathematics content knowledge.

Mathematics involves highly technical language that can be very difficult for students to master and can greatly hinder their progress and enjoyment of mathematics. Freeman and Crawford (2008) explained that the obvious reason for this is that many mathematical terms are new to students and are hardly ever used outside of math class

(e.g., quotient, multiplication, square root, and coefficient). Another reason is that many mathematical terms are familiar to students but have very different meanings in everyday language, and the mathematical definitions of these terms are much more specific and complex than their everyday definitions (e.g., square, power, right, steep, and consistent) (Freeman & Crawford, 2008). This can mislead students to think they understand these terms and the concepts they represent long before they actually do, which in turn can lead to misconceptions that students must overcome before they can grasp these concepts. This was exemplified in a study of English language learners discussing math problems in English (Moschkovich, 2007). In her study, Moschkovich (2007) illustrated that two native Spanish speakers were discussing which line is steeper than the other in English. In the middle of the discussion, the student who struggled with the concept switched the language into Spanish. However, she used “steep” in English throughout the discussion. Moschkovich’s (2007) interpretations of this phenomena were (a) the student did not know the word “steep” in Spanish; (b) the student was struggling with the concept of “steepness.” The interpretations show that the student had insufficient vocabulary or lacks conceptual understanding of the terminology. If a student does not understand what is being taught in a math class, it will be difficult for him or her to move beyond language and master math content and skills, no matter how talented he or she actually is (Freeman and Crawford, 2008). Unfortunately, an analysis of actual statewide exams reveals that even academic content tests are linguistically complex and use words that are likely unknown to ELLs (Menken, 2010). Specifically, Menken (2010) analyzed a New York state math test and found that in this mathematics test, 9.38% of the vocabulary was

extra-formal and only 77.90% of the vocabulary was among the most commonly used words in English. Many of the extra-formal terms in this case were specific to mathematical content, which is extremely difficult for English language learners as they must be learned through a language that they do not reasonably have mastery of yet.

Besides the impact of vocabulary, English language learners often experience comprehension difficulties that make it difficult to solve word problems, and these difficulties increase in the early primary grades as word problems become more linguistically and conceptually complex (Adoniou and Qing, 2014). Word problems often involve both problem-solving procedurals and narrative contexts. The main purpose of word problems is to make learning relevant and applicable in real life. However, this may introduce new information and contexts that are unfamiliar to ELLs, distracting them from the mathematical tasks in the word problems (Adoniou and Qing, 2014).

Cummins (1979) emphasized the importance of understanding the distinction between social language and cognitive academic language. Many text types specific to the mathematical domain have not been encountered or analyzed in the English classroom; they will only be encountered in the context of the mathematical domain and must be taught within the mathematical context. It is thus critical to teach both content and language in the subject content classroom for ELLs, given that teaching mathematics to ELLs not only requires making sure students understand the content of mathematics lessons, but also ensuring that they have the language needed to understand the instruction and to express their mastery of mathematical concepts in both oral and written formats.

Word order or syntax in mathematical word sentences can be converted, which may cause confusion for ELLs as well. In mathematics, students read sentences from left to right, but the order in which they must respond to a sentence is often from right to left. For example, writing an algebraic expression for “7 fewer than x ” requires learners to start writing the expression from right to left, as “ $x-7$.” In this case, the mathematical expression is inverted compared to the word sentence, which may cause confusion for ELLs.

Another source of ambiguity for ELLs is a frequent misunderstanding word problems. In the language literature, such a misunderstanding is called a miscue. Miscues occur when the meaning of the language is interpreted differently from the way it was used (Di Pietro & Ern, 2012). For example, “the sum of 3 and x is at least 12.” In this example, “at least” means the minimum amount is 12, which implies the sum of 3 and x should be greater than or equal to 12. However, many students will use “less than” since “least” is the superlative form of “less.”

For ELLs, the words of math are very challenging, but mathematical symbols and visual tools are equally challenging. Familiarity with symbols and visual representations is key to engaging in mathematical problem solving as they are often used as contextual cues to comprehend textual problems, and sometimes as supporting information necessary to the problem. It is common sense to assume that visualization tools are self-explanatory, but similar visualization tools may be interpreted differently depending on the discipline. For example, number lines in Mathematics look similar to timelines in History but they have completely different meanings (Adoniou & Qing, 2014). Similarly,

mathematical symbols are sometimes interpreted differently in other countries (Freeman & Crawford, 2008; Adams, 2003). For example, the decimal separator in some countries is a dot (e.g., United States, China, and Japan) and in others, is a comma (e.g., Albania and Austria).

In addition to linguistic challenges, some ELLs face significant challenges in classroom participation. In particular, English language learners who are still developing their language proficiency in the language of instruction may only be able to participate in small group discussions to a limited extent (Turner et al., 2013). Even if they understand mathematical concepts, they are likely to have difficulty explaining their reasoning to others because of language limitations. Given that English language learners “may participate only marginally in group discussions” (Turner et al., 2013, p. 200), more research is needed on instructional practices that support equitable participation in mathematics in second language settings. Nathan and Knuth (2003) noted that facilitating mathematical discussion can be a challenging process for most teachers because they often have to ensure that students participate in the discussion while maintaining mathematical accuracy. In addition, teachers may find it more difficult to facilitate discussion to support English language learners in all relevant instructional contexts (Banse et al., 2017). The review in the following section focuses on the challenges faced by ELLs in participating in mathematical discussions.

Recent research reveals that English language learners have difficulty connecting ideas to conceptual explanations (Banes et al., 2018). Banes et al. (2018) conducted a mixed-methods study to examine the relationship between classroom discussion and

student performance among English language learners (ELLs) in 20 third- and fourth-grade classrooms in Northern California. The ratio of ELLs to non-ELLs in these classrooms was approximately 1:1. The experimental results suggest that ELLs make few connections between ideas and conceptual explanations compared to other characteristics. Students make connections between ideas and conceptual explanations correspond to the quality and content of the talk during the discussion, requiring the teacher to respond to students' statements in real time and to connect them to other mathematics, which may be difficult to achieve (Banes et al., 2018). This difficulty can be even greater in a second language learning context. Because of language limitations, ELLs may not fully express themselves, and teachers may not fully understand what ELLs are talking about to the extent that they can appropriately and timely guide students to make connections between ideas and conceptual explanations. There is also the possibility that ELLs may not be able to understand teachers' guidance and scaffolding and miss the opportunity to make connections between ideas and concepts.

Similar to the findings of Banes and colleagues (2018), Turner and colleagues (2013) claimed that their research team rarely observed ELLs making mathematical claims and mathematical connections. Turner and colleagues (2013) focused on seven 4th-5th grade Latino/a, native Spanish-speaking students regarding their participation in an after-school mathematics program organized around solving and discussing mathematical problems. Their analysis was focused on instructional interactions between the teacher and students. They found that students were significantly less likely to make mathematical claims and mathematical connections. In addition, most students (6 out of

7) took more of an active problem-solving role in small group discussions than in whole group discussions. Small group discussions provide better opportunities for equitable participation than whole class discussions because they ensure that every student has an opportunity to speak.

Challenges of Teachers in Mathematics Teaching in an ESOL Context

ELLs encounter a variety of challenges when engaging in mathematical discussions, and indeed, teachers also experience various challenges when facilitating classroom discussions, and these challenges are likely to have a negative impact on student participation. For many teachers, including very experienced teachers, the challenge is to make classroom discourse an integral part of their teaching (Walshaw & Anthony, 2008). Moschkovich (1999) noted that revoicing mathematical statements of students is difficult to accomplish, especially when working with ELLs. It can be difficult, or even impossible, to distinguish which aspects of a student's false statement are the result of the student's conceptual misunderstanding or the result of the student's English language proficiency. However, the analysis of the study suggests that if the goal is to support students' participation in mathematical discussions, identifying sources of error is not as important as listening to students and discovering what they are saying about mathematics (Moschkovich, 1999).

Besides revoicing difficulties, teachers face many other challenges in facilitating mathematical discourse. Nathan and Knuth (2003) reported changes in the classroom practice of an experienced middle school mathematics teacher in the first two years as a result of her involvement in a professional development program. From the 1st year data,

it is clear that the teacher was the central figure in all verbal interactions involving the whole class. Therefore, students rarely engaged in communication unless asked, relying primarily on the teacher rather than other students for information, and students rarely used discourse as a means of constructing their own ideas, testing hypotheses, or questioning other students' ideas. Although the participating teacher valued her students' ideas, often inviting them to contribute their own, the results still revealed a highly teacher-directed form of discourse. Thus, students in a class are less likely to speak directly to each other if the teacher is the center of the conversation. In such a learning environment, classroom conversations tend to involve the teacher addressing a particular student or the class as a whole. The teacher acts as an authority on mathematics and, to some extent, on social behavior. After reviewing the first year's lesson videos, the participating teacher decided to reconsider some of her classroom strategies to provide students with more opportunities for peer interaction and learning through participation. In the second year, the participating teacher consciously removed herself from her central role in the mathematical discussion; instead, she often invited students to speak. The data from the 2nd year showed an increase in interaction between students, but since no mathematical authority was involved, “there was often a lack of rigorous argumentation and evidence in the discussions, and a lack of convergence toward acceptable mathematical ideas and conventions” (Nathan & Knuth, 2003, p. 198). In such discussions, the participating teacher simply regulated the exchange of ideas between the respective students but failed to do more to make social scaffolding work. The team analysis concluded that the participating teacher needed to encourage students to discuss

the ideas of others and to critically evaluate the validity of these ideas. When there is a significant omission, misconception, or conflict of opinions in the group discussion, the teacher should fill in the gaps by joining the group and guiding the students as appropriate. From this study we can see that it is important but also difficult to gauge the level of teacher involvement. In the first year, too much involvement by the participating teachers led to a lack of student participation in the discussion. While in the second year, too little involvement of the participating teachers led to a decrease in the quality of the discussion.

Banse and colleagues (2017) conducted a comparative case study using secondary data to examine how two teachers attempted to facilitate discussion in a fourth-grade classroom where more than half of the class were English language learners. Their findings indicate that teachers used a number of effective discussion practices to support ELLs in mathematics instruction, including a variety of display problems, repetition, elaboration of student responses, and vocabulary use, but there was a distinct lack of referential questions. Referential questions are open-ended questions that tend to lead students to longer, more complex responses. Students usually have to use their own reasoning when answering referential questions rather than simply describing a mathematical procedure. The data from the study (Banse et al., 2017) revealed that neither teacher used referential questions as a basis for classroom discussion. Previous researchers have also determined that teachers tend to ask demonstrative questions rather than referential questions (Banse et al., 2017). The lack of referential questions suggests that concept-rich discussions are missing from the classroom. Thus, while one of the

teachers' efforts to create opportunities for students to interact with the mathematical content was commendable, the lack of referential questions limited opportunities for students to engage in mathematical reasoning. As I mentioned earlier, mathematical reasoning is essential for mathematics learning, demonstrating that mathematics teachers asking referential questions is an important component of facilitating class discourse.

Although a growing body of research suggests that classroom discussions can support students' mathematical learning and contribute to the development of positive mathematical identities (Empson, 2003; Turner et al., 2013), research suggests that the ways in which students benefit from classroom discussions vary from person to person (Baxter, Woodward, & Olson, 2001). Furthermore, ELLs need opportunities to engage in rich mathematical discourse in addition to focusing on vocabulary and procedures to recognize their current and potential competencies (Truxaw, 2020). Hence, it is crucial to study different pedagogic methods so that each student can effectively participate in class discussions. In the following section, I provide a comprehensive critical review of the strategic efforts that mathematics teachers make to address classroom discourse in order to improve student achievement. Specifically, I discuss some examples (from empirical studies) of how teachers could successfully add positive and meaningful discourse to their mathematics classrooms.

Supports for Equitable Participation in Mathematical Discussion

Classroom discourse requires linguistic skills so students with lower English proficiency tend to avoid participating in math discussions (Banes et al., 2018). One of the explanations for this phenomenon is oral discussions require high cognitive demand

(Baxter et al., 2001). It is worth mentioning that, in their study, Banes and colleagues (2018) found that English language learners not only participated in math discussions, but they also benefited from classroom discourse. More importantly, their results suggested that a student's status as an English language learner does not influence classroom discussion. Based on the literature, the authors identified five key characteristics of effective mathematical discussion: diversity of approaches, opportunities to speak, equitable participation, explanations, and connections between ideas. Among the five key features, a variety of approaches and equitable participation scored the highest. The data showed that most students were able to use more than one strategy or representation to solve a problem at least some of the time in the observed courses, and that most students exhibited independent thinking at least once during the course in most classes. More importantly, the interaction between ELL Status and Discussion was not statistically significant, which indicates that in such math classes, English language learners benefit from class discussions as much as those who are native speakers of English.

Though mathematical discourses are also beneficial to ELLs (Banes et al., 2018), they often position themselves as lacking participation skills, so they often opt out of classroom discussions (Turner et al., 2013). However, Turner and colleagues (2013) showed that with teacher scaffolding, ELLs are also able and willing to participate in mathematical discussions. The authors found that all seven participating students were willing to respond to the teacher's invitation to: share their problem-solving strategies, provide valid mathematical justifications for their strategies or ideas, and provide reflections on their peers' ideas or strategies. More importantly, the research team

analyzed what the participating teacher did to counteract English language learners' marginalization and promote ways for ELLs to participate in and make meaningful and substantive contributions to classroom discussions. First, the participating teacher recognized multiple resources for discursive action, including using students' native language as an intellectual resource and allowing students to take on active problem-solving roles, such as explaining their strategies and considering the ideas of their peers. Second, the teacher valued strategy as a different kind of contribution. For example, when a student struggled to follow her groupmates' reasoning, the teacher encouraged the student to continue to explore strategies that made sense to her. When that student shared her strategy, the teacher said, "[d]id you all follow her strategy? Does anyone have questions for Yasmine? Because it's important that we all learn from her strategy" (p. 216). The teacher explicitly positioned the student's strategy as a different way of approaching a problem and called on other students to listen to her thinking. Moreover, the teacher clarified ideas while maintaining the ELL's active problem-solving role. For example, after one student's initial explanation, the teacher invited other students to ask questions if they did not understand what the student had just explained. This strategy interactively positioned the student's thinking as worthy of consideration and positioned him as capable of clarifying his own ideas. Unlike the participating teachers in Nathan and Knuth's (2003) study, who did not add mathematical authority involvement to help students correct their misconceptions, the participating teachers in Turner et al.'s (2013) study intervened at times to remind students of key aspects of their interpretations.

To achieve the goal of rich discourse in the mathematics classroom, teachers need to set up practices that enable students to engage in mathematical discussions. Wood (2002) conducted a secondary analysis, which examined the patterns of interaction that exist in traditional and reform classrooms from the perspective of teacher practice. By examining these patterns of interaction, Wood (2002) attempted to provide a better understanding of what it means to reconstruct a classroom culture for differentiated learning in mathematics and to provide suggestions on how teachers might achieve this goal. The results of the study showed that the type and manner of questioning by teachers play a dominant role in the differences in students' thinking and reasoning skills. In addition, the patterns of interactions established by teachers and students influenced the nature of participations. Thus, it is likely that the differences in classroom discourse are due in large part to specific differences in the expectations of engagement established by teachers and the cognitive demands of teacher questioning on students' mathematical reasoning. The results of the study again demonstrated that teacher's questioning plays a critical role in classroom discourse.

Another important component of mathematical discourse is student interaction. To provide a safe and appropriate environment for students to interact, teachers must establish norms and support students throughout the dialogue process (Gresham & Shannon, 2017). In their article, Gresham and Shannon (2017) described how one first grade teacher incorporated opportunities for communicative reasoning and understanding of mathematical concepts into her math lessons. The teacher allowed students to share their ideas, thoughts, and reasons for solving a problem; then, with her facilitation, she

guided students to compare their results with those of their peers. With the comparison, she prompted them to think about what they noticed about the problem and how they solved it. This led to the expectation of solving the problem in their own way; explaining their answers and problem-solving strategies; and judging the mathematical reasonableness of their answers as a class. Additionally, the teacher constantly reminded students of the importance of listening to each other's ideas and strategies. By asking questions and having students discuss how to solve problems, the teacher designed discourse skills to build simple skills on increasingly complex ones. Students actively listened to one another and showed agreement or disagreement by explaining their reasoning (Gresham & Shannon, 2017). In short, comparing different problem-solving strategies and ideas can promote mathematical discourse and reasoning, but this study again shows the important role of teacher questioning.

It is worth mentioning that teachers who guide classroom discussions with more teacher-centered discourse may still benefit ELLs (Banse et al., 2017). In the study conducted by Banse and colleagues (2017), one of the teachers was more likely to use revoicing to expand on students' responses; however, the other teacher was adept at prompting students to expand on their previous responses. Thus, revoicing can be either teacher-led or student-articulated in order to expand students' responses. Interestingly, Banse and colleagues (2017) noted that the teacher who facilitated class discussion with more teacher-centered talk still benefited her ELLs' academic language development because "this type of constant teacher language and vocabulary modeling coupled with a simpler rate of speech may help ELLs both comprehend the discussion and gradually

master mathematical language” (p. 206). Hence, teachers can choose the appropriate method (teacher-led or student-articulated) based on the English language proficiency and content knowledge level of the ELLs. Furthermore, the results of this study reveal that display questions provide excellent opportunities for ELLs to participate. Display questions are more likely to be closed-ended questions whose answers are predictable to the teacher (Banse et al., 2017). The answers to these questions are usually factual and often short, such as a word or a number. Banse et al. (2017) claimed that, “[t]hese questions are not as conceptually deep as referential questions. However, these questions do create an opportunity for feedback loops between the teacher and students” (p. 203). These questions may provide excellent opportunities for ELLs to interact with the class at their level of proficiency. Students whose English language proficiency is not yet developed are more likely able to answer display questions and thus become participants in class discussion.

Moschkovich (1999) examined classroom discussions in a third-grade math class to illustrate how one teacher supported English language learners' participation in math discussions and to describe how students participated by discussing math situations in different ways. The results reveal several pedagogical strategies that can support English language learners' engagement in mathematical discussions, including: establishing and modeling consistent norms for discussion, restating students' contributions in mathematical terminologies, building on and extending students' stated ideas, and revoicing students' meanings in simple yet authoritative mathematical language. It is noteworthy that the teacher did not focus so much on vocabulary development as they did

on mathematical content and argumentation when explaining, clarifying, and rephrasing what students said. This teacher looked beyond students' grammatical or vocabulary errors, listened and attempted to understand what they had to say. He focused the discussion on the mathematical content provided by the students. To ensure that the discussion remained math-oriented, he asked students to clarify, accept, and build their responses, as well as to revoice other students' statements. The lesson presented in the study showed that English language learners not only participated in class discussions, but also worked hard to learn important mathematical content during those discussions. Moschkovich (1999) noted that it is difficult to carefully consider student contributions in the moment. Thus, the author suggested that teachers can take time after the discussion to reflect on the mathematical content contributed by students and design follow-up lessons to address that content. Although engaging in mathematical classroom discussions can be challenging for English language learners, it is only when students have the opportunity to participate in mathematical discussions that they are likely to discover the mathematical content to which they contributed. Only then will teachers know how much students understood and what content they misunderstood.

In conclusion, teacher questioning plays an important role in classroom discussion. Display questions provide excellent opportunities for English language learners to interact with the class at their level of proficiency and thus become participants in class discussion. Referential questions provide concept-rich discussions and opportunity for mathematical reasoning. In addition, creating a social environment that enables teacher-student and student-student interactions is also essential. Such a

social environment enables students to create their own knowledge and develop mathematical meaning as they listen carefully to the explanations of others or explain and justify their ideas to others. As they learn to speak the mathematical language, they transform their thinking of the mathematical concepts.

Having discussed possible ways to enhance mathematical discourse and equitable participation, I now turn to comparison as a possible solution to the problem solving and mathematical literacy challenges encountered by ELLs. Past research has shown that comparison allows learners to create patterns and to notice key structural features that help transfer knowledge to new situations, including within the domain of mathematics (Rittle-Johnson, Star, & Durkin, 2009; Schwartz & Bransford, 1998; Gentner et al., 2003).

Comparison as a Mechanism for Learning

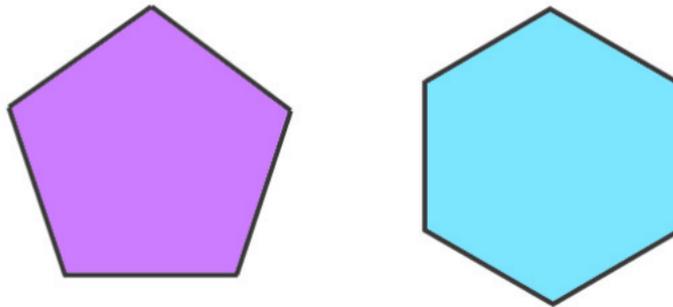
Research has highlighted comparison as an effective learning tool that improves learning in a variety of domains (Rittle-Johnson & Star, 2007; Namy & Gentner, 2002; Kurtz et al., 2001). There are many advantages in comparing analogical examples. First, comparing two examples assists learners' understanding of the fundamental features common to both (Loewenstein, Thompson & Gentner, 1999). Gentner et al. (2003) found that even if neither example is well understood by learners, comparing examples can still be extremely informative. For example, when learners are given two shapes to compare (see Figure 1), they may not know what the two shapes are, but they can still get a lot of information by comparing these two shapes. A student may notice the different colors, number of sides, number of angles, number of vertices, degree of interior angles, angles,

and so forth. Having this information helps learners understand the fundamental features that are common to both shapes. In traditional problem-solving, information is being mapped from the background knowledge to the novel situation (Gentner et al., 2003). Prior knowledge plays an important role in the traditional problem-solving approaches. The lack of problem-solving schemas can be an obstacle for ELL learners. However, comparison minimizes the interference of prior knowledge. To further understand this point, Gentner and colleagues (2003) claimed that “comparison is not being used to facilitate transfer of a well-learned piece of prior knowledge but rather to highlight and clarify a new concept” (p. 394). Thus, comparison enables those who lack prior knowledge (i.e., ELL students) to engage in mathematical learning.

Furthermore, comparing two examples allows learners to pay attention to structural commonalities rather than surface features. In other words, drawing an analogical comparison leads to a structural alignment, which highlights the shared relational elements of the two examples (Gentner et al., 2003). We can see how

Figure 2.1

Shapes for Comparison



relational elements of the two examples (Gentner et al., 2003). We can see how analogical comparison leads to structural alignment through the following two equations: $5x + 2 = 15$ and $5(x + 2) = 15$. In this comparison, the numbers are similar and so attention is given to structural differences between the equations. In other words, students should notice that one equation has parentheses but the other does not. Thus, comparison helps learners pay attention to structure and meaning in the symbols.

More importantly, generalizations can be easily identified when comparing worked examples, as opposed to studying examples sequentially. Gentner et al. (2003) claimed “whatever is understood about one example can serve to shed light on the other...[comparison] focuses learners on precisely those aspects that generalize across cases” (p. 394). For example, through comparing the two shapes in Feature 1, it is easy for learners to notice that as the number of sides increases, the measure of each interior angle increases. This generalization is less likely to be discovered without comparison.

Research showed the power of analogical learning and the advantage of comparing two cases over studying them separately (Gentner et al., 2003), and having students compare and contrast problem solving procedures is promoted by many countries in the world (Star et al., 2015). Many researchers have discovered having students compare worked examples reliably improved mathematics learning (Rittle-Johnson & Star, 2009; Rittle-Johnson & Star, 2007; Rittle-Johnson et al., 2009; Pashler et al., 2007). Allowing students to benefit from comparing, reflecting, and discussing multiple approaches to problem solving has been a central tenet of mathematics teaching and learning reform for the past 20 years (Rittle-Johnson & Star, 2007). However, such

an effective comparison strategy is not supported by average U.S. teachers across domains (Richland et al., 2007). In the following, I review some empirical studies of worked example comparisons and present specifically the advantages of comparing worked examples in the mathematics domain.

Comparison in Mathematics Learning

Learning through analogical approach promotes learning, overcomes the disadvantages and weaknesses of a single strategy, and helps learners to be more flexible and effective in applying them (Croße & Renkl, 2006). More importantly, the analogical approach allows learners to integrate and apply what they have learned to new contexts. However, this approach is prone to cognitive overload because analogical learning requires learners to integrate different solutions in their minds, thus stuffing valid information to gain coherent understanding of the learning materials (Croße & Renkl, 2006). Research over the past 30 years has shown that studying worked examples helps to reduce the cognitive load on students (Sweller & Cooper, 1985; Cooper & Sweller, 1987; Sweller, 1988; Sweller, 1989). However, worked examples are limited in developing insight into far-reaching mathematical interrelationships (Croße & Renkl, 2006). In this case, combining the compared method with worked examples avoids cognitive overload, thus facilitating learning (Croße & Renkl, 2006) and transferring what has been learned to new contexts (Rittle-Johnson, Star, & Durkin, 2009; Schwartz & Bransford, 1998; Gentner et al., 2003).

Research has shown that reducing working memory load is important for learning under different instructional conditions (Sweller, 1988; Paas & Ayres, 2014). Working

memory plays an essential role in learning new concepts, so how can teachers support their students by reducing their working memory load in the mathematics classroom? Richland et al. (2012) discussed how to encourage learners to use prior knowledge structures to reduce working memory processing load. They suggested that the connections between prior knowledge and the new concepts can be made through comparing two different concepts or problems, converting verbal concepts to visual representations, making side by side comparisons, identifying related and unrelated problem elements, and developing visual imagery. In such ways, students are able to relate the procedures they are taught with the original concepts that could assist them in comprehending mathematics as a consistent, meaningful system. According to the authors, once students have relational thinking and alignment between the prior knowledge and the new concepts, they have adequate knowledge to reduce cognitive load.

In mathematics instruction, comparing worked examples is powerful and promising for enhancing mathematics learning (Rittle-Johnson, Star, & Durkin, 2017). To illustrate, research found that comparing worked example pairs leads to greater procedural flexibility and procedural knowledge (Rittle-Johnson & Star, 2007). In the study, the researchers randomly assigned 7th grade students (N=70) to learn about solving multistep linear equations using either comparing worked example pairs side-by-side at the same time or reflecting on the methods one at a time. In this study, the comparison focused on comparing different problem-solving methods to a single problem where one method is better than the other under certain circumstances. The authors followed a

pretest-intervention-posttest design. The findings of the study revealed that those who compared methods acquired greater procedural knowledge and procedural flexibility. The authors explained that a compared approach contributes to the growth of procedural knowledge and procedural flexibility because it facilitates students to explore and use alternative solutions. The authors found that students in the comparison condition were twice as likely to change their original problem-solving methods and use the demonstrated shortcut method to solve practice problems, and this flexibility carried over to the posttest. These findings suggest that the comparison of multiple problem-solving methods helps students move away from rigid dependence on a single solution method and become more adaptive and flexible in using effective methods (Rittle-Johnson & Star, 2007). By comparing multiple methods, students are better able to explain why the shortcut step is a good way to solve a particular equation. As a result, students are able to use shortcuts more often and more flexibly. This also helps to improve the accuracy of solutions.

Previous research has demonstrated that studying correct and incorrect solutions to a problem has many benefits, including the development of expert, flexible, conceptual mathematical knowledge (Booth et al. 2013; Rittle-Johnson et al. 2009; Rittle-Johnson & Star 2007). To examine the benefits of studying both correct and incorrect solutions, Booth and colleagues (2013) randomly assigned 64 eighth-grade Algebra I students to one of the three example-based conditions: correct examples only, incorrect examples only, and both correct and incorrect examples. Students were asked to describe how each example problem was solved and why the solution was correct or incorrect. The results

showed that students gained the most conceptual knowledge by studying correct and incorrect solutions. Surprisingly, however, students who learned only the correct condition did not show a significant increase in procedural fluency compared to their peers in the other two conditions. However, in ordinary classrooms, most teachers only teach students the correct way to solve problems. Therefore, it is crucial to investigate a pedagogy that reconciles the traditional approach (i.e., teaching only the correct method).

Comparing multiple solution methods (i.e., comparing a traditional and unconventional method) may be a good way to address the shortcomings of teaching with a single correct solution approach. That is because comparing multiple methods promotes attention to and adoption of unconventional methods by directing attention to accuracy and efficiency of problem solving (Croße & Renkl, 2006; Rittle-Johnson et al., 2009). However, students benefit from the worked example comparison approach varies from person to person. For example, Rittle-Johnson and colleagues (2009) found that students with high prior knowledge of algebraic procedures benefit the most from comparing worked examples but learn less from sequential study of examples. In this study, seventh and eighth graders (N=236) learn to solve equations by comparing different problem-solving methods to the same problem or comparing different types of problems solved with the same method. The authors argued that learners with previous experience in a domain are able to use their existing knowledge structures to analyze and finish tasks without overloading their working memory. During comparison, when students are familiar with one method, they can learn the new method by comparing it to the familiar method and making inferences about how the unfamiliar method works. Thus, learning

from comparison requires students having adequate prior knowledge with one of the methods (Rittle-Johnson, Star, & Durkin, 2017). The authors also found that students with high prior knowledge are more flexible with methods, because they are more likely to choose and apply the most efficient method when solving problems. This conclusion on the flexibility of those with high prior knowledge is similar to Sweller's (1988) schema construction. High prior knowledge enables learners to quickly and unconsciously select appropriate information leading to the goal because they can recognize each problem state. This is how adequate schemas, as described by Sweller (1988), help problem solvers to take appropriate cognitive actions. Both high prior knowledge and schema construction help problem solvers to reduce working memory load and be more expert-like.

To examine how novices learn mathematics effectively from the comparing worked example method, Rittle-Johnson, Star, and Durkin (2012) conducted a follow-up study in which they slowed down the pace of instruction. Compared to the previous study (Rittle-Johnson et al., 2009), this time students spent more time learning less materials, meaning that there were relatively fewer examples and explanatory prompts, and an additional 30 minutes of intervention time. The research team worked with 198 eighth graders who received little to no instruction in solving equations (lower level of prior knowledge). Students were randomly assigned to one of the three conditions: immediate comparison of procedures (i.e., worked examples were presented in pairs, with one method is more efficient than the other on both Day 1 and Day 2), delayed comparison of procedures (i.e., studying different problems solved with the same method on Day 1 and

comparing two worked examples with one method is better than the other on Day 2), or delayed exposure to multiple procedures (i.e., studying worked examples without comparison on both days). The results of this study showed that the effect of the comparison method condition did not interact with prior knowledge. Furthermore, the compared condition resulted in greater procedural flexibility than sequential learning, regardless of students' prior knowledge. Thus, with appropriate instructional pacing and scaffolding, learners with less prior knowledge were able to learn from comparing two unfamiliar worked examples. One explanation is that with adequate support, novices are able to compare two unfamiliar examples in terms of the respective characteristics of the problem itself, the solution steps, the answer, and which method is more efficient. Nonetheless, the authors emphasized that learning mathematics by comparing two unfamiliar examples seems to be more difficult than learning mathematics by comparing with known methods, and therefore requires more pedagogical support (Rittle-Johnson et al., 2012; Rittle-Johnson et al., 2017).

Similarly, in a study exploring the feasibility of classroom implementation, Newton, Star, and Lynch (2010) demonstrated that high school students with little prior algebraic knowledge were able to learn multiple problem-solving strategies by comparing worked examples in a three-week, researcher-led algebra course. In this small (N=6) algebra remediation/review course, students regularly compared and contrasted worked examples of algebra problems to promote flexibility in problem-solving strategies. The results of the experiment showed that comparing and contrasting worked examples enables students to learn multiple problem-solving methods. Interestingly, even after

learning multiple methods, some students generally preferred to use the familiar, general method, even though they indicated a preference for the alternative method. They would not switch between methods unless the initial approach proved difficult (e.g., if it involved fractions). This may be because students with low prior knowledge tend to “choose a method that maximizes the accuracy of their solution and/or that is most clear or understandable to them” (Newton et al., 2010, p. 34). Students with low prior knowledge may be more confident and comfortable with a familiar method, unless the method proves difficult.

Moreover, Star and colleagues (2015) created a full-year-long supplemental curriculum—side by side worked example pairs (WEPs) and evaluated the impact of teachers’ use of the WEPs curriculum materials. They used a waitlist control design, such that 44 teachers and their students ($n=945$) were randomly assigned to use the intervention WEPs materials, with 32 teachers and their students ($n= 698$) assigned as the control group. All treatment teachers were asked to attend a 35 hours summer professional development institute to become familiar with the WEPs curriculum materials and learn the desired approaches to implement the materials. Treatment teachers were asked to follow as closely as possible the implementation model discussed in the summer professional development, but they were given considerable flexibility in choosing which supplemental curriculum materials to use and how and when to integrate them with the existing curriculum. The findings of the study showed students’ overall, procedural, conceptual, and flexibility knowledge did not differ significantly after the offer of the intervention. However, not all teachers in the treatment group used the

materials regularly. Through further analysis of the data, Star and colleagues (2015) found that increasing students' exposure to intervention materials had a slightly significant positive effect on students' procedural knowledge scores. This effect may reflect the fact that all four comparison types focus on comparing problem solving procedures and should therefore support procedural knowledge.

There were several factors that may explain why the offer of the intervention did not significantly enhance student outcome. First, as noted many treatment teachers implemented the intervention materials much less frequently than intended. Treatment teachers were asked to use the intervention curriculum at least once per week, but 48% of treatment teachers reported using it five times or fewer throughout the year. As Star and colleagues (2015) explained, providing access to intervention materials is unlikely to affect student learning if the intervention is not delivered. Second, in the analysis, one-third of the students had missing data for at least one variable. To manage missing data, the researchers employed multiple imputation, which would more or less affect the results. Third, eight teachers' classes were not included due to lack of posttest administration. The results of this study (Star et al., 2015) reveal the challenge of supporting teachers over time in integrating supplemental comparison materials into the existing curriculum. Beyond these limitations, this study nonetheless provides evidence that comparison can help improve some of the difficulties that students have in Algebra I. It also shows that teachers can strengthen students' procedural knowledge in Algebra through well-designed comparison materials.

Although much empirical support has been provided for the use of comparison cases in student learning in mathematics, research has shown that mathematics teachers do not always make good use of comparison to support instruction (Schenke & Richland, 2017). They often provide too little instructional support to ensure that students pay attention to the meaning of the comparison cases and draw connections to the teacher's intentions (Richland et al., 2007). Schenke and Richland (2017) gave 80 pre-service mathematics teachers a problem and two student solutions (i.e., one of which was correct and the other was incorrect). These participants were asked to design a short lesson using this information provided. The participants were then asked to teach the lesson they had designed to imaginary students. The research team videotaped and coded the instructional activities, and the results show that less than half of the participants (43%) explicitly compared or contrasted the two solution strategies provided. Even for those who did compare, they still did not use additional support strategies to help students pay attention to the key elements of the comparison. The results reveal that participants did not have an awareness of using different student solutions as opportunities for discussion. Therefore, future research should focus on how teachers can make good use of compared methods and the feasibility of these methods to improve learning efficiency.

Worked Example Pairs in the ESOL Context

English language learners experience additional challenges in mastering mathematical concepts when compared to their native English-speaking peers, suggesting the need for more effective instructional strategies in the ESOL classroom. Learning by comparing worked examples may be one of these strategies. Comparison is a powerful

learning tool in cognitive science, and comparing worked examples has shown promising results in mathematics classrooms in a number of previous studies (Rittle-Johnson et al., 2009; Rittle-Johnson et al., 2012; Star et al., 2015). Although comparing worked examples has not been experimented with in ESOL classrooms, based on research on the language specific to mathematical content for English language learners (Kersaint et al., 2008), I hypothesize that the comparing worked examples approach to learning mathematics can also work in an ESOL context. In this section, I analyze the potential benefits of the comparing worked examples approach for ELLs in terms of mathematics learning.

Research on language specific to mathematics content for ELLs shows that an effective second language acquisition environment includes the following two general characteristics: 1. providing sufficient and various opportunities for reading, listening, speaking and writing, and 2. supporting learners to take risks, negotiate, and construct meaning, and seek content knowledge reinterpretation (Garcia & Gonzalez, 1995; Moschkovich, 2012). Research also shows that math instruction in an ESOL context should treat language as a resource (Moschkovich, 2000) and address much more than vocabulary as ELLs learn English (Moschkovich, 1999, 2002). Therefore, to best support ELLs' development of written and oral communication skills, instruction should provide opportunities for students to apply mathematical language and negotiate meaning.

The WEPs align well with research recommendations for ELLs. Worked examples present solution methods step-by-step, which enables ELLs to describe the problem-solving procedures and compare and contrast the two methods presented even if

they do not know how to solve the problems. In the previous WEPs studies (Star et al., 2015; Rittle-Johnson, Star, & Durkin, 2012; Rittle-Johnson, Star, & Durkin, 2017; Rittle-Johnson & Star, 2009; Rittle-Johnson & Star, 2007; Rittle-Johnson et al., 2009), the researchers provided the detailed descriptions of problem-solving procedures that aligned with the solution steps. Such design not only provides learners opportunity to engage in mathematical reading but also facilitates the understanding of solution steps. Student are asked to describe each method, compare them, and then draw conclusions based on the comparison. In conclusion, WEPs have potential to help with creating an effective, acquisition-rich second language learning environment because they provide various opportunities for speaking, listening, writing, and reading. Such an environment allows for meaning construction and knowledge reinterpretations (Garcia & Gonzalez, 1995).

There are gaps in the previous WEPs model in terms of the ESOL literature. One of the most important goals of mathematics instruction for ELLs is “to support all students, regardless of their proficiency in English, in participating in discussions that focus on important mathematical concepts and reasoning” (Moschkovich, 2012, p18). However, the previous WEPs model may not effectively support ELLs’ participation in mathematical reasoning and communication before they develop their proficiency in English. That is because the previous WEPs model does not accommodate any strategies that help ELLs with language acquisition, which will most likely cause them to be left behind (Brown et al., 2009). Many ELLs do not have adequate academic language to complete complex academic tasks (Donnelly & Roe, 2010). Hence, instruction should

incorporate strategies that best support ELLs' language acquisition (Brown et al., 2009; Moschkovich, 2012).

Additionally, research has shown that students' home languages positively affect their math and science learning, regardless of whether or not the teacher speaks these languages. When students are allowed to use their native language in the classroom, both their subject content performance and English-language development improved (Kang & Pham, 1995). Previous models of WEPs have not directly provided opportunities for ELLs to explore new concepts through their native language. However, it is easy to incorporate this strategy into WEPs, such as providing opportunities to look up unfamiliar terms in their native language, having students write down mathematical reasoning or problem-solving procedures in their native language, and allowing students to discuss mathematical problems in their native language in small groups. As Lems (2017) argued, "when learners activate their prior knowledge before engaging in reading, writing, or any kind of academic activity, it's easier for them to hook into many topic areas and respond positively" (p. 21). Lems (2017) also demonstrated that "educators need to consider children's home languages and dialects as assets" (p.19). Crawford (1995) revealed that students can shift their skills in content areas that they learned in their first languages to the second language. Although ELLs engaging in discussion and writing in their native language may not be able to fully express their comprehension in English, this does not mean they do not understand. In most cases, learners understand the mathematical content but because of language limitations they may not be able to fully express their understanding.

A final consideration is that ELLs are often confused about what the questions ask and how to organize their responses in sentences. However, the previous WEPs model have not implemented structured language practice to provide scaffolding for ELLs to communicate using academic language (Sonoma County Office of Education, 2006). Donnelly and Roe (2010) suggested using sentence frames (e.g., The similarity between the two methods is _____.) to help ELLs overcome these limitations. Sentence frames enable ELLs to focus more on the academic content, because they do not need to think about how to form answers on their own. In addition, ELLs are often confused about what is being asked, and by providing them with a sentence frame, they will better understand what is being asked.

To summarize, research has demonstrated that comparison is a best practice in mathematics education (Rittle-Johnson et al., 2017). However, research is lacking on the potential for comparison worked examples to improve learning in ESOL context classrooms. The current study seeks to fill this gap. As a result, the current study aims at examining how the use of comparing worked examples affects English language learners' learning outcome.

Research Questions

In the current study, four questions were asked:

1. To what extent does the use of MLS-WEPs impact ELLs' mathematical performance?
2. Is the effectiveness of the MLS-WEPs intervention related to ELLs' English Language proficiency?

3. Does incorporating MLS-WEPs to the existing curriculum provide ELLs more opportunities to engage in mathematical listening, reading, writing, and speaking?
4. To what extent does the integration of MLS-WEPs impact teachers' use of deep questions and promote equitable participation?

My hypothesis for Research Question 1 is that the use of MLS-WEPs will enhance ELLs' mathematics performance. I base this hypothesis on results from the recent associated research on WEPs (Newton & Star, 2013; Rittle-Johnson & Star, 2007; 2009; 2011; Rittle-Johnson, Star, & Durkin, 2009; 2012; 2017; Star et al., 2015; Richland, Zur, & Holyoak, 2007; Richland, Stigler, & Holyoak, 2012) and research on language specific to mathematics content for ELLs (Garcia & Gonzalez, 1995; Moschkovich, 2012). For Research Question 2, my hypothesis is that the effectiveness of the MLS-WEPs intervention is related to ELLs' English Language proficiency. Various research confirmed mathematical performance is positively correlated to language proficiency (Abedi et al., 2005; Barton, 2008; Planas & Setati, 2009). For Research Question 3, I hypothesize that, with the incorporation of MLS-WEPs, ELLs will have more opportunities to engage in mathematical listening, reading, writing, and speaking. The MLS-WEPs curriculum purposefully provides ELLs with scaffolding in mathematical listening, reading, writing, and speaking. My hypothesis for Research Question 4 is that the intervention materials will enable teachers ask more deep questions and promote equitable participation because the MLS-WEPs curriculum contains many prompt questions that may help teachers generate more in-depth questions. Deep questions include both Type 2 (i.e., Such questions can generally be answered within a

sentence and typically have a clear right or wrong answer.) and Type 3 questions (i.e., Such questions are referential questions that often require longer answers, and generally do not have a preestablished or right/wrong answer.). The comparison approach is extremely informative in that it allows students to generate explanations and provides learners with more mathematical information to discuss (Gentner et al., 2003; Kurtz et al., 2001; Thompson & Gentner, 1999). Thus, it may provide teachers better opportunities to ask deep questions and promote equitable participation.

CHAPTER 3

METHODOLOGY

Introduction

The current study aims at investigating if Modified for Language Support-Worked Example Pairs (MLS-WEs) effectively improve ELLs' mathematical problem solving and literacy skills. Paas and Renkl (2003) claimed, a worked example is one of the earliest techniques that improves learning by reducing cognitive load. In the current study, I developed worked example pairs focusing on enhancing ELLs' mathematical literacy skills, in order to ultimately improve their problem solving and reasoning skills. Newton and Star (2013) claimed that comparing worked examples side-by-side is "an effective way to teach multiple strategies for solving problems. Side-by-side presentation facilitates comparison of these different strategies, which can help students notice and understand important mathematical concepts" (p. 88). The side-by-side worked examples designed in the current study used discussion prompts, step-by-step problem-solving descriptions and procedures, native languages, think-pair-share activities, and sentence frames to improve ELLs' mathematical literacy skills.

Pilot Study

Although several studies on the comparison of worked examples have been done previously, research is lacking on the potential for comparison in the ESOL context. For this reason, I have done a pilot study to explore how to effectively implement WEs in an ESOL context. In the pilot study, I worked with ELLs using WEs in one-on-one

after school tutorials. The data that I collected from the pilot study revealed that, compared to their native speaking peers, ELLs face extra challenges. The ELLs seemed overwhelmed when they were first presented with the material, as if they did not understand the mathematical concepts. However, after working with them closely, I realized that they actually could understand the underlying concepts, but could not comprehend some of the text in which the concepts were presented because of their still-developing English proficiency. Therefore, I encouraged them to look online or use a dictionary to check the meanings of the unfamiliar terms in their native languages. Not surprisingly, after looking at the definitions in their native languages, they were able to give mathematical examples to show their comprehension. Therefore, it is extremely important to provide ELLs' opportunity to build background knowledge and comprehend unfamiliar terms using their home languages (Lems et al., 2017).

Moreover, when answering short response questions, ELLs usually did not answer all the questions that were asked. For example, when asked to describe some similarities and differences between two methods, many ELLs only answered the differences between the two methods but neglected the similarities. Lastly, many learners' sentences were often unclear. For instance, when asked to state a general rule describing what they have learned from comparing the two examples presented, one student wrote, "you can solve these problems with any solution, but you must be able to understand or you can find it simple and not wrong." After further communicating with the student, I learned that she tried to express three points. First, you can solve these kinds of problems using either method. No matter which method you choose, the solutions are both correct.

Second, some problems are easier to solve using one method than the other. Third, in order to determine which method is simpler, you must understand the numbers given in the problem. Many learners told me that they tried to clearly express their thoughts. However, due to their limited understanding of English language rules and vocabulary, they were afraid of making errors; thus, they decided to write in a “safe way.” Lems et al. (2017) claimed that there are various reasons that ELLs make mistakes and errors in writing: “they may over generalize language rules, be unsure of rhetorical or text structures, be unclear about what they want to say or lack the vocabulary to say it” (p. 244). After carefully diagnosing ELLs’ challenge, I tried sentence frames to help my learners with answering short response questions. Since then, ELLs were able to produce clearer sentences. Sentence frames enable students to focus more on the academic content, because they do not need to think about how to form answers on their own. Additionally, ELLs are often confused about what questions ask and with a sentence frame, they understand what is being asked better. My goal is to use sentence frames to help ELLs express themselves clearly and confidently.

Current Study

In order to provide ELLs with feasible WEPs materials, modifications of several areas were made to the previous model (Newton & Star’s, 2013; Star et al.’s, 2015). Such modifications were made in order to engage ELLs in mathematical learning while they are still developing proficiency in English. First, the current model provided learners opportunities to engage in discussion or writing using native languages and present the final product in English to the whole group. Second, ELLs with low English language

proficiency had freedom to choose their preferred language to describe their problem-solving procedures and answer the prompt questions. Third, to improve ELLs' mathematical writing skills, I implemented sentence frames. My goal is to use sentence frames to help ELLs express themselves clearly and confidently. In addition, I rephrased the prompt questions to adjust their language complexity. Finally, I asked students to provide specific mathematical examples before answering the related conceptual understanding questions. For example, before asking ELLs to describe the definition of "like terms," I first asked students to provide mathematical example of like terms (e.g., $3x$ and $5x$).

Participants and Settings

Data for this study were collected in a large K-12 urban school district with two schools. At the time of the study, according to the district website, there are more than 15,000 English language learners who come from more than 130 countries and speak more than 100 languages. The ELL population is approximately 7.4% of the total student population. Before the study began, I sent out a recruitment letter to all the math teachers in the district who teach sheltered algebra classes. And two ESOL math teachers agreed to participate in the current study ($N_{\text{Student1}} = 98$; $N_{\text{Class1}} = 3$; $N_{\text{Student2}} = 49$; $N_{\text{Class2}} = 4$). The average class size of Teacher 2 ($N_{\text{Teacher2}} = 12$) was smaller than that of Teacher 1 ($N_{\text{Teacher1}} = 33$). Students who did not complete the consent form or missed any of the assessments (i.e., Unit 1 Pretest, Unit 1 Posttest, Unit 2 Pretest, and Unit 2 Posttest), and survey were excluded from the study. In the final sample, there were 47 students from Teacher 1's three sheltered algebra classes and 31 students from Teacher 2's four

sheltered algebra classes. All classes were mixed grade level classes from 9th to 11th grade. Students spoke different native languages, according to the participating teachers, Portuguese, Spanish, and Chinese were the three most popular native languages spoken by Teacher 1’s students and Spanish and French were the two most popular native languages spoken by Teacher 2’s students. Each student in the same class had a different level of language proficiency. See Table 3.2 for student English language proficiency. Both teachers were very experienced teachers. See Table 3.1 for teacher demographics. The study was conducted during COVID 19, therefore, all lectures in this study were delivered in a virtual environment.

Table 3.1

Teacher Demographics

	Teacher 1	Teacher2
Year of teaching experience	30	19
ESOL certificate	No	Yes
Undergraduate degree	Secondary math education	Mathematics
Graduate degree	Equivalence in Education	Math Education
QTEL (Quality Teaching for English Learners) training	Yes (twice)	Yes (twice)
English language learner	No	Yes

Table 3.2

Student English Language Proficiency Levels

	Level 1	Level 2	Level 3	Level 4	Level 5
Teacher 1	6	9	15	10	7
Teacher 2	7	5	10	4	5

Materials

I adapted the Worked Example Pair curriculum (Durkin et al., under review) developed by the Leveraging Comparison and Explanation of Multiple Strategies (CEMS) to Improve Algebra Learning Project for English Language Learners. The intervention materials contain four types of worked example comparisons: (1) *Which is better?* (2) *Why does it work?* (3) *Which is correct?* (4) *How do they differ?* (Newton & Star, 2013; Star et al. 2015). First, *Which is better?* WEPs (see Appendix A) demonstrate two different approaches to solving a problem, and both methods are correct, where one method is more efficient or easier than the other under certain circumstances. Similarly, *Why does it work?* WEPs (see Appendix B) illustrate two different correct methods of solving the same problem but with one of the two methods illuminating the conceptual rationale. Next, *Which is correct?* WEPs (see Appendix C) present a problem solved in two different ways, where one method illustrates a common error. Finally, *How do they differ?* WEPs (see Appendix D) show two different problems solved in similar ways, which illuminates an underlying mathematical concept or feature.

I adjusted the materials and instructional strategies according to students' reflection and feedback in the pilot study. Specifically, I added sentence frames to the

discussion prompts and short response questions. In this way, students only needed to focus on what they want to say or write, rather than the language rules and/or the questions asked. Teachers were recommended to provide students time to look up unknown terms in their native languages. Teachers were also suggested to group students by their native languages for think-pair-share activities. In this way, students could discuss their ideas in their native languages and share the results with the whole class in English. This enabled those who have low English language proficiency to participate in classroom discussion. Finally, teachers should allow those who have very low English language proficiency to write in their native languages and then translate it into English.

Students who were in the experimental group were given a pair of worked examples (the intervention materials), while students in the control group were taught the same topics using problem-solving methods (teachers demonstrated how to solve the problems step by step). The WEPs explorations were scaffolded using three kinds of prompts: *Understand*, *Compare*, and *Make Connections* (Star et al., 2015). In order to complete the task, students first needed to read the problem-solving descriptions and comprehend each step presented in the worked examples (*Understand*). This activity is a good way to practice mathematical reading. Because the step-by-step problem-solving procedures are provided, students can comprehend the descriptions by matching them with the steps presented. After comprehending the procedures of each worked example, students were then asked to describe the similarities and differences between the two approaches that were illustrated in the worked examples (*Compare*). Next, students shared their ideas in pairs or wrote down their ideas independently and modified their

answers after a whole group discussion. Finally, students were asked to summarize what they had learned by comparing the two examples (*Make Connections*). In this phase, students first shared their ideas out loud, the teacher then gave direct instruction, and after that, students summarized the big idea in their own words.

The intervention materials were supplements to the existing curriculum. Therefore, they were implemented after the corresponding topics were introduced. To ensure the effectiveness of using MLS-WEPs, teachers were encouraged to implement various language supports. For example, before implementing the MLS-WEPs, teachers may guide students in both the treatment and control groups to comprehend the mathematical concepts and/or ideas using various strategies, such as manipulative tools, visual images, assistance of L1, and concrete examples, to name a few. The control groups should receive the same language support, except for the MLS-WEPs materials.

Teachers were asked to select at least two units from the intervention bank, which comprises Unit 1 Linear Equations, Unit 2 Functions, Unit 3 System Equations, Unit 4 Polynomials, and Unit 5 Radicals. The participating teachers decided to implement the Linear Equation and Function units. To ensure both units had control and intervention data, after discussion, Teacher 1 implemented the intervention material of the Linear Equation unit in her classes, while Teacher 2 taught the same unit without using the MLS-WEPs. As they moved to the Function unit, Teacher 2 implemented the intervention material in her classes, while Teacher 1 taught the same unit without the use of the MLS-WEPs.

Design

To ensure credibility and reliability, I used a waitlist control design so that both the control and treatment groups received the intervention materials (Elliott & Brown, 2002). The design is illustrated in Table 3.3. At the beginning of each topic, both groups were given a researcher-designed math content knowledge pretest. The treatment group were then taught using the MLS-WEPs curriculum and the control group were taught the same topics using the business-as-usual instruction (i.e., teacher demonstrate the step-by-step problem-solving procedures). At the end of the intervention, both groups were given a content-based posttest, which was the same as the pretest.

Table 3.3

Design Illustration

Teacher 1					
Unit 1			Unit 2		
Topic 1 pretest	intervention	Topic 1 posttest	Topic 2 pretest	Control condition	Topic 2 posttest
Teacher 2					
Unit 1			Unit 2		
Topic 1 pretest	Control condition	Topic 1 posttest	Topic 2 pretest	intervention	Topic 2 Posttest

Measures

Lesson Audio

The participating teachers were asked to make an audio recording each time they taught the selected topics, either using the intervention or control materials. Teacher 1 submitted a total of 12 audiotapes ($N_{\text{duration}} = 368$ minutes; $N_{\text{class}} = 3$; $N_{\text{topic}} = 4$) for the intervention condition (Unit 1) and 3 audiotapes ($N_{\text{duration}} = 120$ minutes; $N_{\text{class}} = 3$; $N_{\text{topic}} = 6$) for the control condition (Unit 2). Teacher 2 submitted a total of 8 audiotapes ($N_{\text{duration}} = 244$ minutes; $N_{\text{class}} = 4$; $N_{\text{topic}} = 6$) for the control condition (Unit 1) and 20 audiotapes ($N_{\text{duration}} = 452$ minutes; $N_{\text{class}} = 4$; $N_{\text{topic}} = 5$) for the intervention condition (Unit 2). Those lesson recordings provided data on mathematical literacy engagement and teacher questioning.

Survey

The participating teachers administered the student survey at the beginning of the intervention. In the survey, ELLs were asked to rate their English language proficiency, provide the English language art course that they were taking, and rate their level of comfort with the intervention materials. The student survey provided students' English language proficiency. The English proficiency levels used in the current study were derived from the students' self-report in the student survey. To ensure that the self-reported English proficiency levels in this study were as accurate as possible, I included their level of comfort with English materials, the level of English classes they were taking, and the recommendations of their teachers.

Assessment

To measure student learning and effectiveness of the intervention materials, I developed unit assessments that were consistent with the selected topics. A sample assessment is attached in Appendix E. Thus, the teaching and learning aligned with the assessments. For each unit, the pretest and posttest were the same, whereas the assessments were different across units. For example, Unit 1 was about linear equations, so its assessment was only focused on linear equations. Unit 2 was about functions, so its assessment was only focused on functions. Each item in the assessment contained two parts: a. multiple choice and b. explanation of how to get the answer. Unit 1 assessment contained seven items. Of these, one was solving one step equation, one was simplifying expression, and five were solving multi-step equations, all of which were designed to assess procedural knowledge. Procedural knowledge is the ability to perform a series of actions to solve a problem, including transferring a known procedure to a new problem (Rittle-Johnson & Star, 2007). Unit 2 assessment contained eight items. Of these, two were examining conceptual understanding of function (e.g., Students needed to determine and explain whether the given relation is a function or not). Conceptual knowledge means having a certain understanding of concepts in a field and the interconnections of the concepts or ideas in that field (Star et al., 2015). The other six items were designed to assess procedural knowledge of function: one was evaluating a function, two were finding y-intercept or equation when given two points, three were about graphs of linear functions. The unit assessments were designed to tap two components of mathematics performance – problem solving ability and verbal explanation skill. At the beginning of

the intervention, the participating teachers reviewed the students' assessment questions, and both teachers had no problems with the questions asked.

The teacher participants administered a 30-minute pretest at the beginning of the intervention and a 30-minute posttest at the end of the intervention. Pre- and post- tests were given each time a new topic was taught. For example, when the Linear Equation unit was taught, the Linear Equation unit pre- and post- tests were given, and when the Function unit was taught, the Function unit pre- and post- tests were given. Teachers were asked to select two units from the intervention bank, so students received two sets of pre- and post- assessments.

Professional Development and Implementation

Originally, I planned to provide a week-long professional development for the participating teachers so that they could appropriately present and scaffold new instructional strategies. However, due to many external factors (e.g., teacher availabilities, lack of compensation, difficult consent process etc.), instead of providing teachers with a week-long professional development, prior to the start of the study, I met with teachers individually for three hours to discuss the project and how to implement MLS-WEPs. During the one-on-one professional development, participating teachers were taught how to teach using MLS-WEPs materials as if they were the students. But participant teachers were informed that this was one of the ways to incorporate MLS-WEPs, and the participating teachers had flexibility to make adjustments according to their teaching styles and the learning abilities and styles of their students. In addition to the three hours of professional development, I also communicated with both teachers on a

regular basis (at least once a week). All materials were given to the teachers so they just needed to be able to implement them. During PD, both teachers reported that they did not generally use comparisons as a teaching method and were not told not to use comparison approaches in the control condition.

Data Analysis

The data analysis methods are illustrated in Table 3.4. The content-based assessments were analyzed separately by units. The mean and standard deviation of each topic for both pre- and post- tests are shown in Table 3.5 and Table 3.6. To compare ELLs' knowledge between groups at the beginning of the intervention, a sample t-test was performed. To determine whether the intervention produces significant differences in overall mathematical knowledge, a one-way ANCOVA was conducted. To examine knowledge gains from the pretest to posttest in each condition, a paired samples t-test for both the treatment and control groups was performed.

Coding

Assessment

Both the multiple choice and explanation parts on the pretest and posttest were scored for accuracy of the answer. Students received one point for correctly selecting each the multiple-choice part. And each explanation item was scored on a 1-point scale (1 points for fully correct explanation, .5 point for partially correct explanation, 0 point for

Table 3.4

Data Analysis Illustration

QR	Data	Analysis
<p>1. To what extent does the use of MLS-WEPs impact ELLs' mathematical performance?</p>	<ul style="list-style-type: none"> • Pretest scores • Posttest scores 	<p>Knowledge at the beginning of the intervention.</p> <ul style="list-style-type: none"> • Mean • Standard deviation • Sample t-test <p>Knowledge Gains from pretest to posttest within</p> <ul style="list-style-type: none"> • To determine whether the intervention produces significant differences in overall mathematical knowledge, a one-way ANOVA was conducted. • To examine knowledge gains from the pretest to posttest in each condition, a paired samples t-test for both the treatment and control groups was performed. <p>Quality of Explanation</p> <ul style="list-style-type: none"> • To examine the quality of written explanations in the assessments, the explanation part of the assessments was coded.
<p>2. Is the effectiveness of the MLS-WEPs intervention related to ELLs' English Language proficiency?</p>	<ul style="list-style-type: none"> • Assessment scores • English language proficiency level 	<ul style="list-style-type: none"> • Pearson's r <p>To examine the effect of English language proficiency on MLS-WEPs intervention, a one-way ANCOVA was conducted, controlling for prior knowledge.</p>

Table 3.4*Continued*

QR	Data	Analysis
3. Does incorporating MLS-WEPs to the existing curriculum provide ELLs more opportunities to engage in mathematical listening, reading, writing, and speaking?	<ul style="list-style-type: none"> Lesson audios 	To determine whether the intervention provided English learners with more opportunities to listen, read, write, and speak mathematics, I coded the lesson recordings and compared such opportunities provided by the control and treatment groups.
4. Does integrating MLS-WEPs into the curriculum improve teachers' use of deep questions and promotion of equitable participation?	<ul style="list-style-type: none"> Lesson audios 	I coded the lesson recordings to determine whether or not the intervention enhances teachers' attempts to ask deep questions and encourage students to engage in mathematical talk and to interact with other students.

Table 3.5*Means and (Standard Deviations) for Unit 1*

Condition	Unit 1 Pre-test			Unit 1 Post-test		
	MC	Explanation	Total	MC	Explanation	Total
Treatment (Teacher 1)	5.06 (1.89)	1.21 (1.89)	6.30 (2.98)	6.30 (1.49)	3.40 (2.74)	9.70 (3.52)
Control (Teacher 2)	5.45 (1.93)	2.47 (2.49)	7.89 (3.67)	5.61 (2.14)	2.48 (2.71)	8.38 (3.95)

Table 3.6*Means and (Standard Deviations) for Unit 2*

Condition	Unit 2 Pre-test			Unit 2 Post-test		
	MC	Explanation	Total	MC	Explanation	Total
Treatment (Teacher 2)	3.00 (1.75)	0.58 (0.88)	3.58 (2.19)	4.84 (2.28)	2.32 (2.35)	7.16 (4.40)
Control (Teacher 1)	3.81 (1.93)	1.11 (1.43)	4.92 (2.88)	4.38 (2.07)	1.33 (1.68)	5.71 (3.31)

blank, irrelevant response, or relevant but incorrect response). In addition to scoring accuracy, students' explanations were coded into six categories for further analysis: fully correct explanation, partially correct explanation, concept relevant but incorrect, concept irrelevant explanation, uninterpretable explanation, and blank. Examples of coding will be presented in the next chapter.

To determine the correlations between ELLs' mathematical performance (i.e., problem solving score and explanation score) and their English language proficiency, I computed Pearson's r (Pearson's correlation coefficient). To examine whether the MLS-WEPs intervention effects are related to ELLs' English language proficiency, a one-way ANCOVA was conducted, controlling for prior knowledge.

Lesson Audio

To examine whether incorporating MLS-WEPs to the existing curriculum facilitates teachers to provide ELLs with more opportunities to engage in mathematical listening, reading, writing, and speaking, and to what extent the integration of MLS-WEPs impacted teachers' attempts to ask deep questions and increase student participation, I coded the lesson audios using an inductive approach. To analyze the data,

I coded the lesson audios along broad open coding categories related to teacher questionings, equitable participations, and mathematical literacy engagements. I first viewed all the lesson audios and realized that both teachers used similar strategies, asked similar questions, and followed similar routines for all classes while teaching the same materials. Thus, I decided to focus on one class for each teacher. One-third of the lessons were double-coded by a second coder who did not know the participating teachers and students.

After viewing all the audiotapes, I decided to focus on the first class of both teachers. I first transcribed all the audiotapes of Teacher 1's and Teacher 2's first periods. I then coded the data sources from sentence to sentence and sought "a collection of instances from the data, hoping that issue-relevant meanings will emerge" (Creswell, 2012, p. 206). All of my coding process was based on what I saw in the data. I did not utilize any literature to guide me with any additional coding categories. Through several rounds of coding processes, I was able to determine the emerged themes, which are teacher questioning (i.e., teacher uses referential, display, and probing questions), interaction (i.e., teacher-to-student, student-to-student, student-to-teacher), mathematical literacy engagement (i.e., how and how often did students engage in mathematical reading, writing, speaking, and listening), waiting time (i.e., the length of time between teacher asking a question and students responding to that question). I then performed the second round of coding using these four codes. During the second coding, mathematics communication behaviors (e.g., explain their thinking, restate another student, ask for clarification, extend other's thought, agree/disagree, correct other students' responses etc)

emerged as a recurring theme. In the final codebook, five sets of codes were included: teacher questioning, interaction, mathematical literacy engagement, waiting time, and mathematics communication behaviors. More detailed code examples will be provided in the Result chapter.

CHAPTER 4

RESULTS

This study was undertaken in order to gain a better understanding of pedagogical strategies that enhance ELLs' mathematical achievement and classroom participation. The results are reported in four sections corresponding to the four research questions: the impact of using MLS-WEPs on ELLs' mathematical performance, whether the effectiveness of MLS-WEPs intervention is associated with ELLs' English language proficiency, the influence of MLS-WEPs on English language learners' opportunities to engage in mathematical listening, reading, writing, and speaking, and the impact of MLS-WEPs on teachers' use of deep questions and promotion of equitable participation.

QR 1. To What Extent Does the Use of MLS-WEPs Impact ELLs' Mathematical Performance?

Knowledge at the Beginning of the Unit 1 Intervention

The intervention occurred after students had completed the relevant topics. Thus, they had some prior knowledge while taking the pretests. At the beginning of the intervention (i.e., during Unit 1 pretest), there was no significant difference between treatment and control groups for the multiple-choice part, $t(76) = -0.879$, $p = .382$, although the control group ($M_{mc} = 5.45$, $SD_{mc} = 1.93$) attained slightly higher scores than those of the treatment group ($M_{mc} = 5.06$, $SD_{mc} = 1.89$). Yet for the explanation part, compared to students in the treatment group ($M_{explanation} = 1.21$, $SD_{mc} = 1.89$), students in the control group ($M_{explanation} = 2.47$, $SD_{explanation} = 2.49$) demonstrated significantly higher scores, $t(52.2) = -2.392$, $p = .020$. In other words, the control group students performed better on both the multiple-choice (not significant) and explanation (significant) parts

than those in the treatment group. Thus, it was necessary to control for this initial difference in explanations in subsequent analyses of performance at the posttest. Should a significant difference emerge favoring the treatment group, such a finding would mean the treatment was able to not only overcome this initial difference but also improvement performance beyond that.

Knowledge Gains from Unit 1 Pretest to Posttest

In the Unit 1 posttest, the control group's superior performance vanished. There was no longer a significant effect for conditions in both the multiple-choice part, $t(48.9) = 1.552$, $p = .099$, and explanation part, $t(76) = 1.348$, $p = .182$, of the Unit 1 posttest. It may be argued that the control group had a higher starting point, so it had less room to grow. In fact, as shown in Table 3.5, the treatment group ($M_{mc} = 6.03$, $SD_{mc} = 1.49$; $M_{explanation} = 3.40$, $SD_{explanation} = 2.74$) outperformed the control group ($M_{mc} = 5.61$, $SD_{mc} = 2.14$; $M_{explanation} = 2.48$, $SD_{explanation} = 2.71$) in the Unit 1 posttest.

To further confirm the effectiveness of the intervention, a one-way analysis of covariance (ANCOVA) was conducted to examine the relationship between conditions (i.e., control and treatment) and Unit 1 posttest scores, adjusting for Unit 1 pretest scores. For the multiple-choice part, the assumptions of homogeneity of variance and linearity held, but the assumption of homogeneity of regression was violated, indicating that ANCOVA may be an inappropriate method of analysis and the results should be interpreted with caution. The results from the analysis indicated that the multiple-choice part of the Unit 1 posttest differed significantly by condition, $F(1, 74) = 23.80$, $p < .001$, controlling for pretest score. Additionally, the post-hoc test revealed that students in the treatment condition outperformed students in the control condition in the multiple-choice

part of the Unit 1 post-test by 0.975 points (Cohen's $d = 0.0237$). For the explanation part of Unit 1, the assumptions of linearity and homogeneity of regression held, but the assumption of homogeneity of variance was violated. Likewise, the results from the analysis indicated that the post-test explanation part of Unit 1 differed significantly by condition, $F(1, 74) = 29.20, p < .001$. The post-hoc test showed that students in the treatment condition scored 1.96 points higher than students in the control condition on the posttest explanation part of Unit 1 (Cohen's $d = 0.0426$).

To examine knowledge gains from the pretest to posttest in each condition, I conducted a paired samples t-test for both the treatment and control groups. As illustrated in Table 4.1, students in the comparison condition showed significant gains in both the multiple-choice part, $t(46) = -4.91, p < .001$, and explanation part, $t(46) = -7.76, p < .001$. On the other hand, students in the control condition made no significant gains, neither in the multiple-choice part, $t(30) = -0.378, p = .378$, nor in the explanation part, $t(30) = -0.133, p = .895$, although the results of the posttest ($M_{mc} = 5.61, SD_{mc} = 2.14; M_{explanation} = 2.48, SD_{explanation} = 2.71$) reached higher scores than those of the pretest ($M_{mc} = 5.45, SD_{mc} = 1.93; M_{explanation} = 2.47, SD_{explanation} = 2.49$). However, these differences are negligible.

Knowledge at the Beginning of the Unit 2 Intervention

When Unit 2 was introduced at a later date, the wait-listed participants (Teacher 2 and her students) who were in the Unit 1 control condition, now received the Unit 2 intervention. This time, Teacher 1 and her students who were in the Unit 1 treatment condition were assigned as the control group. At the beginning of the Unit 2 intervention

Table 4.1***Unit 1 Paired Sample T-Test***

Condition	Unit 1 Multiple Choice			Unit 1 Explanation		
	statistic	df	p	statistic	df	p
Treatment (Teacher 1)	-4.91	46.0	<.001	-7.76	46.0	<.001
Control (Teacher 2)	-0.378	30.0	.378	-0.133	30.0	.895

(i.e., Unit 2 pretest), there was no significant effect by condition in the multiple-choice part, $t(76) = 1.88$, $p = .064$. However, compared to the participants (Note: In Unit 2, they were in the treatment group.) in the Unit 1 control group ($M = 0.58$, $SD = 0.88$), the students (Note: In Unit 2, they were in the control group.) who received the Unit 1 intervention materials demonstrated significantly better scores in the explanation part ($M = 1.11$, $SD = 1.43$) of the Unit 2 pretest, $t(75.7) = 2.01$, $p = .048$. This may be due to the transfer of explanatory skills learned in the intervention materials in Unit 1 to different mathematical concepts.

Knowledge Gains from Unit 2 Pretest to Posttest

Similar to Unit 1, a one-way analysis of covariance (ANCOVA) was conducted to examine the relationship between conditions and Unit 2 posttest scores, adjusting for Unit 2 pretest scores. For the multiple-choice part of Unit 2, the assumptions of homogeneity of variance, homogeneity of regression, and linearity held. The results of the analysis indicated that there was no significant effect by condition in the multiple-choice part of the Unit 2 posttest, $F(1, 74) = 2.0194$, $p = .160$. For the explanation part of Unit 2, the assumptions of linearity and homogeneity of regression held, but the assumption of homogeneity of variance was violated, indicating that ANCOVA may be an inappropriate

analysis and the results should be interpreted with caution. The results from the analysis indicated that the explanation part of the Unit 2 posttest differed significantly across conditions, $F(1, 74) = 5.56, p < .021$. The post-hoc test revealed that students in the treatment condition outperformed students in the control group in the explanation part of the Unit 2 post-test by 1.65 points (Cohen's $d = 0.0347$), indicating that worked example comparison is an effective tool for improving the mathematical explanatory skills of ELLs.

To examine knowledge gains from the pretest to posttest by condition, a paired samples t-test was conducted for both the treatment and control groups. As illustrated in Table 4.2, students in the comparison condition made a significant gain from pretest ($M_{mc} = 3, SD_{mc} = 1.75; M_{explanation} = 0.58, SD_{explanation} = 0.88$) to posttest ($M_{mc} = 4.48, SD_{mc} = 2.28; M_{explanation} = 2.32, SD_{explanation} = 2.35$) in both the multiple-choice part, $t(30) = -5.16, p < .001$, and explanation part, $t(30) = -4.89, p < .001$. These results are similar to the results in Unit 1. On the other hand, students in the control condition had no significant gain in the explanation part, $t(46) = -1.27, p = .209$, despite the results from the posttest ($M = 1.33, SD = 1.68$) reached higher scores than those of the pretest ($M = 1.11, SD = 1.43$). However, these differences were minimal. Unlike the results of Unit 1, students in the control condition showed a significant improvement in the multiple-choice part of the Unit 2 posttest ($M = 4.38, SD = 2.07$) compared to the pretest ($M = 3.81, SD = 1.93$), $t(46) = -2.51, p = .016$. This may be because students in the control group scored higher on the multiple-choice part of Unit 1 and therefore had little room for growth. Yet in Unit 2, students in the control group scored lower on the pretest and therefore had more room for growth. Or because the business-as-usual instruction also foster some

growth in Unit 2. The results of Unit 2 showed that the comparison of worked examples not only improved ELLs' mathematical written explanation skills, but also enabled ELLs to transfer such skills to different mathematical concepts.

Table 4.2

Unit 2 Paired Sample T-Test

Condition	Unit 2 Multiple Choice			Unit 2 Explanation		
	statistic	df	p	statistic	df	p
Treatment (Teacher 2)	-5.16	30	< .001	-4.89	30	< .001
Control (Teacher 1)	-2.51	46	.016	-1.27	46	.209

Quality of Explanation

To examine the quality of written explanations in the assessments, I coded the explanation part of the assessments. The coding scheme focused on six general types of responses: fully correct explanations, partially correct explanations, conceptually relevant but incorrect explanations, conceptually irrelevant explanations, uninterpretable explanations, and blank. The inter-rater reliability of one-third of the data from a second coder exceeded 85%. Examples of each coding and the results of the analysis are presented in Table 4.3 and Table 4.4. The results of this analysis showed that the quality of ELLs' explanations improved in the post-assessments in both comparison and sequential conditions. However, the comparison condition had a greater gain in terms of the explanation quality. In addition, at the beginning of the intervention, many students did not know how to do mathematical reasoning in writing. A representative explanation at the beginning of the intervention was, "I used my pen and pencil calculate." The

Table 4.3

Sample Explanations

Explanation codes	Sample explanations for the following question Determine if the points in the table could represent a linear function. <table border="1"><tr><td>x</td><td>4</td><td>6</td><td>7</td></tr><tr><td>y</td><td>10</td><td>20</td><td>25</td></tr></table>	x	4	6	7	y	10	20	25
x	4	6	7						
y	10	20	25						
Fully correct	“because when I make the graph the line is straight and it does not have curves. The three points made a completely straight line.” “I calculated the rate of change between points and they are all the same. So, it is linear function.”								
Partially correct	“I noticed that 5 were not in the table so I put it in. Now the change in the y is constant as x goes up.”								
relevant but incorrect	“Because is not matching the exact same number example x it going by 2 and 1 and y it going by 10 and 5.”								
Irrelevant	“I used my brain, pen, and paper find the answer.” “Because the teacher teach me before.”								
Uninterpretable	“I did $k=4/10$ and I divided both by the smaller numbers.” “Yes because for the linear function you need 2 points and you 4, 6, 10, 20.”								
blank	n/a								

Table 4.4***Quality of Explanation Results (in Percentage)***

	Unit 1				Unit 2			
	treatment		control		treatment		Control	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
Fully correct	22.77	66.71	31.25	32.59	6.25	26.17	17.19	23.44
Partially correct	5.36	4.46	5.80	3.57	1.56	2.34	2.34	2.34
relevant but incorrect	1.79	8.92	0	1.34	2.34	11.72	11.72	7.03
Irrelevant	18.93	4.02	17.86	15.63	11.72	1.95	2.34	2.34
Uninterpretable	5.80	6.70	5.00	4.61	4.21	7.93	8.20	5.86
Blank	45.35	9.19	40.09	42.26	73.92	49.89	58.21	58.99

intervention materials appeared to help ELLs better understand how to reason in writing than those of in the control materials. The number of concept-irrelevant explanations reduced in the treatment condition and this advantage was retained until the end of the Unit 2 intervention, even though they did not receive the intervention materials in Unit 2. Yet, the number of concept-irrelevant explanations remained almost the same in the control condition. Uninterpretable explanations are the responses that the coder could not understand. This issue may be caused by a lack of mathematical concept knowledge and language or English language proficiency. In both Unit 1 and Unit 2, there was an unexpected increase in the uninterpretable explanation category in the treatment group and a slight decrease in the control group. One explanation may be that the intervention materials motivated ELLs to perform mathematical explanations even though they may not have the ability to do so. Furthermore, the number of blank explanations reduced substantially in the treatment group, while there was almost no change in the control group. Overall, the intervention materials improved the quality of English language learners' written explanations.

QR 2. Is the Effectiveness of the MLS-WEPs Intervention Related to ELLs' English Language Proficiency?

The correlation matrix (Table 4.6) showed that ELLs' mathematical performance was positively correlated with their English language proficiency, and this association was stronger in the explanation part than in the multiple-choice part. These findings are consistent with the existing literature (Ellerton & Clarkson, 1996; Clarkson & Galbraith, 1992; Jarrett, 1999). In addition, the scores on each part of the posttest were highly correlated with those on the corresponding part of the pretest, implying that ELLs' performance in mathematics was highly dependent on prior knowledge. To examine the

effect of English language proficiency on MLS-WEPs intervention, a one-way ANCOVA was conducted, controlling for prior knowledge. The assumption checks and results of ANCOVA are presented in Table 4.5. The results of the analysis indicated that there was no significant effect of English language proficiency level on the Unit 1 multiple choice, Unit 1 explanation, Unit 2 multiple choice, or Unit 2 explanation parts after controlling for prior knowledge, $F_{\text{Unit 1_MC}}(4, 41) = 0.513, p = .727$, $F_{\text{Unit 1_Explain}}(4, 41) = 1.65, p = .181$, $F_{\text{Unit 2_MC}}(4, 25) = 2.31, p = .086$, and $F_{\text{Unit 2_Explain}}(8, 21) = 1.210, p = .336$. The marginal means plots of the post assessments for both teachers' classes by English language proficiency are illustrated in Figures 4.1, 4.2, 4.3, and 4.4. The results of post hoc tests for the Unit 1 multiple choice, Unit 1 explanation, Unit 2 multiple choice, and Unit 2 explanation parts are shown in Table 4.7, 4.8, 4.9, and 4.10. These results all suggest that the effectiveness of the MLS-WEPs intervention did not vary by ELLs' English language proficiency after controlling for prior knowledge, implying that students of different language proficiency levels benefited equally from the MLS-WEPs intervention.

QR 3. Does Incorporating MLS-WEPs to the Existing Curriculum Provide ELLs More Opportunities to Engage in Mathematical Listening, Reading, Writing, and Speaking?

The lesson recordings were transcribed in order to better understand how MLS-WEPs supported mathematical listening, reading, writing, and speaking. After the first round of the coding process, I decided not to include the listening domain in this analysis because whenever someone is involved in a mathematical discussion, they are given the opportunity to listen. In addition, it is difficult to examine the quality of the listening input, especially when the coding resource was audiotapes since I could not see students'

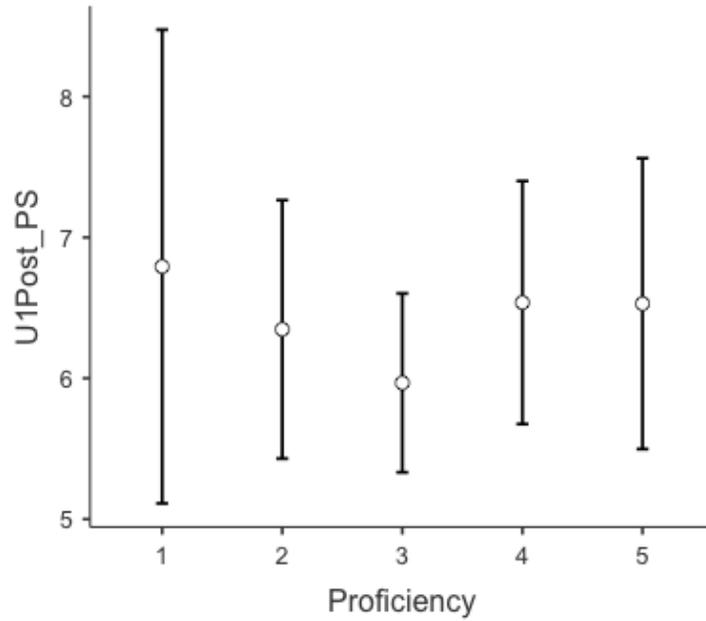
Table 4.5***The Effect of English Language Proficiency on MLS-WEPs***

Teacher	Units	Assumption Checks			Statistics			
		Homogeneity of variance	Homogeneity of regression	linearity	F value	df 1	Df 2	p-value
1	Unit 1_MC	held	held	held	0.513	4	41	.727
1	Unit 1_Explain	held	held	held	1.65	4	41	.181
2	Unit 2_MC	held	violated	held	2.31	4	25	.086
2	Unit 2 Explain	held	held	held	1.210	8	21	.336

Note: Unit 1_MC represents the multiple-choice part in the Unit 1 posttest. Unit 1_Explain represents the explanation part in the Unit 1 posttest. Unit 2_MC represents the multiple-choice part in the Unit 2 posttest. Unit 2_Explain represents the explanation part in the Unit 2 posttest.

Figure 4.1

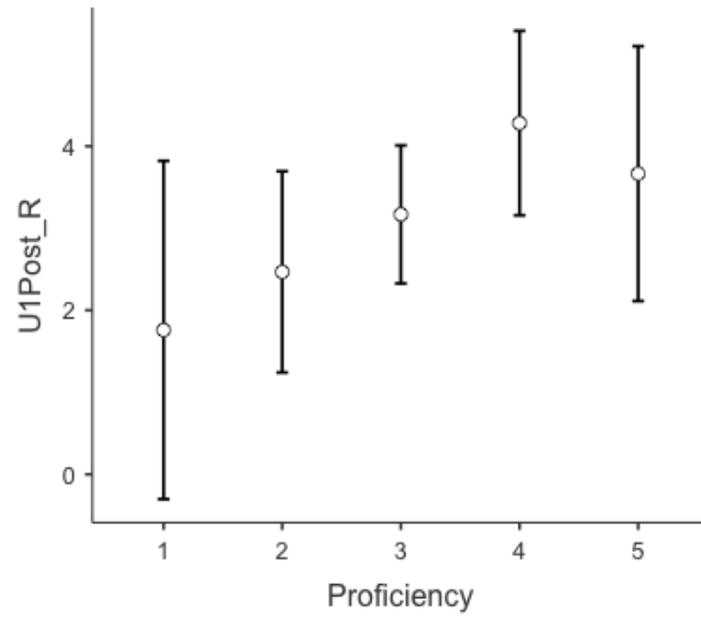
Marginal Means Plots (Teacher 1, Multiple Choice Part)



Note: U1Post_PS represents multiple choice part of Unit 1.

Figure 4.2

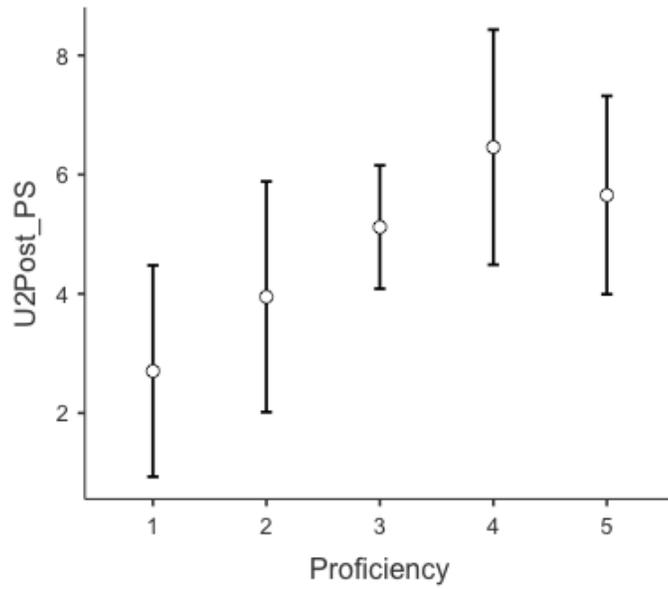
Marginal Means Plots (Teacher 1, Explanation Part)



Note: U1Post_R represents explanation part of Unit 1.

Figure 4.3

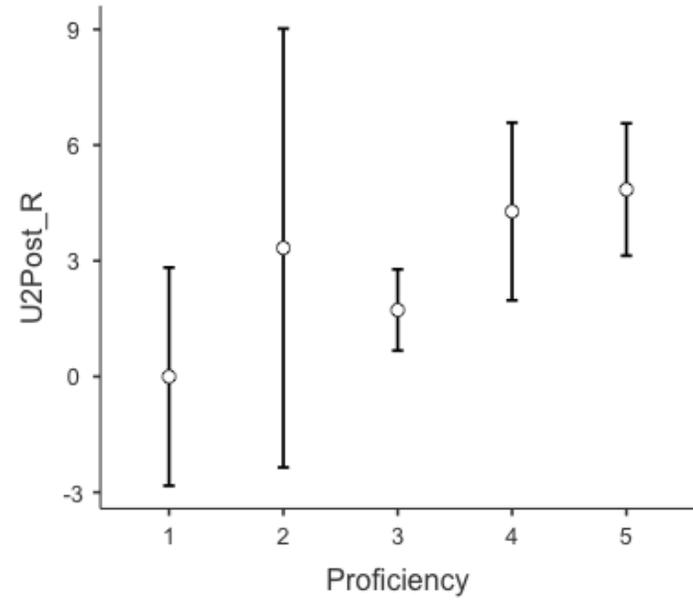
Marginal Means Plots (Teacher 2, Multiple Choice Part)



Note: U2Post_PS represents multiple choice part of Unit 2.

Figure 4.4

Marginal Means Plots (Teacher 2, Explanation Part)



Note: U2Post_R represents explanation part of Unit 2.

Table 4.6***Correlation Matrix***

	1	2	3	4	5	6	7	8	9
proficiency	-								
1. U1Pre_MC	0.483***	-							
2. U1Pre_R	0.567***	0.339***	-						
3. U1Post_MC	0.449***	0.644***	0.252**	-					
4. U1Post_R	0.594***	0.349***	0.724***	0.406***	-				
5. U2Pre_MC	0.296***	0.392***	0.065	0.302***	0.165	-			
6. U2Pre_R	0.465***	0.342***	0.394***	0.231**	0.547***	0.441***	-		
7. U2Post_MC	0.462***	0.533***	0.346***	0.430***	0.254**	0.592***	0.397***	-	
8. U2Post_R	0.628***	0.480***	0.609***	0.321***	0.481***	0.255**	0.533***	0.674***	-

*** p<0.01, ** p<0.05, * p<0.1

Note: Pre_MC represents the multiple-choice part in the pretest. Pre_R represents the explanation part in the pretest. Post_MC represents the multiple-choice part in the posttest. Post_R represents the explanation part in the posttest.

Table 4.7*Post Hoc Comparison-Proficiency (Unit 1 Multiple Choice Part)*

Comparison		Mean	Standard				
Proficiency	Proficiency	Difference	Error	df	t	P_{turkey}	Cohen's d
1	2	0.446	0.914	41	0.488	.988	0.010
	3	0.826	0.899	41	0.919	.888	0.019
	4	0.256	0.961	41	0.266	.999	0.006
	5	0.263	1.007	41	0.262	.999	0.006
2	3	0.380	0.557	41	0.682	.959	0.011
	4	-0.190	0.635	41	-0.300	.998	0.005
	5	-0.183	0.697	41	-0.262	.999	0.005
3	4	-0.570	0.527	41	-1.083	.814	0.017
	5	-0.563	0.596	41	-0.944	.878	0.016
4	5	0.008	0.656	41	0.012	1.000	0.000

Table 4.8*Post Hoc Comparison-Proficiency (Unit 1 Explanation Part)*

Comparison		Mean Difference	Standard Error	df	t	P_{turkey}	Cohen's d
Proficiency	Proficiency						
1	2	-0.708	1.154	41	-0.613	.972	0.014
	3	-1.409	1.087	41	-1.296	.695	0.029
	4	-2.523	1.181	41	-2.136	.225	0.049
	5	-1.906	1.343	41	-1.420	.619	0.035
2	3	-0.701	0.715	41	-0.980	.862	0.018
	4	-1.815	0.847	41	-2.143	.222	0.042
	5	-1.198	1.055	41	-1.135	.787	0.025
3	4	-1.114	0.707	41	-1.575	.522	0.028
	5	-0.498	0.913	41	-0.545	.982	0.011
4	5	0.616	0.906	41	0.681	.960	0.014

Table 4.9*Post Hoc Comparison-Proficiency (Unit 2 Multiple Choice Part)*

Comparison		Mean	Standard				
Proficiency	Proficiency	Difference	Error	df	t	P_{turkey}	Cohen's d
1	2	-1.246	1.204	25	-1.035	.837	0.037
	3	-2.416	1.018	25	-2.373	.156	0.077
	4	-3.756	1.366	25	-2.749	.074	0.104
	5	-2.952	1.204	25	-2.453	.134	0.087
2	3	-1.170	1.083	25	-1.080	.815	0.036
	4	-2.510	1.409	25	-1.781	.406	0.068
	5	-1.706	1.259	25	-1.355	.660	0.049
3	4	1.340	1.060	25	-1.264	.715	0.042
	5	-0.536	0.944	25	-0.568	.979	0.018
4	5	0.804	1.228	25	0.654	.964	0.023

Table 4.10*Post Hoc Comparison-Proficiency (Unit 2 explanation Part)*

Comparison		Mean Difference	Standard Error	df	t	P_{turkey}	Cohen's d
Proficiency	Proficiency						
1	2	-3.333	3.054	21	-1.091	.809	0.062
	3	-1.727	1.451	21	-1.190	.757	0.046
	4	-4.279	1.753	21	-2.441	.143	0.104
	5	-4.848	1.590	21	-3.049	.043	0.124
2	3	1.606	2.781	21	0.577	.997	0.031
	4	-0.946	2.950	21	-0.321	.998	0.018
	5	-1.515	2.856	21	-0.530	.983	0.029
3	4	-2.552	1.218	21	-2.096	.258	0.075
	5	-3.121	0.968	21	-3.225	.030	0.102
4	5	-0.569	1.380	21	-0.412	.993	0.016

facial expressions and gestures. The code for Reading is intended to capture the extent to which the teacher creates an opportunity for students to engage in mathematical reading (e.g., read problem solving descriptions, mathematical reasoning and explanation texts, word problems, prompt questions etc). The code for Writing is intended to capture the extent to which the teacher creates an opportunity for students to engage in mathematical writing (e.g., describe problem solving procedures, explain mathematical rationales, compare and contrast different methods, critique problem solving methods, answer a question proposed etc). The code for Speaking is intended to capture instances of ELLs' speaking (e.g., students orally response to mathematical questions from the teacher, other students, and the prompt questions listed in the materials etc).

In both units, the control group covered more topics than the treatment group. In order for the results to be comparable, only the topics covered by both conditions were included in this analysis. I reviewed all the selected audiotapes and tallied the number of reading, writing, and speaking opportunities. The inter-rater reliability of one-third of the data from a second coder exceeded 90%. As shown in Table 4.11, the treatment group was provided more opportunities for mathematical reading, writing, and speaking. However, students in the control group had little opportunity to read, write, and discuss in mathematics. The large difference between the two conditions in the access to reading, writing, and speaking may be attributed to the design of the intervention materials. Further analysis of the lesson recordings revealed that the opportunities for reading, writing, and speaking created by the teachers for ELLs in the intervention group were primarily provided on the intervention materials. For example, teachers gave students

time to read the problem-solving steps shown on the materials and had them share the similarities and differences in what they read. Teachers also usually gave students enough time to read and comprehend questions before discussing them in depth. The teachers gave students class time to write down their answers after the discussion as well. Sometimes the teachers gave students time to write down briefly what they wanted to discuss before the discussion, and after the class discussion, students were given time to revise their original answers.

Table 4.11

Opportunities for Reading, Writing, and Speaking

	Unit 1		Unit 2	
	treatment	Control	Treatment	Control
Reading	7	0	6	0
Writing	9	0	9	0
Speaking	30	13	24	7

Note: The reading, writing, and speaking opportunities were coded by frequency.

QR 4. Does Integrating MLS-WEPs Into the Curriculum Improve Teachers’ Use of Deep Questions and Promote Equitable Participation?

After viewing all the audiotapes, I realized that both teachers used similar strategies, asked similar questions, and followed similar routines for all classes while teaching the same materials. Therefore, I decided to focus my analysis on one class from each teacher. The teacher's questions and responses to students' answers seemed more natural when teaching the first class than in the subsequent class. Therefore, I decided to focus on the first class of both teachers. I first transcribed all the audiotapes of Teacher

1's and Teacher 2's first periods. I then coded the data sources from sentence to sentence and sought "a collection of instances from the data, hoping that issue-relevant meanings will emerge" (Creswell, 2012, p. 206). After the initial round of coding, three themes emerged: teacher questioning (i.e., teacher uses referential, display, and probing questions), waiting time, and interaction (i.e., teacher-to-student, student-to-student, student-to-teacher). I then performed the second round of coding using these three codes. During the second coding, mathematics communication behaviors (e.g., explain their thinking, restate another student's response, ask for clarification, expand the ideas of other students, agree/disagree, correct other students' responses) emerged as a recurring theme. In the final codebook, four sets of codes were included: teacher questioning, interaction, waiting time, and mathematics communication behaviors. Among these four sets of codes, interaction, waiting time, and mathematics communication behaviors are indicators of equitable participation.

To examine whether the integration of MLS-WEPs into the curriculum improved teachers' use of deep questions and promoted equitable participation, I developed a rubric (Table 4.12) based on four features: teacher questioning, interaction, waiting time, and mathematical communication behaviors. The inter-rater reliability of one-third of the data from a second coder exceeded 88%.

To analyze the data, I explored patterns of discussion features across conditions. The results showed that in either condition, Teacher 1 provided little wait time, so few students had time to process, whereas Teacher 2 provided more wait time in either condition, so more students had time to process the questions she asked. This suggests

that the intervention materials did not seem to be associated with the amount of wait time after the teacher asked questions. Similar to this was the interaction model, where Teacher 1's classes were primarily teacher-to-student interaction regardless of conditions, while Teacher 2's classes were a balanced combination of teacher-to-student and student-to-student interaction regardless of conditions. Likewise, as shown on Table 4.13, teachers used similar methods to engage ELLs in mathematical participation regardless of conditions. Specifically, Teacher 1 increased ELLs' classroom participation primarily by asking students for clarification and prompting questions. Notably, this behavior was much less frequent in the control condition. Teacher 2 implemented a variety of strategies to increase ELLs' classroom participation in both conditions. For example, Teacher 2 asked many clarifying questions throughout the lesson, invited students to explain their thinking, and encouraged them to extend the thinking of others. In addition, she had students submit their responses in the same place so that students could view each other's responses. In this way, she could provide immediate constructive feedback and students could agree with each other, ask clarifying questions, and correct others' or their own responses. As a result, more students participated in discussions in Teacher 2's classroom, while the same few students always participated in discussions in Teacher 1's classroom.

It is worth noting that both teachers focused primarily on procedural understanding in the control condition and on procedural and conceptual understanding and flexibility in the comparison condition. For instance, when simplifying the expression $2(x+1) + 3(x+6)$, the control teacher asked, "How to simplify this expression?", which was focused on the understanding of the procedurals. On the other

hand, the treatment teacher asked: “How did Emma simplify this expression?” “What are like terms?” “What were the like terms in Emma’s ‘distribute first’ way?” “Which method is correct? For the incorrect method, what needs to be done to make it correct?” This happened even with examples designed for conceptual understanding. For example, to determine if a given relation could represent a function, the control teacher asked: “How do I know if it is a function?” The treatment teacher asked, “How did Riley determine if the relation was a function? How did Gloria determine if the relation was a function?” “Why do both methods work?” “What is the definition of function?” “How did Riley’s method relate to the definition?” “How did Gloria’s method relate to the definition?” “Which method is better?” “Which method will you use if you asked to solve the problem?” The impact of promoting flexibility appeared to be inconsistent throughout the intervention. For example, in solving $5(x+3) = 20$, Teacher 1 often expressed that she will teach multiple methods in the future. She stated, “I taught you the ‘distribute first’ method only because it always works. But now I also like the ‘composite variable’ method because it works nicely under specific conditions. Dividing 5 from both sides makes the problem become easier. However, this method does not always work. It only works when the constant in the right side is divisible by the coefficient in the front.” However, in another example, she expressed dislike for an alternative method because it does not always work, even though the alternative method was more efficient. More research is needed on how to motivate teachers to teach multiple methods (i.e., flexibility).

Table 4.12***Teacher Questioning and Equitable Participation Codebook***

	1	2	3	4
Teacher questioning	Little or no questioning	Mostly Type 1 questions	Mostly Type 2 questions	Mostly Type 3 questions
Interaction	Teacher is the only speaker	Primarily teacher-to-student	Primarily student-to-student	A balanced mix of teacher-to-student and student-to-student
Waiting time	Teacher provides no waiting time	Teacher provides little wait time so that a few students have time to process	Teacher provides some wait time so that some students have time to process	Teacher provides sufficient wait time so that most students have time to process
Mathematical communication behavior	No whole group discussion takes place	Teacher provides a single approach to promote mathematical communication	Teacher provides at least 3 approaches to promote mathematical communication	Teacher provides a variety of approaches (>3) to promote mathematical communication.

Note: Teacher questioning is intended to capture the extent to which the teacher creates opportunities for students to engage in deep and sustained mathematical thinking via questioning. Type 1 questions are yes/no questions, or questions that can be answered with a single word or a single number. Type 2 questions can generally be answered within a sentence and typically have a clear right or wrong answer. Type 3 questions are referential questions that often require longer answers, and generally do not have a pre-established or right/wrong answer.

Table 4.13

Mathematical Communication Behaviors, by Condition

	Treatment	Control
Teacher 1		
Strategies for Teacher-to-Student Interaction	<ul style="list-style-type: none"> • Teacher asked for clarification and prompt questions. • Students explained their thinking 	<ul style="list-style-type: none"> • Teacher asked for clarification and prompt questions. • Students explained their thinking
Strategies for Student-to-Student Interaction	No attempts observed	No attempts observed
Teacher 2		
Strategies for Teacher-to-Student Interaction	<ul style="list-style-type: none"> • Teacher asked for clarification • Students explained their thinking • Teacher extended students' thought • Teacher provided instantaneous feedback, • Students submitted their responses in the same place so the teacher can provide instantaneous feedback 	<ul style="list-style-type: none"> • Teacher asked for clarification • Students explained their thinking • Teacher extended students' thought • Teacher provided instantaneous feedback, • Students submitted their responses in the same place so the teacher can provide instantaneous feedback
Strategies for Student-to-Student Interaction	<ul style="list-style-type: none"> • Students asked for clarification • Students explained their thinking • Student extended other's thought • Students submitted their responses in the same place so students can agree or ask for clarification questions, or correct others' responses. 	<ul style="list-style-type: none"> • Students asked for clarification • Students explained their thinking • Student extended other's thought • Students submitted their responses in the same place so students can agree or ask for clarification questions, or correct others' responses.

From the examples presented above, it is obvious that the intervention materials helped teachers ask additional and deeper questions. As shown in Table 4.14, when teachers taught using the intervention materials, they appeared to ask more questions, especially Type 2 questions. Further analysis of the teachers' questions indicated that most of the Type 2 questions asked by the teachers were listed in the intervention materials. On the other hand, teachers asked far fewer questions, especially for Teacher 1, when they were taught without the intervention materials, and a very limited Type 2 questions were asked when the intervention materials were not used. For Teacher 2, the number of Type 1 questions remained almost constant across conditions, while the number of Type 2 questions decreased by more than half in the control condition compared to the treatment condition. Unfortunately, in both conditions, neither teachers asked Type 3: referential questions. Overall, it appeared from the results of the analysis that the intervention materials only had a significant effect on teacher questioning, but not on equitable participation.

Table 4.14

Teacher Questioning, by Condition

	Treatment			Control		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
Teacher 1	16	13	0	2	2	0
Teacher 2	9	21	0	8	10	0

CHAPTER 5

DISCUSSION AND CONCLUSION

The purpose of this study was to investigate how Modified for Language Support-Worked Examples Pairs (MLS-WEPs) adapted based on best practices described in the research on learning for ELLs could be used to help English language learners learn mathematics effectively. The results indicated that worked example comparison not only enhanced ELLs' ability to solve mathematical problems, but also improved their written explanation skills and enabled them to transfer such skills to different mathematical concepts. More importantly, the data suggested when controlling ELLs' prior knowledge, the effectiveness of the MLS-WEPs intervention did not vary by their English language proficiency. In addition, this study demonstrated that the MLS-WEPs intervention materials provided ELLs with more opportunities to read, write, and speak in mathematics and enabled teachers to ask more and deeper questions. However, worked example comparisons did not appear to motivate the participant teachers to promote equitable participation in mathematics classrooms. These findings provide direct empirical support for the need for reforming mathematics teaching and learning in an ESOL context. This information is important given that all other comparable studies have been conducted in mainstream, native English-speaking classrooms. In this section, I present some of the implications of this study for practice and opportunities for future research. I followed the same structure as the results section, organizing the discussion around the four research questions.

QR 1. To What Extent Does the Use of MLS-WEPs Impact ELLs' Mathematical Performance?

Comparing worked examples appeared to support ELLs' progress in mathematical problem-solving skills, which is consistent with prior literature demonstrating that worked example comparison enhances students' procedural knowledge (Rittle-Johnson & Star, 2009; Rittle-Johnson & Star, 2007; Rittle-Johnson et al., 2009; Pashler et al., 2007). Comparing multiple problem-solving methods can facilitate students' exploration and use of alternatives. For example, when students know multiple methods, they either use shortcut methods to ensure accuracy (Rittle-Johnson & Star, 2007) or use alternative methods as a backup (e.g., students may choose to use an alternative method when the traditional method becomes too difficult) (Newton, Star, & Lynch, 2010). Comparing correct and incorrect methods is also promising for mathematical learning because it helps learners develop expert thinking, flexibility, and conceptual mathematical knowledge (Booth et al. 2013; Rittle-Johnson et al. 2009; Rittle-Johnson & Star 2007). In addition, worked examples effectively provide ELLs with solutions to problems, which helps to reduce their cognitive load (Sweller, 2011). Recall that in Chapter 1, I explained that when students learn mathematics in a language that they are not proficient in, they will experience an increased likelihood of heavier cognitive load due to having to use more working memory to decipher different parts of their non-native language. Comparing worked examples enables students to develop relational thinking and to align prior knowledge with new concepts (Richland et al., 2012). Once they have acquired sufficient knowledge, their cognitive load will be reduced, thus contributing to learning.

Based on the results, the comparison of worked examples with detailed descriptions of the problem-solving procedures led to a significant improvement in the quality of the explanations. The intervention materials provided the detailed descriptions of problem-solving procedures that are aligned with the solution steps. Such design not only provided learners with the opportunity to engage in mathematical reading, but also facilitated understanding of the solution steps and served as a model for ELLs on how to describe the problem-solving procedures. Recall that at the beginning of the intervention, more than 50% of ELLs left the explanation part blank or provided responses that were not related to the concept. One of the reasons for the significant decrease in these two types of responses after using the intervention materials may be the written model provided in the intervention materials. In addition, the intervention materials provide rich opportunities for ELLs to explain mathematical problems in written form and to revise their responses after discussion. This design treated language as a resource that provides ample and varied opportunities to read, write, listen, and speak in mathematics, which best supports the development of English language learners' written and oral communication skills (Moschkovich, 2000). An unexpected finding was that the number of uninterpretable explanations increased in the treatment group, while the number of such explanations decreased slightly in the control group. One explanation may be that the intervention materials motivated ELLs to perform mathematical explanations even though they may not have the ability to do so. More research is needed to confirm this conjecture.

More importantly, this study showed that ELLs who received the Unit 1 intervention materials showed significantly higher scores on the explanation part of the

Unit 2 pretest compared to students who did not previously receive the intervention materials. Unit 1 and Unit 2 covered two completely different mathematical concepts. It appeared that once ELLs knew how to perform written explanations, this skill stayed with them. The strength of the written explanation skill continues to be retained across different mathematical concepts, possibly due to the transfer of previously learned explanation skills. However, why and how these skills can be transferred to different concepts remains unexplained. Indeed, it also remains unknown whether such skills are transferable to any mathematical concept. Therefore, more research is needed around the written explanation in the mathematical domain.

QR 2. Is the Effectiveness of the MLS-WEPs Intervention Related to ELLs' English Language Proficiency?

The results revealed that although ELLs' mathematical performance was positively correlated with their English language proficiency (see Table 4.5), the effectiveness of the MLS-WEPs intervention did not vary by ELLs' English language proficiency after controlling for prior knowledge (see Table 4.6), implying that students of different language proficiency levels benefited equally from the MLS-WEPs intervention. In other words, the MLS-WEPs intervention enabled ELLs to learn mathematical concepts and describe mathematical procedurals in writing while they are still developing their English language proficiency. These findings contradict Abedi, Leon, and Mirocha's (2005) claim that ELLs' performance on content-based assessments is highly positively correlated with their English language proficiency. It seemed the design of the MLS-WEPs intervention can reduce the interference of language limitations and help ELLs with low English proficiency benefit as much as those with high English language proficiency. The MLS-WEPs intervention materials productively supported

ELLs' engagement in mathematical reading, writing, speaking, and reasoning, regardless of their proficiency in English, which allowed mathematics instruction to focus on rigorous concepts and reasoning, such focus is one of the most important goals of mathematics instruction in an ESOL context (Moschkovich, 2012).

The MLS-WEPs materials benefited English language learners of all English language levels equally, perhaps because these materials not only aligned well with the principles from the literature on mathematics learning in an ESOL context but also provided instructional strategies for scaffolding instruction and accommodating linguistic needs. First, MLS-WEPs provided a variety of opportunities for students to explore math concepts through listening, reading, writing, and speaking. The worked examples to be compared were presented with detailed problem-solving descriptions and students were prompted to read each description before comparing them. MLS-WEPs also provided learners with various opportunities to apply mathematical language in writing and speaking. In order to guide students to focus on specific aspects of comparison, different types of discussion prompts were implemented. For example, after studying problem solving descriptions and steps, students may be prompted to discuss the similarities and differences between the two methods, identify how each method works, or determine the effectiveness of each method and so forth. Open-ended writing prompts were also included to facilitate writing skills and develop reasoning and critical thinking skills. For instance, students may be asked to answer in what case one method is better than the other. In addition, students were exposed to mathematical listening since the comparison process included multiple interactions (teacher-student, student-student, student-teacher etc.). Second, the MLS-WEPs curriculum gave ELLs sufficient time to use their English

productively and negotiate meaning through both writing and speaking. Moschkovich (2012) mentioned that ELLs “need time and support for moving from expressing their reasoning and arguments in imperfect form” (p. 22). Throughout the intervention, students had multiple opportunities to revise their responses in writing: think-pair-share discussion, whole group discussion, and direct instruction. These processes enable ELLs to negotiate the meaning, modify the language and comprehend the text (Kersaint et al., 2008). All these activities help ELLs develop mathematics literacy and problem-solving skills (Boscolo & Mason, 2001).

QR 3. Does Incorporating MLS-WEPs to the Existing Curriculum Provide ELLs More Opportunities to Engage in Mathematical Listening, Reading, Writing, and Speaking?

Consistent with the hypothesized association, the intervention materials provided more opportunities for ELLs to read, write, and speak mathematics compared to the control materials. Further analysis of the lesson recordings revealed that the reading, writing, and speaking opportunities created by the teachers for ELLs in the treatment condition were primarily provided on the intervention materials. The data provided a new insight into the relationship between curriculum materials and opportunities to read, write, and speak. When curriculum materials addressed opportunities for reading, writing, and speaking, it appeared that teachers naturally and easily created these opportunities for students, but when these curriculum materials were not available, teachers easily missed opportunities to engage students in mathematical literacy practices. Recall that both teachers attended the week-long QTEL training twice (Teacher 1's last participation was in the summer of 2020 and Teacher 2's most recent participation was in the summer of 2018). As someone who completed the QTEL training in the summer of 2019, I learned

that QTEL is a unique professional development program that provides educators with strategies that they need to help all students, especially English language learners, achieve college and career readiness, and that it emphasizes the importance of including all four language domains in the teaching of mathematics. Throughout the training, teachers learn a variety of strategies to create opportunities for English language learners to read, write, speak, and listen. However, the results of the current study indicate that without curriculum materials that address these opportunities, teachers may not provide adequate opportunities for ELLs to read, write, and speak mathematics, even if they are well trained.

QR 4. Does Integrating MLS-WEPs Into the Curriculum Improve Teachers' Use of Deep Questions and Promote Equitable Participation?

The data indicated that the intervention materials had a significant effect on teacher questioning, but not on equitable participation. In particular, both teachers asked more Type 1 and Type 2 questions when teaching with the intervention materials. And most of the Type 2 questions they asked were listed in the intervention materials. These findings again indicate the importance of curriculum materials. Unfortunately, neither teacher asked any referential questions throughout the study. These results are consistent with previous research in that teachers tend to ask demonstrative questions (i.e., Type 1 and Type 2 questions) rather than referential questions (i.e., Type 3 questions) (Banse et al., 2017). Referential questions play a significant role in conceptually rich discussions in the classroom because students have to do their own reasoning when answering such questions (Banse et al., 2017), while the lack of referential questions can limit opportunities for students to reason mathematically. The current study showed that when the prompt questions were included in the curriculum materials, teachers were more

likely to ask those questions. Therefore, further research is needed to examine whether including referential questions in curriculum materials motivates teachers to ask such questions, promotes conceptually rich discussions, and engages students in mathematical reasoning.

The results showed no change in the promotion of equitable participation (i.e., Teachers implement a variety of strategies to engage most or all students in class discussions or activities.) between the two teachers across conditions, suggesting that the intervention materials may not have effect on equitable participation. Specifically, Teacher 2 tended to promote equitable participation with or without the use of intervention materials, whereas Teacher 1 implemented only few strategies to promote equitable participation regardless of the condition. Classroom participation allows students to learn how others think, to reflect on their own shortcomings, and to clarify and modify ways to think for themselves. Pedagogical practice needs to effectively provide opportunities for student dialogue in mathematics because discussion in mathematics classrooms is highly beneficial for the reception of mathematical knowledge (Nathan & Knuth, 2003; Ball, 1991). Discussion gives learners more opportunities to take on active problem-solving roles and influences their inclination to identify themselves as competent and proactive learners in mathematics (Turner et al., 2013). However, some students seem to have an advantage over others in whole class discussions and ELLs with low language proficiency tend to opt out of classroom discussions (Turner et al., 2013). It is important that teachers often invite other classroom members to participate in discussions, as such invitations provide opportunities for students to explore their own mathematical ideas in a meaningful way, to elaborate on their own mathematical ideas,

and to investigate the ideas offered by others (Nathan & Knuth, 2003). Future research should focus on how to instruct teachers in designing effective teaching methods to increase equitable participation (Turner et al., 2013).

Finally, the MLS-WEPs materials appeared to have led teachers to focus on procedural and conceptual understanding and flexibility in mathematics instruction. However, the impact of promoting flexibility did not appear to be consistent throughout the intervention. A teacher sometimes indicated that she would teach different problem-solving methods in future instruction because although some of the methods presented within the MLS-WEPs were not generalizable, they were efficient and accurate under specific conditions. However, sometimes the teacher expressed dislike for the alternative short cut methods that were only applicable under certain conditions. Flexibility refers to a problem solver's knowledge of multiple solutions and the capability and tendency to selectively choose the most appropriate solution for a given problem and problem-solving goal (Star & Newton, 2009). Given that flexibility is a necessary component of students' mathematical ability (Kilpatrick et al., 2001; Star & Newton, 2009), more research is needed on how to motivate and prepare teachers of English language learners to teach multiple approaches (i.e., flexibility).

Limitation

There are several major limitations to this study that can be addressed in future research. The primary limitation to generalizing these results is the small sample size of teacher participants. It is critical that these results be replicated with a larger number of teachers with diverse teaching styles. Second, this study was conducted during COVID-19, and all lectures were delivered online; though small group discussion was an

important component in the design of the intervention materials, students received very limited opportunities for collaborative group discussion and exploration of mathematical concepts. Third, I was unable to obtain official ACCESS scores (i.e., ELLs' English language proficiency level scores) for the participant students because the staff member who prepared the data was out of the office during COVID-19. Therefore, the English language proficiency levels used in this study were derived from students' self-reports, thus accuracy may be compromised. However, I included their level of comfort with English materials and the level of English classes they were taking to ensure that the self-reported English proficiency levels in this study were as accurate as possible. Moreover, the teachers reviewed the level of self-reports and made recommendations as needed. In addition, student performance contained measures of individual students as well as measures of classrooms in which the students were grouped. Hence, it may be more appropriate to use a multilevel model to address the intact classrooms. Fourth, only 53.06% of enrolled students were included in the final data analysis, due to failure to submit consent forms, missed assessments, surveys, and absenteeism. The teachers also experienced a lot during COVID 19, and the teaching strategies presented throughout the study may not fully reflect their usual teaching qualities. Finally, the design of this study could be improved for better inter-condition comparisons. The control and treatment materials covering the same topic were taught by different teachers (e.g., the first unit of intervention material was taught by Teacher 1, whereas its control material was taught by Teacher 2), so that this analysis included two variables: the teacher and the intervention material. Therefore, variations due to differences in teachers should also be taken into account when analyzing the impact of the intervention materials. However, the two

teachers had similar professional backgrounds. For example, both teachers were certified in secondary mathematics, had attended two QTEL trainings, and had many years of teaching experience (i.e., 30 and 19 years). More importantly, both teachers were instructed to implement interventions and control materials in the same manner. Therefore, the differences caused by the teachers were likely to be minor.

Conclusion

The results of this study suggest that Modified for Language Support-Worked Examples Pairs (MLS-WEPs) can help ELLs learn mathematics effectively in several ways. First, MLS-WEPs enhanced ELLs' ability to solve mathematical problems and improve their written explanatory skills, and also enabled them to transfer this ability to different mathematical concepts. More importantly, the effectiveness of the MLS-WEPs intervention did not appear to vary by ELLs' English language proficiency. This is an exciting finding because it demonstrated that learners with low English language proficiency can learn math well before they develop their English proficiency. In addition, this study uncovered that MLS-WEPs provide students with more opportunities to read, write, and speak, as well as that it is a promising way to assist teachers in asking more in-depth questions. More importantly, this study showed that the content included in the materials has a significant impact on the quality of teachers' instruction. When materials include opportunities to read, write, and speak, the participant teachers were more likely to incorporate these opportunities into their instruction. However, worked example comparisons did not appear to motivate participating teachers to promote equitable participation in mathematics classrooms. Classroom participation not only allows students to learn how others think and to reflect on and modify their own

deficiencies, but more importantly it plays a very important role in the language development of English language learners. Therefore, more research is needed on mathematics teaching practices that support equitable participation in an ESOL context.

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APPENDIX A

WHICH IS BETTER? WEP

Which is better?

Topic 2.6

Riley and Gloria were asked to graph the equation $3x - 2y = 6$.

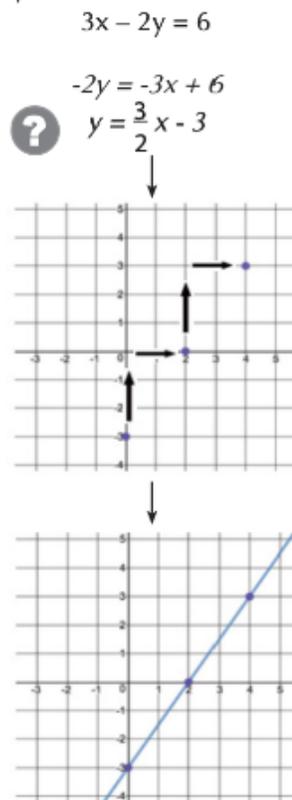
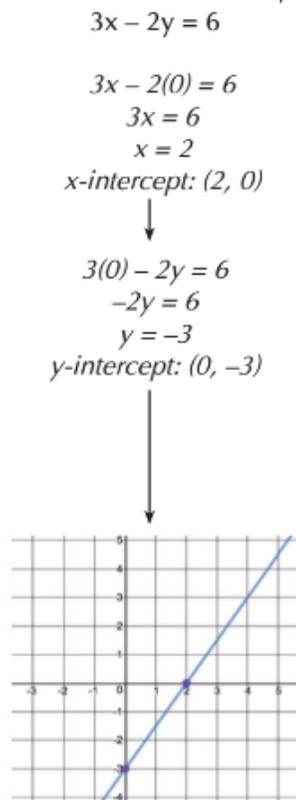
Riley's "x- and y-intercepts" way

Gloria's "slope-intercept" way

First I found the x-intercept by plugging in 0 for y.

Then I found the y-intercept by plugging in 0 for x.

I plotted the intercepts and connected them.



I solved for y to put the equation in $y = mx + b$ form.

I graphed the y-intercept of -3 then used rise over run to get more points.

I connected the points to get the line.



How did Riley graph the line? Why did Gloria solve the equation for y as a first step?

To graph the line, Riley first _____; then _____; next _____.

Gloria solve the equation for y as a first step because _____.



Which method is better? Why?



_____ is better because _____.

APPENDIX B

WHY DOES IT WORK? WEP

Why does it work?

Topic 2.1

Riley and Gloria were given the set of ordered pairs
 $\{(-3, 6), (2, 5), (3, 1), (2, 4), (5, 1)\}$,
and asked to determine if the relation is a function.

Riley's "make a table" way

Gloria's "graph and vertical line test" way

I made a table.

I saw that 2 in the domain is paired with both a 5 and a 4 in the range.

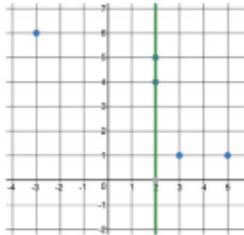
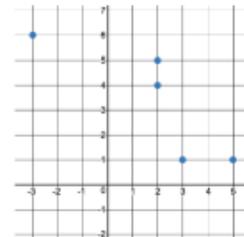
This means the relation is not a function.

x (domain)	y (range)
-3	6
2	5
2	4
3	1
5	1

x (domain)	y (range)
-3	6
2	→ 5
2	→ 4
3	1
5	1



Not a function



Not a function



I graphed the ordered pairs.

I found a vertical line that intersected two of the points.

This means the relation is not a function.

How did Riley determine if the relation was a function? How did Gloria determine if the relation was a function?

? To determine if the relation was a function, Riley first _____; then _____; finally _____.
 To determine if the relation was a function, Gloria first _____; then _____; finally _____.

↔ Why do both methods work? Why does the vertical line test tell us the same thing as the table of values?

Both methods work because _____. Vertical line test and the table of value tell us the same thing because _____.

APPENDIX C

WHICH IS CORRECT? WEP

Which is correct?

Topic 1.6

Emma and Layla were asked to simplify the expression $2(x + 1) + 3(x + 6)$

Emma's "distribute first" way	Layla's "combine like terms" way
<div style="border: 1px solid black; border-radius: 15px; padding: 10px; width: fit-content; margin-left: 20px;"> <p><i>First, I distributed the parentheses.</i></p> <p><i>Then, I combined like terms to get $5x + 20$.</i></p> </div> <div style="text-align: center; margin-top: 20px;"> $2(x + 1) + 3(x + 6)$ $2x + 2 + 3x + 18$ <p style="text-align: center;">↓</p> $5x + 20$ </div> <div style="text-align: center; margin-top: 20px;"> </div>	<div style="border: 1px solid black; border-radius: 15px; padding: 10px; width: fit-content; margin-right: 20px;"> <p><i>First, I combined what was in the parentheses.</i></p> <p><i>Then, I distributed the parentheses to get $10x + 35$.</i></p> </div> <div style="text-align: center; margin-top: 20px;"> $2(x + 1) + 3(x + 6)$ $5(2x + 7)$ <p style="text-align: center;">↓</p> $10x + 35$ </div> <div style="text-align: center; margin-top: 20px;"> </div>

What were the like terms in Emma's "distribute first" way? What were the like terms combined in Layla's "combine like terms" way?

- ? The like terms in Emma's way were _____. The like terms in Layla's way is _____.
- ← Which method is correct? For the incorrect method, what needs to be done to make it correct?
- _____ method is correct. To correct the incorrect method, we need to _____.

APPENDIX D

HOW DO THEY DIFFER? WEP

How do they differ?

Topic 1.3

Emma was asked to solve $2x = 10$; Layla was asked to solve $2(x + 1) = 10$

Emma's "use x as the variable" way	Layla's "use (x + 1) as the variable" way
<div style="border: 1px solid black; border-radius: 15px; padding: 10px; width: 80%; margin: 0 auto;"> <p style="font-size: small;">I divided both sides by 2.</p> <p style="font-size: small;">I got 5.</p> </div>	<div style="border: 1px solid black; border-radius: 15px; padding: 10px; width: 80%; margin: 0 auto;"> <p style="font-size: small;">First, I divided both sides by 2.</p> <p style="font-size: small;">Then I subtracted 1 from both sides.</p> <p style="font-size: small;">I got 4.</p> </div>
$2x = 10$ $\frac{2x}{2} = \frac{10}{2}$ \downarrow $x = 5$	$2(x + 1) = 10$ $\frac{2(x + 1)}{2} = \frac{10}{2}$ \downarrow $x + 1 = 5$ \downarrow $x = 4$
	

What is the first step of Layla's "use (x + 1) as the variable" way? Why is Layla's way called the "use (x + 1) as the variable" way?

? The first step of Layla's "use (x+1) as the variable" way is _____. It is called the "use of (x+1) as the variable" way because _____.

↔ How are the two ways similar? How are the two ways different?
 One similarity between the two method is _____.
 One difference between the two method is _____.

APPENDIX E
SAMPLE ASSESSMENT

Pre-test: Topic 2. Functions

Choose a correct answer and explain how you got your answers.

_____ 1. Determine if the relationship is a function: $\{(-5, 7), (3, 5), (8, 1), (3, 4), (5, 1)\}$

- A. Yes B. No

Explain how you know if the given relation is a function.

_____ 2. Determine if the points in the table could represent a linear function.

x	4	6	7
y	10	20	25

- A. Yes B. No

Explain how you know if the points in the table could represent a linear function.

_____ 3. If $f(x) = 5x + 3$, find $f(4)$.

- A. 5 B. 3 C. 23 D. 8

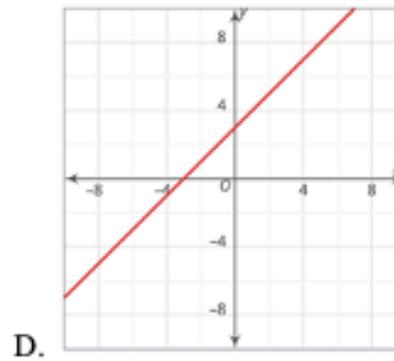
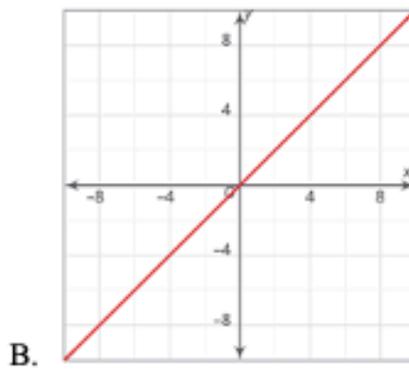
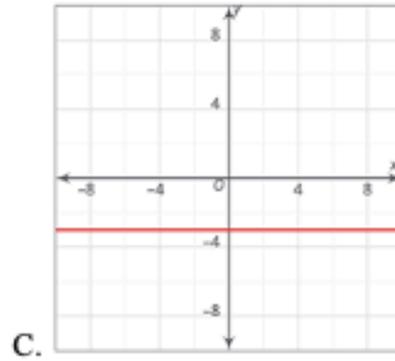
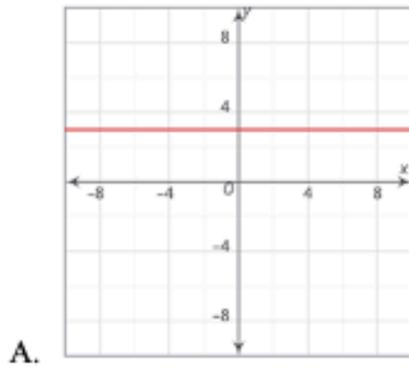
Explain how you got your answer.

_____ 4. Find the slope of the line passing through (5, 8) and (4, -2).

- A. 10 B. $\frac{1}{10}$ C. 6 D. -10

Explain how you got your answer.

_____ 5. Which is the correct graph for $y = 3$.



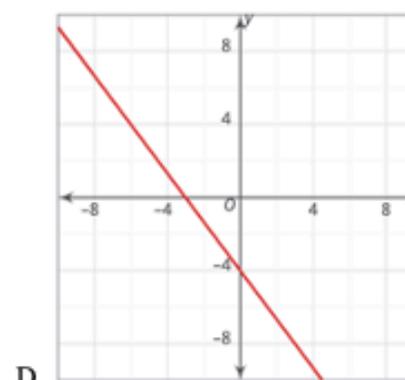
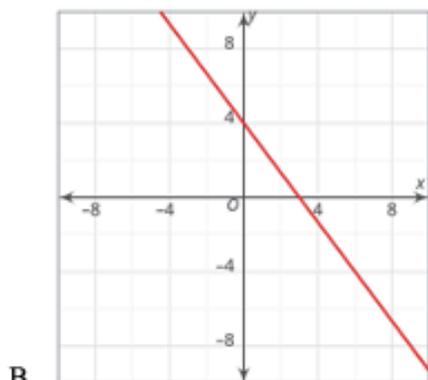
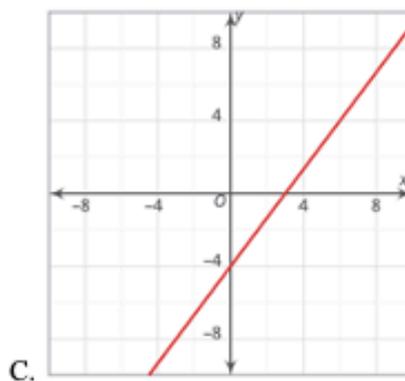
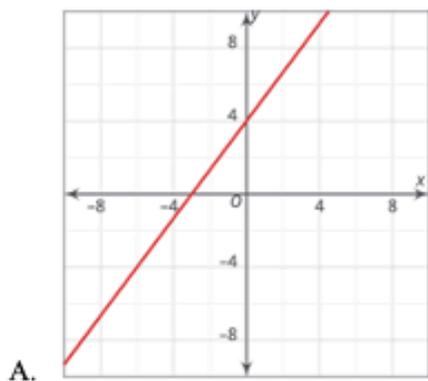
Explain how you got your answer.

_____ 6. Find the y-intercept of the line connecting the two points $(1, 5)$ and $(-2, -1)$

- A. $(0, 3)$ B. $(3, 0)$ C. $(2, 0)$ D. $(0, 2)$

Explain how you got your answer.

_____ 7. Which is the correct graph for $4x - 3y = 12$



Explain how you got your answer.

_____ 8. Write an equation for the line through $(2, 4)$ and $(6, 12)$ using point-slope form.

A. $y - 2 = 2(x - 4)$

B. $y + 4 = 2(x + 2)$

C. $y - 4 = 2(x - 2)$

D. $y - 2 = -2(x - 4)$

Explain how you got your answer.