QUANTUM RANDOM WALK ON FRACTALS

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DOCTOR OF PHILOSOPHY

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by
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ABSTRACT

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Professor Wei-Shih Yang, Chair

Quantum walks are the quantum mechanical analogue of classical random walks. Discrete time quantum walks have been introduced and studied mostly on the line $\mathbb{Z}$ or higher dimensional space $\mathbb{Z}^d$ but rarely defined on graphs with fractal dimensions because the coin operator depends on the position and the Fourier transform on the fractals is not defined. Inspired by its nature of classical walks, different quantum walks will be defined by choosing different shift and coin operators. When the coin operator is uniform, the results of classical walks will be obtained upon measurement at each step. Moreover, with measurement at each step, our results reveal more information about the classical random walks.

In this dissertation, two graphs with fractal dimensions will be considered. The first one is Sierpinski gasket, a degree-4 regular graph with Hausdorff dimension of $d_f = \ln 3/\ln 2$. The second is the Cantor graph derived like Cantor set, with Hausdorff dimension of $d_f = \ln 2/\ln 3$. The definitions and amplitude functions of the quantum walks will be introduced. The main part of this dissertation is to derive a recursive formula to compute the amplitude Green functions. The exiting probability will be computed and compared with the classical results. When the generation of graphs goes to infinity, the recursion of the walks will be investigated and the convergence rates will be obtained and compared with the classical counterparts.

ACKNOWLEDGEMENTS

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To my dear wife Claire and my parents.

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CHAPTER 1

INTRODUCTION

Quantum walks are the quantum analogue of classical random walks. Discrete time quantum walks have been introduced and studied mostly on the line $\mathbb{Z}$ or higher dimensional space $\mathbb{Z}^d$. Central limit theorems for quantum random walks with partial decoherence have been proved in the quantum version; see [2, 8]. Some model involving potential barrier has been studied in [7] and some applications in $\mathbb{Z}^d$ using recursive formula to solve return probability have been studied in [1]. See [4] and [5] for a comprehensive review of quantum walks.

Fractals have many applications in medicine and computper vision, however quantum walks on graphs with fractal deimensions have rarely been studied or even defined because the coin operator depends on the position and the Fourier transform on the fractals is not defined. Therefore computing the hitting probability and expection of passage time using combinatorics or Fourier on $\mathbb{Z}^d$ is not working. In my dissertation, I focus on the quantum random walks on graphs with fractal dimensions.

First we focus our attention on the Sierpinski gasket, a fractal set with the overall shape of an triangle, generated recursively into more triangles with the same shape. Let $F^{(n)}$ be the n^{th} order Sierpinski gasket, see the following graph for $n = 0, 1, 2$. And define $F^{(\infty)} = \cup_{n=0}^{\infty} F^{(n)}$.

The n^{th} order Sierpinski gasket $F^{(n)}$ is a degree-4 regular graph except for

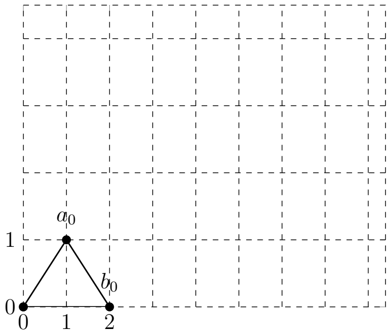

Figure 1.1: 0^{th} order Sierpinski Gasket $F^{(0)}$

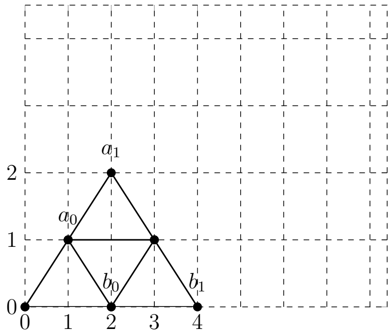

Figure 1.2: 1^{st} order Sierpinski Gasket $F^{(1)}$

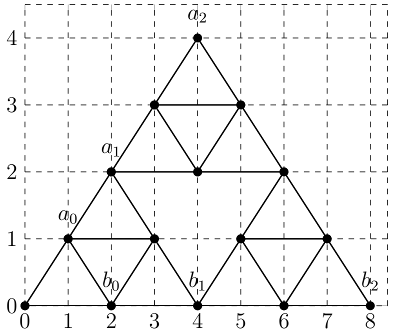

Figure 1.3: 2^{nd} order Sierpinski Gasket $F^{(2)}$

overall corners. Let V_n be the number of nodes for $F^{(n)}$. Then $V_0 = 3$, $V_1 = 6$, $V_2 = 15$ and it is easy to check that $V_n = 3V_{n-1} - 3 = 3^{n+1} - 3^n - 3^{n-1} - \cdots - 3$ $= \frac{3(3^n+1)}{2}$. Let S_n be the number of nodes in the x direction, for example, $S_0 = 2$, $S_1 = 3$, $S_2 = 5$, so $S_n = 2S_{n-1} - 1 = 2^n + 1$. By the definition of Hausdorff dimension d_f,

$$V_n \sim S_n^{d_f}$$

we have $d_f = \lim_{n\to\infty} \frac{\ln V_n}{\ln S_n} = \frac{\ln 3}{\ln 2}$, which is bigger than dimension of the line and smaller than that of the plane. Some basic properties about classical random walks on the Sierpinski gasket have been studied in [11]. And a self-avoiding random walk on the Sierpinski gasket has been introduced in [14].

To define the quantum random walk on the Sierpinski gasket, we use the idea introduced in [6]. Some combinatorics techniques were applied in $\mathbb{Z}$ [12, 13], however, they can not be applied in $\mathbb{Z}^2$ or fractals. In this dissertation we use the path integral approach as in [1]. By solving the Dirichlet problem, we generalize a iterative formula, based on which we compute the hitting probabilities and the expected hitting time.

Some strange behaviors about quantum random walks are observed in this dissertation. For example, the quantum random walk starting with $|\mathbf{0}^0\rangle$ will not exit at $|\mathbf{a_1}^5\rangle$ from the right bottom direction. And with probabiliy less than 1, the particle will ever exit the boundary of $F^{(n)}$ and its reflection part. When the coin operator is uniform, the evolution operator is no longer unitary, and it becomes the classical walk.

The second part of my thesis is about the quantum walks on the Cantor set. The Cantor set is built by removing the middle thirds and iterating the process. If one bulids the Cantor set by growing the fractal, then for generation G, it is considered as two generation G-1 connected by a removed(void) set of the same length see Figure 1.4. So it is easy to check that $d_f = \lim_{G\to\infty} \frac{\ln V_G}{\ln S_G}$ $= \lim_{G\to\infty} \frac{\ln 2^{G+1}}{\ln(3^G+1)} = \frac{\ln 2}{\ln 3}$.

Figure 1.4: Cantor set with G =1,2,3, while the sites inside the red line are in the removed (void) set.

Inspired by [9, 10], we define a quantum walk on this set. By generating a iterative formula, we prove the recursion of the quantum random walk and compare the rate with the classical case.

The method of using path integral to find the iterative Green amplitude functions can be extended to other fractals with self-similar property. The iteration of the Green amplitude functions from n^{th} level to $(n+1)^{th}$ level can be obtained by solving the Dirichlet problem on 1^{st} level of the graph.

In this dissertation we analyze in detail the quantum random walks on the Sierpinski gasket and the Cantor set. We define and solve the Green amplitude functions iteratively when the level of the fractal graph goes to infinity. We obtained exact recursive formulas of the Green amplitude functions, based on which the hitting probabilites and expection of the first-passage time are calculated. And we obtained that when the level of these two fractal graph goes to infinity, with probability 1 that the quantum random walks will return origin, i.e., the quantum walks on these two fractal graph are recurrent.

This dissertation is organized as follows. In Chapter 2, we first introduce the classical random walks on the Sierpinski gasket and the Cantor graph, we also discuss the hitting time and diffusion, and then obtain some properties for the classical random walks.

In Chapter 3, we present the basic concepts of quantum random walks on the Sierpinski gasket and explain how to choose coin operater which is different from the case in $\mathbb{Z}^d$. We then introduce the amplitude functions on the graph and define the corresopning amplitude Green functions. The main computation part is to derive the amplitude Green functions iteratively using symmetry and self-similar property. And we calculate the exit probability using Parseval's theorem. Second we find the expection of the first-passage time on the Sierpinski gasket using these amplitude Green functions. By choosing the coin operator uniformly, we obtain the same results for the classical random walks as that in Chapter 2.

In Chapter 4, we define the quantum random walks on the Cantor set. This new definition depends on the the site location. If the partical resides on the void set, it keeps the dirction till it hits the other end, and if the partical resides on the Cantor set, it follows the rule of quantum random walks on $\mathbb{Z}^1$. Following the scheme as in Chapter 3, we discuss the limiting behavior of the hitting probabilities as G goes to infinity.

CHAPTER 2

RANDOM WALKS

2.1 Random walks on Sierpinski gasket

The n^{th} order Sierpinski gasket $F^{(n)}$ is a degree-4 regular graph except overall corners with Hausdorff dimension of $d_f = \ln 3/\ln 2$, which is bigger than dimension of the line and smaller than that of the plane. The graph of $F^{(2)}$ is shown in Figure 2.1. For the n^{th} order Sierpinski gasket $F^{(n)}$ embedded in the two-dimensional plane, we can extend $F^{(n)}$ as following: if we think of $\mathbf{0}, \mathbf{a_n}, \mathbf{b_n}$ as boundary points of $F^{(n)}$, then we call reflection of $F^{(n)}$ denoted by $F^{(n)'}$ with boundary points $\mathbf{0}, \mathbf{a'_n}, \mathbf{b'_n}$, here $\mathbf{a'_n} = -\mathbf{a_n} = (-2^n, -2^n)$ and $\mathbf{b'_n} = (2^n, -2^n)$, see Figure 2.2.

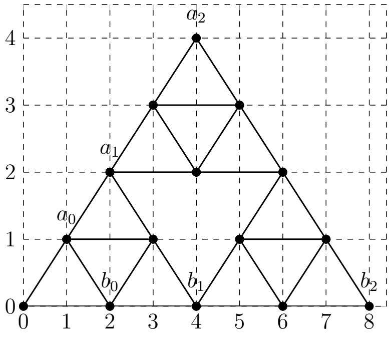

Figure 2.1: 2^{nd} order Sierpinski Gasket $F^{(2)}$

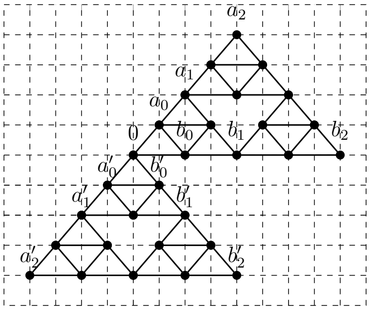

Figure 2.2: $F^{(2)} \cup F^{(2)'}$

Let us introduce the random walk on generations of Sierpinski Gasket. The walker can reside on any site of the lattice, at each time it only moves to one of its neighbors with equal probability 1/4. A path w is defined by $w = (w_0, \ldots, w_m)$, where $w_t = \mathbf{x}_t \in F^{(n)} \cup F^{(n)'}$, and $\|\mathbf{x}_{t+1} - \mathbf{x}_t\| = 2$, where $\|\mathbf{x} - \mathbf{y}\| = |x_1 - y_1| + |x_2 - y_2|$. Each step w_t is updated from one of four neighbors of w_{t-1} with equal probability 1/4. The length of w is defined by $|w| = m$.

Definition 2.1 *Let* $\mathring{F}^{(n)} = \{x \in F^{(n)}; x \neq \mathbf{0}, \mathbf{a_n}, \mathbf{b_n}\}$, *and* $\tau^{(n)} = \inf\{t \geq 1; w_t \notin \mathring{F}^{(n)} \cup \mathring{F}^{(n)'}\}$ *be the first exiting time of* $\mathring{F}^{(n)} \cup \mathring{F}^{(n)'}$. *Let*

$$W_1^{(n)} = \{w = (w_0, \ldots, w_{\tau^{(n)}}), w_0 = \mathbf{0}, w_{\tau^{(n)}} = \mathbf{a_n}\}$$

the set of path with initial $\mathbf{0}$ *first exit of* $\mathring{F}^{(n)} \cup \mathring{F}^{(n)'}$ *at* $\mathbf{a_n}$. *And similarly, let*

$$W_0^{(n)} = \{w = (w_0, \ldots, w_{\tau^{(n)}}), w_0 = \mathbf{0}, w_{\tau^{(n)}} = \mathbf{0}\}$$

Definition 2.2 *Let*

$$S_0(w) = \#\{i; w_{i+1} = w_i, i = 0, 1, ..., \tau^{(1)} - 1\}$$

be the number of self-loops. And

$$S_1(w) = \#\{i; \|w_{i+1} - w_i\| = 2, i = 0, 1, ..., \tau^{(1)} - 1\}.$$

Let

$$\Phi_0^{(n)}(\alpha,\beta) = \sum_{w\in W_0^{(n)}} \alpha^{S_0(w)}\beta^{S_1(w)}, \text{ and } \Phi_1^{(n)}(\alpha,\beta) = \sum_{w\in W_1^{(n)}} \alpha^{S_0(w)}\beta^{S_1(w)}.$$

By assigning $\alpha = 0$, and $\beta = 1/4$ we have the hitting probability

$$P^{(n)}(w_{\tau^{(n)}} = \mathbf{0} | w_0 = \mathbf{0}) = \Phi_0^{(n)}(0, 1/4),$$

$$P^{(n)}(w_{\tau^{(n)}} = \mathbf{a_n} | w_0 = \mathbf{0}) = \Phi_1^{(n)}(0, 1/4).$$

In general we can obtain a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\Phi_0^{(n+1)}, \Phi_1^{(n+1)}) = T(\Phi_0^{(n)}, \Phi_1^{(n)}).$$

Using the self-similar property of the Sierpinski gasket, we have

Lemma 2.1 *Let*

$$\Phi_0^{(n)}(\alpha,\beta) = \sum_{w\in W_0^{(n)}} \alpha^{S_0(w)}\beta^{S_1(w)},$$

and

$$\Phi_1^{(n)}(\alpha,\beta) = \sum_{w\in W_1^{(n)}} \alpha^{S_0(w)}\beta^{S_1(w)}.$$

Then $\Phi_0^{(0)}(\alpha,\beta) = \alpha, \Phi_1^{(0)}(\alpha,\beta) = \beta,$

$$\Phi_0^{(n+1)}(\alpha,\beta) = \Phi_0^{(1)}(\Phi_0^{(n)}(\alpha,\beta), \Phi_1^{(n)}(\alpha,\beta)),$$

and

$$\Phi_1^{(n+1)}(\alpha,\beta) = \Phi_1^{(1)}(\Phi_0^{(n)}(\alpha,\beta), \Phi_1^{(n)}(\alpha,\beta)).$$

Theorem 2.1 *The recursive relation between* $(\Phi_0^{(n+1)}, \Phi_1^{(n+1)})$ *and* $(\Phi_0^{(n)}, \Phi_1^{(n)})$ *is*

$$\Phi_0^{(n+1)} = \Phi_0^{(1)}(\Phi_0^{(n)}, \Phi_1^{(n)}) = \Phi_0^{(n)} + \frac{4(\Phi_1^{(n)})^2(1-\Phi_0^{(n)})}{(1-\Phi_0^{(n)}+\Phi_1^{(n)})(1-\Phi_0^{(n)}-2\Phi_1^{(n)})}$$

and

$$\Phi_1^{(n+1)} = \Phi_1^{(1)}(\Phi_0^{(n)}, \Phi_1^{(n)}) = \frac{(\Phi_1^{(n)})^2(1-\Phi_0^{(n)}+2\Phi_1^{(n)})}{(1-\Phi_0^{(n)}+\Phi_1^{(n)})(1-\Phi_0^{(n)}-2\Phi_1^{(n)})}.$$

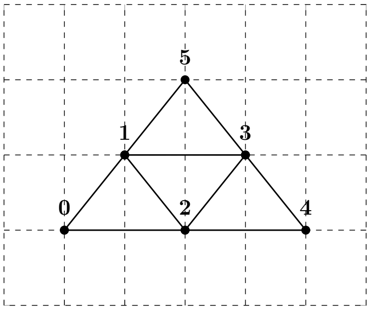

Figure 2.3: relabel on $F^{(1)}$

Proof 2.1 *To find* $\Phi_0^{(1)}(\alpha,\beta)$ *and* $\Phi_1^{(1)}(\alpha,\beta)$*, consider the following graph on* $F^{(1)}$ *with vertices relabeled. For* $F^{(1)'}$*, we just apply on* $F^{(1)}$ *by symmetry.*

Let

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix} \end{array},$$

$h_k = P(w_{\tau^{(1)}} = \mathbf{5} | w_0 = \mathbf{k})$*, and* $g_k = P(w_{\tau^{(1)}} = \mathbf{0} | w_0 = \mathbf{k})$ *for* $k = 1, 2, 3$*. Then*

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \frac{1}{I_3 - A} \begin{bmatrix} \beta \\ 0 \\ \beta \end{bmatrix},$$

$$\Phi_1^{(1)}(\alpha,\beta) = \beta h_1 + \beta h_2.$$

And

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \frac{1}{I_3 - A} \begin{bmatrix} \beta \\ \beta \\ 0 \end{bmatrix},$$

$$\Phi_0^{(1)}(\alpha,\beta) = \alpha + \beta g_1 + \beta g_2 + \beta h_1 + \beta h_3 = \alpha + 4\beta g_1.$$

By computing the inverse, we have

$$(I_3 - A)^{-1} = \frac{1}{(1-\alpha+\beta)^2(1-\alpha-2\beta)} \begin{bmatrix} (1-\alpha)^2-\beta^2 & \beta^2+(1-\alpha)\beta & \beta^2+(1-\alpha)\beta \\ \beta^2+(1-\alpha)\beta & (1-\alpha)^2-\beta^2 & \beta^2+(1-\alpha)\beta \\ \beta^2+(1-\alpha)\beta & \beta^2+(1-\alpha)\beta & (1-\alpha)^2-\beta^2 \end{bmatrix}$$

So

$$\Phi_0^{(1)}(\alpha,\beta) = \alpha + \frac{4\beta^2(1-\alpha)}{(1-\alpha+\beta)(1-\alpha-2\beta)}$$

and

$$\Phi_1^{(1)}(\alpha,\beta) = \frac{\beta^2(1-\alpha+2\beta)}{(1-\alpha+\beta)(1-\alpha-2\beta)}$$

Therefore, we obtain the map $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\alpha,\beta) = (T_0(\alpha,\beta), T_1(\alpha,\beta))$:

$$(\Phi_0^{(n+1)}, \Phi_1^{(n+1)}) = T(\Phi_0^{(n)}, \Phi_1^{(n)}).$$

where

$$\Phi_0^{(n+1)} = T_0(\Phi_0^{(n)}, \Phi_1^{(n)}) = \Phi_0^{(n)} + \frac{4(\Phi_1^{(n)})^2(1-\Phi_0^{(n)})}{(1-\Phi_0^{(n)}+\Phi_1^{(n)})(1-\Phi_0^{(n)}-2\Phi_1^{(n)})}$$

and

$$\Phi_1^{(n+1)} = T_1(\Phi_0^{(n)}, \Phi_1^{(n)}) = \frac{(\Phi_1^{(n)})^2(1-\Phi_0^{(n)}+2\Phi_1^{(n)})}{(1-\Phi_0^{(n)}+\Phi_1^{(n)})(1-\Phi_0^{(n)}-2\Phi_1^{(n)})}.$$

Let $\alpha = 0$ and $\beta = \frac{1}{4}$ we have

$$\Phi_1^{(1)}(0,\frac{1}{4}) = 0.15, \quad \Phi_0^{(1)}(0,\frac{1}{4}) = 0.4$$

$$\Phi_1^{(2)}(0,\frac{1}{4}) = 0.09, \quad \Phi_0^{(2)}(0,\frac{1}{4}) = 0.64$$

$$\Phi_1^{(3)}(0,\frac{1}{4}) = 0.054, \quad \Phi_0^{(3)}(0,\frac{1}{4}) = 0.784$$

$$\vdots$$

Corollary 2.1 *For initial probability distribution* $\Phi_0^{(0)} = 0, \Phi_1^{(0)} = \frac{1}{4}$, *we have*

$$\Phi_0^{(n)} + 4\Phi_1^{(n)} = 1,$$

moreover

$$\Phi_1^{(n+1)} = 0.6\Phi_1^{(n)} \textit{ and } \Phi_0^{(n+1)} = 0.4 + 0.6\Phi_0^{(n)}.$$

So $\lim_{n\to\infty} \Phi_1^{(n)} = 0$ and $\lim_{n\to\infty} \Phi_0^{(n)} = 1$. The random walk is recurrent.

2.1.1 Hitting time and diffusion

Let $T^{(n)} = \inf\{t \geq 0; w_t \in \partial(F^{(n)} \cup F^{(n)'})\}, \partial(F^{(n)} \cup F^{(n)'}) := \{\mathbf{a_n}, \mathbf{b_n}, \mathbf{a'_n}, \mathbf{b'_n}\}$ be the first-passage time taken to exit $F^{(n)} \cup F^{(n)'}$ at the four vertices.

Theorem 2.2 *$E(T^{(n+1)}|w_0 = \mathbf{0}) = 5E(T^{(n)}|w_0 = \mathbf{0})$.*

Proof 2.2 *Consider the following graph on $F^{(1)} \cup F^{(1)'}$ with vertices relabeled. Let $r = 1/4$ and*

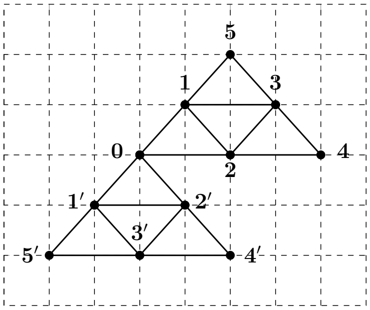

Figure 2.4: relabel on $F^{(1)} \cup F^{(1)'}$

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 1' \\ 2' \\ 3' \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 0 & r & r & 0 & r & r & 0 \\ r & 0 & r & r & 0 & 0 & 0 \\ r & r & 0 & r & 0 & 0 & 0 \\ 0 & r & r & 0 & 0 & 0 & 0 \\ r & 0 & 0 & 0 & 0 & r & r \\ r & 0 & 0 & 0 & r & 0 & r \\ 0 & 0 & 0 & 0 & r & r & 0 \end{pmatrix} \end{array},$$

Let $H_k^{(1)} = E(T^{(1)}|w_0 = \mathbf{k})$ for $k = 0, 1, 2, 3, 1', 2', 3'$. By

$$H_i^{(1)} = 1 + \sum A_{ij} H_j^{(1)},$$

we have

$$\begin{bmatrix} H_0^{(1)} \\ H_1^{(1)} \\ H_2^{(1)} \\ H_3^{(1)} \\ H_{1'}^{(1)} \\ H_{2'}^{(1)} \\ H_{3'}^{(1)} \end{bmatrix} = \frac{1}{I_7 - A} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

So $H_0^{(1)} = 5$.

This can be generated to any n *by*

$$H_i^{(n+1)} = H_0^{(n)} + \sum A_{ij} H_j^{(n+1)},$$

So $H_0^{(n+1)} = 5H_0^{(n)}$.

The diffusion exponent d_w is defined as

$$\sigma(t) \sim t^{\frac{1}{d_w}}. \tag{2.1}$$

where $\sigma(t)$ is the standard deviation of the random walk at time t. (2.1) can be written as

$$T \sim L^{d_w},$$

where L is the length from $\mathbf{0}$ to $\mathbf{a_n}$ [11]. So from $F^{(n)} \cup F^{(n)'}$ to $F^{(n+1)} \cup F^{(n+1)'}$, $L \to 2L$ and $T \to 5T$, and by $5T \sim (2L)^{d_w}$, we have $d_w = \frac{\ln 5}{\ln 2}$.

2.2 Random walks on the Cantor graph

Let us introduce the random walk on generations of the Cantor set. The particle can reside on either the Cantor set or the void set. When the particle enters to the void set from left (right), it keeps going left (right) without changing direction until it reaches the other edge of the void set itself. If the particle is on the Cantor set, it performs a standard random walk with 1/2 probability to move either left or right to the neighbor.

Figure 2.5: Cantor set with G =1,2,3

Figure 2.6: Cantor Graph with G=1,2,3

An equivalent approach is to associate the Cantor set to a directed Cantor graph, obtained by replacing the voids with bubbles of appropriate length whose sites are connected by directed links as above. One can introduce the above random walk on the graph as Figure 2.6 [9].

Let $H_p^{(G)}$ denote the Gth generation of Cantor graph and $H_p^{(G)'}$ denote its reflection on the origion. A path w is defined by $w = (w_0, \ldots, w_m)$, where $w_t \in H_p^{(G)} \cup H_p^{(G)'}$ walks as in the graph.

For the Gth generation, let $\tau^{(G)} = \inf\{t \geq 1; w_t \in \partial H_p^{(G)} \cup \partial H_p^{(G)'}\}$, $\partial H_p^{(G)} \cup \partial H_p^{(G)'} = \{-3^G, 0, 3^G\}$. The probability that a random walk starts with x and hits $y \in \partial H_p^{(G)} \cup \partial H_p^{(G)'}$ is given by

$$P^{(G)}(x, y) = \sum_{t=1}^{\infty} P(w_0 = x, w_t = y, \tau^{(G)} = t).$$

The Green function that a random walk starts with x and hits $y \in \partial H_p^{(G)} \cup$

$\partial H_p^{(G)'}$ is given by

$$g^{(G)}(z)(x,y) = \sum_{t=1}^{\infty} z^t P(w_0 = x, w_t = y, \tau^{(G)} = t).$$

Notice $g^{(G)}(1)(x,y) = P^{(G)}(x,y)$.
For simplicity, let $\alpha^{(G)}(z) = g^{(G)}(z)(0,0)$ and $\beta^{(G)}(z) = g^{(G)}(z)(0,3^G)$.

First let's decide the initial condition on $H_p^{(1)}$, then $H_p^{(1)'}$ is obtained by symmetry:
$g^{(1)}(z)(0,0) = 2(rz) * g^{(1)}(z)(1,0)$ and $g^{(1)}(z)(0,3) = (rz) * g^{(1)}(z)(1,3)$, where

$$\begin{bmatrix} g^{(1)}(z)(1,0) \\ g^{(1)}(z)(2,0) \end{bmatrix} = \frac{1}{I_2 - \begin{bmatrix} 0 & rz \\ rz & 0 \end{bmatrix}} \begin{bmatrix} rz \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2z}{z-4} \\ -\frac{z^2}{z-4} \end{bmatrix}.$$

By symmetry, $g^{(1)}(z)(1,3) = g^{(1)}(z)(2,0)$. So we have

$$\alpha^{(1)}(z) = -\frac{2z^2}{z-4}, \qquad \beta^{(1)}(z) = -\frac{z^3}{2(z-4)}.$$

Theorem 2.3 *The recursive relation between* $(\alpha^{(G+1)}(z), \beta^{(G+1)}(z))$ *and* $(\alpha^{(G)}(z), \beta^{(G)}(z))$ *is*

$$\alpha^{(G+1)}(z) = \alpha^{(G)}(z) - \frac{4\beta^{(G)}(z)^2(\alpha^{(G)}(z) - 2)}{\alpha^{(G)}(z)^2 - 4\alpha^{(G)}(z) - z^{2\times 3^G} + 4},$$

$$\beta^{(G+1)}(z) = \frac{2\beta^{(G)}(z)^2 z^3}{\alpha^{(G)}(z)^2 - 4\alpha^{(G)}(z) - z^{2\times 3^G} + 4}.$$

Proof 2.3 *Since* $H_p^{(G+1)}$ *is obtained by connecting two* $H_p^{(G)}$ *by the void set of the same length, assuming we know* $g^{(G)}(z)(0,0)$ *and* $g^{(G)}(z)(0,3^G)$*, consider the following graph on* $H_p^{(G+1)}$ *with sites relabeled where* $A = 3^G, B = 2 \times 3^G, C = 3^{G+1}$.

Figure 2.7: relabel on $H_p^{(G+1)}$

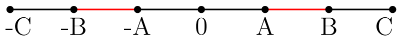

Figure 2.8: $H_p^{(G+1)} \cup H_p^{(G+1)'}$

$\alpha^{(G+1)}(z) = \alpha^{(G)}(z) + \beta^{(G)}(z) g^{(G+1)}(z)(A,0)$ *and* $\beta^{(G+1)}(z) = \beta^{(G)}(z) g^{(G+1)}(z)(A,C)$.

$$\begin{bmatrix} g^{(G+1)}(z)(A,0) \\ g^{(G+1)}(z)(B,0) \end{bmatrix} = \frac{1}{I_2 - \begin{bmatrix} r\alpha^{(G)}(z) & rz^{3^G} \\ rz^{3^G} & r\alpha^{(G)}(z) \end{bmatrix}} \begin{bmatrix} \beta^{(G)}(z) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2\beta^{(G)}(z)(\alpha^{(G)}(z)-2)}{\alpha^{(G)}(z)^2-4\alpha^{(G)}(z)-z^{2\times 3^G}+4} \\ \frac{2\beta^{(G)}(z)z^{3^G}}{\alpha^{(G)}(z)^2-4\alpha^{(G)}(z)-z^{2\times 3^G}+4} \end{bmatrix}.$$

By symmetry, $g^{(G)}(z)(A,C) = g^{(G)}(z)(B,0)$. *So we have*

$$\alpha^{(G+1)}(z) = \alpha^{(G)}(z) - \frac{4\beta^{(G)}(z)^2(\alpha^{(G)}(z)-2)}{\alpha^{(G)}(z)^2 - 4\alpha^{(G)}(z) - z^{2\times 3^G} + 4},$$

$$\beta^{(G+1)}(z) = \frac{2\beta^{(G)}(z)^2 z^3}{\alpha^{(G)}(z)^2 - 4\alpha^{(G)}(z) - z^{2\times 3^G} + 4}.$$

Let $z = 1$, $a^{(G)} = \alpha^{(G)}(1) = P^{(G)}(0,0)$ and $b^{(G)} = \beta^{(G)}(1) = P^{(G)}(0,3^G)$. The initial probability $a^{(1)} = \alpha^{(1)}(1) = 2/3$ and $b^{(1)} = \beta^{(1)}(1) = 1/6$.

Corollary 2.2 *For initial probability* $a^{(1)} = 2/3$ *and* $b^{(1)} = 1/6$*,*

$$a^{(G+1)} = a^{(G)} - \frac{4(b^{(G)})^2(a^{(G)}-2)}{(a^{(G)}-1)(a^{(G)}-3)}, \quad b^{(G+1)} = \frac{2(b^{(G)})^2}{(a^{(G)}-1)(a^{(G)}-3)}.$$

$$P^{(1)}(0,0) = 2/3, \qquad P^{(1)}(0,3) = 1/6.$$

$$P^{(2)}(0,0) = 6/7, \qquad P^{(2)}(0,3^2) = 1/14.$$

$$P^{(3)}(0,0) = 14/15, \qquad P^{(3)}(0,3^3) = 1/30.$$

$$P^{(4)}(0,0) = 30/31, \qquad P^{(4)}(0,3^4) = 1/62.$$

$$\vdots$$

It is easy to see that $a^{(G)} + 2b^{(G)} = 1$ for $G = 1, 2, 3...$ and $b^{(G+1)} = \frac{b^{(G)}}{2(1+b^{(G)})}$. So $\lim_{G\to\infty} b^{(G)} = 0$ and $\lim_{G\to\infty} a^{(G)} = 1$. So the recurrence is also obtained.

CHAPTER 3

QUANTUM WALKS ON SIERPINSKI GASKET

3.1 Notation and Definition

For the n^{th} order Sierpinski gasket $F^{(n)}$ embedded in the two-dimensional plane, it is spanned by $|\mathbf{x}\rangle = |x_1, x_2\rangle$ with $0 \le x_1 \le 2^{n+1}$, and $0 \le x_2 \le 2^n$ restricted to be on the gasket. Let H_c be the coin space with computational basis $\{|\mathbf{e}_i\rangle, 0 \le i \le 5\}$. So we can define $|\mathbf{e}_0\rangle = |2,0\rangle$, $|\mathbf{e}_1\rangle = |1,1\rangle$, $|\mathbf{e}_2\rangle = |-1,1\rangle$, $|\mathbf{e}_3\rangle = |-2,0\rangle$, $|\mathbf{e}_4\rangle = |-1,-1\rangle$, $|\mathbf{e}_5\rangle = |1,-1\rangle$ as in Figure 3.1. But for each $\mathbf{x} \in F^{(n)}$ only four of these basis vectors are used, so define $out(\mathbf{x})$ to be the set of these four basis vectors. For $\mathbf{a_n}$ and $\mathbf{b_n}$, we can take $F^{(n)}$ as part of $F^{(n+1)}$, so $out(\mathbf{a_n}) = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_5\}$ and $out(\mathbf{b_n}) = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. As for $\mathbf{0}$, we can extend $F^{(n)}$ as following: if we think $\mathbf{0}, \mathbf{a_n}, \mathbf{b_n}$ as boundary points of $F^{(n)}$, then we call reflection of $F^{(n)}$ denoted by $F^{(n)'}$ with boundary points $\mathbf{0}, \mathbf{a'_n}, \mathbf{b'_n}$, here $|\mathbf{a'_n}\rangle = |-\mathbf{a_n}\rangle = |-2^n, -2^n\rangle$ and $|\mathbf{b'_n}\rangle = |2^n, -2^n\rangle$, see Figure 2.2. Therefore, $out(\mathbf{0}) = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_5\}$.

Let $H_p^{(n)} = Span\{|\mathbf{x}\rangle; \mathbf{x} \in F^{(n)} \cup F^{(n)'}\}$ be the position subspace.

The state space is defined by $H^{(n)} = Span\{|\bar{\mathbf{x}}\rangle = |\mathbf{x}\rangle \otimes |\mathbf{e}_i\rangle, \mathbf{x} \in F^{(n)} \cup F^{(n)'}, \mathbf{e}_i \in out(\mathbf{x})\}$. $H^{(n)}$ is a subspace of $H_p^{(n)} \otimes H_c$. For simplicity, throughout this paper, we write the state space as $H^{(n)} = Span\{|\bar{\mathbf{x}}\rangle = |\mathbf{x}^i\rangle, \mathbf{x} \in F^{(n)} \cup$

$F^{(n)'}, \mathbf{e}_i \in out(\mathbf{x})\}$.

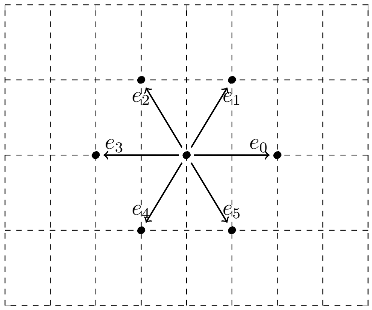

Figure 3.1: computational basis

Let $F^{(\infty)} = \cup_{n=0}^{\infty} F^{(n)}$ and $F^{(\infty)'} = \cup_{n=0}^{\infty} F^{(n)'}$, then the state space can be extended as $H^{(\infty)} = Span\{|\mathbf{x}^i\rangle, \mathbf{x} \in F^{(\infty)} \cup F^{(\infty)'}, \mathbf{e}_i \in out(\mathbf{x})\}$.

The shift operator $S : H^{(\infty)} \to H^{(\infty)}$ is defined by $S|\mathbf{x}^k\rangle = |\mathbf{y}^j\rangle$, for $j = k + 3 \ (mod\ 6)$, where $\mathbf{y} = \mathbf{x} + \mathbf{e}_k$.

Let G be the 4×4 Grover matrix (g_{ij}), where $g_{ij} = \begin{cases} -1/2 & if\ i = j \\ 1/2 & if\ i \neq j \end{cases}$.

So

$$G = r \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \text{where } r = \frac{1}{2}.$$

Let $G_{\mathbf{x}} : H^{(\infty)} \to H^{(\infty)}$ be a local operator defined by

$$G_{\mathbf{x}}(|\mathbf{y}^i\rangle) = \sum_{j, where\ \mathbf{e}_j \in out(\mathbf{y})} g_{ji}|\mathbf{x}^j\rangle \delta_{\mathbf{xy}}, \forall \mathbf{y} \in F^{(\infty)} \cup F^{(\infty)'}, \mathbf{e}_i \in out(\mathbf{y}).$$

Define $\tilde{G} = \sum_{\mathbf{x}} G_{\mathbf{x}}$. The evolution operator for the quantum random walk is defined by $U = S\tilde{G}$. Then $U : H^{(\infty)} \to H^{(\infty)}$ is a unitary operator.

Let $\psi_0 \in H^{(\infty)}$ and $\psi_t = U^t \psi_0$. The sequence $\{\psi_t\}_0^{\infty}$ is called a quantum random walk with the initial state ψ_0.

Let $\psi_t = \sum_{\mathbf{e}_i \in out(\mathbf{x})} \sum_{\mathbf{x} \in F^{(\infty)} \cup F^{(\infty)'}} \psi_t(\mathbf{x}, i)|\mathbf{x}^i\rangle$ be the quantum random walk at time t, where $\psi_t(\mathbf{x}, i)$ is the coefficient at $|\mathbf{x}^i\rangle$. Let $p_t(\mathbf{x}, i) = |\psi_t(\mathbf{x}, i)|^2$

be the probability that the particle is at state $|\mathbf{x}^i\rangle$ at time t, and $p_t(\mathbf{x}) = \sum_{\mathbf{e}_i \in out(\mathbf{x})} p_t(\mathbf{x}, i)$ be the probability that the particle is found at state $|\mathbf{x}\rangle$ at time t.

3.2 The Path Integral

Our formulation of path integral is described as follows. For the state space $Span\{\mathbf{x}^i, \mathbf{x} \in F^{(\infty)} \cup F^{(\infty)'}, \mathbf{e_i} \in out(\mathbf{x})\}$, a path w is defined by $w = (w_0, \ldots, w_m)$, where $w_t = \mathbf{x}_t^{k_t}$, and $\|\mathbf{x}_{t+1} - \mathbf{x}_t\| = 2$ where $\|\mathbf{x} - \mathbf{y}\| = |x_1 - y_1| + |x_2 - y_2|$. The length of w is defined by $|w| = m$. Let $\Omega^m = \{w; |w| = m\}$.

Definition 3.1 *(Amplitude function)*

Let

$$\phi(\mathbf{x}^i, \mathbf{y}^j) = \begin{cases} -r & if\ \|\mathbf{x} - \mathbf{y}\| = 2, j = i + 3\ (mod\ 6) \\ r & if\ \|\mathbf{x} - \mathbf{y}\| = 2, j \neq i + 3\ (mod\ 6), \mathbf{y} = \mathbf{x} + \mathbf{e}_k, j = k + 3\ (mod\ 6) \\ 0 & otherwise \end{cases}$$

The amplitude function for $w \in \Omega^m$ is defined as

$$\Psi(w) = \prod_{t=0}^{m-1} \phi(w_t, w_{t+1}). \tag{3.1}$$

Definition 3.2 *Let $\Gamma \subseteq \Omega^m$ .Then the amplitude function of a Γ is defined by*

$$\Psi(\Gamma) = \sum_{w \in \Gamma} \Psi(w). \tag{3.2}$$

Let $\Omega = \cup_{m=0}^{\infty} \Omega^m$. For $\Gamma \in \Omega$ with $\Gamma^m = \Gamma \cap \Omega^m$, we also define

$$\Psi(\Gamma) = \sum_{m=0}^{\infty} \Psi(\Gamma^m).$$

For any $\psi \in H^{(\infty)}$, we shall write $\psi = \sum_{\mathbf{e}_i \in out(\mathbf{x})} \sum_{\mathbf{x} \in F^{(\infty)} \cup F^{(\infty)'}} \psi(\mathbf{x}, i) |\mathbf{x}^i\rangle$. Suppose $\psi_0 = |\mathbf{x}^i\rangle$, and $\psi_t = U^t \psi_0$, then

$$\psi_t(\mathbf{y}, j) = \Psi(w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j).$$

Let $\tau^{(n)} = \inf\{t \geq 1; w_t \in \partial F^{(n)} \cup \partial F^{(n)'}\}$, where $\partial F^{(n)} \cup \partial F^{(n)'} = \{\mathbf{0}, \mathbf{a_n}, \mathbf{b_n}, \mathbf{a'_n}, \mathbf{b'_n}\}$. Define the set of path that start with $\mathbf{x}^i$ and exit from $\mathbf{y}^j$, where $\mathbf{x}, \mathbf{y} \in \partial F^{(n)} \cup \partial F^{(n)'}$ at time t as $\{w = (w_0, \ldots, w_{\tau^{(n)}}), w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j, \tau^{(n)} = t\}$.

The amplitude Green function for quantum random walk is defined by

$$g^{(n)}(z)(\mathbf{x}, \mathbf{y})^i_j = \sum_{t=1}^{\infty} z^t \Psi(w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j, \tau^{(n)} = t). \tag{3.3}$$

The probability that a quantum random walk starts with $\mathbf{0}^i$ and exits from $\mathbf{y}^j$ is given by

$$P^{(n)}(\mathbf{x}, \mathbf{y})^i_j = \sum_{t=1}^{\infty} |\Psi(w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j, \tau^{(n)} = t)|^2. \tag{3.4}$$

So by Parseval's theorem,

$$P^{(n)}(\mathbf{x}, \mathbf{y})^i_j = \frac{1}{2\pi} \int_0^{2\pi} |g^{(n)}(e^{i\theta})(\mathbf{x}, \mathbf{y})^i_j|^2 d\theta. \tag{3.5}$$

On $F^{(n)}$, consider the exiting amplitude Green matrix $g^{(n)}(z)$ as

$$\begin{array}{c|cccccccccccc}
 & \mathbf{0}^0 & \mathbf{0}^1 & \mathbf{0}^4 & \mathbf{0}^5 & \mathbf{a_n}^0 & \mathbf{a_n}^1 & \mathbf{a_n}^4 & \mathbf{a_n}^5 & \mathbf{b_n}^0 & \mathbf{b_n}^1 & \mathbf{b_n}^2 & \mathbf{b_n}^3 \\
\hline
\mathbf{0}^0 & g^{(n)}(z)(\mathbf{0},\mathbf{0})^0_0 & & \cdots & & & & & & \cdots & & & g^{(n)}(z)(\mathbf{0},\mathbf{b_n})^0_3 \\
\mathbf{0}^1 & & & & & & & & & & & & \\
\mathbf{0}^4 & & & & & & & & & & & & \\
\mathbf{0}^5 & \vdots & & \ddots & & & & & & & & & \vdots \\
\mathbf{a_n}^0 & & & & & & & & & & & & \\
\mathbf{a_n}^1 & & & & & & & & & & & & \\
\mathbf{a_n}^4 & & & & & & & & & & & & \\
\mathbf{a_n}^5 & \vdots & & & & & & & & \ddots & & & \vdots \\
\mathbf{b_n}^0 & & & & & & & & & & & & \\
\mathbf{b_n}^1 & & & & & & & & & & & & \\
\mathbf{b_n}^2 & & & & & & & & & & & & \\
\mathbf{b_n}^3 & g^{(n)}(z)(\mathbf{b_n},\mathbf{0})^3_0 & & \cdots & & & & & & \cdots & & & g^{(n)}(z)(\mathbf{b_n},\mathbf{b_n})^3_3
\end{array}.$$

The matrix $g^{(n)}(z)$ contains 12×12 entries, with each one defined as in (3.3). In fact, $g^{(n)}(z)$ only requires six variables instead of 12×12 ones. The argument is as follows:

Let $\Theta^{(n)}(z)(\mathbf{x}, \mathbf{y})^i_j$ be the same path amplitude Green function as $g^{(n)}(z)(\mathbf{x}, \mathbf{y})^i_j$ without applying the first ratation $\tilde{G}$. Then $\Theta^{(n)}(z)$ has only six different vari-

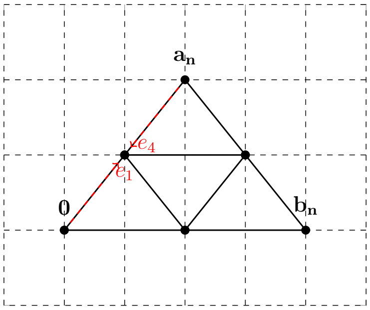

Figure 3.2: $\theta_1^{(n)}$

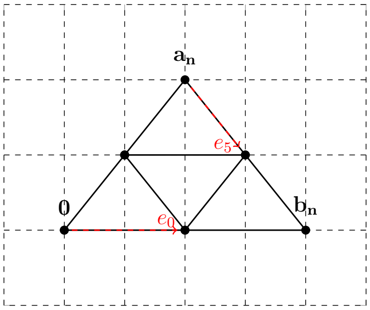

Figure 3.3: $\theta_2^{(n)}$

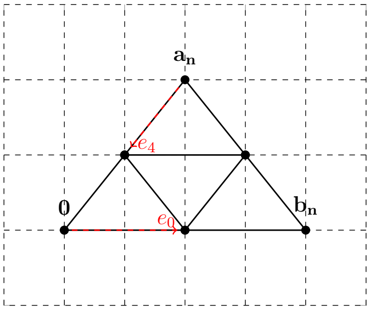

Figure 3.4: $\theta_3^{(n)}$

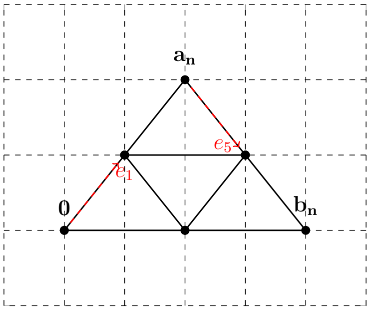

Figure 3.5: $\theta_4^{(n)}$

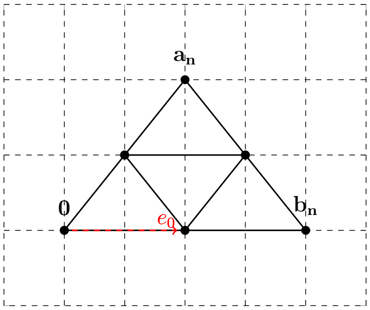

Figure 3.6: $\theta_5^{(n)}$

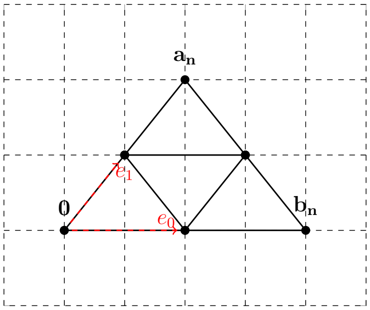

Figure 3.7: $\theta_6^{(n)}$

ables which are corresponding to the graph from Figure 3.2 to Figure 3.7. The rest will be one of these six variables by the symmetry of $F^{(n)}$. For example, $\Theta^{(n)}(z)(\mathbf{a_n},\mathbf{b_n})_2^5 = \Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n})_4^1$ and $\Theta^{(n)}(z)(\mathbf{b_n},\mathbf{0})_0^2 = \Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n})_4^0$.

$$\theta_1^{(n)}(z) = \Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n})_4^1$$

$$\theta_2^{(n)}(z) = \Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n})_5^0$$

$$\theta_3^{(n)}(z) = \Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n})_4^0$$

$$\theta_4^{(n)}(z) = \Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n})_5^1$$

$$\theta_5^{(n)}(z) = \Theta^{(n)}(z)(\mathbf{0},\mathbf{0})_4^4$$

$$\theta_6^{(n)}(z) = \Theta^{(n)}(z)(\mathbf{0},\mathbf{0})_4^5$$

Therefore

$$\Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n}) = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & \theta_3^{(n)} & \theta_2^{(n)} \\ 0 & 0 & \theta_1^{(n)} & \theta_4^{(n)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

and

$$\Theta^{(n)}(z)(\mathbf{0},\mathbf{0}) = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} \theta_5^{(n)} & \theta_6^{(n)} & 0 & 0 \\ \theta_6^{(n)} & \theta_5^{(n)} & 0 & 0 \\ 0 & 0 & \theta_5^{(n)} & \theta_6^{(n)} \\ 0 & 0 & \theta_6^{(n)} & \theta_5^{(n)} \end{pmatrix} \end{array}.$$

Notice that $g^{(n)}(z)(\mathbf{0},\mathbf{a_n})^i_j = [G\Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n})]_{ij}$, so we have

$$\begin{aligned} g^{(n)}(\mathbf{0},\mathbf{a_n}) &= \tilde{G}\Theta^{(n)}(z)(\mathbf{0},\mathbf{a_n}) \\ &= \frac{1}{2} \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & \theta_1^{(n)} - \theta_3^{(n)} & \theta_4^{(n)} - \theta_2^{(n)} \\ 0 & 0 & \theta_3^{(n)} - \theta_1^{(n)} & \theta_2^{(n)} - \theta_4^{(n)} \\ 0 & 0 & \theta_1^{(n)} + \theta_3^{(n)} & \theta_2^{(n)} + \theta_4^{(n)} \\ 0 & 0 & \theta_1^{(n)} + \theta_3^{(n)} & \theta_2^{(n)} + \theta_4^{(n)} \end{pmatrix} \end{array}, \end{aligned}$$

and

$$\begin{aligned} g^{(n)}(\mathbf{0},\mathbf{0}) &= \tilde{G}\Theta^{(n)}(z)(\mathbf{0},\mathbf{0}) \\ &= \frac{1}{2} \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} \theta_6^{(n)} - \theta_5^{(n)} & \theta_5^{(n)} - \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} \\ \theta_5^{(n)} - \theta_6^{(n)} & \theta_6^{(n)} - \theta_5^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} \\ \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_6^{(n)} - \theta_5^{(n)} & \theta_5^{(n)} - \theta_6^{(n)} \\ \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} - \theta_6^{(n)} & \theta_6^{(n)} - \theta_5^{(n)} \end{pmatrix} \end{array}. \end{aligned}$$

Let $u_1^{(n)} = \frac{1}{2}(\theta_1^{(n)} + \theta_3^{(n)})$, $u_2^{(n)} = \frac{1}{2}(\theta_2^{(n)} + \theta_4^{(n)})$, $u_3^{(n)} = \frac{1}{2}(\theta_4^{(n)} - \theta_2^{(n)})$, $u_4^{(n)} = \frac{1}{2}(\theta_1^{(n)} - \theta_3^{(n)})$, $u_5^{(n)} = \frac{1}{2}(\theta_6^{(n)} - \theta_5^{(n)})$, and $u_6^{(n)} = \frac{1}{2}(\theta_6^{(n)} + \theta_5^{(n)})$. We can get all the blocks of $g^{(n)}(z)$ with variables $u_i^{(n)}(z)$ for $i = 1, ..., 6$ as following:

$$g^{(n)}(\mathbf{0}, \mathbf{a_n}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & u_4^{(n)} & u_3^{(n)} \\ 0 & 0 & -u_4^{(n)} & -u_3^{(n)} \\ 0 & 0 & u_1^{(n)} & u_2^{(n)} \\ 0 & 0 & u_1^{(n)} & u_2^{(n)} \end{pmatrix} \end{array}$$

$$g^{(n)}(\mathbf{a_n}, \mathbf{0}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} u_2^{(n)} & u_1^{(n)} & 0 & 0 \\ u_2^{(n)} & u_1^{(n)} & 0 & 0 \\ -u_3^{(n)} & -u_4^{(n)} & 0 & 0 \\ u_3^{(n)} & u_4^{(n)} & 0 & 0 \end{pmatrix} \end{array}$$

$$g^{(n)}(\mathbf{0}, \mathbf{b_n}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & -u_3^{(n)} & -u_4^{(n)} \\ 0 & 0 & u_3^{(n)} & u_4^{(n)} \\ 0 & 0 & u_2^{(n)} & u_1^{(n)} \\ 0 & 0 & u_2^{(n)} & u_1^{(n)} \end{pmatrix} \end{array}$$

$$g^{(n)}(\mathbf{b_n}, \mathbf{0}) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} u_1^{(n)} & u_2^{(n)} & 0 & 0 \\ u_1^{(n)} & u_2^{(n)} & 0 & 0 \\ u_4^{(n)} & u_3^{(n)} & 0 & 0 \\ -u_4^{(n)} & -u_3^{(n)} & 0 & 0 \end{pmatrix} \end{array}$$

$$g^{(n)}(\mathbf{a_n}, \mathbf{b_n}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & u_1^{(n)} & u_2^{(n)} \\ 0 & 0 & u_1^{(n)} & u_2^{(n)} \\ 0 & 0 & u_4^{(n)} & u_3^{(n)} \\ 0 & 0 & -u_4^{(n)} & -u_3^{(n)} \end{pmatrix} \end{array}$$

$$g^{(n)}(\mathbf{b_n},\mathbf{a_n}) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & u_2^{(n)} & u_1^{(n)} \\ 0 & 0 & u_2^{(n)} & u_1^{(n)} \\ 0 & 0 & -u_3^{(n)} & -u_4^{(n)} \\ 0 & 0 & u_3^{(n)} & u_4^{(n)} \end{pmatrix} \end{array}$$

$$g^{(n)}(\mathbf{0},\mathbf{0}) = \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & u_5^{(n)} & -u_5^{(n)} & u_6^{(n)} & u_6^{(n)} \\ 1 & -u_5^{(n)} & u_5^{(n)} & u_6^{(n)} & u_6^{(n)} \\ 4 & u_6^{(n)} & u_6^{(n)} & u_5^{(n)} & -u_5^{(n)} \\ 5 & u_6^{(n)} & u_6^{(n)} & -u_5^{(n)} & u_5^{(n)} \end{array}$$

$$g^{(n)}(\mathbf{a_n},\mathbf{a_n}) = \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & u_5^{(n)} & -u_5^{(n)} & u_6^{(n)} & u_6^{(n)} \\ 1 & -u_5^{(n)} & u_5^{(n)} & u_6^{(n)} & u_6^{(n)} \\ 4 & u_6^{(n)} & u_6^{(n)} & u_5^{(n)} & -u_5^{(n)} \\ 5 & u_6^{(n)} & u_6^{(n)} & -u_5^{(n)} & u_5^{(n)} \end{array}$$

$$g^{(n)}(\mathbf{b_n},\mathbf{b_n}) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & u_5^{(n)} & -u_5^{(n)} & u_6^{(n)} & u_6^{(n)} \\ 1 & -u_5^{(n)} & u_5^{(n)} & u_6^{(n)} & u_6^{(n)} \\ 2 & u_6^{(n)} & u_6^{(n)} & u_5^{(n)} & -u_5^{(n)} \\ 3 & u_6^{(n)} & u_6^{(n)} & -u_5^{(n)} & u_5^{(n)} \end{array}$$

As for $F^{(n)'}$, we can obtain $g^{(n)'}$ by symmetry of $g^{(n)}$. Our goal is to find $g^{(n)}$ iteratively with those six variables $u_i^{(n)}, i = 1, ..., 6$. Here we can think $g^{(n)}$ as a function of six variables $u_i^{(n)}, i = 1, ..., 6$ instead of a 12×12 matrix. By applying the self-similar property again, we have

Lemma 3.1 *$g^{(n+1)} = g(g^{(n)})$, for $n \geq 0$, where $g = g^{(1)}$, the amplitude Green function on $F^{(1)}$.*

To find g, for convenience of notations we consider the following graph on $F^{(1)}$ with vertices relabeled and $g(z)(\mathbf{x},\mathbf{y})_j^i = \sum_{t=1}^{\infty} z^t \Psi(w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j, \tau^{(1)} = t)$, where $\tau^{(1)} = \inf\{t \geq 1; w_t \in \{\mathbf{0},\mathbf{4},\mathbf{5}\}\}$.

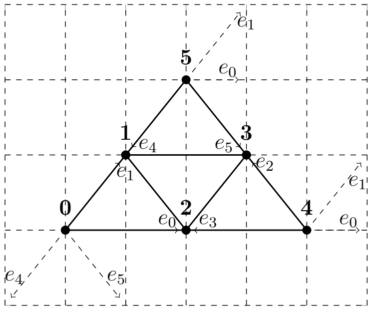

Figure 3.8: relabel on $F^{(1)}$

$$
\begin{aligned}
g^{(n+1)}(\mathbf{0},\mathbf{5}) &= g(\mathbf{0},\mathbf{5}) \\
&= g^{(n)}(\mathbf{0},\mathbf{1})g(\mathbf{1},\mathbf{5}) + g^{(n)}(\mathbf{0},\mathbf{2})g(\mathbf{2},\mathbf{5}),
\end{aligned}
$$

where

$$
\begin{bmatrix} g(\mathbf{1},\mathbf{5}) \\ g(\mathbf{2},\mathbf{5}) \\ g(\mathbf{3},\mathbf{5}) \end{bmatrix} = \frac{1}{I_{12} - [g^{(n)}]|_{1,2,3}} \begin{bmatrix} g^{(n)}(\mathbf{1},\mathbf{5}) \\ g^{(n)}(\mathbf{2},\mathbf{5}) \\ g^{(n)}(\mathbf{3},\mathbf{5}) \end{bmatrix}. \tag{3.6}
$$

For $i = 0, 1, 4, 5,\ j = 0, 1,$

$$
\begin{aligned}
g^{(n+1)}(\mathbf{0},\mathbf{0})^i_j &= g(\mathbf{0},\mathbf{0})^i_j \\
&= g^{(n)}(\mathbf{0},\mathbf{0})^i_j + [g^{(n)}(\mathbf{0},\mathbf{1})g(\mathbf{1},\mathbf{0})]^i_j + [g^{(n)}(\mathbf{0},\mathbf{2})g(\mathbf{2},\mathbf{0})]^i_j,
\end{aligned}
$$

where

$$
\begin{bmatrix} g(\mathbf{1},\mathbf{0}) \\ g(\mathbf{2},\mathbf{0}) \\ g(\mathbf{3},\mathbf{0}) \end{bmatrix} = \frac{1}{I_{12} - [g^{(n)}]|_{1,2,3}} \begin{bmatrix} g^{(n)}(\mathbf{1},\mathbf{0}) \\ g^{(n)}(\mathbf{2},\mathbf{0}) \\ g^{(n)}(\mathbf{3},\mathbf{0}) \end{bmatrix}. \tag{3.7}
$$

For $i = 0, 1, 4, 5,\ j = 4, 5,$

$$
g^{(n+1)}(\mathbf{0},\mathbf{0})^i_j = g(\mathbf{0},\mathbf{0})^i_j = g^{(n)}(\mathbf{5},\mathbf{5})^i_j + [g^{(n)}(\mathbf{5},\mathbf{1})g(\mathbf{1},\mathbf{5})]^i_j + [g^{(n)}(\mathbf{5},\mathbf{3})g(\mathbf{3},\mathbf{5})]^i_j,
$$

where

$$\begin{bmatrix} g(\mathbf{1},\mathbf{5}) \\ g(\mathbf{2},\mathbf{5}) \\ g(\mathbf{3},\mathbf{5}) \end{bmatrix} = \frac{1}{I_{12} - [g^{(n)}]|_{1,2,3}} \begin{bmatrix} g^{(n)}(\mathbf{1},\mathbf{5}) \\ g^{(n)}(\mathbf{2},\mathbf{5}) \\ g^{(n)}(\mathbf{3},\mathbf{5}) \end{bmatrix}. \tag{3.8}$$

Let's first find $g^{(1)}$ base on $g^{(0)}$ by applying the formulas above:
Step 1. Determine $u^{(0)}$.

For $F^{(0)}$,

$$g^{(0)}(\mathbf{0},\mathbf{a_0}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & rz & 0 \\ 0 & 0 & -rz & 0 \\ 0 & 0 & rz & 0 \\ 0 & 0 & rz & 0 \end{pmatrix} \end{array},$$

$$g^{(0)}(\mathbf{0},\mathbf{0}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Comparing with $g^{(n)}(\mathbf{0},\mathbf{a_0})$ and $g^{(n)}(\mathbf{0},\mathbf{0})$ with $n = 0$, we have

$$u_1^{(0)} = rz,$$

$$u_2^{(0)} = 0,$$

$$u_3^{(0)} = 0,$$

$$u_4^{(0)} = rz,$$

$$u_5^{(0)} = 0,$$

$$u_6^{(0)} = 0.$$

By the same argument as $g^{(n)}$, we do not have to find all nine 4×4 block matrices in $g^{(0)}$. We just think $g^{(0)} = g^{(0)}(u_1^{(0)}, ..., u_6^{(0)})$ as the transition function on $F^{(0)}$.

Step 2. Extend $g^{(0)}$ to be a transition function on $F^{(1)}$ by translations. Here we have $[g^{(0)}]|_{1,2,3} =$

$$\begin{array}{c|cccccccccccc|}
 & \mathbf{1}^0 & \mathbf{1}^1 & \mathbf{1}^4 & \mathbf{1}^5 & \mathbf{2}^2 & \mathbf{2}^3 & \mathbf{2}^0 & \mathbf{2}^1 & \mathbf{3}^4 & \mathbf{3}^5 & \mathbf{3}^2 & \mathbf{3}^3 \\
\mathbf{1}^0 & u_5^{(0)} & -u_5^{(0)} & u_6^{(0)} & u_6^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & -u_3^{(0)} & -u_4^{(0)} \\
\mathbf{1}^1 & -u_5^{(0)} & u_5^{(0)} & u_6^{(0)} & u_6^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_3^{(0)} & u_4^{(0)} \\
\mathbf{1}^4 & u_6^{(0)} & u_6^{(0)} & u_5^{(0)} & -u_5^{(0)} & u_4^{(0)} & u_3^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} \\
\mathbf{1}^5 & u_6^{(0)} & u_6^{(0)} & -u_5^{(0)} & u_5^{(0)} & -u_4^{(0)} & -u_3^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} \\
\mathbf{2}^2 & 0 & 0 & -u_3^{(0)} & -u_4^{(0)} & u_5^{(0)} & -u_5^{(0)} & u_6^{(0)} & u_6^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 \\
\mathbf{2}^3 & 0 & 0 & u_3^{(0)} & u_4^{(0)} & -u_5^{(0)} & u_5^{(0)} & u_6^{(0)} & u_6^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 \\
\mathbf{2}^0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_6^{(0)} & u_6^{(0)} & u_5^{(0)} & -u_5^{(0)} & u_4^{(0)} & u_3^{(0)} & 0 & 0 \\
\mathbf{2}^1 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_6^{(0)} & u_6^{(0)} & -u_5^{(0)} & u_5^{(0)} & -u_4^{(0)} & -u_3^{(0)} & 0 & 0 \\
\mathbf{3}^4 & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & -u_3^{(0)} & -u_4^{(0)} & u_5^{(0)} & -u_5^{(0)} & u_6^{(0)} & u_6^{(0)} \\
\mathbf{3}^5 & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_3^{(0)} & u_4^{(0)} & -u_5^{(0)} & u_5^{(0)} & u_6^{(0)} & u_6^{(0)} \\
\mathbf{3}^2 & u_4^{(0)} & u_3^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_6^{(0)} & u_6^{(0)} & u_5^{(0)} & -u_5^{(0)} \\
\mathbf{3}^3 & -u_4^{(0)} & -u_3^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_6^{(0)} & u_6^{(0)} & -u_5^{(0)} & u_5^{(0)}
\end{array}.$$

Notice we switch $\mathbf{2}^0, \mathbf{2}^1$ with $\mathbf{2}^2, \mathbf{2}^3$ and $\mathbf{3}^2, \mathbf{3}^3$ with $\mathbf{3}^4, \mathbf{3}^5$ for rows and columns. The reason of doing this is after switching, this matrix has the following block structure.

$$\left[\begin{array}{c|c|c}
A & B & C \\ \hline
C & A & B \\ \hline
B & C & A
\end{array}\right]$$

And so is $I - [g^{(0)}]|_{1,2,3}$ and its inverse. According to this reordering,

$$g^{(0)}(\mathbf{1},\mathbf{5}) = \begin{array}{c|cccc|}
 & 0 & 1 & 4 & 5 \\
0 & 0 & 0 & u_4^{(0)} & u_3^{(0)} \\
1 & 0 & 0 & -u_4^{(0)} & -u_3^{(0)} \\
4 & 0 & 0 & u_1^{(0)} & u_2^{(0)} \\
5 & 0 & 0 & u_1^{(0)} & u_2^{(0)}
\end{array},$$

$$g^{(0)}(\mathbf{2},\mathbf{5}) = O_{4\times 4},$$

$$g^{(0)}(\mathbf{3},\mathbf{5}) = \begin{array}{c|cccc|}
 & 0 & 1 & 4 & 5 \\
4 & 0 & 0 & u_2^{(0)} & u_1^{(0)} \\
5 & 0 & 0 & u_2^{(0)} & u_1^{(0)} \\
2 & 0 & 0 & -u_3^{(0)} & -u_4^{(0)} \\
3 & 0 & 0 & u_3^{(0)} & u_4^{(0)}
\end{array},$$

$$g^{(0)}(\mathbf{0},\mathbf{2}) = \begin{array}{cc} & \begin{array}{cccc} 2 & 3 & 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} -u_3^{(0)} & -u_4^{(0)} & 0 & 0 \\ u_3^{(0)} & u_4^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \end{pmatrix} \end{array},$$

$$g^{(0)}(\mathbf{0},\mathbf{1}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & u_4^{(0)} & u_3^{(0)} \\ 0 & 0 & -u_4^{(0)} & -u_3^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \end{pmatrix} \end{array}.$$

Pluging in the values of $u_1^{(0)}, ..., u_6^{(0)}$ and applying (3.6), we have

$$g(\mathbf{1},\mathbf{5}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & rz & -\frac{r^2z^2}{1+rz} \\ 0 & 0 & -\frac{2r^3z^3}{1+rz} - rz & r^2z^2 + \frac{r^3z^3}{1+rz} \\ 0 & 0 & -\frac{2r^3z^3}{1+rz} + rz & r^2z^2 + \frac{r^3z^3}{1+rz} \\ 0 & 0 & rz & \frac{r^2z^2}{1+rz} \end{pmatrix} \end{array},$$

$$g(\mathbf{2},\mathbf{5}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & -\frac{r^2z^2}{1+rz} & \frac{r^2z^2}{1+rz} \\ 0 & 0 & r^2z^2 + \frac{r^3z^3}{1+rz} & r^2z^2 + \frac{r^3z^3}{1+rz} \\ 0 & 0 & r^2z^2 + \frac{r^3z^3}{1+rz} & r^2z^2 + \frac{r^3z^3}{1+rz} \\ 0 & 0 & \frac{r^2z^2}{1+rz} & -\frac{r^2z^2}{1+rz} \end{pmatrix} \end{array},$$

$$g(\mathbf{3},\mathbf{5}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 4 \\ 5 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & \frac{r^2z^2}{1+rz} & rz \\ 0 & 0 & r^2z^2 + \frac{r^3z^3}{1+rz} & -\frac{2r^3z^3}{1+rz} + rz \\ 0 & 0 & r^2z^2 + \frac{r^3z^3}{1+rz} & -\frac{2r^3z^3}{1+rz} - rz \\ 0 & 0 & -\frac{r^2z^2}{1+rz} & rz \end{pmatrix} \end{array}.$$

Then

$$g^{(1)}(\mathbf{0},\mathbf{a_1}) = g(\mathbf{0},\mathbf{5}) = g^{(0)}(\mathbf{0},\mathbf{1})g(\mathbf{1},\mathbf{5}) + g^{(0)}(\mathbf{0},\mathbf{2})g(\mathbf{2},\mathbf{5}) \tag{3.9}$$

$$= \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & -r^3z^3 + r^2z^2 - \frac{3r^4z^4}{1+rz} & 0 \\ 0 & 0 & r^3z^3 - r^2z^2 + \frac{3r^4z^4}{1+rz} & 0 \\ 0 & 0 & r^3z^3 + r^2z^2 - \frac{r^4z^4}{1+rz} & 2r^3z^3 + \frac{2r^4z^4}{1+rz} \\ 0 & 0 & r^3z^3 + r^2z^2 - \frac{r^4z^4}{1+rz} & 2r^3z^3 + \frac{2r^4z^4}{1+rz} \end{pmatrix} \end{array}. \tag{3.10}$$

Next, we have

$$g^{(0)}(\mathbf{0},\mathbf{0}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} u_5^{(0)} & -u_5^{(0)} & u_6^{(0)} & u_6^{(0)} \\ -u_5^{(0)} & u_5^{(0)} & u_6^{(0)} & u_6^{(0)} \\ u_6^{(0)} & u_6^{(0)} & u_5^{(0)} & -u_5^{(0)} \\ u_6^{(0)} & u_6^{(0)} & -u_5^{(0)} & u_5^{(0)} \end{pmatrix} \end{array},$$

$$g^{(0)}(\mathbf{5},\mathbf{5}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} u_5^{(0)} & -u_5^{(0)} & u_6^{(0)} & u_6^{(0)} \\ -u_5^{(0)} & u_5^{(0)} & u_6^{(0)} & u_6^{(0)} \\ u_6^{(0)} & u_6^{(0)} & u_5^{(0)} & -u_5^{(0)} \\ u_6^{(0)} & u_6^{(0)} & -u_5^{(0)} & u_5^{(0)} \end{pmatrix} \end{array},$$

$$g^{(0)}(\mathbf{1},\mathbf{0}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ -u_3^{(0)} & -u_4^{(0)} & 0 & 0 \\ u_3^{(0)} & u_4^{(0)} & 0 & 0 \end{pmatrix} \end{array},$$

$$g^{(0)}(\mathbf{2},\mathbf{0}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 0 \\ 1 \end{array} & \begin{pmatrix} u_4^{(0)} & u_3^{(0)} & 0 & 0 \\ -u_4^{(0)} & -u_3^{(0)} & 0 & 0 \\ u_1^{(0)} & u_2^{(0)} & 0 & 0 \\ u_1^{(0)} & u_2^{(0)} & 0 & 0 \end{pmatrix} \end{array},$$

$$g^{(0)}(\mathbf{3},\mathbf{0}) = O_{4\times 4},$$

$$g^{(0)}(\mathbf{5},\mathbf{1}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ -u_3^{(0)} & -u_4^{(0)} & 0 & 0 \\ u_3^{(0)} & u_4^{(0)} & 0 & 0 \end{pmatrix} \end{array},$$

$$g^{(0)}(\mathbf{5},\mathbf{3}) = \begin{array}{cc} & \begin{array}{cccc} 4 & 5 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_4^{(0)} & u_3^{(0)} \\ 0 & 0 & -u_4^{(0)} & -u_3^{(0)} \end{pmatrix} \end{array}.$$

Pluging in the values of $u_1^{(0)}, ..., u_6^{(0)}$ then applying (3.7) and (3.8), we have

$$g(\mathbf{1},\mathbf{0}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} \frac{r^2z^2}{1+rz} & rz & 0 & 0 \\ r^2z^2 + \frac{r^3z^3}{1+rz} & -\frac{2r^3z^3}{1+rz} + rz & 0 & 0 \\ r^2z^2 + \frac{r^3z^3}{1+rz} & -\frac{2r^3z^3}{1+rz} - rz & 0 & 0 \\ -\frac{r^2z^2}{1+rz} & rz & 0 & 0 \end{pmatrix} \end{array},$$

$$g(\mathbf{2},\mathbf{0}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 0 \\ 1 \end{array} & \begin{pmatrix} rz & -\frac{r^2z^2}{1+rz} & 0 & 0 \\ -\frac{2r^3z^3}{1+rz} - rz & r^2z^2 + \frac{r^3z^3}{1+rz} & 0 & 0 \\ -\frac{2r^3z^3}{1+rz} + rz & r^2z^2 + \frac{r^3z^3}{1+rz} & 0 & 0 \\ rz & \frac{r^2z^2}{1+rz} & 0 & 0 \end{pmatrix} \end{array},$$

$$
g(\mathbf{3},\mathbf{0}) = \begin{array}{c} \\ 4 \\ 5 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & 4 & 5 \end{array} \\
\begin{pmatrix}
-\frac{r^2z^2}{1+rz} & \frac{r^2z^2}{1+rz} & 0 & 0 \\
r^2z^2 + \frac{r^3z^3}{1+rz} & r^2z^2 + \frac{r^3z^3}{1+rz} & 0 & 0 \\
r^2z^2 + \frac{r^3z^3}{1+rz} & r^2z^2 + \frac{r^3z^3}{1+rz} & 0 & 0 \\
\frac{r^2z^2}{1+rz} & -\frac{r^2z^2}{1+rz} & 0 & 0
\end{pmatrix} \end{array}.
$$

Therefore

$$
g^{(1)}(\mathbf{0},\mathbf{0}) = g(\mathbf{0},\mathbf{0}) \tag{3.11}
$$

$$
= \begin{array}{c|cccc}
 & 0 & 1 & 4 & 5 \\
\hline
0 & r^3z^3 + r^2z^2 + \frac{3r^4z^4}{1+rz} & -r^3z^3 - r^2z^2 - \frac{3r^4z^4}{1+rz} & r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz} & r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz} \\
1 & -r^3z^3 - r^2z^2 - \frac{3r^4z^4}{1+rz} & r^3z^3 + r^2z^2 + \frac{3r^4z^4}{1+rz} & r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz} & r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz} \\
4 & r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz} & r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz} & r^3z^3 + r^2z^2 + \frac{3r^4z^4}{1+rz} & -r^3z^3 - r^2z^2 - \frac{3r^4z^4}{1+rz} \\
5 & r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz} & r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz} & -r^3z^3 - r^2z^2 - \frac{3r^4z^4}{1+rz} & r^3z^3 + r^2z^2 + \frac{3r^4z^4}{1+rz}
\end{array}. \tag{3.12}
$$

Comparing the pattern of $g^{(n)}(\mathbf{0},\mathbf{a_n})$ and $g^{(n)}(\mathbf{0},\mathbf{0})$ with (3.10) and (3.12), we obtain the following theorem.

Theorem 3.1 *Given initial* $u_1^{(0)} = rz, u_2^{(0)} = 0, u_3^{(0)} = 0, u_4^{(0)} = rz, u_5^{(0)} = 0, u_6^{(0)} = 0$*, we have*

$u_1^{(1)}(z) = r^3z^3 + r^2z^2 - \frac{r^4z^4}{1+rz},$

$u_2^{(1)}(z) = 2r^3z^3 + \frac{2r^4z^4}{1+rz},$

$u_3^{(1)}(z) = 0,$

$u_4^{(1)}(z) = -r^3z^3 + r^2z^2 - \frac{3r^4z^4}{1+rz},$

$u_5^{(1)}(z) = r^3z^3 + r^2z^2 + \frac{3r^4z^4}{1+rz},$

and $u_6^{(1)}(z) = r^3z^3 - r^2z^2 - \frac{r^4z^4}{1+rz}.$

By (3.5), we can have the exit probabilities

$$P^{(1)}(\mathbf{0},\mathbf{a_1}) = \frac{1}{2\pi}\int_0^{2\pi} \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0 & 0 & |u_4^{(1)}(e^{i\theta})|^2 & |u_3^{(1)}(e^{i\theta})|^2 \\ 1 & 0 & 0 & |u_4^{(1)}(e^{i\theta})|^2 & |u_3^{(1)}(e^{i\theta})|^2 \\ 4 & 0 & 0 & |u_1^{(1)}(e^{i\theta})|^2 & |u_2^{(1)}(e^{i\theta})|^2 \\ 5 & 0 & 0 & |u_1^{(1)}(e^{i\theta})|^2 & |u_2^{(1)}(e^{i\theta})|^2 \end{array} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0 & 0 & \frac{\sin^2\theta}{5+4\cos\theta} & 0 \\ 1 & 0 & 0 & \frac{\sin^2\theta}{5+4\cos\theta} & 0 \\ 4 & 0 & 0 & \frac{1}{2}\frac{1+\cos\theta}{5+4\cos\theta} & \frac{1}{2}\frac{1+\cos\theta}{5+4\cos\theta} \\ 5 & 0 & 0 & \frac{1}{2}\frac{1+\cos\theta}{5+4\cos\theta} & \frac{1}{2}\frac{1+\cos\theta}{5+4\cos\theta} \end{array} d\theta$$

$$= \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0 & 0 & \frac{1}{8} & 0 \\ 1 & 0 & 0 & \frac{1}{8} & 0 \\ 4 & 0 & 0 & \frac{1}{12} & \frac{1}{12} \\ 5 & 0 & 0 & \frac{1}{12} & \frac{1}{12} \end{array},$$

and

$$P^{(1)}(\mathbf{0},\mathbf{0}) = \frac{1}{2\pi}\int_0^{2\pi} \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} |u_5^{(1)}(e^{i\theta})|^2 & |u_5^{(1)}(e^{i\theta})|^2 & |u_6^{(1)}(e^{i\theta})|^2 & |u_6^{(1)}(e^{i\theta})|^2 \\ |u_5^{(1)}(e^{i\theta})|^2 & |u_5^{(1)}(e^{i\theta})|^2 & |u_6^{(1)}(e^{i\theta})|^2 & |u_6^{(1)}(e^{i\theta})|^2 \\ |u_6^{(1)}(e^{i\theta})|^2 & |u_6^{(1)}(e^{i\theta})|^2 & |u_5^{(1)}(e^{i\theta})|^2 & |u_5^{(1)}(e^{i\theta})|^2 \\ |u_6^{(1)}(e^{i\theta})|^2 & |u_6^{(1)}(e^{i\theta})|^2 & |u_5^{(1)}(e^{i\theta})|^2 & |u_5^{(1)}(e^{i\theta})|^2 \end{pmatrix} \end{array} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} \frac{1}{4} - \frac{\sin^2\theta}{5+4\cos\theta} & \frac{1}{4} - \frac{\sin^2\theta}{5+4\cos\theta} & \frac{1}{4}\frac{1}{5+4\cos\theta} & \frac{1}{4}\frac{1}{5+4\cos\theta} \\ \frac{1}{4} - \frac{\sin^2\theta}{5+4\cos\theta} & \frac{1}{4} - \frac{\sin^2\theta}{5+4\cos\theta} & \frac{1}{4}\frac{1}{5+4\cos\theta} & \frac{1}{4}\frac{1}{5+4\cos\theta} \\ \frac{1}{4}\frac{1}{5+4\cos\theta} & \frac{1}{4}\frac{1}{5+4\cos\theta} & \frac{1}{4} - \frac{\sin^2\theta}{5+4\cos\theta} & \frac{1}{4} - \frac{\sin^2\theta}{5+4\cos\theta} \\ \frac{1}{4}\frac{1}{5+4\cos\theta} & \frac{1}{4}\frac{1}{5+4\cos\theta} & \frac{1}{4} - \frac{\sin^2\theta}{5+4\cos\theta} & \frac{1}{4} - \frac{\sin^2\theta}{5+4\cos\theta} \end{pmatrix} \end{array} d\theta$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{8} & \frac{1}{8} \end{pmatrix} \end{array}.$$

Suppose $\psi_0 = |\mathbf{0}^0\rangle$, then distribution of $\tau^{(1)}$ is as follows.

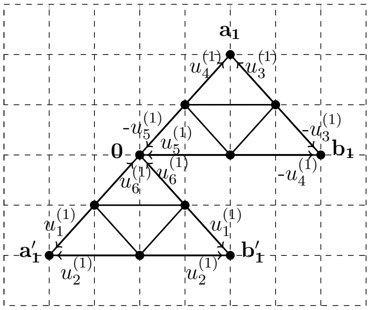

Figure 3.9: amplitude distribution of $\tau^{(1)}$ on $F^{(1)} \cup F^{(1)'}$

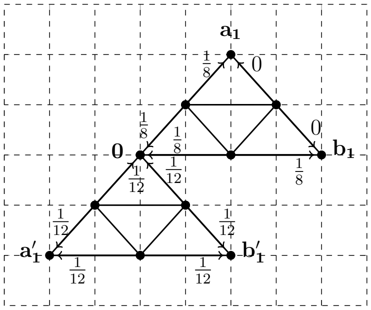

Figure 3.10: probability distribution of $\tau^{(1)}$ on $F^{(1)} \cup F^{(1)'}$

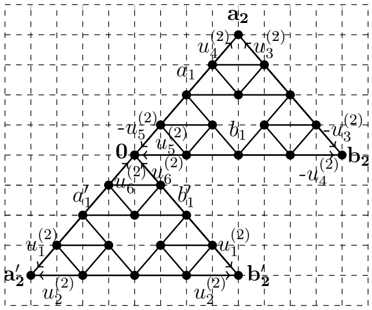

Figure 3.11: amplitude distribution of $\tau^{(2)}$ on $F^{(2)} \cup F^{(2)'}$

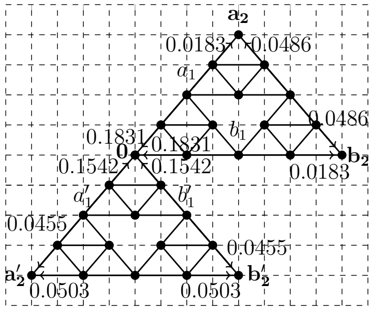

Figure 3.12: probability distribution of $\tau^{(2)}$ on $F^{(2)} \cup F^{(2)'}$

For $u^{(2)}$, just use the same recursive formula by plugging $u^{(1)}$ instead of $u^{(0)}$, we have
$u_1^{(2)}(z) = -(z^2(2z^9+3z^8+6z^7+6z^6+7z^5+8z^4+6z^3+4z^2))/(2(4z^{11}-4z^{10}+15z^9-23z^8+33z^7-50z^6+56z^5-66z^4+52z^3-64z^2+16z-32))$,
$u_2^{(2)}(z) = -(z^2(3z^8+6z^7+10z^6+11z^5+8z^4+4z^3))/(2(4z^{11}-4z^{10}+15z^9-23z^8+33z^7-50z^6+56z^5-66z^4+52z^3-64z^2+16z-32))$,
$u_3^{(2)}(z) = -(z^2(3z^7+2z^6+z^5-2z^4-4z^3))/(2(4z^{10}+4z^9+19z^8+7z^7+36z^6+4z^5+54z^4+12z^3+64z^2+16z+32))$,
$u_4^{(2)}(z) = -(z^2(2z^8+z^7+4z^6+z^5-2z^4-2z^3-4z^2))/(2(4z^{10}+4z^9+19z^8+7z^7+36z^6+4z^5+54z^4+12z^3+64z^2+16z+32))$,
$u_5^{(2)}(z) = (8z^{12}+4z^{11}+28z^{10}+z^9+45z^8+2z^7+60z^6+20z^5+52z^4+16z^3+16z^2)/(2(4z^{10}+4z^9+19z^8+7z^7+36z^6+4z^5+54z^4+12z^3+64z^2+16z+32))$,
and $u_6^{(2)}(z) = (-8z^{13}+4z^{12}-24z^{11}+29z^{10}-39z^9+61z^8-58z^7+64z^6-52z^5+44z^4-16z^3+16z^2)/(2(4z^{11}-4z^{10}+15z^9-23z^8+33z^7-50z^6+56z^5-66z^4+52z^3-64z^2+16z-32))$.

The corresponding exit probabilities are:

$\frac{1}{2\pi}\int_0^{2\pi} |u_1^{(1)}(e^{i\theta})|^2 d\theta = 0.0455$,

$\frac{1}{2\pi}\int_0^{2\pi} |u_2^{(1)}(e^{i\theta})|^2 d\theta = 0.0503$,

$\frac{1}{2\pi}\int_0^{2\pi}|u_3^{(1)}(e^{i\theta})|^2 d\theta = 0.0486,$

$\frac{1}{2\pi}\int_0^{2\pi}|u_4^{(1)}(e^{i\theta})|^2 d\theta = 0.0183,$

$\frac{1}{2\pi}\int_0^{2\pi}|u_5^{(1)}(e^{i\theta})|^2 d\theta = 0.1831,$

$\frac{1}{2\pi}\int_0^{2\pi}|u_6^{(1)}(e^{i\theta})|^2 d\theta = 0.1542.$

In general, we can find a function $T : \mathbb{C}^6 \to \mathbb{C}^6$,

$$(u_1^{(n+1)}, u_2^{(n+1)}, u_3^{(n+1)}, u_4^{(n+1)}, u_5^{(n+1)}, u_6^{(n+1)}) = T(u_1^{(n)}, u_2^{(n)}, u_3^{(n)}, u_4^{(n)}, u_5^{(n)}, u_6^{(n)}).$$

Equation (3.6) can be generalized on $F^{(n+1)}$ as

$$\begin{aligned} u_3^{(n+1)} &= [g^{(n)}(\mathbf{0},\mathbf{1})g(\mathbf{1},\mathbf{5})]_5^0 + [g^{(n)}(\mathbf{0},\mathbf{2})g(\mathbf{2},\mathbf{5})]_5^0 \\ &= u_4^{(n)}g(\mathbf{1},\mathbf{5})_5^4 + u_3^{(n)}g(\mathbf{1},\mathbf{5})_5^5 - u_4^{(n)}g(\mathbf{2},\mathbf{5})_5^3 - u_3^{(n)}g(\mathbf{2},\mathbf{5})_5^2 \\ &= u_4^{(n)}(g(\mathbf{1},\mathbf{5})_5^4 - g(\mathbf{2},\mathbf{5})_5^3) + u_3^{(n)}(g(\mathbf{1},\mathbf{5})_5^5 - g(\mathbf{2},\mathbf{5})_5^2) \end{aligned}$$

$$\begin{aligned} u_4^{(n+1)} &= [g^{(n)}(\mathbf{0},\mathbf{1})g(\mathbf{1},\mathbf{5})]_4^0 + [g^{(n)}(\mathbf{0},\mathbf{2})g(\mathbf{2},\mathbf{5})]_4^0 \\ &= u_4^{(n)}g(\mathbf{1},\mathbf{5})_4^4 + u_3^{(n)}g(\mathbf{1},\mathbf{5})_4^5 - u_4^{(n)}g(\mathbf{2},\mathbf{5})_4^3 - u_3^{(n)}g(\mathbf{2},\mathbf{5})_4^2 \\ &= u_4^{(n)}(g(\mathbf{1},\mathbf{5})_4^4 - g(\mathbf{2},\mathbf{5})_4^3) + u_3^{(n)}(g(\mathbf{1},\mathbf{5})_4^5 - g(\mathbf{2},\mathbf{5})_4^2) \end{aligned}$$

$$\begin{aligned} u_5^{(n+1)} &= g^{(n)}(\mathbf{0},\mathbf{0})_0^0 + [g^{(n)}(\mathbf{0},\mathbf{1})g(\mathbf{1},\mathbf{0})]_0^0 + [g^{(n)}(\mathbf{0},\mathbf{2})g(\mathbf{2},\mathbf{0})]_0^0 \\ &= u_5^{(n)} + u_4^{(n)}g(\mathbf{1},\mathbf{0})_0^4 + u_3^{(n)}g(\mathbf{1},\mathbf{0})_0^5 - u_4^{(n)}g(\mathbf{2},\mathbf{0})_0^3 - u_3^{(n)}g(\mathbf{2},\mathbf{0})_0^2 \\ &= u_5^{(n)} + u_4^{(n)}(g(\mathbf{1},\mathbf{0})_0^4 - g(\mathbf{2},\mathbf{0})_0^3) + u_3^{(n)}(g(\mathbf{1},\mathbf{0})_0^5 - g(\mathbf{2},\mathbf{0})_0^2) \end{aligned}$$

Similarly, on $F^{(n+1)'}$ we have

$$\begin{aligned} u_1^{(n+1)} &= [g^{(n)}(\mathbf{0},\mathbf{1'})g(\mathbf{1'},\mathbf{5'})]_1^0 + [g^{(n)}(\mathbf{0},\mathbf{2'})g(\mathbf{2'},\mathbf{5'})]_1^0 \\ &= u_1^{(n)}g(\mathbf{1'},\mathbf{5'})_1^1 + u_2^{(n)}g(\mathbf{1'},\mathbf{5'})_1^0 + u_1^{(n)}g(\mathbf{2'},\mathbf{5'})_1^2 + u_2^{(n)}g(\mathbf{2'},\mathbf{5'})_1^3 \\ &= u_1^{(n)}(g(\mathbf{1'},\mathbf{5'})_1^0 + g(\mathbf{2'},\mathbf{5'})_1^2) + u_2^{(n)}(g(\mathbf{1'},\mathbf{5'})_1^0 + g(\mathbf{2'},\mathbf{5'})_1^3) \end{aligned}$$

$$\begin{aligned} u_2^{(n+1)} &= [g^{(n)}(\mathbf{0},\mathbf{1}')g(\mathbf{1}',\mathbf{5}')]_0^0 + [g^{(n)}(\mathbf{0},\mathbf{2}')g(\mathbf{2}',\mathbf{5}')]_0^0 \\ &= u_1^{(n)}g(\mathbf{1}',\mathbf{5}')_0^1 + u_2^{(n)}g(\mathbf{1}',\mathbf{5}')_0^0 + u_1^{(n)}g(\mathbf{2}',\mathbf{5}')_0^2 + u_2^{(n)}g(\mathbf{2}',\mathbf{5}')_0^3 \\ &= u_1^{(n)}(g(\mathbf{1}',\mathbf{5}')_0^0 + g(\mathbf{2}',\mathbf{5}')_0^2) + u_2^{(n)}(g(\mathbf{1}',\mathbf{5}')_0^0 + g(\mathbf{2}',\mathbf{5}')_0^3) \end{aligned}$$

$$\begin{aligned} u_6^{(n+1)} &= g^{(n)}(\mathbf{0},\mathbf{0})_4^0 + [g^{(n)}(\mathbf{0},\mathbf{1}')g(\mathbf{1}',\mathbf{0})]_4^0 + [g^{(n)}(\mathbf{0},\mathbf{2}')g(\mathbf{2}',\mathbf{0})]_4^0 \\ &= u_6^{(n)} + u_1^{(n)}g(\mathbf{1}',\mathbf{0})_4^1 + u_2^{(n)}g(\mathbf{1}',\mathbf{0})_4^0 + u_1^{(n)}g(\mathbf{2}',\mathbf{0})_4^2 + u_2^{(n)}g(\mathbf{2}',\mathbf{0})_4^3 \\ &= u_6^{(n)} + u_1^{(n)}(g(\mathbf{1}',\mathbf{0})_4^0 + g(\mathbf{2}',\mathbf{0})_4^2) + u_2^{(n)}(g(\mathbf{1}',\mathbf{0})_4^0 + g(\mathbf{2}',\mathbf{0})_4^3) \end{aligned}$$

Substituting z by $e^{i\theta}$, we have

$$|u_1^{(n)}(e^{i\theta})|^2 + |u_2^{(n)}(e^{i\theta})|^2 + |u_6^{(n)}(e^{i\theta})|^2 = \frac{1}{4}. \tag{3.13}$$

$$|u_3^{(n)}(e^{i\theta})|^2 + |u_4^{(n)}(e^{i\theta})|^2 + |u_5^{(n)}(e^{i\theta})|^2 = \frac{1}{4}. \tag{3.14}$$

Also we have the limiting behavior as follows.

$$\lim_{n\to\infty} P^{(n)}(\mathbf{0},\mathbf{a_n'})_1^0 = \frac{1}{2\pi}\int_0^{2\pi} |u_1^{(n)}(e^{i\theta})|^2 d\theta = 0.$$

$$\lim_{n\to\infty} P^{(n)}(\mathbf{0},\mathbf{a_n'})_0^0 = \frac{1}{2\pi}\int_0^{2\pi} |u_2^{(n)}(e^{i\theta})|^2 d\theta = 0.$$

$$\lim_{n\to\infty} P^{(n)}(\mathbf{0},\mathbf{a_n})_4^0 = \frac{1}{2\pi}\int_0^{2\pi} |u_3^{(n)}(e^{i\theta})|^2 d\theta = 0.$$

$$\lim_{n\to\infty} P^{(n)}(\mathbf{0},\mathbf{a_n})_5^0 = \frac{1}{2\pi}\int_0^{2\pi} |u_4^{(n)}(e^{i\theta})|^2 d\theta = 0.$$

$$\lim_{n\to\infty} P^{(n)}(\mathbf{0},\mathbf{0})_0^0 = \frac{1}{2\pi}\int_0^{2\pi} |u_5^{(n)}(e^{i\theta})|^2 d\theta = \frac{1}{4}.$$

$$\lim_{n\to\infty} P^{(n)}(\mathbf{0},\mathbf{0})_4^0 = \frac{1}{2\pi}\int_0^{2\pi} |u_6^{(n)}(e^{i\theta})|^2 d\theta = \frac{1}{4}.$$

In summary, these results show the following theorem.

Theorem 3.2 *The quantum random walk on Sierpinshi gasket is recurrent.*

3.3 hitting time

Let $\partial(F^{(n)} \cup F^{(n)'}) := \{\mathbf{a_n}, \mathbf{b_n}, \mathbf{a'_n}, \mathbf{b'_n}\}$ and $T^{(n)} = \inf\{t \geq 1; w_t \in \partial(F^{(n)} \cup F^{(n)'})\}$ be the first-passage time taken to exit $F^{(n)} \cup F^{(n)'}$ at the four vertices.

The corresponding amplitude Green function for quantum random walk is defined by

$$g_1^{(n)}(z)(\mathbf{x}, \mathbf{y})_j^i = \sum_{t=1}^{\infty} z^t \Psi(w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j, T^{(n)} = t). \tag{3.15}$$

The probability that a quantum random walk starts with $\mathbf{0}^i$ and exits from $\mathbf{y}^j$ is given by

$$P_1^{(n)}(\mathbf{0}, \mathbf{y})_j^i = \sum_{t=1}^{\infty} |\Psi(w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j, T^{(n)} = t)|^2. \tag{3.16}$$

In this section, we shall use the result of amplitude Green functions $u_i^{(n)}$ in previous section to find the expection of first-passage time $E(T^{(n)})$ given $w_0 = \mathbf{0}^i$.

$$\begin{aligned} E(T^{(n)}) &= \sum_{\mathbf{y} \in \partial(F^{(n)} \cup F^{(n)'})} \sum_{\mathbf{e}_j \in out(\mathbf{y})} \sum_{t=1}^{\infty} t P_1^{(n)}(\mathbf{0}, \mathbf{y}, T^{(n)} = t)_j^i \\ &= \sum_{\mathbf{y} \in \partial(F^{(n)} \cup F^{(n)'})} \sum_{\mathbf{e}_j \in out(\mathbf{y})} \sum_{t=1}^{\infty} t |\Psi(w_0 = \mathbf{0}^i, w_t = \mathbf{y}^j, T^{(n)} = t)|^2 \\ &= \sum_{\mathbf{y} \in \partial(F^{(n)} \cup F^{(n)'})} \sum_{\mathbf{e}_j \in out(\mathbf{y})} \frac{1}{2\pi} \int_0^{2\pi} \sum_{t=1}^{\infty} e^{i\theta t} t \Psi(w_0 = \mathbf{0}^i, w_t = \mathbf{y}^j, T^{(n)} = t) \sum_{t=1}^{\infty} e^{-i\theta t} \Psi(w_0 = \mathbf{0}^i, w_t = \mathbf{y}^j, T^{(n)} = t) d\theta \\ &= \sum_{\mathbf{y} \in \partial(F^{(n)} \cup F^{(n)'})} \sum_{\mathbf{e}_j \in out(\mathbf{y})} \frac{1}{2\pi} \int_0^{2\pi} (\partial_s g_1^{(n)}(e^{s+i\theta})(\mathbf{0}, \mathbf{y})_j^i |_{s=0}) g_1^{(n)}(e^{-i\theta})(\mathbf{0}, \mathbf{y})_j^i d\theta. \end{aligned}$$

So we have the following formula for expection of hitting time

$$E(T^{(n)}) = \sum_{y \in \partial(F^{(n)} \cup F^{(n)'})} \sum_{\mathbf{e}_j \in out(\mathbf{y})} \frac{1}{2\pi} \int_0^{2\pi} (\partial_s g_1^{(n)}(e^{s+i\theta})(\mathbf{0}, \mathbf{y})_j^i |_{s=0}) g_1^{(n)}(e^{-i\theta})(\mathbf{0}, \mathbf{y})_j^i d\theta. \tag{3.17}$$

Given $u_i^{(n)}$ for $i = 1, ..., 6$, we will first compute the matrix $g_1^{(n+1)}(z)(\mathbf{0}, \mathbf{5})$. Then the Green function from $\mathbf{0}$ to other vertices can be obtained by symmetry.

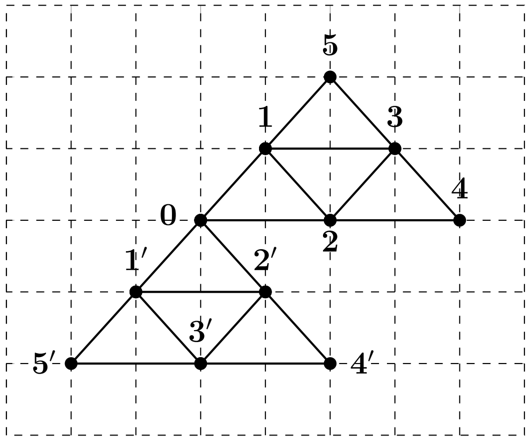

Figure 3.13: relable of vertices on $F^{(1)} \cup F^{(1)'}$

Consider the relabled figure agian as Figure 3.13, we have

$$\begin{bmatrix} g_1^{(n+1)}(z)(\mathbf{0},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{1},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{2},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{3},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{1'},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{2'},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{3'},\mathbf{5}) \end{bmatrix} = \frac{1}{I_{28} - [g_1^{(n)}(z)]|_{0,1,2,3,1',2',3'}} \begin{bmatrix} g_1^{(n)}(z)(\mathbf{0},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{1},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{2},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{3},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{1'},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{2'},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{3'},\mathbf{5}) \end{bmatrix}, \tag{3.18}$$

where $[g_1^{(n)}(z)]|_{0,1,2,3,1',2',3'}$ is the transition matrix in terms of $u_i^{(n)}$, for $i = 1, ..., 6$.

For $n = 0$, by solving the above equation, we obtain

$$g_1^{(1)}(z)(\mathbf{0},\mathbf{5}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & \frac{z^2(z^2+z-2)}{2(z^3-z^2-4)} & \frac{z^5}{2(z^3-z^2-4)} \\ 0 & 0 & \frac{z^2(z^2-z+2)}{2(z^3-z^2-4)} & \frac{z^5}{2(z^3-z^2-4)} \\ 0 & 0 & \frac{-z^2(z+2)}{2(z^3-z^2-4)} & \frac{-z^3(z+2)}{2(z^3-z^2-4)} \\ 0 & 0 & \frac{-z^2(z+2)}{2(z^3-z^2-4)} & \frac{-z^3(z+2)}{2(z^3-z^2-4)} \end{pmatrix} \end{array}. \tag{3.19}$$

If we start $w_0 = \mathbf{0}^0$, then each exit Green amplitude function can be obtained from above matirx (3.17). For example $g_1^{(1)}(z)(\mathbf{0},\mathbf{4})^0_3 = g_1^{(1)}(z)(\mathbf{0},\mathbf{5})^1_4 =$

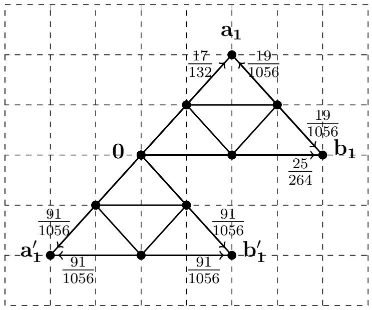

Figure 3.14: distribution of $T^{(1)}$ on $F^{(1)} \cup F^{(1)'}$ with $w_0 = \mathbf{0}^0$

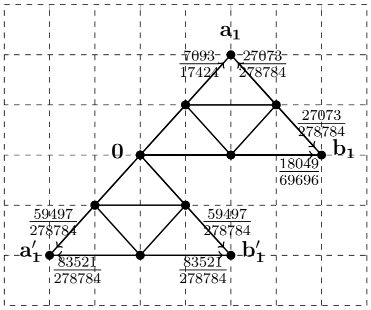

Figure 3.15: distribution of $E(T^{(1)})$ on $F^{(1)} \cup F^{(1)'}$ with $w_0 = \mathbf{0}^0$

$\frac{z^2(z^2-z+2)}{2(z^3-z^2-4)}$ and $g_1^{(1)}(z)(\mathbf{0}, \mathbf{5'})_1^0 = g_1^{(1)}(z)(\mathbf{0}, \mathbf{4})_4^4 = \frac{-z^2(z+2)}{2(z^3-z^2-4)}$.

So by Parseval's theorem again,

$$P_1^{(n)}(\mathbf{x}, \mathbf{y})_j^i = \frac{1}{2\pi}\int_0^{2\pi} |g_1^{(n)}(e^{i\theta})(\mathbf{x}, \mathbf{y})_j^i|^2 d\theta,$$

we have

$$P_1^{(1)}(z)(\mathbf{0}, \mathbf{5}) = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & \frac{17}{132} & \frac{19}{1056} \\ 0 & 0 & \frac{25}{264} & \frac{19}{1056} \\ 0 & 0 & \frac{91}{1056} & \frac{91}{1056} \\ 0 & 0 & \frac{91}{1056} & \frac{91}{1056} \end{pmatrix} \end{array}. \tag{3.20}$$

Using (3.17), we have $E(T^{(1)}) = \frac{2173}{1152}$.

One notices that unlike $\tau^{(1)}$, if we sum the exit probablities for $T^{(1)}$, we have $P_1(T^{(1)} < \infty) = \frac{319}{528} < 1$. So we obtain

$$E(T^{(1)}|T^{(1)} < \infty) = \frac{2173}{696} = 3.121.$$

3.4 Uniform Coin Operator

When the coin operator is uniform, then U is no longer a unitary operator and in this case, the quantum random walk becomes the classical random walk. The amplitude function becomes the transition probability. The result coincides with the classical one obtained in Section 2.1. Let

$$G = r \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

where $r = \dfrac{1}{4}$. Here $\Theta^{(n)}(z)$ is the same as in Section 3.2. The amplitude Green function before the first rotation. So in this case,

$$g^{(n)}(\mathbf{0}, \mathbf{a_n}) = G \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0 & 0 & \theta_3^{(n)} & \theta_2^{(n)} \\ 1 & 0 & 0 & \theta_1^{(n)} & \theta_4^{(n)} \\ 4 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{array} = \frac{1}{4} \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0 & 0 & \theta_1^{(n)} + \theta_3^{(n)} & \theta_4^{(n)} + \theta_2^{(n)} \\ 1 & 0 & 0 & \theta_3^{(n)} + \theta_1^{(n)} & \theta_2^{(n)} + \theta_4^{(n)} \\ 4 & 0 & 0 & \theta_1^{(n)} + \theta_3^{(n)} & \theta_2^{(n)} + \theta_4^{(n)} \\ 5 & 0 & 0 & \theta_1^{(n)} + \theta_3^{(n)} & \theta_2^{(n)} + \theta_4^{(n)} \end{array},$$

and

$$g^{(n)}(\mathbf{0}, \mathbf{0}) = G \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & \theta_5^{(n)} & \theta_6^{(n)} & 0 & 0 \\ 1 & \theta_6^{(n)} & \theta_5^{(n)} & 0 & 0 \\ 4 & 0 & 0 & \theta_5^{(n)} & \theta_6^{(n)} \\ 5 & 0 & 0 & \theta_6^{(n)} & \theta_5^{(n)} \end{array}$$

$$= \frac{1}{4} \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & \theta_6^{(n)} + \theta_5^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} \\ 1 & \theta_5^{(n)} + \theta_6^{(n)} & \theta_6^{(n)} + \theta_5^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} \\ 4 & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_6^{(n)} + \theta_5^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} \\ 5 & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_5^{(n)} + \theta_6^{(n)} & \theta_6^{(n)} + \theta_5^{(n)} \end{array}.$$

By replacing $u_1^{(n)} = \frac{1}{4}(\theta_1^{(n)} + \theta_3^{(n)})$, $u_2^{(n)} = \frac{1}{4}(\theta_2^{(n)} + \theta_4^{(n)})$, $u_3^{(n)} = \frac{1}{4}(\theta_5^{(n)} + \theta_6^{(n)})$, so each iteration we only have three variables compared with six in the quantuam case.

$$[g^{(0)}]|_{1,2,3} = \begin{array}{c|cccccccccccc|} & \mathbf{1}^0 & \mathbf{1}^1 & \mathbf{1}^4 & \mathbf{1}^5 & \mathbf{2}^2 & \mathbf{2}^3 & \mathbf{2}^0 & \mathbf{2}^1 & \mathbf{3}^4 & \mathbf{3}^5 & \mathbf{3}^2 & \mathbf{3}^3 \\ \mathbf{1}^0 & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} \\ \mathbf{1}^1 & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} \\ \mathbf{1}^4 & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} \\ \mathbf{1}^5 & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} \\ \mathbf{2}^2 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 \\ \mathbf{2}^3 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 \\ \mathbf{2}^0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 \\ \mathbf{2}^1 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_1^{(0)} & u_2^{(0)} & 0 & 0 \\ \mathbf{3}^4 & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} \\ \mathbf{3}^5 & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} \\ \mathbf{3}^2 & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} \\ \mathbf{3}^3 & u_1^{(0)} & u_2^{(0)} & 0 & 0 & 0 & 0 & u_2^{(0)} & u_1^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} & u_3^{(0)} \end{array} . \tag{3.21}$$

It follows from the same procedure as in Section 3.2, for $F^{(0)}$,

$$g^{(0)}(\mathbf{0}, \mathbf{a_0}) = \begin{array}{c|cccc|} & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & rz & 0 \\ 1 & 0 & 0 & rz & 0 \\ 4 & 0 & 0 & rz & 0 \\ 5 & 0 & 0 & rz & 0 \end{array},$$

$$g^{(0)}(\mathbf{0}, \mathbf{0}) = \begin{array}{c|cccc|} & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{array},$$

so $u_1^{(0)} = rz$, $u_2^{(0)} = 0$, $u_3^{(0)} = 0$.

$$g^{(1)}(\mathbf{0}, \mathbf{a_1}) = g^{(0)}(\mathbf{0}, \mathbf{1})g(\mathbf{1}, \mathbf{5}) + g^{(0)}(\mathbf{0}, \mathbf{2})g(\mathbf{2}, \mathbf{5}), \tag{3.22}$$

where

$$\begin{bmatrix} g(\mathbf{1}, \mathbf{5}) \\ g(\mathbf{2}, \mathbf{5}) \\ g(\mathbf{3}, \mathbf{5}) \end{bmatrix} = \frac{1}{I_{12} - [g^{(0)}]|_{1,2,3}} \begin{bmatrix} g^{(0)}(\mathbf{1}, \mathbf{5}) \\ g^{(0)}(\mathbf{2}, \mathbf{5}) \\ g^{(0)}(\mathbf{3}, \mathbf{5}) \end{bmatrix}, \tag{3.23}$$

and

$$g^{(0)}(\mathbf{0},\mathbf{1}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \end{pmatrix} \end{array}, \tag{3.24}$$

$$g^{(0)}(\mathbf{0},\mathbf{2}) = \begin{array}{cc} & \begin{array}{cccc} 2 & 3 & 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \end{pmatrix} \end{array}, \tag{3.25}$$

$$g^{(0)}(\mathbf{1},\mathbf{5}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \\ 0 & 0 & u_1^{(0)} & u_2^{(0)} \end{pmatrix} \end{array}, \tag{3.26}$$

$$g^{(0)}(\mathbf{2},\mathbf{5}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$g^{(0)}(\mathbf{3},\mathbf{5}) = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{array}{c} 4 \\ 5 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & u_2^{(0)} & u_1^{(0)} \\ 0 & 0 & u_2^{(0)} & u_1^{(0)} \\ 0 & 0 & u_2^{(0)} & u_1^{(0)} \\ 0 & 0 & u_2^{(0)} & u_1^{(0)} \end{pmatrix} \end{array}. \tag{3.27}$$

Plug in the values of $u_1^{(0)}, u_2^{(0)}, u_3^{(0)}$, we have

$$g^{(1)}(\mathbf{0}, \mathbf{a_1}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & -\frac{z^2}{2(z^2+2z-8)} & -\frac{z^3}{4(z^2+2z-8)} \\ 0 & 0 & -\frac{z^2}{2(z^2+2z-8)} & -\frac{z^3}{4(z^2+2z-8)} \\ 0 & 0 & -\frac{z^2}{2(z^2+2z-8)} & -\frac{z^3}{4(z^2+2z-8)} \\ 0 & 0 & -\frac{z^2}{2(z^2+2z-8)} & -\frac{z^3}{4(z^2+2z-8)} \end{pmatrix} \end{array}.$$

Similarly, for $i = 0, 1, 4, 5,\ j = 0, 1,$

$$g^{(1)}(\mathbf{0}, \mathbf{0})^i_j = g^{(0)}(\mathbf{0}, \mathbf{0})^i_j + [g^{(0)}(\mathbf{0}, \mathbf{1})g(\mathbf{1}, \mathbf{0})]^i_j + [g^{(0)}(\mathbf{0}, \mathbf{2})g(\mathbf{2}, \mathbf{0})]^i_j, \tag{3.28}$$

where

$$\begin{bmatrix} g(\mathbf{1}, \mathbf{0}) \\ g(\mathbf{2}, \mathbf{0}) \\ g(\mathbf{3}, \mathbf{0}) \end{bmatrix} = \frac{1}{I_{12} - [g^{(0)}]|_{1,2,3}} \begin{bmatrix} g^{(0)}(\mathbf{1}, \mathbf{0}) \\ g^{(0)}(\mathbf{2}, \mathbf{0}) \\ g^{(0)}(\mathbf{3}, \mathbf{0}) \end{bmatrix}. \tag{3.29}$$

For $i = 0, 1, 4, 5,\ j = 4, 5,$

$$g^{(1)}(\mathbf{0}, \mathbf{0})^i_j = g(0)(\mathbf{0}, \mathbf{0})^i_j + [g^{(0)}(\mathbf{5}, \mathbf{1})g(\mathbf{1}, \mathbf{5})]^i_j + [g^{(0)}(\mathbf{5}, \mathbf{3})g(\mathbf{3}, \mathbf{5})]^i_j, \tag{3.30}$$

where

$$g^{(0)}(\mathbf{1}, \mathbf{0}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \\ u_2^{(0)} & u_1^{(0)} & 0 & 0 \end{pmatrix} \end{array}, \tag{3.31}$$

$$g^{(0)}(\mathbf{2}, \mathbf{0}) = \begin{array}{c} \\ 2 \\ 3 \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} u_1^{(0)} & u_2^{(0)} & 0 & 0 \\ u_1^{(0)} & u_2^{(0)} & 0 & 0 \\ u_1^{(0)} & u_2^{(0)} & 0 & 0 \\ u_1^{(0)} & u_2^{(0)} & 0 & 0 \end{pmatrix} \end{array}, \tag{3.32}$$

$$g^{(0)}(\mathbf{3},\mathbf{0}) = \begin{array}{c@{\,}c} & \begin{matrix}0&1&4&5\end{matrix}\\ \begin{matrix}4\\5\\2\\3\end{matrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\end{array},$$

$$g^{(0)}(\mathbf{0},\mathbf{0}) = \begin{array}{c@{\,}c} & \begin{matrix}0&1&4&5\end{matrix}\\ \begin{matrix}0\\1\\4\\5\end{matrix} & \begin{pmatrix}u_3^{(0)}&u_3^{(0)}&u_3^{(0)}&u_3^{(0)}\\u_3^{(0)}&u_3^{(0)}&u_3^{(0)}&u_3^{(0)}\\u_3^{(0)}&u_3^{(0)}&u_3^{(0)}&u_3^{(0)}\\u_3^{(0)}&u_3^{(0)}&u_3^{(0)}&u_3^{(0)}\end{pmatrix}\end{array}, \tag{3.33}$$

$$g^{(0)}(\mathbf{5},\mathbf{1}) = \begin{array}{c@{\,}c} & \begin{matrix}0&1&4&5\end{matrix}\\ \begin{matrix}0\\1\\2\\3\end{matrix} & \begin{pmatrix}u_2^{(0)}&u_1^{(0)}&0&0\\u_2^{(0)}&u_1^{(0)}&0&0\\u_2^{(0)}&u_1^{(0)}&0&0\\u_2^{(0)}&u_1^{(0)}&0&0\end{pmatrix}\end{array}, \tag{3.34}$$

$$g^{(0)}(\mathbf{5},\mathbf{3}) = \begin{array}{c@{\,}c} & \begin{matrix}4&5&2&3\end{matrix}\\ \begin{matrix}2\\3\\4\\5\end{matrix} & \begin{pmatrix}0&0&u_1^{(0)}&u_2^{(0)}\\0&0&u_1^{(0)}&u_2^{(0)}\\0&0&u_1^{(0)}&u_2^{(0)}\\0&0&u_1^{(0)}&u_2^{(0)}\end{pmatrix}\end{array}. \tag{3.35}$$

Plug in the values of $u_1^{(0)}, u_2^{(0)}, u_3^{(0)}$, we have

$$g^{(1)}(\mathbf{0},\mathbf{0}) = \begin{array}{c@{\,}c} & \begin{matrix}0&1&4&5\end{matrix}\\ \begin{matrix}0\\1\\4\\5\end{matrix} & \begin{pmatrix}-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}\\-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}\\-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}\\-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}&-\frac{z^2}{2(z^2+2z-8)}\end{pmatrix}\end{array}.$$

So $u_1^{(1)} = -\frac{z^2}{2(z^2+2z-8)}, u_2^{(0)} = -\frac{z^3}{4(z^2+2z-8)}, u_3^{(0)} = -\frac{z^2}{2(z^2+2z-8)}$. Let $z = 1$, we have the transition probability

$$\Psi^{(1)}(\mathbf{0},\mathbf{a_1}) = \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0 & 0 & 0.1 & 0.05 \\ 1 & 0 & 0 & 0.1 & 0.05 \\ 4 & 0 & 0 & 0.1 & 0.05 \\ 5 & 0 & 0 & 0.1 & 0.05 \end{array}, \quad \Psi^{(1)}(\mathbf{0},\mathbf{0}) = \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0.1 & 0.1 & 0.1 & 0.1 \\ 1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 4 & 0.1 & 0.1 & 0.1 & 0.1 \\ 5 & 0.1 & 0.1 & 0.1 & 0.1 \end{array}.$$

We iterate the above calculation by pluging $u^{(1)}$ instead of $u^{(0)}$ into (3.21)-(3.35) to obtain $u^{(2)}$.

$$\Psi^{(2)}(\mathbf{0},\mathbf{a_1}) = \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0 & 0 & 0.05 & 0.04 \\ 1 & 0 & 0 & 0.05 & 0.04 \\ 4 & 0 & 0 & 0.05 & 0.04 \\ 5 & 0 & 0 & 0.05 & 0.04 \end{array}, \quad \Psi^{(2)}(\mathbf{0},\mathbf{0}) = \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0.16 & 0.16 & 0.16 & 0.16 \\ 1 & 0.16 & 0.16 & 0.16 & 0.16 \\ 4 & 0.16 & 0.16 & 0.16 & 0.16 \\ 5 & 0.16 & 0.16 & 0.16 & 0.16 \end{array}.$$

$$\Psi^{(3)}(\mathbf{0},\mathbf{a_1}) = \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0 & 0 & 0.028 & 0.026 \\ 1 & 0 & 0 & 0.028 & 0.026 \\ 4 & 0 & 0 & 0.028 & 0.026 \\ 5 & 0 & 0 & 0.028 & 0.026 \end{array}, \quad \Psi^{(3)}(\mathbf{0},\mathbf{0}) = \begin{array}{c|cccc} & 0 & 1 & 4 & 5 \\ \hline 0 & 0.196 & 0.196 & 0.196 & 0.196 \\ 1 & 0.196 & 0.196 & 0.196 & 0.196 \\ 4 & 0.196 & 0.196 & 0.196 & 0.196 \\ 5 & 0.196 & 0.196 & 0.196 & 0.196 \end{array}.$$

Suming each row, we get the probability the same as the classical case.

In general, we have a map $T : \mathbb{R}^3 \to \mathbb{R}^3$,

$$(u_1^{(n+1)}, u_2^{(n+1)}, u_3^{(n+1)}) = T(u_1^{(n)}, u_2^{(n)}, u_3^{(n)}).$$

where

$$u_1^{(n+1)} = T_1(u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$
$$= \frac{-(u_1^{(n)}+u_2^{(n)})(2(u_2^{(n)})^2+2u_1^{(n)}u_2^{(n)}+u_1^{(n)}-4u_1^{(n)}u_3^{(n)})}{(2(u_1^{(n)})^2+4u_1^{(n)}u_2^{(n)}-4u_1^{(n)}u_3^{(n)}+u_1^{(n)}+2(u_1^{(n)})^2-4u_2^{(n)}u_3^{(n)}+u_2^{(n)}-16(u_3^{(n)})^2+8u_3^{(n)}-1)}.$$
$$u_2^{(n+1)} = T_2(u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$
$$= \frac{-(u_1^{(n)}+u_2^{(n)})(2(u_1^{(n)})^2+2u_1^{(n)}u_2^{(n)}+u_2^{(n)}-4u_2^{(n)}u_3^{(n)})}{(2(u_1^{(n)})^2+4u_1^{(n)}u_2^{(n)}-4u_1^{(n)}u_3^{(n)}+u_1^{(n)}+2(u_1^{(n)})^2-4u_2^{(n)}u_3^{(n)}+u_2^{(n)}-16(u_3^{(n)})^2+8u_3^{(n)}-1)}.$$
$$u_3^{(n+1)} = T_3(u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$$
$$= 6(u_1^{(n)})^2u_3^{(n)}-(u_1^{(n)})^2+12u_1^{(n)}u_2^{(n)}u_3^{(n)}-2u_1^{(n)}u_2^{(n)}-4u_1^{(n)}(u_3^{(n)})^2+u_1^{(n)}u_3^{(n)}+6(u_2^{(n)})^2u_3^{(n)}-(u_2^{(n)})^2-4u_2^{(n)}(u_3^{(n)})^2$$
$$+u_2^{(n)}u_3^{(n)}-16(u_3^{(n)})^3+8(u_3^{(n)})^2-u_3^{(n)}/(2(u_1^{(n)})^2+4u_1^{(n)}u_2^{(n)}-4u_1^{(n)}u_3^{(n)}+u_1^{(n)}+2(u_1^{(n)})^2-4u_2^{(n)}u_3^{(n)}+u_2^{(n)}-$$
$$16(u_3^{(n)})^2+8u_3^{(n)}-1).$$

We have thus obtained

Theorem 3.3 *For initial probability distribution* $u_1^{(0)}|_{z=1} = \frac{1}{4}, u_2^{(0)}|_{z=1} = 0, u_3^{(0)}|_{z=1} = 0$*, we have*

$$u_1^{(n)}|_{z=1} + u_2^{(n)}|_{z=1} + u_3^{(n)}|_{z=1} = \frac{1}{4}.$$

Moreover

$$u_1^{(n+1)}|_{z=1} = 0.4u_1^{(n)}|_{z=1} + 0.2u_2^{(n)}|_{z=1}, u_2^{(n+1)}|_{z=1} = 0.2u_1^{(n)}|_{z=1} + 0.4u_2^{(n)}|_{z=1},$$

$$u_3^{(n+1)}|_{z=1} = 0.1 + 0.6u_3^{(n)}|_{z=1}.$$

Corollary 3.1 $\lim_{n\to\infty} u_1^{(n)}|_{z=1} = u_2^{(n)}|_{z=1} = 0$ *and* $\lim_{n\to\infty} u_3^{(n)}|_{z=1} = \frac{1}{4}$.

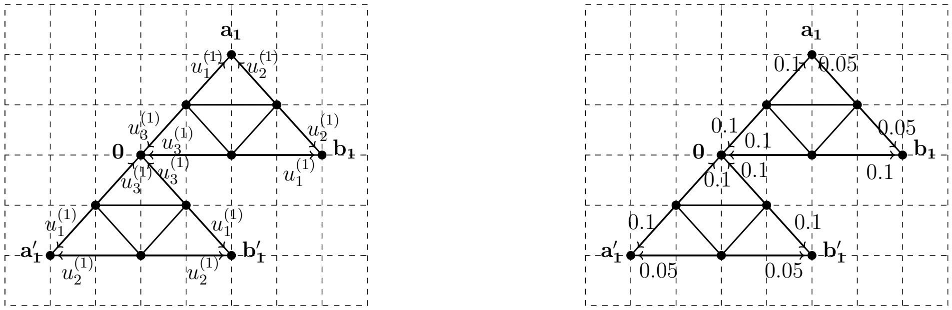

Figure 3.16: Green function and probability distribution of τ on $F^{(1)} \cup F^{(1)'}$

Also for the uniform coin operator, let's retrieve the hitting time as in Section 3.3. Let $T^{(n)} = \inf\{t \geq 1; w_t \in \partial(F^{(n)} \cup F^{(n)'})\}, \partial(F^{(n)} \cup F^{(n)'}) := \{\mathbf{a_n}, \mathbf{b_n}, \mathbf{a'_n}, \mathbf{b'_n}\}$ be the first-passage time taken to exit $F^{(n)} \cup F^{(n)'}$ at the four vertices.

The corresponding amplitude Green function for quantum random walk is defined by

$$g_1^{(n)}(z)(\mathbf{x}, \mathbf{y})_j^i = \sum_{t=1}^{\infty} z^t \Psi(w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j, T = t).$$

The probability that a quantum random walk starts with $\mathbf{0}^i$ and exits from $\mathbf{y}^j$ is given by

$$P_1^{(n)}(\mathbf{0}, \mathbf{y})_j^i = \sum_{t=1}^{\infty} \Psi(w_0 = \mathbf{x}^i, w_t = \mathbf{y}^j, T = t).$$

In this section, we shall use the result of Green functions $u_i^{(n)}$ in previous section to find the expection of first-passage time $E(T^{(n)})$ given $w_0 = \mathbf{0}^i$.

$$\begin{aligned}
E(T^{(n)}) &= \sum_{\mathbf{y}\in\partial(F^{(n)}\cup F^{(n)'})} \sum_{\mathbf{e}_j\in out(\mathbf{y})} \sum_{t=1}^{\infty} tP_1^{(n)}(\mathbf{0},\mathbf{y},T=t)_j^i \\
&= \sum_{\mathbf{y}\in\partial(F^{(n)}\cup F^{(n)'})} \sum_{\mathbf{e}_j\in out(\mathbf{y})} \sum_{t=1}^{\infty} t\Psi(w_0=\mathbf{0}^i, w_t=\mathbf{y}^j, T=t) \\
&= \sum_{\mathbf{y}\in\partial(F^{(n)}\cup F^{(n)'})} \sum_{\mathbf{e}_j\in out(\mathbf{y})} (\partial_z g_1^{(n)}(z)(\mathbf{0},\mathbf{y})_j^i|_{z=1})
\end{aligned}$$

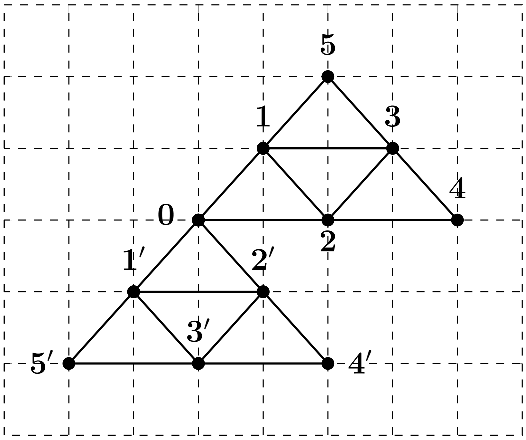

Figure 3.17: relable of vertices on $F^{(1)} \cup F^{(1)'}$

Given $u_i^{(n)}$ for $i = 1,2,3$, let's first compute the matrix $g_1^{(n+1)}(z)(\mathbf{0},\mathbf{5})$.

$$\begin{bmatrix} g_1^{(n+1)}(z)(\mathbf{0},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{1},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{2},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{3},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{1'},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{2'},\mathbf{5}) \\ g_1^{(n+1)}(z)(\mathbf{3'},\mathbf{5}) \end{bmatrix} = \frac{1}{I_{28} - [g_1^{(n)}(z)]|_{0,1,2,3,1',2',3'}} \begin{bmatrix} g_1^{(n)}(z)(\mathbf{0},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{1},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{2},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{3},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{1'},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{2'},\mathbf{5}) \\ g_1^{(n)}(z)(\mathbf{3'},\mathbf{5}) \end{bmatrix}, \qquad (3.36)$$

where $[g_1^{(n)}(z)]|_{0,1,2,3,1',2',3'}$ is the transition matrix in terms of $u_i^{(n)}$ for $i = 1,2,3$.

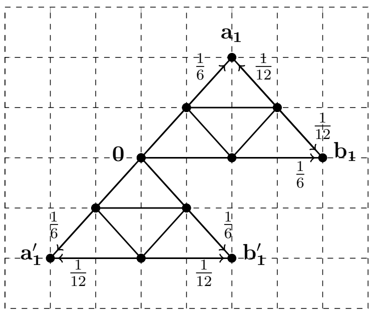

Figure 3.18: distribution of $T^{(1)}$ on $F^{(1)} \cup F^{(1)'}$ with $w_0 = \mathbf{0}^0$

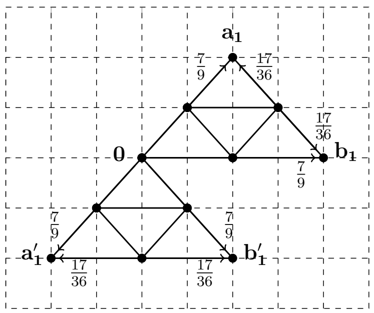

Figure 3.19: distribution of $E(T^{(1)})$ on $F^{(1)} \cup F^{(1)'}$ with $w_0 = \mathbf{0}^0$

For $n = 0$, by solving the above equation, we obtain

$$g_1^{(1)}(z)(\mathbf{0},\mathbf{5}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & -\frac{z^2}{2(3z^2+2z-8)} & -\frac{z^3}{4(3z^2+2z-8)} \\ 0 & 0 & -\frac{z^2}{2(3z^2+2z-8)} & -\frac{z^3}{4(3z^2+2z-8)} \\ 0 & 0 & -\frac{z^2}{2(3z^2+2z-8)} & -\frac{z^3}{4(3z^2+2z-8)} \\ 0 & 0 & -\frac{z^2}{2(3z^2+2z-8)} & -\frac{z^3}{4(3z^2+2z-8)} \end{pmatrix} \end{array}.$$

Since this is classical random walk, each vertex has the same hitting probability by letting $z = 1$.

$$P_1^{(1)}(z)(\mathbf{0},\mathbf{5}) = \begin{array}{c} \\ 0 \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & \frac{1}{6} & \frac{1}{12} \end{pmatrix} \end{array}.$$

Unlike quantum random walk, if we sum the exit probablities for T, then $P_1(T < \infty) = 1$. By taking derivative of $g_1^{(1)}(z)(\mathbf{0},\mathbf{5})$ and pluging $z = 1$, we have $E(T^{(1)}) = 5$ agreeing Theorem 2.2. Therefore we have obtained the hitting probability and expected hitting time from each direction, which generalize Theorem 2.1 and Theorem 2.2.

CHAPTER 4

QUANTUM WALKS ON THE CANTOR SET

4.1 Notation and Definition

The Cantor Graph is introduced in Section 2.2. For generation G, we can think of it as two generation G-1 connected by a removed (void) set of the same length. Let $H_p^{(G)}$ be the position subspace spanned by $|x\rangle$ with x in the site of the lattice while $H_p^{(G)'}$ denote its reflection about the origion, and H_c be the coin space with computational basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$. But for each x in the lattice only two of these basis are used, so define $out(x)$ to be the set of these two basis. If a site x belongs to the Cantor set C, then $out(x) = \{|1\rangle, |2\rangle\}$; if a site belongs to the void set $\mathring{C}$, then $out(x) = \{|3\rangle, |4\rangle\}$. In the above definition, we can think of $|1\rangle$ and $|3\rangle$ as the positive directions (to the right), while $|2\rangle$ and $|4\rangle$ as the negative directions (to the left).

The state space is defined by $H^{(G)} = Span\{|x^i\rangle, x \in H_p^{(G)} \cup H_p^{(G)'}, i \in out(x)\}$.

Let $H_p^{(\infty)} = \cup_{G=1}^{\infty} H_p^{(G)}$ and $H_p^{(\infty)'} = \cup_{G=1}^{\infty} H_p^{(G)'}$, then the state space can be extended as $H^{(\infty)} = Span\{|x^i\rangle, x \in H_p^{(\infty)} \cup H_p^{(\infty)'}, i \in out(x)\}$.

The shift operator $S : H^{(\infty)} \to H^{(\infty)}$ is defined by $S|x^k\rangle = |y^j\rangle$, where

For $x, y \in C$, if $k = 1$, then $y = x + 1$ and $j = 1$. If $k = 2$, then $y = x - 1$ and $j = 2$.

For $x, y \in \overset{\circ}{C}$, if $k = 3$, then $y = x + 1$ and $j = 3$. If $k = 4$, then $y = x - 1$ and $j = 4$.

For $x \in C$, $y \in \overset{\circ}{C}$, if $k = 1$, then $y = x + 1$ and $j = 3$. If $k = 2$, then $y = x - 1$ and $j = 4$.

For $x \in \overset{\circ}{C}$, $y \in C$, if $k = 3$, then $y = x + 1$ and $j = 1$. If $k = 4$, then $y = x - 1$ and $j = 2$.

The coin operator $A : H_c \to H_c$ is a unitary operator,

$$A = \begin{pmatrix} r & r & 0 & 0 \\ r & -r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $r = \dfrac{1}{\sqrt{2}}$.

Then the evolution operator for the quantum random walk is defined by $U = S(I_p \otimes A)$, where I_p denotes the identity operator on $H_p^{(\infty)} \cup H_p^{(\infty)'}$.

Proposition 4.1 *The evolution operator U is unitary.*

Proof 4.1 *U is unitary by directly checking that $\langle U\psi, U\phi \rangle = \langle \psi, \phi \rangle$, for ψ and ϕ in Cantor or void sets.*

Let $\psi_0 \in H^{(\infty)}$ and $\psi_t = U^t \psi_0$. The sequence $\{\psi_t\}_0^\infty$ is called a quantum random walk with the initial state ψ_0.

Let $\psi_t = \sum_{i \in out(x)} \sum_x \psi_t(x, i) |x^i\rangle$ be the quantum random walk at time t, where $\psi_t(x, i)$ is the coefficient at $|x^i\rangle$. Let $p_t(x, i) = |\psi_t(x, i)|^2$ be the probability that the particle is at state $|x^i\rangle$ at time t, and $p_t(x) = \sum_{i \in out(x)} p_t(x, i)$ be the probability that the particle is found at state $|x\rangle$ at time t.

4.2 The Path Integral

Our formulation of path integral is described as Section 3.2. For the state space $Span\{x^i, x \in H_p^{(\infty)} \cup H_p^{(\infty)'}, i \in out(x)\}$, a path w is defined by $w = (w_0, \ldots, w_m)$, where $w_t = x_t^{k_t}$, and $|x_{t+1} - x_t| = 1$. The length of w is defined by $|w| = m$. Let $\Omega^m = \{w; |w| = m\}$.

Definition 4.1 *(Amplitude function) Let*

$$\phi(x^i, y^j) = \begin{cases} -r & if\ x \in C, |x-y| = 1, i = 2\ and\ j = 2\ or\ 4. \\ r & if\ x \in C, |x-y| = 1, i = 1\ or\ i = 2, j = 1\ or\ 3. \\ 1 & if\ x \in \mathring{C}, y = x+1\ for\ i = 3, j = 3\ or\ 1.\ Or\ y = x-1\ for\ i = 4, j = 4\ or\ j = 2. \\ 0 & otherwise. \end{cases}$$

The amplitude function for $w \in \Omega^m$ is defined as

$$\Psi(w) = \prod_{t=0}^{m-1} \phi(w_t, w_{t+1}), \tag{4.1}$$

Definition 4.2 *Let $\Gamma \subseteq \Omega^m$ .Then the amplitude function of a Γ is defined by*

$$\Psi(\Gamma) = \sum_{w \in \Gamma} \Psi(w). \tag{4.2}$$

Let $\Omega = \cup_{m=0}^{\infty} \Omega^m$. For $\Gamma \in \Omega$ with $\Gamma^m = \Gamma \cap \Omega^m$, we also define

$$\Psi(\Gamma) = \sum_{m=0}^{\infty} \Psi(\Gamma^m).$$

For any $\psi \in H^{(\infty)}$, we shall write $\psi = \sum_{i \in out(x)} \sum_x \psi(x,i)|x^i\rangle$. Suppose $\psi_0 = |x^i\rangle$, and $\psi_t = U^t \psi_0$, then

$$\psi_t(y, j) = \Psi(w_0 = x^i, w_t = y^j)$$

Let $\tau^{(G)} = \inf\{t \geq 1; w_t \in \partial H_p^{(G)} \cup \partial H_p^{(G)'}\}, \partial H_p^{(G)} \cup \partial H_p^{(G)'} = \{-3^G, 0, 3^G\}$. Define the set of path that start with x^i and exit from y^j,where $x, y \in \partial H_p^{(G)}$ at time t as $\{w = (w_0, \ldots, w_\tau), w_0 = x^i, w_t = y^j, \tau^{(G)} = t\}$.

The amplitude Green function for quantum random walk is defined by

$$f^{(G)}(z)(x,y)_j^i = \sum_{t=1}^{\infty} z^t \Psi(w_0 = 0^i, w_t = y^j, \tau^{(G)} = t). \tag{4.3}$$

The probability that a quantum random walk starts with x^i and exits from y^j is given by

$$P^{(G)}(x,y)_j^i = \sum_{t=1}^{\infty} |\Psi(w_0 = x^i, w_t = y^j, \tau^{(G)} = t)|^2. \tag{4.4}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |f^{(G)}(e^{i\theta})(x,y)_j^i|^2 d\theta. \tag{4.5}$$

On $H_p^{(G)}$, consider

$$f^{(G)}(z)(0,0) = \begin{array}{c c} & \begin{array}{cc} 0^1 & 0^2 \end{array} \\ \begin{array}{c} 0^1 \\ 0^2 \end{array} & \begin{pmatrix} f^{(G)}(z)(0,0)_1^1 & f^{(G)}(z)(0,0)_2^1 \\ f^{(G)}(z)(0,0)_1^2 & f^{(G)}(z)(0,0)_2^2 \end{pmatrix} \end{array},$$

$$f^{(G)}(z)(0,3^G) = \begin{array}{c c} & \begin{array}{cc} (3^G)^1 & (3^G)^2 \end{array} \\ \begin{array}{c} 0^1 \\ 0^2 \end{array} & \begin{pmatrix} f^{(G)}(z)(0,3^G)_1^1 & f^{(G)}(z)(0,3^G)_2^1 \\ f^{(G)}(z)(0,3^G)_1^2 & f^{(G)}(z)(0,3^G)_2^2 \end{pmatrix} \end{array}.$$

By the definition of the amplitude function, $f^{(G)}(z)(0,3^G)_2^1 = f^{(G)}(z)(0,3^G)_2^2 = 0$, since it can only first arrive at 3^G with type 1. Also by symmetry $f^{(G)}(z)(0,0)_2^1 = f^{(G)}(z)(0,0)_2^2 = f^{(G)}(z)(0,0)_1^1 = -f^{(G)}(z)(0,0)_1^2$, $f^{(G)}(z)(0,3^G)_1^1 = f^{(G)}(z)(0,3^G)_1^2$. Let $u_1^{(G)}(z) = f^{(G)}(z)(0,0)_1^1$ and $u_2^{(G)}(z) = f^{(G)}(z)(0,3^G)_1^1$, then

$$f^{(G)}(z)(0,0) = \begin{array}{c c} & \begin{array}{cc} 0^1 & 0^2 \end{array} \\ \begin{array}{c} 0^1 \\ 0^2 \end{array} & \begin{pmatrix} u_1^{(G)}(z) & u_1^{(G)}(z) \\ -u_1^{(G)}(z) & u_1^{(G)}(z) \end{pmatrix} \end{array}, \tag{4.6}$$

$$f^{(G)}(z)(0,3^G) = \begin{array}{c c} & \begin{array}{cc} (3^G)^1 & (3^G)^2 \end{array} \\ \begin{array}{c} 0^1 \\ 0^2 \end{array} & \begin{pmatrix} u_2^{(G)}(z) & 0 \\ u_2^{(G)}(z) & 0 \end{pmatrix} \end{array}. \tag{4.7}$$

Now let's solve $u_i^{(G)}, i = 1,2$ iteratively.

Let's start with the initial step on $H_p^{(1)}$ as below, see Figure 4.1. The Green amplitude function for $H_p^{(1)'}$ is the same as for $H_p^{(1)}$ if $w_0 = 0^1$, and the Green amplitude function for $H_p^{(1)'}$ is negative of the Green amplitude function for $H_p^{(1)}$ if $w_0 = 0^2$.

$$f^{(1)}(0,0)_2^i = [f^{(0)}(0,1)f^{(1)}(1,0)]_2^i, \tag{4.8}$$

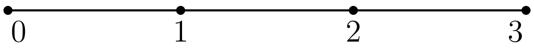

Figure 4.1: $H_p^{(1)}$

$$f^{(1)}(0,3) = f^{(0)}(0,1) f^{(1)}(1,3), \tag{4.9}$$

where

$$\begin{bmatrix} f^{(1)}(1,0) \\ f^{(1)}(2,0) \end{bmatrix} = \frac{1}{I_4 - [f^{(1)}]|_{1,2}} \begin{bmatrix} f^{(0)}(1,0) \\ f^{(0)}(2,0) \end{bmatrix}, \quad \begin{bmatrix} f^{(1)}(1,3) \\ f^{(1)}(2,3) \end{bmatrix} = \frac{1}{I_4 - [f^{(1)}]|_{1,2}} \begin{bmatrix} f^{(0)}(1,3) \\ f^{(0)}(2,3) \end{bmatrix}.$$

And

$$[f^{(1)}]|_{1,2} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 0 & rz & 0 \\ 0 & 0 & rz & 0 \\ 0 & rz & 0 & 0 \\ 0 & -rz & 0 & 0 \end{pmatrix} \end{array},$$

$$\begin{bmatrix} f^{(0)}(1,0) \\ f^{(0)}(2,0) \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & rz \\ 0 & -rz \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array},$$

$$\begin{bmatrix} f^{(0)}(1,3) \\ f^{(0)}(2,3) \end{bmatrix} = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ rz & 0 \\ rz & 0 \end{pmatrix} \end{array}.$$

$$f^{(0)}(0,1) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} rz & 0 \\ rz & 0 \end{pmatrix} \end{array}.$$

So we get

$$f^{(1)}(1,0) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & \frac{\sqrt{2}z(z^2-1)}{z^2-2} \\ 0 & \frac{\sqrt{2}z}{z^2-2} \end{pmatrix} \end{array},$$

$$f^{(1)}(1,3) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \frac{-z^2}{z^2-2} & 0 \\ \frac{-z^2}{z^2-2} & 0 \end{pmatrix} \end{array}.$$

Therefore by (4.8) and (4.9),

$$f^{(1)}(0,0) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \frac{z^4-z^2}{z^2-2} & \frac{z^4-z^2}{z^2-2} \\ -\frac{z^4-z^2}{z^2-2} & \frac{z^4-z^2}{z^2-2} \end{pmatrix} \end{array},$$

$$f^{(1)}(0,3) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \frac{-\sqrt{2}z^3}{2(z^2-2)} & 0 \\ \frac{-\sqrt{2}z^3}{2(z^2-2)} & 0 \end{pmatrix} \end{array}.$$

Then by comparing (4.6) and (4.7) with $n = 1$, we have $u_1^{(1)} = \frac{z^4-z^2}{z^2-2}$, and $u_2^{(1)} = \frac{-\sqrt{2}z^3}{2(z^2-2)}$.

Since $H_p^{(G+1)}$ is obtained by connecting two $H_p^{(G)}$ by the void set of the same length, assuming we know $f^{(G)}(z)(0,0)$ and $f^{(G)}(z)(0,3^G)$, consider the following graph on $H_p^{(G+1)}$ with sites relabeled where $A = 3^G, B = 2 \times 3^G, C = 3^{G+1}$.

Figure 4.2: relabel of $H_p^{(G+1)}$

$$f^{(G+1)}(0,0)_2^i = f^{(G)}(0,0)_2^i + [f^{(G)}(0,A)f^{(G+1)}(A,0)]_2^i, \tag{4.10}$$

$$f^{(G+1)}(0,C) = f^{(G)}(0,A)f^{(G+1)}(A,C), \tag{4.11}$$

where

$$\begin{bmatrix} f^{(G+1)}(A,0) \\ f^{(G+1)}(B,0) \end{bmatrix} = \frac{1}{I_4 - [f^{(G+1)}]|_{A,B}} \begin{bmatrix} f^{(G)}(A,0) \\ f^{(G)}(B,0) \end{bmatrix},$$

$$\begin{bmatrix} f^{(G+1)}(A,C) \\ f^{(G+1)}(B,C) \end{bmatrix} = \frac{1}{I_4 - [f^{(G+1)}]|_{A,B}} \begin{bmatrix} f^{(G)}(A,C) \\ f^{(G)}(B,C) \end{bmatrix},$$

$$[f^{(G+1)}]|_{A,B} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array} & \begin{pmatrix} u_1^{(G)} & 0 & rz^{(3^G)} & 0 \\ -u_1^{(G)} & 0 & rz^{(3^G)} & 0 \\ 0 & rz^{(3^G)} & 0 & u_1^{(G)} \\ 0 & -rz^{(3^G)} & 0 & u_1^{(G)} \end{pmatrix} \end{array}.$$

Also,

$$\begin{bmatrix} f^{(G)}(A,0) \\ f^{(G)}(B,0) \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & u_2^{(G)} \\ 0 & -u_2^{(G)} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array},$$

$$\begin{bmatrix} f^{(G)}(A,C) \\ f^{(G)}(B,C) \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ u_2^{(G)} & 0 \\ u_2^{(G)} & 0 \end{pmatrix} \end{array}.$$

And

$$f^{(G)}(0,0) = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} u_1^{(G)} & u_1^{(G)} \\ -u_1^{(G)} & u_1^{(G)} \end{pmatrix} \end{array},$$

$$f^{(G)}(0,A) = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} u_2^{(G)} & 0 \\ u_2^{(G)} & 0 \end{pmatrix} \end{array}.$$

So we get

$$f^{(G+1)}(A,0) = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & \frac{2u_2^{(G)}(u_1^{(G)}+z^{2\times 3^G}-2u_1^{(G)}z^{2\times 3^G}-1)}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} \\ 0 & \frac{-2u_2^{(G)}(u_1^{(G)}-1)}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} \end{pmatrix} \end{array},$$

$$f^{(G+1)}(A,C) = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \frac{-\sqrt{2}u_2^{(G)}z^{3^G}}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} & 0 \\ \frac{\sqrt{2}u_2^{(G)}z^{3^G}(2u_1^{(G)}-1)}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} & 0 \end{pmatrix} \end{array}.$$

Therefore by (4.10) and (4.11), we have

$$f^{(G+1)}(0,0) =$$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & u_1^{(G)} + \frac{(u_2^{(G)})^2(2u_1^{(G)}+2z^{2\times 3^G}-4u_1^{(G)}z^{2\times 3^G}-2)}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} & u_1^{(G)} + \frac{(u_2^{(G)})^2(2u_1^{(G)}+2z^{2\times 3^G}-4u_1^{(G)}z^{2\times 3^G}-2)}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} \\ 2 & -u_1^{(G)} - \frac{(u_2^{(G)})^2(2u_1^{(G)}+2z^{2\times 3^G}-4u_1^{(G)}z^{2\times 3^G}-2)}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} & u_1^{(G)} + \frac{(u_2^{(G)})^2(2u_1^{(G)}+2z^{2\times 3^G}-4u_1^{(G)}z^{2\times 3^G}-2)}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} \end{array}.$$

$$f^{(G+1)}(0,3^{(G+1)}) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \frac{-\sqrt{2}(u_2^{(G)})^2z^{3^G}}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} & 0 \\ 2 & \frac{-\sqrt{2}(u_2^{(G)})^2z^{3^G}}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2} & 0 \end{array}.$$

So we have iteration as below:

$u_1^{(G+1)} = u_1^{(G)} + \frac{(u_2^{(G)})^2(2u_1^{(G)}+2z^{2\times 3^G}-4u_1^{(G)}z^{2\times 3^G}-2)}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2}$,
$u_2^{(G+1)} = \frac{-\sqrt{2}(u_2^{(G)})^2z^{3^G}}{4u_1^{(G)}+z^{2\times 3^G}+4(u_1^{(G)})^2z^{2\times 3^G}-2(u_1^{(G)})^2-4u_1^{(G)}z^{2\times 3^G}-2}$.

With initial condition $u_1^{(1)} = \frac{z^4-z^2}{z^2-2}$, and $u_2^{(1)} = \frac{-\sqrt{2}z^3}{2(z^2-2)}$, $z = e^{it}$ for $t \in [0, 2\pi]$:
i) When $t = 0$, $u_1^{(G)}(1) = 0, u_2^{(G)}(1) = \frac{\sqrt{2}}{2}$, for $G = 1, 2, 3....$
ii) When $t = \pi$,$u_1^{(G)}(-1) = 0, u_2^{(G)}(-1) = -\frac{\sqrt{2}}{2}$, for $G = 1, 2, 3....$
iii) $|u_1^{(G)}(e^{it})|^2 + |u_2^{(G)}(e^{it})|^2 = \frac{1}{2}$ for $G = 0, 1, 2, ...$ and $t \in [0, 2\pi]$.

Now let's compare the quantum results with Corollary 2.2.

For $G = 1$, $u_1^{(1)} = \frac{z^4-z^2}{z^2-2}$, and $u_2^{(1)} = \frac{-\sqrt{2}z^3}{2(z^2-2)}$, so

$$P^{(1)}(0,0) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \frac{1}{3} & \frac{1}{3} \\ 2 & \frac{1}{3} & \frac{1}{3} \end{array}, \qquad P^{(1)}(0,3) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \frac{1}{6} & 0 \\ 2 & \frac{1}{6} & 0 \end{array}.$$

For $G = 2$,

$u_1^{(2)} = -\frac{-4z^{16}+10z^{14}-12z^{12}+9z^{10}-9z^8+12z^6-10z^4+4z^2}{4z^{14}-12z^{12}+17z^{10}-14z^8+12z^6-16z^4+16z^2-8}$,
$u_2^{(2)} = \frac{-\sqrt{2}z^9}{8z^{14}-24z^{12}+34z^{10}-28z^8+24z^6-32z^4+32z^2-16}$,

$$P^{(2)}(0,0) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.3809 & 0.3809 \\ 2 & 0.3809 & 0.3809 \end{array}, \qquad P^{(2)}(0,3^2) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.1191 & 0 \\ 2 & 0.1191 & 0 \end{array}.$$

For $G = 3$,

$$P^{(3)}(0,0) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.4222 & 0.4222 \\ 2 & 0.4222 & 0.4222 \end{array}, \qquad P^{(3)}(0,3^3) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.0778 & 0 \\ 2 & 0.0778 & 0 \end{array}.$$

For $G = 4$,

$$P^{(4)}(0,0) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.4471 & 0.4471 \\ 2 & 0.4471 & 0.4471 \end{array}, \qquad P^{(3)}(0,3^4) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.0529 & 0 \\ 2 & 0.0529 & 0 \end{array}.$$

And we have the limiting behavior as follows.

$$\lim_{G\to\infty} P^{(G)}(\mathbf{0},\mathbf{0})_2^1 = \frac{1}{2\pi}\int_0^{2\pi} |u_1^{(G)}(e^{i\theta})|^2 d\theta = 1/2.$$

$$\lim_{G\to\infty} P^{(G)}(\mathbf{0},\mathbf{3^G})_1^1 = \frac{1}{2\pi}\int_0^{2\pi} |u_2^{(G)}(e^{i\theta})|^2 d\theta = 0.$$

These results show that the following theorem.

Theorem 4.1 *The quantum random walk on Cantor set is recurrent.*

The following graph compares the hitting probability of the quantum random walk with that of the Classical random walk at 3^G, where x-axis denotes the generation G.

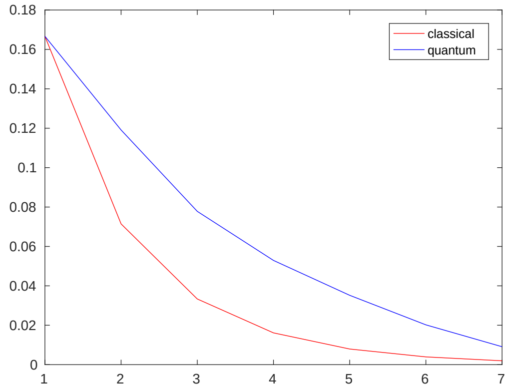

Figure 4.3: comparsion of hitting probability at 3^G

CHAPTER 5

CONCLUSION

We have defined two quantum random walks in this dissertation. One is so called flip-flop quantum walk on the Sierpinski gasket and the other is the quantum walk without flip of types on the Cantor set. For each one, the recrusive formulas of the Green amplitude functions have been developed and the corresponding exit probabilities have been calculated. As the generation goes to infinity, the hitting distribution for Sierpinski gasket converges faster than the classical case since the flip-flop property, it has more trending to go back, but the hitting distribution for the Cantor set converges slower than the classical random walk because it has more trending to expand like the quantum walk in $\mathbb{Z}$.

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