

**RECURSIVELY GENERATING FORMALITY QUASI-  
ISOMORPHISMS WITH APPLICATIONS TO  
DEFORMATION QUANTIZATION**

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by  
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# ABSTRACT

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Dr. Vasily Dolgushev, Chair

Formality quasi-isomorphisms  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}$  are a necessary component of the machinery used in deformation quantization to produce quantized algebras of observables, however they are often constructed via transcendental methods, resulting in computational difficulties and quasi-isomorphisms defined over extensions of  $\mathbb{Q}$ . We will show that these formality quasi-isomorphisms can be "demystified" for a large class of dg-operads, by showing that they can be constructed recursively via an algorithm that builds them from systems of linear equations over  $\mathbb{Q}$ , given certain assumptions on  $H^\bullet(\mathcal{O})$ .

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# CHAPTER 1

## INTRODUCTION

In the transition from classical to quantum physics we move from an algebra of observables which is commutative, to one that is merely associative, with commutators of formerly commuting observables equal to a multiple of a small parameter  $\hbar$ . One approach to obtaining the quantum algebra is to quantize the classical one. Deformation quantization provides a rigorous construction obtaining a noncommutative algebra from commutative ones arising as the smooth functions on a manifold [3] [4] [12] [23] [8]. The products in these noncommutative algebras depend on a formal parameter  $\epsilon$  which, when set to 0, recover the original commutative algebra.

The quantized algebra of a commutative algebra  $A$  is obtained via a quasi-isomorphism giving formality of Hochschild cochains on  $A$ , however, when one tries to construct this quasi-isomorphism by directly solving the necessary equation (which is linear on graded components), one finds that there is a space of obstructions to this process. For  $A$  the smooth functions on a manifold, we find that by viewing Hochschild cochains on this algebra as an algebra over a certain resolution of the operad  $\text{Ger}$ , the space of obstructions is 0 (i.e. there are no obstructions). To give  $\text{HH}^\bullet(A)$  this algebraic structure, we need a formality theorem for operads showing that an operad (the operad  $\text{Br}$  here, but in another approach the operad  $B_\infty$  [19] [31] [15]) governing algebraic structure on Hochschild is formal. This is equivalent to the existence of a quasi-isomorphism from the resolution of



Ger to this operad.

Our goal here is first to provide an algorithmic method for recursively generating the necessary quasi-isomorphism of operads. Previous methods of constructing these maps required the use of transcendental methods (e.g. configuration space integrals) [26] [32] [36] [21] [24] and so this new recursive algorithm allows for a direct computation which demystifies these maps. The algorithm is furthermore a general method that can apply to other important examples. In particular, there are extended forms of deformation quantization that correspond to additional structure on Hochschild cohomology [5] [25] [33] [36]. In one such example, one takes into account an additional differential and an action of Hochschild homology on cohomology, giving the structure of a calculi i.e. an algebra over an operad calc. For this case we prove a version of Koszulity, which gives a resolution that satisfies the conditions necessary to allow the use of the algorithm.

In Chapters 2 and 3 we begin with reminders on the basics of operads, highlighting details used later and allowing us to fix notation. Chapter 2 reviews operads in general and 3 reviews the homological algebra of operads. In Chapters 4 and 5 we give the motivating background for the application of the recursive algorithm in the most basic case. Chapter 4 recalls the solution to Deligne's conjecture, providing formality of operad Br. Chapter 5 reviews the basics of deformation quantization and explains the use of this formality theorem in this setting. The new material is found in Chapters 6 and 7. Chapter 6 provides the recursive algorithm as well as conditions under which it will successfully recursively generate a formality quasi-isomorphism. Chapter 7 discusses the operad calc, proving Koszulity and thereby showing that the results of the previous section apply in this case.

## 1.1 Conventions

In general, we will work over a field  $\mathbb{K}$  of characteristic 0, except in Chapter 6, where we sometimes specify that we work over  $\mathbb{Q}$ , and use  $\mathbb{K}$  to represent some field extension of  $\mathbb{Q}$ .

We will need the category  $\text{GrVect}$  of graded vector spaces and the category  $\text{dgVect}$  of differential graded vector spaces. The former category we will sometimes view as the full subcategory of the latter whose objects are those differential graded vector spaces with the 0 differential. On  $\text{dgVect}$  and its subcategory  $\text{GrVect}$ , we have shift functors given by

$$(\mathfrak{s}V)^\bullet = V^{\bullet-1}, \quad (\mathfrak{s}^{-1}V)^\bullet = V^{\bullet+1}. \quad (1.1.1)$$

These categories both have a symmetric monoidal structure with monoidal product given by

$$(V \otimes W)^n = \bigoplus_{i+j=n} V^i \otimes_{\mathbb{K}} W^j \quad (1.1.2)$$

with differential given by insisting that the differential on  $V \otimes W$  is a derivation of the tensor product. The symmetric structure is given by

$$\tau: V \otimes W \rightarrow W \otimes V \quad (1.1.3)$$

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \quad (1.1.4)$$

where  $|v|$  is the degree of  $v$ . This will give rise to the usual Koszul sign rules in operads and other kinds of structure built on objects in  $\text{dgVect}$ .

We will also use differential graded Lie algebras, which are simply Lie algebra objects in  $\text{dgVect}$ .

# CHAPTER 2

## OPERADS

To a beginner, the definition of an operad is, though clear in its details, opaque in its overall meaning. Though for some the best way may be to muddle through, (and through its use, slowly understand its meaning) I suggest first considering the canonical example from which all general properties of operads are derived.

For an object  $A$  in a symmetric monoidal category, one can always consider the collection  $\{\text{Hom}(A^{\otimes n}, A)\}_{n=0}^{\infty}$ . Operads are the abstraction of the structure and properties of such collections. What structure do we have? We can always precompose with a sequence of twists coming from the symmetry of the monoidal product, represented by an element of  $\sigma \in S_n$ , the symmetric group,  $\sigma: A^{\otimes n} \rightarrow A^{\otimes n}$ . We can also take monoidal products, and then compose, e.g. if  $f_i: A^{\otimes n_i} \rightarrow A$ ,  $i = 1, \dots, k$ , and  $g: A^{\otimes k} \rightarrow A$ , then we can compose, giving  $g \circ (f_1 \otimes \dots \otimes f_k): A^{\otimes n_1 + \dots + n_k} \rightarrow A$ . These compositions satisfy a number of properties, e.g. there are several versions of associativity satisfied, the simplest of which gives that if  $f, g, h: A \rightarrow A$ ,  $(f \circ g) \circ h = f \circ (g \circ h)$ . Finally, we have the identity  $\mathbb{1}: A \rightarrow A$ . This structure also always satisfies certain properties, e.g.  $g \circ (\mathbb{1} \otimes \dots \otimes \mathbb{1}) = g$ . Working out all of these properties will give the axioms of an operad. By replacing a single object with an ordered set of objects  $\{A_i\}_{i=1}^n$  and maps from products of these to one of them, we move to the notion of a colored operad (each color corresponding to an object).

Luckily for those writing about operads, these properties can, with some preparation,

all be bundled together as those of a monoid in a certain monoidal category, as I will do in the general definition below, however to aid the beginner, I will also give a more concrete definition (when working over the category of differential graded vector spaces, as this is the setting in which we will mostly work). More details on the definitions of operads can be found in [10] (we mostly use the notation from here), [27] or [13].

## 2.1 General Definition

For the general definition of an operad, we work over a symmetric monoidal category  $(\mathcal{C}, \otimes)$  with all countable colimits, and a 0 object with  $C \otimes 0$  naturally isomorphic to 0 for all  $C \in \mathcal{C}$  (in what follows we will mainly work with operads over the category of differential graded vector spaces  $\text{dgVect}_{\mathbb{K}}$  over a field  $\mathbb{K}$  of characteristic 0). In this setting we can construct a category  $\text{Coll}(\mathcal{C})$  (written  $\text{Coll}_{\mathbb{K}}$  in the case where  $\mathcal{C} = \text{dgVect}_{\mathbb{K}}$ ) of collections in  $\mathcal{C}$ .

**Definition 2.1.1.** For a finite totally ordered set  $\Xi$  the category of  $\Xi$ -colored collections over  $\mathcal{C}$ , written  $\text{Coll}^{\Xi}(\mathcal{C})$ , or just  $\text{Coll}(\mathcal{C})$  leaving  $\Xi$  implicit, is the category of functors to  $\mathcal{C}$  from the category with objects  $|\Xi| \times \mathbb{Z}_{\geq 0}^{|\Xi|}$  and morphisms

$$\text{Hom}((l, (m_i)_{i \in \Xi}), (k, (n_i)_{i \in \Xi})) = \begin{cases} \prod_{i \in \Xi} S_{n_i} & m_i = n_i \ \forall i \in \Xi \text{ and } l = k \\ \emptyset & \text{otherwise} \end{cases} \quad (2.1.1)$$

for  $l, k \in \Xi$ . For  $A \in \text{Coll}(\mathcal{C})$ , we write the image of  $(l, \{n_i\}_{i \in l})$  as  $A^l(\{n_i\}_{i \in l})$ , or as just  $A(n)$  in the case where  $|\Xi| = 1$ .

*Remark 2.1.1.* We could, without theoretical issues, allow the cardinality of  $\Xi$  to be infinite, however, for all our examples  $\Xi$  will have a cardinality of at most 2.

The category  $\text{Coll}(\mathcal{C})$  can be upgraded to a monoidal category via the plethysm bifunctor. To give a definition for this bifunctor, first we define a groupoid  $\text{Tree}$  and some subgroupoids.

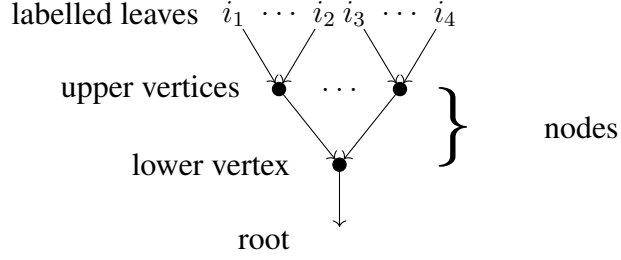
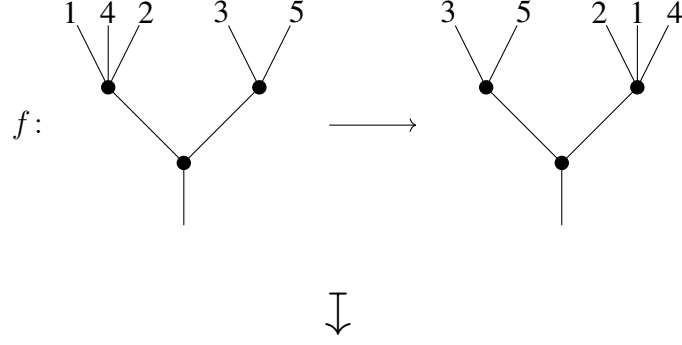


Figure 2.1: Notation for trees. Orientation of edges plays no role except in terminology (i.e. it is fixed by specifying the root).

**Definition 2.1.2.** For a finite totally ordered set  $\Xi$ , an  $\Xi$ -colored tree is rooted planar tree with a function from the set of edges to  $\Xi$  (the coloring of edges) and for each  $l \in \Xi$  an injection from  $\{1, \dots, n_l\}$  to the set of leaves with adjacent edge colored  $l$  (labelling of the leaves). We call the unlabelled non-root vertices nodes, and say that a tree is colored  $l$  if the edge adjacent to the vertex is colored  $l$ . An isomorphism of such trees is an isomorphism of the underlying rooted planar tree that preserves coloring and labelling. A nonplanar isomorphism the same, but not necessarily preserving the planar structure.

Tree is the groupoid whose objects are isomorphism classes of  $\Xi$ -colored trees. Morphisms are nonplanar isomorphisms of  $\Xi$ -colored trees. We write  $\text{Tree}((l, \{n_i\}_{i \in \Xi}))$  for the full subgroupoid of trees colored  $l$  with  $n_i$  labelled leaves for each color  $i \in \Xi$ , and sometimes leave off the  $l$  if we want to include trees of all colors. We write  $2\text{Tree}$  (and  $2\text{Tree}((l, \{n_i\}_{i \in \Xi}))$  when we want to specify the number of labelled leaves and the tree color) for the full subgroupoid whose objects are planar rooted trees where all paths from a leaf to the root pass through at most two nodes, and each path from a labelled leaf to the root passes through exactly two nodes. We will refer to vertices and edges of trees in  $2\text{Tree}$  as indicated in Figure 2.1.

For each  $\{n_i\}_{i \in \Xi}, n_i \geq 0$ , and each pair of collections  $A, B \in \text{Coll}(\mathcal{C})$  there is a functor



$$\sigma_f := (\mathbb{1} \otimes \tau) \circ ((12) \otimes (123) \otimes \mathbb{1}): A(2) \otimes B(3) \otimes B(2) \rightarrow A(2) \otimes B(2) \otimes B(3)$$

Figure 2.2: How  $F_{A,B,5}$  acts on the morphism  $f$  in  $2\text{Tree}(5)$ .  $\tau$  is the twist  $B(3) \otimes B(2) \rightarrow B(2) \otimes B(3)$  from the symmetric monoidal structure of  $\mathcal{C}$ .

$F_{A,B,(l,\{n_i\}_{i \in \Xi})}: 2\text{Tree}((l, \{n_i\}_{i \in \Xi})) \rightarrow \mathcal{C}$  given by

$$F_{A,B,(l,\{n_i\}_{i \in \Xi})}(t) = A((l, \{m_i\}_{i \in \Xi})) \otimes \bigotimes_{u \in U} B((l_u, \{(k_u)_i\}_{i \in \Xi})) \quad (2.1.2)$$

where  $U$  is the set of incoming edges of the lower vertex of  $t$ ,  $m_i$  is the number of edges of color  $i$  in  $U$ ,  $l_u$  is the color of  $u \in U$  and  $(k_u)_i$  is the number of incoming edges of the upper vertex adjacent to  $u$  of color  $i$ . Isomorphisms are sent to the appropriate compositions of twists via the symmetric monoidal product of  $\mathcal{C}$  with products of actions of symmetric groups on  $A(l)$  and the  $B(u_i)$  (see Figure 2.2 for an example in one color). We will write  $\sigma_f := F_{A,B,(l,\{n_i\}_{i \in \Xi})}(f)$ .

**Definition 2.1.3.** The *plethysm bifunctor* is given on collections  $A, B \in \text{Coll}(\mathcal{C})$  as

$$(A \circ B)((l, \{n_i\}_{i \in \Xi})) := \text{Colim } F_{A,B,(l,\{n_i\}_{i \in \Xi})}. \quad (2.1.3)$$

The action of  $\prod_{l \in \Xi} S_{n_l}$  on the labels of trees in  $2\text{Tree}$  gives an action of  $S_{n_l}$  on this colimit.

**Example 2.1.1.** Let  $\mathbb{1}$  be the unit of  $\otimes$ . Then, we give a unit for  $\circ$  in  $\text{Coll}^{\Xi}(\mathcal{C})$ , which we will also call  $\mathbb{1}$ , as follows

$$\mathbb{1}^l(\{n_i\}_{i \in \Xi}) = \begin{cases} \mathbb{1} & n_l = 1, n_i = 0 \text{ for } i \neq l \\ 0 & \text{otherwise.} \end{cases} \quad (2.1.4)$$

The isomorphisms  $\mathbb{1} \circ A \cong A$  and  $A \circ \mathbb{1} \cong A$  come respectively from the maps

$$\mathbb{1}^l(0, \dots, 1, \dots, 0) \otimes A^l(\{n_i\}_{i \in \Xi}) = \mathbb{1} \otimes A^l(\{n_i\}_{i \in \Xi}) \cong A^l(\{n_i\}_{i \in \Xi})$$

and

$$\begin{aligned} A^l(\{n_i\}_{i \in \Xi}) \otimes \mathbb{1}^l(0, \dots, 1, \dots, 0) \otimes \dots \otimes \mathbb{1}^l(0, \dots, 1, \dots, 0) \\ = A^l(\{n_i\}_{i \in \Xi}) \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1} \cong A^l(\{n_i\}_{i \in \Xi}). \end{aligned}$$

**Definition 2.1.4.** An *operad* over  $\mathcal{C}$  is a monoid in the monoidal category  $(\text{Coll}(\mathcal{C}), \circ)$ . Maps of operads are maps of collections respecting this monoidal structure.

*Remark 2.1.2.* There is a clash of terminology related to this definition. In a monoid we have a *multiplication* map and a *unit* map, whereas these represent a *composition* map and *identity* map(s). When we specify to operads over  $\text{dgVect}$ , we will refer to composition maps and refer to the unit map, but the image of  $1 \in \mathbb{K}$  will be called the identity.

**Example 2.1.2.** We can upgrade  $\mathbb{1} \in \text{Coll}(\mathcal{C})$  to an operad using the natural map  $\mathbb{1} \circ \mathbb{1} \cong \mathbb{1}$  and  $\mathbb{1} = \mathbb{1}$  as the unit. This is the initial object in the category of operads.

**Example 2.1.3.** Let  $(\mathcal{C}, \otimes) = (\text{Vect}_{\mathbb{K}}, \otimes)$ . Then let  $\text{As}(n) = \mathbb{K}S_n$  (where our convention here is that  $\mathbb{K}S_0 = 0$ ). Then  $\text{As}$  defines a collection. To give  $\text{As}$  the structure of a monoid, we define  $\mathbb{1} \rightarrow \text{As}$  via the isomorphism  $\mathbb{1}(1) \cong \mathbb{K}S_1$ , and the map  $\text{As} \circ \text{As} \rightarrow \text{As}$  via what follows.

Consider the first the embedding  $S_{l_1} \times \dots \times S_{l_k} \hookrightarrow S_{l_1 + \dots + l_k}$  given by relabelling and second the map  $S_k \hookrightarrow S_{l_1 + \dots + l_k}$  given by permuting blocks  $\{1, \dots, l_1\}, \{l_1 + 1, \dots, l_1 + l_2\}, \dots, \{l_1 + \dots + l_{k-1} + 1, \dots, l_1 + \dots + l_k\}$ . For every  $t \in 2\text{Tree}(l_1 + \dots + l_k)$ , we have an element  $\sigma_t \in S_{l_1 + \dots + l_k}$  given by the planar order of the leaves. Thus, for  $t \in 2\text{Tree}$  we have a map  $\text{As}(k) \otimes \text{As}(l_1) \otimes \text{As}(l_k) \rightarrow \text{As}(l_1 + \dots + l_k)$  given on a simple tensor by  $\sigma \otimes \sigma_1 \otimes \dots \otimes \sigma_k \mapsto \sigma_t(\sigma_1, \dots, \sigma_k) \sigma(\sigma_1, \dots, \sigma_k)^{-1} \sigma_t^{-1}$  multiplying in  $S_{l_1 + \dots + l_k}$ .

The value of operads comes from the way they define categories of algebras. To describe this, first we describe the endomorphism operad, already hinted at above. For this, we require that  $\mathcal{C}$  has an internal hom, as will be the case in all categories of interest.

**Definition 2.1.5.** The *endomorphism operad* of  $\{A_i\}_{i \in \Xi} \in \mathcal{C}$  is given by

$$\text{End}_A^l(\{n_i\}_{i \in \Xi}) := \text{Hom}\left(\bigotimes_{i \in \Xi} A_i^{\otimes n_i}, A_l\right) \quad (2.1.5)$$

with the obvious action of  $\prod_{i \in \Xi} S_{n_i}$  and monoid structure given by composition.

**Definition 2.1.6.** An *algebra over an operad*  $\mathcal{O}$  is a set of objects  $A = \{A_i\}_{i \in \Xi} \subset \mathcal{C}$  along with a map of operads  $\mathcal{O} \rightarrow \text{End}_A$ . A map of algebras  $A \rightarrow B$  over an operad  $\mathcal{O}$  is set of maps in the underlying category  $A_i \rightarrow B_i, i \in \Xi$  inducing a map of operads  $\text{End}_A \rightarrow \text{End}_B$  which commutes with the maps defining the algebras.

**Example 2.1.4.** The category of algebras in over  $\text{As}$  is exactly the category of (non-unital) algebras over  $\mathbb{K}$ , where the product is given by the image of the identity in  $\text{As}(2) = \mathbb{K}S_2$ .

So, less formally, an algebra  $A$  over an operad  $\mathcal{O}$  is a set of objects in  $\mathcal{C}$  together with operations parametrized by  $\mathcal{O}$  with compatibilities required by the relations in  $\mathcal{O}$ .

One very useful construction is the free operad on a collection.

**Definition 2.1.7.** For  $P$  a collection, we get a functor  $F_P: \text{Tree} \rightarrow \mathcal{C}$  given by

$$F_P(t) = \bigotimes_{n \in \text{Nodes}(t)} P(l_n, \{(m_n)_i\}_{i \in \Xi}) \quad (2.1.6)$$

where  $l_n$  is the color of the edge below  $n$  and  $(m_n)_i$  the number of edges of color  $i$  above  $n$ . Then the *free operad* on the collection  $P$  is

$$\mathbb{O}\mathbb{P}(P) = \text{Colim}(F_P). \quad (2.1.7)$$

This has the structure of a collection by its grading by tree color and number of labelled leaves of each color. The unit is given by the isomorphisms  $\mathbb{1}^l \cong F_P(t_{\mathbb{1}}^l)$  where  $t_{\mathbb{1}}^l$  is the tree of color  $l$  with zero nodes and one labelled leaf. Composition is given by grafting trees by identifying edges attached to labelled leaves to edges attached to roots of the same color.



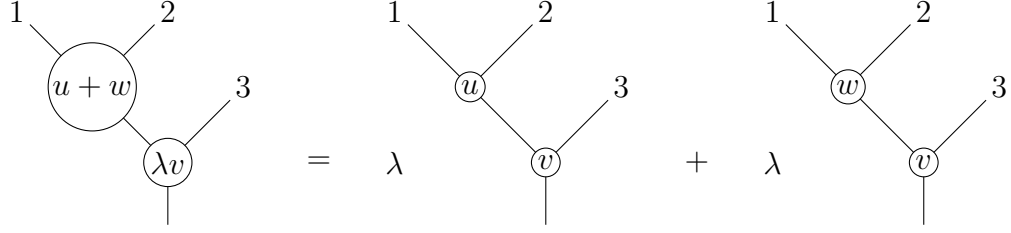


Figure 2.3: An example of an identity in  $F_P(t)$  where  $P \in \text{Coll}(\text{Vect}_{\mathbb{K}})$ ,  $u, v, w \in P$  and  $\lambda \in \mathbb{K}$ .

*Remark 2.1.3.* To understand this construction, it is useful to think of labelling the nodes of a tree with elements of  $P$  with the correct arities and colors (of course, this does not necessarily make sense for an arbitrary choice of  $\mathcal{C}$ ). For example, When  $\mathcal{C} = \text{Set}$ , we can exactly view  $F_P(t)$  as the set of labellings of  $t$  by elements of  $P$  in a way compatible with colors and arities. When  $\mathcal{C} = \text{Vect}$  or  $\mathcal{C} = \text{dgVect}$  we need to modify this by identifying each tree where a node is labelled by a linear combination of elements with linear combinations of trees where the that node is labelled by the elements in that linear combination (see Figure 2.3).

## 2.2 Concrete Definition

In this section we work over  $\text{dgVect}_{\mathbb{K}}$ , and will simplify definitions by working only with one-colored operads. Generalization to multiple colors is straightforward.

Let  $\text{Tree}_2$  be the full subgroupoid of  $2\text{Tree}$  whose objects have 2 nodes. As in  $\text{Tree}$  and  $2\text{Tree}$ ,  $\text{Tree}_2(n)$  denotes the subset with  $n$  leaves. Also we write  $\text{Tree}_2^{i,j}$  (or  $\text{Tree}_2^{i,j}(i+j-1)$ ) for the full subgroupoid of trees with  $i$  incoming edges on the lower node, and  $j$  incoming edges on the higher node.

**Definition 2.2.1.** An operad  $\mathcal{O}$  (over  $\text{dgVect}_{\mathbb{K}}$ , this will be implicit from now on, unless otherwise stated) is a sequence of differential graded vector spaces  $\{\mathcal{O}(n)\}_{n=0}^{\infty}$  ( $n$  is called the

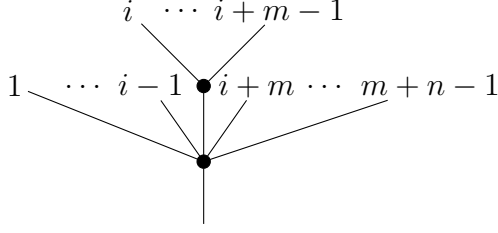


Figure 2.4:  $t_i$  in Definition 2.2.1.

arity”) with an action of  $S_n$  on  $\mathcal{O}(n)$  for each  $n$ , composition maps  $\mu_t: \mathcal{O}(n) \otimes \mathcal{O}(m) \rightarrow \mathcal{O}(m+n-1)$  for each  $t \in \text{Tree}_2^{n,m}$ , and a unit map  $u: \mathbb{K} \rightarrow \mathcal{O}(1)$ . For  $\phi, \psi \in \mathcal{O}$ , we will often write  $\phi \circ_i \psi$  for  $\mu_{t_i}(\phi, \psi)$  where  $t_i$  is (for the correct choice of  $m, n$ ) a representative of the tree shown in Figure 2.2.1. These maps must satisfy the following axioms for  $\phi \in \mathcal{O}(m), \psi \in \mathcal{O}(n), \rho \in \mathcal{O}(p), \sigma \in S_{m+n-1}, f: t \rightarrow t'$  in  $\text{Tree}_2$ :

- $(\phi \circ_i \psi) \circ_j \rho = \begin{cases} \phi \circ_i (\psi \circ_{j-i+1} \rho) & i \leq j \leq i+n-1 \\ (-1)^{|\psi||\rho|} (\phi \circ_{j-n+1} \rho) \circ_i \psi & i+n \leq j \\ (-1)^{|\phi||\rho|} (\phi \circ_j \rho) \circ_{i+p-1} \psi & i > j \end{cases}$
- $\sigma(\mu_t(\phi \otimes \psi)) = \mu_{\sigma(t)}(\phi \otimes \psi)$
- $\mu_{t'}(\phi \otimes \psi) = \mu_t((\phi \otimes \psi) \circ F_{\mathcal{O}}(f))$
- $\phi \circ_i u(1) = \phi$
- $u(1) \circ_1 \phi = \phi$

**Example 2.2.1.** Any operad over  $\text{GrVect}$  can be viewed as an operad over  $\text{dgVect}$  with 0 differential, and an operad over  $\text{Vect}$  can be viewed as an operad over  $\text{GrVect}$  concentrated in degree 0, and hence as a operad over  $\text{dgVect}$ .

**Example 2.2.2.** The operad  $\text{Ger}$  is the operad generated by  $\beta, \mu \in \text{Ger}(2)$  with  $|\beta| = -1, |\mu| = 0$  satisfying relations:

1.  $(1, 2)\mu = \mu$
2.  $\mu \circ_1 \mu = \mu \circ_2 \mu$
3.  $(1, 2)\beta = \beta$
4.  $\beta \circ_1 \beta + (1, 2, 3)\beta \circ_1 \beta + (1, 3, 2)\beta \circ_1 \beta = 0$
5.  $\beta \circ_2 \mu = \mu \circ_1 \beta + (1, 2)\mu \circ_2 \beta$

*Remark 2.2.1.* Note that because we work over  $\text{dgVect}$  where the twist is given by  $a \otimes b \mapsto -b \otimes a$ , we will get the usual Koszul sign rules. E.g. if  $A$  is a Ger-algebra with  $\{\bullet, \bullet\}$  the image of  $\beta$ , then the third relation implies that, for  $a, b \in A$

$$\{a, b\} = (-1)^{|a||b|}\{b, a\}. \quad (2.2.1)$$

A pseudo-operad is a collection with composition maps satisfying the first three axioms required in Definition 2.2.1 for the composition maps of an operad, however, no unit is required.

## 2.3 Cooperads

Operads have a dual version called a cooperad where all maps are reversed.

**Definition 2.3.1.** A *cooperad* is a comonoid with respect to the plethysm bifunctor. In particular, if  $\mathcal{P}$  is a cooperad, it is a collection with cocompositions  $\Delta_t: \mathcal{P}(m+n-1) \rightarrow \mathcal{P}(m) \otimes \mathcal{P}(n)$  for each  $t \in \text{Tree}_2^{m,n}$ , and actions of  $S_n$  on each  $\mathcal{P}(n)$ , which satisfy axioms dual to those in 2.2.1.

The linear duals of operads give examples of cooperads, i.e. for an operad  $\mathcal{O}$ ,  $\mathcal{O}^*$  given by

$$\mathcal{O}^*(n) = \mathcal{O}(n)^* := \text{Hom}(\mathcal{O}(n), \mathbb{K}) \quad (2.3.1)$$

is a cooperad. Dually, for a cooperad  $\mathcal{P}$ ,  $\mathcal{P}^*$  is an operad. Dual to the definition of pseudo-operads, pseudo-cooperads are defined as cooperad except without the requirement for a counit.

# CHAPTER 3

## HOMOLOGICAL ALGEBRA OF OPERADS

Working over the category of differential graded vector spaces allows us to take the (co)homology of an operad. Because we required all maps defining an operad to be maps of differential graded vector spaces, and (co)homology is a monoidal functor of this category, the resulting collection is itself an operad.

In this section we will recall relevant tools from homological algebra of operads. For a more detailed discussion see [10], [27] or [13].

### 3.1 Cobar/Bar Construction

In this section we will describe a pair of adjoint functors between certain categories of operads and cooperads, which will be used to find good resolutions of operads.

**Definition 3.1.1.** Let  $\mathbb{1}$  be the collection which is the identity with respect to the plethysm functor. An *augmented operad* is an operad  $\mathcal{O}$  along with a map  $\mathcal{O} \rightarrow \mathbb{1}$  which composed with the unit map (viewed as a map  $\mathbb{1} \rightarrow \mathcal{O}$ ) gives the identity map. We write  $\mathcal{O}_\circ$  for the kernel of the augmentation. A *coaugmented cooperad* is a cooperad  $\mathcal{C}$  along with a map  $\mathbb{1} \rightarrow \mathcal{C}$  which gives the identity map when precomposed with the counit. We write  $\mathcal{C}_\circ$  for

the cokernel of the coaugmentation.

**Example 3.1.1.** If we have an operad  $\mathcal{O}$  given by a set of generators and relations such that none of the relations involve the unit of  $\mathcal{O}$ , then we have an augmentation  $\mathcal{O} \rightarrow \mathbb{1}$  given by sending all generators to 0 and the unit to  $1 \in \mathbb{1}(1)$ . Different sets of generators and relations that do not involve the unit will give the same augmentation.

From here on, all (co)operads will be (co)augmented. All particular examples of the sort discussed in the Example 3.1.1 will be given the augmentation discussed there.

The (co)kernel of the (co)augmentation of an (co)augmented (co)operad is a pseudo-(co)operad.

**Definition 3.1.2.** A coaugmented cooperad  $\mathcal{C}$  is *conilpotent* if for each  $c \in \mathcal{C}_o$ , there is an  $n > 0$  such that any product of  $n$  cocomposition maps on  $\mathcal{C}_o$  takes  $c$  to 0.

Now, consider the free operad  $\mathbb{O}\mathbb{P}(s\mathcal{C}_o)$  on a conilpotent cooperad  $\mathcal{C}$ . We can define a differential as the sum of two differentials  $\partial_1$  and  $\partial_2$  using the cooperad structure. They will be derivations with respect to composition, and so we need only define them on generators. So, for  $X \in \mathcal{C}_o(n)$ , we define

$$\partial_1(X) := -s\partial_{\mathcal{C}}s^{-1}X \quad (3.1.1)$$

$$\partial_2(X) := - \sum_{t \in \pi_0 \text{Tree}_2(n)} \mu_t((s \otimes s)\Delta_t(s^{-1}X)) \quad (3.1.2)$$

where  $\mu_t$  takes place in the free operad. A direct computation shows that these and their sum square to 0. Similarly we can define a differential on  $\mathbb{O}\mathbb{P}^*(s\mathcal{C}_o)$ .

**Definition 3.1.3.** For a conilpotent cooperad  $\mathcal{C}$ ,  $\text{Cobar}(\mathcal{C})$  is the operad with the underlying collection and compositions the same as in  $\mathbb{O}\mathbb{P}(s\mathcal{C}_o)$ , but with differential  $\partial_1 + \partial_2$ . For an operad  $\mathcal{O}$ , the dual construction gives  $\text{Bar}(\mathcal{O})$ .

The bijection that gives the adjunction will be defined using the convolution dgla:

**Definition 3.1.4.** For an operad  $\mathcal{O}$  and a cooperad  $\mathcal{C}$ , the underlying vector space of the convolution dgla is

$$\text{Conv}(\mathcal{C}, \mathcal{O}) := \prod_{n \geq 0} \text{Hom}_{S_n}(\mathcal{C}_o(n), \mathcal{O}_o(n)). \quad (3.1.3)$$

The a pre-Lie bracket is given for  $X \in \mathcal{C}(n)$  by

$$f \bullet g(X) := \sum_{t \in \pi_0 \text{Tree}_2(n)} \mu_t(f \otimes g) \Delta_t(X) \quad (3.1.4)$$

the commutator of which gives the Lie bracket, and the differential by the sum of the differentials coming from  $\mathcal{O}$  and  $\mathcal{C}$  (see [10] for full details).

Now, the following theorem establishes the adjunction

**Theorem 3.1.1.** *Morphisms of operads  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}$  are in bijection with Maurer-Cartan elements of  $\text{Conv}(\mathcal{C}, \mathcal{O})$  which in turn are in bijection with morphism of cooperads  $\mathcal{C} \rightarrow \text{Bar}(\mathcal{O})$ .*

*Proof.* Because the  $\text{Cobar}(\mathcal{C})$  and  $\text{Bar}(\mathcal{O})$  are free and cofree respectively, the relevant morphisms are determined by maps  $\mathcal{C} \rightarrow \mathcal{O}$ . The required compatibilites translate to the MC-equation. □

**Proposition 3.1.2.**  *$\mathcal{C} \rightarrow \text{Bar}(\mathcal{O})$  is a quasi-isomorphism if and only if  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}$  is a quasi-isomorphism.*

*Proof.* See [27]. □

## 3.2 Koszul Resolutions

We call resolutions of the form  $\text{Cobar}(\mathcal{C})$  for a cooperad  $\mathcal{C}$  Cobar resolutions. We will see in what follows that they have especially good properties. The counit of the adjunction  $\text{Cobar} \text{Bar}(\mathcal{O}) \rightarrow \mathcal{O}$  is a quasi-isomorphism because the adjunction preserve quasi-isomorphisms, which thus provides a resolution (the Cobar – Bar resolution) of  $\mathcal{O}$  by an

operad which is free as the underlying operad over the category of graded vector spaces. This shows the existence of a Cobar resolution for every operad  $\mathcal{O}$ , however, it is often too large to work with effectively. In some settings, one may shrink this complex down by replacing  $\text{Bar}(\mathcal{O})$  with a quasi-isomorphic subcomplex. This section details a class of these cases which we will call "Koszul". Our version of this theory will be a slight generalization of the theory found in [17] and [27], as we will replace the weight grading used there by any appropriate choice of grading.

Suppose we have an operad  $\mathcal{O}$  over  $\text{dgVect}$  (so, with 0 differential and possibly with more than one color) with an additional strictly positive grading  $\mathcal{W}$  compatible with the operad structure. Then we can define

**Definition 3.2.1.** The  $\mathcal{W}$ -syzygy grading (we will leave the  $\mathcal{W}$  implicit in the future, and simply write the syzygy grading) on  $\text{Bar}(\mathcal{O})$  is given by declaring that a decorated tree with  $k$  nodes decorated by  $v_1, \dots, v_k$  is in the  $n$ th syzygy grading if  $v_i \in \mathcal{W}^{l_i}$  with  $l_1 + \dots + l_k = n + k$ . We write  $S^k$  for the  $k$ th level of the syzygy grading.

The differential on  $\text{Bar}(\mathcal{O})$  is such that it raises the syzygy degree by 1, and so we can view  $\text{Bar}(\mathcal{O})$  as the cochain complex

$$S^0 \xrightarrow{\partial} S^1 \xrightarrow{\partial} S^2 \xrightarrow{\partial} \dots \quad (3.2.1)$$

Now, suppose that the cohomology of the complex is concentrated in degree 0. Then the cohomology is

$$H^\bullet(\text{Bar}(\mathcal{O})) = \text{Ker}(S^0 \xrightarrow{\partial} S^1). \quad (3.2.2)$$

Then we have

$$H^\bullet(\text{Bar}(\mathcal{O})) \xrightarrow{\sim} \text{Bar}(\mathcal{O}) \quad (3.2.3)$$

and so using the Cobar-Bar adjunction we have

$$\text{Cobar}(H^\bullet(\text{Bar}(\mathcal{O}))) \xrightarrow{\sim} \mathcal{O}. \quad (3.2.4)$$

We therefore give the following definitions:



**Definition 3.2.2.** The *Koszul dual cooperad* of an operad  $\mathcal{O}$  with grading  $\mathcal{W}$  is  $\mathcal{O}^i := \text{Ker}(S^0 \xrightarrow{\partial} S^1)$ , and the *Koszul dual operad* is  $\mathcal{O}^! := (\mathcal{O}^i)^*$ . In the case where  $H^\bullet(\text{Bar}(\mathcal{O})) = \mathcal{O}^i$  we say that the operad is *Koszul* (see [27] for a variant of this definition in the case where the grading is given by a quadratic presentation).

**Proposition 3.2.1.** *The operad Ger is Koszul with respect to the grading by arity.*

Thus, we get the following proposition.

**Proposition 3.2.2.**  *$\mathcal{O}$  is a Koszul operad if and only if the map*

$$\text{Cobar}(\mathcal{O}^i) \rightarrow \mathcal{O}$$

*is a quasi-isomorphism of operads.*

Now for an operad  $\mathcal{O}$ , assume that  $\mathcal{O}$  is given by generators and homogeneous weight 2 relations, and the grading  $\mathcal{W}$  is the weight grading with respect to this presentation (such an operad is called *quadratic*). Then  $S^0$  is the free cooperad  $\mathbb{O}P^*(\mathfrak{s}V)$  where  $V$  is the collection of generators of  $\mathcal{O}$ , and  $S^1$  is spanned by trees decorated by (shifted) generators in all but one vertex, which is decorated by a (twice shifted) weight 2 element of  $\mathcal{O}$ . We therefore can split  $S^0$  and  $S^1$  as shown below, with  $\mathbb{T}^k$  the functor that takes the a collection to the subcollection of the free (co)operad spanned by trees with  $k$  nodes decorated by elements of the original collection, so that  $\mathbb{O}P(V) = \bigoplus_k \mathbb{T}^k(V)$ .

$$\begin{array}{ccccccc} S^0 & = & \mathbb{K} & \oplus & \mathbb{T}^1(\mathfrak{s}V) & \oplus & \mathbb{T}^2(\mathfrak{s}V) & \oplus & \mathbb{T}^3(\mathfrak{s}V) & \dots \\ \downarrow \partial & = & \downarrow \partial & \oplus & \downarrow \partial & \oplus & \downarrow \partial & \oplus & \downarrow \partial & \dots \\ S^1 & = & \mathbb{K} & \oplus & \mathbb{T}^1(\mathfrak{s}V) & \oplus & \mathbb{T}^2(\mathfrak{s}V)/\mathfrak{s}^2R & \oplus & \mathbb{T}^1(\mathfrak{s}V) \otimes \mathbb{T}^2(\mathfrak{s}V)/\mathfrak{s}^2R & \dots \\ & & & & & & & & \oplus & \mathbb{T}^2(\mathfrak{s}V)/\mathfrak{s}^2R \otimes \mathbb{T}^1(\mathfrak{s}V) & \dots \end{array}$$

where the general term is  $\mathbb{T}^k(\mathfrak{s}V) \xrightarrow{\partial} \bigoplus_{i=1}^{k-1} \mathbb{T}^{i-1}(\mathfrak{s}V) \otimes \mathbb{T}^2(\mathfrak{s}V)/\mathfrak{s}^2R \otimes \mathbb{T}^{k-i-1}(\mathfrak{s}V)$ . Thus, we see that  $\mathcal{O}^i$  is universal among cooperads  $\mathcal{C}$  satisfying the following commutative diagram

(where the top map is restricted to be a map of cooperads, but the other maps are maps of vector spaces)

$$\begin{array}{ccc}
 \mathcal{C} & \longrightarrow & \mathbb{O}\mathbb{P}^*(\mathfrak{s}V) \\
 \downarrow & & \downarrow \\
 \mathbb{0} & \longrightarrow & \mathbb{T}^2(\mathfrak{s}V)/\mathfrak{s}^2R
 \end{array}$$

with the map on the right given by projection onto  $\mathbb{T}^2(\mathfrak{s}V)$  followed by the quotient map. Its dual,  $\mathcal{O}^\dagger$ , is therefore (by looking at the dual of the above diagram) universal among operads  $\mathcal{P}$  satisfying the following diagram (where the map on the right is restricted to be a map of operads)

$$\begin{array}{ccc}
 \mathfrak{s}^{-2}R^\perp & \longrightarrow & \mathbb{O}\mathbb{P}(\mathfrak{s}^{-1}V^*) \\
 \downarrow & & \downarrow \\
 \mathbb{0} & \longrightarrow & \mathcal{P}
 \end{array}$$

where  $R^\perp$  is the image of the map  $(\mathbb{T}^2(V)/R)^* \rightarrow (\mathbb{O}\mathbb{P}^*(V))^* = \mathbb{O}\mathbb{P}(V^*)$  which is the (shifted) dual of the map in the above diagram. This, however, just means that  $\mathcal{O}^\dagger$  is the operad generated by  $\mathfrak{s}^{-1}V^*$  with relations  $\mathfrak{s}^{-2}R^\perp$ .

The above discussion is summarized by the following proposition.

**Proposition 3.2.3.** *For a Koszul operad  $\mathcal{O}$  whose grading is from a presentation with generators  $V$  with homogeneous weight 2 relations  $R$ ,  $\mathcal{O}^\dagger$  is the operad with generated by  $\mathfrak{s}^{-1}V^*$  with relations  $\mathfrak{s}^{-2}R^\perp$  and we have the Cobar resolution*

$$\text{Cobar}(\mathcal{O}^\dagger) \xrightarrow{\sim} \mathcal{O} \tag{3.2.5}$$

given for a basis  $B_V$  of  $V$  by

$$\sum_{v \in B_V} v \otimes v \in \prod_n \mathcal{O}(n) \otimes \mathcal{O}_\circ^\dagger(n) = \text{Conv}(\mathcal{O}_\circ^\dagger, \mathcal{O}). \tag{3.2.6}$$

A useful tool for showing that an operad is Koszul is the Koszul complex:

**Definition 3.2.3.** The *Koszul complex* of a (colored) operad  $\mathcal{O}$  with grading  $\mathcal{W}$  is the complex  $\mathcal{O} \circ \mathcal{O}^i$  where  $\circ$  is the plethysm bifunctor which gives collections their symmetric monoidal structure, and with differential given by the sum of compositions with the root elements in the trees defining elements in  $\mathcal{O}^i$ .

**Lemma 3.2.4.** *A quadratic operad  $\mathcal{O}$  is Koszul if its Koszul complex has cohomology  $\mathbb{K}\mathbb{1}$ .*

*Proof.* We begin by attempting to show that the cohomology of  $\text{Bar}(\mathcal{O})$  is concentrated in syzygy degree 0. Note that  $\text{Bar}(\mathcal{O})$  is split as a cochain complex by the total weight grading

$$\text{Bar}(\mathcal{O}) = \bigoplus_{n \geq 0} \text{Bar}(\mathcal{O})_n. \quad (3.2.7)$$

We proceed by induction on the total weight. The total weight 0 and weight 1 parts are contained entirely in  $S^0$ , and so there is nothing to prove.

Now, for the general case, we begin by considering the filtration of the total weight  $n$  complex by the weight of the root vertex. We will show that the associated spectral sequence converges to a subspace in (the image of) the syzygy degree 0 part of the complex. For the 0th page:

$$E^0 = \text{Gr}(\text{Bar}(\mathcal{O})_n) = \bigoplus_{\substack{i+j=n \\ j < n}} \mathcal{O}_i \otimes \left( \bigoplus_{j_1+\dots+j_l=j} \bigotimes_{k=1}^l \text{Bar}(\mathcal{O})_{j_k} \right) \quad (3.2.8)$$

where the differential acts only on the  $\text{Bar}(\mathcal{O})$  part. Thus, by induction, the first page is

$$E^1 = \bigoplus_{\substack{i+j=n \\ j < n}} \mathcal{O}_i \otimes \left( \bigoplus_{j_1+\dots+j_l=j} \bigotimes_{k=1}^l H^\bullet(\text{Bar}(\mathcal{O})_{j_k}) \right) \quad (3.2.9)$$

$$= \bigoplus_{\substack{i+j=n \\ j < n}} \mathcal{O}_i \otimes \left( \bigoplus_{j_1+\dots+j_l=j} \bigotimes_{k=1}^l \mathcal{O}_{j_k}^i \right). \quad (3.2.10)$$

where the differential comes from the terms of the original differential on  $\text{Bar}(\mathcal{O})_n$  that correspond to contractions of the outgoing edges of the root vertex. We also see that the

spectral sequence abuts here, as the differential will never increase the root weight by more than 1 (because  $\mathcal{O}^i$  is in the subspace of  $\text{Bar}(\mathcal{O})$  given by trees decorated by generators of  $\mathcal{O}$ ).

So, we see that  $\mathcal{O}$  is Koszul if the righthand side of equation (3.2.9) has cohomology in its  $\mathcal{O}$ -weight 1 part, but comparing this to the Koszul complex, we see

$$(\mathcal{O} \circ \mathcal{O}^i)_n = \text{Cone}(\mathcal{O}_n^i \xrightarrow{\partial_{\mathbb{1}}} E^1) \quad (3.2.11)$$

where  $\partial_{\mathbb{1}}$  acts as the term of the differential in the Koszul complex that acts nontrivially on  $\mathcal{O}_n^i$  viewed as an element in  $\mathbb{K}\mathbb{1} \circ \mathcal{O}_n^i$ . The image of  $\partial_{\mathbb{1}}$  is contained in the  $\mathcal{O}$ -weight 1 subspace, completing the proof.  $\square$

We can also use the following, justifying the use of the word "dual" in "Koszul dual".

**Lemma 3.2.5.** *If  $\mathcal{O}$  is Koszul and arity-wise finite dimensional, then  $\mathcal{O}^!$  is also Koszul and  $(\mathcal{O}^!)^! \cong \mathcal{O}$ .*

*Proof.* We note that  $\mathcal{O}^i$  is the linear span of some set of decorated trees. We give this a grading by number of nodes, and hence give its dual  $\mathcal{O}^!$  a  $\mathcal{W}$ -grading necessary to get the Koszul dual. We will use the properties of the Cobar-Bar adjunction (specifically, it preserves quasi-isomorphisms) along with the formula

$$\text{Bar}(\mathcal{P})^* \cong \text{Cobar}(\mathcal{P}^*) \quad (3.2.12)$$

for any arity-wise finite dimensional dg-operad  $\mathcal{P}$ . For the operad structure we see that the dual of a cofree cooperad on some collection is a free operad on the dual of that collection using the universal properties of cofree cooperads and free operads. The internal differential from  $\partial_{\mathcal{P}}$  clearly dualizes correctly, and the external differential from the composition in  $\mathcal{P}$  dualizes correctly, because the dual of the composition on  $\mathcal{P}$  is the cocomposition on  $\mathcal{P}^*$ , from which the external differential on  $\text{Cobar}(\mathcal{P}^*)$  is built.

Now, consider the quasi-isomorphism  $\mathcal{O}^* \rightarrow \text{Bar}(\mathcal{O}^!)$  obtained using these formulae as shown in Figure 3.1. The image of this map is in syzygy degree 0 because after dualizing

$$\begin{array}{ccc}
\mathcal{O}^i & \xrightarrow{\sim} & \text{Bar}(\mathcal{O}) \\
& \Updownarrow \text{dualize} & \\
\text{Bar}(\mathcal{O})^* & \xrightarrow{\sim} & \mathcal{O}^! \\
& \Updownarrow \text{formula (3.2.12)} & \\
\text{Cobar}(\mathcal{O}^*) & \xrightarrow{\sim} & \mathcal{O}^! \\
& \Updownarrow \text{Bar-Cobar adjunction} & \\
\mathcal{O}^* & \xrightarrow{\sim} & \text{Bar}(\mathcal{O}^!)
\end{array}$$

Figure 3.1: Use of the formulae in the proof of 3.2.5. We can always construct these maps, but arity-wise finite dimensionality is required to ensure that they are quasi-isomorphisms.

we see that of the trees with only one node, only those with syzygy degree 0 (i.e., their decoration is in  $\mathcal{W}_1\mathcal{O}$ ) have a nonzero image and the Cobar – Bar adjunction requires that  $\mathcal{O}^* \rightarrow \text{Cobar}(\mathcal{O}^*) \rightarrow \mathcal{O}^!$  is the same as  $\mathcal{O}^* \rightarrow \text{Bar}(\mathcal{O}^!) \rightarrow \mathcal{O}^!$ . Thus, we see that a syzygy degree 0 subcomplex of  $\text{Bar}(\mathcal{O}^!)$  is quasi-isomorphic to it, and hence we have that  $\mathcal{O}^!$  is Koszul, and because  $\mathcal{O}^*$  has no differential,  $\mathcal{O}^* = (\mathcal{O}^!)^i$ , giving us the desired result.

See also [27] and [17] for the proof in the quadratic case. □

### 3.3 Formality of Operads

In this section we will discuss formality of operads, which will be an important concept in the following sections.

First, we define an equivalence relation on operads.

**Definition 3.3.1.** Two operads  $\mathcal{O}$  and  $\mathcal{P}$  are *weakly equivalent* if there is a sequence of operads  $\{\mathcal{O}_i\}_{i=0}^n$  with  $\mathcal{O} = \mathcal{O}_0$ ,  $\mathcal{P} = \mathcal{O}_n$  and a quasi-isomorphism between each of  $\mathcal{O}_i$  and  $\mathcal{O}_{i+1}$  (in either direction) for  $i = 0, \dots, n - 1$ . I.e. weak equivalence is the smallest

equivalence relation on operads containing all quasi-isomorphisms.

**Definition 3.3.2.** An operad  $\mathcal{O}$  is *formal* if it is weakly equivalent to its (co)homology.

So, what can we say about two weakly equivalent operads? For one, they have isomorphic (co)homologies, but more than that, we will see that in the filtered case, we can resolve them via Cobar of the same cooperad.

**Definition 3.3.3.** A cooperad  $\mathcal{C}$  is *filtered* if there is a cocomplete ascending filtration of differential graded vector spaces

$$0 = \mathcal{F}^0\mathcal{C} \subset \mathcal{F}^1\mathcal{C} \subset \dots \subset \mathcal{C} \quad (3.3.1)$$

such that

$$\Delta_t(\mathcal{F}^m\mathcal{C}) \subset \bigoplus_{i+j=m} \mathcal{F}^i\mathcal{C} \otimes \mathcal{F}^j\mathcal{C} \quad (3.3.2)$$

for all  $t \in \text{Tree}_2$ .

**Lemma 3.3.1.** Given a quasi-isomorphism of operads  $f: \mathcal{O}_2 \rightarrow \mathcal{O}_1$ , and Cobar resolution  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}_1$ , for  $\mathcal{C}$  a filtered cooperad, then the dotted arrow in the following diagram exists, and is a quasi-isomorphism:

$$\begin{array}{ccc} & & \mathcal{O}_2 \\ & \nearrow \text{dotted} & \downarrow \sim \\ \text{Cobar}(\mathcal{C}) & \xrightarrow{\sim} & \mathcal{O}_1 \end{array}$$

**Theorem 3.3.2.** If two operads  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are weakly equivalent, and one has a Cobar resolution  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}_i$  by a filtered cooperad  $\mathcal{C}$ , then there is a cobar resolution  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}_{\tilde{i}}$ ,  $\{i, \tilde{i}\} = \{1, 2\}$ .

In particular,

**Corollary 3.3.3.** If  $\mathcal{O}$  is formal, and its (co)homology  $\mathcal{H}$  has a Cobar resolution  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{H}$  via a filtered cooperad  $\mathcal{C}$ , then there is a Cobar resolution  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}$ .

# CHAPTER 4

## DELIGNE'S CONJECTURE

Deligne's conjecture (now a theorem) asked about algebraic structures on the Hochschild cochain complex. Its cohomology was known to have the structure of a Gerstenhaber algebra (i.e. it was an algebra over the operad  $\text{Ger}$ ), but what kind of algebraic structure on cochains gives rise to the Gerstenhaber structure? There was a natural candidate in chains on the little disks operad (the (negatively graded) homology of this operad is  $\text{Ger}$ ), however, how this could act on cochains was unclear. The resolution to this conjecture was the recognition that Hochschild cochains carry the structure of a braces algebra, and more importantly, that the operad  $\text{Br}$  governing braces algebras is weakly equivalent to chains on the little disks operad. In this section we explain all the players in this story, and give an outline of the proof of weak equivalence.

### 4.1 Hochschild (Co)homology

Throughout this section  $A$  will be an associative algebra over  $\mathbb{K}$ .

**Definition 4.1.1.** The *Hochschild chains* of  $A$  are given as

$$C_n(A) := A \otimes A^{\otimes n} \tag{4.1.1}$$

with differential given as  $d = \sum_{i=0}^n (-1)^i d_i$  for

$$d_0(a \otimes a_1 \otimes \dots \otimes a_n) = aa_1 \otimes \dots \otimes a_n \quad (4.1.2)$$

$$d_i(a \otimes a_1 \otimes \dots \otimes a_n) = a \otimes a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n \quad i \in \{1, \dots, n-1\} \quad (4.1.3)$$

$$d_n(a \otimes a_1 \otimes \dots \otimes a_n) = a_n a \otimes a_1 \otimes \dots \otimes a_{n-1}. \quad (4.1.4)$$

The homology, called the *Hochschild homology* of  $A$ , is written  $\mathrm{HH}_\bullet(A)$ .

**Definition 4.1.2.** The *Hochschild cochains* of  $A$  are given as the dual of Hochschild chains, or equivalently,

$$C^n(A) := \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, A) \quad (4.1.5)$$

with differential given as  $\partial = \sum_{i=0}^{n+1} (-1)^i \partial_i$  for

$$\partial_0(f)(a_1 \otimes \dots \otimes a_{n+1}) = a_1 f(a_2 \otimes \dots \otimes a_{n+1}) \quad (4.1.6)$$

$$\partial_i(f)(a_1 \otimes \dots \otimes a_{n+1}) = f(a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}) \quad i \in \{1, \dots, n\} \quad (4.1.7)$$

$$\partial_{n+1}(f)(a_1 \otimes \dots \otimes a_{n+1}) = f(a_1 \otimes \dots \otimes a_n) a_{n+1}. \quad (4.1.8)$$

The cohomology, called *Hochschild cohomology* of  $A$  is written  $\mathrm{HH}^\bullet(A)$ .

*Remark 4.1.1.* We shorten the usual notations  $C_n(A, A)$  et al. because for us coefficients of Hochschild (co)homology will always be in the algebra itself.

## 4.2 Algebraic structures for Hochschild (co)homology

The operad  $\mathrm{Ger}$  is generated by two operations  $\mu, \beta \in \mathrm{Ger}(2)$  (i.e. multiplication and the bracket). Therefore, to specify the Gerstenhaber algebra structure on  $\mathrm{HH}^\bullet(A)$ , we need only specify how to multiply two elements and take their bracket (and show that the resulting operations satisfy some relations). These operations on  $\mathrm{HH}^\bullet(A)$  were originally defined in [14].

**Multiplication in  $\mathrm{HH}^\bullet$ :** Suppose  $f \in C^m(A), g \in C^n(A)$ . Then we define

$$(f \cup g)(a_1, \dots, a_{m+n}) = f(a_1, \dots, a_m) g(a_{m+1}, \dots, a_{m+n}). \quad (4.2.1)$$



It is easily checked that  $\partial$  is a derivation of this product, and so it descends to a product on  $\mathrm{HH}^\bullet(A)$ .

**Bracket in  $\mathrm{HH}^\bullet$ :** Again, suppose  $f \in C^m(A), g \in C^n(A)$ . Then define,

$$f\{g\}(a_1, \dots, a_{m+n-1}) := \sum_{i=1}^m (-1)^{ni} f(a_1, \dots, a_{i-1}, g(a_i, \dots, a_{i+n-1}), a_{i+m}, \dots, a_{m+n-1}) \quad (4.2.2)$$

we can then defined the bracket as the (graded) commutator of this operation:

$$[f, g] := f\{g\} - (-1)^{(|f|-1)(|g|-1)} g\{f\}. \quad (4.2.3)$$

This bracket is shifted graded antisymmetric, satisfies the shifted graded Jacobi identity and  $\partial$  is a derivation of the bracket, so  $C^\bullet(A)$  is a shifted dgla, and  $\mathrm{HH}^\bullet(A)$  is a shifted graded Lie algebra.

**Proposition 4.2.1.** *The following equation holds, showing that the cup product is commutative on  $\mathrm{HH}^\bullet(A)$*

$$f \cup g - (-1)^{|f||g|} g \cup f = -(-1)^{|f||g|} \partial(f\{g\}) - \partial(f)\{g\} - (-1)^{|f|-1} f\{\partial g\}. \quad (4.2.4)$$

Furthermore, for  $f \in \mathrm{HH}^\bullet(A)$ ,  $[f, \bullet]$  is a derivation of the product. Therefore, we get that  $\beta \mapsto [\bullet, \bullet], \mu \mapsto \bullet \cup \bullet$  defines a map of operads  $\mathrm{Ger} \rightarrow \mathrm{End}_{\mathrm{HH}^\bullet(A)}$ , defining a Gerstenhaber algebra structure on  $\mathrm{HH}^\bullet(A)$ .

On  $\mathrm{HH}_\bullet(A)$ , for the case where  $A$  is unital, we have the additional structure of a differential  $\delta$  defined on chains as:

$$\delta(a_0 \otimes \dots \otimes a_n) = \sum_{i \geq 0} (-1)^{n-i} a_i \otimes \dots \otimes a_n \otimes 1 \otimes a_0 \otimes \dots \otimes a_{i-1}. \quad (4.2.5)$$

This anti-commutes with the usual differential,  $d$ , and so is a differential on  $\mathrm{HH}_\bullet(A)$ .

We also have contraction and the Lie derivative marrying the structures on  $\mathrm{HH}_\bullet(A)$  and  $\mathrm{HH}^\bullet(A)$ : For  $f \in \mathrm{HH}^n(A)$ , we have  $i_f: \mathrm{HH}_m(A) \rightarrow \mathrm{HH}_{m-n}(A)$  given by

$$i_f(a_0 \otimes \dots \otimes a_m) = f(a_1 \otimes \dots \otimes a_{n+1}) a_0 \otimes \dots \otimes a_m. \quad (4.2.6)$$

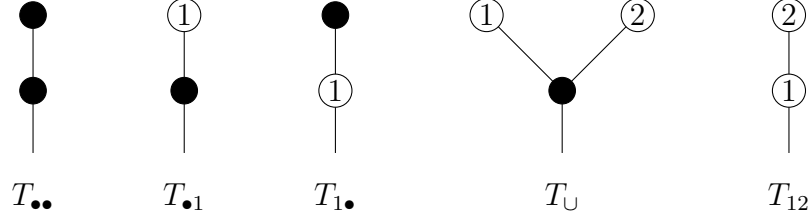


Figure 4.1: Some useful trees.  $T_U$  and  $T_{12}$  are in the set of trees spanning  $\text{Br}$ ,  $T_{\bullet\bullet}$ ,  $T_{\bullet 1}$  and  $T_{1\bullet}$  are not, because they have neutral vertices with less than 3 adjacent edges.

We define the Lie derivative by the commutator  $l_f = [\delta, i_f]$ .

These structures on  $(\text{HH}^\bullet(A), \text{HH}_\bullet(A))$  come together to form a calculi, i.e. an algebra over a 2-colored operad  $\text{calc}$ , as will be discussed further in Chapter 7.

### 4.3 Algebraic structures on cochains

In this section we introduce an operad  $\text{Br}$  governing an algebraic structure on Hochschild cochains extending that given above, whose cohomology is Ger [11].

$\text{Br}(n)$  will be the linear span of trees with two kinds of vertices: neutral and labelled. Each tree will have  $n$  labelled vertices which are labelled by  $\{1, \dots, n\}$ , and any number of neutral vertices, where neutral vertices are required to have at least 2 incoming edges. The trees will have a planar structure, and will be viewed up to planar isomorphism respecting the labelling.  $S_n$  will act by permuting labels. See Figure 4.1 for examples of such trees. To be in the spanning set for  $\text{Br}$  we will further require that each neutral vertex have at least 3 adjacent edges.

To compose two trees, say  $t_1 \circ_i t_2$ , we follow this procedure:

1. Relabel vertices in  $t_2$  by adding  $i - 1$  to each label, and relabel vertices in  $t_1$  by adding one less than the arity of  $t_2$  to each label greater than  $i$ .
2. Remove all subtrees in  $t_1$  attached via their root to labelled vertex  $i$ .

3. Attach this sequence of subtrees to  $t_2$  in all possible ways retaining the sequential order, giving a set of trees.
4. Attach each of these trees to what remains of  $t_1$  by replacing the vertex labelled  $i$  with the lowest vertex of each tree.
5.  $t_1 \circ_i t_2$  is the sum of these trees with sign given by the sign of the permutation of the planar order of edges performed in this process.

The differential is given by inserting the sum  $T_{1\bullet} + T_{\bullet 1}$  into each labelled vertex and then inserting  $T_{\bullet\bullet}$  into each neutral vertex, discarding all trees not in the set spanning Br. See Figure 4.3 for an example.

Above we gave two operations on cochains, the cup product and  $(f, g) \mapsto f\{g\}$ . By computation we see that these satisfy the same relations as the elements  $T_{\cup}$ , and  $T_{12}$  in Br.

**Theorem 4.3.1.**  $C^\bullet(A)$  is an algebra over the operad Br, such that the tree  $T_{\cup}$  acts as the cup product and  $T_{12}$  acts as  $(f, g) \mapsto f\{g\}$ .

## 4.4 The little disks operad

The little disks operad is a topological operad (i.e. an operad over  $(\text{Top}, \times)$ ). Via the (negatively graded) singular chains functor, this will give us an operad over  $\text{dgVect}$ .

Let  $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ . A standard embedding  $D^2 \hookrightarrow D^2$  is an embedding which is obtained by first dilating by  $r > 0$  and then translating. Similarly, a standard embedding  $D^2 \coprod \dots \coprod D^2 \hookrightarrow D^2$  is a standard embedding on each component such that the images are disjoint.

$\text{LD}_2(n)$  is then the space of standard embeddings of disjoint unions of  $n$  copies of  $D^2$  into  $D^2$ . Permuting components gives an action of  $S_n$ , making  $\text{LD}_2$  into a collection.

For  $1 \leq i \leq n$ , and  $f \in \text{LD}_2(m)$  we have a map  $\iota_{f,i}: \coprod_{j=1}^m D^2 \hookrightarrow \coprod_{j=1}^n D^2$  by embedding via  $f$  into the  $k$ th component. We also have  $f_{<i}: \coprod_{j=1}^{i-1} D^2 \hookrightarrow D^2$ , and

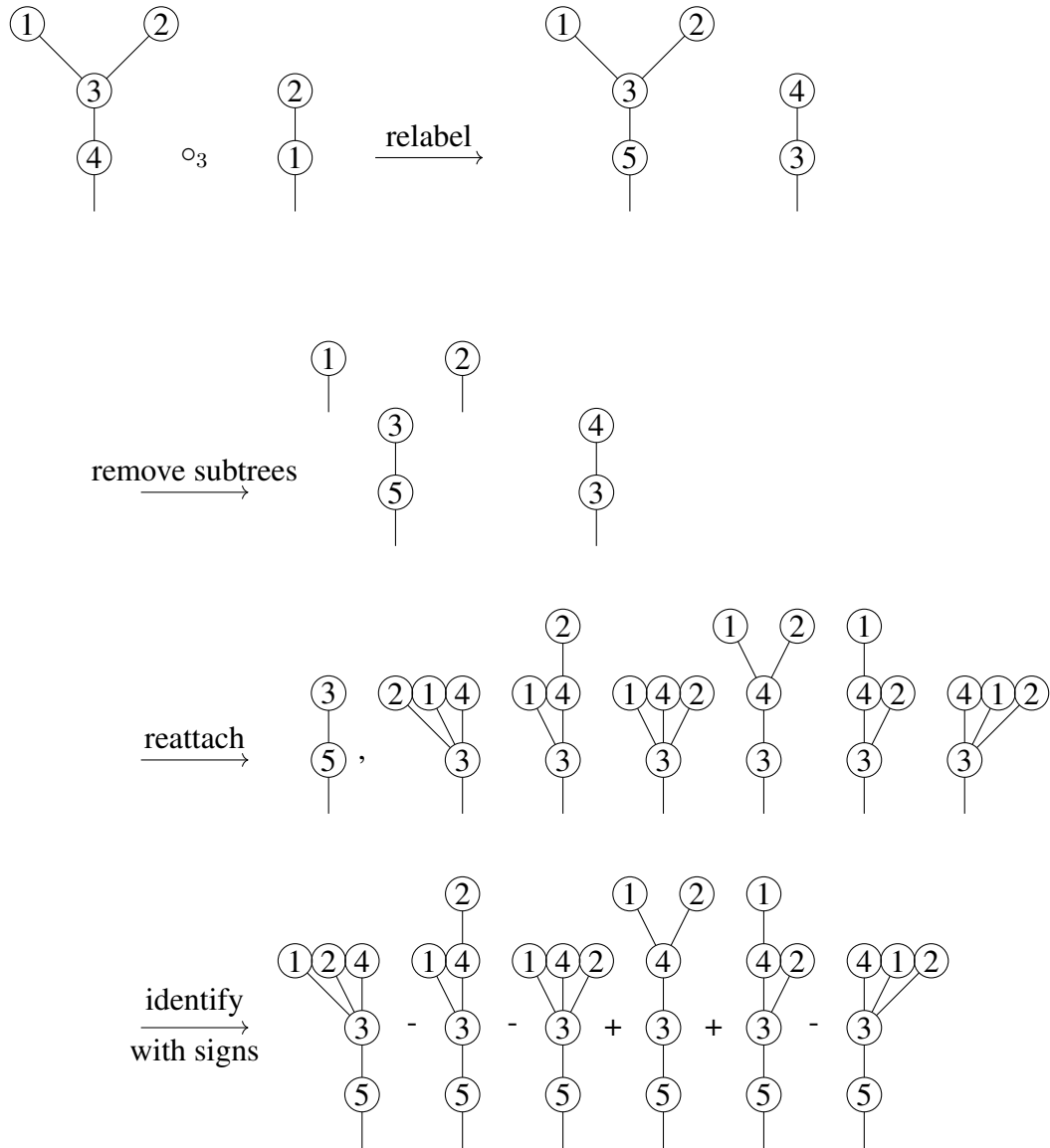


Figure 4.2: An example of composition  $t_1 \circ_3 t_2$  in Br.

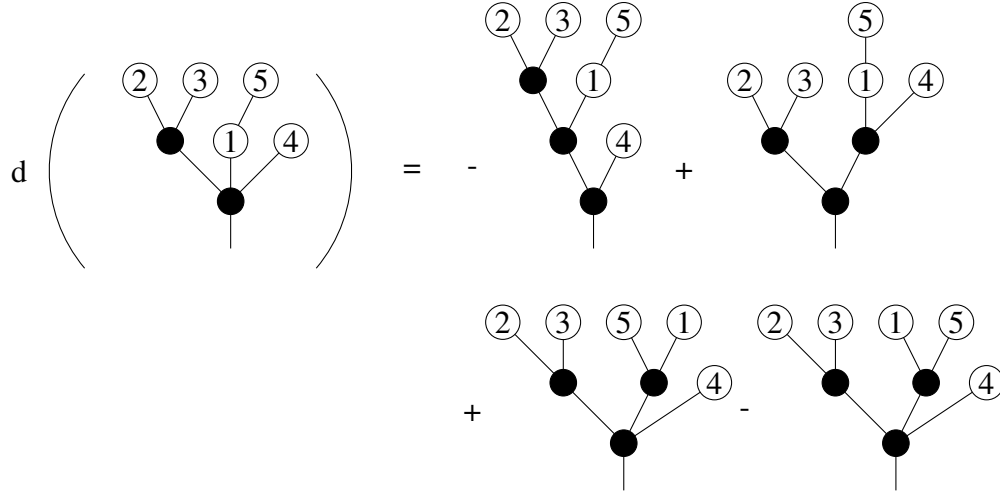


Figure 4.3: An example of applying the differential in  $\text{Br}$ . The first two trees come from inserting  $T_{\bullet\bullet}$  into the lowest neutral vertex, and the second two from inserting  $T_{\bullet 1}$  into the vertex labelled 1. All other trees obtained from insertion are discarded, as they are not in the spanning set.

$f_{>i}: \coprod_{j=i+1}^m D^2 \hookrightarrow D^2$  given by restriction. Then, for  $f \in \text{LD}_2(m)$  and  $g \in \text{LD}_2(n)$  we define  $f \circ_i g$  to be  $f_{<i}$  on the first  $i - 1$  components,  $f \circ_{\iota_{g,i}}$  on components  $i$  through  $i + n - 1$ , and  $f_{>k}$  on components  $i + n$  through  $m + n - 1$ . See Figure 4.4 for a graphical example of composition in  $\text{LD}_2$ .

There is a similar topological operad called the Fulton-Macpherson operad  $\text{FM}$  which is homotopy equivalent to  $\text{LD}_2$ .  $\text{FM}(n)$  is a compactification of a version the configuration space of  $n$  points in the real plane,  $\text{Conf}_2(n)$  such that we allow points to approach one another, but keep track of the direction of approach. It is a manifold with corners. As an operad over  $\text{Set}$  it is defined as the free operad on the collection  $\widetilde{\text{Conf}}_2$  where  $\widetilde{\text{Conf}}_2(n)$  is the quotient of  $\text{Conf}_2(n)$  by the group consisting of affine transformations  $v \mapsto u + \lambda v$  where  $\lambda > 0$ . Topologically, we get this operad by taking the closures of the images of

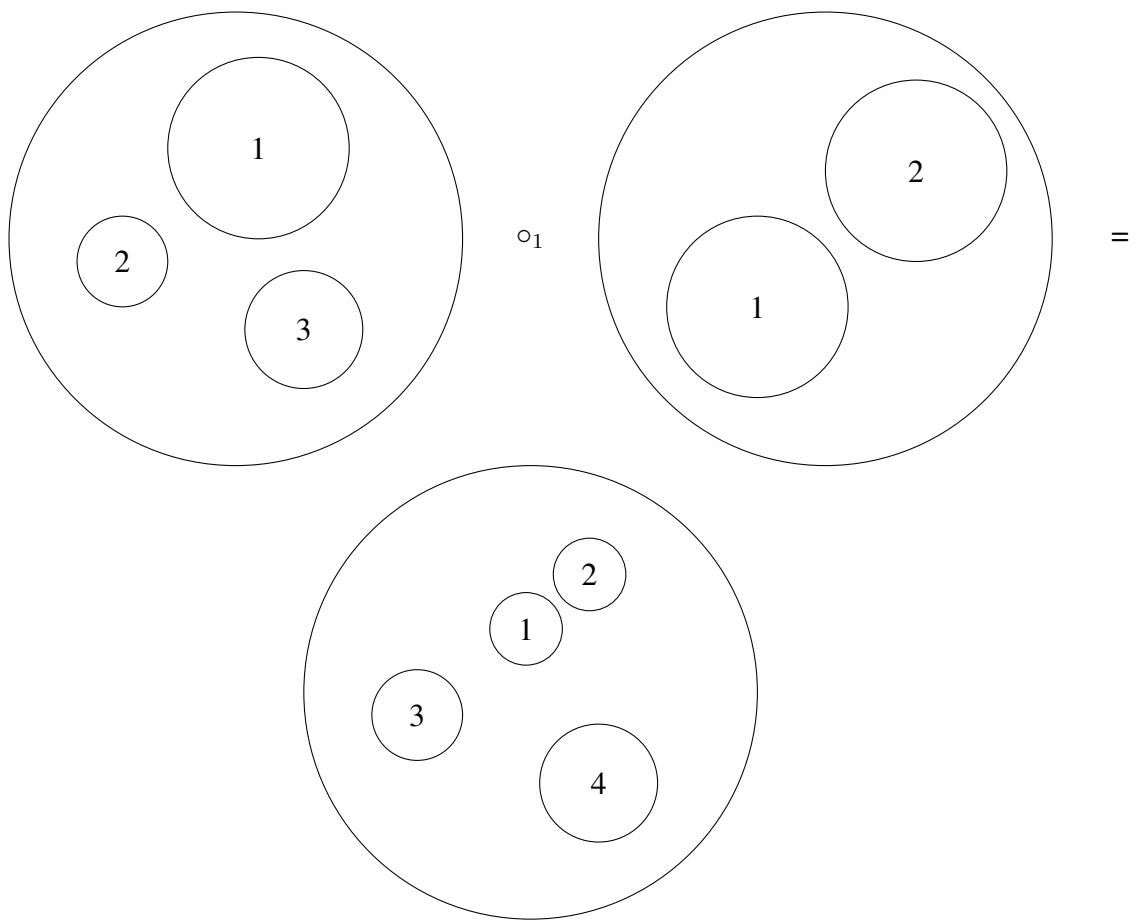


Figure 4.4: An example of composition in  $LD_2$ .

embeddings  $\tilde{\text{Conf}}_2(n) \hookrightarrow (S^1)^{n(n-1)/2} \times [0, \infty)^{n(n-1)(n-2)}$  via

$$(x_1, \dots, x_n) \mapsto \left( \left( \frac{x_i - x_j}{|x_i - x_j|} \right)_{i < j}, \left( \frac{|x_i - x_j|}{|x_i - x_k|} \right)_{|\{i,j,k\}|=3} \right). \quad (4.4.1)$$

A third version of this operad is  $\text{FM}'$ , which is the same as  $\text{FM}$ , except we replace points by disjoint disks of any nonnegative radius (a disk of size 0 is a point). Composition into one of the positive radius disks is done by doing an affine transformation of the configuration to the largest size fitting within the disk. Composition into a point is done by inserting as in  $\text{FM}$ . To formally define the topology on  $\text{FM}'$ , we let  $D(n)$  be the space of standard maps of  $n$  disks into the plane, where a map is standard when each restriction to a single disk is either a standard embedding or a map to a point, and all images of these restrictions are disjoint. We write elements in the form  $((x_1, r_1), \dots, (x_n, r_n))$  where  $x_i$  is the image of the center of the  $i$ th disk, and  $r_i$  is the radius of the image of the  $i$ th disk (0 if it is a point). Then we get  $\tilde{D}(n)$  by quotienting out by affine transformations. We consider the map  $\tilde{D}(n) \rightarrow A(n) := (S^1)^{n(n-1)/2} \times [0, \infty)^{n(n-1)(n-2)} \times [0, \infty)^n / (0, \infty)$  given by

$$((x_1, r_1), \dots, (x_n, r_n)) \mapsto \left( \left( \frac{x_i - x_j}{|x_i - x_j|} \right)_{i < j}, \left( \frac{|x_i - x_j| - (r_i + r_j)}{|x_i - x_k| - (r_i + r_k)} \right)_{|\{i,j,k\}|=3}, (r_i)_{i=1}^n \right). \quad (4.4.2)$$

The image of this map lies in the subspace  $A(n) \setminus Z(n)$  where  $Z(n)$  is the subset where the  $(i, j, k)$  term of  $[0, \infty)^{n(n-1)(n-2)}$  is 0, but either the  $i$ th,  $j$ th, or  $k$ th term of  $[0, \infty)^n / (0, \infty)$  is nonzero.  $\text{FM}'(n)$  is the closure of  $\tilde{D}(n)$  in this subspace.

Both  $\text{LD}_2$  and  $\text{FM}$  are suboperads of  $\text{FM}'$ .

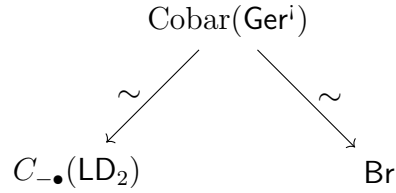
## 4.5 Outline of the proof of Deligne's conjecture

There are multiple proofs of Deligne's conjecture. In this section we outline the approach of Kontsevich and Soibelman [24]. This proof uses the action of  $\text{Br}$  on  $\text{HH}^\bullet(A)$ , but it does not directly give a quasi-isomorphism from  $\text{Cobar}(\text{Ger}^i)$  to  $\text{Br}$ , but instead uses a  $\text{Cobar} - \text{Bar}$  resolution of  $\text{Br}$ . However, we can fix this because the formality of the little

disks operad [26] [32] ensures that  $\text{Br}$  is formal, and hence, via Theorem 3.3.2, there exists a quasi-isomorphism from  $\text{Cobar}(\text{Ger}^i)$  to  $\text{Br}$ . Second, we can refer to the proof of Deligne’s conjecture by Tamarkin [31], which passes through  $\text{Cobar}(\text{Ger}^i)$  directly (though it uses another operad  $B_\infty$  acting on  $\text{HH}^\bullet(A)$ ).

Ultimately we get the statement we want:

**Theorem 4.5.1.** *(Negatively graded) chains on the little disks operad, and  $\text{Br}$  are both formal, giving the following diagram of quasi-isomorphisms.*



*Proof.* (outline, following Kontsevich. For details not included here, see [24]).

1. FM and  $\text{LD}_2$ : First we show that the operad FM is homotopy equivalent to the operad  $\text{LD}_2$ , via the operad  $\text{FM}'$ . The equivalences are from the natural embeddings  $\text{LD}_2 \hookrightarrow \text{FM}' \hookrightarrow \text{FM}$ .
2. Chains: The homotopy equivalence of FM and  $\text{LD}_2$  induces a quasi-isomorphism of (negatively graded) singular chains on each. The operad of semi-algebraic chains on FM ( $C^{\text{semi-alg-}\bullet}(\text{FM})$ ) is in turn quasi-isomorphic to the operad of singular chains on FM.
3.  $\text{Cobar}(\text{Bar}(\text{Br}))$ : Next we construct a quasi-isomorphism from  $\text{Cobar}(\text{Bar}(\text{Br})) \rightarrow C^{\text{semi-alg-}\bullet}(\text{FM})$  by showing that there are no obstructions to constructing MC elements of  $\text{Conv}(\text{Bar}(\text{Br}), C^{\text{semi-alg-}\bullet}(\text{FM}))$ .

□



# CHAPTER 5

## DEFORMATION QUANTIZATION

Deformation quantization is concerned with finding noncommutative products in a formal parameter  $\epsilon$  that deform a given commutative product in the sense that they specialize to the commutative product when we set  $\epsilon = 0$ . The solution to Deligne's conjecture gives us a way to find such deformations. In particular,

**Definition 5.0.1.** A *star-product*  $\star$  on a polynomial algebra  $A$  is a  $\mathbb{R}$ -linear and continuous (with respect to the topology coming from the  $\epsilon$ -grading) associative product on  $A[[\epsilon]]$  such that for  $a, b \in A$ ,  $a \star b = ab + B_1(a, b)\epsilon + B_2(a, b)\epsilon^2 + \dots$  where the  $B_i$ 's are bilinear operators.

For a given star-product,  $\star$ , the anti-symmetric part  $B_1^-$  of  $B_1$  gives a Poisson structure  $\alpha$  on  $\text{Spec}(A)$  by  $B_1^-(a, b) = \langle \alpha, da \otimes db \rangle$ . On the other hand, if we start with a Poisson structure on  $\text{Spec}(A)$ , can we get a star-product? The answer is yes, and that is the topic of this section. Throughout this section we will take  $A = SV$  for a real vector space  $V$ .

### 5.1 DGLAs and Maurer-Cartan Elements

First we find a way of representing sets of such Poisson structures and star-product as dglas, following the usual philosophy of deformation theory.

The most important tool in deformation theory is the dgla (differential graded Lie algebra), from which we get Maurer-Cartan elements.

**Definition 5.1.1.** A filtered dgla is a dgla  $L$  with a complete filtration  $\mathcal{F}$  of the form

$$L = \mathcal{F}^1 L \supset \mathcal{F}^2 L \supset \dots \quad (5.1.1)$$

which is compatible with the bracket.

**Definition 5.1.2.** A *Maurer-Cartan element* of a dgla  $L$  is a degree 1 element  $\alpha$  satisfying

$$\text{Curv}(\alpha) := \partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0. \quad (5.1.2)$$

We write the set of Maurer-Cartan elements as  $\text{MC}(L)$ . Additionally, if  $L$  is filtered, we say  $\alpha, \alpha' \in \text{MC}(L)$  are *equivalent* Maurer-Cartan elements if there is some degree 0 element  $\xi \in L$  such that

$$d + \mathbf{ad}(\alpha') = e^{\mathbf{ad}(\xi)}(d + \mathbf{ad}(\alpha))e^{-\mathbf{ad}(\xi)}. \quad (5.1.3)$$

We write  $\pi_0(L)$  for the set of Maurer-Cartan elements modulo equivalence.

Maps of dglas induce maps between sets of Maurer-Cartan elements and these sets modulo equivalence. In fact, we have the following:

**Proposition 5.1.1.** *The above constructions  $\text{MC}$  and  $\pi_0$  are functorial (from the categories of dglas and filtered dglas respectively), and  $\pi_0$  takes quasi-isomorphisms to bijections if the restriction of the quasi-isomorphism to each filtered piece is also a quasi-isomorphism.*

## 5.2 Two DGLAs

This section will concern two dglas filtered by a formal variable  $\epsilon$  and the objects that they are used to model. The first dgla is  $C^\bullet(A)$ , and the objects it models are star-products on  $A$ .

Given a star-product  $\star$  on  $A$ , with, for  $a, b \in A$ ,

$$a \star b = ab + \sum_{i>0} B_i(a, b)\epsilon^i. \quad (5.2.1)$$

We note that  $B_i \in C^2(A)$ .

We also need an equivalence relation on star-products.

**Definition 5.2.1.** Two star-products  $\star, \star'$  are equivalent if there exists a  $\mathbb{K}[[\epsilon]]$ -linear map  $T: A[[\epsilon]] \rightarrow A[[\epsilon]]$  such that for  $a \in A$ ,  $T_i \in C^1(A)$ ,

$$T(a) = a + \sum_{i>0} T_i(a)\epsilon^i \quad (5.2.2)$$

and  $T(a \star b) = T(a) \star' T(b)$  for all  $a, b \in A$ .

We get the following proposition:

**Proposition 5.2.1.** *Star-products on an  $A = SV$  for a real vector space  $V$  modulo equivalence of star products are in bijection with  $\pi_0(\epsilon C^\bullet(A)[[\epsilon]])$ .*

Our second dgla will be  $\Lambda_A \text{Der}(A)$  with the 0 differential. The objects we will associate with these are formal Poisson structures:

**Definition 5.2.2.** A *formal Poisson structure* is a Poisson structure  $\pi = \sum_{i>0} \pi_i \epsilon^i$  on  $A[[\epsilon]]$ .

By writing out the equations satisfied by Poisson structures in this case, we observe that:

**Observation 5.2.2.** *Formal Poisson structures on  $A = SV$  modulo formal diffeomorphism are in bijection with  $\pi_0(\epsilon \Lambda_A \text{Der}(A)[[\epsilon]])$ .*

Having these sets in bijection with star-products and formal Poisson structures will allow us, in the next section, to obtain a bijection between them.

## 5.3 Using Operadic Formality to get Formality of Algebras

In this section we show how formality and Deligne's conjecture are used to prove Theorem 5.3.3.

**Theorem 5.3.1.**  $\Lambda_A \text{Der}(A)$  is isomorphic to  $\text{HH}^\bullet(A)$ .

*Proof.* See [20] for the proof. Here we simply record the map

$$v_1 \wedge \dots \wedge v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \cup \dots \cup v_{\sigma(n)}. \quad (5.3.1)$$

□

**Theorem 5.3.2.** *There exist a zig-zag of quasi-isomorphisms of  $\text{Cobar}(\text{Ger}^i)$ -algebras between  $C^\bullet(A)$  and  $\Lambda_A \text{Der}(A)$ .*

*Proof.* See [31], [19] and [9] for the proof. Deligne's conjecture gives the  $\text{Cobar}(\text{Ger}^i)$ -algebra structure on  $C^\bullet(A)$ . It is necessary to use this algebra structure because the proof is completed by showing that there are no obstructions to formality of  $C^\bullet(A)$  as a  $\text{Cobar}(\text{Ger}^i)$ -algebra (and hence the required quasi-isomorphisms will be maps of  $\text{Cobar}(\text{Ger}^i)$ -algebras). The HKR map provides initial data needed for this construction. □

**Theorem 5.3.3.** *Star-products on  $A = SV$  modulo isomorphisms are in bijection with formal Poisson structures on  $A$  modulo formal diffeomorphisms.*

*Proof.* Because the two dglas  $\Lambda_A \text{Der}(A)$  and  $C^\bullet(A)$  are weakly equivalent, the quasi-isomorphisms giving this equivalence gives bijections of sets of Maurer-Cartan elements modulo equivalence. By the bijections obtained in Section 5.2, this gives a bijection between these two types of algebraic structure. See also [23], [1] and [22]. □

Thus, because the two dglas  $\Lambda_A \text{Der}(A)$  and  $C^\bullet(A)$  are weakly equivalent, the quasi-isomorphisms giving this equivalence gives bijections of sets of Maurer-Cartan elements modulo equivalence, proving Theorem 5.3.3.

# CHAPTER 6

## RECURSIVELY GENERATING QUASI-ISOMORPHISMS

In this section we give an algorithm for recursively generating certain kinds of quasi-isomorphisms which we call formality quasi-isomorphisms.

**Definition 6.0.1.** Let  $\mathcal{O}$  be a formal operad with homology  $\mathcal{H}$ , and  $\mathcal{C}$  a cooperad such that we have a quasi-isomorphism of operads  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{H}$ . A *formality quasi-isomorphism* is a quasi-isomorphism of operads  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}$ .

Recall from Corollary 3.3.3 that the existence of a formality quasi-isomorphism is guaranteed when  $\mathcal{C}$  is filtered, however, there are several issues. First, the quasi-isomorphism may not be over  $\mathbb{Q}$ , but instead over some field extension of  $\mathbb{Q}$ . Second, there is no guarantee that we will be able to do any computations with this map, as its definition will depend on the quasi-isomorphisms in the zigzag of maps realizing the formality of  $\mathcal{O}$ , which may not be explicitly defined. For example, in the case of  $\mathcal{H} = \text{Ger}$  and  $\mathcal{O} = \text{Br}$ , these maps depend on the use of transcendental methods.

The algorithm we will give improves this situation in both of these senses. It defines a quasi-isomorphism over  $\mathbb{Q}$ , and can give us the images of particular elements in terms of chosen bases.

Throughout this section we will work over  $\mathbb{Q}$ , with  $\mathbb{K}$  a field extension. When we want to work over  $\mathbb{K}$  we will use the tensor  $\bullet \otimes \mathbb{K}$ , which we think of as a functor to a category defined over  $\mathbb{K}$  (so, for example, formality of some  $A \otimes \mathbb{K}$  means formality in the category over  $\mathbb{K}$ ).

## 6.1 Assumptions

To begin, we need to define a class of (co)operads that will be amenable to the tools used in the algorithm. We will need a grading on the cooperad compatible with the dg-cooperad structure (**A1** below) and finite dimensionality conditions (**A1** and **A2**). We will also need  $\mathcal{C}$  to be part of a "nice" Cobar resolution of  $\mathcal{H}$  (**A3**), and of course we will need formality of  $\mathcal{O}$  (**A4**). Throughout  $P$  will be the cokernel of the coaugmentation of  $\mathcal{C}$ .

**A1** There is a grading  $\mathcal{G}$  on  $P$  into finite dimensional graded pieces. This grading is compatible with the dg-pseudo-cooperad structure, i.e.,

$$P = \bigoplus_{k \geq 1} \mathcal{G}^k P \quad (6.1.1)$$

for a tree  $\mathbf{t}$  with  $q$  nodes,

$$\Delta_{\mathbf{t}}(\mathcal{G}^m P) \subset \bigoplus_{l_1 + \dots + l_q = m} \mathcal{G}^{l_1} P \otimes \mathcal{G}^{l_2} P \otimes \dots \otimes \mathcal{G}^{l_q} P \quad (6.1.2)$$

and,

$$\partial(\mathcal{G}^k P) \subset \mathcal{G}^{k-1} P \quad (6.1.3)$$

with  $\mathcal{G}^0 P := 0$ .

**A2** The graded components of  $\mathcal{O}(n)$  are finite dimensional for all  $n$ .

**A3**  $\text{Cobar}(\mathcal{C})$  is a resolution of  $\mathcal{H}$ , i.e. there exists a quasi-isomorphism

$$\rho: \text{Cobar}(\mathcal{C}) \rightarrow \mathcal{H} \quad (6.1.4)$$

and also  $\mathcal{H}$  is generated by the image of  $s\mathcal{G}^1 P$  under  $\rho$ , and  $\rho$  is 0 on  $s\mathcal{G}^k$ ,  $k > 1$ .

**A4**  $\mathcal{O} \otimes \mathbb{K}$  is formal.

$\mathcal{C}$  and  $\mathcal{O}$  will satisfy these assumptions throughout. We will write  $F_\beta$  for the formality quasi-isomorphism guaranteed to exist by Corollary 3.3.3, and  $\beta$  for the corresponding MC element.

*Remark 6.1.1.* The existence of  $F_\beta$  will be needed only to prove the existence of solutions to certain linear systems of equations, i.e. it will be needed to show that the algorithm will produce a result. When implementing the algorithm one need not know anything about  $F_\beta$ . In fact, it could be implemented in the case where  $\mathcal{O}$  is not known to be formal (and hence  $F_\beta$  is not known to exist). If  $\mathcal{O}$  is not formal, the algorithm will fail by producing a system of linear equations with no solution at some level of the recursion. Hence, the algorithm could also be used to prove non-formality if all other assumptions are met.

*Remark 6.1.2.* Assumption **A3** implies that  $H^\bullet(F_\beta)$  gives a bijection from cohomology classes in  $\mathcal{S}\mathcal{G}^1P$  to generators of  $\mathcal{H}$ . With this and the finite dimensionality assumption **A2**, we get that  $F_\beta$  is a quasi-isomorphism.

These assumptions should be viewed as an operadic version of the Sullivan minimal model of commutative algebras [30], with the one difference that we require a grading rather than just a filtration (in particular they will be met in the quadratic or linear-quadratic Koszul setting). It may be possible to extend this result to the filtered case, however, because the applications all are suitably graded, there is no need to complicate the result. The simplest case where these assumptions are met is given as Example 6.1.1 below.

**Example 6.1.1.** Suppose we have  $\mathcal{C}_o(0) = \mathcal{C}_o(1) = 0$  and the differential  $\partial_{\mathcal{C}} = 0$ . Then, if  $\mathcal{C}(n)$  and  $\mathcal{O}(n)$  are finite dimensional for all  $n$ , the grading by arity satisfies assumption **A1** and assumption **A2** is also satisfied (by definition). In particular, for  $\mathcal{C} = \text{Ger}^\vee$  the Koszul dual of the Gerstenhaber operad, and  $\mathcal{O} = \text{Br}$ , all assumptions are met when we grade by arity. See 6.4 for more details.

Under these assumptions, we can give some additional notation.



**Definition 6.1.1.** We say  $\alpha \in \text{Conv}(P, \mathcal{O})$  is an  $n$ th sprout if

$$\text{Curv}(\alpha)(X) = 0 \tag{6.1.5}$$

for all  $X \in \mathcal{G}^k P$  for  $k \leq n$ . We call this equation the  $n$ th sprout equation.

Additionally, for  $\alpha \in \text{Conv}(P, \mathcal{O})$ , we write  $\alpha_k := \alpha|_{\mathcal{G}^k P}$ , and

$$\alpha = \sum_{k=1}^{\infty} \alpha_k. \tag{6.1.6}$$

## 6.2 The main theorem

Throughout, we will assume **A1-A4**. The algorithm for building formality quasi-isomorphisms will work by taking an  $n$ th sprout and building an  $(n + 1)$ th sprout. To do this we take the restriction of the  $n$ th sprout equation to  $\oplus_{k=1}^{n-1} \mathcal{G}^k P$  as an initial condition, and notice that in this case, the condition to be a solution of the MC equation reduces to an inhomogenous linear system of equations. The main result is the existence result needed to ensure that this system of equations has a solution.

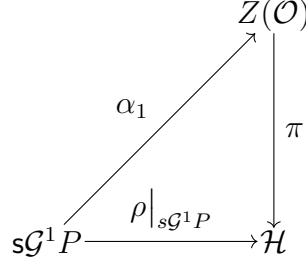
**Theorem 6.2.1.** (*The main theorem*) For  $n \geq 2$ , given an  $(n - 1)$ th sprout  $\alpha = \alpha_1 + \dots + \alpha_n \in \text{Conv}(P, \mathcal{O})$ , where the following commutes,

$$\begin{array}{ccc}
 & & Z(\mathcal{O}) \\
 & \nearrow^{\alpha_1} & \downarrow \pi \\
 s\mathcal{G}^1 P & \xrightarrow{\rho|_{s\mathcal{G}^1 P}} & \check{\mathcal{H}}
 \end{array}$$

then there exists  $\tilde{\alpha} = \tilde{\alpha}_1 + \dots + \tilde{\alpha}_{n+1}$ , an  $(n + 1)$ th sprout in  $\text{Conv}(P, \mathcal{O})$  with  $\alpha_k = \tilde{\alpha}_k$  for  $k < n$ .

We can (and would like to) do better for  $n = 1$ .

**Proposition 6.2.2.** *Given an 1st sprout  $\alpha \in \text{Conv}(P, \mathcal{O})$ , with*



*there is a 2nd sprout  $\tilde{\alpha} \in \text{Conv}(P, \mathcal{O})$  with  $\tilde{\alpha}_1 = \alpha_1$ .*

Because the above commutative diagram completely determines  $\alpha_1$ , we get:

**Corollary 6.2.3.** *(of Proposition 6.2.2, Theorem 6.2.1) We can recursively construct quasi-isomorphisms  $\text{Cobar}(\mathcal{C}) \rightarrow \mathcal{O}$ .*

*Remark 6.2.1.* The fact that the diagram commutes in Theorem 6.2.1 implies that the recursively constructed MC element represents a quasi-isomorphism by the same argument that tells us our original  $\beta$  represents a quasi-isomorphism.

*Remark 6.2.2.* Figure 6.1 shows how the recursive construction proceeds. The sprout given by Theorem 6.2.1 at each step is not necessarily extendable to a higher sprout, but because we only need to alter the highest graded piece to extend, there is a growing stable range, the limit of which gives a genuine MC element.

The main theorem above is proven using the following lemma in which the (possibly transcendental)  $\beta$ , is adjusted by acting via a derivation of  $\text{Cobar}(\mathcal{C}) \otimes \mathbb{K}$  constructed from the difference of  $\beta$  and  $\alpha$  on one side, and by a homotopy equivalence on the other.

**Lemma 6.2.4.** *Given an  $n$ th sprout  $\alpha \in \text{Conv}(P, \mathcal{O})$ , with  $\alpha_k = \beta_k$  for  $k < l$  for some  $l < n$ , there is a MC element  $\tilde{\beta} \in \text{Conv}(P, \mathcal{C}) \otimes \mathbb{K}$  with  $\tilde{\beta}_k = \alpha_k$  for  $k \leq l$ .*

To prove Lemma 6.2.4 we use the following technical lemma.

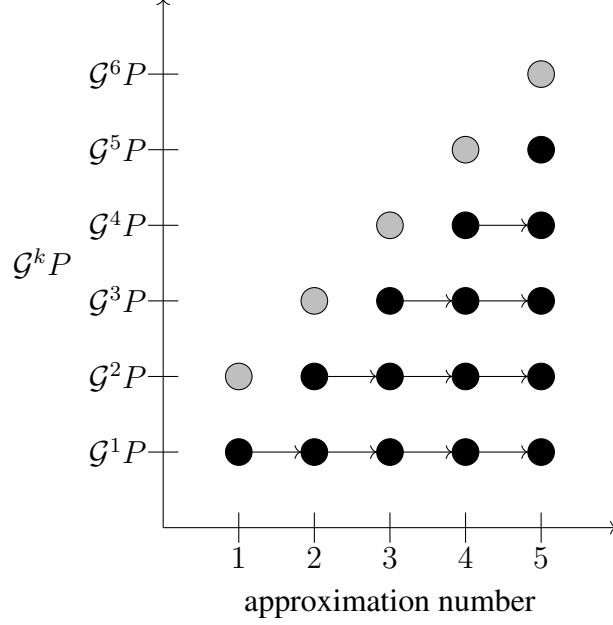


Figure 6.1: The first five successive sprouts where the black dots represent terms in the stable range, and arrows indicate equality of the corresponding terms.

**Lemma 6.2.5.** For  $k > 2$ , and  $\psi \in \text{Hom}(\mathcal{G}^k P, \mathcal{H})$  viewed as a vector in  $\text{Conv}(P, \mathcal{H})$  by setting  $\psi|_{\mathcal{G}^l P} = 0$  for  $l \neq k$ , with

$$\psi \circ \partial_{\mathcal{C}} + [\rho|_{\mathfrak{s}\mathcal{G}^1 P}, \psi] = 0 \quad (6.2.1)$$

there exists a locally nilpotent derivation  $\mathcal{D} \in \text{Der}(\text{Cobar}(\mathcal{C}))$  which is closed with respect to the differential on  $\text{Der}(\text{Cobar}(\mathcal{C}))$ , for which

$$\rho \circ \mathcal{D}|_{\mathfrak{s}\mathcal{G}^k P} = \psi \quad (6.2.2)$$

and for which

$$\mathcal{D}|_{\mathfrak{s}\mathcal{G}^{<k} P} = 0. \quad (6.2.3)$$

## 6.3 Proofs

*Proposition 6.2.2.* We need to find  $\tilde{\alpha}_2$  such that

$$\partial_{\mathcal{O}}\tilde{\alpha}_2(X) + \alpha_1(\partial_{\mathcal{C}}X) + [\alpha_1, \alpha_1](X) = 0 \quad (6.3.1)$$

for all  $X \in \mathcal{G}^2P$ . We note that replacing  $\tilde{\alpha}_2$  with  $\beta_2$  solves this equation in  $\text{Conv}(P, \mathcal{O}) \otimes \mathbb{K}$ , and so by the finite dimensionality assumptions **A1** and **A2**, there exists a solution  $\tilde{\alpha}_2$  in  $\text{Conv}(P, \mathcal{O})$ . □

*Lemma 6.2.4.* Consider  $\gamma_l = \beta_l - \alpha_l$ . Because

$$\partial_{\mathcal{O}}\kappa(X) + \alpha_{l-1}(\partial_{\mathcal{C}}X) + \frac{1}{2} \sum_{i+j=l} [\alpha_i, \alpha_j](X) = 0 \quad (6.3.2)$$

for  $X \in \mathcal{G}^lP$  is satisfied for  $\kappa = \alpha_l$  and for  $\kappa = \beta_l$ , we get

$$\partial_{\mathcal{O}} \circ \gamma_l = 0. \quad (6.3.3)$$

Thus we can consider the composition  $\psi_l := \pi \circ \gamma_l$ .

$$\partial_{\mathcal{O}}\chi(X) + \kappa(\partial_{\mathcal{C}}X) + [\alpha_1, \kappa](X) + \frac{1}{2} \sum_{\substack{i+j=l+1 \\ i,j < l}} [\alpha_i, \alpha_j](X) = 0 \quad (6.3.4)$$

is satisfied by  $\chi = \alpha_{l+1}$ ,  $\kappa = \alpha_l$ , and by  $\chi = \beta_{l+1}$ ,  $\kappa = \beta_l$ . Subtracting these two variants of (6.3.4) we get

$$\psi_l \circ \partial_{\mathcal{C}} + [\alpha^{\mathcal{H}}, \psi_l] = 0 \quad (6.3.5)$$

where  $\alpha^{\mathcal{H}} := \pi \circ \alpha_1 = \rho|_{\mathfrak{s}\mathcal{G}^1P}$ . Thus, we use Lemma 6.2.5 (proof to follow) to get a derivation  $\mathcal{D}$ , which can be exponentiated to an automorphism of  $\text{Cobar}(\mathcal{C})$  because it is locally nilpotent.

We set  $F_{\tilde{\beta}} := F_{\beta} \circ \exp(\mathcal{D})$ , and let  $\tilde{\beta}$  be the corresponding MC element. Because  $\mathcal{D}$  acts as 0 on  $\mathcal{G}^{<l}P$ , we get that,

$$\tilde{\beta}_k - \alpha_k = 0 \quad (6.3.6)$$

for  $k < l$ .  $\tilde{\beta}_l - \alpha_l$  is exact because  $\mathcal{D}|_{\mathfrak{sG}^l P}$  acts as  $\psi_l = \beta_l - \alpha_l$ . Thus, we can adjust  $\tilde{\beta}$  to an equivalent MC element in  $\text{Conv}(P, \mathcal{O}) \otimes \mathbb{K}$  with

$$\tilde{\beta}(X) - \alpha(X) = 0 \quad (6.3.7)$$

for all  $X \in \mathfrak{G}^k P$ ,  $k \leq l$ .

□

*Lemma 6.2.5.* Because  $\text{Cobar}(\mathcal{C})$  is freely generated by  $sP$ , the derivations on  $\text{Cobar}(\mathcal{C})$  are in bijection with elements of  $\text{Conv}(P, \text{Cobar}(\mathcal{C}))$  via restriction of the domain. We will construct an element of this latter set, defining it on each graded level recursively, with the property that the image is in a lower syzygy degree (considering  $sP$  as trees with one node decorated by elements of  $sP$ ), and hence the corresponding derivation is locally nilpotent. We write  $\partial_{\text{Der}}$  for the differential on  $\text{Conv}(P, \text{Cobar}(\mathcal{C}))$  corresponding to that on  $\text{Der}(\text{Cobar}(\mathcal{C}))$  (i.e.  $\partial_{\text{Der}} = \partial_{\text{Cobar}} + \partial_{\mathcal{C}} + [\alpha_{\mathbb{1}}, ]$  where  $\alpha_{\mathbb{1}}$  is the MC element corresponding to the identity map). We will also use the operator  $c = \partial_{\mathcal{C}} + [\alpha_{\mathbb{1}}, ]$ , and a chosen splitting  $h$  of  $\partial: \text{Cobar}(\mathcal{C}) \rightarrow Z(\text{Cobar}(\mathcal{C}))$  (where  $Z$  is used to indicate cocycles) guaranteed to exist because we are working over a field.

Because  $\rho(\mathfrak{sG}^1 P)$  generates  $\mathcal{H}$ , we can choose  $\Psi_k: \mathfrak{G}^k P \rightarrow \text{Cobar}(\mathcal{C})$  with  $\rho \circ \Psi_k = \psi$ , and with the image of  $\Psi_k$  in syzygy degree 0. We let  $\Psi^{(k)} \in \text{Conv}(P, \text{Cobar}(\mathcal{C}))$  be defined by

$$\Psi^{(k)}(X) = \begin{cases} 0 & X \in \mathfrak{sG}^l P, l \neq k \\ \Psi_k(X), & X \in \mathfrak{sG}^k P. \end{cases} \quad (6.3.8)$$

$\Psi_k$  takes vectors in syzygy degree  $k - 1$  to syzygy degree 0, and so  $\Psi^{(k)}$  decreases syzygy degree by  $k - 1$ . It also satisfies

$$\partial_{\text{Der}}(\Psi^{(k)})|_{\mathfrak{G}^{\leq k} P} = 0 \quad (6.3.9)$$

and

$$\text{Im}(c(\Psi^{(k)}))|_{\mathfrak{G}^{k+1} P} \subset Z(\text{Cobar}(\mathcal{C})). \quad (6.3.10)$$

Now, suppose for some  $m \geq k$  we have

$$\Psi^{(m)} = \Psi_k + \Psi_{k+1} + \dots + \Psi_m \in \text{Conv}(P, \text{Cobar}(\mathcal{C}))$$

which decreases syzygy degree by  $k - 1$ , satisfying

$$\partial_{\text{Der}}(\Psi^{(m)})|_{\mathcal{G}^{\leq m}P} = 0 \quad (6.3.11)$$

and

$$\text{Im}(c(\Psi^{(m)}))|_{\mathcal{G}^{m+1}P} \subset Z(\text{Cobar}(\mathcal{C})). \quad (6.3.12)$$

Then we can define  $\Psi_{m+1}: \mathcal{G}^{m+1}P \rightarrow \text{Cobar}(\mathcal{C})$  by  $\Psi_{m+1} := -h(c(\Psi^{(m)}))|_{\mathcal{G}^{m+1}P}$ . We then see that, because  $h$  increases syzygy degree by 1 and  $c(f)$  decreases syzygy degree by one more than  $f$  for any  $f$ ,  $\Psi_{m+1}$  decreases syzygy degree by  $k - 1$  because  $\Psi^{(m)}$  does. Because  $h$  is a splitting of  $\partial_{\text{Cobar}}$ , we get that

$$\partial_{\text{Der}}(\Psi^{(m+1)})|_{\mathcal{G}^{\leq m+1}P} = 0 \quad (6.3.13)$$

we also compute

$$\partial_{\text{Cobar}}(c(\Psi^{(m+1)}))|_{\mathcal{G}^{m+2}P} = 0 \quad (6.3.14)$$

using this and the MC equation for  $\alpha_1$ , we compute that

$$\text{Im}(c(\Psi^{(m+1)}))|_{\mathcal{G}^{m+2}P} \subset Z(\text{Cobar}(\mathcal{C})). \quad (6.3.15)$$

Thus, continuing recursively, we build  $\Psi = \Psi_k + \Psi_{k+1} + \dots$  as desired. Because  $\rho|_{\mathcal{G}^{>1}P} = 0$ , and  $\mathfrak{S}(\Psi_{k+i})$  is in syzygy degree  $> 0$  (it takes vectors in syzygy degree  $k + i - 1$  to syzygy degree  $i > 0$ ),  $\rho \circ \Psi^{(m)} = \psi$ .  $\square$

*Theorem 6.2.1.* First use Lemma 6.2.4 repeatedly to replace  $\beta \mapsto \tilde{\beta}$  with  $\tilde{\beta}_k = \alpha_k$  for  $k > n$ . We want to find  $\tilde{\alpha} := \tilde{\alpha}_1 + \tilde{\alpha}_2 + \dots + \tilde{\alpha}_{n+1}$ . Thus, we need

$$\text{Curv}(\tilde{\alpha})(X) = 0 \quad (6.3.16)$$

for all  $X \in \mathcal{G}^k P$ ,  $k \leq n+1$ . Because we require the first  $n-1$  terms to be  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ , this is already satisfied for  $k \leq n-1$ . For  $k = n$  we get the equation

$$\partial_{\mathcal{O}} \tilde{\alpha}_n(X) + \alpha_{n-1}(\partial_{\mathcal{C}} X) + \frac{1}{2} \sum_{i+j=n} [\alpha_i, \alpha_j](X) = 0. \quad (6.3.17)$$

For  $k = n+1$  we get

$$\partial_{\mathcal{O}} \tilde{\alpha}_{n+1}(X) + \tilde{\alpha}_n(\partial_{\mathcal{C}} X) + [\alpha_1, \tilde{\alpha}_n](X) + \frac{1}{2} \sum_{\substack{i+j=n+1 \\ i,j < n}} [\alpha_i, \alpha_j](X) = 0 \quad (6.3.18)$$

and so equations (6.3.17) and (6.3.18) form a linear system of equations in  $\tilde{\alpha}_n$ , and  $\tilde{\alpha}_{n+1}$  which has a solution over  $\mathbb{K}$  given by  $\tilde{\beta}_n, \tilde{\beta}_{n+1}$ . By the finite dimensionality assumptions in **A1** and **A2**, the existence of a solution over  $\mathbb{K}$  implies the existence of a solution over  $\mathbb{Q}$ .  $\square$

## 6.4 Gerstenhaber algebras

We set  $\mathcal{H} = \text{Ger}$ , the operad governing Gerstenhaber algebras,  $\mathcal{O} = \text{Br}$  the braces operad, and  $\mathcal{C} = \text{Ger}^i$ . The differential on  $\text{Ger}^i$  is 0, and so we use the grading given by arity. Noting that

$$\text{Conv}(\text{Ger}_o^i, \text{Br}) = \prod_{n>1} \text{Br}(n) \otimes s^{2-2n} \text{Ger}(n), \quad (6.4.1)$$

there is a MC element  $\beta \in \text{Conv}(\text{Ger}_o^i, \text{Br})$  representing a quasi-isomorphism which begins with the following in  $\text{Br}(2) \otimes \text{Ger}(2)$

$$\alpha_2 = \left( \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} + \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{2} \end{array} \right) \otimes b_1 b_2 + \begin{array}{c} \textcircled{1} \quad \textcircled{2} \\ \diagdown \quad / \\ \bullet \\ | \end{array} \otimes \{b_1, b_2\}$$

where  $\{b_1, b_2\}$  and  $b_1 b_2$  are respectively the bracket and product in  $\text{Ger}$ .

We consider this element as a 1st sprout in  $\text{Conv}(\text{Ger}_o^i, \text{Br})$ , and use Proposition 6.2.2 to extend to a 2nd sprout. Then we can recursively construct a MC element representing a

quasi-isomorphism  $\text{Cobar}(\text{Ger}^i) \rightarrow \text{Br}$ . In Figure 6.2, we show an element  $\alpha_3 \in \text{Br}(3) \otimes \text{Ger}(3)$  which, when added to  $\alpha_2$  gives a 2nd sprout which can be extended to a 3rd sprout.  $\alpha_3$  computed using a python implementation of the algorithm outlined in this section.

$$\begin{aligned}
 \alpha_3 = & \frac{1}{2} \begin{array}{c} \textcircled{2} \quad \textcircled{3} \\ \diagdown \quad / \\ \textcircled{1} \\ | \end{array} \otimes b_1\{b_2, b_3\} - \frac{1}{3} \begin{array}{c} \textcircled{1} \textcircled{2} \textcircled{3} \\ | \quad | \quad | \\ \bullet \\ | \end{array} \otimes \{b_1, \{b_2, b_3\}\} \\
 & - \frac{1}{6} \begin{array}{c} \textcircled{1} \textcircled{2} \textcircled{3} \\ | \quad | \quad | \\ \bullet \\ | \end{array} \otimes \{b_2, \{b_1, b_3\}\} - \frac{1}{12} \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \quad \textcircled{3} \\ \diagdown \quad / \\ \bullet \\ | \end{array} \otimes \{b_1, \{b_2, b_3\}\} \\
 & - \frac{1}{12} \begin{array}{c} \textcircled{3} \\ | \\ \textcircled{1} \quad \textcircled{2} \\ \diagdown \quad / \\ \bullet \\ | \end{array} \otimes \{b_2, \{b_1, b_3\}\}
 \end{aligned}$$

Figure 6.2:  $\alpha_2 + \alpha_3$  is a 2nd MC sprout which can be extended to a 3rd MC sprout.



# CHAPTER 7

## THE OPERAD GOVERNING

### CALCULI

#### 7.1 Definitions of the calculi operad and related operads

In this section we define the 2-colored operad  $\text{calc}$ , a related operad  $\text{calc}^\diamond$  and homogenized versions thereof. We will use  $\{\mathfrak{c}, \mathfrak{a}\}$  as the set of colors with the order  $\mathfrak{c} < \mathfrak{a}$ .  $\text{Ger}_n$  will be the free Gerstenhaber algebra on the symbols  $\{v_1, \dots, v_n\}$  given degree 0. We begin by defining an auxiliary associative algebra.

**Definition 7.1.1.** For a Gerstenhaber algebra  $V$ , let  $\mathcal{Y}(V)$  be the associative algebra generated by the symbols  $\delta, l_v, i_v$ , for  $v \in V$ , with degrees  $|l_v| = |v| - 1$ ,  $|i_v| = |v|$  and  $|\delta| = -1$ , with relations

$$i_{vw} = i_v i_w \tag{7.1.1}$$

$$l_{\{v,w\}} = -(-1)^{|v|} l_v l_w - (-1)^{(|v|+1)|w|} l_w l_v \tag{7.1.2}$$

$$i_{\{v,w\}} = l_v i_w - (-1)^{|w|(|v|+1)} i_w l_v \tag{7.1.3}$$

$$l_{vw} = l_v i_w + (-1)^{|v|} i_v l_w \tag{7.1.4}$$

$$\delta^2 = 0 \tag{7.1.5}$$

$$\delta \circ i_v - (-1)^{|v|} i_v \circ \delta = l_v \tag{7.1.6}$$

for all  $v, w \in V$ .

We write  $\mathcal{Y}_n$  for the subspace of  $\mathcal{Y}(\text{Ger}_n)$  spanned by monomials where each of  $\{v_1, \dots, v_n\}$  appear exactly once. We define the underlying collection of  $\text{calc}$  via the following:

$$\begin{aligned} \text{calc}^c(n, 0) &= \text{Ger}(n) \\ \text{calc}^c(n, k) &= 0 && \text{if } k \neq 0 \\ \text{calc}^a(n, k) &= 0 && \text{if } k \neq 1 \\ \text{calc}^a(n, 1) &= \mathcal{Y}_n && \text{if } n \neq 0 \\ \text{calc}^a(0, 1) &= \mathcal{Y}_0 \oplus \mathbb{K} \end{aligned}$$

Composition in the color  $c$  only will just be the composition coming from  $\text{Ger}$ . Compositions of the form  $\text{calc}^a(n, 1) \otimes \text{calc}^c(m, 0) \rightarrow \text{calc}^a(m+n, 1)$  will come from insertions of polynomials with relabellings coming from  $r_i: \{1, \dots, n\} \uplus \{1, \dots, m\} \xrightarrow{\cong} \{1, \dots, m+n\}$  given by

$$r_i(k) = \begin{cases} k & \text{if } k \in \{1, \dots, n\}, k < i \\ k + m & \text{if } k \in \{1, \dots, n\}, k \geq i \\ k + i - 1 & \text{if } k \in \{1, \dots, m\}. \end{cases} \tag{7.1.7}$$

Compositions of the form  $\text{calc}^a(m, 1) \otimes \text{calc}^a(n, 1) \rightarrow \text{calc}^a(m+n, 1)$  will come from the map  $\mathcal{Y}_m \otimes \mathcal{Y}_n \rightarrow \mathcal{Y}_{m+n}$  coming from multiplication in  $\mathcal{Y}(\text{Ger}_{m+n})$  via the relabelling  $r: \{1, \dots, m\} \uplus \{1, \dots, n\} \xrightarrow{\cong} \{1, \dots, m+n\}$  given by

$$r(k) = \begin{cases} k & \text{if } k \in \{1, \dots, n\} \\ k + n & \text{if } k \in \{1, \dots, m\}. \end{cases} \quad (7.1.8)$$

Algebras over the operad  $\text{calc}$  are called *calculi*.

A related operad will be called  $\text{calc}^\diamond$  and will be the same as  $\text{calc}$  except that  $\delta$  will be replaced by  $u$  with  $|u| = 2$ , relations (7.1.5) and (7.1.6) will be replaced by

$$ui_v = i_v u \quad (7.1.9)$$

$$ul_v = l_v u \quad (7.1.10)$$

for all  $v \in V$ , and we will add a differential  $\partial^\diamond$ . We will write  $\mathcal{Y}^u(V)$  for the associative algebra with  $u$  replacing  $\delta$  and the differential  $\partial^\diamond$  given on generators as

$$\partial^\diamond(i_v) = ul_v \quad (7.1.11)$$

$$\partial^\diamond(l_v) = 0 \quad (7.1.12)$$

$$\partial^\diamond(u) = 0 \quad (7.1.13)$$

for all  $v \in V$ .

**Proposition 7.1.1.** *The equations (7.1.11), (7.1.12) and (7.1.13) define a differential on  $\mathcal{Y}^u(V)$  by extending as a derivation with respect to multiplication.*

*Proof.* Since the algebra  $\mathcal{Y}^u(V)$  is generated by  $i_v$ ,  $l_v$ , and  $u$ , our goal is to show that the ideal generated by the relations is closed with respect to  $\partial^\diamond$ .

For the relation  $i_{v_1 v_2} - i_{v_1} i_{v_2}$  we have

$$\begin{aligned} \partial^\diamond(i_{v_1 v_2} - i_{v_1} i_{v_2}) &= ul_{v_1 v_2} - ul_{v_1} i_{v_2} - (-1)^{|v_1|} i_{v_1} ul_{v_2} = \\ &u(l_{v_1 v_2} - l_{v_1} i_{v_2} - (-1)^{|v_1|} i_{v_1} l_{v_2}) - (-1)^{|v_1|} (i_{v_1} u - ui_{v_1}) l_{v_2}. \end{aligned}$$

The expression  $\partial^\diamond(l_{v_1 v_2} - l_{v_1} i_{v_2} - (-1)^{|v_1|} i_{v_1} l_{v_2})$  can be rewritten as

$$\partial^\diamond(l_{v_1 v_2} - l_{v_1} i_{v_2} - (-1)^{|v_1|} i_{v_1} l_{v_2}) = (-1)^{|v_1|} l_{v_1} u l_{v_2} - (-1)^{|v_1|} u l_{v_1} l_{v_2} =$$

$$-(-1)^{|v_1|}(ul_{v_1} - l_{v_1}u)l_{v_2}.$$

For the relation  $i_{\{v_1, v_2\}} - l_{v_1}i_{v_2} + (-1)^{|v_2|(|v_1|+1)}i_{v_2}l_{v_1}$  we have

$$\begin{aligned} \partial^\diamond(i_{\{v_1, v_2\}} - l_{v_1}i_{v_2} + (-1)^{|v_2|(|v_1|+1)}i_{v_2}l_{v_1}) &= ul_{\{v_1, v_2\}} + (-1)^{|v_1|}l_{v_1}ul_{v_2} + (-1)^{|v_2|(|v_1|+1)}ul_{v_2}l_{v_1} = \\ &u(l_{\{v_1, v_2\}} + (-1)^{|v_1|}l_{v_1}l_{v_2} + (-1)^{|v_2|(|v_1|+1)}l_{v_2}l_{v_1}) - (-1)^{|v_1|}(ul_{v_1} - l_{v_1}u)l_{v_2}. \end{aligned}$$

For the remaining relations, the verification is straightforward.  $\square$

Thus  $\partial^\diamond$  defines a differential on  $\text{calc}^\diamond$ .

The goal will be to show that the map  $\text{Cobar}(\text{calc}^{\diamond*}) \rightarrow \text{calc}$  given as the MC element  $v_1v_2 \otimes \{v_1, v_2\} + \{v_1, v_2\} \otimes v_1v_2 + l_{v_1} \otimes i_{v_1} + i_{v_1} \otimes l_{v_1} + \delta \otimes u$  in

$$\text{Conv}((\text{calc}^\diamond)_\circ, \text{calc}) = \left( \prod_{k=0}^{\infty} \text{calc}(k) \otimes (\text{calc}^\diamond)_\circ(k) \right)^{S_k} \quad (7.1.14)$$

is a quasi-isomorphism. However, we can replace these by the associated graded operads with respect to the weight filtration with the first level of the filtration spanned (as a vector space) by  $\mathbb{1}, v_1v_1, \{v_1, v_2\}, i_{v_1}, l_{v_1}$  and either  $\delta$  or  $u$ . We call these the homogenized version of the operads and write them as  $\text{calc}_h$  and  $\text{calc}^\diamond_h$ .

Concretely, to go from  $\text{calc}$  to  $\text{calc}_h$  we replace relation (7.1.6) with the relations:

$$\delta i_v = (-1)^{|v|}i_v\delta \quad (7.1.15)$$

$$\delta l_v = -(-1)^{|v|}l_v\delta \quad (7.1.16)$$

Note that we can define a surjective map  $\mathbb{O}\mathbb{P}(V) \rightarrow \text{calc}_h$  where  $V$  is the graded collection generated by  $\mu, \beta \in V^c(2, 0), l, i \in V^a(1, 1), \delta \in V^a(0, 1)$ , where the  $S_2$  action on  $\mu$  and  $\beta$  is trivial, by sending each generator to its representative in  $\text{calc}_h$ :

$$\mu \mapsto v_1v_2$$

$$\beta \mapsto \{v_1, v_2\}$$

$$l \mapsto l_{v_1}$$

$$\begin{aligned}
i &\mapsto i_{v_1} \\
\delta &\mapsto \delta
\end{aligned}$$

The kernel,  $R$ , of this map consists of linear combinations of trees with two nodes, i.e., it consists of homogeneous quadratic relations. This gives a generators and relations presentation of  $\text{calc}_h$ .

To go from  $\text{calc}^\diamond$  to  $\text{calc}^\diamond_h$ , we replace the differential by the 0 differential.

## 7.2 Some calculi

The pair  $(\text{HH}^\bullet(A), \text{HH}_\bullet(A))$  form a calculi for  $A$  a unital commutative algebra with the differential defined as in Section 4.2, and with contraction and the Lie derivative as  $i_v$  and  $l_v$  respectively. I.e. for  $f \in \text{HH}^n(A)$ , and  $a = a_0 \otimes \dots \otimes a_m \in \text{HH}_m(A)$ ,  $m \geq n$

$$i_f(a) = a_0 f(a_1 \otimes \dots \otimes a_n) \otimes a_{n+1} \otimes \dots \otimes a_m \quad (7.2.1)$$

and

$$\begin{aligned}
l_f(a) &= \sum_i \pm a_0 \otimes \dots \otimes a_i \otimes f(a_{i+1} \otimes \dots) \otimes a_{i+n+1} \otimes \dots + \\
&+ \sum_j \pm f(a_{m-j+1} \otimes \dots \otimes a_m \otimes a_0 \otimes \dots) \otimes \dots \otimes a_{m-j}
\end{aligned} \quad (7.2.2)$$

where signs are determined by usual Koszul sign rules. Direct computation shows that relations 7.1.1, 7.1.2, 7.1.3, 7.1.4, 7.1.5 and 7.1.6 hold.

Another calculi is given by the pair  $(\Lambda_A \text{Der}(A), \Omega_{A/\mathbb{K}}^*)$ . An extension of the Hochschild-Kostant-Rosenberg isomorphism, given on  $\text{HH}_\bullet(A) \rightarrow \Omega_{A/\mathbb{K}}^*$  by

$$a_0 \otimes \dots \otimes a_n \mapsto \frac{1}{n!} a_0 da_1 \dots da_n \quad (7.2.3)$$

gives an isomorphism of calculi for  $A = SV$ .

### 7.3 Computing the Koszul Dual

In this section we aim to show the following proposition

**Proposition 7.3.1.**  $(\text{calc}_h)^! \cong \text{calc}_h^\diamond$  via an isomorphism of operads given in  $\text{Conv}((\text{calc}_h)^!, \text{calc}_h^\diamond)$  by

$$v_1, v_2 \otimes s\mu + v_1 v_2 \otimes s\beta + i_{v_1} \otimes sl + l_{v_1} \otimes si + u \otimes \delta$$

where  $\{\mu, \beta, l, i, \delta\}$  are generators in the presentation of  $\text{calc}_h$  as discussed in Section 7.1.

*Proof.* We are in the setting of the Section 3.2, with  $V$  given as in Section 7.1, and  $R$  defined by the basis given by the elements:

$$\mu \circ_1 \mu - \sigma(\mu \circ_1 \mu) \tag{7.3.1}$$

$$\sigma(\mu \circ_1 \mu) - \sigma^2(\mu \circ_1 \mu) \tag{7.3.2}$$

$$\beta \circ_1 \beta + \sigma(\beta \circ_1 \beta) + \sigma^2(\beta \circ_1 \beta) \tag{7.3.3}$$

$$\beta \circ_1 \mu - \sigma(\mu \circ_1 \beta) + \sigma^2(\mu \circ_1 \beta) \tag{7.3.4}$$

$$\sigma(\beta \circ_1 \mu) - \sigma^2(\mu \circ_1 \beta) + \mu \circ_1 \beta \tag{7.3.5}$$

$$\sigma^2(\beta \circ_1 \mu) - \mu \circ_1 \beta + \sigma(\mu \circ_1 \beta) \tag{7.3.6}$$

$$i \circ i - i \circ \mu \tag{7.3.7}$$

$$\tau(i \circ i) - i \circ \mu \tag{7.3.8}$$

$$l \circ \beta + l \circ l + \tau(l \circ l) \tag{7.3.9}$$

$$l \circ \mu - l \circ i - i \circ l \tag{7.3.10}$$

$$l \circ \beta - l \circ i + \tau(i \circ l) \tag{7.3.11}$$

$$\tau(l \circ i) + \tau(i \circ l) - l \circ i - i \circ l \tag{7.3.12}$$

$$i \circ \delta - \delta \circ i \tag{7.3.13}$$

$$l \circ \delta + \delta \circ l \quad (7.3.14)$$

$$\delta \circ \delta \quad (7.3.15)$$

where the compositions are taken in the free operad  $\mathbb{O}\mathbb{P}(V)$ , and  $\sigma$  is the element of  $S_3$  sending  $k \mapsto k + 1 \pmod{3}$ , and  $\tau$  is the nontrivial element of  $S_2$ .

We simplify the computation of  $R^\perp$  by decomposing  $\mathbb{T}^2(V)$ , the weight 2 part of  $\text{calc}_h$ , and the projection between them as shown below:

$$\begin{array}{ccccccccc} \mathbb{T}^2(V) & = & T_{Com} & \oplus & T_{Lie} & \oplus & T_{Ger} & \oplus & T_{ComRep} & \oplus & T_{LieRep} \\ \downarrow \pi & = & \downarrow \pi_{Com} & \oplus & \downarrow \pi_{Lie} & \oplus & \downarrow \pi_{Ger} & \oplus & \downarrow \pi_{ComRep} & \oplus & \downarrow \pi_{LieRep} \\ \text{calc}_h^2 & = & L_{Com} & \oplus & L_{Lie} & \oplus & L_{Ger} & \oplus & L_{ComRep} & \oplus & L_{LieRep} \\ & & & & & & \oplus & T_{GerRep} & \oplus & T_\delta & \oplus & T_{\delta^2} \\ & & & & & & \oplus & \downarrow \pi_{GerRep} & \oplus & \downarrow \pi_\delta & \oplus & \downarrow 0 \\ & & & & & & \oplus & L_{GerRep} & \oplus & L_\delta & \oplus & 0 \end{array}$$

The following table gives ordered bases of the vector spaces in the decomposition above, as well as the matrix representing the map in those bases.

From here we compute  $R^\perp$  directly as the subcollection of  $V^*$  spanned by the following, where we use the dual basis  $\{\mu^*, \beta^*, l^*, i^*, \delta^*\}$  to the basis  $\{\mu, \beta, l, i, \delta\}$  of  $V$ . Letting  $W$  be the collection spanned by  $\{v_1 v_2, \{v_1, v_2\}, l_{v_1}, i_{v_1}, u\}$ , which generates  $\text{calc}_h^\diamond$ , the isomorphism  $s^{-1}V^* \rightarrow W$  given in  $W \otimes sV$  as

$$v_1, v_2 \otimes s\mu + v_1 v_2 \otimes s\beta + i_{v_1} \otimes sl + l_{v_1} \otimes si + u \otimes \delta$$

defines an isomorphism  $\mathbb{O}\mathbb{P}(s^{-1}V^*) \xrightarrow{\sim} \mathbb{O}\mathbb{P}(W)$  which takes  $R^\perp$  to the relations of  $\text{calc}_h^\diamond$ . Thus it descends to an isomorphism  $(\text{calc}_h)^\dagger \xrightarrow{\sim} \text{calc}_h^\diamond$ .  $\square$

Label	Com	Lie	Ger	ComRep	LieRep	GerRep	$\delta$	$\delta^2$
$T_{\text{Label}}$	$\mu \circ_1 \mu,$ $(\mu \circ_1 \mu)^\sigma,$ $(\mu \circ_1 \mu)^{\sigma^2}$	$\beta \circ_1 \beta,$ $(\beta \circ_1 \beta)^\sigma,$ $(\beta \circ_1 \beta)^{\sigma^2}$	$\mu \circ_1 \beta, (\mu \circ_1 \beta)^\sigma,$ $(\mu \circ_1 \beta)^{\sigma^2}, \beta \circ_1 \mu,$ $(\beta \circ_1 \mu)^\sigma, (\beta \circ_1 \mu)^{\sigma^2}$	$i \circ \mu,$ $i \circ i,$ $(i \circ i)^\tau$	$l \circ \beta,$ $l \circ l,$ $(l \circ l)^\tau$	$l \circ \mu, l \circ i,$ $(l \circ i)^\tau, i \circ \beta$ $i \circ l, (i \circ l)^\tau$	$\delta \circ l, l \circ \delta,$ $\delta \circ i, i \circ \delta$	$\delta \circ \delta$
$L_{\text{Label}}$	$v_1 v_2 v_3$	$\{v_1, v_2\}, v_3\},$ $\{v_2, v_3\}, v_1\}$	$\{v_1, v_2\} v_3,$ $\{v_2, v_3\} v_1,$ $\{v_3, v_1\} v_2$	$i_{v_1 v_2}$	$l_{v_1} l_{v_2},$ $l_{v_2} l_{v_1}$	$l_{v_1} i_{v_2},$ $i_{v_1} l_{v_2},$ $i_{v_2} l_{v_1}$	$\delta l_{v_1},$ $\delta i_{v_1}$	$\emptyset$
$\pi_{\text{Label}}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	0

Table 7.1: Computation of  $\pi : \mathbb{T}^2(V) \rightarrow \text{calc}_h^2$  on a direct sum decomposition.



## 7.4 A resolution of the calculi operad

The theorem we will prove is:

**Theorem 7.4.1.** *The map*

$$\text{Cobar}(\text{calc}^{\diamond*}) \rightarrow \text{calc}$$

given by

$$v_1 v_2 \otimes \{v_1, v_2\} + \{v_1, v_2\} \otimes v_1 v_2 + l_{v_1} \otimes i_{v_1} + i_{v_1} \otimes l_{v_1} + \delta \otimes u$$

in  $\text{Conv}(\text{calc}^{\diamond\circ}, \text{calc})$  is a quasi-isomorphism.

By the standard spectral sequence argument, we can replace these by their homogenized versions. Then, by the preceding sections, all that is left to show is that  $\text{calc}_h$  is Koszul. To do so, we will first analyze  $\text{calc}_h$  via the suboperads:  $\text{LieMod}$  generated by  $l_{v_1}, \{v_1, v_2\}$ ,  $\text{ComMod}$  generated by  $i_{v_1}, v_1 v_2$  and  $\mathcal{D}$  generated by  $\delta$ . Also, we write  $\mathcal{U}$  for the suboperad of  $\text{calc}^{\diamond}$  generated by  $u$ .

**Proposition 7.4.2.** *We have the following isomorphisms of collections*

1.  $\text{calc}_h \cong \text{ComMod} \circ \text{LieMod} \circ \mathcal{D}$ .
2.  $\text{calc}_h^i \cong \mathcal{D}^i \circ \text{LieMod}^i \circ \text{ComMod}^i$

*Proof.* For **(a)**  $\text{calc}_h^c \cong \text{Ger} \cong \text{Com} \circ \text{Lie} \cong \text{ComMod}^c \circ \text{LieMod}^c \circ \mathcal{D}^c$  ( $\mathcal{D}^c$  only contains multiples of the identity). For  $\text{calc}_h^a$  we can rewrite relation (7.1.3) as

$$l_v i_w = i_{\{v,w\}} + (-1)^{|w|(|v|+1)} i_w l_v \tag{7.4.1}$$

and then substituting this into relation (7.1.4) we also get

$$l_{vw} = i_{\{v,w\}} + (-1)^{|w|(|v|+1)} i_w l_v + (-1)_v^i l_w. \tag{7.4.2}$$

Thus, using these two equations together with the fact for  $\text{calc}_h^c$ , we can commute all elements of  $\text{ComMod}$  past those of  $\text{LieMod}$ . **(b)** follows the exact same argument as **(a)** by considering  $\text{calc}_h^!$ , replacing relation (7.1.3) and relation (7.1.4) with their analogs, and then taking duals.  $\square$

Furthermore,

**Proposition 7.4.3.** *The operads  $\text{ComMod}$ ,  $\text{LieMod}$ ,  $\mathcal{D}$  and  $\mathcal{U}$  are Koszul.*

*Proof.* By duality (see Lemma 3.2.5), we can just prove this for  $\mathcal{D}$  and  $\text{ComMod}$ , as the computation in Section 7.3 tells us (in particular) that  $\mathcal{D}^! = \mathcal{U}$  and  $\text{ComMod}^! = \text{LieMod}$ .

For  $\mathcal{D}$ , there is nothing to prove, as  $\text{Bar}(\mathcal{D})$  is already entirely contained in syzygy degree 0.

For  $\text{ComMod}$ , in color  $c$ , this is the same as Koszulity of  $\text{Com}$ , which is well-known (see [27]). In color  $a$ , first note that we can write any non-identity element as  $i_\alpha$  for  $\alpha$  in  $\text{Com}$ . Filtering the  $a$  part of the Koszul complex by  $i$ -weight in  $\text{ComMod}^i$  (viewed as a suboperad of  $\text{Bar}(\text{ComMod})$ ) we get a spectral sequence with

$$E^0 = (\mathbb{K}1^a \otimes (\text{ComMod}^i)^a) \oplus (\mathbb{K}i \otimes (\text{ComMod}^i)^a \otimes (\text{Com} \circ \text{Com}^i)) \quad (7.4.3)$$

where the first summand comes from the subspace containing only the color  $a$  identity in  $\text{ComMod}$  and has the 0 differential and on the second term the differential acts as the differential on the Koszul complex of  $\text{Com}$ . By the Koszulity of  $\text{Com}$ , the Koszul complex goes to the identity when we take cohomology, giving

$$E^1 = \mathbb{K}1^a \otimes (\text{ComMod}^i)^a \oplus \mathbb{K}i \otimes (\text{ComMod}^i)^a \quad (7.4.4)$$

$$\cong \mathbb{K}1^a \otimes \mathbb{K}1^a \oplus \mathbb{K}1 \otimes (\text{ComMod}^i)_\circ^a \oplus \mathbb{K}i \otimes (\text{ComMod}^i)^a \quad (7.4.5)$$

where  $(\text{ComMod}^i)_\circ$  is the cokernel of the coaugmentation, and the differential takes elements in the second summand to the third, so the cohomology is the identity, as desired.  $\square$

Finally,

**Theorem 7.4.4.**  $\text{calc}_h$  is Koszul.

*Proof.* We write the Koszul complex as

$$\text{ComMod} \circ \text{LieMod} \circ \mathcal{D} \circ \mathcal{D}^i \circ \text{LieMod}^i \circ \text{ComMod}^i. \quad (7.4.6)$$

Viewing  $\text{ComMod}^i$  and  $\text{LieMod}^i$  as suboperads of  $\text{Bar}(\text{calc}_h)$ , we get we filter by the sum of  $\text{ComMod}$ -weight and  $\text{LieMod}$ -weight, and the differential on the associated graded is simply that of the Koszul complex of  $\mathcal{D}$ , by Proposition 7.4.3 giving us cohomology equal to

$$\text{ComMod} \circ \text{LieMod} \circ \text{LieMod}^i \circ \text{ComMod}^i. \quad (7.4.7)$$

We then filter similarly by  $\text{ComMod}$ -weight in  $\text{ComMod}^i$ , giving the Koszul complex of  $\text{ComMod}$  after taking cohomology (again by Proposition 7.4.3, and then finally we use Proposition 7.4.3) directly to give that the total cohomology is the identity.

□

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