HARMONIC ANALYSIS METHODS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS IN UNIFORMLY RECTIFIABLE AND INFINITESIMALLY FLAT AHLFORS REGULAR DOMAINS

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ABSTRACT

The work in this Ph.D. thesis lies at the intersection of Harmonic Analysis, Partial Differential Equations, and Geometric Measure Theory. It focuses on the study of boundary integral operators associated with elliptic boundary value problems on non-smooth domains via singular integral methods. The results are joint work with Dorina Mitrea (Baylor University), Irina Mitrea, and Marius Mitrea (Baylor University) (cf. [2], [3], and [4]), and the thesis is structured around four interconnected themes:

A. *Overdetermined boundary value problems for the Laplacian on uniformly rectifiable domains*;

B. *Overdetermined boundary value problems for second-order, homogeneous, constant complex coefficient elliptic systems on uniformly rectifiable domains*;

C. *Overdetermined boundary value problems for the Stokes system of linearized hydrostatics on uniformly rectifiable domains*;

D. *Fredholm Theory for the Dirichlet boundary value problem for Δ^3 on infinitesimally flat Ahlfors regular domains in $\mathbb{R}^n$.*

For themes **A** and **B**, this dissertation contains a well-posedness theory for overdetermined boundary value problems in bounded uniformly rectifiable domains with boundary data in L^p-based function spaces. A key ingredient to obtaining such results is introducing a Cauchy-like singular integral operator adapted to second-order homogeneous elliptic systems and the overdetermined setting. See also [2].

For theme **C**, the well-posedness theory for the overdetermined boundary problem for the Stokes system of linearized hydrostatics requires the introduction of a new L^p-based *divergence-free* Lebesgue-Whitney function space, along with a pair of Cauchy-like integral operators specifically adapted to the Stokes system. Additionally, this work contains a connection between the Cauchy integrals for the Stokes system of linearized hydrostatics and those for the Lamé system of linearized elastostatics as one of the Lamé parameters tends to infinity. See also [3].

Finally, for theme **D**, this dissertation contains a *distinguished* coefficient tensor for the polyharmonic operator Δ^3 in all dimensions of the Euclidean space. It also provides an argument showing that the associated singular integral operator for the Dirichlet problem for Δ^3 in infinitesimally flat Ahlfors regular domains is compact on L^p Lebesgue-Whitney function spaces for all $p \in (1, \infty)$, thus opening the door for the employment of Fredholm Theory for the solvability of the Dirichlet Problem. All developments presented are from ongoing joint in [4].

To my family,

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CHAPTER 1

INTRODUCTION

Elliptic boundary value problems arise naturally in modeling a wide range of physical phenomena, including electrostatics, elasticity, steady-state incompressible fluid flow, and electromagnetism. A powerful tool for treating such problems is the method of boundary layer potential, through which matters are reduced to solving a boundary integral equation involving a singular integral operator naturally associated with the domain, and the underlying partial differential equation. This method does not rely on positivity and is suitable for systems of equations.

For the Laplacian, E. Fabes, M. Jodeit, and N. Rivière in [14] established well-posedness results for the Dirichlet and Neumann problems in bounded C^1 domains with boundary data in L^p, for any $p \in (1, \infty)$, and G. Verchota in [34] proved well-posedness results for the Dirichlet and Neumann problems in arbitrary bounded Lipschitz domains with L^2 boundary data. Subsequently, B. Dahlberg and C. Kenig in [10] extended these results to boundary data in L^p, with the range of p depending on the Lipschitz character of the domain. Moving on to second-order systems, the Dirichlet problem for the Lamé system of elastostatics in arbitrary Lipschitz domains with boundary data in L^2 was first treated in [12] by B. Dahlberg, C. Kenig and G. Verchota and the Stokes system of linearized hydrostatics was treated in [15] by E. Fabes, C. Kenig and G. Verchota. Much more is known today through the work of R. Brown, D. Jerison, S. Mayboroda, J. Pipher, Z. Shen, see for instance [7],

[20], [27], [32]. For a comprehensive treatise on solvability and well-posedness results of boundary value problems for second-order elliptic systems in uniformly rectifiable domains, see the Geometric Harmonic Analysis series [22]-[26] by D. Mitrea, I. Mitrea, and M. Mitrea, and the references therein.

In addition to Dirichlet and Neumann problems, another physically relevant class of boundary value problems is the overdetermined boundary value problem. Loosely speaking, an overdetermined boundary value problem is formulated as the task of finding a function that solves a certain partial differential equation in the interior of a given region, and this function and all its first-order partial derivatives take prescribed, compatible values at the boundary. A concrete example of an overdetermined boundary value problem is the task of finding the electrostatic potential u within a charge-free region $\Omega \subset \mathbb{R}^3$ with prescribed electrostatic potential and electric field at the boundary. In this setting, the electrostatic potential satisfies the partial differential equation (cf. J. D. Jackson [19])

$$\Delta u = 0 \quad \text{in } \Omega, \tag{1.1}$$

while the boundary conditions mentioned above are equivalent to

$$\text{prescribing both } \ u\big|_{\partial\Omega} \ \text{ and } \ (\nabla u)\big|_{\partial\Omega}, \tag{1.2}$$

since the gradient of the electrostatic potential is, up to a multiplicative constant, the electric field. The *overdetermined* nature of the problem comes from the boundary conditions (1.2). For the Dirichlet problem one solely prescribes $u\big|_{\partial\Omega}$, and for the Neumann problem one solely prescribes $\langle \nu, (\nabla u)\big|_{\partial\Omega}\rangle$, where ν is the outward unit normal vector to Ω, while in (1.2) one prescribes both $u\big|_{\partial\Omega}$ and $(\nabla u)\big|_{\partial\Omega}$.

This dissertation contains a systematic treatment for overdetermined second-order elliptic boundary value problems using boundary layer potential methods, in the class of uniformly rectifiable domains, which form the largest possible class of domains that have a befitting L^p theory, $p \in (1, \infty)$, for the boundary integrals associated with certain elliptic systems. These results are from joint work with Dorina Mitrea, Irina Mitrea, and Marius Mitrea in [2, 3].

This dissertation also contains recent progress in the implementation of Fredholm Theory for the treatment of the Dirichlet problem for the polyharmonic operator Δ^3 in infinitesimally flat Ahlfors regular domains in $\mathbb{R}^n$ with prescribed boundary data in the space of Lebesgue-Whitney array with integrability exponent $p \in (1, \infty)$. This is part of an ongoing joint work with Dorina Mitrea, Irina Mitrea, and Marius Mitrea [4].

Chapter 2 contains background material from geometric measure theory, partial differential equations, harmonic analysis, and an overview of the classical Cauchy integral operator.

Chapter 3 covers the overdetermined boundary value problem for the Laplacian operator in bounded uniformly rectifiable domains in the Euclidean space. It includes integral representation formulas, jump relations, well-posedness for the overdetermined boundary value problem for the Laplacian in bounded uniformly rectifiable domains, and classical Hardy spaces associated with the Laplacian operator. This chapter also contains a discussion about the formulation and Fredholm solvability of the overdetermined Riemann-Hilbert problem for the Laplacian operator in uniformly rectifiable domains with compact boundary.

Chapter 4 contains the analysis of the overdetermined boundary value problem for second-order, homogeneous, constant complex coefficients $M \times M$ weakly elliptic systems in bounded uniformly rectifiable domains. We establish integral representation formulas, jump relations, well-posedness for the overdetermined boundary value problem in bounded uniformly rectifiable domains, and classical Hardy spaces associated with these systems.

In Chapter 5 we consider the overdetermined boundary value problem for the Stokes system of linearized hydrostatics in bounded uniformly rectifiable domains. For the treatment of this boundary value problem, one introduces a new function space for the boundary data, namely, the space of divergence-free Lebesgue-Whitney arrays, and the integral representation formulas are modified to take into account the particularities of the Stokes system of linearized hydrostatics. This chapter contains integral representation formulas, jump relations, well-posedness for the overdetermined boundary value problem in bounded uniformly rectifiable domains, and classical Hardy spaces associated with the Stokes system.

Chapter 6 covers Fredholm Theory for the Dirichlet problem for Δ^3 in infinitesimally flat Ahlfors regular domains with prescribed data in Lebesgue-Whitney array spaces with $p \in (1, \infty)$. The progress registered here consists in the construction a *distinguished* coefficient tensor for the operator Δ^3. The importance of such an object resides in the fact that this choice leads to double multi-layers of chord-dot-normal type, a brand of singular integral operators naturally associated with the PDE which is sensitive to the flatness of the domain under consideration. Indeed, availed of this key object, using machinery developed in G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17] and in D. Mitrea, I. Mitrea, and M. Mitrea [26], Fredholm solvability of the Dirichlet problem for Δ^3 is established in the geometric setting of infinitesimally flat Ahlfors regular domains in arbitrary dimensions.

CHAPTER 2

PRELIMINARIES

2.1 Geometric Measure Theory

Fix $n \in \mathbb{N}$ with $n \geq 2$. Throughout, $\mathscr{L}^n$ denotes the n-dimensional Lebesgue measure, while $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. Call a given closed set $\Sigma \subset \mathbb{R}^n$ Ahlfors regular provided

$$\mathcal{H}^{n-1}(B(x,r) \cap \Sigma) \approx r^{n-1}, \tag{2.1}$$

uniformly in $x \in \Sigma$ and $r \in (0, \text{diam}(\Sigma))$.

Following G. David and S. Semmes (cf.,e.g., [13] and the discussion in D. Mitrea, I. Mitrea, and M. Mitrea [22]), call a closed set $\Sigma \subset \mathbb{R}^{n-1}$ a uniformly rectifiable set (or UR, for short) if Σ is an Ahlfors regular set with the property that there exist positive constants ε, M such that for each $x \in \Sigma$ and each $r \in (0, \text{diam}(\Sigma))$, it is possible to find a Lipschitz map $\varphi : B_r^{n-1} \to \mathbb{R}^n$ (where B_r^{n-1} denotes a $(n-1)$-dimensional ball) with $||\nabla\varphi||_{L^\infty} \leq M$ such that

$$\mathcal{H}^{n-1}\left(\Sigma \cap B(x,r) \cap \varphi(B_r^{n-1})\right) \geq \varepsilon r^{n-1}. \tag{2.2}$$

Say that an open set $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular domain provided the topological boundary $\partial\Omega$ is Ahlfors regular, and $\mathcal{H}^{n-1}(\partial\Omega\backslash\partial_*\Omega) = 0$. Here, $\partial_*\Omega$ denotes the geometric measure theoretic boundary of Ω, defined as

$$\partial_*\Omega := \Big\{ x \in \partial\Omega : \limsup_{r\to 0^+} \frac{\mathscr{L}^n(B(x,r)\cap\Omega)}{r^n} > 0, \\ \limsup_{r\to 0^+} \frac{\mathscr{L}^n(B(x,r)\setminus\Omega)}{r^n} > 0 \Big\}. \tag{2.3}$$

A set $\Omega \subset \mathbb{R}^n$ is a set of locally finite perimeter provided it is Lebesgue measurable and $\mathcal{H}^{n-1}(\partial_*\Omega \cap K) < \infty$ for each compact set $K \subset \mathbb{R}^n$. It follows from the definitions that every Ahlfors regular domain is a set of locally finite perimeter. Thus if $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular domain and we denote by $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ the surface measure on $\partial\Omega$, then a classical result of E. De Giorgi and H. Federer (cf. the discussion in D. Mitrea, I. Mitrea, and M. Mitrea [22]) implies that

$$\nabla\chi_\Omega = -\nu\sigma \quad \text{in } [\mathcal{D}'(\mathbb{R}^n)]^n, \tag{2.4}$$

where χ_Ω denotes the characteristic function of Ω, and $\nu \in [L^\infty(\partial\Omega,\sigma)]^n$ is a vector field, hereafter referred to as the geometric measure theoretic (GMT) outward unit normal to Ω.

Next, say that $\Omega \subset \mathbb{R}^n$ is a uniformly rectifiable domain (hereafter abbreviated as UR domain, cf. [22]) if Ω is an Ahlfors regular domain for which $\partial\Omega$ is a UR set. This is the most general class of sets in $\mathbb{R}^n$ for which there is a suitable Calderón–Zygmund theory for implementing singular integral methods in elliptic boundary value problems [24].

Let $\Omega \subset \mathbb{R}^n$ be an open set and fix $\kappa \in (0,\infty)$. Then the nontangential approach region of aperture κ at a point $x \in \partial\Omega$ is defined as

$$\Gamma_\kappa(x) := \big\{y \in \Omega : |x-y| < (1+\kappa)\text{dist}(y,\partial\Omega)\big\}. \tag{2.5}$$

The nontangential maximal operator $\mathcal{N}_\kappa$ acts on any given $\mathscr{L}^n$-measurable function $u : \Omega \to \mathbb{C}$ according to

$$\left(\mathcal{N}_\kappa u\right)(x) := ||u||_{L^\infty(\Gamma_\kappa(x),\mathscr{L}^n)} \quad \text{for each } x \in \partial\Omega, \tag{2.6}$$

and the nontangential limit of a function u at the point $x \in \partial\Omega$ exists, and its value is $A \in \mathbb{C}$, provided

$$\begin{aligned}&\text{for every } \varepsilon > 0 \text{ there exists some } r > 0 \text{ such that}\\ &|u(y) - A| < \varepsilon \text{ for } \mathscr{L}^n\text{-a.e. point } y \in \Gamma_\kappa(x) \cap B(x,r).\end{aligned}$$

When this limit exists, we denote it by $\left(u\big|_{\partial\Omega}^{\kappa-nt}\right)(x)$. In [22] has been proved that if $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular domain, then for each $\kappa \in (0,\infty)$ one has $x \in \overline{\Gamma_\kappa(x)}$ for σ-a.e. point $x \in \partial\Omega$.

More generally, define the m-th order nontangential boundary multitrace of a function $u \in W^{m,1}_{\text{loc}}(\Omega)$ (cf. G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17]) by

$$\text{Tr}_m^{\kappa-nt} u := \{(\partial^\gamma u)\big|_{\partial\Omega}^{\kappa-nt}\}_{|\gamma|\leq m}, \tag{2.7}$$

assuming that the above nontangential limits exist σ-a.e. on $\partial\Omega$.

2.2 Partial Differential Equations

Fix $n, M \in \mathbb{N}$ with $n \geq 2$ and consider a second-order, homogeneous, complex constant coefficient, $M \times M$ system L in $\mathbb{R}^n$, that is, L is given by (summation over repeated indices is assumed here and elsewhere)

$$L = (a_{rs}^{\alpha\beta}\partial_r\partial_s)_{1\leq\alpha,\beta\leq M}. \tag{2.8}$$

The system L is said to be weakly elliptic if for each non-zero $\xi \in \mathbb{R}^n$ the characteristic matrix

$$L(\xi) := -(a_{rs}^{\alpha\beta}\xi_r\xi_s)_{1\leq\alpha,\beta\leq M} \text{ is invertible.} \tag{2.9}$$

The Laplacian is a scalar second-order, homogeneous, complex constant coefficient which is weakly elliptic. Indeed, the characteristic polynomial for the Laplacian is given by

$$|\xi|^2 \text{ for each } \xi \in \mathbb{R}^n, \tag{2.10}$$

hence it satisfies the condition in (2.9) for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Consider a pair of complex numbers $(\mu, \lambda) \in \mathbb{C}^2$. The Lamé operator with Lamé moduli μ and λ is the differential operator that acts on $\mathbb{C}^n$-valued functions $u = (u^\alpha)_{1\leq\alpha\leq n}$ in $\mathbb{R}^n$ as

$$L_{\mu,\lambda}u := \Delta u + (\mu+\lambda)\nabla \text{div } u = \left(\Delta u^\alpha + (\mu+\lambda)\partial_\alpha\partial_\beta u^\beta\right)_{1\leq\alpha\leq n}. \tag{2.11}$$

is an example of a second-order, homogeneous, complex constant coefficient $n \times n$ weakly elliptic in $\mathbb{R}^n$ whenever the Lamé moduli satisfy $\mu \neq 0$ and $2\mu + \lambda \neq 0$ (cf. D. Mitrea, I. Mitrea, and M. Mitrea [24, Proposition 1.4.4]).

Following D. Mitrea, I. Mitrea, and M. Mitrea [26, Chapter 1], let $\mathfrak{A}_L$ denote the collection of all coefficient tensors associated with L, that is, the set $\mathfrak{A}_L$ contains all "blocks of matrices" of complex numbers $A = (a_{rs}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ that satisfies (2.8). There are infinitely many coefficient tensors associated with a given second-order system.

As shown in D. Mitrea [21, Theorem 11.1], every homogeneous, constant complex coefficient, weakly elliptic $M \times M$ system of even order admits a canonical fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$. Below, we restate the theorem for the case of second-order, homogeneous, weakly elliptic systems with constant complex coefficients. Unless explicitly stated otherwise, the summation convention over repeated indices is in effect.

Theorem 2.2.1. *Fix $n, M \in \mathbb{N}$, with $n \geq 2$, and let*

$$L = \left(a_{rs}^{\alpha\beta}\partial_r\partial_s\right)_{1\leq\alpha,\beta\leq M} \tag{2.12}$$

be a second-order homogeneous $M\times M$ system in $\mathbb{R}^n$, with complex constant coefficients, which is weakly elliptic in the sense of (2.9)*. That is, the characteristic matrix*

$$L(\xi)=-\left(a_{rs}^{\alpha\beta}\xi_r\xi_s\right)_{1\leq\alpha,\beta\leq M},\quad\forall\xi=(\xi_r)_{1\leq r\leq n}\in\mathbb{R}^n \tag{2.13}$$

is invertible for every non-zero $\xi\in\mathbb{R}^n$.

Consider the $M\times M$ matrix-valued function $E=(E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ defined for each point $x\in\mathbb{R}^n\setminus\{0\}$ by

$$E(x):=\begin{cases}-\dfrac{\Delta_x^{(n-1)/2}}{4(2\pi i)^{n-1}}\left\{\displaystyle\int_{S^{n-1}}|\langle x,\xi\rangle|[L(\xi)]^{-1}\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\right\} & \text{if } n \text{ is odd,}\\[2ex] \dfrac{\Delta_x^{(n-2)/2}}{(2\pi i)^n}\left\{\displaystyle\int_{S^{n-1}}\ln|\langle x,\xi\rangle|[L(\xi)]^{-1}\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\right\} & \text{if } n \text{ is even.}\end{cases} \tag{2.14}$$

In relation to the $\mathbb{C}^{M\times M}$-valued function (2.14)*, the following properties hold:*

1. For each $\alpha,\beta\in\{1,\ldots,M\}$ one has

$$E_{\alpha\beta}\in C^\infty\left(\mathbb{R}^n\setminus\{0\}\right)\cap L^1_{\mathrm{loc}}\left(\mathbb{R}^n,\mathscr{L}^n\right), \tag{2.15}$$

$$E_{\alpha\beta}(-x)=E_{\alpha\beta}(x)\text{ for all }x\in\mathbb{R}^n\setminus\{0\} \tag{2.16}$$

In fact, each entry $E_{\alpha\beta}$ is a real-analytic function in $\mathbb{R}^n\setminus\{0\}$. Moreover, each $E_{\alpha\beta}$ is an even tempered distribution in $\mathbb{R}^n$ (induced via integration against Schwartz functions).

2. If for each $y \in \mathbb{R}^n$ one denotes by δ_y Dirac's delta distribution with mass at y in $\mathbb{R}^n$, then in the sense of tempered distributions in $\mathbb{R}^n$ one has

$$L_x[E(x-y)] = \delta_y(x) I_{M\times M}, \quad \forall y \in \mathbb{R}^n \tag{2.17}$$

where $I_{M\times M}$ is the $M \times M$ identity matrix, and the subscript x indicates that the operator L in (2.17) *is applied to each column of the matrix $E(x-y)$ in the variable x. Componentwise, for each $\alpha, \beta \in \{1, \ldots, M\}$ one therefore has*

$$a_{rs}^{\alpha\gamma} \partial_{x_r} \partial_{x_s} \left[E_{\gamma\beta}(x-y)\right] = \delta_{\alpha\beta}\delta_y(x) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \delta_y(x) & \text{if } \alpha = \beta, \end{cases} \qquad x, y \in \mathbb{R}^n \tag{2.18}$$

3. For each multi-index $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \geq 1$, the tempered distribution $\partial^\gamma E$ is positive homogeneous of degree $2 - n - |\gamma|$ in $\mathbb{R}^n$. This is also true for $|\gamma| = 0$ provided $n \geq 3$, i.e., the tempered distribution E is positive homogeneous of degree $2 - n$ in $\mathbb{R}^n$ if $n \geq 3$.

Finally, corresponding to $n = 2$ and $|\gamma| = 0$, one may express

$$E(x) = \Phi(x) - \frac{\ln|x|}{4\pi^2} \int_{S^1} [L(\xi)]^{-1} d\mathcal{H}^1(\xi), \ \ \forall x \in \mathbb{R}^2 \setminus \{0\} \tag{2.19}$$

where $\Phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{C}^{M\times M}$, given by

$$\Phi(x) := -\frac{1}{4\pi^2} \int_{S^1} \ln \left| \left\langle \frac{x}{|x|}, \xi \right\rangle \right| [L(\xi)]^{-1} d\mathcal{H}^1(\xi), \ \ \forall x \in \mathbb{R}^2 \backslash \{0\} \tag{2.20}$$

is a function of class C^∞ and positive homogeneous of degree 0 in $\mathbb{R}^2 \setminus \{0\}$.

4. For each $\gamma \in \mathbb{N}_0^n$ there exists a finite constant $C_\gamma > 0$ such that for each $x \in \mathbb{R}^n \setminus \{0\}$ one has

$$|(\partial^\gamma E)(x)| \leq \begin{cases} \dfrac{C_\gamma}{|x|^{n+|\gamma|-2}} & \text{if either } n \geq 3, \text{ or } n = 2 \text{ and } |\gamma| > 0, \\ & \text{or } n = 2 \text{ and } \displaystyle\int_{S^1} [L(\xi)]^{-1} \mathrm{d}\mathcal{H}^1(\xi) = 0, \\ C_0(1 + |\ln|x||) & \text{whenever } n = 2 \text{ and } |\gamma| = 0. \end{cases} \tag{2.21}$$

5. Let 'hat' denote the Fourier transform in $\mathbb{R}^n$. Then $\widehat{E}$ is a tempered distribution in $\mathbb{R}^n$ (which is positive homogeneous of degree -2 if $n \geq 3$), whose restriction to $\mathbb{R}^n \setminus \{0\}$ is a (matrix-valued) function of class C^∞. In fact,

$$\widehat{E}(\xi) = [L(\xi)]^{-1} \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.22}$$

We highlight that, in particular, if $E = (E_{\alpha\beta})_{1 \leq \alpha, \beta \leq M}$ is the canonical fundamental solution of a second-order, homogeneous, weakly elliptic systems, with constant complex coefficients, then E is a matrix whose entries are smooth and even in $\mathbb{R}^n \setminus \{0\}$,

$$\partial_r E(x) \text{ is smooth and positive homogeneous of degree } 1 - n \text{ on } \mathbb{R}^n \setminus \{0\}, \tag{2.23}$$

and the Fourier transform of E in $\mathbb{R}^n$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and it satisfies

$$\widehat{E}(\xi) = [L(\xi)]^{-1} \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.24}$$

2.3 Harmonic Analysis

The work of G. David and S. Semmes (cf. [13]) has shown that uniformly rectifiable sets are the broadest geometrical setting in which the Calderón–Zygmund theory maintains much of its original strength as it does in the classical Euclidean setting. Here, we include two results of this nature that are key for our forthcoming discussions. A proof of the statement below is available in D. Mitrea, I. Mitrea, and M. Mitrea [24, Chapter 2].

Theorem 2.3.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a UR domain, and consider a function $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$ which is odd and positive homogeneous function of order $1-n$. Fix an integrability exponent $p \in (1,\infty)$ and define the singular integral operator acting on $f \in L^p(\partial\Omega,\sigma)$ according to*

$$\mathscr{T} f(x) := \int_{\partial\Omega} k(x-y) f(y)\, d\sigma(y) \quad \text{for every } x \in \Omega \tag{2.25}$$

along with its principal value version defined as the limit

$$T f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} k(x-y) f(y)\, d\sigma(y) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.26}$$

Then the operators $\mathscr{T}$ and T satisfy the following properties:

1. For each function $f \in L^p(\partial\Omega,\sigma)$ the limit defining $Tf(x)$ in formula (2.26) *exists at σ-a.e. point $x \in \partial\Omega$, and the induced operator*

$$T : L^p(\partial\Omega,\sigma) \to L^p(\partial\Omega,\sigma) \tag{2.27}$$

is well-defined, linear, and bounded.

2. For each aperture parameter $\kappa \in (0,\infty)$ there exists a positive constant $C \in (0,\infty)$ such that

$$||\mathcal{N}_\kappa(\mathscr{T} f)||_{L^p(\partial\Omega,\sigma)} \leq C||f||_{L^p(\partial\Omega,\sigma)} \tag{2.28}$$

for every $f \in L^p(\partial\Omega,\sigma)$.

3. For each aperture parameter $\kappa \in (0,\infty)$ and each $f \in L^p(\partial\Omega,\sigma)$

$$\mathscr{T} f\big|^{\kappa-nt}_{\partial\Omega}(x) = \frac{1}{2i}\hat{k}(\nu(x))f(x) + Tf(x) \quad \text{at } \sigma\text{-a.e. point } x \in \partial\Omega, \tag{2.29}$$

where ν is the outward unit normal to Ω and $\hat{k}$ stands for the Fourier transform of the function k in $\mathbb{R}^n$.

Another key result we shall use in our forthcoming discussions in this dissertation is the one that treats singular integral operators whose integral kernels have an algebraic structure described in D. Mitrea, I. Mitrea, and M. Mitrea [24, Proposition 1.2.1].

Proposition 2.3.2. *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a UR domain with surface measure σ and denote by $\nu = (\nu_1,\ldots,\nu_n)$ the geometric measure theoretic outward unit normal vector to Ω. Next, consider a complex-valued function $b \in L^1_{\rm loc}(\mathbb{R}^n,\mathscr{L}^n)$ with the property that $b\big|_{\mathbb{R}^n\setminus\{0\}} \in \mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})$, and such that ∇b is odd and positive homogeneous of degree $1-n$. Finally, for each pair of indices $j,k \in \{1,\ldots,n\}$ introduce the*

operators acting on each function $f \in L^1\left(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\right)$ *according to*

$$\mathcal{T}_{jk}f(x) := \int_{\partial\Omega} \left\{\nu_j(y)\left(\partial_k b\right)(x-y) - \nu_k(y)\left(\partial_j b\right)(x-y)\right\} f(y)d\sigma(y), \quad \forall x \in \Omega \tag{2.30}$$

along with its principal value version defined as the limit

$$T_{jk}f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \left\{\nu_j(y)\left(\partial_k b\right)(x-y) - \nu_k(y)\left(\partial_j b\right)(x-y)\right\} f(y)d\sigma(y) \tag{2.31}$$

for σ*-a.e.* $x \in \partial\Omega$*. Then for each aperture parameter* $\kappa \in (0,\infty)$

$$\mathcal{T}_{jk}\big|_{\partial\Omega}^{\kappa-nt}(x) = T_{jk}f(x) \quad \text{at } \sigma\text{-a.e. point } x \in \partial\Omega. \tag{2.32}$$

Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain. For each $j,k \in \{1,\dots,n\}$ consider the first-order tangential derivative operator $\partial_{\tau_{jk}}$ acting on an arbitrary function $\phi \in C^1(\mathbb{R}^n)$ via

$$\partial_{\tau_{jk}}\phi := \nu_j(\partial_k\phi)\big|_{\partial\Omega} - \nu_k(\partial_j\phi)\big|_{\partial\Omega} \quad \sigma\text{-a.e. on } \partial\Omega. \tag{2.33}$$

Here, $\nu = (\nu_1,\dots,\nu_n)$ is the geometric measure theoretic outward unit normal vector to Ω.

Given indices $j,k\in\{1,\dots,n\}$ and an integrability exponent $p\in(1,\infty)$, say that a function $f\in L^p(\partial\Omega,\sigma)$ is such that $\partial_{\tau_{jk}}f$ belongs to $L^p(\partial\Omega,\sigma)$ provided there exists a function $f_{jk}\in L^p(\partial\Omega,\sigma)$ such that

$$\int_{\partial\Omega}(\partial_{\tau_{jk}}\phi)\,f\,d\sigma=-\int_{\partial\Omega}\phi\,(f_{jk})\,d\sigma\quad\text{for every }\phi\in C^1_0(\mathbb{R}^n).\tag{2.34}$$

Define $L^p_1(\partial\Omega,\sigma)$, the L^p-based Sobolev space of order 1 on $\partial\Omega$, as the collection of all functions $f\in L^p(\partial\Omega,\sigma)$ such that $\partial_{\tau_{jk}}f$ defined in (2.34) belongs to $L^p(\partial\Omega,\sigma)$ for all the indices $j,k\in\{1,\dots,n\}$. This is a Banach space when equipped with the norm

$$||f||_{L^p_1(\partial\Omega,\sigma)}:=||f||_{L^p(\partial\Omega,\sigma)}+\sum_{j,k}||\partial_{\tau_{jk}}f||_{L^p_1(\partial\Omega,\sigma)}.\tag{2.35}$$

Below, we restate two results regarding properties of the boundary Sobolev spaces, which will be used throughout this dissertation. The first is an integration by parts formula in D. Mitrea, I. Mitrea, and M. Mitrea [23, Proposition 11.1.14].

Proposition 2.3.3. *Assume $\Omega\subset\mathbb{R}^n$ is a set of locally finite perimeter and abbreviate $\sigma_*:=\mathcal{H}^{n-1}\lfloor\partial_*\Omega$. Consider $f\in\mathrm{Lip}_c(\partial_*\Omega)$, and $g\in L^p_{1,\mathrm{loc}}(\partial_*\Omega,\sigma_*)$ with $p\in[1,\infty]$, and fix a pair of indices $j,k\in\{1,\dots,n\}$. Then*

$$f\in L^\infty_1(\partial_*\Omega,\sigma_*),\ \ \partial_{\tau_{jk}}f\in L^\infty(\partial_*\Omega,\sigma_*),\ \ \operatorname{supp}\left(\partial_{\tau_{jk}}f\right)\subseteq\operatorname{supp}f\tag{2.36}$$

and

$$\int_{\partial_*\Omega} \left(\partial_{\tau_{jk}} f\right) g \, d\sigma_* = -\int_{\partial_*\Omega} f \left(\partial_{\tau_{jk}} g\right) d\sigma_* \tag{2.37}$$

The second result, which we restate here, is the fact that the boundary Sobolev spaces just introduced in are modules over the space of compactly supported Lipschitz functions D. Mitrea, I. Mitrea, and M. Mitrea [23, Proposition 11.1.15].

Proposition 2.3.4. *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a set of locally finite perimeter and abbreviate $\sigma_* := \mathcal{H}^{n-1} \lfloor \partial_*\Omega$. Then, given any pair of functions $f \in \mathrm{Lip}_c(\partial_*\Omega)$, and $g \in L^p_{1,\mathrm{loc}}(\partial_*\Omega, \sigma_*)$ with $p \in [1, \infty]$, it follows that $fg \in L^p_1(\partial_*\Omega, \sigma_*)$, and for each $j, k \in \{1, \ldots, n\}$ the following product rule holds:*

$$\partial_{\tau_{jk}}(fg) = \left(\partial_{\tau_{jk}} f\right) g + f \left(\partial_{\tau_{jk}} g\right) \quad \text{at } \sigma_*\text{-a.e. on } \partial_*\Omega. \tag{2.38}$$

Further properties of the boundary Sobolev spaces just introduced are contained in D. Mitrea, I. Mitrea, and M. Mitrea [23].

Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain. For $p \in (1, \infty)$, $m, M \in \mathbb{N}$ with $m \geq 1$, say that a family

$$\dot{f} := \left\{ f_\gamma : \gamma \in \mathbb{N}_0^n, \, |\gamma| \leq m - 1 \right\} \tag{2.39}$$

of $\mathbb{C}^M$-valued functions in $[L^p(\partial\Omega,\sigma)]^M$, indexed by multi-indices $\gamma\in\mathbb{N}_0^n$ which are of order $|\gamma|\leq m-1$, is a Lebesgue-Whitney array if it satisfies the following compatibility conditions (abbreviated as CC)

$$\dot{f}\in \mathrm{CC} \iff \begin{cases} \partial_{\tau_{jk}} f_\gamma = \nu_j f_{\gamma+e_k} - \nu_k f_{\gamma+e_j} \ \ \sigma\text{-a.e. on } \partial\Omega \\ \\ \text{for all } |\gamma|\leq m-2 \ \text{ and all } \ 1\leq j,k\leq n, \end{cases} \tag{2.40}$$

where $\partial_{\tau_{jk}}$ denotes the tangential derivative operator. Also $\{e_k\}_{1\leq k\leq n}$ stands for the canonical basis in $\mathbb{N}_0^n$. With this in hand, define the L^p Lebesgue-Whitney space of order $m-1$ of $\mathbb{C}^M$-valued functions as

$$\mathrm{WA}^M_{m-1}[L^p(\partial\Omega,\sigma)] := \{\dot{f}\in \mathrm{CC} \,:\, f_\gamma\in[L^p(\partial\Omega,\sigma)]^M \text{ for } |\gamma|\leq m-1\}, \tag{2.41}$$

equipped with the norm

$$||\dot{f}||_{\mathrm{WA}^M_{m-1}[L^p(\partial\Omega,\sigma)]} := \sum_{|\gamma|\leq m-1} ||f_\gamma||_{[L^p(\partial\Omega,\sigma)]^M}. \tag{2.42}$$

Note that for each $p\in[1,\infty]$, the L^p Lebesgue-Whitney spaces are Banach spaces when equipped with the norm in (2.42). We refer to G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17] (cf. I. Mitrea and M. Mitrea [28] for the theory on Lipschitz domains) for further properties of these function spaces.

Let $\Omega \subset \mathbb{R}^n$ be a UR domain. For every second-order, homogeneous, complex constant coefficient, $M \times M$ system L in $\mathbb{R}^n$, and every coefficient tensor $A \in \mathfrak{A}_L$, introduce the boundary-to-domain double layer potential operator, $\mathscr{D}_A$, acting on $\mathbb{C}^M$-valued functions $f = (f^\alpha)_{1\leq\alpha\leq M} \in \Big[L^1(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}})\Big]^M$ as

$$\begin{aligned}\mathscr{D}_A f(x) &:= -\left(\int_{\partial\Omega} \nu_s(y)(\partial_r E_{\gamma\alpha})(x-y)a_{rs}^{\alpha\beta} f^\beta(y)d\sigma(y)\right)_{1\leq\gamma\leq n} \\ &= -\int_{\partial\Omega} \nu_s(y)(\partial_r E_L)(x-y)A_{rs}f(y)d\sigma(y),\end{aligned} \tag{2.43}$$

for $x \in \mathbb{R}^n \setminus \partial\Omega$, where $A_{rs} := (a_{rs}^{\alpha\beta})_{1\leq\alpha,\beta\leq M}$. Also, introduce the boundary-to-boundary double layer, K_A, acting on $\mathbb{C}^M$-valued in the same weighted L^1 function space as

$$K_A f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_s(y)(\partial_r E_L)(x-y)A_{rs}f(y)d\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.44}$$

Introduce the boundary-to-domain single layer potential, $\mathscr{S}_L$, acting on $\mathbb{C}^M$-valued functions $f = (f^\alpha)_{1\leq\alpha\leq M}$ such that

$$\begin{cases} f \in \Big[L^1(\partial\Omega, (1+\ln|y|)\sigma(y))\Big]^M & \text{if } n = 2, \\ f \in \Big[L^1(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-2}})\Big]^M & \text{if } n \geq 3, \end{cases} \tag{2.45}$$

as the integral operator

$$\mathscr{S}_L f(x) := \left(\int_{\partial\Omega} E_{\gamma\alpha}(x-y)f^\alpha(y)d\sigma(y)\right)_{1\leq\gamma\leq n}$$

$$= \int_{\partial\Omega} E_L(x-y) f(y) d\sigma(y), \tag{2.46}$$

for $x \in \mathbb{R}^n \setminus \partial\Omega$, and the boundary-to-boundary single layer, S_L, as

$$S_L f(x) := \int_{\partial\Omega} E_L(x-y) f(y) d\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.47}$$

Note that these boundary-to-domain potentials are a mechanism for generating null solutions for the weakly elliptic system L. For completeness, we restate below some of the mapping properties of these operators when acting on vector-valued functions with components in $L^p(\partial\Omega, \sigma)$. A proof may be found in D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 1.5.1].

Theorem 2.3.5. *Fix an exponent $p \in (1, \infty)$ along with $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a UR domain. For some $M \in \mathbb{N}$, consider a coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M \\ 1\leq r,s\leq n}}$ with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$*

$$L_A = (a_{rs}^{\alpha\beta}\partial_r\partial_s)_{1\leq\alpha,\beta\leq M} \quad \textit{is weakly elliptic.} \tag{2.48}$$

Finally, consider the boundary layer potentials $\mathscr{D}_A$, K_A, $\mathscr{S}_L$, and S_L associated with A and Ω as in (2.43)*,* (2.44)*,* (2.46) *and* (2.47)*. Then, in relation to these operators, the following properties hold.*

1. For each function $f \in L^p(\partial\Omega, \sigma)$ the limit defining $K_A f(x)$ in formula (2.44) *exists at σ-a.e. point $x \in \partial\Omega$, and the induced operator*

$$K_A : L^p(\partial\Omega, \sigma) \to L^p(\partial\Omega, \sigma) \tag{2.49}$$

is well-defined, linear, and bounded.

2. For each aperture parameter $\kappa \in (0, \infty)$ there exists a positive constant $C \in (0, \infty)$ depending only on $\partial\Omega$, A, n, κ, and p such that

$$||\mathcal{N}_\kappa(\mathscr{D}_A f)||_{L^p(\partial\Omega,\sigma)} \leq C||f||_{L^p(\partial\Omega,\sigma)} \tag{2.50}$$

for every $f \in L^p(\partial\Omega, \sigma)$, and

$$||\mathcal{N}_\kappa(\mathscr{D}_A f)||_{L^p} + ||\mathcal{N}_\kappa(\nabla\mathscr{D}_A f)||_{L^p(\partial\Omega,\sigma)} \leq C||f||_{L^p_1(\partial\Omega,\sigma)} \tag{2.51}$$

for every $f \in L^p_1(\partial\Omega, \sigma)$.

3. For each aperture parameter $\kappa \in (0, \infty)$ and each $f \in L^p(\partial\Omega, \sigma)$

$$\mathscr{D}_A f\big|^{\kappa-nt}_{\partial\Omega}(x) = \frac{1}{2}f(x) + K_A f(x) \quad \text{at } \sigma\text{-a.e. point } x \in \partial\Omega. \tag{2.52}$$

4. Assume $\partial\Omega$ is bounded. Then the boundary-to-boundary single layer, S_L, induces an operator

$$S_L : L^p(\partial\Omega, \sigma) \to L^p_1(\partial\Omega, \sigma) \tag{2.53}$$

that is well-defined, linear, and bounded.

5. Further assume Ω is a bounded UR domain. Then for each aperture parameter $\kappa \in (0,\infty)$ there exists a positive constant $C \in (0,\infty)$ depending only on $\partial\Omega$, n, κ, and p such that

$$||\mathcal{N}_\kappa(\mathscr{S}_L f)||_{L^p(\partial\Omega,\sigma)} + ||\mathcal{N}_\kappa(\nabla\mathscr{S}_L f)||_{L^p(\partial\Omega,\sigma)} \leq C||f||_{L^p(\partial\Omega,\sigma)} \tag{2.54}$$

for every $f \in L^p(\partial\Omega,\sigma)$.

We highlight that a more complete statement on the mapping properties of these operators, including different functions spaces, is contained in D. Mitrea, I. Mitrea, and M. Mitrea [25].

2.4 The Classical Cauchy Integral Operator

Let us briefly recall the classical Cauchy integral operator and review some of its properties.

Let $\Omega \subset \mathbb{C}$ be a UR domain. Recall that the complex arc-length on the geometric measure theoretic boundary of the set Ω is defined as

$$d\zeta := i\nu\, d\sigma = i\nu\, d\mathcal{H}^1\lfloor\partial\Omega, \tag{2.55}$$

where ν denotes the GMT outward unit normal to Ω.

With the identification $\mathbb{C} \equiv \mathbb{R}^2$, consider the Cauchy-Riemann operator $\bar{\partial}$ given by

$$\bar{\partial} := \frac{1}{2}(\partial_x + i\partial_y). \tag{2.56}$$

For an UR domain $\Omega \subset \mathbb{C}$, introduce the boundary-to-domain Cauchy operator acting on any function $f \in L^1(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|})$ as

$$\mathscr{C}f(z) := \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\nu(\zeta)d\sigma(\zeta) \tag{2.57}$$

for $z \in \mathbb{C} \setminus \partial\Omega$. Note that the above integral is absolutely convergent, and in fact, for each $f \in L^1(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|})$

$$\bar{\partial}(\mathscr{C}f) \equiv 0 \text{ in } \mathbb{C} \setminus \partial\Omega. \tag{2.58}$$

Under the same geometric assumptions, introduce the boundary-to-boundary Cauchy operator acting on any function $f \in L^1(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|})$ as

$$Cf(z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{\substack{\zeta\in\partial\Omega \\ |\zeta - z|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}d\zeta = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{\substack{\zeta\in\partial\Omega \\ |\zeta - z|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\nu(\zeta)d\sigma(\zeta) \tag{2.59}$$

for σ-a.e. point $z \in \partial\Omega$. Below we state the results in D. Mitrea, I. Mitrea, and M. Mitrea [25], which highlight remarkable properties of the Cauchy operators introduced in (2.57) and (2.59).

Theorem 2.4.1. *Let $\Omega \subset \mathbb{C}$ be a UR domain with compact boundary and surface measure σ. Fix an integrability exponent $p \in (1, \infty)$ and an aperture parameter $\kappa \in (0, \infty)$. Then the following properties hold:*

1. For each function $u : \Omega \to \mathbb{C}$ with

$$u \in C^1(\Omega) \ \text{ and } \ \bar{\partial} u = 0 \ \text{ in } \Omega, \tag{2.60}$$

and satisfying the size condition $\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma)$, the κ-nontangential trace $u\big|_{\partial\Omega}^{\kappa-nt}(z)$ is well-defined at σ-a.e. point $z \in \partial\Omega$, $u\big|_{\partial\Omega}^{\kappa-nt} \in L^p(\partial\Omega, \sigma)$, and

$$u = \mathscr{C}(u\big|_{\partial\Omega}^{\kappa-nt}) \ \text{ in } \Omega. \tag{2.61}$$

2. If I denotes the identity operator, then for $f \in L^p(\partial\Omega, \sigma)$,

$$(\mathscr{C} f)\big|_{\partial\Omega}^{\kappa-nt} = \left(\frac{1}{2} I + C\right) f \quad \sigma\text{-a.e. on } \partial\Omega. \tag{2.62}$$

3. There exists a constant $C \in (0, \infty)$ such that

$$\|\mathcal{N}_\kappa(\mathscr{C} f)\|_{L^p(\partial\Omega,\sigma)} \leq C \|f\|_{L^p(\partial\Omega,\sigma)} \tag{2.63}$$

for every $f \in L^p(\partial\Omega, \sigma)$.

4. The boundary-to-boundary Cauchy operator is bounded from $L^p(\partial\Omega,\sigma)$ into itself, and satisfies

$$C^2=\frac{1}{4}I \ \text{ on } L^p(\partial\Omega,\sigma). \tag{2.64}$$

Next, let $\Omega\subset\mathbb{C}$ be a bounded UR domain with surface measure σ, $p\in(1,\infty)$ be an integrability exponent and $\kappa\in(0,\infty)$ be an aperture parameter. Recall that the domain Hardy space of holomorphic functions in Ω is given by

$$\mathscr{H}^p(\Omega):=\{u:u\in C^1(\Omega),\,\bar\partial u=0 \ \text{ and } \ \mathcal{N}_\kappa u\in L^p(\partial\Omega,\sigma)\}. \tag{2.65}$$

This is a Banach space when equipped with the norm

$$\|u\|_{\mathscr{H}^p(\Omega)}:=\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,\sigma)}. \tag{2.66}$$

Note that the results in D. Mitrea, I. Mitrea, and M. Mitrea [25], ensure that the definitions in (2.65)-(2.66) are independent of the aperture parameter $\kappa\in(0,\infty)$. Moreover, one has the following Fatou-type Theorem

Theorem 2.4.2. *Let $\Omega\subset\mathbb{C}$ be a UR domain with surface measure σ and fix an integrability exponent $p\in(1,\infty)$. Then for each function $u\in\mathscr{H}^p(\Omega)$ and for each aperture parameter $\kappa\in(0,\infty)$ there holds*

$$u\big|_{\partial\Omega}^{\kappa-nt} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \text{ and } u\big|_{\partial\Omega}^{\kappa-nt}\in L^p(\partial\Omega,\sigma). \tag{2.67}$$

Moving on, recall that the boundary Hardy space is given by

$$\mathscr{H}^p(\partial\Omega,\sigma) := \{f \in L^p(\partial\Omega,\sigma) : \exists u \in \mathscr{H}^p(\Omega) \text{ such that } f = u\big|_{\partial\Omega}^{\kappa-nt} \ \sigma\text{-a.e. on } \partial\Omega\}. \tag{2.68}$$

It follows from Theorem 2.4.2 that the boundary Hardy space is a subspace of $L^p(\partial\Omega,\sigma)$. Define the linear operator $P : L^p(\partial\Omega,\sigma) \to L^p(\partial\Omega,\sigma)$ as

$$P := \frac{1}{2}I + C, \tag{2.69}$$

where C is the boundary-to-boundary Cauchy operator. Thanks to the result in Theorem 2.4.1 stating that $C^2 = \frac{1}{4}I$ as operators on $L^p(\partial\Omega,\sigma)$, it follows that the operator P is a linear projection on $L^p(\partial\Omega,\sigma)$. Moreover, it can be shown that (cf. [25])

$$\mathscr{H}^p(\partial\Omega,\sigma) = P(L^p(\partial\Omega,\sigma)). \tag{2.70}$$

In fact, if $\Omega \subset \mathbb{C}$ is a bounded UR domain and $p \in (1,\infty)$ is an integrability exponent, then the boundary-to-domain Cauchy operator is such that

$$\mathscr{C} : \mathscr{H}^p(\partial\Omega,\sigma) \to \mathscr{H}^p(\Omega) \text{ is an isomorphism.} \tag{2.71}$$

In this context, the κ-nontangential trace and the boundary-to-domain Cauchy operator are inverses.

CHAPTER 3

OVERDETERMINED BOUNDARY VALUE PROBLEM FOR THE LAPLACIAN IN UNIFORMLY RECTIFIABLE DOMAINS

In this chapter we discuss the overdetermined boundary value problem for the Laplacian in uniformly rectifiable domains with compact boundary and prescribed boundary data in L^p-based function spaces.

In section 3.1 we introduce and analyze a Cauchy-like operator associated with the Laplacian and its basic properties. We also show that the Cauchy operator associated with the Laplacian may be used for writing integral representation formulas for harmonic functions in uniformly rectifiable domains with compact boundaries.

In section 3.2 we formulate and establish well-posedness results for the overdetermined boundary value problem for the Laplacian in uniformly rectifiable domains with compact boundaries and prescribed boundary data in Lebesgue-Whitney spaces with integrability exponent $p \in (1, \infty)$.

In section 3.3 we discuss classical Hardy spaces associated with the Laplacian operator and study equivalent formulations of the overdetermined boundary value problem.

Finally, in section 3.4 we formulate and discuss the overdetermined Riemann-Hilbert problem associated with the Laplacian operator in uniformly rectifiable domains with compact boundaries, and we show that under suitable assumptions on the twisting function we have Fredholm solvability for this problem.

We highlight that throughout the discussion, the Cauchy operator associated with the Laplacian introduced in section 3.1 plays a key role in obtaining the above mentioned results. All the results presented in this chapter are based on joint work with Dorina Mitrea, Irina Mitrea, and Marius Mitrea in [2].

3.1 Cauchy Operators and Integral Representation Formula for Null Solutions of the Laplacian

Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain with compact boundary and outward unit normal $\nu = (\nu_1, \ldots, \nu_n)$. For each integrability exponent $p \in (1, \infty)$ and for each coefficient tensor $A \in \mathfrak{A}_\Delta$ we define the boundary-to-domain Cauchy operator associated with A, denoted by $\dot{\mathscr{C}}_A$, as the linear operator acting on arrays $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ and producing a scalar function according to the formula

$$\dot{\mathscr{C}}_A \dot{f}(x) := \mathscr{D}_A(f_0)(x) - \mathscr{S}_\Delta(\nu_r A_{rs} f_{e_s})(x) \tag{3.1}$$

for all $x \in \mathbb{R}^n \setminus \partial\Omega$. Here, 0 stands for the zero multi-index in $\mathbb{N}_0^n$ and $\{e_s\}_{1 \leq s \leq n}$ stands for the canonical basis in $\mathbb{N}_0^n$, and $\dot{f} = (f_0, f_{e_1}, \ldots, f_{e_n})$.

As a first goal, we wish to monitor the dependence of the boundary-to-domain Cauchy operators introduced in (3.1) of the choice of coefficient tensor A used to represent the Laplacian operator.

Following D. Mitrea, I. Mitrea and M. Mitrea [26, Chapter 1], consider for the moment $n = 2$ and the following two matrices A_0 and A_1:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \tag{3.2}$$

A direct computation shows that $A_0, A_1 \in \mathfrak{A}_\Delta$, hence for each of these matrices we may associate a boundary-to-domain Cauchy operator.

More precisely, let $\Omega \subset \mathbb{R}^2$ be a UR domain with compact boundary, and outward unit normal vector $\nu = (\nu_1, \nu_2)$. Fix next an integrability exponent $p \in (1, \infty)$, and consider $\dot{f} = (f_0, f_{e_1}, f_{e_2}) \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$. Then the boundary-to-domain Cauchy operator associated with A_0 is given by the formula

$$\begin{aligned} \dot{\mathscr{C}}_{A_0} \dot{f}(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle \nu(y), y - x \rangle}{|x - y|^2} f_0(y) d\sigma(y) \\ - \frac{1}{2\pi} \int_{\partial\Omega} \log|x - y| \, \nu_r(y) f_{e_r}(y) d\sigma(y) \end{aligned} \tag{3.3}$$

for all $x \in \mathbb{R}^2 \setminus \partial\Omega$.

With the identification $\mathbb{R}^2 \equiv \mathbb{C}$, $z = x_1 + ix_2$, and $\zeta = y_1 + iy_2$, the boundary-to-domain Cauchy operator associated with A_1 is given by

$$\begin{aligned} \dot{\mathscr{C}}_{A_1} \dot{f}(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f_0(\zeta)}{z - \zeta} d\zeta \\ - \frac{1}{2\pi} \int_{\partial\Omega} \log|z - \zeta| \big(f_{e_1}(\zeta) + i f_{e_2}(\zeta)\big) \overline{\nu(\zeta)} d\sigma(\zeta) \end{aligned} \tag{3.4}$$

for all $z \in \mathbb{C} \setminus \partial\Omega$, where $i\nu(\zeta)d\sigma(\zeta) = d\zeta$ is the complex arc-length measure.

Formally, equations (3.3) and (3.4) are distinct and their writings involve different layer potential-type operators. However, as we will show in the following lemma, one of the features of the boundary-to-domain Cauchy operator introduced above is that it is independent of the choice of coefficient tensor, that is, the Cauchy operator introduced in (3.1) is intrinsic to the PDE, in this case the Laplacian.

Lemma 3.1.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain with compact boundary. Let $p \in (1, \infty)$ be an integrability exponent and consider $A, B \in \mathfrak{A}_\Delta$. Then for every $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ there holds*

$$\dot{\mathscr{C}}_A \dot{f}(x) = \dot{\mathscr{C}}_B \dot{f}(x) \ \textit{for all } x \in \mathbb{R}^n \setminus \partial\Omega. \tag{3.5}$$

Proof. Consider an arbitrary $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$. To conclude that (3.5) holds, we must show that

$$\mathscr{D}_A(f_0)(x) - \mathscr{S}_\Delta(\nu_r A_{rs} f_{e_s})(x) = \mathscr{D}_B(f_0)(x) - \mathscr{S}_\Delta(\nu_r B_{rs} f_{e_s})(x) \tag{3.6}$$

for all $x \in \mathbb{R}^n \setminus \partial\Omega$.

Note that (3.6) is equivalent to the expression

$$\mathscr{D}_A(f_0)(x) - \mathscr{D}_B(f_0)(x) = \mathscr{S}_\Delta(\nu_r A_{rs} f_{e_s})(x) - \mathscr{S}_\Delta(\nu_r B_{rs} f_{e_s})(x). \tag{3.7}$$

Our goal is to show that (3.7) holds for all $x \in \mathbb{R}^n \setminus \partial\Omega$.

Fix $x \in \mathbb{R}^n \setminus \partial\Omega$. For the given pair $A, B \in \mathfrak{A}_\Delta$, introduce the matrix $D = \{D_{rs}\}_{1\leq r,s\leq n}$ as

$$D_{rs} := A_{rs} - B_{rs} \text{ for every } 1 \leq r, s \leq n. \tag{3.8}$$

Then D is an antisymmetric $n \times n$ matrix (cf. D. Mitrea, I. Mitrea, and M. Mitrea [26, (1.1.38)]), that is,

$$D_{rs} = -D_{sr} \text{ for every } 1 \leq r, s \leq n, \tag{3.9}$$

and in particular $D_{rr} = 0$ for every $1 \leq r \leq n$. With this piece of notation, we can rewrite the left-hand side of (3.7) as

$$\begin{aligned} \mathscr{D}_A(f_0)(x) - \mathscr{D}_B(f_0)(x) &= -\int_{\partial\Omega} \nu_s(y)(\partial_r E_\Delta)(x-y) D_{rs} f_0(y) d\sigma(y) \\ &= \int_{\partial\Omega} \nu_s(y)\partial_r^y[E_\Delta(x-y)] D_{rs} f_0(y) d\sigma(y). \end{aligned} \tag{3.10}$$

Using that $D_{rr} = 0$ for every $1 \leq r \leq n$ and explicitly writing the summation over repeated indices give us that the right-hand side on the last line in (3.10) is equal to

$$\sum_{1\leq r<s\leq n} \int_{\partial\Omega} \Big(\nu_s(y)\partial_r^y[E_\Delta(x-y)] D_{rs} + \nu_r(y)\partial_s^y[E_\Delta(x-y)] D_{sr}\Big) f_0(y) d\sigma(y). \tag{3.11}$$

The antisymmetry of the matrix D as in (3.9), the identity $\nu_s\partial_r - \nu_r\partial_s = \partial_{\tau_{sr}}$, and integration by parts imply that (3.11) may be written as

$$\sum_{1\leq r<s\leq n}\int_{\partial\Omega}\Big(\nu_s(y)\partial_r^y[E_\Delta(x-y)]D_{rs}+\nu_r(y)\partial_s^y[E_\Delta(x-y)]D_{sr}\Big)f_0(y)d\sigma(y)$$

$$=\sum_{1\leq r<s\leq n}\int_{\partial\Omega}(\nu_s(y)\partial_r^y-\nu_r(y)\partial_s^y)[E_\Delta(x-y)]D_{rs}f_0(y)d\sigma(y)$$

$$=\sum_{1\leq r<s\leq n}\int_{\partial\Omega}\partial_{\tau_{sr}(y)}[E_\Delta(x-y)]D_{rs}f_0(y)d\sigma(y)$$

$$=\sum_{1\leq r<s\leq n}\int_{\partial\Omega}E_\Delta(x-y)D_{rs}\partial_{\tau_{rs}(y)}[f_0(y)]d\sigma(y)$$

$$=\sum_{1\leq r<s\leq n}\mathscr{S}_\Delta(D_{rs}\partial_{\tau_{rs}}f_0)(x). \tag{3.12}$$

The conclusion is that the left-hand side of (3.7) is given by the expression in the last line of (3.12).

For the right-hand side of (3.7), write

$$\mathscr{S}_\Delta(\nu_rA_{rs}f_{e_s})(x)-\mathscr{S}_\Delta(\nu_rB_{rs}f_{e_s})(x)=\mathscr{S}_\Delta(\nu_rD_{rs}f_{e_s})(x). \tag{3.13}$$

Explicitly writing the summation over repeated indices and the antisymmetry of the matrix D as in (3.9) imply that

$$\begin{aligned}\mathscr{S}_\Delta(\nu_rD_{rs}f_{e_s})(x)&=\sum_{1\leq r<s\leq n}\mathscr{S}_\Delta(\nu_rD_{rs}f_{e_s}+\nu_sD_{sr}f_{e_r})(x)\\&=\sum_{1\leq r<s\leq n}\mathscr{S}_\Delta(D_{rs}(\nu_rf_{e_s}-\nu_sf_{e_r}))(x)\end{aligned} \tag{3.14}$$

The compatibility conditions of the array $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ in (2.40) gives that

$$\nu_r f_{e_s} - \nu_s f_{e_r} = \partial_{\tau_{rs}} f_0 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.15}$$

thus the last line in (3.14) is equal to

$$\sum_{1\leq r<s\leq n} \mathscr{S}_\Delta(D_{rs}\partial_{\tau_{rs}} f_0)(x). \tag{3.16}$$

Since (3.12) is equal to (3.7), it follows from our previous conclusion that (3.7) holds pointwise for arbitrary $x \in \mathbb{R}^n \setminus \partial\Omega$, and this finishes the proof. □

From now on, given any $A \in \mathfrak{A}_\Delta$, we shall call the boundary-to-domain Cauchy operator introduced in (3.1) the boundary-to-domain Cauchy operator associated with the Laplacian. The next proposition contains some of its basic properties.

Proposition 3.1.2. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain with compact boundary, outward unit normal vector $\nu = (\nu_1, \ldots, \nu_n)$ and surface measure σ. Consider an integrability exponent $p \in (1,\infty)$ and the coefficient tensor $A = (A_{rs})_{1\leq r,s\leq n} \in \mathfrak{A}_\Delta$. Then for each array $\dot{f} = (f_0, f_{e_1}, \ldots, f_{e_n}) \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ the complex-valued function $\dot{\mathscr{C}}_A\dot{f}$ satisfies*

$$\dot{\mathscr{C}}_A\dot{f} \in C^\infty(\mathbb{R}^n \setminus \partial\Omega) \ \text{ and } \ \Delta(\dot{\mathscr{C}}_A\dot{f}) = 0 \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \tag{3.17}$$

and for each $l \in \{1, \dots, n\}$ there holds

$$\partial_l \dot{\mathscr{C}}_A \dot{f}(x) = \mathscr{D}_A(f_{e_l})(x) + \int_{\partial\Omega} \partial_{\tau_{rl}(y)}[E_\Delta(x-y)] A_{rs} f_{e_s}(y) d\sigma(y) \tag{3.18}$$

for all $x \in \mathbb{R}^n \setminus \partial\Omega$.

Moreover, given an aperture parameter $\kappa \in (0, \infty)$, the following holds. If Ω is bounded, then there is a positive constant $C = C(\Omega, p, \kappa)$ such that for every Lebesgue-Whitney array $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$

$$\sum_{|\gamma| \leq 1} ||\mathcal{N}_\kappa(\partial^\gamma \dot{\mathscr{C}}_A \dot{f})||_{L^p(\partial\Omega,\sigma)} \leq C ||\dot{f}||_{\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]}. \tag{3.19}$$

If Ω is unbounded, that is, if Ω is an exterior UR domain, then there is a positive constant $C = C(\Omega, p, \kappa)$ such that for every Lebesgue-Whitney array $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$

$$\sum_{|\gamma| = 1} ||\mathcal{N}_\kappa(\partial^\gamma \dot{\mathscr{C}}_A \dot{f})||_{L^p(\partial\Omega,\sigma)} \leq C ||\dot{f}||_{\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]}. \tag{3.20}$$

Additionally, there is a constant $c \in \mathbb{C}$ such that

$$\dot{\mathscr{C}}_A \dot{f}(x) = c E_\Delta(x) + \mathcal{O}(|x|^{1-n}) \ \text{ as } |x| \to \infty, \tag{3.21}$$

and for every $\varepsilon > 0$ there is a positive constant $C_\varepsilon = C(\varepsilon, \Omega, p, \kappa)$ such that

$$||\mathcal{N}_\kappa^\varepsilon(\dot{\mathscr{C}}_A \dot{f})||_{L^p(\partial\Omega,\sigma)} \leq C_\varepsilon ||\dot{f}||_{\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]}. \tag{3.22}$$

Proof. The claims in (3.17) are an immediate consequence of the definition and the properties of the canonical fundamental solution for the Laplacian. To prove (3.18) consider $l \in \{1, \ldots, n\}$ and for $x \in \mathbb{R}^n \setminus \partial\Omega$ compute

$$\partial_l \dot{\mathscr{C}}_A \dot{f}(x) = \partial_l \mathscr{D}_A(f_0)(x) - \partial_l \mathscr{S}_\Delta(\nu_r A_{rs} f_{e_s})(x). \tag{3.23}$$

Consider the first term on the right-hand side of (3.23). Since E_Δ is the canonical fundamental solution of the Laplacian operator, and $A \in \mathfrak{A}_\Delta$, it follows that

$$(A_{rs}\partial_r\partial_s E_\Delta)(x-y) = (\Delta E_\Delta)(x-y) = 0 \ \text{ for } x \in \mathbb{R}^n \setminus \partial\Omega \text{ and } y \in \partial\Omega. \tag{3.24}$$

Thus using the identity $\nu_s \partial_l = \partial_{\tau_{sl}} + \nu_l \partial_s$, equation (3.24), and integration by parts allow us to write

$$\begin{aligned}
\partial_l \mathscr{D}_A(f_0)(x) &= -\int_{\partial\Omega} \nu_s(y)[(\partial_l\partial_r E_\Delta)(x-y)]A_{rs}f_0(y)d\sigma(y) \\
&= \int_{\partial\Omega} \nu_s(y)\partial_l^y[(\partial_r E_\Delta)(x-y)]A_{rs}f_0(y)d\sigma(y) \\
&= \int_{\partial\Omega} \big(\partial_{\tau_{sl}(y)} + \nu_l(y)\partial_s^y\big)[(\partial_r E_\Delta)(x-y)]A_{rs}f_0(y)d\sigma(y) \\
&= \int_{\partial\Omega} \partial_{\tau_{sl}(y)}[(\partial_r E_\Delta)(x-y)]A_{rs}f_0(y)d\sigma(y) \\
&= \int_{\partial\Omega} (\partial_r E_\Delta)(x-y)A_{rs}\partial_{\tau_{ls}(y)}[f_0(y)]d\sigma(y).
\end{aligned} \tag{3.25}$$

The compatibility condition of the array $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ as in (2.40) gives us that

$$\partial_{\tau_{ls}(y)}[f_0(y)] = \nu_l(y) f_{e_s}(y) - \nu_s(y) f_{e_l}(y), \tag{3.26}$$

thus the last line of (3.25) is equal to

$$\partial_l \mathscr{D}_A(f_0)(x) = \mathscr{D}_A(f_{e_l})(x) + \int_{\partial\Omega} \nu_l(y)(\partial_r E_\Delta)(x-y) A_{rs} f_{e_s}(y) d\sigma(y). \tag{3.27}$$

One also readily verifies that

$$\partial_l \mathscr{S}_\Delta(\nu_r A_{rs} f_{e_s})(x) = \int_{\partial\Omega} \nu_r(y)(\partial_l E_\Delta)(x-y) A_{rs} f_{e_s}(y) d\sigma(y). \tag{3.28}$$

Inserting (3.27) and (3.28) into (3.23), and using the identity $\partial_{\tau_{rl}} = \nu_r \partial_l - \nu_l \partial_r$ imply the result in (3.18).

For the boundedness of the nontangential maximal functions stated in (3.19), (3.20) and (3.22), note that the definitions in (2.42) and the result in (3.18), in concert with the Calderón-Zygmund theory in UR domains in [25, Theorem 1.5.1] imply that there is a constant $C \in (0, \infty)$ depending on $A \in \mathfrak{A}_\Delta$, the UR constants of $\partial\Omega$, and $p \in (1, \infty)$, such that for each $l \in \{1, \ldots, n\}$

$$||\mathcal{N}_\kappa(\partial_l \dot{\mathscr{C}}_A \dot{f})||_{L^p(\partial\Omega,\sigma)} \leq C ||\dot{f}||_{\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]}. \tag{3.29}$$

By Lemma 3.1.1 the boundary-to-domain Cauchy operator is independent of the choice of coefficient tensor in $\mathfrak{A}_{\Delta}$, so the same argument as above holds for every $A \in \mathfrak{A}_{\Delta}$. This allows us to take the constant on the last line of (3.29) to be independent of A, hence proving the desired inequalities.

To prove (3.21) consider $c \in \mathbb{C}$ given by

$$c := -\int_{\partial\Omega} \nu_r(y) A_{rs} f_{e_s}(y)\, d\sigma(y). \tag{3.30}$$

Then for every $x \in \mathbb{R}^n \setminus \partial\Omega$

$$\dot{\mathscr{C}}_A \dot{f}(x) - cE_{\Delta}(x) = \mathscr{D}_A f_0(x) - \int_{\partial\Omega} \big[E_{\Delta}(x-y) - E_{\Delta}(x)\big] \nu_r(y) A_{rs} f_{e_s}(y)\, d\sigma(y). \tag{3.31}$$

Since $\partial\Omega$ is compact, it follows from the mean value inequality, properties of the canonical fundamental solution for the Laplacian, and (3.31) that

$$\dot{\mathscr{C}}_A \dot{f}(x) - cE_{\Delta}(x) = \mathcal{O}(|x|^{1-n}) \quad \text{as } |x| \to \infty \tag{3.32}$$

and this finishes the proof. □

Another remarkable property of the boundary-to-domain Cauchy operator associated with the Laplacian is reproduces harmonic functions. More precisely, the following integral representation formula allows one to recover a harmonic function from its nontangential traces at the boundary (under appropriate assumptions on nontangential traces and size). Recall the operator $\mathrm{Tr}_1^{\kappa-nt}$ from (2.7).

Theorem 3.1.3. *Fix $n \in \mathbb{N}$ with $n \geq 2$, an integrability exponent $p \in (1,\infty)$ and an aperture parameter $\kappa \in (0,\infty)$. Let $\Omega \subset \mathbb{R}^n$ be UR domain with compact boundary. Suppose $u : \Omega \to \mathbb{C}$ satisfies*

$$u \in C^2(\Omega) \text{ and } \Delta u = 0 \text{ in } \Omega, \tag{3.33}$$

and is such that

$$u|_{\partial\Omega}^{\kappa-nt} \text{ and } \nabla u|_{\partial\Omega}^{\kappa-nt} \text{ exist } \sigma\text{-a.e. on } \partial\Omega. \tag{3.34}$$

Additionally, if Ω is bounded, then suppose

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma) \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma), \tag{3.35}$$

and if Ω is unbounded, then suppose

$$\begin{cases} \text{For every } \varepsilon > 0,\ \mathcal{N}_\kappa^\varepsilon u \in L^p(\partial\Omega,\sigma) \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma) \\ \text{There is } c \in \mathbb{C} \text{ such that } u(x) = cE_\Delta(x) + \mathcal{O}(|x|^{1-n}) \text{ as } |x| \to \infty. \end{cases} \tag{3.36}$$

Then for all $x \in \Omega$

$$u(x) = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt}u)(x). \tag{3.37}$$

Proof. This theorem follows as a consequence of D. Mitrea, I. Mitrea, and M. Mitrea [24, Theorem 1.5.1], hence we need to verify that the hypotheses are met.

First, note that if $\Omega \subset \mathbb{R}^n$ is an UR domain with compact boundary, then it is in particular a lower Ahlfors regular domain, and the surface measure σ is a doubling measure.

Second, the considerations in (3.33) and (3.34) are stronger than the ones stated in [24, (1.5.2)]. Moreover, the integrability and decay conditions in (3.35) and (3.36), in concert with the fact that $\partial\Omega$ is compact, imply that conditions of D. Mitrea, I. Mitrea, and M. Mitrea [24, (1.5.3), (1.5.5) and (1.5.9)] are met.

The conclusion is that u satisfies the hypotheses of [24, Theorem 1.5.1], which finishes the proof. □

Moving on, let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and fix an integrability exponent $p \in (1,\infty)$ along with a coefficient tensor $A = (A_{rs})_{1\leq r,s\leq n} \in \mathfrak{A}_\Delta$. Define the boundary-to-boundary Cauchy operator as the map that acts on a Lebesgue-Whitney array $\dot{f} = (f_0, f_{e_1}, \ldots, f_{e_n}) \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ and produces the family of functions

$$\dot{C}_A\dot{f} := \{(\dot{C}_A\dot{f})_0, (\dot{C}_A\dot{f})_{e_1}, \ldots, (\dot{C}_A\dot{f})_{e_n}\}, \tag{3.38}$$

where, for σ-a.e. point $x \in \partial\Omega$, with $\nu = (\nu_1, \ldots, \nu_n)$ the outward unit normal to Ω,

$$(\dot{C}_A \dot{f})_0(x) := K_A f_0(x) - S_\Delta(\nu_r A_{rs} f_{e_s})(x), \tag{3.39}$$

while for each $l \in \{1, \ldots, n\}$ and for σ-a.e. point $x \in \partial\Omega$

$$(\dot{C}_A \dot{f})_{e_l}(x) := K_A f_{e_l}(x) + \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \partial_{\tau_{rl}(y)}[E_\Delta(x-y)] A_{rs} f_{e_s}(y) d\sigma(y). \tag{3.40}$$

A number of basic properties of this operator and its relation to the boundary-to-domain Cauchy operator associated with the Laplacian are contained in the next proposition.

Proposition 3.1.4. *Fix $n \in \mathbb{N}$ with $n \geq 2$, an exponent $p \in (1, \infty)$ and an aperture parameter $\kappa \in (0, \infty)$. Let $\Omega \subset \mathbb{R}^n$ be a UR domain with compact boundary. Then for every $A \in \mathfrak{A}_\Delta$ the following properties hold:*

1: For every $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ and σ-a.e. on $\partial\Omega$, the first-order nontangential boundary trace of the function $\dot{\mathscr{C}}_A \dot{f}$ exists and satisfy

$$\mathrm{Tr}_1^{\kappa-nt}(\dot{\mathscr{C}}_A \dot{f}) = (\tfrac{1}{2} I + \dot{C}_A)\dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.41}$$

2: The operator $\dot{C}_A$ is bounded from $\mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ to itself. More precisely,

$$\dot{C}_A : \mathrm{WA}_1[L^p(\partial\Omega, \sigma)] \to \mathrm{WA}_1[L^p(\partial\Omega, \sigma)] \tag{3.42}$$

is well-defined, linear, and bounded.

Proof. To prove 1 we note that by definition and (3.38)-(3.40) we need to show that

$$\dot{\mathscr{C}}_A \dot{f}\big|_{\partial\Omega}^{\kappa-nt} = \tfrac{1}{2} f_0 + (\dot{C}_A \dot{f})_0 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.43}$$

and that for each $l \in \{1, \ldots, n\}$

$$\partial_l \dot{\mathscr{C}}_A \dot{f}\big|_{\partial\Omega}^{\kappa-nt} = \tfrac{1}{2} f_{e_l} + (\dot{C}_A \dot{f})_{e_l} \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.44}$$

To check that (3.43) holds we use D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 1.5.1] to conclude that the nontangential boundary trace of the second term on the right-hand side of (3.39) is given by

$$\mathscr{S}_\Delta(\nu_r A_{rs} f_{e_s})\big|_{\partial\Omega}^{\kappa-nt}(x) = S_\Delta(\nu_r A_{rs} f_{e_s})(x) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.45}$$

For the first term on the right-hand side of (3.39) we use the jump formulas in D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 1.5.1] for the double layer associated with $A \in \mathfrak{A}_\Delta$ to obtain

$$\mathscr{D}_A(f_0)\big|_{\partial\Omega}^{\kappa-nt}(x) = \tfrac{1}{2} f_0(x) + K_A f_0(x) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.46}$$

Therefore (3.43) is a consequence of (3.45) and (3.46). To prove that (3.44) holds we use Proposition 3.1.2 to write

$$\partial_l \dot{\mathscr{C}}_A \dot{f}(x) = \mathscr{D}(f_{e_l})(x) + \int_{\partial\Omega} \partial_{\tau_{rl}(y)}[E_\Delta(x-y)] A_{rs} f_{e_s}(y) d\sigma(y) \ \text{ for } x \in \Omega. \tag{3.47}$$

By the jump formula in D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 1.5.1] for the double layer associated with $A \in \mathfrak{A}_\Delta$ we have that the nontangential boundary trace of the first term on the right-hand side of (3.47) is given by

$$\mathscr{D}_A(f_{e_l})\big|_{\partial\Omega}^{\kappa-nt}(x) = \tfrac{1}{2} f_{e_l}(x) + K_A f_{e_l}(x) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.48}$$

For the second term, note that the integral kernel of this operator satisfies the hypotheses in D. Mitrea, I. Mitrea, and M. Mitrea [25, Proposition 1.2.1], hence its nontangential boundary trace is given by

$$\begin{aligned} &\left(\int_{\partial\Omega} \partial_{\tau_{rl}(y)}[E_\Delta(\cdot - y)] A_{rs} f_{e_s}(y) d\sigma(y) \right) \Big|_{\partial\Omega}^{\kappa-nt}(x) \\ &= \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \partial_{\tau_{rl}(y)}[E_\Delta(x-y)] A_{rs} f_{e_s}(y) d\sigma(y) \ \ \sigma\text{-a.e. on } \partial\Omega, \end{aligned} \tag{3.49}$$

and this finishes the proof of 1.

To prove 2 we first check that for an arbitrary $\dot{f} = (f_0, f_{e_1}, \ldots, f_{e_n}) \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$, the family $\{\dot{C}_A \dot{f}_\gamma\}_{|\gamma|\leq 1}$ satisfies the compatibility conditions in (2.40). More precisely, given $j, l \in \{1, \ldots, n\}$ we need to verify that for σ-a.e. $x \in \partial\Omega$

$$\partial_{\tau_{jl}(x)}[(\dot{C}_A \dot{f})_0(x)] = \nu_j(x)(\dot{C}_A \dot{f})_{e_l}(x) - \nu_l(x)(\dot{C}_A \dot{f})_{e_j}(x). \tag{3.50}$$

Using the results in 1, the definition of the tangential derivative operator, and the compatibility condition for the array $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$, we may write

$$\begin{aligned}
\partial_{\tau_{jl}}(\dot{C}_A \dot{f})_0 &= \partial_{\tau_{jl}}(\dot{\mathscr{C}}_A \dot{f}\big|_{\partial\Omega}^{\kappa-nt}) - \tfrac{1}{2}\partial_{\tau_{jl}} f_0 \\
&= \nu_j \left(\partial_l \dot{\mathscr{C}}_A \dot{f}\big|_{\partial\Omega}^{\kappa-nt}\right) - \nu_l \left(\partial_j \dot{\mathscr{C}}_A \dot{f}\big|_{\partial\Omega}^{\kappa-nt}\right) - \tfrac{1}{2}\partial_{\tau_{jl}} f_0 \\
&= \nu_j \left(\tfrac{1}{2} f_{e_l} + (\dot{C}_A \dot{f})_{e_l}\right) - \nu_l \left(\tfrac{1}{2} f_{e_j} + (\dot{C}_A \dot{f})_{e_j}\right) - \tfrac{1}{2}\partial_{\tau_{jl}} f_0 \\
&= \nu_j (\dot{C}_A \dot{f})_{e_l} - \nu_l (\dot{C}_A \dot{f})_{e_j} + \tfrac{1}{2}\left(\nu_j f_{e_l} - \nu_l f_{e_j} - \partial_{\tau_{jl}} f_0\right) \\
&= \nu_j (\dot{C}_A \dot{f})_{e_l} - \nu_l (\dot{C}_A \dot{f})_{e_j},
\end{aligned} \tag{3.51}$$

for σ-a.e. on $\partial\Omega$, hence (3.50) holds, hence the family $\{\dot{C}_A \dot{f}_\gamma\}_{|\gamma|\leq 1}$ is an array.

Secondly, to conclude that the boundary-to-boundary Cauchy operator $\dot{C}_A$ is a well-defined, linear, bounded operator from $\mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ to itself note that the singular integral operators involved in the definition of the boundary-to-boundary Cauchy operator $\dot{C}_A$ are all treated in D. Mitrea, I. Mitrea, and M. Mitrea [25, Chapter 1], and thus

$$(\dot{C}_A \dot{f})_0 := K_A f_0 - S_\Delta(\nu_r A_{rs} f_{e_s}) \tag{3.52}$$

is a well-defined, linear, bounded operator from $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ to $L^p_1(\partial\Omega,\sigma)$ and for each $l\in\{1,\ldots,n\}$

$$(\dot{C}_A\dot{f})_{e_l}=K_Af_{e_l}+\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\|\cdot-y|>\varepsilon}}\partial_{\tau_{rl}(y)}[E_\Delta(\cdot-y)]A_{rs}f_{e_s}(y)d\sigma(y) \tag{3.53}$$

is a well-defined, linear, bounded operator from $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ to $L^p(\partial\Omega,\sigma)$.

All in all, we conclude that the boundary-to-boundary Cauchy operator $\dot{C}_A$ is a well-defined, linear, bounded operator from $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ to itself and this finishes the proof.

□

We are now ready to formulate and establish well-posedness for the overdetermined boundary value problem associated with the Laplacian, which is the topic of our next section.

3.2 Formulation and Well-posedness for the Overdetermined Boundary Value Problem

Fix $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and $\kappa \in (0, \infty)$ be an aperture parameter. For each exponent $p \in (1, \infty)$ we formulate the overdetermined boundary value problem for the Laplacian in Ω with prescribed boundary data $\dot{f} \in \text{WA}_1[L^p(\partial\Omega, \sigma)]$ as the task of finding a complex-valued function u in Ω satisfying

$$(\text{OBVP}_{\Delta,p}) \begin{cases} u \in C^2(\Omega), \ \ \Delta u = 0 \ \text{ in } \Omega, \\ \\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega, \sigma), \, \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\ \\ \text{Tr}_1^{\kappa - nt} u = \dot{f} \in \text{WA}_1[L^p(\partial\Omega, d\sigma)]. \end{cases} \tag{3.54}$$

As discussed in the Introduction, for $n = 3$ the boundary value problem $(\text{OBVP}_{\Delta,p})$ stated above is equivalent to finding the electrostatic potential in a charge-free region Ω with both electrostatic potential and electric field prescribed at the boundary. Therefore

$$\begin{aligned} &\text{A solution } u \text{ for } (\text{OBVP}_{\Delta,p}) \text{ exists provided the components of} \\ &\text{the array } \dot{f} \text{ satisfy an additional relation given by the PDE.} \end{aligned} \tag{3.55}$$

Indeed, the content of the next theorem characterizes the space of admissible boundary data associated with the boundary value problem $(\text{OBVP}_{\Delta,p})$. This can be interpreted as a rigorous mathematical statement of the comment in (3.55).

Theorem 3.2.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$, an exponent integrability $p \in (1, \infty)$ and an aperture parameter $\kappa \in (0, \infty)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain.*

Then the space of admissible boundary data for ($OBVP_{\Delta,p}$) in Ω, that is, the set

$$\big\{ \mathrm{Tr}_1^{\kappa-nt} u : u \in C^2(\Omega),\ \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega, \sigma) \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma) \big\}, \tag{3.56}$$

may be described as

$$\big\{ \dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)] : \dot{C}_A \dot{f} = \tfrac{1}{2} \dot{f} \big\}, \tag{3.57}$$

where $\dot{C}_A$ is the boundary-to-boundary Cauchy operator associated with the Laplacian.

Remark 3.2.2. Before starting the proof of the theorem, let us remark that the set in (3.56) is well-defined. To this end, suppose u is a function satisfying the properties in (3.56). Then by the Fatou-type theorem D. Mitrea, I. Mitrea and M. Mitrea [24, Theorem 3.3.4] the first-order nontangential boundary trace of u exists σ-a.e. on $\partial\Omega$, that is,

$$\mathrm{Tr}_1^{\kappa-nt} u = (u|_{\partial\Omega}^{\kappa-nt}, \nabla u|_{\partial\Omega}^{\kappa-nt}) \text{ exist } \sigma\text{-a.e. on } \partial\Omega. \tag{3.58}$$

Moreover, the assumption

$$\mathcal{N}_\kappa(u) \in L^p(\partial\Omega, \sigma) \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma) \tag{3.59}$$

ensures that $\mathrm{Tr}_1^{\kappa-nt} u$ belongs to $\mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$.

Proof. The first claim is that the set in (3.56) is contained in (3.57). To see that, note that if u has the properties in (3.56), then by (3.58) the nontangential boundary trace of u exists σ-a.e. on $\partial\Omega$ and u satisfies the requirements in Theorem 3.1.3. Thus the integral representation formula in (3.37) imply that $u = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt}u)$. Taking nontangential boundary traces and using Proposition 3.1.4 we obtain

$$\mathrm{Tr}_1^{\kappa-nt}u = (\tfrac{1}{2}I + \dot{C}_A)\mathrm{Tr}_1^{\kappa-nt}u \implies \tfrac{1}{2}\mathrm{Tr}_1^{\kappa-nt}u = \dot{C}_A\mathrm{Tr}_1^{\kappa-nt}u, \tag{3.60}$$

which proves the claim.

The second claim is that the set in (3.57) is a subset of the one in (3.56). Indeed, consider an element $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ of the set in (3.57). By Proposition 3.1.2 the function $\dot{\mathscr{C}}_A\dot{f}$ is a harmonic function satisfying the properties

$$\mathcal{N}_\kappa(\dot{\mathscr{C}}_A\dot{f}) \in L^p(\partial\Omega,\sigma) \text{ and } \mathcal{N}_\kappa(\nabla\dot{\mathscr{C}}_A\dot{f}) \in L^p(\partial\Omega,\sigma). \tag{3.61}$$

Therefore $\mathrm{Tr}_1^{\kappa-nt}(\dot{\mathscr{C}}_A\dot{f})$ is an element of the set in (3.56). By Proposition 3.1.4 its first-order nontangential boundary trace satisfies

$$\mathrm{Tr}_1^{\kappa-nt}(\dot{\mathscr{C}}_A\dot{f}) = (\tfrac{1}{2}I + \dot{C}_A)\dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.62}$$

Since $\dot{f}$ is in the set defined in (3.57), it follows that $\dot{C}_A\dot{f} = \frac{1}{2}\dot{f}$. Therefore from (3.62) we conclude that $\mathrm{Tr}_1^{\kappa-nt}(\dot{\mathscr{C}}_A\dot{f}) = \dot{f}$, and this proves the second claim. These two claims together prove the theorem. □

With this machinery, we are able to obtain a well-posedness result for the overdetermined boundary value problem for the Laplacian in bounded uniformly rectifiable domains in the Euclidean setting with boundary data in Lebesgue-Whitney spaces for arbitrary integrability exponent $p \in (1, \infty)$.

Theorem 3.2.3. *Fix $n \in \mathbb{N}$ with $n \geq 2$, an exponent $p \in (1, \infty)$ and an aperture parameter $\kappa \in (0, \infty)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and let $A \in \mathfrak{A}_\Delta$. Then the overdetermined boundary value problem for the Laplacian in Ω with boundary data $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$*

$$(\textit{OBVP}_{\Delta,p}) \begin{cases} u \in C^2(\Omega), \ \Delta u = 0 \ \textit{in } \Omega, \\ \\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega, \sigma), \ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\ \\ \mathrm{Tr}_1^{\kappa - nt} u = \dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, d\sigma)]. \end{cases}$$

is solvable if and only if the boundary data is admissible, i.e., it satisfies

$$\dot{C}_A \dot{f} = \tfrac{1}{2} \dot{f} \ \ \sigma\textit{-a.e. on } \partial\Omega, \tag{3.63}$$

where $\dot{C}_A$ is the boundary-to-boundary Cauchy operator associated with the Laplacian.

Moreover, the $(OBVP_{\Delta,p})$ in Ω is well-posed in the sense of Hadamard, that is, for each prescribed data from the space of admissible boundary data there exists a unique solution u and this solution depends continuously on the boundary data. Additionally, we may represent the solution as

$$u = \dot{\mathscr{C}}_A \dot{f} \ \text{ in } \Omega, \tag{3.64}$$

where $\dot{\mathscr{C}}_A$ is the boundary-to-domain Cauchy operator associated with the Laplacian

Proof. To prove solvability, note that if u is a solution of $(\text{OBVP}_{\Delta,p})$, then by Theorem 3.2.1

$$\frac{1}{2}\text{Tr}_1^{\kappa-nt} u = \dot{C}_A \text{Tr}_1^{\kappa-nt} u. \tag{3.65}$$

Conversely, if $\dot{f} \in \text{WA}_1[L^p(\partial\Omega)]$ is admissible, then by Theorem 3.2.1 there exists u in Ω that is a solution of $(\text{OBVP}_{\Delta,p})$ with boundary data $\dot{f} \in \text{WA}_1[L^p(\partial\Omega)]$.

For uniqueness note that if u_1 and u_2 are two solutions of the $(\text{OBVP}_{\Delta,p})$ for the Laplacian in Ω with boundary data $\dot{f} \in \text{WA}_1[L^p(\partial\Omega, \sigma)]$, then the function $v = u_1 - u_2$ is a solution of the $(\text{OBVP}_{\Delta,p})$ for the Laplacian in Ω with boundary data equal to zero, hence the integral representation formula in Theorem 3.1.3 implies that $v = 0$ in Ω.

Finally, it follows from the integral representation formula in Theorem 3.1.3 that if $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ is an admissible boundary data, then the function $u = \dot{\mathscr{C}}_A\dot{f}$ is the solution of the $(\mathrm{OBVP}_{\Delta,p})$ in Ω. By Proposition 3.1.2 the nontangential maximal function of all derivatives up to order one of $\dot{\mathscr{C}}_A\dot{f}$ are in $L^p(\partial\Omega)$ and its norm is bounded above by a positive constant multiple of the $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$-norm of the boundary data, and this finishes the proof. □

Using the richness of information and structure arising from the function space framework used to describe the boundary data, together with properties of the Cauchy operators associated with the Laplacian, the integral representation formula in Theorem 3.1.3 and the powerful Calderón–Zygmund theory in uniformly rectifiable domains in D. Mitrea, I. Mitrea, and M. Mitrea [25] we obtain the jump formulas stated in Proposition 3.1.4. These results allow us to characterize the space of admissible boundary data in Theorem 3.2.1, and thus conclude that for every $p \in (1,\infty)$, the overdetermined boundary value problem $(\mathrm{OBVP}_{\Delta,p})$ is well-posed for arbitrary bounded uniformly rectifiable domains in $\mathbb{R}^n$ if and only if the boundary data is admissible.

3.3 Hardy Spaces and Equivalent Formulations of the Overdetermined Boundary Value Problem in Uniformly Rectifiable Domains

The Cauchy operators discussed in the previous section are connected to the study of Hardy spaces associated with the Laplacian. These function spaces may be used in conjunction with an appropriate packing operator to construct a dictionary between the overdetermined boundary value problem ($\text{OBVP}_{\Delta,p}$) and different formulations of overdetermined boundary value problems for the Laplacian with L^p-based boundary data in uniformly rectifiable domains.

To set the stage, fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1, \infty)$ be an integrability exponent and $\kappa \in (0, \infty)$ be an aperture parameter. Recall that the domain Hardy space associated with the Laplacian in Ω is given by

$$\mathscr{H}^p(\Omega, \Delta) := \big\{u \in C^2(\Omega) : \Delta u = 0 \,\text{in}\, \Omega, \\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega, \sigma), \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma)\big\}. \tag{3.66}$$

This is a vector space, and we may endow it with the norm

$$||u||_{\mathscr{H}^p(\Omega,\Delta)} := ||\mathcal{N}_\kappa(u)||_{L^p(\partial\Omega,\sigma)} + ||\mathcal{N}_\kappa(\nabla u)||_{L^p(\partial\Omega,\sigma)}. \tag{3.67}$$

We claim that the domain Hardy space $\mathscr{H}^p(\Omega, \Delta)$ endowed with the norm in (3.67) is a Banach space.

Indeed, note that for all $p \in (1,\infty)$, it follows from Remark 3.2.2 that the nontangential boundary trace map

$$\begin{aligned} \mathrm{Tr}_1^{\kappa-nt} : \mathscr{H}^p(\Omega,\Delta) &\to \mathrm{WA}_1[L^p(\partial\Omega,\sigma)] \\ u &\mapsto \mathrm{Tr}_1^{\kappa-nt} u \end{aligned} \tag{3.68}$$

is a well-defined, linear, and bounded map. Fix $A \in \mathfrak{A}_\Delta$ coefficient tensor for the Laplacian and note that by Proposition 3.1.2 the boundary-to-domain Cauchy operator associated with the Laplacian

$$\begin{aligned} \dot{\mathscr{C}}_A : \mathrm{WA}_1[L^p(\partial\Omega,\sigma)] &\to \mathscr{H}^p(\Omega,\Delta) \\ \dot{f} &\mapsto \dot{\mathscr{C}}_A \dot{f} \end{aligned} \tag{3.69}$$

is a well-defined, linear, and bounded operator.

Now, suppose $\{u_j\}_{j\in\mathbb{N}} \subset \mathscr{H}^p(\Omega,\Delta)$ is a Cauchy sequence. Then $\{\mathrm{Tr}_1^{\kappa-nt} u_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$. Since $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ is a Banach space, there exists a Lebesgue-Whitney array $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ such that

$$\mathrm{Tr}_1^{\kappa-nt} u_j \longrightarrow \dot{f} \ \text{ in } \mathrm{WA}_1[L^p(\partial\Omega,\sigma)] \text{ as } j \to \infty. \tag{3.70}$$

Let $u = \dot{\mathscr{C}}_A \dot{f}$. Then by (3.69) we have $u \in \mathscr{H}^p(\Omega,\Delta)$.

Moreover, it follows from the integral representation formula in Theorem 3.1.3 that for each $j \in \mathbb{N}$ we have

$$u_j = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt} u_j), \tag{3.71}$$

and by (3.69) we conclude that

$$\begin{aligned} ||u_j - u||_{\mathscr{H}^p(\Omega,\Delta)} &= ||\dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt} u_j - \dot{f})||_{\mathscr{H}^p(\Omega,\Delta)} \\ &\leq C||\mathrm{Tr}_1^{\kappa-nt} u_j - \dot{f}||_{\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]}, \end{aligned} \tag{3.72}$$

which combined with (3.70) proves the claim.

Define the linear operator $\dot{P}_A : \mathrm{WA}_1[L^p(\partial\Omega,\sigma)] \to \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ as

$$\dot{P}_A \dot{f} := (\tfrac{1}{2} I + \dot{C}_A)\dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.73}$$

where $\dot{C}_A$ is the boundary-to-boundary Cauchy operator associated with the Laplacian.

By Proposition 3.1.4 the map $\dot{P}_A$ is well-defined, linear, and bounded. Furthermore, for each aperture parameter $\kappa \in (0,\infty)$ the nontangential boundary trace map satisfies

$$\mathrm{Tr}_1^{\kappa-nt}\big(\dot{\mathscr{C}}_A \dot{f}\big) = \dot{P}_A \dot{f} \ \ \text{for every } \dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]. \tag{3.74}$$

The next proposition establishes further analytical properties of the boundary-to-boundary Cauchy operator associated with the Laplacian and the map $\dot{P}_A$.

Theorem 3.3.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain with surface measure σ. Let $p \in (1, \infty)$ be an integrability exponent and $\kappa \in (0, \infty)$ be an aperture parameter. Let $A \in \mathfrak{A}_\Delta$. Then the boundary-to-boundary Cauchy operator associated with the Laplacian satisfies*

$$(\dot{C}_A)^2 = \tfrac{1}{4} I \ \text{ on } \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]. \tag{3.75}$$

Moreover, the map $\dot{P}_A$ defined in (3.73) *is a continuous linear projection on* $\mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$.

Proof. To prove the claim in (3.75), let $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ be an arbitrary Lebesgue-Whitney array. Then the function $\dot{\mathscr{C}}_A \dot{f}$ satisfies the conditions on Theorem 3.1.3, hence the integral representation formula implies

$$\dot{\mathscr{C}}_A \dot{f} = \dot{\mathscr{C}}_A \big(\mathrm{Tr}_1^{\kappa - nt} \big(\dot{\mathscr{C}}_A \dot{f} \big) \big) \ \text{ in } \Omega. \tag{3.76}$$

It follows from Proposition 3.1.4 that the nontangential boundary trace of (3.76) yields the equality

$$(\tfrac{1}{2} I + \dot{C}_A) \dot{f} = (\tfrac{1}{2} I + \dot{C}_A)^2 \dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.77}$$

which implies the result. As a consequence of (3.77) we readily verify that $\dot{P}_A = \dot{P}_A^2$ on $\mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$, thus finishing the proof. □

Given any $A \in \mathfrak{A}_\Delta$, define the boundary Hardy space associated with the Laplacian on $\partial\Omega$

$$\mathscr{H}^p(\partial\Omega, \Delta) := \dot{P}_A \big(\mathrm{WA}_1[L^p(\partial\Omega, \sigma)] \big), \tag{3.78}$$

endowed with the norm from $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$. With this piece of notation, we have the following corollary that connects the boundary Hardy space associated with the Laplacian and the space of admissible boundary data for the overdetermined boundary value problem $(\mathrm{OBVP}_{\Delta,p})$ in Ω.

Corollary 3.3.2. *Fix $n\in\mathbb{N}$ with $n\geq 2$. Let $\Omega\subset\mathbb{R}^n$ be a bounded UR domain, $p\in(1,\infty)$ be an integrability exponent, $\kappa\in(0,\infty)$ be an aperture parameter and $A\in\mathfrak{A}_\Delta$ be a coefficient tensor for the Laplacian.*

Then the space of admissible boundary data for $(OBVP_{\Delta,p})$ in Ω introduced in Theorem 3.2.1, that is, the function space

$$\Big\{\mathrm{Tr}_1^{\kappa-nt}u : u\in C^2(\Omega),\ \Delta u=0 \text{ in } \Omega,\\ \mathcal{N}_\kappa(u)\in L^p(\partial\Omega,\sigma) \text{ and } \mathcal{N}_\kappa(\nabla u)\in L^p(\partial\Omega,\sigma)\Big\},\tag{3.79}$$

defined in (3.56) *coincides with the boundary Hardy space associated with the Laplacian on $\partial\Omega$*

$$\mathscr{H}^p(\partial\Omega,\Delta)=\dot{P}_A\big(\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]\big),\tag{3.80}$$

defined in (3.78)*.*

Proof. It follows from (3.57) in Theorem 3.2.1 that an element in the space of admissible boundary data for $(\mathrm{OBVP}_{\Delta,p})$ in Ω are arrays $\dot{f}\in\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ that satisfy the identity

$$\dot{C}_A\dot{f}=\tfrac{1}{2}\dot{f},\tag{3.81}$$

where $\dot{C}_A$ is the boundary-to-boundary Cauchy operator associated with the Laplacian. Therefore by the definition in (3.73) we readily verify that for these arrays we have

$$\dot{P}_A \dot{f} = (\tfrac{1}{2} I + \dot{C}_A)\dot{f} = \dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.82}$$

thus $\dot{f} \in \mathscr{H}^p(\partial\Omega, \Delta)$ by (3.78).

Conversely, suppose $\dot{f} \in \mathscr{H}^p(\partial\Omega, \Delta)$. Then by (3.73) and the definition in (3.78) there exists $\dot{g} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ such that

$$\dot{f} = \dot{P}_A \dot{g} = (\tfrac{1}{2} I + \dot{C}_A)\dot{g}. \tag{3.83}$$

By Theorem 3.3.1 we have that

$$\dot{C}_A \dot{f} = \dot{C}_A(\tfrac{1}{2} I + \dot{C}_A)\dot{g} = (\tfrac{1}{2}\dot{C}_A + \dot{C}_A^2)\dot{g} = (\tfrac{1}{2}\dot{C}_A + \tfrac{1}{4} I)\dot{g} = \tfrac{1}{2}\dot{f}, \tag{3.84}$$

hence elements in $\mathscr{H}^p(\partial\Omega, \Delta)$ are in the set defined in (3.57) in Theorem 3.2.1 and this finishes the proof. □

We also have the following result regarding these Hardy spaces and the nontangential boundary trace map.

Theorem 3.3.3. *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1, \infty)$ be an integrability exponent, and $\kappa \in (0, \infty)$ be an aperture parameter. Then the nontangential boundary trace map $\mathrm{Tr}_1^{\kappa - nt}$ is an isomorphism from $\mathscr{H}^p(\Omega, \Delta)$ to $\mathscr{H}^p(\partial\Omega, \Delta)$.*

Proof. It follows from Remark 3.2.2 and the integral representation formula in Theorem 3.1.3 that

$$u \in \mathscr{H}^p(\Omega, \Delta) \implies u = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt} u) \text{ in } \Omega, \tag{3.85}$$

thus by Proposition 3.1.4

$$\mathrm{Tr}_1^{\kappa-nt} u = \dot{P}_A(\mathrm{Tr}_1^{\kappa-nt} u). \tag{3.86}$$

This shows that $u \mapsto \mathrm{Tr}_1^{\kappa-nt} u$ is a well-defined, linear, and bounded map from $\mathscr{H}^p(\Omega, \Delta)$ to $\mathscr{H}^p(\partial\Omega, \Delta)$.

To see that $\mathrm{Tr}_1^{\kappa-nt}$ is injective, note that if u_1 and u_2 are functions in $\mathscr{H}^p(\Omega, \Delta)$, then the linearity of the nontangential boundary trace and the boundary-to-domain Cauchy operator associated with the Laplacian, together with (3.85) give us that

$$u_1 - u_2 = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt}(u_1 - u_2)) \text{ in } \Omega. \tag{3.87}$$

Hence $\mathrm{Tr}_1^{\kappa-nt} u_1 = \mathrm{Tr}_1^{\kappa-nt} u_2$ σ-a.e. on $\partial\Omega$ implies $u_1 = u_2$ in $\mathscr{H}^p(\Omega, \Delta)$.

To see that $\mathrm{Tr}_1^{\kappa-nt}$ is surjective it is enough to show that for a given Lebesgue-Whitney array $\dot{f} \in \mathscr{H}^p(\partial\Omega, \Delta)$ there is an element in $\mathscr{H}^p(\Omega, \Delta)$ whose nontangential boundary trace is $\dot{f}$.

It follows from the definition that for $\dot{f} \in \mathscr{H}^p(\partial\Omega, \Delta)$ there exists a Lebesgue-Whitney array $\dot{g} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ such that

$$\dot{f} = \dot{P}_A \dot{g} \text{ in } \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]. \tag{3.88}$$

Consider the function $\dot{\mathscr{C}}_A\dot{g}$. Then by Proposition 3.1.2 this is an element of $\mathscr{H}^p(\Omega,\Delta)$, and Proposition 3.1.4 in concert with (3.88) imply

$$\mathrm{Tr}_1^{\kappa-nt}(\dot{\mathscr{C}}_A\dot{g}) = \dot{P}_A\dot{g} = \dot{f}, \tag{3.89}$$

thus finishing the proof. □

Remark 3.3.4. It follows from the proof of the previous proposition that if $\Omega \subset \mathbb{R}^n$ is a bounded UR domain and $p \in (1,\infty)$ is an integrability exponent, then the boundary-to-domain Cauchy operator associated with the Laplacian is such that

$$\dot{\mathscr{C}}_A : \mathscr{H}^p(\partial\Omega,\Delta) \to \mathscr{H}^p(\Omega,\Delta) \text{ is an isomorphism.} \tag{3.90}$$

In this context, the nontangential boundary trace and the boundary-to-domain Cauchy operator associated with the Laplacian are inverses.

Our next goal is to show that different formulations of the overdetermined boundary value problem for the Laplacian in bounded uniformly rectifiable domains with boundary data in L^p-based function spaces are equivalent to the boundary problem $(\mathrm{OBVP}_{\Delta,p})$ discussed in the previous section.

To set the stage, fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain. Consider an integrability exponent $p \in (1,\infty)$ and a matrix $B \in \mathbb{C}^{n\times n}$.

Define the packing operator associated with B acting on each Lebesgue-Whitney array $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ as

$$\dot{\Theta}_B \dot{f} := (f_0, \nu_r B_{rs} f_{e_s}). \tag{3.91}$$

Note that

$$\dot{\Theta}_B : \mathrm{WA}_1[L^p(\partial\Omega, \sigma)] \to L^p(\partial\Omega, \sigma) \oplus L^p(\partial\Omega, \sigma) \tag{3.92}$$

is a well-defined, bounded, and linear map.

Define $\mathfrak{A}_{\text{inv}}(n, 1, \Omega)$ as the collection of $n \times n$ matrices such that

$$[B_{rs}\nu_r\nu_s]^{-1} \text{ is well-defined and belongs to } L^\infty(\partial\Omega, \sigma), \tag{3.93}$$

where ν is the geometric outward unit normal to Ω.

It follows from D. Mitrea, I. Mitrea, and M. Mitrea [26, Definition 1.1.2] that the inclusion $\mathfrak{A}_{\text{WE}}(n, 1) \subset \mathfrak{A}_{\text{inv}}(n, 1, \Omega)$ holds for every $\Omega \subset \mathbb{R}^n$ UR domain. Indeed, note that if $B \in \mathfrak{A}_{\text{WE}}(n, 1)$, then

$$B_{rs}\xi_r\xi_s \neq 0 \text{ for every non-zero } \xi \in \mathbb{R}^n. \tag{3.94}$$

Since the expression on the left-hand side of (3.94) is a polynomial of degree two in $\xi \in \mathbb{R}^n$, the map $\xi \mapsto |B_{rs}\xi_r\xi_s|$ attains a maximum and a minimum on the sphere $\xi \in \mathbb{S}^{n-1}$. Thus there are constants $c_1, c_2 \in (0,\infty)$ such that

$$c_1 \leq |[B_{rs}\nu_r\nu_s]^{-1}| \leq c_2 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.95}$$

which ensures that $B \in \mathfrak{A}_{\text{inv}}(n,1,\Omega)$.

We then have the following mapping properties for the packing operator.

Proposition 3.3.5. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. Consider an exponent $p \in (1,\infty)$ and a matrix $B \in \mathfrak{A}_{\text{inv}}(n,1,\Omega)$. Then the packing operator defined in* (3.91)-(3.92) *is injective.*

Moreover, given a pair of functions $(g_0, g_1) \in L^p(\partial\Omega,\sigma) \oplus L^p(\partial\Omega,\sigma)$ such that

$$(g_0,g_1) \in \dot{\Theta}_B(\mathbf{WA}_1[L^p(\partial\Omega,\sigma)]), \tag{3.96}$$

the family $\{f_\gamma\}_{|\gamma|\leq 1}$ defined, σ-a.e. on $\partial\Omega$, as

$$\begin{cases} f_0 := g_0, \\ f_{e_j} := [B_{kl}\nu_k\nu_l]^{-1}(\nu_j g_1 + \nu_r B_{rs}\partial_{\tau_{sj}} g_0) \ \textit{for every } j \in \{1,\dots,n\}, \end{cases} \tag{3.97}$$

and satisfying

$$\begin{cases} f_0 \in L_1^p(\partial\Omega, \sigma), \\ \\ f_{e_j} \in L^p(\partial\Omega, \sigma) \ \textit{for every } 1 \leq j \leq n, \end{cases} \tag{3.98}$$

is the unique Lebesgue-Whitney array such that $\dot{\Theta}_B \dot{f} = (g_0, g_1)$.

Hence for every $B \in \mathfrak{A}_{\text{inv}}(n, 1, \Omega)$*, the packing operator* $\dot{\Theta}_B$ *assembles Lebesgue-Whitney arrays in a linear, bounded, and reversible fashion.*

Proof. To prove the injectivity suppose that $\dot{f}, \dot{h} \in \text{WA}_1[L^p(\partial\Omega)]$ are such that

$$\dot{\Theta}_B \dot{f} = \dot{\Theta}_B \dot{h}. \tag{3.99}$$

Then by definition, we must have

$$\nu_r B_{rs} f_{e_s} = \nu_r B_{rs} h_{e_s} \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.100}$$

and $f_0 = h_0$ in $L_1^p(\partial\Omega, \sigma)$. In particular, σ-a.e. on $\partial\Omega$,

$$\partial_{\tau_{sj}} f_0 = \partial_{\tau_{sj}} h_0 \ \ \text{for every } 1 \leq s, j \leq n. \tag{3.101}$$

It follows from the compatibility conditions of the array $\dot{f} \in \text{WA}_1[L^p(\partial\Omega)]$ in (2.40) that for each $j \in \{1, \ldots, n\}$ we have

$$\nu_j \nu_r B_{rs} f_{e_s} + \nu_r B_{rs} \partial_{\tau_{sj}} f_0 = \nu_j \nu_r B_{rs} f_{e_s} + \nu_r B_{rs} (\nu_s f_{e_j} - \nu_j f_{e_s}) = \nu_r B_{rs} \nu_s f_{e_j}, \tag{3.102}$$

thus if $B \in \mathfrak{A}_{\text{inv}}(n,1,\Omega)$, then (3.102) implies that

$$f_{e_j} = [B_{kl}\nu_k\nu_l]^{-1}(\nu_j\nu_r B_{rs} f_{e_s} + \nu_r B_{rs}\partial_{\tau_{sj}} f_0) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.103}$$

This result, together with (3.100)-(3.101) and a similar calculation for the Whitney array $\dot{h} \in \text{WA}_1[L^p(\partial\Omega,\sigma)]$ allow us to conclude that $\dot{f} = \dot{h}$, as we wanted to show.

To prove the second part, suppose we are given $(g_0, g_1) \in \dot{\Theta}_B(\text{WA}_1[L^p(\partial\Omega,\sigma)])$, and consider the family $\{f_\gamma\}_{|\gamma|\leq 1}$ satisfying (3.97) and (3.98).

Due to the conditions in (3.98), we know that

$$\begin{cases} f_0 \in L^p_1(\partial\Omega,\sigma), \\ \\ f_{e_j} \in L^p(\partial\Omega,\sigma) \ \text{ for each } 1 \leq j \leq n, \end{cases} \tag{3.104}$$

hence to conclude the result it remains to prove that the family $\{f_\gamma\}_{|\gamma|\leq 1}$ is a Lebesgue-Whitney array in $\text{WA}_1[L^p(\partial\Omega,\sigma)]$, that is, we need to check if this family satisfy the compatibility conditions in (2.40). More precisely, it remains to be shown that for every $1 \leq i,j \leq n$

$$\partial_{\tau_{ij}} f_0 = \nu_i f_{e_j} - \nu_j f_{e_i} \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.105}$$

To see that (3.105) holds, note that (3.97) implies that $f_0 = g_0$ in $L^p_1(\partial\Omega,\sigma)$ and for each $1 \leq i,j \leq n$ and σ-a.e. on $\partial\Omega$

$$\nu_i f_{e_j} - \nu_j f_{e_i} = [B_{kl}\nu_k\nu_l]^{-1}(\nu_j\nu_i g_1 + \nu_r B_{rs}\nu_i\partial_{\tau_{sj}} f_0 - \nu_i\nu_j g_1 - \nu_r B_{rs}\nu_j\partial_{\tau_{si}} f_0)$$

$$= [B_{kl}\nu_k\nu_l]^{-1}(\nu_r B_{rs})(\nu_i \partial_{\tau_{sj}} f_0 - \nu_j \partial_{\tau_{si}} f_0). \tag{3.106}$$

Now, set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$, and fix an aperture parameter $\kappa \in (0, \infty)$. It follows from G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17, Theorem 1.6] that there are functions $u_\pm : \Omega_\pm \to \mathbb{C}$ such that

$$\begin{aligned} &u_\pm \in C^\infty(\Omega_\pm),\ u_- \text{ vanishes in a neighborhood of infinity, both} \\ &\mathcal{N}_\kappa(u_\pm) \text{ and } \mathcal{N}_\kappa(\nabla u_\pm) \text{ belong to } L^p(\partial\Omega, \sigma), \text{ the nontangential} \\ &\text{traces of } u_\pm \text{ and } \nabla u_\pm \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \end{aligned} \tag{3.107}$$

and

$$u_+\big|_{\partial\Omega}^{\kappa-nt} - u_-\big|_{\partial\Omega}^{\kappa-nt} = f_0 \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.108}$$

These results, in concert with G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17, Proposition 2.13], imply that for every $1 \leq j, s \leq n$ and σ-a.e. on $\partial\Omega$

$$\begin{aligned} \partial_{\tau_{sj}} f_0 &= \partial_{\tau_{sj}}(u_+\big|_{\partial\Omega}^{\kappa-nt} - u_-\big|_{\partial\Omega}^{\kappa-nt}) \\ &= \partial_{\tau_{sj}}(u_+\big|_{\partial\Omega}^{\kappa-nt}) - \partial_{\tau_{sj}}(u_-\big|_{\partial\Omega}^{\kappa-nt}) \\ &= (\nu_s \partial_j u_+\big|_{\partial\Omega}^{\kappa-nt} - \nu_j \partial_s u_+\big|_{\partial\Omega}^{\kappa-nt}) - (\nu_s \partial_j u_-\big|_{\partial\Omega}^{\kappa-nt} - \nu_j \partial_s u_-\big|_{\partial\Omega}^{\kappa-nt}) \\ &= \nu_s(\partial_j u_+\big|_{\partial\Omega}^{\kappa-nt} - \partial_j u_-\big|_{\partial\Omega}^{\kappa-nt}) - \nu_j(\partial_s u_+\big|_{\partial\Omega}^{\kappa-nt} - \partial_s u_-\big|_{\partial\Omega}^{\kappa-nt}). \end{aligned} \tag{3.109}$$

Thus the right-hand side in the last line of (3.109), G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17, Proposition 2.13], and (3.108) imply that for each collection $1 \leq i, j, s \leq n$ and σ-a.e. on $\partial\Omega$ we may write

$$\begin{aligned}
&\nu_i \partial_{\tau_{sj}} f_0 - \nu_j \partial_{\tau_{si}} f_0 \\
&= \nu_i \nu_s (\partial_j u_+\big|_{\partial\Omega}^{\kappa-nt} - \partial_j u_-\big|_{\partial\Omega}^{\kappa-nt}) - \nu_j \nu_s (\partial_i u_+\big|_{\partial\Omega}^{\kappa-nt} - \partial_i u_-\big|_{\partial\Omega}^{\kappa-nt}) \\
&= \nu_s (\partial_{\tau_{ij}} u_+\big|_{\partial\Omega}^{\kappa-nt} - \partial_{\tau_{ij}} u_-\big|_{\partial\Omega}^{\kappa-nt}) \\
&= \nu_s \partial_{\tau_{ij}} (u_+\big|_{\partial\Omega}^{\kappa-nt} - u_-\big|_{\partial\Omega}^{\kappa-nt}) \\
&= \nu_s \partial_{\tau_{ij}} f_0.
\end{aligned} \tag{3.110}$$

Inserting the right-hand side of the last line in (3.110) into (3.106) allows us to conclude that

$$\nu_i f_{e_j} - \nu_j f_{e_i} = [B_{kl}\nu_k\nu_l]^{-1}(\nu_r B_{rs})(\nu_s \partial_{\tau_{ij}} f_0) = \partial_{\tau_{ij}} f_0 \tag{3.111}$$

σ-a.e. on $\partial\Omega$, thus establishing (3.105) and finishing the proof. □

Remark 3.3.6. In general, without assumptions on the matrix B, the packing operator $\dot{\Theta}_B$ is not injective.

Indeed, fix $n = 2$ and an integrability exponent $p \in (1, \infty)$. Consider the smooth domain $\Omega := \{x \in \mathbb{R}^2 : |x| < 1\}$ and the matrix

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.112}$$

Note that $\nu(x) = (x_1, x_2)$ for every $x \in \partial\Omega$ and that

$$B_{rs}\nu_r\nu_s = 0 \text{ for every } x \in \partial\Omega. \tag{3.113}$$

Moreover, for every Lebesgue-Whitney array $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ there holds

$$\dot{\Theta}_B\dot{f} = (f_0, \nu_r B_{rs} f_{e_s}) = (f_0, \nu_1 f_{e_2} - \nu_2 f_{e_1}) = (f_0, \partial_{\tau_{12}} f_0). \tag{3.114}$$

Consequently, the Whitney arrays given by $\dot{f}(x) = (0, 0, 0)$ and $\dot{g}(x) = (0, x_1, x_2)$ at every $x \in \partial\Omega$ are such that $\dot{f} \neq \dot{g}$ in $\mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$, but

$$\dot{\Theta}_B\dot{f} = \dot{\Theta}_B\dot{g}. \tag{3.115}$$

The previous proposition plays a key role in studying other formulations of the overdetermined boundary value problem for the Laplacian.

Indeed, fix $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and $\kappa \in (0, \infty)$ be an aperture parameter. For each integrability exponent $p \in (1, \infty)$ and each matrix $B \in \mathfrak{A}_{\text{inv}}(n, 1, \Omega)$, we formulate the L^p-based overdetermined boundary value problem for the Laplacian in Ω associated with B and boundary data $(g_0, g_1) \in L^p(\partial\Omega, \sigma) \oplus L^p(\partial\Omega, \sigma)$

as the task of finding a complex-valued function u in Ω satisfying

$$(L^p\text{-OBVP}_{\Delta,B})\begin{cases} u\in C^2(\Omega),\ \ \Delta u=0\ \text{ in }\Omega, \\ \\ \mathcal{N}_\kappa(u)\in L^p(\partial\Omega,\sigma),\ \mathcal{N}_\kappa(\nabla u)\in L^p(\partial\Omega,\sigma), \\ \\ u\big|_{\partial\Omega}^{\kappa-nt}=g_0\in L^p(\partial\Omega,\sigma),\ \ \partial_\nu^B u\big|_{\partial\Omega}^{\kappa-nt}=g_1\in L^p(\partial\Omega,\sigma), \end{cases} \tag{3.116}$$

where $\partial_\nu^B u\big|_{\partial\Omega}^{\kappa-nt}:=\nu_r B_{rs}\partial_s u\big|_{\partial\Omega}^{\kappa-nt}$ is the conormal derivative of u associated with the matrix B.

We then have the following result regarding the boundary value problem $(L^p\text{-OBVP}_{\Delta,B})$ and its connections with Hardy spaces associated with the Laplacian and the overdetermined boundary value problem $(\text{OBVP}_{\Delta,p})$.

Theorem 3.3.7. *Retain the geometrical and analytical context from the previous paragraph and bring in the packing operator associated with B defined in* (3.91)-(3.92)*. Then the boundary value problem $(L^p\text{-OBVP}_{\Delta,B})$ is solvable if and only if the boundary data $(g_0,g_1)\in L^p(\partial\Omega,\sigma)\oplus L^p(\partial\Omega,\sigma)$ is such that*

$$(g_0,g_1)\in\dot{\Theta}_B(\mathscr{H}^p(\partial\Omega,\Delta)). \tag{3.117}$$

Moreover, the boundary value problem (L^p-*OBVP*$_{\Delta,B}$) *with prescribed boundary data* $(g_0, g_1) \in \dot{\Theta}_B(\mathscr{H}^p(\partial\Omega, \Delta))$ *admits a unique solution, and we may represent the solution as*

$$u = \dot{\mathscr{C}}_A \dot{f} \ \text{ in } \Omega, \tag{3.118}$$

where $\dot{f} \in \mathscr{H}^p(\partial\Omega, \Delta)$ *is the unique array such that* $\dot{\Theta}_B \dot{f} = (g_0, g_1)$*, and* $\dot{\mathscr{C}}_A$ *is the boundary-to-domain Cauchy operator associated with the Laplacian.*

Finally, $u : \Omega \to \mathbb{C}$ *is the solution of the boundary value problem* (L^p-*OBVP*$_{\Delta,B}$) *with boundary data* $(g_0, g_1) \in L^p(\partial\Omega, \sigma) \oplus L^p(\partial\Omega, \sigma)$ *if and only if the function* u *is the solution of the overdetermined boundary value problem* (*OBVP*$_{\Delta,p}$) *with boundary data* $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$*, where* $\dot{\Theta}_B \dot{f} = (g_0, g_1)$.

Proof. First, we prove the solvability result stated in the theorem.

Assume u is a solution of the boundary value problem (L^p-OBVP$_{\Delta,B}$) with boundary data $(g_0, g_1) \in L^p(\partial\Omega) \oplus L^p(\partial\Omega)$. Then by (3.66) we have that $u \in \mathscr{H}^p(\Omega, \Delta)$, and Theorem 3.3.3 ensures that

$$\mathrm{Tr}_1^{\kappa - nt} u \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \text{ and } \mathrm{Tr}_1^{\kappa - nt} u \in \mathscr{H}^p(\partial\Omega, \Delta). \tag{3.119}$$

By (3.91)-(3.92) we conclude that

$$(g_0, g_1) = \dot{\Theta}_B \mathrm{Tr}_1^{\kappa - nt} u \implies (g_0, g_1) \in \dot{\Theta}_B(\mathscr{H}^p(\partial\Omega, \Delta)). \tag{3.120}$$

Conversely, assume $(g_0, g_1) \in \dot{\Theta}_B(\mathscr{H}^p(\partial\Omega, \Delta))$, and let $\dot{f} \in \mathscr{H}^p(\partial\Omega, \Delta)$ be such that

$$(g_0, g_1) = \dot{\Theta}_B \dot{f}. \tag{3.121}$$

Since $\dot{f} \in \mathscr{H}^p(\partial\Omega, \Delta)$, it follows from Corollary 3.3.2 that $\dot{f}$ is an admissible boundary data for the overdetermined boundary value problem $(\text{OBVP}_{\Delta,p})$, and by Theorem 3.2.3 there exists a unique function $u : \Omega \to \mathbb{C}$ such that $u \in \mathscr{H}^p(\Omega, \Delta)$ and

$$\text{Tr}_1^{\kappa-nt} u = \dot{f}. \tag{3.122}$$

Together with (3.121) we conclude that the same function u is a solution of the boundary value problem $(L^p\text{-OBVP}_{\Delta,B})$ with boundary data (g_0, g_1), which proves the solvability result stated in the theorem.

To prove uniqueness, note that since $B \in \mathfrak{A}_{\text{inv}}(n, 1, \Omega)$, it follows from Proposition 3.3.5 that for a given pair $(g_0, g_1) \in \dot{\Theta}_B(\mathscr{H}^p(\partial\Omega, \Delta))$ there exists a unique Lebesgue-Whitney array $\dot{f} \in \mathscr{H}^p(\partial\Omega, \Delta)$ such that (3.121) holds.

Consequently, if u_1 and u_2 are solutions of $(L^p\text{-OBVP}_{\Delta,B})$ with the same boundary data, then the function $v = u_1 - u_2$ is a solution of $(L^p\text{-OBVP}_{\Delta,B})$ with zero boundary data and by (3.119), Proposition 3.3.5, and Theorem 3.3.3 we conclude that $v = 0$ in Ω.

The final remark follows from the representation formula in (3.118) and the proofs of the first parts of this theorem. □

For the remainder of this section, we expand the discussion on the class of coefficient tensors $\mathfrak{A}_{\text{inv}}(n, 1, \Omega)$ defined according to (3.93).

Note that this definition depends on the geometry of the domain $\Omega \subset \mathbb{R}^n$. Indeed, consider the open square $\Omega = (0, 1)^2 \subset \mathbb{R}^2$ and let $B = (B_{rs})_{1 \leq r,s \leq 2}$ be any 2×2 matrix. Note that the outward unit normal to Ω is, at almost every point $x \in \partial\Omega$, given by $(\pm 1, 0)$ or $(0, \pm 1)$. Thus, a direct computation using (3.93) shows that B is an element of $\mathfrak{A}_{\text{inv}}(2, 1, \Omega)$ if and only if B_{11} and B_{22} are non-zero.

In particular, the matrix

$$B = \begin{pmatrix} 1 & 1 \\ \\ 1 & 1 \end{pmatrix} \tag{3.123}$$

is an element of $\mathfrak{A}_{\text{inv}}(2, 1, \Omega)$ that is not weakly elliptic, that is, $B \notin \mathfrak{A}_{\text{WE}}(2, 1)$. To see that B is not weakly elliptic, note that the vector $\xi = (1, -1)$ is such that the symbol $B_{rs}\xi_r\xi_s$ is equal to zero since

$$\begin{aligned} B_{rs}\xi_r\xi_s &= B_{11}\xi_1\xi_1 + B_{12}\xi_1\xi_2 + B_{21}\xi_2\xi_1 + B_{22}\xi_2\xi_2 \\ &= 1 + (-1) + (-1) + 1 = 0. \end{aligned} \tag{3.124}$$

Moving on, we show that under suitable geometrical considerations on the domain $\Omega \subset \mathbb{R}^n$, the classes $\mathfrak{A}_{\text{inv}}(n, 1, \Omega)$ and $\mathfrak{A}_{\text{WE}}(n, 1)$ coincide.

Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. As in D. Mitrea, I. Mitrea, and M. Mitrea [22, (5.6.13)], consider the reduced boundary, denoted by $\partial^*\Omega$, as the subset of the topological boundary $\partial\Omega$ satisfying

$$\begin{aligned} &\partial^*\Omega \text{ consists of points } x \in \partial\Omega \text{ such that } 0 < \sigma(B(x,r)) < \infty \text{ for} \\ &\text{each } r \in (0,\infty), \text{ the formula } \lim_{r\to 0^+} \fint_{B(x,r)} \nu d\sigma = \nu(x) \text{ holds,} \\ &\text{and } |\nu(x)| = 1. \end{aligned} \tag{3.125}$$

Note that if $\Omega \subset \mathbb{R}^n$ is a bounded C^1 domain, then the reduced boundary is the same as the topological boundary. Moreover, the outward unit normal $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ defines a surjective map. This last property is what we shall explore in the following proposition.

Proposition 3.3.8. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain. Let $\partial^*\Omega$ denote its reduced boundary. Then a sufficient condition for $\mathfrak{A}_{\text{WE}}(n,1) = \mathfrak{A}_{\text{inv}}(n,1,\Omega)$ is that the map $\nu : \partial^*\Omega \to \mathbb{S}^{n-1}$ is surjective.*

Proof. As observed after (3.93), we have $\mathfrak{A}_{\text{WE}}(n,1) \subset \mathfrak{A}_{\text{inv}}(n,1,\Omega)$ for every $\Omega \subset \mathbb{R}^n$ UR domain.

For the other inclusion, let $B \in \mathfrak{A}_{\text{inv}}(n,1,\Omega)$. By definition, there exists $M \in (0,\infty)$ such that

$$\frac{1}{M} \leq |B_{rs}\nu_r\nu_s| \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.126}$$

Thus, under the assumption that $\nu : \partial^*\Omega \to \mathbb{S}^{n-1}$ is surjective, and the fact that

$$\xi \longmapsto |B_{rs}\xi_r\xi_s| \ \text{ is continuous,} \tag{3.127}$$

we conclude that

$$|B_{rs}\xi_r\xi_s| = |\xi|^2 \left| B_{rs} \frac{\xi_r}{|\xi|} \frac{\xi_s}{|\xi|} \right| \geq \frac{|\xi|^2}{M}, \tag{3.128}$$

for every non-zero $\xi \in \mathbb{R}^n$ and this finishes the proof. □

As mentioned earlier, bounded C^1 domains are a particular class of domains for which we have $\mathfrak{A}_{\text{WE}}(n,1) = \mathfrak{A}_{\text{inv}}(n,1,\Omega)$. If the domain Ω is a hemisphere, then we can show that $\mathfrak{A}_{\text{WE}}(n,1) = \mathfrak{A}_{\text{inv}}(n,1,\Omega)$ even though $\nu : \partial^*\Omega \to \mathbb{S}^{n-1}$ is not surjective. This follows from the fact that the polynomial $\xi \mapsto B_{rs}\xi_r\xi_s$ is even, hence for a given $\xi \in \mathbb{R}^n$ there holds

$$B_{rs}\xi_r\xi_s \neq 0 \iff B_{rs}(-\xi_r)(-\xi_s) \neq 0. \tag{3.129}$$

Let $R\xi := -\xi$ be the reflection map. If the mapping $\nu : \partial^*\Omega \to \mathbb{S}^{n-1}$ is such that $\mathbb{S}^{n-1} = \nu(\partial^*\Omega) \bigcup R(\nu(\partial^*\Omega))$, we also conclude that $\mathfrak{A}_{\text{WE}}(n,1) = \mathfrak{A}_{\text{inv}}(n,1,\Omega)$.

To establish what are the necessary conditions on the domain Ω for which the equality $\mathfrak{A}_{\text{WE}}(n,1) = \mathfrak{A}_{\text{inv}}(n,1,\Omega)$ holds requires further investigation.

3.4 Overdetermined Riemann-Hilbert Problem

This section discusses a formulation of the Riemann-Hilbert boundary value problem in the overdetermined setting for the Laplacian operator.

To set the stage, fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. It follows from D. Mitrea, I. Mitrea, and M. Mitrea [23, Proposition 11.1.14] that for every $\Phi_0 \in \text{Lip}(\partial\Omega)$ and for each pair of indices $j, k \in \{1, \ldots, n\}$, we have

$$\Phi_0 \in L^\infty_1(\partial\Omega, \sigma) \text{ and } \partial_{jk}\Phi_0 \in L^\infty(\partial\Omega, \sigma). \tag{3.130}$$

Define the function space of bounded Whitney arrays $\text{BWA}_1(\partial\Omega, \sigma)$ as the collection of all families $\dot{\Phi} := \{\Phi_0, \Phi_{e_1}, \ldots, \Phi_{e_n}\}$ satisfying the compatibility conditions (CC) in (2.40), that is,

$$\dot{\Phi} \in \text{CC} \iff \begin{cases} \partial_{\tau_{jk}}\Phi_0 = \nu_j\Phi_{e_k} - \nu_k\Phi_{e_j} \ \sigma\text{-a.e. on } \partial\Omega \\ \\ \text{for all } 1 \leq j, k \leq n, \end{cases} \tag{3.131}$$

and with $\Phi_0 \in \text{Lip}(\partial\Omega)$, and $\Phi_{e_j} \in L^\infty(\partial\Omega, \sigma)$ for each $j \in \{1, \ldots, n\}$. Equip this space with the norm

$$||\dot{\Phi}||_{\text{BWA}_1(\partial\Omega,\sigma)} := \sum_{|\gamma|\leq 1} ||\Phi_\gamma||_{L^\infty(\partial\Omega,\sigma)}. \tag{3.132}$$

Note that this function space is a vector space over $\mathbb{C}$ when equipped with pointwise addition and scalar multiplication. Moreover, inclusion $\text{BWA}_1(\partial\Omega, \sigma) \subset \text{WA}_1[L^\infty(\partial\Omega, \sigma)]$ holds.

For a pair $\dot{\Phi}, \dot{\Psi} \in \text{BWA}_1(\partial\Omega, \sigma)$, define the array product as the binary operation

$$\dot{\Phi} \odot \dot{\Psi} := \{\Phi_0\Psi_0, \Phi_{e_1}\Psi_0 + \Phi_0\Psi_{e_1}, \ldots, \Phi_{e_n}\Psi_0 + \Phi_0\Psi_{e_n}\}. \tag{3.133}$$

We claim that $\dot{\Phi} \odot \dot{\Psi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$. Indeed, note that $(\Phi \odot \Psi)_0 \in \mathrm{Lip}(\partial\Omega)$ and that $(\Phi \odot \Psi)_{e_k} \in L^\infty(\partial\Omega, \sigma)$ for each $k \in \{1, \ldots, n\}$. As for the compatibility conditions (3.131) we have

$$\begin{aligned}
\partial_{\tau_{jk}}[(\Phi \odot \Psi)_0] &= \partial_{\tau_{jk}}(\Phi_0 \Psi_0) \\
&= (\partial_{\tau_{jk}} \Phi_0)\Psi_0 + \Phi_0(\partial_{\tau_{jk}} \Psi_0) \\
&= (\nu_j \Phi_{e_k} - \nu_k \Phi_{e_j})\Psi_0 + \Phi_0(\nu_j \Psi_{e_k} - \nu_k \Psi_{e_j}) \\
&= \nu_j(\Phi_{e_k}\Psi_0 + \Phi_0 \Psi_{e_k}) - \nu_k(\Phi_{e_j}\Psi_0 + \Phi_0 \Psi_{e_j}) \\
&= \nu_j[(\Phi \odot \Psi)_{e_k}] - \nu_k[(\Phi \odot \Psi)_{e_j}], \qquad (3.134)
\end{aligned}$$

where in the second line we used D. Mitrea, I. Mitrea, and M. Mitrea [23, Proposition 11.1.15], so $\mathrm{Lip}(\partial\Omega)$ is a module over $L^\infty_1(\partial\Omega, \sigma)$, and in the third line we use the compatibility conditions for $\dot{\Phi}$ and $\dot{\Psi}$, and from the fourth line to the last line we used the definition of $\dot{\Phi} \odot \dot{\Psi}$.

Proposition 3.4.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. Then $(\mathrm{BWA}_1(\partial\Omega, \sigma), +, \odot)$ is an associative commutative algebra over $\mathbb{C}$, that is, for every $\dot{\Phi}, \dot{\Psi}, \dot{\eta} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ and for every $a, b \in \mathbb{C}$ there holds*

$$\dot{\Phi} \odot \dot{\Psi} = \dot{\Psi} \odot \dot{\Phi} \qquad (3.135)$$

$$(\dot{\Phi} \odot \dot{\Psi}) \odot \dot{\eta} = \dot{\Phi} \odot (\dot{\Psi} \odot \dot{\eta}) \qquad (3.136)$$

$$(\dot{\Phi} + \dot{\Psi}) \odot \dot{\eta} = \dot{\Phi} \odot \dot{\eta} + \dot{\Psi} \odot \dot{\eta} \tag{3.137}$$

$$(a\dot{\Phi}) \odot (b\dot{\Psi}) = (ab)(\dot{\Phi} \odot \dot{\Psi}) \tag{3.138}$$

Proof. The commutativity follows from the commutativity of the pointwise product of complex-valued functions. More precisely, we have

$$\Phi_0\Psi_0 = \Psi_0\Phi_0 \text{ everywhere on } \partial\Omega, \tag{3.139}$$

and for every $j \in \{1, \ldots, n\}$

$$\Phi_0\Psi_{e_j} = \Psi_{e_j}\Phi_0 \text{ and } \Psi_0\Phi_{e_j} = \Phi_{e_j}\Psi_0 \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.140}$$

Inserting this in (3.133) gives us (3.135).

To prove the associativity, note that by definition we have

$$(\dot{\Phi} \odot \dot{\Psi})_0\eta_0 = \Phi_0\Psi_0\eta_0 = \Phi_0(\dot{\Psi} \odot \dot{\eta})_0 \tag{3.141}$$

and for every $j \in \{1, \ldots, n\}$

$$\begin{aligned}(\dot{\Phi} \odot \dot{\Psi})_{e_j}\eta_0 + (\dot{\Phi} \odot \dot{\Psi})_0\eta_{e_j} &= (\Phi_{e_j}\Psi_0 + \Phi_0\Psi_{e_j})\eta_0 + \Phi_0\Psi_0\eta_{e_j} \\ &= \Phi_{e_j}\Psi_0\eta_0 + \Phi_0(\Psi_{e_j}\eta_0 + \Psi_0\eta_{e_j}) \\ &= \Phi_{e_j}(\dot{\Psi} \odot \dot{\eta})_0 + \Phi_0(\dot{\Psi} \odot \dot{\eta})_{e_j}. \end{aligned} \tag{3.142}$$

The conclusion is that (3.136) holds.

To prove the distributivity property, we use the distributivity and associativity of the algebra of complex-valued functions to get that

$$((\dot{\Phi}+\dot{\Psi})\odot\dot{\eta})_0=(\Phi_0+\Psi_0)\eta_0=\Phi_0\eta_0+\Psi_0\eta_0=(\dot{\Phi}\odot\dot{\eta})_0+(\dot{\Psi}\odot\dot{\eta})_0, \tag{3.143}$$

and for every $j\in\{1,\ldots,n\}$

$$\begin{aligned}((\dot{\Phi}+\dot{\Psi})\odot\dot{\eta})_{e_j}&=(\Phi_{e_j}+\Psi_{e_j})\eta_0+(\Phi_0+\Psi_0)\eta_{e_j}\\&=(\Phi_{e_j}\eta_0+\Phi_0\eta_{e_j})+(\Psi_{e_j}\eta_0+\Psi_0\eta_{e_j})\\&=(\dot{\Phi}\odot\dot{\eta})_{e_j}+(\dot{\Psi}\odot\dot{\eta})_{e_j},\end{aligned} \tag{3.144}$$

hence (3.137) holds.

Finally, the compatibility with scalars in (3.138) follows from the compatibility with scalars for complex-valued functions, and this finishes the proof. □

Consider the array $\dot{\mathbf{1}}:=\{1,0,\ldots,0\}$. Note that $\dot{\mathbf{1}}\in\mathrm{BWA}_1(\partial\Omega,\sigma)$ and that for every $\dot{\Phi}\in\mathrm{BWA}_1(\partial\Omega,\sigma)$ there holds

$$\dot{\mathbf{1}}\odot\dot{\Phi}=\dot{\Phi}\odot\dot{\mathbf{1}}=\dot{\Phi}, \tag{3.145}$$

hence $\dot{\mathbf{1}}$ is the multiplicative identity in the algebra $(\mathrm{BWA}_1(\partial\Omega,\sigma),+,\odot)$.

Say that an array $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ has a multiplicative inverse with respect to the array product if there exists $\dot{\Psi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ such that

$$\dot{\Phi} \odot \dot{\Psi} = \dot{\Psi} \odot \dot{\Phi} = \dot{\mathbf{1}}. \tag{3.146}$$

The next proposition gives necessary and sufficient conditions for the existence of multiplicative inverses.

Proposition 3.4.2. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. An element $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ has a multiplicative inverse with respect to the array product if and only if*

$$\Phi_0 \neq 0 \ \text{ everywhere on } \partial\Omega. \tag{3.147}$$

Moreover, the multiplicative inverse of $\dot{\Phi}$, denoted by $\dot{\Phi}^{-1}$, is given by

$$\dot{\Phi}^{-1} := \left\{ \frac{1}{\Phi_0}, \frac{-\Phi_{e_1}}{\Phi_0^2}, \ldots, \frac{-\Phi_{e_n}}{\Phi_0^2} \right\}. \tag{3.148}$$

Proof. We start by showing the first implication stated in the proposition. Suppose that $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ has a multiplicative inverse with respect to the array product. Then there exists $\dot{\Psi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ such that (3.146) holds. In particular, $\Psi_0 \in \mathrm{Lip}(\partial\Omega)$ and

$$\Phi_0\Psi_0 = 1 \implies \Phi_0 \neq 0 \ \text{ everywhere on } \partial\Omega. \tag{3.149}$$

For the converse, we shall also prove that the multiplicative inverse is given by the expression on the right-hand side of (3.148).

Suppose $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ is such that Φ_0 is non-zero everywhere on $\partial\Omega$. Let $\dot{\Psi}$ be a family of functions defined as the right-hand side of (3.148), that is,

$$\dot{\Psi} := \{\Psi_0, \Psi_{e_1}, \ldots, \Psi_{e_n}\} = \left\{ \frac{1}{\Phi_0}, \frac{-\Phi_{e_1}}{\Phi_0^2}, \ldots, \frac{-\Phi_{e_n}}{\Phi_0^2} \right\}. \tag{3.150}$$

We claim that $\dot{\Psi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ and that $\dot{\Psi}$ is the multiplicative inverse of $\dot{\Phi}$ with respect to the array product.

To prove the first claim, first note that since Φ_0 is non-zero everywhere on $\partial\Omega$, the function Ψ_0 is well-defined. Moreover, since $\partial\Omega$ is compact, the function Φ_0 has a minimum in $\partial\Omega$ and therefore there exists a positive constant $C \in (0, \infty)$ such that

$$|\Psi_0(x)| = \left| \frac{1}{\Phi_0(x)} \right| \leq C \ \text{ for every } x \in \partial\Omega. \tag{3.151}$$

Furthermore, for every $x, y \in \partial\Omega$

$$|\Psi_0(x) - \Psi_0(y)| = \left| \frac{1}{\Phi_0(x)} - \frac{1}{\Phi_0(y)} \right| \leq C^2 |\Phi_0(x) - \Phi_0(y)|, \tag{3.152}$$

which implies that $\Psi_0 \in \text{Lip}(\partial\Omega)$. It is also follows from (3.151) and the definition that $\Psi_{e_j} \in L^\infty(\partial\Omega, \sigma)$ for every $j \in \{1, \ldots, n\}$. To show that the family $\dot{\Psi}$ satisfies the compatibility conditions, we use D. Mitrea, I. Mitrea, and M. Mitrea [23, Proposition 11.1.15], so $\text{Lip}(\partial\Omega)$ is a module over $L^\infty_1(\partial\Omega, \sigma)$, and we obtain that for every pair $j, k \in \{1, \ldots, n\}$

$$0 = \partial_{\tau_{jk}}(\Phi_0\Psi_0) = \partial_{\tau_{jk}}(\Phi_0)\Psi_0 + \Phi_0\partial_{\tau_{jk}}(\Psi_0). \tag{3.153}$$

Bringing in the compatibility conditions for the array $\dot{\Phi}$ and rearranging terms in (3.153) we conclude that

$$\begin{aligned}\partial_{\tau_{jk}}(\Psi_0) &= -\frac{1}{\Phi_0}\partial_{\tau_{jk}}(\Phi_0)\Psi_0 = -\frac{1}{\Phi_0^2}\partial_{\tau_{jk}}(\Phi_0) \\ &= -\frac{1}{\Phi_0^2}[\nu_j\Phi_{e_k} - \nu_k\Phi_{e_j}] = \nu_j\Psi_{e_k} - \nu_k\Psi_{e_j},\end{aligned} \tag{3.154}$$

so the conclusion is that $\dot{\Psi} \in \text{BWA}_1(\partial\Omega, \sigma)$. A direct computation shows that

$$\Phi_0\Psi_0 = 1 \text{ everywhere on } \partial\Omega, \tag{3.155}$$

and for every $j \in \{1, \ldots, n\}$

$$\Phi_{e_j}\Psi_0 + \Phi_0\Psi_{e_j} = 0 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.156}$$

hence $\dot{\Phi} \odot \dot{\Psi} = \dot{\mathbf{1}}$, and this finishes the proof. □

Moving on, our next goal is to define and study the array multiplication operator. Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. For each integrability exponent $p \in [1, \infty]$, and for each bounded Whitney array $\dot{\Phi} \in \text{BWA}_1(\partial\Omega, \sigma)$, define the array multiplication operator $M_{\dot{\Phi}}$, acting on each Lebesgue-Whitney array $\dot{f} \in \text{WA}_1[L^p(\partial\Omega, \sigma)]$, as

$$M_{\dot{\Phi}}\dot{f} := \{\Phi_0 f_0, \Phi_{e_1} f_0 + \Phi_0 f_{e_1}, \ldots, \Phi_{e_n} f_0 + \Phi_0 f_{e_n}\}. \tag{3.157}$$

The next proposition explores the mapping properties of the array multiplication operator.

Proposition 3.4.3. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, and $p \in [1, \infty]$ be an integrability exponent.*

Then for each $\dot{\Phi} \in \text{BWA}_1(\partial\Omega, \sigma)$ the array multiplication operator $M_{\dot{\Phi}}$ defined in (3.157) *is such that*

$$M_{\dot{\Phi}} : \text{WA}_1[L^p(\partial\Omega, \sigma)] \to \text{WA}_1[L^p(\partial\Omega, \sigma)] \tag{3.158}$$

is well-defined, linear, and bounded.

Moreover, for $\dot{\Phi}, \dot{\Psi} \in \text{BWA}_1(\partial\Omega, \sigma)$ there holds

$$M_{\dot{\Phi}} M_{\dot{\Psi}} = M_{\dot{\Phi} \odot \dot{\Psi}} \ \text{ on } \text{WA}_1[L^p(\partial\Omega, \sigma)]. \tag{3.159}$$

Consequently, if $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ *has a multiplicative inverse with respect to the array product, then* $M_{\dot{\Phi}}$ *is a continuous bijection with continuous inverse and we have*

$$M_{\dot{\Phi}}^{-1} = M_{\dot{\Phi}^{-1}}. \tag{3.160}$$

Proof. First we show that $M_{\dot{\Phi}}\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$. Note that from D. Mitrea, I. Mitrea, and M. Mitrea [23, Proposition 11.1.15] we obtain the regularity $\Phi_0 f_0 \in L_1^p(\partial\Omega, \sigma)$ and that for each $j, k \in \{1, \ldots, n\}$ the product rule

$$\partial_{\tau_{jk}}(\Phi_0 f_0) = \partial_{\tau_{jk}}(\Phi_0) f_0 + \Phi_0 \partial_{\tau_{jk}}(f_0) \tag{3.161}$$

holds σ-a.e. on $\partial\Omega$. Since $\Phi_{e_j} \in L^\infty(\partial\Omega, \sigma)$ and $f_{e_j} \in L^p(\partial\Omega, \sigma)$, we conclude that

$$(M_{\dot{\Phi}}\dot{f})_0 \in L_1^p(\partial\Omega, \sigma), \text{ and } (M_{\dot{\Phi}}\dot{f})_{e_j} \in L^p(\partial\Omega, \sigma) \text{ for each } j \in \{1, \ldots, n\}. \tag{3.162}$$

It remains to check that $M_{\dot{\Phi}}\dot{f}$ satisfies the compatibility conditions.

It follows from the compatibility conditions of the arrays $\dot{\Phi}$ and $\dot{f}$, and the product rule (3.161) that

$$\begin{aligned}
\partial_{\tau_{jk}}[(M_{\dot{\Phi}}\dot{f})_0] &= \partial_{\tau_{jk}}(\Phi_0) f_0 + \Phi_0 \partial_{\tau_{jk}}(f_0) \\
&= (\nu_j \Phi_{e_k} - \nu_k \Phi_{e_j}) f_0 + \Phi_0(\nu_j f_{e_k} - \nu_k f_{e_j}) \\
&= \nu_j(\Phi_{e_k} f_0 + \Phi_0 f_{e_k}) - \nu_k(\Phi_{e_j} f_0 + \Phi_0 f_{e_j})
\end{aligned}$$

$$= \nu_j (M_{\dot{\Phi}} \dot{f})_{e_k} - \nu_k (M_{\dot{\Phi}} \dot{f})_{e_j} \tag{3.163}$$

holds σ-a.e. on $\partial\Omega$, and this shows that $M_{\dot{\Phi}} \dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$.

It follows from the definition that the map $M_{\dot{\Phi}}$ is linear on $\mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$. For boundedness, we note that

$$\begin{aligned}
&||M_{\dot{\Phi}} \dot{f}||_{\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]} = \sum_{|\gamma|\leq 1} ||(M_{\dot{\Phi}} \dot{f})_\gamma||_{L^p(\partial\Omega,\sigma)} \\
&= ||\Phi_0 f_0||_{L^p(\partial\Omega,\sigma)} + \sum_{j=1}^{n} ||\Phi_{e_j} f_0 + \Phi_0 f_{e_j}||_{L^p(\partial\Omega,\sigma)} \\
&\leq ||\Phi_0 f_0||_{L^p(\partial\Omega,\sigma)} + \sum_{j=1}^{n} ||\Phi_{e_j} f_0||_{L^p(\partial\Omega,\sigma)} + \sum_{j=1}^{n} ||\Phi_0 f_{e_j}||_{L^p(\partial\Omega,\sigma)} \\
&\leq \left(\sum_{|\gamma|\leq 1} ||\Phi_\gamma||_{L^\infty(\partial\Omega,\sigma)} \right) ||f_0||_{L^p(\partial\Omega,\sigma)} + ||\Phi_0||_{L^\infty(\partial\Omega,\sigma)} \left(\sum_{|\gamma|=1} ||f_\gamma||_{L^p(\partial\Omega,\sigma)} \right) \\
&\leq \left(\sum_{|\gamma|\leq 1} ||\Phi_\gamma||_{L^\infty(\partial\Omega,\sigma)} \right) \left(\sum_{|\gamma|\leq 1} ||f_\gamma||_{L^p(\partial\Omega,\sigma)} \right) \\
&= ||\dot{\Phi}||_{\mathrm{BWA}_1(\partial\Omega,\sigma)} ||\dot{f}||_{\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]}
\end{aligned} \tag{3.164}$$

hence $M_{\dot{\Phi}} : \mathrm{WA}_1[L^p(\partial\Omega, \sigma)] \to \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$ is well-defined, linear and bounded as we wanted to show.

To prove the composition formula in (3.159) let $\dot{\Phi}, \dot{\Psi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ and consider an arbitrary $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$. Recall that by (3.133) we have

$$(\dot{\Phi} \odot \dot{\Psi})_0 = \Phi_0 \Psi_0 \tag{3.165}$$

and for every $j \in \{1, \ldots, n\}$

$$(\dot{\Phi} \odot \dot{\Psi})_{e_j} = \Phi_{e_j}\Psi_0 + \Phi_0\Psi_{e_j}, \tag{3.166}$$

and by (3.157)

$$M_{\dot{\Psi}}\dot{f} = \{\Psi_0 f_0, \Psi_{e_1} f_0 + \Psi_0 f_{e_1}, \ldots, \Psi_{e_n} f_0 + \Psi_0 f_{e_n}\}. \tag{3.167}$$

Therefore we may write

$$(M_{\dot{\Phi}\odot\dot{\Psi}}\dot{f})_0 = (\dot{\Phi} \odot \dot{\Psi})_0 f_0 = \Phi_0\Psi_0 f_0 = \Phi_0(M_{\dot{\Psi}}\dot{f})_0 = (M_{\dot{\Phi}}M_{\dot{\Psi}}\dot{f})_0 \tag{3.168}$$

and for every $j \in \{1, \ldots, n\}$

$$\begin{aligned}
(M_{\dot{\Phi}\odot\dot{\Psi}}\dot{f})_{e_j} &= (\dot{\Phi} \odot \dot{\Psi})_{e_j} f_0 + (\dot{\Phi} \odot \dot{\Psi})_0 f_{e_j} \\
&= (\Phi_{e_j}\Psi_0 + \Phi_0\Psi_{e_j}) f_0 + \Phi_0\Psi_0 f_{e_j} \\
&= \Phi_{e_j}\Psi_0 f_0 + \Phi_0(\Psi_{e_j} f_0 + \Psi_0 f_{e_j}) \\
&= \Phi_{e_j}(M_{\dot{\Psi}}\dot{f})_0 + \Phi_0(M_{\dot{\Psi}}\dot{f})_{e_j} \\
&= (M_{\dot{\Phi}}M_{\dot{\Psi}}\dot{f})_{e_j},
\end{aligned} \tag{3.169}$$

hence (3.168) and (3.169) give us (3.159).

The last part is a consequence of the composition formula (3.159), thus

$$M_{\dot{\Phi}} M_{\dot{\Phi}^{-1}} = M_{\dot{\Phi}\odot\dot{\Phi}^{-1}} = M_{\dot{\mathbf{i}}}, \tag{3.170}$$

and

$$M_{\dot{\mathbf{i}}}\dot{f} = \dot{f} \text{ for every } \dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]. \tag{3.171}$$

Since $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ is a Banach space, follows as a consequence of the Open Mapping theorem. □

We now formulate and study the Riemann-Hilbert boundary value problem for the Laplacian in this overdetermined setting.

Fix $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and $\kappa \in (0,\infty)$ be an aperture parameter. Let $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$.

For each integrability exponent $p \in (1,\infty)$ and for each $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega,\sigma)$, we formulate the Riemann-Hilbert overdetermined boundary value problem for the Laplacian in Ω with twisting coefficient $\dot{\Phi}$ and transmission boundary data $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ as the task of finding a pair of complex-valued function (u_+,u_-) in $\Omega_\pm$ satisfying

$$(L^p\text{-RH}_{\Delta,\dot{\Phi}})\begin{cases} u_\pm \in C^2(\Omega_\pm), \ \ \Delta u_\pm = 0 \ \text{ in } \Omega_\pm, \\ \\ \mathcal{N}_\kappa(u_+) \in L^p(\partial\Omega,\sigma),\, \mathcal{N}_\kappa(\nabla u_+) \in L^p(\partial\Omega,\sigma), \\ \\ \mathcal{N}_\kappa(u_-) \in L^p(\partial\Omega,\sigma) \text{ if } n \geq 3, \ \mathcal{N}^\varepsilon_\kappa(u_-) \in L^p(\partial\Omega,\sigma) \ \forall \varepsilon > 0 \text{ if } n = 2, \\ \\ \mathcal{N}_\kappa(\nabla u_-) \in L^p(\partial\Omega,\sigma),\, u_-(x) = cE_\Delta(x) + \mathcal{O}(|x|^{1-n}) \ \text{ as } |x| \to \infty, \\ \\ M_{\dot{\Phi}}\mathrm{Tr}_1^{\kappa-nt}u_+ - \mathrm{Tr}_1^{\kappa-nt}u_- = \dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,d\sigma)]. \end{cases} \tag{3.172}$$

Note that by the Fatou-type theorem [24, Theorem 3.3.4] it follows that any pair of complex-valued functions $u_\pm \in C^2(\Omega_\pm)$ satisfying the above size conditions are such that

$$\begin{aligned} &\mathrm{Tr}_1^{\kappa-nt}u_\pm \text{ exist } \sigma\text{-a.e. on } \partial\Omega \text{ and are well-defined elements of} \\ &\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]. \end{aligned} \tag{3.173}$$

Therefore the boundary condition

$$M_{\dot{\Phi}}\mathrm{Tr}_1^{\kappa-nt}u_+ - \mathrm{Tr}_1^{\kappa-nt}u_- = \dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega,d\sigma)]. \tag{3.174}$$

is well-defined.

The strategy we shall implement to study solvability results regarding the boundary value problem formulated in $(L^p\text{-RH}_{\Delta,\dot{\Phi}})$ is as follows:

Let $\dot{g} \in \text{WA}_1[L^p(\partial\Omega,\sigma)]$ be an arbitrary Lebesgue-Whitney array. Consider the pair of functions $u_\pm = \dot{\mathscr{C}}_A\dot{g}$. Then by Proposition 3.1.2 this pair satisfies both the PDE and size conditions in the formulation of the Riemann-Hilbert boundary value problem $(L^p\text{-RH}_{\Delta,\dot{\Phi}})$.

As for the boundary condition in (3.174), note that the jump formulas imply that taking $u_\pm = \dot{\mathscr{C}}_A\dot{g}$ reduces the boundary condition to the family of singular integral equations

$$M_{\dot{\Phi}}\dot{P}_A\dot{g} + (I-\dot{P}_A)\dot{g} = \dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{3.175}$$

where $\dot{P}_A$ is the projection operator associated with the Laplacian introduced in (3.73). Introduce the linear operator $\tilde{T}_{A,\dot{\Phi}}$, acting on Lebesgue-Whitney arrays $\text{WA}_1[L^p(\partial\Omega,\sigma)]$, as

$$\tilde{T}_{A,\dot{\Phi}}\dot{g} := M_{\dot{\Phi}}\dot{P}_A\dot{g} + (I-\dot{P}_A)\dot{g}. \tag{3.176}$$

Then by Proposition 3.1.4 we have that $\tilde{T}_{A,\dot{\Phi}}$ is a well-defined, linear, and bounded operator from $\text{WA}_1[L^p(\partial\Omega,\sigma)]$ into itself, and equation (3.175) becomes

$$\tilde{T}_{A,\dot{\Phi}}\dot{g} = \dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{3.177}$$

The conclusion is that solvability results of the Riemann-Hilbert boundary value problem $(L^p\text{-RH}_{\Delta,\dot{\Phi}})$ may be obtained via boundary layer methods using the Cauchy integrals associated with the Laplacian and the analytical properties of the operator $\tilde{T}_{A,\dot{\Phi}}$.

To study $\tilde{T}_{A,\dot{\Phi}}$, consider the following linear operators on $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$

$$Q^{+}_{A,\dot{\Phi}} := I + (I - \dot{P}_A)M_{\dot{\Phi}}\dot{P}_A, \tag{3.178}$$

$$Q^{-}_{A,\dot{\Phi}} := I - (I - \dot{P}_A)M_{\dot{\Phi}}\dot{P}_A, \tag{3.179}$$

and

$$T_{A,\dot{\Phi}} := \dot{P}_A M_{\dot{\Phi}}\dot{P}_A + (I - \dot{P}_A). \tag{3.180}$$

Then we have the following proposition.

Proposition 3.4.4. *Fix $n \in \mathbb{N}$ with $n \geq 2$, an exponent $p \in (1,\infty)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. Then for every $A \in \mathfrak{A}_\Delta$ and for each array $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega,\sigma)$ we have*

1: The operators $Q^{\pm}_{A,\dot{\Phi}}$ are well-defined, linear, bounded, and invertible operators, with

$$Q^{+}_{A,\dot{\Phi}}Q^{-}_{A,\dot{\Phi}} = Q^{-}_{A,\dot{\Phi}}Q^{+}_{A,\dot{\Phi}} = I \ \text{ on } \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]. \tag{3.181}$$

2: The operator $T_{A,\dot{\Phi}}$ is well-defined, linear and bounded. Moreover,

$$T_{A,\dot{\Phi}} = \tilde{T}_{A,\dot{\Phi}}Q^{-}_{A,\dot{\Phi}} \ \text{ on } \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]. \tag{3.182}$$

Consequently, the operator $\tilde{T}_{A,\dot{\Phi}}$ *is invertible/Fredholm on* $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$ *if and only if* $T_{A,\dot{\Phi}}$ *is invertible/Fredholm on* $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$*, and we have*

$$\mathrm{ind}(\tilde{T}_{A,\dot{\Phi}}) = \mathrm{ind}(T_{A,\dot{\Phi}}), \tag{3.183}$$

where $\mathrm{ind}(T_{A,\dot{\Phi}})$ *denotes the Fredholm index.*

Proof. To prove 1 we compute

$$Q^{+}_{A,\dot{\Phi}} Q^{-}_{A,\dot{\Phi}} = \big(I + (I - \dot{P}_A) M_{\dot{\Phi}} \dot{P}_A\big)\big(I - (I - \dot{P}_A) M_{\dot{\Phi}} \dot{P}_A\big). \tag{3.184}$$

Recall that from Theorem 3.3.1 we have that the map $\dot{P}_A$ is a linear projection, so in particular $\dot{P}_A(I - \dot{P}_A) = 0$. Using these properties to expand the right-hand side of (3.184) gives us

$$Q^{+}_{A,\dot{\Phi}} Q^{-}_{A,\dot{\Phi}} = I - (I - \dot{P}_A) M_{\dot{\Phi}} \dot{P}_A + (I - \dot{P}_A) M_{\dot{\Phi}} \dot{P}_A = I, \tag{3.185}$$

which is the desired result. A similar computation holds for $Q^{-}_{A,\dot{\Phi}} Q^{+}_{A,\dot{\Phi}}$ and this finishes the proof of 1.

To prove 2 we compute

$$\tilde{T}_{A,\dot{\Phi}} Q^{-}_{A,\dot{\Phi}} = \big(M_{\dot{\Phi}} \dot{P}_A + (I - \dot{P}_A)\big)\big(I - (I - \dot{P}_A) M_{\dot{\Phi}} \dot{P}_A\big). \tag{3.186}$$

Theorem 3.3.1 implies $\dot{P}_A(I-\dot{P}_A)=0$ and $(I-\dot{P}_A)(I-\dot{P}_A)=I-\dot{P}_A$, therefore the right-hand side of (3.186) may be written as

$$M_{\dot{\Phi}}\dot{P}_A+(I-\dot{P}_A)-(I-\dot{P}_A)M_{\dot{\Phi}}\dot{P}_A=\dot{P}_AM_{\dot{\Phi}}\dot{P}_A+(I-\dot{P}_A). \tag{3.187}$$

All in all, (3.186) together with (3.187) imply the result in (3.182). From the result in 1 we know that $Q^{-}_{A,\dot{\Phi}}$ is invertible on $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$, hence the result in 2 follows and this finishes the proof. □

Moving on, let $\dot{\Phi},\dot{\Psi}\in\mathrm{BWA}_1(\partial\Omega,\sigma)$. It follows from Theorem 3.3.1 that $\dot{P}_A$ is such that $\dot{P}_A^2=\dot{P}_A$, and so $\dot{P}_A(I-\dot{P}_A)=0$. Recall that Proposition 3.4.3 implies the formula $M_{\dot{\Phi}}M_{\dot{\Psi}}=M_{\dot{\Phi}\odot\dot{\Psi}}$. Therefore,

$$\begin{aligned}T_{A,\dot{\Phi}}T_{A,\dot{\Psi}}&=(\dot{P}_AM_{\dot{\Phi}}\dot{P}_A+(I-\dot{P}_A))(\dot{P}_AM_{\dot{\Psi}}\dot{P}_A+(I-\dot{P}_A))\\&=\dot{P}_AM_{\dot{\Phi}}\dot{P}_AM_{\dot{\Psi}}\dot{P}_A+(I-\dot{P}_A),\end{aligned} \tag{3.188}$$

and

$$\begin{aligned}T_{A,\dot{\Phi}\odot\dot{\Psi}}&=\dot{P}_AM_{\dot{\Phi}\odot\dot{\Psi}}\dot{P}_A+(I-\dot{P}_A)\\&=\dot{P}_AM_{\dot{\Phi}}M_{\dot{\Psi}}\dot{P}_A+(I-\dot{P}_A).\end{aligned} \tag{3.189}$$

This implies that for every $\dot{\Phi}, \dot{\Psi} \in \text{BWA}_1(\partial\Omega, \sigma)$

$$\begin{aligned} T_{A,\dot{\Phi}} T_{A,\dot{\Psi}} - T_{A,\dot{\Phi}\odot\dot{\Psi}} &= \dot{P}_A M_{\dot{\Phi}} \dot{P}_A M_{\dot{\Psi}} \dot{P}_A - \dot{P}_A M_{\dot{\Phi}} M_{\dot{\Psi}} \dot{P}_A \\ &= \dot{P}_A M_{\dot{\Phi}} \dot{P}_A M_{\dot{\Psi}} \dot{P}_A - \dot{P}_A \dot{P}_A M_{\dot{\Phi}} M_{\dot{\Psi}} \dot{P}_A \\ &= \dot{P}_A (M_{\dot{\Phi}} \dot{P}_A - \dot{P}_A M_{\dot{\Phi}}) M_{\dot{\Psi}} \dot{P}_A \\ &= \dot{P}_A [M_{\dot{\Phi}}, \dot{P}_A] M_{\dot{\Psi}} \dot{P}_A \\ &= \dot{P}_A [M_{\dot{\Phi}}, \dot{C}_A] M_{\dot{\Psi}} \dot{P}_A \end{aligned} \tag{3.190}$$

where in the second line we used once again that $\dot{P}_A^2 = \dot{P}_A$, in the fourth line we introduced the commutator

$$[M_{\dot{\Phi}}, \dot{P}_A] := M_{\dot{\Phi}} \dot{P}_A - \dot{P}_A M_{\dot{\Phi}}, \tag{3.191}$$

and the last line we used that the commutator is linear and a multiple of the identity commutes with any well-defined, linear, and bounded operator on $\text{WA}_1[L^p(\partial\Omega, \sigma)]$, hence $[M_{\dot{\Phi}}, \dot{P}_A] = [M_{\dot{\Phi}}, \dot{C}_A]$.

Now, suppose that we are given $\dot{\Phi} \in \text{BWA}_1(\partial\Omega, \sigma)$ which is invertible with respect to the array product. If $\dot{\Psi} = \dot{\Phi}^{-1}$, then the last line of (3.189) becomes

$$T_{A,\dot{\Phi}\odot\dot{\Phi}^{-1}} = \dot{P}_A^2 + (I - \dot{P}_A) = I \text{ on } \text{WA}_1[L^p(\partial\Omega, \sigma)], \tag{3.192}$$

and from the computations in (3.190) we obtain that

$$T_{A,\dot{\Phi}}T_{A,\dot{\Phi}^{-1}} - I = \dot{P}_A[M_{\dot{\Phi}}, \dot{C}_A]M_{\dot{\Phi}^{-1}}\dot{P}_A \text{ on } \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]. \tag{3.193}$$

The conclusion is that

> If the commutator $[M_{\dot{\Phi}}, \dot{C}_A]$ is zero, then $T_{A,\dot{\Phi}}$ is invertible on $\mathrm{WA}_1[L^p(\partial\Omega)]$, and if the commutator $[M_{\dot{\Phi}}, \dot{C}_A]$ is a compact operator, then $T_{A,\dot{\Phi}}$ is a Fredholm operator on $\mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$. (3.194)

Therefore (3.194), in concert with Proposition 3.4.4, implies that under the assumption that $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega,\sigma)$ has a multiplicative inverse, we may obtain solvability results for the Riemann-Hilbert boundary value problem formulated in (L^p-$\mathrm{RH}_{\Delta,\dot{\Phi}}$) from analytical properties of the commutator $[M_{\dot{\Phi}}, \dot{C}_A]$.

Consider the action of the commutator $[M_{\dot{\Phi}}, \dot{C}_A]$ on an array $\dot{g} \in \mathrm{WA}_1[L^p(\partial\Omega,\sigma)]$. Unraveling the definitions gives us an array defined, for σ-a.e. point $x \in \partial\Omega$, as

$$\begin{aligned}([M_{\dot{\Phi}}, \dot{C}_A]\dot{g})_0(x) &= ([\Phi_0, K_A]g_0)(x) - ([\Phi_0, S_\Delta](\nu_s A_{rs} g_{es}))(x) \\ &\quad - S_\Delta(\nu_r A_{rs}\Phi_{e_s} g_0)(x),\end{aligned} \tag{3.195}$$

and for each $j \in \{1,\dots,n\}$, and for σ-a.e. point $x \in \partial\Omega$,

$$([M_{\dot{\Phi}}, \dot{C}_A]\dot{g})_{e_j}(x) = ([\Phi_0, K_A]g_{e_j})(x) + ([\Phi_0, T_{rj}](A_{rs}g_{e_s}))(x)$$

$$- \Phi_{e_j}(x)(K_A g_0)(x) - K_A(\Phi_{e_j} g_0)(x)$$

$$- \Phi_{e_j}(x) S_\Delta(\nu_r A_{rs} g_{e_s})(x). \tag{3.196}$$

Note that since these are all scalar Calderón-Zygmund operators it follows that for every $g \in L^p(\partial\Omega, \sigma)$ and for σ-a.e. point $x \in \partial\Omega$

$$([K_A, \Phi_0]g)(x) = \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \{\Phi_0(x) - \Phi_0(y)\} \nu_s(y) (\partial_r E_\Delta)(x-y) A_{rs} g(y) d\sigma(y), \tag{3.197}$$

and

$$([\Phi_0, S_\Delta]g)(x) = \int_{\partial\Omega} \{\Phi_0(x) - \Phi_0(y)\} E_\Delta(x-y) g(y) d\sigma(y). \tag{3.198}$$

Additionally,

$$([\Phi_0, T_{rj}]g)(x) = \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \{\Phi_0(x) - \Phi_0(y)\} \partial_{\tau_{rl}(y)} [E_\Delta(x-y)] g(y) d\sigma(y). \tag{3.199}$$

Hence the following result regarding Fredholm solvability of the overdetermined Riemann-Hilbert boundary value problem (L^p-RH$_{\Delta,\dot{\Phi}}$) holds

Theorem 3.4.5. *Fix $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $\kappa \in (0, \infty)$ be an aperture parameter, and let $p \in (1, \infty)$ be an integrability exponent.*

Let $\Omega_+ := \Omega$ *and* $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$*, and let* $\dot{\Phi} \in \mathrm{BWA}_1(\partial\Omega, \sigma)$ *be a bounded array which is invertible with respect to the array product, that is, assume that*

$$\Phi_0(x) \neq 0 \ \ \textit{for every } x \in \partial\Omega. \tag{3.200}$$

Then the overdetermined Riemann-Hilbert boundary value problem for the Laplacian in Ω *with twisting coefficient* $\dot{\Phi}$ *and transmission boundary data* $\dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$

$$(L^p\text{-}RH_{\Delta,\dot{\Phi}}) \begin{cases} u_\pm \in C^2(\Omega_\pm), \ \ \Delta u_\pm = 0 \ \ \textit{in } \Omega_\pm, \\ \\ \mathcal{N}_\kappa(u_+) \in L^p(\partial\Omega, \sigma), \, \mathcal{N}_\kappa(\nabla u_+) \in L^p(\partial\Omega, \sigma), \\ \\ \mathcal{N}_\kappa(u_-) \in L^p(\partial\Omega, \sigma) \ \textit{if } n \geq 3, \ \mathcal{N}^\varepsilon_\kappa(u_-) \in L^p(\partial\Omega, \sigma) \ \forall \varepsilon > 0 \ \textit{if } n = 2, \\ \\ \mathcal{N}_\kappa(\nabla u_-) \in L^p(\partial\Omega, \sigma), \, u_-(x) = cE_\Delta(x) + \mathcal{O}(|x|^{1-n}) \ \ \textit{as } |x| \to \infty, \\ \\ M_{\dot{\Phi}} \mathrm{Tr}_1^{\kappa-nt} u_+ - \mathrm{Tr}_1^{\kappa-nt} u_- = \dot{f} \in \mathrm{WA}_1[L^p(\partial\Omega, d\sigma)], \end{cases} \tag{3.201}$$

is Fredholm solvable.

Proof. It follows from Theorem 2.4.5 and Lemma 2.4.6 in S. Hofmann, M. Mitrea, and M. Taylor [18] that for every $p \in (1, \infty)$ the operator in (3.197) is compact from $L^p_1(\partial\Omega, \sigma)$ to itself, and from $L^p(\partial\Omega, \sigma)$ to itself. Additionally, the operator (3.198) is compact from $L^p(\partial\Omega, \sigma)$ into $L^p_1(\partial\Omega, \sigma)$. Finally, the operator (3.199) is compact from $L^p(\partial\Omega, \sigma)$ to itself.

Hence the conclusion is that the commutator operator $[M_{\dot{\Phi}}, \dot{C}_A]$ is a compact operator on $\mathrm{WA}_1[L^p(\partial\Omega, \sigma)]$. Hence by (3.194) one has Fredholm solvability of $(L^p\text{-RH}_{\Delta,\dot{\Phi}})$. $\square$

It remains to be understood under which general circumstances one would be able to obtain well-posedness of the Riemann-Hilbert problem $(L^p\text{-RH}_{\Delta,\dot{\Phi}})$. Moreover, it is also a question os interest to understand under which general geometric assumptions we may consider different function spaces for the twisting coefficients in the formulation of the Riemann-Hilbert boundary value problem.

CHAPTER 4

OVERDETERMINED BOUNDARY VALUE PROBLEM FOR SECOND-ORDER HOMOGENEOUS ELLIPTIC SYSTEMS IN UNIFORMLY RECTIFIABLE DOMAINS

In this chapter, we analyze the overdetermined boundary value problem for general second-order, homogeneous, constant complex coefficient, weakly elliptic $M \times M$ systems L in $\mathbb{R}^n$ and boundary data in L^p-based function spaces in bounded uniformly rectifiable domains.

In section 4.1 we introduce and analyze a Cauchy-like operator associated with second-order, homogeneous, constant complex coefficient, weakly elliptic systems $M \times M$ in $\mathbb{R}^n$ and its basic properties. This includes integral representation formulas and jump formulas in bounded uniformly rectifiable domains.

In section 4.2 we formulate and establish well-posedness results for the overdetermined boundary value problem for the system L in bounded uniformly rectifiable domains and prescribed boundary data in Lebesgue-Whitney array function spaces with integrability exponent $p \in (1, \infty)$.

In section 4.3 we discuss classical Hardy spaces associated with the elliptic systems in bounded uniformly rectifiable domains.

All the results presented in this chapter are based on joint work with Dorina Mitrea, Irina Mitrea, and Marius Mitrea in [2].

4.1 Cauchy Operator and Integral Representation Formula for Null Solutions of Second-order Homogeneous Elliptic Systems

Fix $n, M \in \mathbb{N}$ with $n \geq 2$ and consider a homogeneous, second-order, complex constant coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain with surface measure σ and consider a coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ associated with L, that is,

$$L = (a_{rs}^{\alpha\beta}\partial_r\partial_s)_{1\leq\alpha,\beta\leq M}. \tag{4.1}$$

For each $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ define the boundary-to-domain Cauchy operator associated with L as

$$\dot{\mathscr{C}}_A\dot{f}(x) := \mathscr{D}_A(f_0)(x) - \mathscr{S}_L(\nu_r A_{rs} f_{e_s})(x) \ \text{ for } x \in \mathbb{R}^n \setminus \partial\Omega, \tag{4.2}$$

where 0 stands for the zero multi-index, and $\nu_r A_{rs} f_{e_s} := (\nu_r a_{rs}^{\alpha\beta} f_{e_s}^{\beta})_{1\leq\alpha\leq M}$.

The next lemma shows that this operator is independent of the choice of coefficient tensor used to write the system L.

Lemma 4.1.1 (Independence of choice of coefficient tensor)**.** *Retain the geometrical and analytical context from the previous paragraphs and consider $A, B \in \mathfrak{A}_L$. Then for every $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ there holds*

$$\dot{\mathscr{C}}_A\dot{f}(x) = \dot{\mathscr{C}}_B\dot{f}(x) \ \textit{for } x \in \mathbb{R}^n \setminus \partial\Omega. \tag{4.3}$$

Proof. Fix $p \in (1,\infty)$ and consider an arbitrary $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$. To conclude the result it is enough to show that for $x \in \mathbb{R}^n \setminus \partial\Omega$ there holds

$$\mathscr{D}_A(f_0)(x) - \mathscr{D}_B(f_0)(x) = \mathscr{S}_L(\nu_r A_{rs} f_{e_s})(x) - \mathscr{S}_L(\nu_r B_{rs} f_{e_s})(x). \tag{4.4}$$

If $A, B \in \mathfrak{A}_L$, then the collection $D = (d_{rs}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ given by

$$d_{rs}^{\alpha\beta} := a_{rs}^{\alpha\beta} - b_{rs}^{\alpha\beta} \text{ for } 1 \leq \alpha,\beta \leq M \text{ and } 1 \leq r,s \leq n \tag{4.5}$$

is antisymmetric in the lower indices (cf. [26, (1.1.38)]), that is,

$$d_{rs}^{\alpha\beta} = -d_{sr}^{\alpha\beta} \text{ for every } 1 \leq \alpha,\beta \leq M \text{ and } 1 \leq r,s \leq n. \tag{4.6}$$

More succinctly, the matrices $D_{rs} = \{d_{rs}^{\alpha\beta}\}_{1\leq\alpha,\beta\leq M}$ satisfy

$$D_{rs} = -D_{sr} \text{ for every } 1 \leq r,s \leq n, \tag{4.7}$$

and in particular $D_{rr} = 0$ for every $1 \leq r \leq n$. Now,

$$\mathscr{D}_A(f_0)(x) - \mathscr{D}_B(f_0)(x) = \int_{\partial\Omega} \nu_s(y)\partial_r^y[E_L(x-y)]D_{rs}f_0(y)d\sigma(y). \tag{4.8}$$

On one hand, the antisymmetry of D, the identity $\partial_{\tau_{sr}} = \nu_s \partial_r - \nu_r \partial_s$, and integration by parts imply that (4.8) is equal to

$$\begin{aligned}
&\sum_{1\leq r<s\leq n} \int_{\partial\Omega} \Big(\nu_s(y) \partial_r^y [E_L(x-y)] D_{rs} + \nu_r(y) \partial_s^y [E_L(x-y)] D_{sr} \Big) f_0(y) d\sigma(y) \\
&= \sum_{1\leq r<s\leq n} \int_{\partial\Omega} (\nu_s(y) \partial_r^y - \nu_r(y) \partial_s^y) [E_L(x-y)] D_{rs} f_0(y) d\sigma(y) \\
&= \sum_{1\leq r<s\leq n} \int_{\partial\Omega} \partial_{\tau_{sr}(y)} [E_L(x-y)] D_{rs} f_0(y) d\sigma(y) \\
&= \sum_{1\leq r<s\leq n} \int_{\partial\Omega} E_L(x-y) D_{rs} \partial_{\tau_{rs}(y)} [f_0(y)] d\sigma(y) \\
&= \sum_{1\leq r<s\leq n} \mathscr{S}_L(D_{rs} \partial_{\tau_{rs}} f_0)(x). \qquad (4.9)
\end{aligned}$$

On the other hand, the antisymmetry of D, the identity $\partial_{\tau_{sr}} = \nu_s \partial_r - \nu_r \partial_s$, and the compatibility conditions of the Lebesgue-Whitney array $\dot{f}$ allow us to write

$$\begin{aligned}
&\mathscr{S}_L(\nu_r A_{rs} f_{e_s})(x) - \mathscr{S}_L(\nu_r B_{rs} f_{e_s})(x) = \mathscr{S}_L(\nu_r D_{rs} f_{e_s})(x) \\
&= \sum_{1\leq r<s\leq n} \mathscr{S}_L(\nu_r D_{rs} f_{e_s} + \nu_s D_{sr} f_{e_r})(x) \\
&= \sum_{1\leq r<s\leq n} \mathscr{S}_L(D_{rs}(\nu_r f_{e_s} - \nu_s f_{e_r}))(x) \\
&= \sum_{1\leq r<s\leq n} \mathscr{S}_L(D_{rs} \partial_{\tau_{rs}} f_0)(x). \qquad (4.10)
\end{aligned}$$

Thus (4.9) and (4.10) imply (4.4), hence the desired result. □

The next proposition contains some properties of the boundary-to-domain Cauchy operator introduced in (4.2).

Proposition 4.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and $p \in (1,\infty)$ be an integrability exponent. Let*

$$L = (a_{rs}^{\alpha\beta}\partial_r\partial_s)_{1\leq\alpha,\beta\leq M}$$

be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system. Then for every $A \in \mathfrak{A}_L$ and for each $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ the $\mathbb{C}^M$-valued function $\dot{\mathscr{C}}_A\dot{f}$ satisfy

$$\dot{\mathscr{C}}_A\dot{f} \in [C^\infty(\mathbb{R}^n \setminus \partial\Omega)]^M \ \text{ and } \ L(\dot{\mathscr{C}}_A\dot{f}) = 0 \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \tag{4.11}$$

and for each $l \in \{1,\dots,n\}$ we may explicitly write

$$\partial_l\dot{\mathscr{C}}_A\dot{f}(x) = \mathscr{D}_A(f_{e_l})(x) + \int_{\partial\Omega} \partial_{\tau_{rl}(y)}[E_L(x-y)]A_{rs}f_{e_s}(y)d\sigma(y) \ \text{ for } x \in \mathbb{R}^n \setminus \partial\Omega. \tag{4.12}$$

Moreover, given an aperture parameter $\kappa \in (0,\infty)$ there is a constant $C = C(\Omega, L, p, \kappa)$ such that for every $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$

$$\sum_{|\gamma|\leq 1} ||\mathcal{N}_\kappa(\partial^\gamma\dot{\mathscr{C}}_A\dot{f})||_{L^p(\partial\Omega,\sigma)} \leq C||\dot{f}||_{\mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]}. \tag{4.13}$$

Proof. The claims in (4.11) are an immediate consequence of the definition. To prove the claim in (4.12) fix $l \in \{1, \ldots, n\}$ and $x \in \mathbb{R}^n \setminus \partial\Omega$, then compute

$$\partial_l \dot{\mathscr{C}}_A \dot{f}(x) = \partial_l \mathscr{D}_A(f_0)(x) - \partial_l \mathscr{S}_L(\nu_r A_{rs} f_{e_s})(x), \tag{4.14}$$

Using the identity $\nu_s \partial_l = \partial_{\tau_{sl}} + \nu_l \partial_s$, integration by parts, and that E_L is a fundamental solution for the system L, one obtains

$$\begin{aligned}
\partial_l \mathscr{D}_A(f_0)(x) &= \int_{\partial\Omega} \nu_s(y) \partial_l^y [(\partial_r E_L)(x-y)] A_{rs} f_0(y) d\sigma(y) \\
&= \int_{\partial\Omega} (\partial_{\tau_{sl}(y)} + \nu_l(y) \partial_s^y)[(\partial_r E_L)(x-y)] A_{rs} f_0(y) d\sigma(y) \\
&= \int_{\partial\Omega} (\partial_r E_L)(x-y) A_{rs} \partial_{\tau_{ls}(y)}[f_0(y)] d\sigma(y).
\end{aligned}$$

Hence the compatibility conditions $\partial_{\tau_{ls}} f_0 = \nu_l f_{e_s} - \nu_s f_{e_l}$ imply

$$\partial_l \mathscr{D}_A(f_0)(x) = \mathscr{D}_A(f_{e_l})(x) + \int_{\partial\Omega} \nu_l(y) (\partial_r E_L)(x-y) A_{rs} f_{e_s}(y) d\sigma(y). \tag{4.15}$$

One also readily verifies that

$$\partial_l \mathscr{S}_L(\nu_r A_{rs} f_{e_s})(x) = \int_{\partial\Omega} \nu_r(y) (\partial_l E_L)(x-y) A_{rs} f_{e_s}(y) d\sigma(y). \tag{4.16}$$

Inserting (4.15) and (4.16) into (4.14), and using the identity $\partial_{\tau_{rl}} = \nu_r \partial_l - \nu_l \partial_r$ imply the result in (4.12).

The boundedness of the nontangential maximal function stated in (4.13) is a consequence of the definition in (4.2), the independence of coefficient tensors obtained in Lemma 4.1.1, the result in equation (4.12), and the Calderón-Zygmund theory in UR domains in [25, Theorem 1.5.1]. □

Another remarkable property of the Cauchy integral operator associated with the system L is that it can be used to concisely write an integral representation formula for null solutions of the system. More precisely, the following integral representation formula allows one to recover null solutions of L from the nontangential traces at the boundary (under appropriate assumptions on nontangential traces and size).

Theorem 4.1.3. *Fix $n, M \in \mathbb{N}$ with $n \geq 2$, an exponent $p \in (1, \infty)$ and an aperture parameter $\kappa \in (0, \infty)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and let*

$$L = (a_{rs}^{\alpha\beta}\partial_r\partial_s)_{1\leq\alpha,\beta\leq M} \tag{4.17}$$

be a second-order, homogeneous, constant complex coefficient, weakly elliptic $M \times M$ system. Suppose u is a $\mathbb{C}^M$-valued function satisfying

$$u \in [C^2(\Omega)]^M \ \text{ and } \ Lu = 0 \ \text{ in } \Omega, \tag{4.18}$$

and such that

$$\begin{cases} \mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma) \ \text{ and } \ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma), \\ \\ u|_{\partial\Omega}^{\kappa-nt} \ \text{ and } \ \nabla u|_{\partial\Omega}^{\kappa-nt} \ \text{exist } \sigma\text{-a.e. on } \partial\Omega. \end{cases} \tag{4.19}$$

Then for all $x \in \Omega$

$$u(x) = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt}u)(x). \tag{4.20}$$

Proof. This theorem follows as a consequence of D. Mitrea, I. Mitrea, and M. Mitrea [24, Theorem 1.5.1], hence we need to verify that the hypotheses are met.

First, note that if $\Omega \subset \mathbb{R}^n$ is an UR domain with compact boundary, then it is in particular a lower Ahlfors regular domain, and the surface measure σ is a doubling measure.

Second, the considerations in (4.18) and (4.19) are stronger than the ones stated in [24, (1.5.2) and (1.5.3)]. In concert with the fact that $\partial\Omega$ is compact, it implies that conditions of D. Mitrea, I. Mitrea, and M. Mitrea [24, Theorem 1.5.1] are met, and this finishes the proof. □

Moving on, define the boundary-to-boundary Cauchy operator associated with L as the map that acts on a Lebesgue-Whitney array $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ and produces the family of functions

$$\dot{C}_A\dot{f} := \{(\dot{C}_A\dot{f})_0, (\dot{C}_A\dot{f})_{e_1}, \ldots, (\dot{C}_A\dot{f})_{e_n}\}, \tag{4.21}$$

where, for σ-a.e. point $x \in \partial\Omega$,

$$(\dot{C}_A\dot{f})_0(x) = K_A f_0(x) - S_L(\nu_r A_{rs} f_{e_s})(x), \tag{4.22}$$

while for each $l \in \{1, \ldots, n\}$

$$(\dot{C}_A \dot{f})_{e_l}(x) = K_A f_{e_l}(x) + \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \partial_{\tau_{rl}(y)}[E_L(x-y)] A_{rs} f_{e_s}(y) d\sigma(y). \tag{4.23}$$

Some properties of this operator and its relation to the boundary-to-domain Cauchy operator associated with L are contained in the next proposition.

Proposition 4.1.4. *Fix $n, M \in \mathbb{N}$ with $n \geq 2$, an integrability exponent $p \in (1, \infty)$ and an aperture parameter $\kappa \in (0, \infty)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, and let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system. Then for every $A \in \mathfrak{A}_L$ the following properties hold:*

1: For every $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$ and σ-a.e. on $\partial\Omega$, the first-order nontangential boundary trace of the function $\dot{\mathscr{C}}_A \dot{f}$ exists and satisfy

$$\mathrm{Tr}_1^{\kappa - nt}(\dot{\mathscr{C}}_A \dot{f}) = (\tfrac{1}{2} I + \dot{C}_A)\dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{4.24}$$

2: The operator $\dot{C}_A$ is bounded from $\mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$ to itself. More precisely,

$$\dot{C}_A : \mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)] \to \mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)] \tag{4.25}$$

is well-defined, bounded, and linear.

Proof. To prove 1 note that in light of the definitions, this is equivalent to showing that

$$\dot{\mathscr{C}_A f}\big|_{\partial\Omega}^{\kappa-nt} = \tfrac{1}{2} f_0 + (\dot{C}_A \dot{f})_0 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{4.26}$$

and that for each $l \in \{1, \ldots, n\}$

$$\partial_l \dot{\mathscr{C}_A f}\big|_{\partial\Omega}^{\kappa-nt} = \tfrac{1}{2} f_{e_l} + (\dot{C}_A \dot{f})_{e_l} \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{4.27}$$

To check that (4.26) holds note that by [25, Theorem 1.5.1] the boundary behavior of the term involving the single layer is given by

$$\mathscr{S}_L(\nu_r A_{rs} f_{e_s})\big|_{\partial\Omega}^{\kappa-nt}(x) = S_L(\nu_r A_{rs} f_{e_s})(x) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{4.28}$$

For the first term involving the double layer associated with $A \in \mathfrak{A}_L$ we use the jump formulas in [25, Theorem 1.5.1] to obtain

$$\mathscr{D}_A(f_0)\big|_{\partial\Omega}^{\kappa-nt}(x) = \tfrac{1}{2} f_0(x) + K_A f_0(x) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{4.29}$$

Therefore (4.26) is a consequence of (4.28) and (4.29). To prove that (4.27) holds one uses Proposition 4.1.2 to write

$$\partial_l \dot{\mathscr{C}_A f}(x) = \mathscr{D}(f_{e_l})(x) + \int_{\partial\Omega} \partial_{\tau_{rl}(y)}[E_L(x-y)] A_{rs} f_{e_s}(y) d\sigma(y) \ \text{ for } x \in \Omega. \tag{4.30}$$

By the jump formulas in [25, Theorem 1.5.1] we have that the boundary behavior of the first term in (4.30) is given by

$$\mathscr{D}_A(f_{e_l})\big|_{\partial\Omega}^{\kappa-nt}(x) = \tfrac{1}{2}f_{e_l}(x) + K_A f_{e_l}(x) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{4.31}$$

For the second term, note that the integral kernel of this operator satisfies the hypotheses in [25, Proposition 1.2.1], hence its nontangential boundary trace is given by

$$\begin{aligned}
&\left(\int_{\partial\Omega} \partial_{\tau_{rl}(y)}[E_L(\cdot - y)]A_{rs}f_{e_s}(y)d\sigma(y)\right)\Big|_{\partial\Omega}^{\kappa-nt}(x) \\
&= \lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \partial_{\tau_{rl}(y)}[E_L(x-y)]A_{rs}f_{e_s}(y)d\sigma(y) \ \ \sigma\text{-a.e. on } \partial\Omega,
\end{aligned} \tag{4.32}$$

and this finishes the proof of part 1.

To prove 2 we first check that for $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ the family $\{\dot{C}_A\dot{f}_\gamma\}_{|\gamma|\leq 1}$ satisfies the compatibility conditions. More precisely, given $j,l \in \{1,\dots,n\}$ we need to verify that for σ-a.e. $x \in \partial\Omega$

$$\partial_{\tau_{jl}(x)}[(\dot{C}_A\dot{f})_0(x)] = \nu_j(x)(\dot{C}_A\dot{f})_{e_l}(x) - \nu_l(x)(\dot{C}_A\dot{f})_{e_j}(x). \tag{4.33}$$

Using the results in 1 and the definition of the tangential derivative operator we may write

$$\begin{aligned}
\partial_{\tau_{jl}}(\dot{C}_A\dot{f})_0 &= \partial_{\tau_{jl}}(\dot{\mathscr{C}}_A\dot{f}\big|_{\partial\Omega}^{\kappa-nt}) - \tfrac{1}{2}\partial_{\tau_{jl}}f_0 \\
&= \nu_j\left(\partial_l\dot{\mathscr{C}}_A\dot{f}\big|_{\partial\Omega}^{\kappa-nt}\right) - \nu_l\left(\partial_j\dot{\mathscr{C}}_A\dot{f}\big|_{\partial\Omega}^{\kappa-nt}\right) - \tfrac{1}{2}\partial_{\tau_{jl}}f_0
\end{aligned}$$

$$= \nu_j \left(\tfrac{1}{2} f_{e_l} + (\dot{C}_A \dot{f})_{e_l} \right) - \nu_l \left(\tfrac{1}{2} f_{e_j} + (\dot{C}_A \dot{f})_{e_j} \right) - \tfrac{1}{2} \partial_{\tau_{jl}} f_0$$

$$= \nu_j (\dot{C}_A \dot{f})_{e_l} - \nu_l (\dot{C}_A \dot{f})_{e_j}, \tag{4.34}$$

for σ-a.e. point on $\partial\Omega$, hence (4.33) holds.

To conclude that the boundary-to-boundary Cauchy operator $\dot{C}_A$ is a well-defined, linear, bounded operator from $\mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$ to itself note that the singular integral operators involved in the definition of the boundary-to-boundary Cauchy operator $\dot{C}_A$ are all treated in [25, Chapter 1], and thus

$$(\dot{C}_A \dot{f})_0 := K_A f_0 - S_L(\nu_r A_{rs} f_{e_s}) \tag{4.35}$$

is a well-defined, linear, bounded operator from $\mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$ to $[L_1^p(\partial\Omega, \sigma)]^M$ and for each $l \in \{1, \ldots, n\}$

$$(\dot{C}_A \dot{f})_{e_l} = K_A f_{e_l} + \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |\cdot - y| > \varepsilon}} \partial_{\tau_{rl}(y)} [E_L(\cdot - y)] A_{rs} f_{e_s}(y) d\sigma(y) \tag{4.36}$$

is a well-defined, linear, bounded operator from $\mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$ to $[L^p(\partial\Omega, \sigma)]^M$, and this finishes the proof. □

With this technology, we are ready to formulate and establish well-posedness for the overdetermined boundary value problem associated with a system L, which is the topic of our next section.

4.2 Formulation and Well-posedness result for the Overdetermined Boundary Value Problem for Second-order Homogeneous Elliptic systems

Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and let $\kappa \in (0,\infty)$ be an aperture parameter. For each exponent $p \in (1,\infty)$ we formulate the L^p-Overdetermined boundary value problem for the system L in Ω with prescribed boundary data $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ as the task of finding a $\mathbb{C}^M$-valued function u in Ω satisfying

$$(\mathrm{OBVP}_{L,p}) \begin{cases} u \in [C^2(\Omega)]^M, \ \ Lu = 0 \ \text{ in } \Omega, \\ \\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega,\sigma),\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma), \\ \\ \mathrm{Tr}_1^{\kappa-nt} u = \dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]. \end{cases} \tag{4.37}$$

Our goal is to establish a well-posedness result for the previous boundary value problem.

Proposition 4.1.4 in concert with Theorem 4.1.3 allows us to obtain a characterization for the space of admissible boundary data associated with the $(\mathrm{OBVP}_{L,p})$ formulated in (4.37) in terms of the boundary-to-boundary Cauchy operator associated with the system L.

Theorem 4.2.1. *Fix $n, M \in \mathbb{N}$ with $n \geq 2$, an exponent $p \in (1,\infty)$ and an aperture parameter $\kappa \in (0,\infty)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and let*

$$L = (a_{rs}^{\alpha\beta}\partial_r\partial_s)_{1\leq\alpha,\beta\leq M}$$

be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system. Then the space of admissible boundary data for ($OBVP_{L,p}$) in Ω

$$\big\{\mathrm{Tr}_1^{\kappa-nt}u : u \in [C^2(\Omega)]^M,\ Lu = 0 \text{ in } \Omega,$$
$$\mathcal{N}_\kappa(u) \in L^p(\partial\Omega,\sigma) \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma)\big\}, \tag{4.38}$$

may be described as

$$\big\{\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)] : \dot{C}_A\dot{f} = \tfrac{1}{2}\dot{f}\big\}. \tag{4.39}$$

Proof. The first claim is that the set defined in (4.38) is contained in the set in (4.39). Indeed, if u is a function satisfying the properties in (4.38), then by the Fatou-type theorem [24, Theorem 3.3.4] the first-order nontangential boundary trace of u exist σ-a.e. on $\partial\Omega$. Therefore the function u satisfies the requirements in Theorem 4.1.3 and the integral representation formula implies that $u = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt}u)$. Taking nontangential boundary traces and using Proposition 4.1.4 we obtain

$$\mathrm{Tr}_1^{\kappa-nt}u = (\tfrac{1}{2}I + \dot{C}_A)\mathrm{Tr}_1^{\kappa-nt}u \implies \tfrac{1}{2}\mathrm{Tr}_1^{\kappa-nt}u = \dot{C}_A\mathrm{Tr}_1^{\kappa-nt}u, \tag{4.40}$$

which proves the claim.

The second claim is that the set in (4.39) is a subset of the one in (4.38). Indeed, consider an element $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ of the set in (4.39). By Proposition 4.1.2 the function $\dot{\mathscr{C}}_A\dot{f}$ satisfies the properties in (4.38), and by Proposition 4.1.4 its first-order non-tangential boundary trace satisfies

$$\mathrm{Tr}_1^{\kappa-nt}(\dot{\mathscr{C}}_A\dot{f}) = (\tfrac{1}{2}I + \dot{C}_A)\dot{f}. \tag{4.41}$$

Since $\dot{C}_A\dot{f} = \frac{1}{2}\dot{f}$, it follows from (4.41) that $\mathrm{Tr}_1^{\kappa-nt}(\dot{\mathscr{C}}_A\dot{f}) = \dot{f}$, hence the set defined in (4.39) is contained in the set in (4.38) and this finishes the proof. □

With this machinery, we can obtain a well-posedness result for the L^p-Overdetermined boundary value problem for any second-order, homogeneous, constant complex coefficient, $M\times M$ weakly elliptic system L in bounded uniformly rectifiable domains in the Euclidean setting with boundary data in Lebesgue-Whitney spaces for arbitrary exponent $p \in (1,\infty)$.

Theorem 4.2.2. *Fix $n, M \in \mathbb{N}$ with $n \geq 2$, an exponent $p \in (1,\infty)$ and an aperture parameter $\kappa \in (0,\infty)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and let*

$$L = (a_{rs}^{\alpha\beta}\partial_r\partial_s)_{1\leq\alpha,\beta\leq M}$$

be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system and consider $A \in \mathfrak{A}_L$. Then the overdetermined boundary value problem for L in Ω with boundary data $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$

$$(\textit{OBVP}_{L,p}) \begin{cases} u \in [C^2(\Omega)]^M, \ \ Lu = 0 \ \textit{ in } \Omega, \\ \\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega, \sigma), \, \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\ \\ \mathrm{Tr}_1^{\kappa - nt} u = \dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]. \end{cases}$$

is solvable if and only if the boundary data is admissible, i.e., it satisfies

$$\dot{C}_A \dot{f} = \tfrac{1}{2} \dot{f} \ \ \sigma\textit{-a.e. on } \partial\Omega. \tag{4.42}$$

Moreover, the $(\textit{OBVP}_{L,p})$ for L in Ω is well-posed in the sense of Hadarmard, that is, for each prescribed data from the space of admissible boundary data there exists a unique solution u and this solution depends continuously on the boundary data. Additionally, we may represent the solution as

$$u = \dot{\mathscr{C}}_A \dot{f} \ \textit{ in } \Omega. \tag{4.43}$$

Proof. The first implication of the solvability result is an immediate consequence of Theorem 4.2.1. To prove uniqueness note that if u_1 and u_2 are two solutions of the $(\mathrm{OBVP}_{L,p})$ for L in Ω with boundary data $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$, then the function $v = u_1 - u_2$ is a solution of the $(\mathrm{OBVP}_{L,p})$ for L in Ω with boundary data equal to zero, hence the integral

representation formula in Theorem 4.1.3 implies that $v = 0$ in Ω. Finally, it follows from the Integral representation formula in Theorem 4.1.3 that if $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$ is an admissible boundary data, then the function $u = \dot{\mathscr{C}}_A \dot{f}$ is the solution of the $(\mathrm{OBVP}_{L,p})$ in Ω. By equation (4.13) in Proposition 4.1.2 the nontangential maximal function of all derivatives up to order one of $\dot{\mathscr{C}}_A \dot{f}$ are bounded by the norm of the boundary data, and this finishes the proof. □

Thus we conclude that for every $p \in (1, \infty)$, the overdetermined boundary value problem $(\mathrm{OBVP}_{L,p})$ is well-posed for arbitrary bounded uniformly rectifiable domains in $\mathbb{R}^n$ if and only if the boundary data is admissible.

4.3 Hardy Spaces associated with Second-order Homogeneous Elliptic systems in Uniformly Rectifiable Domains

The Cauchy operators discussed thus far are connected to the study of Hardy spaces associated with elliptic systems L in bounded UR domains.

To set the stage, fix $n, M \in \mathbb{N}$ with $n \geq 2$ and consider a homogeneous, second-order, complex constant coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$. Also, consider $A \in \mathfrak{A}_L$ a coefficient tensor for L.

Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1,\infty)$ be an integrability exponent and $\kappa \in (0,\infty)$ be an aperture parameter. Recall that the domain Hardy space associated with L in Ω is given by

$$\mathscr{H}^p(\Omega,L) := \big\{u \in [C^2(\Omega)]^M : Lu = 0 \text{ in } \Omega,\, \mathcal{N}_\kappa(u) \in L^p(\partial\Omega,\sigma), \\ \text{and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma)\big\}. \tag{4.44}$$

This is a Banach space when endowed with the norm

$$||u||_{\mathscr{H}^p(\Omega,L)} := ||\mathcal{N}_\kappa(u)||_{L^p(\partial\Omega,\sigma)} + ||\mathcal{N}_\kappa(\nabla u)||_{L^p(\partial\Omega,\sigma)}. \tag{4.45}$$

From Proposition 4.1.2 it follows that

$$\dot{\mathscr{C}}_A : \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)] \to \mathscr{H}^p(\Omega,L) \tag{4.46}$$

is a bounded linear operator for all $p \in (1,\infty)$. Moreover, it follows from Fatou-type theorems in [24, Theorem 3.3.4], that an element in $\mathscr{H}^p(\Omega,L)$ has nontangential boundary traces σ-a.e. on $\partial\Omega$.

Define the linear operator $\dot{P}_A : \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)] \to \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ as

$$\dot{P}_A\dot{f} := (\tfrac{1}{2}I + \dot{C}_A)\dot{f} \;\; \sigma\text{-a.e. on } \partial\Omega. \tag{4.47}$$

By Proposition 4.1.4 the map $\dot{P}_A$ is linear and continuous. Furthermore, for any aperture parameter $\kappa \in (0,\infty)$ the nontangential boundary trace map satisfies

$$\mathrm{Tr}_1^{\kappa-nt}\big(\dot{\mathscr{C}}_A\dot{f}\big) = \dot{P}_A\dot{f} \ \text{ for } \dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]. \tag{4.48}$$

The next proposition establishes further analytical properties of the boundary-to-boundary Cauchy operator associated with L and the linear map $\dot{P}_A$.

Theorem 4.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1,\infty)$ an integrability exponent, $\kappa \in (0,\infty)$ an aperture parameter, and $A \in \mathfrak{A}_L$. Then the boundary-to-boundary Cauchy operator associated with L satisfies*

$$(\dot{C}_A)^2 = \tfrac{1}{4}I \ \text{ on } \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]. \tag{4.49}$$

Moreover, the map $\dot{P}_A$ defined in (4.47) *is a continuous linear projection on* $\mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$.

Proof. To prove the claim in (4.49), let $\dot{f} \in \mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$ be an arbitrary Lebesgue-Whitney array. Then the function $\dot{\mathscr{C}}_A\dot{f}$ satisfies the conditions of the Integral representation formula in Theorem 4.1.3, hence

$$\dot{\mathscr{C}}_A\dot{f} = \dot{\mathscr{C}}_A\big(\mathrm{Tr}_1^{\kappa-nt}\big(\dot{\mathscr{C}}_A\dot{f}\big)\big) \ \text{ in } \Omega. \tag{4.50}$$

It follows from Proposition 4.1.4 that the nontangential boundary trace of (4.50) yields the equality

$$(\tfrac{1}{2}I + \dot{C}_A)\dot{f} = (\tfrac{1}{2}I + \dot{C}_A)^2\dot{f} \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{4.51}$$

which implies the result. As a consequence of (4.49) one verifies that $\dot{P}_A = \dot{P}_A^2$ on $\mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$, thus finishing the proof. □

Given any $A \in \mathfrak{A}_L$, define the boundary Hardy space associated with L on $\partial\Omega$

$$\mathscr{H}^p(\partial\Omega, L) := \dot{P}_A\big(\mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]\big), \tag{4.52}$$

endowed with the norm from $\mathrm{WA}_1^M[L^p(\partial\Omega,\sigma)]$. Then we have the following result regarding these Hardy spaces and the nontangential boundary trace operator.

Proposition 4.3.2. *Fix $n, M \in \mathbb{N}$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1,\infty)$ be an integrability exponent, and $\kappa \in (0,\infty)$ be an aperture parameter. Let L be a second-order, homogeneous, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Then the nontangential boundary trace map $\mathrm{Tr}_1^{\kappa-nt}$ is an isomorphism from $\mathscr{H}^p(\Omega, L)$ to $\mathscr{H}^p(\partial\Omega, L)$.*

Proof. It follows from the Integral representation formula in Theorem 4.1.3 that

$$u \in \mathscr{H}^p(\Omega, L) \implies u = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt}u) \ \text{ in } \Omega, \tag{4.53}$$

thus by Proposition 4.1.4

$$\mathrm{Tr}_1^{\kappa-nt} u = \dot{P}_A(\mathrm{Tr}_1^{\kappa-nt} u). \tag{4.54}$$

Thus $\mathrm{Tr}_1^{\kappa-nt}$ is a well-defined, bounded, linear map from $\mathscr{H}^p(\Omega, L)$ to $\mathscr{H}^p(\partial\Omega, L)$.

To see that $\mathrm{Tr}_1^{\kappa-nt}$ is injective, note that if u_1 and u_2 are functions in $\mathscr{H}^p(\Omega, L)$, then the linearity of the nontangential boundary trace and the Integral representation formula in Theorem 4.1.3 yields to

$$u_1 - u_2 = \dot{\mathscr{C}}_A(\mathrm{Tr}_1^{\kappa-nt}(u_1 - u_2)) \text{ in } \Omega. \tag{4.55}$$

Hence $\mathrm{Tr}_1^{\kappa-nt} u_1 = \mathrm{Tr}_1^{\kappa-nt} u_2$ σ-a.e. on $\partial\Omega$ implies $u_1 = u_2$ in $\mathscr{H}^p(\Omega, L)$.

To see that $\mathrm{Tr}_1^{\kappa-nt}$ is surjective it is enough to show that for every Lebesgue-Whitney array $\dot{f} \in \mathscr{H}^p(\partial\Omega, L)$ there is an element in $\mathscr{H}^p(\Omega, L)$ whose nontangential boundary trace is $\dot{f}$. Indeed, it follows from the definition that given $\dot{f} \in \mathscr{H}^p(\partial\Omega, L)$ there exists $\dot{g} \in \mathrm{WA}_1^M[L^p(\partial\Omega, \sigma)]$ such that $\dot{f} = \dot{P}_A \dot{g}$. Consider the function $\dot{\mathscr{C}}_A \dot{g}$. Then by Proposition 4.1.2 this is an element of $\mathscr{H}^p(\Omega, L)$ and by Proposition 4.1.4

$$\mathrm{Tr}_1^{\kappa-nt}(\dot{\mathscr{C}}_A \dot{g}) = \dot{P}_A \dot{g} = \dot{f}, \tag{4.56}$$

and this finishes the proof. □

Remark 4.3.3. It follows from the proof of the previous proposition that if $\Omega \subset \mathbb{R}^n$ is a bounded UR domain and $p \in (1, \infty)$, then the boundary-to-domain Cauchy operator associated with L is such that

$$\dot{\mathscr{C}}_A : \mathscr{H}^p(\partial\Omega, L) \to \mathscr{H}^p(\Omega, L) \text{ is an isomorphism}$$

and in this context, the nontangential boundary trace and the boundary-to-domain Cauchy operator associated with L are inverses of each other.

CHAPTER 5

OVERDETERMINED BOUNDARY VALUE PROBLEM FOR THE STOKES SYSTEM OF LINEARIZED HYDROSTATICS IN UNIFORMLY RECTIFIABLE DOMAINS

In this chapter, we analyze the overdetermined boundary value problem for the Stokes system of linearized hydrostatics with boundary data in L^p-based function spaces in bounded uniformly rectifiable domains.

In section 5.1, we first introduce a new function space of divergence-free Lebesgue-Whitney arrays. Then, we introduce and analyze Cauchy integral operators associated with the Stokes system of hydrostatics. These Cauchy operators are used to write integral representation formulas for null-solutions of the Stokes system in bounded uniformly rectifiable domains.

In section 5.2 we formulate and establish well-posedness results for the overdetermined boundary value problem for the Stokes system of linearized hydrostatics in bounded uniformly rectifiable domains and prescribed boundary data L^p-based function spaces with integrability exponent $p \in (1, \infty)$.

In section 5.3 we discuss classical Hardy spaces associated with the Stokes system of hydrostatics in bounded uniformly rectifiable domains, and study equivalent formulations of the overdetermined boundary value problem for the Stokes system.

Finally, in section 5.4 we discuss connections between the Cauchy integral for the Stokes system of linearized hydrostatics and the Cauchy integral for the Lamé system of elastostatics.

All the results presented in this chapter are based on joint work with Dorina Mitrea, Irina Mitrea, and Marius Mitrea in [3].

5.1 Cauchy Operator and Integral Representation Formula for Null Solutions of the Stokes System

Stokes system of linear hydrostatics admits a canonical fundamental solution (E, q) in $\mathbb{R}^n$ (cf. D. Mitrea [21, Section 11.2]), where, for each $x \in \mathbb{R}^n \setminus \{0\}$, the fundamental matrix E has entries (r, s) given by

$$E_{rs}(x) := \frac{\delta_{rs}}{2} E_{\Delta}(x) - \frac{1}{2\omega_{n-1}} \frac{x_r x_s}{|x|^n}, \tag{5.1}$$

and the pressure vector q is given by

$$q(x) = (q_s(x))_{1\leq s\leq n} := -\frac{1}{\omega_{n-1}} \frac{x}{|x|^n}. \tag{5.2}$$

Note that the fundamental matrix E is symmetric and satisfy

$$\partial_s E_{\bullet s}(x) = 0 \ \text{ pointwise in } \mathbb{R}^n \setminus \{0\} \text{ and in } \mathcal{S}'(\mathbb{R}^n), \tag{5.3}$$

and

$$\Delta E_{\bullet s}(x) = \partial_s q_{\bullet}(x) \text{ pointwise in } \mathbb{R}^n \setminus \{0\} \text{ and in } [\mathcal{S}'(\mathbb{R}^n)]^n. \tag{5.4}$$

Additionally, the pressure vector q satisfies

$$q(x) = -\nabla E_{\Delta}(x) \text{ pointwise in } \mathbb{R}^n \setminus \{0\} \text{ and in } [\mathcal{S}'(\mathbb{R}^n)]^n, \tag{5.5}$$

where E_{Δ} is the canonical fundamental solution of the Laplacian operator.

Following the discussion in D. Mitrea, I. Mitrea, and M. Mitrea [25, Chapter 6], for each fixed $\theta \in \mathbb{C}$, set the coefficient tensor associated with the Stokes system and parameter θ as the collection

$$a_{rs}^{\alpha\beta}(\theta) := \delta_{rs}\delta_{\alpha\beta} + \theta\delta_{r\beta}\delta_{s\alpha}, \text{ for } 1 \leq r, s, \alpha, \beta \leq n. \tag{5.6}$$

Note that if $u \in [C^2(\Omega)]^n$, then

$$L_\theta u = (a_{rs}^{\alpha\beta}(\theta)\partial_r\partial_s u^\beta)_{1\leq\alpha\leq n} = \Delta u + \theta\nabla \text{div } u. \tag{5.7}$$

Hence for $\Omega \subset \mathbb{R}^n$ open subset, and for each fixed $\theta \in \mathbb{C}$, a null-solution of the Stokes system in Ω satisfies

$$L_\theta u - \nabla\pi = 0 \text{ in } \Omega, \text{ and } \text{div } u = 0 \text{ in } \Omega. \tag{5.8}$$

Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain, $\kappa \in (0,\infty)$ be an aperture parameter, and suppose that

$$\nabla u|_{\partial\Omega}^{\kappa-nt} \text{ and } \pi|_{\partial\Omega}^{\kappa-nt} \text{ exist } \sigma\text{-a.e. on } \partial\Omega. \tag{5.9}$$

For each $\theta \in \mathbb{C}$, set the conormal derivative associated with the Stokes system in Ω as

$$\begin{aligned}\partial_\nu^\theta(u|_{\partial\Omega}^{\kappa-nt}, \pi|_{\partial\Omega}^{\kappa-nt}) &:= (\nu_r a_{rs}^{\alpha\beta}(\theta)\partial_s u^\beta|_{\partial\Omega}^{\kappa-nt} - \nu_\alpha \pi|_{\partial\Omega}^{\kappa-nt})_{1\leq\alpha\leq n} \\ &= (\nu_r \partial_r u^\alpha|_{\partial\Omega}^{\kappa-nt} + \theta\nu_r \partial_\alpha u^r|_{\partial\Omega}^{\kappa-nt} - \nu_\alpha \pi|_{\partial\Omega}^{\kappa-nt})_{1\leq\alpha\leq n}\end{aligned} \tag{5.10}$$

σ-a.e. on $\partial\Omega$, where ν is the geometric outward unit normal to Ω.

Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and consider for each function

$$f = (f^k)_{1\leq k\leq n} \in [L^1(\partial\Omega,\sigma)]^n \tag{5.11}$$

the boundary-to-domain single layer associated with the Stokes system

$$\begin{aligned}\mathscr{S}f(x) &= \int_{\partial\Omega} E(x-y)f(y)\,d\sigma(y) \\ &= \left(\int_{\partial\Omega} E_{jk}(x-y)f^k(y)\,d\sigma(y)\right)_{1\leq j\leq n} \quad \text{for } x\in\Omega,\end{aligned} \tag{5.12}$$

and the boundary-to-domain pressure single layer potential

$$\mathcal{Q}f(x) = \int_{\partial\Omega} \langle q(x-y), f(y)\rangle\,d\sigma(y)$$

$$= \int_{\partial\Omega} q_k(x-y) f^k(y)\, d\sigma(y) \quad \text{for } x \in \Omega. \tag{5.13}$$

Consider also, for each fixed $\theta \in \mathbb{C}$, the boundary-to-domain double layer associated with the Stokes system as the operator

$$\mathscr{D}_\theta f(x) = \left(\int_{\partial\Omega} \left[\partial^\theta_{\nu(y)} \{E_{\bullet j}, -q_j\}(x-y) \right]^\top f(y) d\sigma(y) \right)_{1\le j\le n} \quad \text{for } x \in \Omega, \tag{5.14}$$

and the boundary-to-domain pressure single layer potential

$$\mathcal{P}_\theta f(x) = -(1+\theta) \int_{\partial\Omega} [\nu_j(y)(\partial_j q_\bullet)(x-y)]^\top f(y) d\sigma(y) \quad \text{for } x \in \Omega. \tag{5.15}$$

Now, if u is a function satisfying

$$\text{div } u = 0 \ \text{ in } \Omega, \ \text{ and } \ \nabla u|^{\kappa - nt}_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \tag{5.16}$$

then u also satisfies

$$\text{div } u|^{\kappa - nt}_{\partial\Omega} = 0 \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{5.17}$$

Hence for the treatment of the overdetermined boundary value problem for the Stokes system in Ω with L^p-based boundary data, it is natural to consider that the nontangential boundary trace $\big(u\big|^{nt}_{\partial\Omega}, \nabla u\big|^{nt}_{\partial\Omega}\big)$ of the velocity-field u belongs to the divergence-free Lebesgue-Whitney space defined as

$$\mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] := \big\{\dot{f} \in \mathrm{WA}^n_1[L^p(\partial\Omega,\sigma)] : f^r_{e_r} = 0 \ \ \sigma\text{-a.e. on } \partial\Omega\big\}. \tag{5.18}$$

Above $f^r_{e_r}$ denotes the r component of the vector-valued function f_{e_r} and, as before, we sum over the repeated index r, which in this current setting runs over the set $\{1,\dots,n\}$.

For $\Omega \subset \mathbb{R}^n$ a UR domain and for each integrability exponent $p \in (1,\infty)$, we equip the divergence-free Lebesgue-Whitney space with the norm of the space of Lebesgue-Whitney array. In this way, the divergence-free Lebesgue-Whitney space becomes a Banach space. Indeed, suppose we are given a Cauchy sequence $\{\dot{f}_j\}_{j\in\mathbb{N}} \subset \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]$. Then is also a Cauchy sequence in $\mathrm{WA}^n_1[L^p(\partial\Omega,\sigma)]$, and since the later space is a Banach space, there exists an array $\dot{f} \in \mathrm{WA}^n_1[L^p(\partial\Omega,\sigma)]$ such that

$$\dot{f}_j \longrightarrow \dot{f} \ \text{ in } \mathrm{WA}^n_1[L^p(\partial\Omega,\sigma)] \text{ as } j\to\infty. \tag{5.19}$$

In particular, if for each $j \in \mathbb{N}$ we denote by $f_{e_r,j}$ the r-th component of the array $\dot{f}_j$, and by f_{e_r} the r-th component of the array $\dot{f}$, then (5.19) implies that

$$f_{e_r,j} \longrightarrow f_{e_r} \ \text{ in } L^p(\partial\Omega,\sigma) \text{ as } j\to\infty. \tag{5.20}$$

Thus it follows that, up to a subsequence,

$$f^r_{e_r,j} \longrightarrow f^r_{e_r} \ \ \sigma\text{-a.e. on } \partial\Omega \text{ as } j \to \infty. \tag{5.21}$$

By definition

$$f^r_{e_r,j} = 0 \ \ \sigma\text{-a.e. on } \partial\Omega \text{ for each } j \in \mathbb{N}. \tag{5.22}$$

Therefore (5.21) and (5.22) imply that

$$f^r_{e_r} = 0 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.23}$$

which implies $\dot{f} \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]$ as we wanted to show.

and each pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ define, for every $x \in \mathbb{R}^n \setminus \partial\Omega$, the velocity field boundary-to-domain Cauchy operator

$$\dot{\mathscr{U}}_\theta(\dot{f}, g)(x) := \mathscr{D}_\theta(f_0)(x) - \mathscr{S}((\nu_r(f^s_{e_r} + \theta f^r_{e_s}) - \nu_s g)_{1\le s\le n})(x), \tag{5.24}$$

and the pressure boundary-to-domain Cauchy operator

$$\dot{\mathscr{P}}_\theta(\dot{f}, g)(x) := \mathcal{P}_\theta(f_0)(x) - \mathcal{Q}((\nu_r(f^s_{e_r} + \theta f^r_{e_s}) - \nu_s g)_{1\le s\le n})(x). \tag{5.25}$$

Finally, define the Stokes boundary-to-domain Cauchy operator by

$$\dot{\mathscr{C}}_\theta(\dot{f}, g)(x) := \big(\dot{\mathscr{U}}_\theta(\dot{f}, g)(x), \dot{\mathscr{P}}_\theta(\dot{f}, g)(x)\big), \ \text{ for } x \in \mathbb{R}^n \setminus \partial\Omega. \tag{5.26}$$

It will become clear from the arguments in the proofs of the upcoming results the importance of the compatibility conditions and the divergence-free condition of the array $\dot{f} \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]$ in the algebraic and analytical properties of these Cauchy operators.

Lemma 5.1.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ a UR domain with compact boundary. Then for each pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ and for every $\theta_1, \theta_2 \in \mathbb{C}$ there holds*

$$\dot{\mathscr{U}}_{\theta_1}(\dot{f}, g)(x) = \dot{\mathscr{U}}_{\theta_2}(\dot{f}, g)(x) \ \text{ for all } x \in \mathbb{R}^n \setminus \partial\Omega. \tag{5.27}$$

Additionally, for every $\theta \in \mathbb{C}$ and for all $x \in \mathbb{R}^n \setminus \partial\Omega$ there holds

$$\dot{\mathscr{P}}_{\theta}(\dot{f}, g)(x) = \mathscr{D}_{\Delta}(g)(x) + \int_{\partial\Omega} \partial_{\tau_{rs}(y)}[E_{\Delta}(x-y)] f^r_{e_s}(y) d\sigma(y). \tag{5.28}$$

Consequently, the Stokes boundary-to-domain Cauchy operator is independent of the parameter $\theta \in \mathbb{C}$.

Proof. To prove (5.27) note that this is equivalent to showing that

$$\mathscr{D}_{\theta_1}(f_0)(x) - \mathscr{D}_{\theta_2}(f_0)(x) = (\theta_1 - \theta_2)\mathscr{S}(\nu_r f^r_{e_s})(x). \tag{5.29}$$

On the one hand,

$$\mathscr{D}_{\theta_1}(f_0)(x) - \mathscr{D}_{\theta_2}(f_0)(x) = (\theta_2 - \theta_1) \int_{\partial\Omega} \nu_r(y)(\partial_s E_{\bullet r})(x-y) f^s_0(y) d\sigma(y). \tag{5.30}$$

Using the identity $-\nu_r\partial_s = \partial_{\tau_{sr}} - \nu_s\partial_r$, property (5.3) of the fundamental matrix E, and integration by parts, we may write the right-hand side of (5.30) as

$$\begin{aligned}
&(\theta_2-\theta_1)\int_{\partial\Omega}\nu_r(y)(\partial_s E_{\bullet r})(x-y)f_0^s(y)d\sigma(y)\\
&=(\theta_2-\theta_1)\int_{\partial\Omega}\partial_{\tau_{sr}(y)}[E_{\bullet r}(x-y)]f_0^s(y)d\sigma(y)\\
&=(\theta_2-\theta_1)\int_{\partial\Omega}E_{\bullet r}(x-y)\partial_{\tau_{rs}(y)}[f_0^s(y)]d\sigma(y).
\end{aligned}\tag{5.31}$$

Using the compatibility conditions and the divergence-free condition of the Whitney array $\dot{f}\in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]$ we obtain

$$\partial_{\tau_{rs}}f_0^s=\nu_r f_{e_r}^r-\nu_s f_{e_r}^s=-\nu_s f_{e_r}^s \ \ \sigma\text{-a.e. on }\partial\Omega.\tag{5.32}$$

Inserting this into the last term of (5.31) gives us

$$\mathscr{D}_{\theta_1}(f_0)(x)-\mathscr{D}_{\theta_2}(f_0)(x)=(\theta_1-\theta_2)\int_{\partial\Omega}\nu_s(y)E_{\bullet r}(x-y)f_{e_r}^s d\sigma(y)\tag{5.33}$$

On the other hand, by definition

$$(\theta_1-\theta_2)\mathscr{S}(\nu_r f_{e_s}^r)(x)=(\theta_1-\theta_2)\int_{\partial\Omega}\nu_r(y)E_{\bullet s}(x-y)f_{e_s}^r d\sigma(y).\tag{5.34}$$

Rebranding the indices in (5.34) gives us (5.33), hence it implies (5.29).

To obtain the equality in equation (5.28) note that for every $r, s \in \{1, \ldots, n\}$ one has $\partial_s q_r = \partial_r q_s$. Therefore for every $x \in \mathbb{R}^n \setminus \partial\Omega$

$$\begin{aligned}\mathcal{P}_\theta(f_0)(x) &= -(1+\theta)\int_{\partial\Omega} \nu_s(y)(\partial_s q_r(x-y) f_0^r(y) d\sigma(y) \\ &= (1+\theta)\int_{\partial\Omega} \nu_s(y)\partial_r^y[q_s(x-y)] f_0^r(y) d\sigma(y). \end{aligned} \tag{5.35}$$

Using the identity $\nu_s \partial_r = \partial_{\tau_{sr}} + \nu_r \partial_s$, the fact that the pressure vector q is divergence-free, and integration by parts we may write

$$\begin{aligned}&\int_{\partial\Omega} \nu_s(y)\partial_r^y[q_s(x-y)] f_0^r(y) d\sigma(y) \\ &= \int_{\partial\Omega} \partial_{\tau_{sr(y)}}[q_s(x-y)] f_0^r(y) d\sigma(y) \\ &= \int_{\partial\Omega} q_s(x-y)\partial_{\tau_{rs(y)}}[f_0^r(y)] d\sigma(y). \end{aligned} \tag{5.36}$$

Now, the identity (5.32) together with $q_s = -\partial_s E_\Delta$, imply

$$\begin{aligned}\int_{\partial\Omega} q_s(x-y)\partial_{\tau_{rs}(y)}[f_0^r(y)] d\sigma(y) &= \int_{\partial\Omega} q_s(x-y)\nu_r(y) f_{e_s}^r(y) d\sigma(y) \\ &= -\int_{\partial\Omega} \nu_r(y)(\partial_s E_\Delta)(x-y) f_{e_s}^r(y) d\sigma(y) \\ &= \int_{\partial\Omega} \nu_r(y)\partial_s^y[E_\Delta(x-y)] f_{e_s}^r(y) d\sigma(y). \end{aligned} \tag{5.37}$$

The conclusion is that

$$\mathcal{P}_\theta(f_0)(x) = (1+\theta)\int_{\partial\Omega} \nu_r(y)\partial_s^y[E_\Delta(x-y)]f_{e_s}^r(y)d\sigma(y). \tag{5.38}$$

For the term involving the pressure single layer, we use that $q_s = -\partial_s E_\Delta$ to rewrite it as

$$\begin{aligned}
&\mathcal{Q}((\nu_r(f_{e_r}^s + \theta f_{e_s}^r) - \nu_s g)_{1\le s\le n})(x)\\
&= -\int_{\partial\Omega} (\partial_s E_\Delta)(x-y)[\nu_r(y)(f_{e_r}^s(y) + \theta f_{e_s}^r(y)) - \nu_s(y)g(y)]d\sigma(y)\\
&= \int_{\partial\Omega} \nu_r(y)\partial_s^y[E_\Delta(x-y)](f_{e_r}^s(y) + \theta f_{e_s}^r(y))d\sigma(y) - \mathscr{D}_\Delta(g)(x).
\end{aligned} \tag{5.39}$$

Subtracting (5.39) from (5.38) and grouping the appropriate terms using tangential derivatives gives the result in (5.28).

The implication that the Stokes boundary-to-domain Cauchy operator is independent of the parameter $\theta \in \mathbb{C}$ is now a consequence of (5.27)-(5.28) and formula (5.26). □

The next result contains some properties of the boundary-to-domain operators defined above.

Proposition 5.1.2. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain with compact boundary. Let $p \in (1,\infty)$ be an integrability exponent and consider $\theta \in \mathbb{C}$. Then for each pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ the functions defined in* (5.24) *and* (5.25) *satisfy*

$$\begin{aligned} &\dot{\mathscr{U}}_\theta(\dot{f}, g) \in [C^\infty(\mathbb{R}^n \setminus \partial\Omega)]^n, \ \ \dot{\mathscr{P}}_\theta(\dot{f}, g) \in C^\infty(\mathbb{R}^n \setminus \partial\Omega), \\ &\Delta\dot{\mathscr{U}}_\theta(\dot{f}, g) - \nabla\dot{\mathscr{P}}_\theta(\dot{f}, g) = 0 \ \text{ and } \ \mathrm{div}\, \dot{\mathscr{U}}_\theta(\dot{f}, g) = 0 \ \text{ in } \mathbb{R}^n \setminus \partial\Omega. \end{aligned} \tag{5.40}$$

Additionally, for each $l \in \{1,\dots,n\}$

$$\begin{aligned} \partial_l\dot{\mathscr{U}}_\theta(\dot{f}, g)(x) = \mathscr{D}_\theta(f_l)(x) &- \int_{\partial\Omega} \partial_{\tau_{sl}(y)}[E_{\bullet s}(x-y)]g(y)d\sigma(y) \\ &+ \int_{\partial\Omega} \big\{\partial_{\tau_{sl}(y)}[E_{\bullet r}(x-y)] + \theta\partial_{\tau_{rl}(y)}[E_{\bullet s}(x-y)]\big\} f^r_{e_s}(y)d\sigma(y). \end{aligned} \tag{5.41}$$

Moreover, if $\Omega \subset \mathbb{R}^n$ is bounded, then for each aperture parameter $\kappa \in (0,\infty)$ there exists a constant $C = C(\Omega, p, \kappa)$ such that for every pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$

$$\begin{aligned} &\sum_{|\gamma|\leq 1} ||\mathcal{N}_\kappa(\partial^\gamma\dot{\mathscr{U}}_\theta(\dot{f}, g))||_{L^p(\partial\Omega,\sigma)} + ||\mathcal{N}_\kappa(\dot{\mathscr{P}}_\theta(\dot{f}, g))||_{L^p(\partial\Omega,\sigma)} \\ &\qquad \leq C\left(||(\dot{f}, g)||_{\mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]\oplus L^p(\partial\Omega,\sigma)}\right) \end{aligned} \tag{5.42}$$

Proof. The claims in (5.40) are an immediate consequence of the definition.

To prove the result in (5.41) fix $l \in \{1, \ldots, n\}$ and $x \in \mathbb{R}^n \setminus \partial\Omega$, then compute

$$\partial_l \dot{\mathscr{U}}_\theta(\dot{f}, g)(x) = \partial_l \mathscr{D}_\theta(f_0)(x) - \partial_l \mathscr{S}((\nu_r(f^s_{e_r} + \theta f^r_{e_s}) - \nu_s g)_{1\leq s\leq n})(x). \tag{5.43}$$

For the first term in the right-hand side of (5.43), we differentiate under the integral sign to get

$$\begin{aligned}\partial_l \mathscr{D}_\theta(f_0)(x) = &- \int_{\partial\Omega} \nu_r(y)[(\partial_l\partial_r E_{\bullet s})(x-y) + \theta(\partial_l\partial_s E_{\bullet r})(x-y)]f^s_0(y)d\sigma(y) \\ &+ \int_{\partial\Omega} \nu_s(y)(\partial_l q_\bullet)(x-y)f^s_0(y)d\sigma(y).\end{aligned} \tag{5.44}$$

Using the identity $\nu_r\partial_l = \partial_{\tau_{rl}} + \nu_l\partial_r$, and the properties of the fundamental matrix E gives

$$\begin{aligned}-\nu_r(y)(\partial_l\partial_r E_{\bullet s})(x-y) &= \nu_r(y)\partial^y_l(\partial_r E_{\bullet s})(x-y) \\ &= \partial_{\tau_{rl}(y)}(\partial_r E_{\bullet s})(x-y) + \nu_l(y)\partial^y_r(\partial_r E_{\bullet s})(x-y) \\ &= \partial_{\tau_{rl}(y)}(\partial_r E_{\bullet s})(x-y) - \nu_l(y)(\Delta E_{\bullet s})(x-y) \\ &= \partial_{\tau_{rl}(y)}(\partial_r E_{\bullet s})(x-y) - \nu_l(y)(\partial_s q_\bullet)(x-y).\end{aligned} \tag{5.45}$$

Similarly,

$$\begin{aligned}-\nu_r(y)(\partial_l\partial_s E_{\bullet r})(x-y) &= \nu_r(y)\partial^y_l(\partial_s E_{\bullet r})(x-y) \\ &= \partial_{\tau_{rl}(y)}(\partial_s E_{\bullet r})(x-y) + \nu_l(y)\partial^y_r(\partial_s E_{\bullet r})(x-y) \\ &= \partial_{\tau_{rl}(y)}(\partial_s E_{\bullet r})(x-y) - \nu_l(y)\partial^y_s(\partial_r E_{\bullet r})(x-y)\end{aligned}$$

$$= \partial_{\tau_{rl}(y)}(\partial_s E_{\bullet r})(x-y). \tag{5.46}$$

For the term whose integral kernel is first-order derivatives of components of the pressure vector q, we bring in tangential derivatives once again to write

$$\begin{aligned}
\nu_s(y)(\partial_l q_\bullet)(x-y) &= \nu_s(y)\partial_l^y q_\bullet(x-y) \\
&= -\partial_{\tau_{sl}(y)}[q_\bullet(x-y)] - \nu_l(y)\partial_s^y[q_\bullet(x-y)] \\
&= -\partial_{\tau_{sl}(y)}[q_\bullet(x-y)] + \nu_l(y)(\partial_s q_\bullet)(x-y).
\end{aligned} \tag{5.47}$$

Note that the last term in (5.45) is the opposite of the last term in (5.47), so they cancel each other and we conclude that

$$\begin{aligned}
\partial_l \mathscr{D}_\theta(f_0)(x) = &\int_{\partial\Omega} \{\partial_{\tau_{rl}(y)}[(\partial_r E_{\bullet s})(x-y)] + \theta\partial_{\tau_{rl}(y)}[(\partial_s E_{\bullet r})(x-y)]\} f_0^s(y) d\sigma(y) \\
&- \int_{\partial\Omega} \partial_{\tau_{sl}(y)}[q_\bullet(x-y)] f_0^s(y) d\sigma(y).
\end{aligned} \tag{5.48}$$

After integrating by parts, and using the compatibility conditions and the divergence-free condition of the array $\dot{f}$, it follows that

$$\partial_l \mathscr{D}_\theta(f_0)(x) = \mathscr{D}_\theta(f_l)(x) + \int_{\partial\Omega} \nu_l(y)\{(\partial_r E_{\bullet s})(x-y) + \theta(\partial_s E_{\bullet r})(x-y)\} f_{e_r}^s d\sigma(y). \tag{5.49}$$

For the derivative of the term given by the single layer we obtain

$$\partial_l \mathscr{S}((\nu_r(f^s_{e_r} + \theta f^r_{e_s}) - \nu_s g)_{1\leq s\leq n})(x) = -\int_{\partial\Omega} \nu_s(y)(\partial_l E_{\bullet s})(x-y)g(y)d\sigma(y) + \int_{\partial\Omega} \nu_r(y)(\partial_l E_{\bullet s})(x-y)(f^s_{e_r}(y) + \theta f^r_{e_s}(y))d\sigma(y). \quad (5.50)$$

For the first term on the right-hand side of (5.50) we may use tangential derivatives and properties of the fundamental matrix E to write

$$\begin{aligned} -\nu_s(y)(\partial_l E_{\bullet s})(x-y) &= \nu_s(y)\partial^y_l[E_{\bullet s}(x-y)] \\ &= \partial_{\tau_{sl}(y)}[E_{\bullet s}(x-y)] + \nu_l(y)\partial^y_s[E_{\bullet s}(x-y)] \\ &= \partial_{\tau_{sl}(y)}[E_{\bullet s}(x-y)], \end{aligned} \quad (5.51)$$

so that

$$-\int_{\partial\Omega} \nu_s(y)(\partial_l E_{\bullet s})(x-y)g(y)d\sigma(y) = \int_{\partial\Omega} \partial_{\tau_{sl}(y)}[E_{\bullet s}(x-y)]g(y)d\sigma(y) \quad (5.52)$$

For the second term on the right-hand side of (5.50), we may rebrand some of the indices to write it as

$$\int_{\partial\Omega} [\nu_r(y)(\partial_l E_{\bullet s})(x-y) + \theta\nu_s(y)(\partial_l E_{\bullet r})(x-y)]f^s_{e_r}(y)d\sigma(y). \quad (5.53)$$

All in all,

$$\partial_l \mathscr{S}((\nu_r(f^s_{e_r}+\theta f^r_{e_s})-\nu_s g)_{1\leq s\leq n})(x)=\int_{\partial\Omega}\partial_{\tau_{sl}(y)}[E_{\bullet s}(x-y)]g(y)d\sigma(y)$$
$$+\int_{\partial\Omega}[\nu_r(y)(\partial_l E_{\bullet s})(x-y)+\theta\nu_s(y)(\partial_l E_{\bullet r})(x-y)]f^s_{e_r}(y)d\sigma(y). \tag{5.54}$$

Subtracting from (5.49) the result in (5.54), and grouping the appropriate terms using tangential derivatives finishes the proof of (5.41).

Finally, the boundedness of the nontangential maximal operator as in (5.42) is a consequence of the definition in (5.24), the independence of the Stokes boundary-to-domain Cauchy operator with respect to $\theta\in\mathbb{C}$ obtained in Lemma 5.1.1, equations (5.28) and (5.41), all together with the Calderón-Zygmund theory in UR domains designed to accommodate the operators associated with Stokes system in D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 6.2.4]. □

The Cauchy operators introduced in this section may be used to write succinctly an Integral representation formula for the null-solutions of the Stokes system of linearized hydrostatics in non-smooth domains, as shown in the following theorem.

Theorem 5.1.3. *Fix $n\in\mathbb{N}$ with $n\geq 2$, an integrability exponent $p\in(1,\infty)$ and an aperture parameter $\kappa\in(0,\infty)$. Let $\Omega\subset\mathbb{R}^n$ be a UR domain with compact boundary. Suppose $(u,\pi)\in[C^2(\Omega)]^n\oplus C^1(\Omega)$ is a null-solution for the Stokes system in Ω, that is,*

$$\Delta u-\nabla\pi=0 \ \textit{and}\ \operatorname{div}u=0 \ \textit{in}\ \Omega, \tag{5.55}$$

and assume that

$$\mathcal{N}_\kappa(u) \in L^p(\partial\Omega),\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega),\ \textit{and}\ \mathcal{N}_\kappa(\pi) \in L^p(\partial\Omega), \tag{5.56}$$

and

$$u|_{\partial\Omega}^{\kappa-nt},\ \nabla u|_{\partial\Omega}^{\kappa-nt},\ \textit{and}\ \pi|_{\partial\Omega}^{\kappa-nt}\ \textit{exist}\ \sigma\textit{-a.e. on}\ \partial\Omega. \tag{5.57}$$

Then for each $\theta \in \mathbb{C}$ *and for all* $x \in \Omega$ *there holds:*

$$u(x) = \dot{\mathscr{U}}_\theta \left(\mathrm{Tr}_1^{\kappa-nt} u, \pi|_{\partial\Omega}^{\kappa-nt}\right)(x)$$

and

$$\pi(x) = \dot{\mathscr{P}}_\theta \left(\mathrm{Tr}_1^{\kappa-nt} u, \pi|_{\partial\Omega}^{\kappa-nt}\right)(x)$$

Proof. This theorem follows as a consequence of D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 6.2.1], hence we need to verify that the hypotheses are met.

First, note that if $\Omega \subset \mathbb{R}^n$ is an UR domain with compact boundary, then it is in particular a lower Ahlfors regular domain, and the surface measure σ is a doubling measure.

Second, the considerations in (5.55) imply that the functions u and π are smooth. Additionally, the size conditions (5.56) and the existence of the nontangential traces in (5.57), in concert with the assumption that $\partial\Omega$ is compact, ensure that the appropriate conditions in D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 6.2.1] are met, and this finishes the proof. □

Moving on, let $\Omega \subset \mathbb{R}^n$ be a UR domain with compact boundary and fix an exponent $p \in (1,\infty)$. For each $\theta \in \mathbb{C}$ and for every pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ define the velocity field boundary-to-boundary Cauchy operator as the family of functions

$$\dot{U}_\theta(\dot{f}, g) := \{\dot{U}_\theta(\dot{f}, g)_{e_l}\}_{0\leq l\leq n} \tag{5.58}$$

where, for σ-a.e. point $x \in \partial\Omega$,

$$\dot{U}_\theta(\dot{f}, g)_0(x) := K_\theta f_0(x) - S((\nu_r(f^s_{e_r} + \theta f^r_{e_s}) - \nu_s g)_{1\leq s\leq n})(x) \tag{5.59}$$

and for each $l \in \{1,\dots,n\}$ and σ-a.e. point $x \in \partial\Omega$

$$\begin{aligned}\dot{U}_\theta(\dot{f}, g)_{e_l}(x) &:= K_\theta f_{e_l}(x) - \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \partial_{\tau_{sl}(y)}[E_{\bullet s}(x-y)]g(y)d\sigma(y)\\ &+ \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\{\partial_{\tau_{rl}(y)}[E_{\bullet s}(x-y)] + \theta\partial_{\tau_{sl}(y)}[E_{\bullet r}(x-y)]\big\} f^s_{e_r}(y)d\sigma(y).\end{aligned} \tag{5.60}$$

Also, for σ-a.e. point $x \in \partial\Omega$, define the pressure boundary-to-boundary Cauchy operator as

$$\dot{P}_\theta(\dot{f}, g)(x) := K_\Delta g(x) + \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \partial_{\tau_{sr}(y)}[E_\Delta(x-y)] f^s_{e_r}(y)d\sigma(y). \tag{5.61}$$

Finally, define the Stokes boundary-to-boundary Cauchy operator as

$$\dot{C}_\theta(\dot{f}, g)(x) := (\dot{U}_\theta(\dot{f}, g)(x), \dot{P}_\theta(\dot{f}, g)(x)) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{5.62}$$

The next proposition summarizes some analytical properties of the Stokes boundary-to-boundary Cauchy operator.

Proposition 5.1.4. *Fix $n \in \mathbb{N}$ with $n \geq 2$, an exponent $p \in (1, \infty)$, an aperture parameter $\kappa \in (0, \infty)$, and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. Then for every $\theta \in \mathbb{C}$ the following properties hold:*

1: For every pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ and for σ-a.e. on $\partial\Omega$ the first-order nontangential boundary trace of the velocity field boundary-to-domain Cauchy operator exist and satisfy

$$\mathrm{Tr}_1^{\kappa-nt}\big(\dot{\mathscr{U}}_\theta(\dot{f}, g)\big) = \tfrac{1}{2}\dot{f} + \dot{U}_\theta(\dot{f}, g) \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.63}$$

and the nontangential boundary trace of the pressure boundary-to-domain Cauchy operator exists and satisfies

$$\dot{\mathscr{P}}_\theta(\dot{f}, g)\big|_{\partial\Omega}^{\kappa-nt} = \tfrac{1}{2}g + \dot{P}_\theta(\dot{f}, g) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{5.64}$$

Consequently, the nontangential boundary trace of the Stokes boundary-to-domain Cauchy operator $\dot{\mathscr{C}}_\theta(\dot{f}, g)$ exist and satisfy

$$(\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})\big(\dot{\mathscr{C}}_\theta(\dot{f}, g)\big) = \big(\tfrac{1}{2}I + \dot{C}_\theta\big)(\dot{f}, g) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{5.65}$$

2: The operator $\dot{C}_\theta$ is bounded from $\mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]\oplus L^p(\partial\Omega,\sigma)$ *to itself, that is,*

$$\dot{C}_\theta : \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]\oplus L^p(\partial\Omega,\sigma)\to \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]\oplus L^p(\partial\Omega,\sigma) \tag{5.66}$$

is well-defined, linear, and bounded.

Proof. To obtain the result in 1 it is enough to check that the nontangential boundary trace of the pressure Cauchy operator associated with the Stokes system satisfies

$$\dot{\mathscr{P}}_\theta(\dot{f},g)\big|^{\kappa-nt}_{\partial\Omega}=\tfrac{1}{2}g+\dot{P}_\theta(\dot{f},g)\ \ \sigma\text{-a.e. on }\partial\Omega, \tag{5.67}$$

and that the nontangential boundary traces of the velocity-field Cauchy operator associated with the Stokes system satisfy

$$\dot{\mathscr{U}}_\theta(\dot{f},g)\big|^{\kappa-nt}_{\partial\Omega}=\tfrac{1}{2}f_0+\dot{U}_\theta(\dot{f},g)_0\ \ \sigma\text{-a.e. on }\partial\Omega, \tag{5.68}$$

and for each $l\in\{1,\dots,n\}$

$$\partial_l\dot{\mathscr{U}}_\theta(\dot{f},g)\big|^{\kappa-nt}_{\partial\Omega}=\tfrac{1}{2}f_{e_l}+\dot{U}_\theta(\dot{f},g)_{e_l}\ \ \sigma\text{-a.e. on }\partial\Omega. \tag{5.69}$$

The proof of (5.67) is a consequence of equation (5.28) in Proposition 5.1.2 and the jump formula D. Mitrea, I. Mitrea, and M. Mitrea [24, (2.5.4)].

Similarly, equation (5.68) follows from the jump formula in D. Mitrea, I. Mitrea, and M. Mitrea [24, (2.5.4)] and the fact that for every function $f \in L^p(\partial\Omega, \sigma)$ the nontangential boundary trace of the boundary-to-domain single layer acting on f is the boundary-to-boundary single layer acting on f. The result in (5.69) is a consequence of (5.41) in Proposition 5.1.2 and the jump formula in D. Mitrea, I. Mitrea, and M. Mitrea [24, (2.5.4)].

To prove 2, first note that for each pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ the collection $\dot{U}_\theta(\dot{f}, g) = \{\dot{U}_\theta(\dot{f}, g)_\gamma\}_{|\gamma|\leq 1}$ satisfy both the compatibility condition and the divergence-free condition.

To conclude that $\dot{C}_\theta$ defines a bounded linear operator note that the singular integral operators involved in the definition of $\dot{C}_\theta$ are either weakly singular or satisfy the conditions of Calderón–Zygmund theory on Ahlfors regular closed UR sets in D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 6.2.4], hence they define bounded linear operators from $L^p(\partial\Omega, \sigma)$ to itself and from $L^p_1(\partial\Omega, \sigma)$ to itself for every $p \in (1, \infty)$ and this finishes the proof. □

We now move on to the study of the overdetermined boundary value problem for the Stokes system in UR domains with compact boundaries.

5.2 Formulation and Well-posedness result for the Overdetermined Boundary Value Problem for the Stokes System of Hydrostatics

Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and consider an aperture parameter $\kappa \in (0, \infty)$. For each integrability exponent $p \in (1, \infty)$ we formulate the L^p-OBVP for the Stokes system of linearized hydrostatics in Ω with prescribed boundary data $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ as the task of finding a pair of functions (u, π) satisfying

$$(\mathrm{OBVP}_{S,p}) \begin{cases} u \in [C^2(\Omega)]^n, \ \ \pi \in C^1(\Omega), \\ \\ \Delta u - \nabla \pi = 0 \ \text{ and } \ \mathrm{div}\, u = 0 \ \text{ in } \Omega, \\ \\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega), \, \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega), \, \mathcal{N}(\pi) \in L^p(\partial\Omega), \\ \\ \mathrm{Tr}_1^{\kappa - nt} u = \dot{f} \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \ \text{ and } \ \pi\big|^{\kappa-nt}_{\partial\Omega} = g \in L^p(\partial\Omega, \sigma). \end{cases} \tag{5.70}$$

The following result characterizes the space of admissible boundary data for the Stokes system in bounded UR domains.

Theorem 5.2.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$, an integrability exponent $p \in (1, \infty)$, an aperture parameter $\kappa \in (0, \infty)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. Then the space of admissible boundary datum for $(OBVP_{S,p})$ in Ω, that is, the set*

$$\Big\{ (\mathrm{Tr}_1^{\kappa-nt} u, \pi\big|^{\kappa-nt}_{\partial\Omega}) : (u, \pi) \in [C^2(\Omega)]^n \oplus C^1(\Omega), \ \Delta u - \nabla \pi = 0 \text{ in } \Omega$$

$$\text{div } u = 0 \text{ in } \Omega,\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega,\sigma),\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma),$$

$$\text{and } \mathcal{N}_\kappa(\pi) \in L^p(\partial\Omega,\sigma)\big\}, \tag{5.71}$$

may be described as

$$\big\{(\dot{f},g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega)] \oplus L^p(\partial\Omega) : \dot{C}_\theta(\dot{f},g) = \tfrac{1}{2}(\dot{f},g)\big\}. \tag{5.72}$$

Remark 5.2.2. To see that the set in (5.71) is well-defined, suppose (u,π) is a pair of functions satisfying the properties in (5.71). Then by the Fatou-type theorems D. Mitrea, I. Mitrea, and M. Mitrea [25, Theorem 6.3.2] and [24, Theorem 3.1.13], we conclude that the first-order nontangential boundary trace of u exists σ-a.e. on $\partial\Omega$, that is,

$$\mathrm{Tr}_1^{\kappa-nt}u = (u|_{\partial\Omega}^{\kappa-nt}, \nabla u|_{\partial\Omega}^{\kappa-nt}) \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \tag{5.73}$$

and that the nontangential boundary trace of π exists σ-a.e. on $\partial\Omega$. Thus the set in (5.71) is well-defined. Moreover, the assumptions

$$\mathcal{N}_\kappa(u) \in L^p(\partial\Omega,\sigma),\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma),\ \mathcal{N}_\kappa(\pi) \in L^p(\partial\Omega,\sigma) \tag{5.74}$$

ensure that $\pi\big|_{\partial\Omega}^{\kappa-nt} \in L^p(\partial\Omega,\sigma)$, and $\mathrm{Tr}_1^{\kappa-nt}u$ belongs to $\mathrm{WA}^n_1[L^p(\partial\Omega,\sigma)]$. Finally, the PDE condition div $u = 0$ in Ω and the existence of the nontangential boundary trace of ∇u guarantee that $\mathrm{Tr}_1^{\kappa-nt}u$ is a divergence-free array.

Proof. The first claim is that the set in (5.71) is contained in (5.72). To see that, note that if a pair of functions (u,π) has the properties in (5.71), then by Remark 5.2.2 the first-order nontangential boundary trace of u exists σ-a.e. on $\partial\Omega$, and the nontangential boundary trace of π exists σ-a.e. on $\partial\Omega$. Additionally, we have

$$\big(\mathrm{Tr}_1^{\kappa-nt}u,\pi\big|_{\partial\Omega}^{\kappa-nt}\big)\in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]\oplus L^p(\partial\Omega,\sigma). \tag{5.75}$$

Thus the integral representation formula in (5.1.3) implies that

$$(u,\pi)=\dot{\mathscr{C}}_\theta(\mathrm{Tr}_1^{\kappa-nt}u,\pi\big|_{\partial\Omega}^{\kappa-nt}). \tag{5.76}$$

Taking nontangential boundary traces and using Proposition 5.1.4 we obtain

$$\big(\mathrm{Tr}_1^{\kappa-nt}u,\pi\big|_{\partial\Omega}^{\kappa-nt}\big)=\Big(\tfrac{1}{2}I+\dot{C}_\theta\Big)\big(\mathrm{Tr}_1^{\kappa-nt}u,\pi\big|_{\partial\Omega}^{\kappa-nt}\big), \tag{5.77}$$

σ-a.e. on $\partial\Omega$, which proves the claim.

The second claim is that the set in (5.72) is a subset of the one in (5.71). Indeed, consider an element $(\dot{f},g)\in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]\oplus L^p(\partial\Omega,\sigma)$ of the set in (5.72). By Proposition 5.1.2 the pair of functions $\dot{\mathscr{C}}_\theta(\dot{f},g)$ is a null-solution of the Stokes system in Ω and satisfy

$$\mathcal{N}_\kappa(u)\in L^p(\partial\Omega,\sigma),\ \mathcal{N}_\kappa(\nabla u)\in L^p(\partial\Omega,\sigma),\ \mathcal{N}_\kappa(\pi)\in L^p(\partial\Omega,\sigma). \tag{5.78}$$

Moreover, by Proposition 5.1.4 its nontangential boundary trace satisfy

$$(\mathrm{Tr}_1^{\kappa-nt}\oplus\cdot\big|_{\partial\Omega}^{\kappa-nt})\big(\dot{\mathscr{C}}_\theta(\dot f,g)\big)=\big(\tfrac12 I+\dot C_\theta\big)(\dot f,g)\ \ \sigma\text{-a.e. on }\partial\Omega. \tag{5.79}$$

Since $\dot C_\theta(\dot f,g)=\frac12(\dot f,g)$, we conclude that

$$(\mathrm{Tr}_1^{\kappa-nt}\oplus\cdot\big|_{\partial\Omega}^{\kappa-nt})\big(\dot{\mathscr{C}}_\theta(\dot f,g)\big)=(\dot f,g)\ \ \sigma\text{-a.e. on }\partial\Omega, \tag{5.80}$$

thus proving the second claim. These two claims prove the theorem. □

The next theorem concerns the well-posedness of the overdetermined boundary value problem for the Stokes system in bounded uniformly rectifiable domains formulated in $(\mathrm{OBVP}_{S,p})$.

Theorem 5.2.3. *Fix $n\in\mathbb{N}$ with $n\geq 2$, an integrability exponent $p\in(1,\infty)$ and an aperture parameter $\kappa\in(0,\infty)$. Let $\Omega\subset\mathbb{R}^n$ be a bounded UR domain and let $\theta\in\mathbb{C}$. Then the overdetermined boundary value problem for the Stokes system in Ω with boundary data $(\dot f,g)\in\mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]\oplus L^p(\partial\Omega,\sigma)$*

$$(\textit{OBVP}_{S,p})\begin{cases} u\in[C^2(\Omega)]^n,\ \ \pi\in C^1(\Omega),\\[2mm] \Delta u-\nabla\pi=0\ \textit{and}\ \textit{div}\,u=0\ \textit{in}\ \Omega,\\[2mm] \mathcal{N}_\kappa(u)\in L^p(\partial\Omega),\ \mathcal{N}_\kappa(\nabla u)\in L^p(\partial\Omega),\ \mathcal{N}(\pi)\in L^p(\partial\Omega),\\[2mm] \mathrm{Tr}_1^{\kappa-nt}u=\dot f\in\mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]\ \textit{and}\ \pi\big|_{\partial\Omega}^{\kappa-nt}=g\in L^p(\partial\Omega,\sigma), \end{cases} \tag{5.81}$$

is solvable if and only if the boundary data is admissible, i.e., it satisfies

$$\dot{C}_\theta(\dot{f}, g) = \tfrac{1}{2}(\dot{f}, g) \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.82}$$

where $\dot{C}_\theta$ is the Stokes boundary-to-boundary Cauchy operator.

Moreover, the overdetermined boundary value problem $(OBVP_{S,p})$ for the Stokes system of linearized hydrostatics in Ω is well-posed in the sense of Hadarmard, that is, for each prescribed pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ from the space of admissible boundary data there exists a unique solution (u, π) and this solution depends continuously on the boundary data. Additionally, one may represent the solution as

$$(u, \pi) := \dot{\mathscr{C}}_\theta(\dot{f}, g) \ \ \text{in } \Omega. \tag{5.83}$$

where $\dot{\mathscr{C}}_\theta$ is the Stokes boundary-to-domain Cauchy operator.

Proof. To prove solvability, note that if a pair (u, π) is a solution of $(\mathrm{OBVP}_{S,p})$, then by Theorem 5.2.1

$$\frac{1}{2}\big(\mathrm{Tr}_1^{\kappa-nt} u, \pi\big|_{\partial\Omega}^{\kappa-nt}\big) = \dot{C}_\theta\big(\mathrm{Tr}_1^{\kappa-nt} u, \pi\big|_{\partial\Omega}^{\kappa-nt}\big). \tag{5.84}$$

Conversely, if $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ is admissible, then by Theorem 5.2.1 there is a pair (u, π) in Ω that is a solution of the boundary value problem $(\mathrm{OBVP}_{S,p})$.

For uniqueness note that if (u_1, π_1) and (u_2, π_2) are two solutions of the boundary value problem $(\text{OBVP}_{S,p})$ for the Stokes system of linearized hydrostatics in Ω with boundary data $(\dot{f}, g) \in \text{WA}^n_{\text{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$, then the pair

$$(v, w) \in [C^2(\Omega)]^n \oplus C^1(\Omega), \text{ where } v := u_1 - u_2, \ w := \pi_1 - \pi_2, \tag{5.85}$$

is a solution of the $(\text{OBVP}_{S,p})$ with boundary data equal to zero. Thus the integral representation formula in Theorem 5.1.3 implies that $(v, w) = 0$ in Ω.

Finally, it follows from the integral representation formula in Theorem 5.1.3 that if $(\dot{f}, g) \in \text{WA}^n_{\text{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ is an admissible boundary data, then the pair $(u, \pi) := \dot{\mathscr{C}}_\theta(\dot{f}, g)$ is the solution of the $(\text{OBVP}_{S,p})$ in Ω. By Proposition 5.1.2 we conclude that there is a constant $C \in (0, \infty)$ such that

$$\begin{aligned} &\sum_{|\gamma|\leq 1} ||\mathcal{N}_\kappa(\partial^\gamma \dot{\mathscr{U}}_\theta(\dot{f}, g))||_{L^p(\partial\Omega,\sigma)} + ||\mathcal{N}_\kappa(\dot{\mathscr{P}}_\theta(\dot{f}, g))||_{L^p(\partial\Omega,\sigma)} \\ &\qquad \leq C \left(||(\dot{f}, g)||_{\text{WA}^n_{\text{div}}[L^p(\partial\Omega,\sigma)]\oplus L^p(\partial\Omega,\sigma)} \right) \end{aligned} \tag{5.86}$$

and this finishes the proof. □

The characterization the space of admissible boundary data in Theorem 5.2.1 and the integral representation formula in Theorem 5.1.3 allow us to conclude that for every integrability exponent $p \in (1,\infty)$, the overdetermined boundary value problem $(\text{OBVP}_{S,p})$ is well-posed for arbitrary bounded uniformly rectifiable domains in $\mathbb{R}^n$ if and only if the boundary data is admissible.

5.3 Hardy Spaces associated with the Stokes System of Hydrostatics

Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1,\infty)$ be an integrability exponent and $\kappa \in (0,\infty)$ be an aperture parameter. Define the domain Hardy space associated with the Stokes system in Ω as

$$\begin{aligned}\mathscr{H}^p(\Omega,S) := \big\{(u,\pi) \in [C^2(\Omega)]^n \oplus C^1(\Omega) : \Delta u - \nabla\pi = 0, \text{ and div } u = 0 \text{ in } \Omega,\\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega,\sigma),\, \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma),\, \mathcal{N}_\kappa(\pi) \in L^p(\partial\Omega,\sigma)\big\}.\end{aligned} \tag{5.87}$$

This is a vector space, and we may endow it with the norm

$$||(u,\pi)||_{\mathscr{H}^p(\Omega,S)} := \sum_{|\gamma|\leq 1} ||\mathcal{N}_\kappa(\partial^\gamma u)||_{L^p(\partial\Omega,\sigma)} + ||\mathcal{N}_\kappa(\pi)||_{L^p(\partial\Omega,\sigma)}. \tag{5.88}$$

With the norm in (5.88), the domain Hardy space associated with the Stokes system in Ω becomes a Banach space. Indeed, note that from Remark 5.2.2 we get that the nontangential boundary trace map

$$\begin{aligned} \mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt} : \mathscr{H}^p(\Omega, S) &\to \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma) \\ (u, \pi) &\mapsto \big(\mathrm{Tr}_1^{\kappa-nt} u, \pi\big|_{\partial\Omega}^{\kappa-nt}\big) \end{aligned} \tag{5.89}$$

is a well-defined, linear, and bounded map. Moreover, it follows from Proposition 5.1.2 that for each $\theta \in \mathbb{C}$ we have that

$$\dot{\mathscr{C}}_\theta : \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma) \to \mathscr{H}^p(\Omega, S) \tag{5.90}$$

is a well-defined, linear, and bounded operator.

Thus if $\{(u_j, \pi_j)\}_{j\in\mathbb{N}} \subset \mathscr{H}^p(\Omega, S)$ is a Cauchy sequence, then

$$\big\{(\mathrm{Tr}_1^{\kappa-nt} u_j, \pi_j\big|_{\partial\Omega}^{\kappa-nt})\big\}_{j\in\mathbb{N}} \subset \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma) \tag{5.91}$$

is a Cauchy sequence. Since the function space in (5.91) is a Banach space, we conclude there exists $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ such that

$$(\mathrm{Tr}_1^{\kappa-nt} u_j, \pi_j\big|_{\partial\Omega}^{\kappa-nt}) \xrightarrow{j\to\infty} (\dot{f}, g) \ \text{ in } \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma). \tag{5.92}$$

Therefore the pair $(u,\pi) := \dot{\mathscr{C}}_\theta(\dot{f},g)$ is such that $(u,\pi) \in \mathscr{H}^p(\Omega,S)$ and by (5.90) and (5.92) we have

$$(u_j,\pi_j) \xrightarrow{j\to\infty} (u,\pi) \text{ in } \mathscr{H}^p(\Omega,S). \tag{5.93}$$

Moving on, for each $\theta \in \mathbb{C}$, define the linear operator

$$\dot{\Pi}_\theta : \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma) \to \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma) \tag{5.94}$$

as

$$\dot{\Pi}_\theta(\dot{f},g) := (\tfrac{1}{2}I + \dot{C}_\theta)(\dot{f},g) \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{5.95}$$

It follows from Proposition 5.1.4 that the mapping $\dot{\Pi}_\theta$ is a well-defined, linear, and bounded operator, and that for each aperture parameter $\kappa \in (0,\infty)$ and for every pair of functions $(\dot{f},g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ the nontangential boundary trace map satisfies

$$(\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})\big(\dot{\mathscr{C}}_\theta(\dot{f},g)\big) = \dot{\Pi}_\theta(\dot{f},g). \tag{5.96}$$

The next proposition establishes further analytical properties of the Stokes boundary-to-boundary Cauchy operator and the map $\dot{\Pi}_\theta$.

Theorem 5.3.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1,\infty)$ be an integrability exponent, $\kappa \in (0,\infty)$ be an aperture parameter. Also, let $\theta \in \mathbb{C}$. Then the Stokes boundary-to-boundary Cauchy operator satisfies*

$$(\dot{C}_\theta)^2 = \tfrac{1}{4} I \ \text{ on } \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma). \tag{5.97}$$

Consequently, $\dot{\mathrm{II}}_\theta : \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma) \to \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ defines a continuous linear projection.

Proof. To prove (5.97), let $(\dot{f},g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ be an arbitrary element and consider the pair of functions given by $\dot{\mathscr{C}}_\theta(\dot{f},g)$.

It follows from Proposition 5.1.2, Proposition 5.1.4 and the integral representation formula in Theorem 5.1.3 that

$$\dot{\mathscr{C}}_\theta\Big((\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})\big(\dot{\mathscr{C}}_\theta(\dot{f},g)\big)\Big) = \dot{\mathscr{C}}_\theta(\dot{f},g) \ \text{ in } \Omega. \tag{5.98}$$

Taking nontangential boundary traces and using Proposition 5.1.4 gives us

$$(\tfrac{1}{2}I + \dot{C}_\theta)^2(\dot{f},g) = (\tfrac{1}{2}I + \dot{C}_\theta)(\dot{f},g) \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.99}$$

which proves (5.97). The last statement in the proposition follows from definition (5.95) and (5.99). □

Given any $\theta \in \mathbb{C}$, define the boundary Hardy space associated with the Stokes system on $\partial\Omega$ as

$$\mathscr{H}^p(\partial\Omega, S) := \dot{\Pi}_\theta\big(\mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)\big), \tag{5.100}$$

endowed with the norm from $\mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$. Note that this is a Banach space since $\dot{\Pi}_\theta$ is a continuous linear projection. Furthermore, as shown in the following corollary, the boundary Hardy space associated with the Stokes system on $\partial\Omega$ is equivalent to the space of admissible boundary data for the overdetermined boundary value problem $(\mathrm{OBVP}_{S,p})$ in Ω.

Corollary 5.3.2. *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1,\infty)$ be an integrability exponent, $\kappa \in (0,\infty)$ be an aperture parameter and $\theta \in \mathbb{C}$.*

Then the space of admissible boundary data for $(\mathrm{OBVP}_{S,p})$ in Ω introduced in Theorem 5.2.1, that is, the function space

$$\begin{aligned}\Big\{(\mathrm{Tr}_1^{\kappa-nt}u, \pi\big|_{\partial\Omega}^{\kappa-nt}) : (u,\pi) \in [C^2(\Omega)]^n \oplus C^1(\Omega),\ \Delta u - \nabla\pi = 0 \text{ in } \Omega \\ \text{div } u = 0 \text{ in } \Omega,\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega,\sigma),\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma), \\ \text{and } \mathcal{N}_\kappa(\pi) \in L^p(\partial\Omega,\sigma)\Big\},\end{aligned} \tag{5.101}$$

coincides with the boundary Hardy space associated with the Stokes system

$$\mathscr{H}^p(\partial\Omega, S) = \dot{\Pi}_\theta\big(\mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)\big), \tag{5.102}$$

defined in (5.100).

Proof. Let $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ be an admissible boundary data for $(\mathrm{OBVP}_{S,p})$ in Ω. It follows from (5.72) in Theorem 5.2.1 that

$$\dot{C}_\theta(\dot{f}, g) = \frac{1}{2}(\dot{f}, g), \tag{5.103}$$

where $\dot{C}_\theta$ is the Stokes boundary-to-boundary Cauchy operator. Therefore by the definition in (5.95) we verify that

$$\dot{\Pi}_\theta(\dot{f}, g) = (\dot{f}, g) \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.104}$$

thus by (5.100) we have $(\dot{f}, g) \in \mathscr{H}^p(\partial\Omega, S)$.

Conversely, suppose $(\dot{f}, g) \in \mathscr{H}^p(\partial\Omega, S)$. Then by Theorem 5.3.1 we have

$$\dot{\Pi}_\theta(\dot{f}, g) = (\dot{f}, g) \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.105}$$

which implies

$$\dot{C}_\theta(\dot{f}, g) = \frac{1}{2}(\dot{f}, g) \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.106}$$

and by (5.72) in Theorem 5.2.1 we conclude the result. □

The following result connects the domain and the boundary Hardy spaces associated with the Stokes system and the nontangential boundary trace map.

Proposition 5.3.3. *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1, \infty)$ be an integrability exponent, and $\kappa \in (0, \infty)$ be an aperture parameter. Then the nontangential boundary trace map $(\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})$ is an isomorphism from $\mathscr{H}^p(\Omega, S)$ to $\mathscr{H}^p(\partial\Omega, S)$.*

Proof. It follows from Remark 5.2.2 and the integral representation formula in Theorem 5.1.3 that

$$(u, \pi) \in \mathscr{H}^p(\Omega, S) \implies u = \dot{\mathscr{C}}_\theta(\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})(u, \pi) \text{ in } \Omega, \tag{5.107}$$

thus by Proposition 5.1.4

$$(\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})(u, \pi) = \dot{\mathrm{I}}_\theta(\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})(u, \pi). \tag{5.108}$$

This shows that $(u, \pi) \mapsto (\mathrm{Tr}_1^{\kappa-nt} u, \pi\big|_{\partial\Omega}^{\kappa-nt})$ is a well-defined, linear, and bounded map from $\mathscr{H}^p(\Omega, S)$ to $\mathscr{H}^p(\partial\Omega, S)$.

To see that $(\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})$ is injective, note that if (u_1, π_1) and (u_2, π_2) are elements in $\mathscr{H}^p(\Omega, S)$ such that

$$(u_1, \pi_1) = (u_2, \pi_2) \text{ in } \mathscr{H}^p(\Omega, S) \tag{5.109}$$

then the linearity of the nontangential boundary trace and the Stokes boundary-to-domain Cauchy operator, together with (5.107) give us that if

$$(\mathrm{Tr}_1^{\kappa-nt} u_1, \pi_1\big|_{\partial\Omega}^{\kappa-nt}) = (\mathrm{Tr}_1^{\kappa-nt} u_2, \pi_2\big|_{\partial\Omega}^{\kappa-nt}) \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.110}$$

then $(u_1, \pi_1) = (u_2, \pi_2)$ in $\mathscr{H}^p(\Omega, S)$.

To see that $(\mathrm{Tr}_1^{\kappa-nt} \oplus \cdot\big|_{\partial\Omega}^{\kappa-nt})$ is surjective note that by Corollary 5.3.2 we have that for every $(\dot{f}, g) \in \mathscr{H}^p(\partial\Omega, S)$ is an admissible boundary data for the overdetermined boundary value problem $(\mathrm{OBVP}_{S,p})$ in Ω, hence by Theorem 5.2.3 there is a (unique) pair $(u, \pi) \in \mathscr{H}^p(\Omega, S)$ such that

$$(\mathrm{Tr}_1^{\kappa-nt} u, \pi\big|_{\partial\Omega}^{\kappa-nt}) = (\dot{f}, g) \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.111}$$

and this finishes the proof. □

Remark 5.3.4. It follows from the proof of the previous proposition that if $\Omega \subset \mathbb{R}^n$ is a bounded UR domain and $p \in (1, \infty)$ is an integrability exponent, then for each $\theta \in \mathbb{C}$ the Stokes boundary-to-domain Cauchy operator is such that

$$\dot{\mathscr{C}}_\theta : \mathscr{H}^p(\partial\Omega, S) \to \mathscr{H}^p(\Omega, S) \ \text{ is an isomorphism.} \tag{5.112}$$

In this context, the nontangential boundary trace and the Stokes boundary-to-domain Cauchy operator are inverses.

We now study different formulations of the overdetermined boundary value problem for the Stokes system in bounded uniformly rectifiable domains with boundary data in L^p-based function spaces. We show that, under suitable algebraic conditions on the conormal derivative, these formulations are equivalent to the boundary problem $(\mathrm{OBVP}_{S,p})$ discussed in the previous section.

To set the stage, fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a UR domain. Consider an integrability exponent $p \in (1, \infty)$ and $B \in \mathfrak{A}(n, n)$ a coefficient tensor, that is, B is "blocks of matrices" of complex numbers of the form

$$B = (b_{rs}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq n\\ 1\leq r,s\leq n}}. \tag{5.113}$$

Define the Stokes packing operator associated with B, $\dot{\Theta}_B^{\text{Stokes}}$, acting on each pair $(\dot{f}, g) \in \text{WA}^n_{\text{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$ as

$$\dot{\Theta}_B^{\text{Stokes}}(\dot{f}, g) := (f_0, \nu_r B_{rs} f_{e_s} - g\nu), \tag{5.114}$$

where $\nu_r B_{rs} f_{e_s} := \{\nu_r b_{rs}^{\alpha\beta} f_{e_s}^{\beta}\}_{1\leq\alpha\leq n}$ and $g\nu := \{g\nu_\alpha\}_{1\leq\alpha\leq n}$.

Note that

$$\dot{\Theta}_B^{\text{Stokes}} : \text{WA}^n_{\text{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma) \to [L^p(\partial\Omega, \sigma)]^n \oplus [L^p(\partial\Omega, \sigma)]^n \tag{5.115}$$

is a well-defined, linear, and bounded map.

Define $\mathfrak{A}^{\text{Stokes}}_{\text{inv}}(n, \Omega)$ as the collection of coefficient tensors in $\mathfrak{A}(n, n)$ with the additional property that

$$\begin{aligned}&[B_{rs}\nu_r\nu_s]^{-1} \text{ is well-defined, belongs to } [L^\infty(\partial\Omega, \sigma)]^{n\times n}, \text{ and sat-}\\ &\text{isfies } \langle \nu, [B_{rs}\nu_r\nu_s]^{-1}\nu\rangle \neq 0 \ \sigma\text{-a.e. on } \partial\Omega,\end{aligned} \tag{5.116}$$

where ν is the geometric outward unit normal to Ω.

It follows from D. Mitrea, I. Mitrea, and M. Mitrea [26, Definition 1.1.2] that the inclusion $\mathfrak{A}_{\rm LH}(n,n) \subset \mathfrak{A}^{\rm Stokes}_{\rm inv}(n,\Omega)$ holds for every $\Omega \subset \mathbb{R}^n$ UR domain. Indeed, if $B \in \mathfrak{A}_{\rm LH}(n,n)$, then by D. Mitrea, I. Mitrea, and M. Mitrea [26, (1.1.25)] there exists $c > 0$ such that

$$\text{Re}[\langle \nu, [B_{rs}\nu_r\nu_s]^{-1}\nu\rangle] \geq c \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.117}$$

which implies that $B \in \mathfrak{A}^{\rm Stokes}_{\rm inv}(n,\Omega)$.

We then have the following mapping properties for the Stokes packing operator.

Proposition 5.3.5. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain. Consider an integrability exponent $p \in (1,\infty)$ and a matrix $B \in \mathfrak{A}^{\rm Stokes}_{\rm inv}(n,\Omega)$. Then the packing operator*

$$\dot{\Theta}^{\rm Stokes}_B : \text{WA}^n_{\rm div}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma) \to [L^p(\partial\Omega,\sigma)]^n \oplus [L^p(\partial\Omega,\sigma)]^n \tag{5.118}$$

given by

$$\dot{\Theta}^{\rm Stokes}_B(\dot{f},g) := (f_0, \nu_r B_{rs} f_{e_s} - g\nu) \tag{5.119}$$

is injective.

Proof. Suppose $(\dot{f},g) \in \text{WA}^n_{\rm div}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ is in the kernel of the packing operator $\dot{\Theta}^{\rm Stokes}_B$, that is,

$$\dot{\Theta}^{\rm Stokes}_B(\dot{f},g) = (0,0) \ \ \text{ on } [L^p(\partial\Omega,d\sigma)]^n \oplus [L^p(\partial\Omega,d\sigma)]^n. \tag{5.120}$$

This implies that

$$f_0 = 0 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.121}$$

thus

$$\partial_{\tau_{jk}} f_0 = 0 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.122}$$

for every $j, k \in \{1, \dots, n\}$. It remains to show that $f_{e_j} = 0$ σ-a.e. on $\partial\Omega$ for every $j \in \{1, \dots, n\}$ and $g = 0$ σ-a.e. on $\partial\Omega$.

Note that (5.120) implies

$$\nu_r B_{rs} f_{e_s} = g\nu \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{5.123}$$

A similar computation to the one in (3.102) allows us to show that

$$f_{e_j} = [B_{kl}\nu_k\nu_l]^{-1}(\nu_j\nu_r B_{rs} f_{e_s} + \nu_r B_{rs}\partial_{\tau_{sj}} f_0) \ \text{ for every } j \in \{1, \dots, n\} \tag{5.124}$$

Therefore (5.122) and (5.123)-(5.124) imply that for each pair $j, \alpha \in \{1, \dots, n\}$

$$f^{\alpha}_{e_j} = \left([B_{kl}\nu_k\nu_l]^{-1}(\nu_j\, g\nu)\right)_{\alpha} = (\nu_j g)[b^{\alpha\beta}_{kl}\nu_k\nu_l]^{-1}\nu_\beta. \tag{5.125}$$

Thus from (5.125) we conclude that

$$f^{\alpha}_{e_\alpha} = (\nu_\alpha g)[b^{\alpha\beta}_{kl}\nu_k\nu_l]^{-1}\nu_\beta = g\langle\nu, [B_{kl}\nu_k\nu_l]^{-1}\nu\rangle \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{5.126}$$

Since $\dot{f} \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]$, we must have

$$g\langle \nu, [B_{kl}\nu_k\nu_l]^{-1}\nu\rangle = f^{\alpha}_{e_\alpha} = 0 \ \ \sigma\text{-a.e. on } \partial\Omega, \tag{5.127}$$

and under the assumption that $B \in \mathfrak{A}^{\mathrm{Stokes}}_{\mathrm{inv}}(n,\Omega)$ this is equivalent to

$$g = 0 \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{5.128}$$

It then follows from (5.125) that $f_{e_j} = 0$ σ-a.e. on $\partial\Omega$ and this finishes the proof. □

We shall use the previous proposition to study other formulations of the overdetermined boundary value problem for the Stokes system.

Indeed, fix $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and $\kappa \in (0,\infty)$ be an aperture parameter. For each exponent $p \in (1,\infty)$ and each matrix $B \in \mathfrak{A}^{\mathrm{Stokes}}_{\mathrm{inv}}(n,\Omega)$, we formulate the L^p-based overdetermined boundary value problem for the Stokes system in Ω associated with B and with boundary data $(g_0,g_1) \in [L^p(\partial\Omega,\sigma)]^n \oplus [L^p(\partial\Omega,\sigma)]^n$ as the task of finding a pair of functions $u : \Omega \to \mathbb{C}^n$ and $\pi : \Omega \to \mathbb{C}$ satisfying

$$(L^p\text{-OBVP}_{S,B}) \begin{cases} u \in [C^2(\Omega)]^n, \ \ \pi \in C^1(\Omega), \\ \\ \Delta u - \nabla\pi = 0 \ \text{ and } \ \mathrm{div}\, u = 0 \ \text{ in } \Omega, \\ \\ \mathcal{N}_\kappa(u) \in L^p(\partial\Omega,\sigma), \ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma), \ \mathcal{N}(\pi) \in L^p(\partial\Omega,\sigma), \\ \\ u\big|^{\kappa-nt}_{\partial\Omega} = g_0 \in L^p(\partial\Omega,\sigma), \ \ \partial^B_\nu(u\big|^{\kappa-nt}_{\partial\Omega}, \pi\big|^{\kappa-nt}_{\partial\Omega}) = g_1 \in L^p(\partial\Omega,\sigma), \end{cases} \tag{5.129}$$

where $\partial_\nu^B(u\big|_{\partial\Omega}^{\kappa-nt},\pi\big|_{\partial\Omega}^{\kappa-nt}):=\nu_r B_{rs}\partial_s u\big|_{\partial\Omega}^{\kappa-nt}-\pi\big|_{\partial\Omega}^{\kappa-nt}\nu$ is the conormal derivative of (u,π) associated with the Stokes system and matrix B.

We then have the following result regarding the boundary value problem (L^p-OBVP$_{S,B}$) and its connections with Hardy spaces associated with the Stokes system and the overdetermined boundary value problem (OBVP$_{S,p}$).

Theorem 5.3.6. *Retain the geometrical and analytical context from the previous paragraph and bring in the packing operator associated with B defined in* (5.114)-(5.115)*. Then the boundary value problem (L^p-OBVP$_{S,B}$) is solvable if and only if the boundary data $(g_0,g_1)\in[L^p(\partial\Omega)]^n\oplus[L^p(\partial\Omega)]^n$ is such that*

$$(g_0,g_1)\in\dot{\Theta}_B^{\text{Stokes}}(\mathscr{H}^p(\partial\Omega,S)). \tag{5.130}$$

Moreover, the boundary value problem (L^p-OBVP$_{S,B}$) with prescribed boundary data $(g_0,g_1)\in\dot{\Theta}_B(\mathscr{H}^p(\partial\Omega,S))$ admits a unique solution, and one may represent the solution as

$$(u,\pi)=\dot{\mathscr{C}}_\theta(\dot{f},g)\ \text{ in }\Omega, \tag{5.131}$$

where $(\dot{f},g)\in\mathscr{H}^p(\partial\Omega,S)$ is the unique pair such that $\dot{\Theta}_B^{\text{Stokes}}(\dot{f},g)=(g_0,g_1)$, and $\dot{\mathscr{C}}_\theta$ is the boundary-to-domain Cauchy operator associated with the Stokes system.

Finally, a pair (u,π) *is the solution of the boundary value problem formulated in* (L^p-*OBVP*$_{S,B}$) *with boundary data* $(g_0,g_1)\in[L^p(\partial\Omega,\sigma)]^n\oplus[L^p(\partial\Omega,\sigma)]^n$ *if and only if the pair* (u,π) *is the solution of the overdetermined boundary value problem* (*OBVP*$_{S,p}$) *with boundary data* $(\dot f,g)\in\mathscr{H}^p(\partial\Omega,S)$*, where* $\dot\Theta_B^{\text{Stokes}}(\dot f,g)=(g_0,g_1)$.

Proof. First, assume (u,π) is a solution of the boundary value problem (L^p-OBVP$_{\Delta,B}$) with boundary data $(g_0,g_1)\in L^p(\partial\Omega)\oplus L^p(\partial\Omega)$. By definition $(u,\pi)\in\mathscr{H}^p(\Omega,S)$, and Proposition 5.3.3 ensures that

$$(\text{Tr}_1^{\kappa-nt}\oplus\cdot\big|_{\partial\Omega}^{\kappa-nt})(u,\pi)\text{ exists }\sigma\text{-a.e. on }\partial\Omega,\text{ it belongs to }\mathscr{H}^p(\partial\Omega,S).\tag{5.132}$$

The conclusion is that

$$(g_0,g_1)=\dot\Theta_B^{\text{Stokes}}\Big((\text{Tr}_1^{\kappa-nt}\oplus\cdot\big|_{\partial\Omega}^{\kappa-nt})(u,\pi)\Big)\implies(g_0,g_1)\in\dot\Theta_B^{\text{Stokes}}(\mathscr{H}^p(\partial\Omega,S)).\tag{5.133}$$

Conversely, assume $(g_0,g_1)\in\dot\Theta_B^{\text{Stokes}}(\mathscr{H}^p(\partial\Omega,S))$, and consider a pair of functions such that $(\dot f,g)\in\mathscr{H}^p(\partial\Omega,S)$ and

$$(g_0,g_1)=\dot\Theta_B^{\text{Stokes}}(\dot f,g).\tag{5.134}$$

It follows from Corollary 5.3.2 that $(\dot{f}, g)$ is an admissible boundary data for the overdetermined boundary value problem $(\text{OBVP}_{S,p})$, and by Theorem 5.2.3 there exists a unique solution (u, π) for the overdetermind problem $(\text{OBVP}_{S,p})$ with boundary data $(\dot{f}, g)$, and we may write

$$(u, \pi) = \dot{\mathscr{C}}_\theta(\dot{f}, g) \quad \text{in } \Omega. \tag{5.135}$$

Together with (5.134) we conclude that (u, π) is a solution of the boundary value problem $(L^p\text{-OBVP}_{S,B})$ with boundary data (g_0, g_1), which proves the solvability result stated in the theorem.

To prove uniqueness, note that since $B \in \mathfrak{A}_{\text{inv}}^{\text{Stokes}}(n, \Omega)$, it follows from Proposition 5.3.5 that for a given pair $(g_0, g_1) \in \dot{\Theta}_B(\mathscr{H}^p(\partial\Omega, S))$ there exists a unique pair of functions $(\dot{f}, g) \in \mathscr{H}^p(\partial\Omega, S)$ such that

$$\dot{\Theta}_B^{\text{Stokes}}(\dot{f}, g) = (g_0, g_1). \tag{5.136}$$

Consequently, if (u_1, π_1) and (u_2, π_2) are solutions of $(L^p\text{-OBVP}_{S,B})$ with the same boundary data, then the function $v = u_1 - u_2$ and $q = \pi_1 - \pi_2$ is a solution of $(L^p\text{-OBVP}_{S,B})$ with zero boundary data and by Theorem 5.2.3 this implies (v, q) is zero in Ω, and this finishes the proof. □

The above result shows that under suitable conditions over the packing operator and the coefficient tensor B, the boundary value problem $(L^p\text{-OBVP}_{S,B})$ is a reformulation of the overdetermined boundary value problem $(\text{OBVP}_{S,p})$.

5.4 Cauchy Integrals associated with the Lamé System of Elastostatics and the Stokes System of Hydrostatics

This subsection is focused on an asymptotic relationship between the layer potential operators associated with the Lamé system of elastostatic and the boundary-to-domain Cauchy operators associated with the Stokes system of linear hydrostatic, as the Lamé moduli λ tends to infinity.

Recall the following relations between the canonical fundamental solutions for the Stokes system and the fundamental solution for the Lamé system in $\mathbb{R}^n$ (cf. D. Mitrea [21, Chapter 10]). To make matters more precise, given Lamé moduli $(\mu, \lambda) \in \mathbb{C}^2$ with $\mu \neq 0$ and $2\mu + \lambda \neq 0$, denote by $E^{L_{\mu,\lambda}}$ the canonical fundamental solution of the Lamé operator and by (E^S, q) the canonical fundamental solution associated with the Stokes system.

For every $r \in \{1, \ldots, n\}$ we have

$$\lim_{\lambda\to\infty} \partial_s E_{rs}^{L_{\mu,\lambda}} = \lim_{\lambda\to\infty} \partial_s E_{sr}^{L_{\mu,\lambda}} = 0 \text{ pointwise in } \mathbb{R}^n \setminus \{0\} \text{ and in } \mathcal{S}'(\mathbb{R}^n), \tag{5.137}$$

and

$$\lim_{\lambda\to\infty} \lambda\partial_s E_{rs}^{L_{\mu,\lambda}} = \lim_{\lambda\to\infty} \lambda\partial_s E_{sr}^{L_{\mu,\lambda}} = -q_r \text{ pointwise in } \mathbb{R}^n \setminus \{0\} \text{ and in } \mathcal{S}'(\mathbb{R}^n). \tag{5.138}$$

Moreover,

$$\lim_{\lambda\to\infty} E^{L_{\mu,\lambda}} = \frac{1}{\mu}E^S \text{ pointwise in } \mathbb{R}^n \setminus \{0\} \text{ and in } [\mathcal{S}'(\mathbb{R}^n)]^{n^2}, \tag{5.139}$$

and for every pair $(\mu, \lambda) \in \mathbb{C}^2$ with $\mu \neq 0$ and $2\mu + \lambda \neq 0$

$$\partial_r \partial_s E_{rs}^{L_{\mu,\lambda}} = 0 \text{ pointwise in } \mathbb{R}^n \setminus \{0\} \text{ and in } \mathcal{S}'(\mathbb{R}^n). \tag{5.140}$$

Then we have the following theorem for the single layer potential operators associated with the Lamé system and the Stokes system

Theorem 5.4.1. *Fix $n \in \mathbb{N}$, with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain with surface measure σ and $p \in (1, \infty)$ be an integrability index. Then for every $\mu \neq 0$ and every function $f \in [L^p(\partial\Omega)]^n$ there holds:*

$$\lim_{\lambda\to\infty} \mathscr{S}_{L_{\mu,\lambda}} f = \mathscr{S}_{\text{Stokes}} f \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \tag{5.141}$$

$$\lim_{\lambda\to\infty} \operatorname{div} \mathscr{S}_{L_{\mu,\lambda}} f = 0 \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \tag{5.142}$$

and

$$\lim_{\lambda\to\infty} \lambda \operatorname{div} \mathscr{S}_{L_{\mu,\lambda}} f = -\mathcal{Q} f \ \text{ in } \mathbb{R}^n \setminus \partial\Omega. \tag{5.143}$$

Proof. The equality in (5.141) follows from Lebesgue Dominated Convergence Theorem and equation (5.139). Indeed, for every $x \in \mathbb{R}^n \setminus \partial\Omega$ we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \mathscr{S}_{L_{\mu,\lambda}} f(x) &= \lim_{\lambda\to\infty} \int_{\partial\Omega} E_{rs}^{L_{\mu,\lambda}}(x-y) f_s(y) d\sigma(y) \\
&= \int_{\partial\Omega} \Big(\lim_{\lambda\to\infty} E_{rs}^{L_{\mu,\lambda}}(x-y) \Big) f_s(y) d\sigma(y) \\
&= \frac{1}{\mu} \int_{\partial\Omega} E_{rs}^{S}(x-y) f_s(y) d\sigma(y) \\
&= \frac{1}{\mu} \mathscr{S}_{\text{Stokes}} f(x). \qquad (5.144)
\end{aligned}$$

For (5.142) and (5.143), recall that for any pair $(\mu, \lambda) \in \mathbb{C}^2$ such that $\mu \neq 0$ and $2\mu + \lambda \neq 0$, we have

$$\partial_r E_{rs}^{L_{\mu,\lambda}}(x) = \frac{1}{(2\mu+\lambda)\omega_{n-1}} \frac{x_s}{|x|^n} = \frac{-1}{(2\mu+\lambda)} q_s(x) \quad \text{pointwise in } \mathbb{R}^n \setminus \{0\}, \qquad (5.145)$$

where q_s is the s-component of the canonical fundamental solution for the pressure in the Stokes system. Therefore for every $x \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned}
\text{div } \mathscr{S}_{L_{\mu,\lambda}} f(x) &= \int_{\partial\Omega} \partial_r E_{rs}^{L_{\mu,\lambda}}(x-y) f_s(y) d\sigma(y) \\
&= \int_{\partial\Omega} \frac{1}{(2\mu+\lambda)\omega_{n-1}} \frac{(x_s - y_s)}{|x-y|^n} f_s(y) d\sigma(y) \\
&= \frac{-1}{(2\mu+\lambda)} \int_{\partial\Omega} q_s(x-y) f_s(y) d\sigma(y) \\
&= \frac{-1}{(2\mu+\lambda)} \mathcal{Q} f(x). \qquad (5.146)
\end{aligned}$$

Thus the right-hand side of the last line in (5.146) implies that for every $x \in \mathbb{R}^n \setminus \{0\}$

$$\lim_{\lambda\to\infty} \operatorname{div} \mathscr{S}_{L_{\mu,\lambda}} f(x) = 0 \tag{5.147}$$

and

$$\lim_{\lambda\to\infty} \lambda \operatorname{div} \mathscr{S}_{L_{\mu,\lambda}} f(x) = -\mathcal{Q}f(x) \tag{5.148}$$

and this finishes the proof. □

A similar result holds for the asymptotic of the double layer potential operators associated with the Lamé system and the Stokes system.

Theorem 5.4.2. *Fix $n \in \mathbb{N}$, with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1,\infty)$ be an integrability index, and ν be the geometric outward unit normal to Ω. For every $(\mu,\lambda) \in \mathbb{C}^2$, with $\mu \neq 0$, let $\zeta \in \mathbb{C}$ and consider the family $A_{(\mu,\lambda,\zeta)}$ of coefficient tensors associated with the Lamé system*

$$A_{(\mu,\lambda,\zeta)} := (\mu\delta_{rs}\delta_{\alpha\beta} + (\mu+\lambda-\zeta)\delta_{r\alpha}\delta_{s\beta} + \zeta\delta_{r\beta}\delta_{s\alpha})_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq n}}. \tag{5.149}$$

Then for every function $f \in [L^p(\partial\Omega)]^n$ there holds:

$$\lim_{\lambda\to\infty} \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f = \mathscr{D}_{\zeta/\mu} f \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \tag{5.150}$$

$$\lim_{\lambda\to\infty} \operatorname{div} \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f = 0 \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \tag{5.151}$$

and

$$\lim_{\lambda\to\infty} \lambda \,\mathrm{div}\, \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f = -\mu \mathcal{P}_{\zeta/\mu} f \;\; \textit{in}\; \mathbb{R}^n \setminus \partial\Omega. \tag{5.152}$$

Proof. It follows from the definition that for every $x \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned}\mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f(x) = \Bigg(- \int_{\partial\Omega} \big\{ \mu\, \nu_r(y) [\partial_r E_{js}^{L_{\mu,\lambda}}(x-y)] + \zeta\, \nu_r(y) [\partial_s E_{jr}^{L_{\mu,\lambda}}(x-y)] \\ + (\mu+\lambda-\zeta) \nu_s(y) [\partial_r E_{jr}^{L_{\mu,\lambda}}(x-y)] \big\} f_s(y) d\sigma(y) \Bigg)_{1\le j\le n}.\end{aligned} \tag{5.153}$$

It follows from (5.139) that

$$\lim_{\lambda\to\infty} \partial_r E_{js}^{L_{\mu,\lambda}} = \frac{1}{\mu} \partial_r E_{js}^{S} \;\; \text{pointwise in } \mathbb{R}^n \setminus \{0\} \tag{5.154}$$

and (5.145) implies that

$$\lim_{\lambda\to\infty} (\mu+\lambda-\zeta) \partial_r E_{jr}^{L_{\mu,\lambda}} = -q_j \;\; \text{pointwise in } \mathbb{R}^n \setminus \{0\}. \tag{5.155}$$

Hence, by the Lebesgue Dominated Convergence Theorem, we have that for $x \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned}\lim_{\lambda\to\infty} \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f(x) = \Bigg(- \int_{\partial\Omega} \big\{ \nu_r(y) [\partial_r E_{js}^{S}(x-y)] + \frac{\zeta}{\mu} \nu_r(y) [\partial_s E_{jr}^{S}(x-y)] \\ - \nu_s(y) q_j(x-y) \big] \big\} f_s(y) d\sigma(y) \Bigg)_{1\le j\le n}\end{aligned} \tag{5.156}$$

which is equivalent to

$$\lim_{\lambda\to\infty} \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f(x) = \mathscr{D}_{\zeta/\mu} f(x). \tag{5.157}$$

To prove (5.151) and (5.152), note that for every $x \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned}
\text{div } \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f(x) &= \partial_j \left(\mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f(x) \right)_j \\
&= -\int_{\partial\Omega} \big\{ \mu\, \nu_r(y) [\partial_j \partial_r E_{js}^{L_{\mu,\lambda}}(x-y)] + \zeta\, \nu_r(y) [\partial_s \partial_j E_{jr}^{L_{\mu,\lambda}}(x-y)] \\
&\quad + (\mu+\lambda-\zeta) \nu_s(y) [\partial_j \partial_r E_{jr}^{L_{\mu,\lambda}}(x-y)] \big\} f_s(y) d\sigma(y).
\end{aligned} \tag{5.158}$$

It follows from (5.140) that

$$\partial_r \partial_s E_{rs}^{L_{\mu,\lambda}} = 0 \ \text{ pointwise in } \mathbb{R}^n \setminus \{0\}, \tag{5.159}$$

hence

$$\begin{aligned}
\text{div } \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f(x) = -\int_{\partial\Omega} \big\{ & \mu\, \nu_r(y) [\partial_j \partial_r E_{js}^{L_{\mu,\lambda}}(x-y)] \\
& + \zeta\, \nu_r(y) [\partial_s \partial_j E_{jr}^{L_{\mu,\lambda}}(x-y)] \big\} f_s(y)\, d\sigma(y).
\end{aligned} \tag{5.160}$$

Equation (5.160) together with (5.137), and the Lebesgue Dominated Convergence Theorem imply that

$$\lim_{\lambda\to\infty} \text{div } \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f(x) = 0 \ \text{ for every } x \in \mathbb{R}^n \setminus \{0\}, \tag{5.161}$$

which proves (5.151). Also from (5.138) and (5.160), and the Lebesgue Dominated Convergence Theorem we obtain that for every $x \in \mathbb{R}^n \setminus \{0\}$

$$\lim_{\lambda\to\infty} \lambda \,\mathrm{div}\, \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f(x) = \int_{\partial\Omega} \nu_r(y)\big\{\mu\,[\partial_r q_s(x-y)] + \zeta\,[\partial_s q_r(x-y)]\big\} f_s(y)\,d\sigma(y). \tag{5.162}$$

Since $\partial_s q_r = \partial_r q_s$, the right-hand side the last line of (5.162) is equal to

$$(\mu+\zeta)\int_{\partial\Omega} \nu_r(y)[\partial_r q_s(x-y)] f_s(y)\,d\sigma(y) = -\mu\mathcal{P}_{\zeta/\mu} f(x), \tag{5.163}$$

and this finishes the proof. □

As a consequence, we have the following lemma for the asymptotics of the Cauchy integral operator associated with the Lamé system.

Lemma 5.4.3. *Fix $n \in \mathbb{N}$, with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain and $p \in (1,\infty)$ be an integrability exponent. For $\mu \neq 0$, let $A(\mu,\lambda)$ be any coefficient tensor associated with the Lamé operator*

$$L_{\mu,\lambda} = \mu\Delta + (\mu+\lambda)\nabla\mathrm{div}. \tag{5.164}$$

Then for every $\dot{f} \in \mathrm{WA}_1^n[L^p(\partial\Omega,\sigma)]$ there holds

$$\lim_{\lambda\to\infty} \mathrm{div}\left[\dot{\mathscr{C}}_{A(\mu,\lambda)}\dot{f}\right] = \mathscr{D}_\Delta(f^r_{e_r}) \ \text{ in } \mathbb{R}^n \setminus \partial\Omega. \tag{5.165}$$

In particular, if $\dot{f} \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]$, *then*

$$\lim_{\lambda\to\infty} \mathrm{div}\left[\dot{\mathscr{C}}_{A(\mu,\lambda)}\dot{f}\right] = 0 \ \ \textit{in}\ \mathbb{R}^n\setminus\partial\Omega. \tag{5.166}$$

Proof. From Lemma 4.1.1 it follows that for each Lamé moduli $(\mu,\lambda)\in\mathbb{C}^2$ with $\mu\neq 0$ and $2\mu+\lambda\neq 0$ the Cauchy integral associated with the Lamé system is independent of the choice of coefficient tensor. Fix $\zeta\in\mathbb{C}$ and consider the coefficient tensor in (5.149), that is,

$$A_{(\mu,\lambda,\zeta)} := \big(\mu\delta_{rs}\delta_{\alpha\beta}+(\mu+\lambda-\zeta)\delta_{r\alpha}\delta_{s\beta}+\zeta\delta_{r\beta}\delta_{s\alpha}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq n}}. \tag{5.167}$$

Then write for every $x\in\mathbb{R}^n\setminus\partial\Omega$

$$\dot{\mathscr{C}}_A\dot{f}(x) = \mathscr{D}_{A_{(\mu,\lambda,\zeta)}}f_0(x) - \mathscr{S}_{L_{\mu,\lambda}}\big((\nu_r(\mu f^s_{e_r}+\zeta f^r_{e_s})+(\mu+\lambda-\zeta)\nu_s f^r_{e_r})_{1\leq s\leq n}\big)(x). \tag{5.168}$$

It follows from (5.142) and (5.143) in Theorem 5.4.1, equation (5.151) in Theorem 5.4.2, and $q_s=-\partial_s E_\Delta$ that for every $x\in\mathbb{R}^n\setminus\partial\Omega$

$$\begin{aligned}
\lim_{\lambda\to\infty}\mathrm{div}\left[\dot{\mathscr{C}}_{A(\mu,\lambda)}\dot{f}(x)\right] &= -\lim_{\lambda\to\infty}\lambda\,\mathrm{div}\mathscr{S}_{L_{\mu,\lambda}}\big((\nu_s f^r_{e_r})_{1\leq s\leq n}\big)(x)\\
&= \mathcal{Q}\big((\nu_s f^r_{e_r})_{1\leq s\leq n}\big)(x)\\
&= \int_{\partial\Omega}\nu_s(y)q_s(x-y)f^r_{e_r}(y)d\sigma(y)\\
&= -\int_{\partial\Omega}\nu_s(y)(\partial_s E_\Delta)(x-y)f^r_{e_r}(y)d\sigma(y)\\
&= \mathscr{D}_\Delta(f^r_{e_r})(x),
\end{aligned} \tag{5.169}$$

which proves (5.165). The result in (5.166) is now a consequence of the fact that if the array $\dot{f} \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)]$, then $f^r_{e_r}(y) = 0$ at σ-a.e. point $y \in \partial\Omega$. □

Finally, we can establish an asymptotic relationship between the Cauchy integrals associated with the Lamé system, and the one associated with the Stokes system.

Theorem 5.4.4. *Fix $n \in \mathbb{N}$, with $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, $p \in (1,\infty)$ be an integrability index, and ν be the geometric outward unit normal to Ω. For $\mu \neq 0$ and $\lambda \in \mathbb{C}$, let $A(\mu,\lambda)$ be any coefficient tensor for the Lamé system*

$$L_{\mu,\lambda} = \mu\Delta + (\mu+\lambda)\nabla\mathrm{div}. \tag{5.170}$$

Then for all $\zeta \in \mathbb{C}$ and every pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega,\sigma)] \oplus L^p(\partial\Omega,\sigma)$ there holds:

$$\lim_{\lambda\to\infty}\Big[\dot{\mathscr{C}}_{A(\mu,\lambda)}\dot{f} + \mu\mathscr{S}_{L_{\mu,\lambda}}(\nu g)\Big] = \dot{\mathscr{U}}_{\zeta/\mu}(\dot{f}, g) \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \tag{5.171}$$

$$\lim_{\lambda\to\infty}\mathrm{div}\Big[\dot{\mathscr{C}}_{A(\mu,\lambda)}\dot{f} + \mu\mathscr{S}_{L_{\mu,\lambda}}(\nu g)\Big] = 0 \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \tag{5.172}$$

and

$$\lim_{\lambda\to\infty}\lambda\,\mathrm{div}\Big[\dot{\mathscr{C}}_{A(\mu,\lambda)}\dot{f} + \mu\mathscr{S}_{L_{\mu,\lambda}}(\nu g)\Big] = -\mu\dot{\mathscr{P}}_{\zeta/\mu}(\dot{f}, g) \ \text{ in } \mathbb{R}^n \setminus \partial\Omega. \tag{5.173}$$

Proof. It follows from (5.168) that for every $x \in \mathbb{R}^n \setminus \partial\Omega$, for each $\zeta \in \mathbb{C}$ and for every $\dot{f} \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)]$ we have

$$\dot{\mathscr{C}}_{A(\mu,\lambda)}\dot{f}(x) = \mathscr{D}_{A_{(\mu,\lambda,\zeta)}} f_0(x) - \mathscr{S}_{L_{\mu,\lambda}}\big((\nu_r(\mu f^s_{e_r} + \zeta f^r_{e_s}))_{1\leq s\leq n}\big)(x) \tag{5.174}$$

It follows from (5.141) in Theorem 5.4.1 and from (5.150) in Theorem 5.4.2 that

$$\begin{aligned}\lim_{\lambda\to\infty}\Big[\dot{\mathscr{C}}_{A(\mu,\lambda)}\dot{f} + \mu\mathscr{S}_{L_{\mu,\lambda}}(\nu g)\Big] &= \mathscr{D}_{\zeta/\mu} f_0 - \mathscr{S}_{\text{Stokes}}\big((\nu_r(f^s_{e_r} + \zeta/\mu f^r_{e_s}) + \nu_s g)_{1\leq s\leq n}\big) \\ &= \dot{\mathscr{U}}_{\zeta/\mu}(\dot{f}, g) \ \text{ in } \mathbb{R}^n \setminus \partial\Omega, \end{aligned} \tag{5.175}$$

and this proves (5.171).

To prove (5.172) we use (5.142) in Theorem 5.4.1 and (5.165) in Lemma 5.4.3 to obtain the result.

The proof of (5.173) follows from (5.142) in Theorem 5.4.1 and (5.165) in Theorem 5.4.2. □

The conclusion is that for a pair $(\dot{f}, g) \in \mathrm{WA}^n_{\mathrm{div}}[L^p(\partial\Omega, \sigma)] \oplus L^p(\partial\Omega, \sigma)$, the Cauchy integrals for the Lamé system are related to the Cauchy integrals for the Stokes system as the parameter λ tends to infinity.

CHAPTER 6

FREDHOLM THEORY FOR THE DIRICHLET PROBLEM FOR Δ^3 IN INFINITESIMALLY FLAT AR DOMAINS

In this chapter, we discuss Fredholm Theory for the Dirichlet problem for Δ^3 in infinitesimally flat Ahlfors regular domains with prescribed data in Lebesgue-Whitney array spaces with $p \in (1, \infty)$.

Section 6.1 provides a short introduction to the Fredholm Theory approach for higher-order elliptic systems in non-smooth domains.

Section 6.2 presents results contained in D. Mitrea, I. Mitrea, and M. Mitrea [26], which are crucial to the developments presented here.

In section 6.3 we construct an infinity family of coefficient tensors for the polyharmonic operator Δ^3. This construction is important since it allows us to identify one coefficient tensor leading to a double multi-layer of chord-dot-normal type (at the top level). Then using machinery developed in G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17] and in D. Mitrea, I. Mitrea, and M. Mitrea [26], we establish Fredholm solvability of the Dirichlet problem in domains with flatness.

Section 6.4 contains a list of conjectures of a programmatic nature to guide future research directions within the scope of boundary value problems for higher-order elliptic systems in non-smooth domains.

All the results presented in this chapter are based on ongoing joint work with Dorina Mitrea, Irina Mitrea, and Marius Mitrea in [4].

6.1 Introduction

The layer potential method serves as a powerful tool for addressing well-posedness of boundary value problems for elliptic systems in rough domains. The essence of the boundary layer method is the ability to reduce the solvability of a boundary value problem in a given domain $\Omega \subset \mathbb{R}^n$ to the task of solving a boundary integral equation involving a singular integral operator associated with the domain and a coefficient tensor corresponding to the underlying PDE. When this singular integral operator is compact, the boundary integral equation can be treated using Fredholm Theory. While significant progress has been made in studying second-order elliptic systems through this approach, higher-order elliptic systems remain less understood.

The Fredholm Theory approach for the Dirichlet problem for higher-order elliptic operators was first treated in S. Agmon [1] for homogeneous, real constant coefficients, elliptic scalar operators of order $2m$ in bounded domains of class $C^{1+\beta}$ in $\mathbb{R}^2$, with $\beta > \frac{1}{2}$, and with continuous and Hölder continuous boundary data. Cohen and Gosselin [8] obtained well-posedness results for Δ^2 in bounded domains of class C^1 in $\mathbb{R}^2$ and boundary data in L^p-based function spaces for $p \in (1, \infty)$.

Recent advances from D. Mitrea, I. Mitrea, and M. Mitrea have established well-posedness results in bounded infinitesimally flat Ahlfors regular domains in $\mathbb{R}^2$ for homogeneous, real constant coefficients, elliptic scalar operators of order $2m$, and boundary data in L^p-based function spaces for every $p \in (1, \infty)$. D. Mitrea, I. Mitrea, and M. Mitrea have also treated the Dirichlet problem for the biharmonic operator Δ^2 in bounded infinitesimally flat Ahlfors regular domains in any dimension $\mathbb{R}^n$, and boundary data in L^p Lebesgue-Whitney array function spaces for every $p \in (1, \infty)$ for every $p \in (1, \infty)$.

The work contained in this chapter can be seen as a next step in the research program of systematical implementation of Fredholm Theory to establish well-posedness results for the Dirichlet problem for higher-order elliptic systems in non-smooth domains in $\mathbb{R}^n$.

Here, we establish the existence of a coefficient tensor for Δ^3 that gives us singular integral operators that are compact on L^p Lebesgue-Whitney array function spaces for every $p \in (1, \infty)$ whenever the underlying domain $\Omega \subset \mathbb{R}^n$ is a bounded infinitesimally flat Ahlfors regular domain. As a consequence, we obtained Fredholm solvability for the Dirichlet problem for Δ^3 in bounded infinitesimally flat Ahlfors regular domain and boundary data in L^p Lebesgue-Whitney function spaces for every $p \in (1, \infty)$. This is based on ongoing joint work with Dorina Mitrea, Irina Mitrea, and Marius Mitrea [4].

6.2 Infinitesimally Flat AR Domains and Chord-Dot-Normal Singular Integral Operators

First, we introduce the geometrical setting of infinitesimally flat Ahlfors regular (AR) domains. Following D. Mitrea, I. Mitrea, and M. Mitrea [26], fix $n \in \mathbb{N}$ with $n \geq 2$, and consider a bounded AR domain $\Omega \subset \mathbb{R}^n$ with surface measure σ. Denoting by $\Delta(x,r) := B(x,r) \bigcap \partial\Omega$ the surface ball at $x \in \partial\Omega$ with radius $r \in (0, \text{diam}(\partial\Omega))$, recall that the Sarason space $\text{VMO}(\partial\Omega, \sigma)$ is the collection of all functions $f \in \text{BMO}(\partial\Omega, \sigma)$ such that

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \\ r\in(0,R)}} \fint_{\Delta(x,r)} \fint_{\Delta(x,r)} \big|f(y) - f(z)\big| d\sigma(y) d\sigma(z) = 0. \tag{6.1}$$

Set $\nu \in [L^\infty(\partial\Omega, \sigma)]$ to be the geometric measure theoretical outward unit normal to Ω.

Definition 6.2.1. *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be an Ahlfors regular domain with compact boundary, surface measure σ, and geometric outward unit normal $\nu = (\nu_1, \ldots, \nu_n)$. Then Ω is called an infinitesimally flat AR domain provided one has $\nu \in [\text{VMO}(\partial\Omega, \sigma)]^n$.*

Remark 6.2.2. For bounded domains

$$\big\{C^1 \text{ domains}\big\} \subsetneq \big\{\text{infinitesimally flat AR domains}\big\} \subsetneq \big\{\text{UR domains}\big\}. \tag{6.2}$$

Consider the following class of singular integral operators defined on boundaries of UR domains and whose integral kernels depend linearly on the geometric measure theoretic outward unit normal of the domain in question.

Definition 6.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain with surface measure σ and outward unit normal $\nu = (\nu_1, \ldots, \nu_n)$. An integral operator T is said to be a chord-dot-normal singular integral operator if it is of the form*

$$Tf(x) := \text{p.v.} \int_{\partial\Omega} \langle \nu(y)\,, x - y \rangle\, k(x-y) f(y) d\sigma(y) \quad \text{at } \sigma\text{-a.e. point } x \in \partial\Omega,$$

with $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$ even and positive homogeneous of degree $-n$.

Below we restate the compactness result in D. Mitrea, I. Mitrea, and M. Mitrea [26, Corollary 5.2.4].

Theorem 6.2.4. *Let $\Omega \subset \mathbb{R}^n$ be an infinitesimally flat AR domain and let T be a chord-dot-normal singular integral operator. Then T is compact on $L^p(\partial\Omega, \sigma)$ for every $p \in (1, \infty)$.*

This is a key compactness result that will play an important role in the proof of the main theorem of the next section.

6.3 The Dirichlet problem for Δ^3 and Fredholm Theory

Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $m \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded AR domain with surface measure σ. For each aperture parameter $\kappa \in (0, \infty)$ and each function u defined in Ω, recall that

$$\mathrm{Tr}_{m-1}^{\kappa-nt} u := \{(\partial^\gamma u)|_{\partial\Omega}^{\kappa-nt}\}_{|\gamma|\leq m-1}, \tag{6.3}$$

assuming that the above nontangential limits exist σ-a.e. on $\partial\Omega$.

Recall that for each integrability exponent $p \in (1, \infty)$, the Dirichlet problem for Δ^m with boundary data $\dot{f} = \{f_\gamma\}_{\gamma\in\mathbb{N}_0^n:|\gamma|\leq m-1} \in \mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)]$ is given by

$$(\mathrm{Dir}_{\Delta^m,p}) \begin{cases} u \in C^{2m}(\Omega), \ \ \Delta^m u = 0 \ \text{ in } \Omega, \\ \\ \displaystyle\sum_{|\gamma|\leq m-1} \mathcal{N}_\kappa(\partial^\gamma u) \in L^p(\partial\Omega, \sigma), \\ \\ \mathrm{Tr}_{m-1}^{\kappa-nt} u = \dot{f} \in \mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)]. \end{cases} \tag{6.4}$$

For $n, m \in \mathbb{N}$, with $n \geq 2$, recall the canonical fundamental solution for Δ^m in $\mathbb{R}^n$ (cf. D. Mitrea [21, Theorem 7.28]) may be written as

$$E_{\Delta^m}(x) := \begin{cases} c_{m,n}|x|^{2m-n} \ln|x| & \text{if } n \text{ is even and } n \leq 2m, \\ \\ c_{m,n}|x|^{2m-n} & \text{if } n > 2m \text{ or } n \text{ is odd}, \end{cases} \tag{6.5}$$

where $c_{m,n} \in \mathbb{R}$ are given constants.

Recall that A is said to be a *coefficient tensor* for the polyharmonic operator Δ^m provided

$$A := \{A_{\alpha\beta}\}_{|\alpha|=|\beta|=m} \tag{6.6}$$

is a collection of complex numbers indexed by multiindices $\alpha, \beta \in \mathbb{N}_0^n$ of order m such that

$$\Delta^m = \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta} \partial^\beta. \tag{6.7}$$

We denote by $\mathfrak{A}_{\Delta^m}$ the collection of all coefficient tensors for Δ^m. For each $m \in \mathbb{N}$, there are infinitely many coefficient tensors associated with Δ^m.

The mechanism we shall consider for generating null-solutions of the polyharmonic operator Δ^m is the averaged boundary-to-domain multilayer potential introduced in G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17, Theorem 1.1], written below for convenience.

Definition 6.3.1. *Fix $n, m \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain with surface measure σ and outward unit normal $\nu = (\nu_1, \ldots, \nu_n)$. Let $A \in \mathfrak{A}_{\Delta^m}$ be a coefficient tensor for the polyharmonic operator Δ^m in $\mathbb{R}^n$, and let $p \in (1, \infty)$ be an integrability exponent. For each Lebesgue-Whitney array $\dot{f} = \{f_\gamma\}_{\gamma \in \mathbb{N}_0^n : |\gamma| \leq m-1} \in \mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)]$, define the averaged boundary-to-domain multilayer associated with A as*

$$\dot{\mathscr{D}}_A \dot{f}(x) := -\sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\gamma+\delta+e_j=\beta} \frac{\beta!}{|\beta|!} \frac{|\gamma|!}{\gamma!} \frac{|\delta|!}{\delta!} A_{\alpha\beta} \int_{\partial\Omega} \nu_j(y) \big(\partial^{\alpha+\delta} E_{\Delta^m}\big)(x-y) f_\gamma(y) d\sigma(y) \tag{6.8}$$

for all $x \in \Omega$.

It follows from G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17, Theorem 1.1] that

$$\dot{\mathscr{D}}_A \dot{f} \in C^\infty(\Omega) \ \text{ and } \ \Delta^m\big(\dot{\mathscr{D}}_A \dot{f}\big) = 0 \ \text{ in } \Omega. \tag{6.9}$$

For each $\dot{f} = \{f_\gamma\}_{\gamma\in\mathbb{N}_0^n:|\gamma|\leq m-1} \in \mathrm{WA}_{m-1}[L^p(\partial\Omega,\sigma)]$ and each multiindex $\gamma \in \mathbb{N}_0^n$ of order $\leq m-1$, recall that the averaged boundary-to-boundary multilayer associated with a coefficient tensor $A \in \mathfrak{A}_{\Delta^m}$ is the family (cf. G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17, Definition 1.4])

$$\dot{K}_A \dot{f} := \{(\dot{K}_A \dot{f})_\gamma\}_{|\gamma|\leq m-1} \tag{6.10}$$

where, for each multiindex $\gamma \in \mathbb{N}_0^n$ of length $\leq m-1$,

$$\begin{aligned}(\dot{K}_A \dot{f})_\gamma(x) := &\sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\substack{\delta+\eta+e_l=\beta\\ \theta+\omega+e_k=\gamma\\ |\theta|=|\delta|}} \frac{\beta!}{|\beta|!}\frac{\gamma!}{|\gamma|!}\frac{|\delta|!}{\delta!}\frac{|\eta|!}{\eta!}\frac{|\theta|!}{\theta!}\frac{|\omega|!}{\omega!} A_{\alpha\beta} \\ &\times \text{p.v.} \int_{\partial\Omega} \partial_{\tau_{lk}(y)} \left[\left(\partial^{\delta+\omega+\alpha} E_{\Delta^m}\right)(x-y)\right] f_{\theta+\eta}(y)\, d\sigma(y) \\ -\sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\substack{\delta+\eta+e_l=\beta\\ |\delta|\geq|\gamma|}} &\frac{\beta!}{|\beta|!}\frac{|\delta|!}{\delta!} A_{\alpha\beta}\ \text{p.v.} \int_{\partial\Omega} \nu_l(y) \left(\partial^{\delta+\alpha} E_{\Delta^m}\right)(x-y) f_{\gamma+\eta}(y)\, d\sigma(y)\end{aligned} \tag{6.11}$$

for σ-a.e. point $x \in \partial\Omega$.

A key result is the Calderón-Zygmund theory for averaged multilayer layer potentials (cf. G. Hoepfner, P. Liboni, D. Mitrea, I. Mitrea, and M. Mitrea [17, Theorem 1.5])

Theorem 6.3.2. *Fix $n, m \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain with surface measure σ. Let $A \in \mathfrak{A}_{\Delta^m}$ be a coefficient tensor for the polyharmonic operator Δ^m in $\mathbb{R}^n$, and let $p \in (1, \infty)$ be an integrability exponent. For each Whitney array $\dot{f} = \{f_\gamma\}_{\gamma \in \mathbb{N}_0^n : |\gamma| \leq m-1} \in \mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)]$ there holds*

1: $\mathcal{N}(\partial^\gamma \dot{\mathscr{D}}_A \dot{f}) \in L^p(\partial\Omega, \sigma)$, for every array $\dot{f} \in \mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)]$ and every multiindex $|\gamma| \leq m-1$.

2: For every $\dot{f} \in \mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)]$ the expression $\dot{K}_A \dot{f}(x)$ in (6.11) *is well-defined at σ-a.e. point $x \in \partial\Omega$, and*

$$\dot{K}_A : \mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)] \to \mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)] \tag{6.12}$$

is a well-defined, linear and bounded operator. Moreover, the following jump formula holds:

$$\mathrm{Tr}_{m-1}^{\kappa-nt}(\dot{\mathscr{D}}_A \dot{f}) = \Big(\frac{1}{2} I + \dot{K}_A\Big) \dot{f} \quad \sigma\text{-a.e. on } \partial\Omega. \tag{6.13}$$

From now on we focus on the case when $m = 3$. Our goal is to identify a coefficient tensor for Δ^3 such that for every Whitney array $\dot{f} = \{f_\gamma\}_{\gamma \in \mathbb{N}_0^n : |\gamma| \leq 2} \in \mathrm{WA}_2[L^p(\partial\Omega, \sigma)]$ the family $\dot{K}_A \dot{f}$ is composed of singular integral operators which are either weakly singular or the integral kernel has the chord-dot-normal structure in Definition 6.2.3. For this task one needs the following lemma.

Lemma 6.3.3. *Fix $n \in \mathbb{N}$ with $n \geq 2$, and let E_{Δ^3} be the canonical fundamental solution for Δ^3 in $\mathbb{R}^n$. Then there is a dimensional constant $c_n \in \mathbb{R}$ such that for each quadruplet $a, b, c, d \in \{1, \ldots, n\}$ there holds*

$$\begin{aligned}\partial_a\partial_b\partial_c\partial_d^2 E_{\Delta^3}(x) = \frac{c_n}{|x|^{n+4}}\Big\{ &\big(\delta_{ab}x_c + \delta_{ac}x_b + \delta_{bc}x_a + 2\delta_{ab}\delta_{cd}x_d + 2\delta_{ac}\delta_{bd}x_d \\ &+ 2\delta_{ad}\delta_{bc}x_d + 2\delta_{ad}\delta_{bd}x_c + 2\delta_{ad}\delta_{cd}x_b + 2\delta_{bd}\delta_{cd}x_a\big)|x|^4 \\ &- n\big[(1 + 2\delta_{ad} + 2\delta_{bd} + 2\delta_{cd})x_a x_b x_c + x_d^2(\delta_{ab}x_c + \delta_{ac}x_b + \delta_{bc}x_a)\big]|x|^2 \\ &+ n(n+2)x_a x_b x_c x_d^2\Big\}. \end{aligned} \tag{6.14}$$

Proof. Setting $m = 3$ in (6.5) gives us that for each $n \in \mathbb{N}$ with $n \geq 2$

$$E_{\Delta^3}(x) := \begin{cases} c_n|x|^{6-n}\ln|x| & \text{if } n \text{ is even and } n \leq 6, \\ \\ c_n|x|^{6-n} & \text{if } n > 6 \text{ or } n \text{ is odd.} \end{cases} \tag{6.15}$$

First, consider the case when $n > 6$ or n is odd. Then for each $d \in \{1, \ldots, n\}$

$$\partial_d E_{\Delta^3}(x) = c_n(6-n)x_d|x|^{4-n}, \tag{6.16}$$

which, after changing the constant, is equal to

$$\partial_d E_{\Delta^3}(x) = c_n x_d|x|^{4-n}. \tag{6.17}$$

From (6.17) we obtain

$$\partial_d^2 E_{\Delta^3}(x) = c_n\big(|x|^{4-n} + (4-n)x_d^2|x|^{2-n}\big). \tag{6.18}$$

Let $c \in \{1, \ldots, n\}$. Then by (6.18)

$$\partial_c\partial_d^2 E_{\Delta^3}(x) = c_n\big((4-n)x_c|x|^{2-n} + \delta_{cd}(4-n)2x_d|x|^{2-n} + (4-n)(2-n)x_c x_d^2|x|^{-n}\big). \tag{6.19}$$

Since $\delta_{cd}x_d = \delta_{cd}x_c$ and $(4-n)$ is a common factor, we may rebrand the dimensional constant to write

$$\partial_c\partial_d^2 E_{\Delta^3}(x) = c_n\big[(1+2\delta_{cd})x_c|x|^{2-n} + (2-n)x_c x_d^2|x|^{-n}\big]. \tag{6.20}$$

Let $b \in \{1, \ldots, n\}$. Then

$$\begin{aligned}\partial_b\partial_c\partial_d^2 E_{\Delta^3}(x) = c_n\big[&(1+2\delta_{cd})\delta_{bc}|x|^{2-n} + (2-n)(1+2\delta_{cd})x_b x_c|x|^{-n} \\ &+ (2-n)(\delta_{bc}x_d^2 + 2\delta_{bd}x_c x_d)|x|^{-n} + (2-n)(-n)x_b x_c x_d^2|x|^{-n-2}\big].\end{aligned} \tag{6.21}$$

Let $a \in \{1, \ldots, n\}$. Then

$$\begin{aligned}\partial_a\partial_b\partial_c\partial_d^2 E_{\Delta^3}(x) = (2-n)c_n\big[&(1+2\delta_{cd})\delta_{bc}x_a|x|^{-n} + (1+2\delta_{cd})(\delta_{ab}x_c + \delta_{ac}x_b)|x|^{-n} \\ &- n(1+2\delta_{cd})x_a x_b x_c|x|^{-n-2} + (2\delta_{ad}\delta_{bc}x_d + 2\delta_{ad}\delta_{bd}x_c + 2\delta_{ac}\delta_{bd}x_d)|x|^{-n} \\ &- n(\delta_{bc}x_d^2 + 2\delta_{bd}x_c x_d)x_a|x|^{-n-2} - n(\delta_{ab}x_c x_d^2 + \delta_{ac}x_b x_d^2 + 2\delta_{ad}x_b x_c x_d)|x|^{-n-2}\end{aligned}$$

$$+ n(n+2)x_a x_b x_c x_d^2 |x|^{-n-4}\big]. \tag{6.22}$$

Renaming the constant by absorbing the factor $(2-n)$, and rearranging the terms accordingly we obtain (6.14).

Finally, a direct calculation using the fundamental solution for Δ^3 in (6.15) for the cases when $n \in \{2, 4, 6\}$ gives us the same result, hence finishing the proof. □

Moving on, let us discuss the construction of an infinite family of coefficient tensors for Δ^3. Fix $n \in \mathbb{N}$ with $n \geq 2$. Then we may write

$$\Delta^3 = \sum_{1\leq r\leq n} \partial^{6e_r} + 3 \sum_{\substack{1\leq r,s\leq n \\ r\neq s}} \partial^{4e_r+2e_s} + 6 \sum_{\substack{1\leq r,s,t\leq n \\ (r,\,s,\,t) \text{ distinct}}} \partial^{2e_r+2e_s+2e_t}. \tag{6.23}$$

For each pair $(\varphi_1, \varphi_2) \in \mathbb{C}^2$, consider the coefficient tensor

$$A(\varphi_1, \varphi_2) := \{A_{\alpha\beta}\}_{|\alpha|=|\beta|=3} \tag{6.24}$$

as the collection of complex numbers satisfying the following properties: First, assume

$$A_{\alpha\beta} = A_{\beta\alpha} \quad \text{for every } \alpha, \beta \in \mathbb{N}_0^n, \, |\alpha| = |\beta| = 3. \tag{6.25}$$

Second, for every $r, s, t \in \{1, \ldots, n\}$, let

$$A_{(3e_r)(3e_r)} = 1, \tag{6.26}$$

and if the entries in (r,s) are distinct, then let

$$A_{(3e_r)(e_r+2e_s)}=-\varphi_1 \quad \text{and} \quad A_{(2e_r+e_s)(2e_r+e_s)}=3+2\varphi_1. \tag{6.27}$$

If the entries in the triplet (r,s,t) are distinct, then let

$$A_{(2e_r+e_s)(e_s+2e_t)}=-\varphi_2 \quad \text{and} \quad A_{(e_r+e_s+e_t)(e_r+e_s+e_t)}=6+6\varphi_2. \tag{6.28}$$

Otherwise set

$$A_{\alpha\beta}=0 \ \ \alpha,\beta\in\mathbb{N}_0^n, \text{ with } |\alpha|=|\beta|=3. \tag{6.29}$$

A direct computation shows that for every $(\varphi_1,\varphi_2)\in\mathbb{C}^2$

$$\Delta^3=\sum_{|\alpha|=|\beta|=3}\partial^\alpha A_{\alpha\beta}(\varphi_1,\varphi_2)\partial^\beta. \tag{6.30}$$

Among the coefficient tensors in this family, there is one which gives us the chord-dot-normal kernel structure we are looking for. This is the content of the next theorem.

Theorem 6.3.4. *Fix $n\in\mathbb{N}$ with $n\geq 2$ and let $\Omega\subset\mathbb{R}^n$ be a bounded UR domain with surface measure σ and outward unit normal $\nu=(\nu_1,\ldots,\nu_n)$. Consider the family of coefficient tensors $A(\varphi_1,\varphi_2)\in\mathfrak{A}_{\Delta^3}$ defined in* (6.25)-(6.29)*. Then the averaged boundary-to-boundary multilayer associated with $A(\varphi_1,\varphi_2)$ is a chord-dot-normal singular integral operator at top-level $|\gamma|=2$ if and only if $\varphi_1=\varphi_2=3$.*

Proof. Let $\dot{f} \in \mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$ arbitrary. For $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = 2$ we investigate the top-level $(\dot{K}_A\dot{f})_\gamma$ of the boundary-to-boundary multilayer associated with A. It follows from (6.11) that under these considerations one has

$$\begin{aligned}(\dot{K}_A\dot{f})_\gamma(x) := &\sum_{\substack{|\alpha|=3\\|\beta|=3}}\sum_{\substack{\delta+\eta+e_l=\beta\\ \theta+\omega+e_k=\gamma\\ |\theta|=|\delta|}}\frac{\beta!}{|\beta|!}\frac{\gamma!}{|\gamma|!}\frac{|\delta|!}{\delta!}\frac{|\eta|!}{\eta!}\frac{|\theta|!}{\theta!}\frac{|\omega|!}{\omega!}A_{\alpha\beta}\\ &\times \text{p.v.}\int_{\partial\Omega}\partial_{\tau_{lk}(y)}\left[\left(\partial^{\delta+\omega+\alpha}E_{\Delta^3}\right)(x-y)\right]f_{\theta+\eta}(y)d\sigma(y)\\ -\sum_{\substack{|\alpha|=3\\|\beta|=3}}&\sum_{\delta+e_l=\beta}\frac{\beta!}{|\beta|!}\frac{|\delta|!}{\delta!}A_{\alpha\beta}\ \text{p.v.}\int_{\partial\Omega}\nu_l(y)\left(\partial^{\delta+\alpha}E_{\Delta^3}\right)(x-y)f_\gamma(y)d\sigma(y).\end{aligned}\tag{6.31}$$

To prove the statement of the theorem, implement the following **strategy**.

- First, fix $\gamma \in \mathbb{N}_0^n$ such that $|\gamma| = 2$. Then rewrite (6.31) as

$$\left(\dot{K}_A\dot{f}\right)_\gamma(x) = \sum_{|\lambda|=2}T_A^{\gamma,\lambda}(f_\lambda)(x)\ \text{ for } \sigma\text{-a.e. point } x \in \partial\Omega\tag{6.32}$$

  and analyze each term in this sum.

- Second, note that for each $\lambda \in \mathbb{N}_0^n$, the kernel of $T_A^{\gamma,\lambda}$ may be written as

$$\nu_j(y)[P_{j,A}(\nabla)E_{\Delta^3}](x-y)\tag{6.33}$$

  where $P_{j,A}(\nabla)$ are 'packages' of 5th order partial derivatives that depend on the index $j \in \{1,\ldots,n\}$ and the coefficient tensor $A(\varphi_1,\varphi_2)$.

- Finally, look for parameters $(\varphi_1, \varphi_2) \in \mathbb{C}^2$ such that (6.33) is chord-dot-normal, that is,

$$\nu_j(y)[P_{j,A}(\nabla)E_{\Delta^3}](x-y) = \nu_j(y)(x_j - y_j)\frac{P_A(x-y)}{|x-y|^{n+4}}. \tag{6.34}$$

  where P_A is a homogeneous polynomial of 4th order that is independent of the index $j \in \{1, \ldots, n\}$.

Now, to implement the above strategy it is beneficial to separate it into cases since for $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = 2$, there are two possibilities:

- $\gamma = 2e_r$ for some $r \in \{1, \ldots, n\}$
- $\gamma = e_r + e_s$ for some $r, s \in \{1, \ldots, n\}$ with $r \neq s$.

Our goal is to cover all these in order of computational complexity.

Case 1: Consider the case $\gamma = 2e_r$, with $r \in \{1, \ldots, n\}$. Following equation (6.32) write

$$\left(\dot{K}_A \dot{f}\right)_{2e_r}(x) = \sum_{|\lambda|=2} T_A^{2e_r,\lambda}(f_\lambda)(x) \text{ at } \sigma\text{-a.e. point } x \in \partial\Omega \tag{6.35}$$

For this case, we may have the following possible distinct values for the multiindex λ:

$$\gamma = 2e_r \longmapsto \begin{cases} \lambda = 2e_r, \\ \lambda = e_s + e_t, \ \ (r,s,t) \text{ distinct} \\ \lambda = 2e_s, \ \ r \neq s \\ \lambda = e_r + e_s, \ \ r \neq s \end{cases}$$

Subcase 1: For $\gamma = 2e_r$, and $\lambda = 2e_r$, it follows from formulas (6.31) and (6.35) that the integral kernel of $T_A^{2e_r,2e_r}$ is given by

$$\sum_{l=1}^{n}\Bigg\{\partial_{\tau_{lr}(y)}\Big[\Big((2+\varphi_1)\partial^{3e_r+e_l}+\sum_{\substack{t=1\\ t\notin\{r\}}}^{n}\partial^{e_r+e_l+2e_t}\Big)E_{\Delta^3}(x-y)\Big] \\ -\nu_l(y)[(\partial^{e_l}\Delta^2 E_{\Delta^3})(x-y)]\Bigg\}. \tag{6.36}$$

The term in the second line of (6.36) is the integral kernel of the classical boundary-to-boundary harmonic double layer and is given by

$$-\sum_{l=1}^{n}\nu_l(y)[(\partial^{e_l}\Delta^2 E_{\Delta^3})(x-y)] = \frac{\langle \nu(y), y-x\rangle}{|x-y|^n}. \tag{6.37}$$

For the first line in the sum (6.36) we expand the tangential derivatives and write

$$-\sum_{\substack{l=1\\ l\neq r}}^{n}\nu_l(y)\Big((2+\varphi_1)\partial^{4e_r+e_l}+\sum_{\substack{t=1\\ t\notin\{r\}}}^{n}\partial^{2e_r+e_l+2e_t}\Big)E_{\Delta^3}(x-y) \\ +\nu_r(y)\Bigg\{\sum_{\substack{l=1\\ l\neq r}}\Big((2+\varphi_1)\partial^{3e_r+2e_l}+\sum_{\substack{t=1\\ t\notin\{r\}}}^{n}\partial^{e_r+2e_l+2e_t}\Big)E_{\Delta^3}(x-y)\Bigg\} \tag{6.38}$$

Let us focus on the first line of (6.38). From Lemma 6.3.3, one can verify that for $l \neq r$

$$(\partial^{4e_r+e_l}E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}}\Big[3x_l|x|^4 - 6nx_r^2x_l|x|^2 + n(n+2)x_r^4x_l\Big], \tag{6.39}$$

$$(\partial^{2e_r+3e_l}E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}}\Big[3x_l|x|^4 - n(3x_r^2+x_l^2)x_l|x|^2 + n(n+2)x_r^2x_l^3\Big], \tag{6.40}$$

and for $l \in \{1, \ldots, n\}$ with l distinct from r and t

$$(\partial^{2e_r+2e_t+e_l} E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}} \Big[x_l |x|^4 - n(x_r^2 + x_t^2) x_l |x|^2 + n(n+2) x_r^2 x_t^2 x_l \Big]. \quad (6.41)$$

Note that each of the above packages of derivatives of the fundamental solution E_{Δ^3} factors an x_l. Therefore for each $l \in \{1, \ldots, n\}$ with $l \neq r$ we have that the term with $\nu_l(y)(y_l - x_l)$ is multiplied by a homogeneous polynomial of 4th order as in (6.34), and this polynomial is given by

$$\begin{aligned} P_{(\varphi_1,\varphi_2)}(x) = {} & (2+\varphi_1)\big(3|x|^4 - 6nx_r^2|x|^2 + n(n+2)x_r^4\big) \\ & + |x|^4 + 2nx_r^2|x|^2 - n(n+2)x_r^4, \end{aligned} \quad (6.42)$$

which is independent of the component $l \in \{1, \ldots, n\}$. For the term in (6.38) which is multiplied by $\nu_r(y)$ we first use Lemma 6.3.3 to write

$$(\partial^{4e_l+e_r} E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}} \Big[3x_r |x|^4 - 6nx_l^2 x_r |x|^2 + n(n+2) x_l^4 x_r \Big], \quad (6.43)$$

$$(\partial^{3e_r+2e_l} E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}} \Big[3x_r |x|^4 - n(3x_l^2 + x_r^2) x_r |x|^2 + n(n+2) x_l^2 x_r^3 \Big], \quad (6.44)$$

and for $l \in \{1, \ldots, n\}$ with l distinct from r and t

$$(\partial^{e_r+2e_t+2e_l} E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}} \Big[x_r |x|^4 - n(x_l^2 + x_t^2) x_l |x|^2 + n(n+2) x_l^2 x_t^2 x_r \Big]. \quad (6.45)$$

Note that each of the above packages of derivatives of the fundamental solution E_{Δ^3} factors an x_r. Pushing through with the computations one obtains that the numerator of the term multiplied by $\nu_r(y)(y_r - x_r)$ is the same a 4th order homogeneous polynomial as in (6.42), hence this is a chord-dot-normal integral kernel for every $(\varphi_1, \varphi_2) \in \mathbb{C}^2$.

Subcase 2: Consider $\gamma = 2e_r$, and $\lambda = e_s + e_t$ with (r, s, t) distinct. In this case, the multiindices γ and λ do not share any common directions. It follows that this pair of multiindices implies that the terms to be considered in (6.31) and (6.35) have to be such that

$$k = r,\ \omega = e_r,\ \theta = \delta = 0 \text{ and } \eta = e_s + e_t. \tag{6.46}$$

This together with the definition of the coefficient tensor $A(\varphi_1, \varphi_2)$ allows one to conclude that the integral kernel of $T_A^{2e_r, e_s+e_t}$ is given by

$$\begin{aligned}
&\frac{\partial_{\tau_{sr}(y)}}{3}\Big[\Big(-2\varphi_1\partial^{e_r+3e_t} + (6+4\varphi_1)\partial^{e_r+2e_s+e_t} - 2\varphi_2 \sum_{\substack{l=1\\ l\notin\{s,t\}}}^{n} \partial^{e_r+e_t+2e_l}\Big)E_{\Delta^3}(x-y)\Big] \\
&+\frac{\partial_{\tau_{tr}(y)}}{3}\Big[\Big(-2\varphi_1\partial^{e_r+3e_s} + (6+4\varphi_1)\partial^{e_r+e_s+2e_t} - 2\varphi_2 \sum_{\substack{l=1\\ l\notin\{s,t\}}}^{n} \partial^{e_r+e_s+2e_l}\Big)E_{\Delta^3}(x-y)\Big] \\
&+ \sum_{\substack{l=1\\ l\notin\{r,s,t\}}}^{n} \partial_{\tau_{lr(y)}}\Big[\Big((2+2\varphi_2)\partial^{e_r+e_s+e_t+e_l}E_{\Delta^3}\Big)(x-y)\Big].
\end{aligned} \tag{6.47}$$

Expanding the tangential derivatives allows one to write it in the form (6.33). Now, focus on the term multiplied by $\nu_s(y)$, that is,

$$\frac{\nu_s(y)}{3}\Big[\big(2\varphi_1\partial^{2e_r+3e_t}-(6+4\varphi_1)\partial^{2e_r+2e_s+e_t}+2\varphi_2\sum_{\substack{l=1\\ l\notin\{s,t\}}}^{n}\partial^{2e_r+e_t+2e_l}\big)E_{\Delta^3}(x-y)\Big] \quad (6.48)$$

We need to check under which conditions over $(\varphi_1,\varphi_2)\in\mathbb{C}^2$ the terms appearing in the numerator in (6.48) are divisible by x_s-y_s.

It follows from Lemma 6.3.3 that

$$(\partial^{2e_r+3e_t}E_{\Delta^3})(x)=\frac{c_n}{|x|^{n+4}}\Big[3x_t|x|^4-n(3x_r^2+x_t^2)x_t|x|^2+n(n+2)x_r^2x_t^3\Big], \quad (6.49)$$

$$(\partial^{2e_r+2e_s+e_t}E_{\Delta^3})(x)=\frac{c_n}{|x|^{n+4}}\Big[x_t|x|^4-n(x_r^2+x_s^2)x_t|x|^2+n(n+2)x_r^2x_s^2x_t\Big], \quad (6.50)$$

$$(\partial^{4e_r+e_t}E_{\Delta^3})(x)=\frac{c_n}{|x|^{n+4}}\Big[3x_t|x|^4-6nx_r^2x_t|x|^2+n(n+2)x_r^4x_t\Big], \quad (6.51)$$

and for $l\in\{1,\dots,n\}$ with l distinct from r, s, and t

$$(\partial^{2e_r+e_t+2e_l}E_{\Delta^3})(x)=\frac{c_n}{|x|^{n+4}}\Big[x_t|x|^4-n(x_r^2+x_l^2)x_t|x|^2+n(n+2)x_r^2x_l^2x_t\Big]. \quad (6.52)$$

Thus, combining the terms in the numerator in (6.48) which are multiplied by $x_r^4x_t$ and imposing that they add to zero gives us the equation

$$0=2\varphi_1\big[3x_r^4x_t-3nx_r^4x_t\big]-\big(6+4\varphi_1\big)\big[x_r^4x_t-nx_r^4x_t\big]$$

$$+ 2\varphi_2\big[3x_r^4x_t - 6nx_r^4x_t + n(n+2)x_r^4x_t\big] + 2\varphi_2 \sum_{\substack{l=1\\ l\notin\{r,s,t\}}}^{n} \big[x_r^4x_t - nx_r^4x_t\big]$$

$$= x_r^4x_t\big\{6\varphi_1(1-n) - (6+4\varphi_1)(1-n)\big\}. \tag{6.53}$$

Note that in equation (6.53) the terms multiplied by φ_2 add to zero independently of φ_2, and we conclude that (6.53) holds if and only if

$$6\varphi_1 = 6 + 4\varphi_1 \implies \varphi_1 = 3. \tag{6.54}$$

Henceforth fix $\varphi_1 = 3$. Now, combining the terms in the numerator in (6.48) which are multiplied by x_t^5 and imposing that they add to zero gives us the equation

$$0 = 6\big[3x_t^5 - nx_t^5\big] - 18x_t^5 + 6\varphi_2 x_t^5 + 2\varphi_2 \sum_{\substack{l=1\\ l\notin\{r,s,t\}}}^{n} x_t^5$$

$$= x_t^5\big(-6n + 2\varphi_2 n\big), \tag{6.55}$$

which implies that $\varphi_2 = 3$. From here onward set $\varphi_1 = \varphi_2 = 3$.

With these values, the term multiplied by $\nu_s(y)$ in (6.48) is equal to

$$\nu_s(y)\Big[\big(-6\,\partial^{2e_r+2e_s+e_t} + 2\sum_{\substack{l=1\\ l\notin\{s\}}}^{n} \partial^{2e_r+e_t+2e_l}\big)E_{\Delta^3}(x-y)\Big] \tag{6.56}$$

$$= \nu_s(y)(x_s - y_s)\frac{8n\,c_n}{|x-y|^{n+4}}(x-y)^{e_s+e_t}\big[-(n+2)(x-y)^{2e_r} + |x-y|^2\big]. \tag{6.57}$$

A similar computation shows that the term multiplied by $\nu_t(y)$ becomes

$$\nu_t(y)(x_t - y_t)\frac{8n\,c_n}{|x-y|^{n+4}}(x-y)^{e_s+e_t}\big[-(n+2)(x-y)^{2e_r}+|x-y|^2\big], \tag{6.58}$$

and the term multiplied by $\nu_j(y)$, for $j \in \{1,\dots,n\}$ with $j \notin \{r,s,t\}$

$$\nu_j(y)(x_j - y_j)\frac{8n\,c_n}{|x-y|^{n+4}}(x-y)^{e_s+e_t}\big[-(n+2)(x-y)^{2e_r}+|x-y|^2\big]. \tag{6.59}$$

Finally, the term multiplied by $\nu_r(y)$ is given by

$$\nu_r(y)\Big[4\big(-\partial^{3e_r+e_s+e_t}+\sum_{\substack{l=1\\ l\notin\{r\}}}^{n}\partial^{e_r+e_s+e_t+2e_l}\big)E_{\Delta^3}(x-y)\Big]. \tag{6.60}$$

and by Lemma 6.3.3 once again we obtain

$$\nu_r(y)(x_r - y_r)\frac{8n\,c_n}{|x-y|^{n+4}}(x-y)^{e_s+e_t}\big[-(n+2)(x-y)^{2e_r}+|x-y|^2\big]. \tag{6.61}$$

The conclusion is that for $\varphi_1 = \varphi_2 = 3$, and for $\gamma = 2e_r$ and $\lambda = e_s + e_t$, with (r,s,t) distinct, the kernel of $T_A^{2e_r,e_s+e_t}$ is given by

$$\langle \nu(y)\,, x-y\rangle\frac{8n\,c_n}{|x-y|^{n+4}}(x-y)^{e_s+e_t}\big[-(n+2)(x-y)^{2e_r}+|x-y|^2\big], \tag{6.62}$$

hence it has the desired chord-dot-normal structure if only if $\varphi_1 = \varphi_2 = 3$.

Having fixed $\varphi_1 = 3$ and $\varphi_2 = 3$, the other cases are now reduced to verifying if this choice of coefficient tensor gives us chord-dot-normal structure.

Subcase 3 - $\gamma = 2e_r$, and $\lambda = 2e_s$: In this case the multiindices γ and λ do not share any common directions. This pair of multiindices implies that the terms to be considered in (6.31) and (6.35) have to be such that

$$k = r,\ \omega = e_r,\ \theta = \delta = 0 \text{ and } \eta = 2e_s. \tag{6.63}$$

A similar computation as the one in **Subcase 2** using Lemma 6.3.3 shows that the kernel of $T_A^{2e_r,2e_s}$ is given by

$$\langle \nu(y)\,, x-y\rangle \frac{4n\,c_n}{|x-y|^{n+4}}(x-y)^{2e_s}\big[-(n+2)(x-y)^{2e_r}+|x-y|^2\big], \tag{6.64}$$

hence it has the chord-dot-normal structure.

Subcase 4 - $\gamma = 2e_r$, and $\lambda = e_r + e_s$: Now the pair of multiindices γ and λ share a common direction, namely, e_r. This pair of multiindices implies that the terms to be considered in (6.31) and (6.35) have to be such that

$$k = r,\ \ \theta + \omega = e_r,\ \text{ and }\ \theta + \eta = e_r + e_s. \tag{6.65}$$

This makes the manipulation of the sum in (6.31) more challenging since one needs to consider the cases where $\theta = e_r$ or $\theta = 0$. Note that if $\theta = e_r$, then $|\delta| = 1$, $\omega = 0$ and $\eta = e_s$, and if $\theta = 0$, then $\delta = 0$, $\omega = e_r$ and $\eta = e_r + e_s$. The same strategy still holds, and the kernel of $T_A^{2e_r, e_r+e_s}$ is given by

$$\partial_{\tau_{sr}(y)}\Big[\Big(-3\,\partial^{4e_r} + \partial^{4e_s} - \sum_{\substack{l=1\\ l\notin\{r,s\}}}^{n}\big(\partial^{4e_l} - 4\,\partial^{2e_r+2e_l}\big) - \sum_{\substack{l=1\\ l\notin\{r,s\}}}^{n}\sum_{\substack{t=1\\ t\notin\{r,s,l\}}}^{n}\partial^{2e_l+2e_t}\Big)E_{\Delta^3}(x-y)\Big]$$

$$+\sum_{\substack{l=1\\ l\notin\{r,s\}}}^{n}\partial_{\tau_{lr}(y)}\Big[\Big(10\,\partial^{2e_r+e_s+e_l} + 2\sum_{\substack{t=1\\ t\notin\{r\}}}^{n}\partial^{e_s+e_l+2e_t}\Big)E_{\Delta^3}(x-y)\Big] \tag{6.66}$$

Expand the tangential derivatives and focus on the term multiplied by $\nu_l(y)$, for every $l \in \{1, \ldots, n\}$ with $l \notin \{r, s\}$. This term has the form

$$-\nu_l(y)\Big[\big(10\,\partial^{3e_r+e_s+e_l} + 2\sum_{\substack{t=1\\ t\notin\{r\}}}^{n}\partial^{e_r+e_s+e_l+2e_t}\big)E_{\Delta^3}(x-y)\Big] \tag{6.67}$$

It follows from Lemma 6.3.3 that

$$(\partial^{3e_r+e_s+e_l}E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}}\Big[-3nx_rx_sx_l|x|^2 + n(n+2)x_r^3x_sx_l\Big], \tag{6.68}$$

$$(\partial^{e_r+3e_s+e_l}E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}}\Big[-3nx_rx_sx_l|x|^2 + n(n+2)x_rx_s^3x_l\Big], \tag{6.69}$$

$$(\partial^{e_r+e_s+3e_l}E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}}\Big[-3nx_rx_sx_l|x|^2 + n(n+2)x_rx_sx_l^3\Big], \tag{6.70}$$

and for all $t \in \{1, \ldots, n\}$ with t distinct from r, s, l

$$(\partial^{e_r+e_s+e_l+2e_t} E_{\Delta^3})(x) = \frac{c_n}{|x|^{n+4}} \Big[- n x_r x_s x_l |x|^2 + n(n+2) x_r x_s x_l x_t^2 \Big]. \tag{6.71}$$

These equations allow us to show that for $l \in \{1, \ldots, n\}$ with $l \notin \{r, s\}$ the terms multiplied by $\nu_l(y)$ are of the form

$$\nu_l(y)(x_l - y_l) \frac{8n\, c_n}{|x-y|^{n+4}} (x-y)^{e_r+e_s} \big[- (n+2)(x-y)^{2e_r} + 4|x-y|^2 \big]. \tag{6.72}$$

Using the same strategy for the other components gives us that the integral kernel of $T_A^{2e_r, e_r+e_s}$ may be written as

$$\langle \nu(y)\,, x-y \rangle \frac{8n\, c_n}{|x-y|^{n+4}} (x-y)^{e_r+e_s} \big[- (n+2)(x-y)^{2e_r} + 4|x-y|^2 \big], \tag{6.73}$$

which has the chord-dot-normal structure.

This finishes the possible subcases associated with the multiindex $\gamma = 2e_r$.

Case 2: Consider the multiindex $\gamma = e_r + e_s$, with $r, s \in \{1, \ldots, n\}$, $r \neq s$. Following equation (6.32) write

$$\big(\dot{K}_A \dot{f}\big)_{e_r+e_s}(x) = \sum_{|\lambda|=2} T_A^{e_r+e_s,\lambda}(f_\lambda)(x) \;\text{ at } \sigma\text{-a.e. point } x \in \partial\Omega. \tag{6.74}$$

We may have the following possible distinct values for λ:

$$\gamma = e_r + e_s \longmapsto \begin{cases} \lambda = e_r + e_s, \ \ r \neq s \\ \lambda = 2e_t, \ \ (r,s,t) \text{ distinct} \\ \lambda = e_t + e_w, \ \ (r,s,t,w) \text{ distinct} \\ \lambda = 2e_s \text{ or } \lambda = 2e_r, \ \ r \neq s \\ \lambda = e_s + e_t, \text{ or } \lambda = e_r + e_t, \ \ (r,s,t) \text{ distinct} \end{cases} \tag{6.75}$$

Using Lemma 6.3.3 we may verify each of the subcases.

Subcase 1 - $\gamma = e_r + e_s$**, and** $\lambda = e_r + e_s$: A direct computation using Lemma 6.3.3 shows that the operator $T_A^{e_r+e_s, e_r+e_s}$ is given by the sum of the classical boundary-to-boundary harmonic double layer and an operator which is chord-dot-normal.

Subcase 2 - $\gamma = e_r + e_s$**, and** $\lambda = 2e_t$: In this case the multiindices γ and λ do not share any common directions, and for this pair of multiindices the terms to be considered in (6.31) and (6.35) have to be such that

$$\omega + e_k = e_r + e_s, \ \ \theta = \delta = 0, \text{ and } \eta = 2e_t. \tag{6.76}$$

The kernel of $T_A^{e_r+e_s, 2e_t}$ is given by

$$\frac{1}{2} \sum_{\omega + e_k = e_r + e_s} \sum_{\substack{j=1 \\ j \notin \{t\}}}^{n} \partial_{\tau_{jk}(y)} \Big[\Big(3\partial^{\omega + 2e_t + e_j} - \sum_{\substack{l=1 \\ l \notin \{t\}}}^{n} \partial^{\omega + e_j + 2e_l} \Big) E_{\Delta^3}(x-y) \Big]$$

$$+\frac{1}{2}\sum_{\omega+e_k=e_r+e_s}\partial_{\tau_{tk}(y)}\Big[\Big(\partial^{\omega+3e_t}-3\sum_{\substack{l=1\\ l\notin\{t\}}}^{n}\partial^{\omega+e_t+2e_l}\Big)E_{\Delta^3}(x-y)\Big] \tag{6.77}$$

Expand the tangential derivatives and focus on the term multiplied by $\nu_t(y)$. By (6.76) one has $\omega+e_k=e_r+e_s$, and thus

$$-\nu_t(y)\Big[\big(\partial^{e_r+e_s+3e_t}-3\sum_{\substack{l=1\\ l\notin\{t\}}}^{n}\partial^{e_r+e_s+e_t+2e_l}\big)E_{\Delta^3}(x-y)\Big]. \tag{6.78}$$

Using Lemma 6.3.3 one obtains that (6.78) reduces to

$$\nu_t(y)(y_t-x_t)\frac{4n(n+2)\,c_n}{|x-y|^{n+4}}(x-y)^{e_r+e_s+2e_t}. \tag{6.79}$$

Pressing forward with the computations for the other components allows one to conclude that the kernel of $T_A^{e_r+e_s,2e_t}$ is given by

$$\langle\nu(y)\,,y-x\rangle\frac{4n(n+2)\,c_n}{|x-y|^{n+4}}(x-y)^{e_r+e_s+2e_t}. \tag{6.80}$$

Subcase 3 - $\gamma=e_r+e_s$**, and** $\lambda=e_t+e_w$: In this case the multiindices γ and λ do not share any common directions, and for this pair of multiindices the terms to be considered in (6.31) and (6.35) have to be such that

$$\omega+e_k=e_r+e_s,\ \ \theta=\delta=0,\ \text{and}\ \eta=e_t+e_w. \tag{6.81}$$

The computation is similar to the one presented in the previous subcase, and the kernel of $T_A^{e_r+e_s,e_t+e_w}$ is given by

$$\langle \nu(y)\,, x-y\rangle \frac{8n(n+2)\,c_n}{|x-y|^{n+4}}(x-y)^{e_r+e_s+e_t+e_w}. \tag{6.82}$$

Subcase 4 - $\gamma = e_r + e_s$**, and** $\lambda = 2e_s$: Now, the multiidices γ and λ share common directions, hence this case is similar to **Subcase 4** presented in (6.65). The restrictions to be considered in (6.31) and (6.35) are

$$\theta + \omega + e_k = e_r + e_s \text{ and } \theta + \eta = 2e_s. \tag{6.83}$$

Note that the multiindex $\theta \in \mathbb{N}_0^n$ is allowed to be either $\theta = e_s$ or $\theta = 0$. For $\theta = e_s$, one has $k = r$, $\omega = 0$, $|\delta| = 1$ and $\eta = e_s$. On the other hand, for $\theta = 0$ we have $\omega + e_k = e_r + e_s$, $\delta = 0$ and $\eta = 2e_s$. Using Lemma 6.3.3 allows one to conclude that the kernel of $T_A^{e_r+e_s,2e_s}$ is given by

$$\langle \nu(y)\,, x-y\rangle \frac{4n\,c_n}{|x-y|^{n+4}}(x-y)^{e_r+e_s}\big[-(n+2)(x-y)^{2e_s}+3|x-y|^2\big], \tag{6.84}$$

which is chord-dot-normal.

Subcase 5 - $\gamma = e_r+e_s$**, and** $\lambda = e_s+e_t$: The multiidices γ and λ share common directions, and the restrictions to be considered in (6.31) and (6.35) are

$$\theta + \omega + e_k = e_r + e_s \text{ and } \theta + \eta = e_s + e_t. \tag{6.85}$$

Note that the multiindex $\theta \in \mathbb{N}_0^n$ is allowed to be either $\theta = e_s$ or $\theta = 0$. For $\theta = e_s$, one has $k = r$, $\omega = 0$, $|\delta| = 1$ and $\eta = e_t$. On the other hand, for $\theta = 0$ we have $\omega + e_k = e_r + e_s$, $\delta = 0$ and $\eta = e_s + e_t$. Pressing forward and using Lemma 6.3.3 allows one to conclude that the kernel of $T_A^{e_r+e_s,e_s+e_t}$ is given by

$$\langle \nu(y)\,, y - x\rangle \frac{6n\,c_n}{|x-y|^{n+4}}(x-y)^{e_r+e_t}\big[-(n+2)(x-y)^{2e_s} + |x-y|^2\big]. \tag{6.86}$$

This finishes the possible subcases associated with the multiindex $\gamma = e_r + e_s$ and proves the theorem. □

The next theorem regards how the chord-dot-normal kernel structure at top-level ensures that under appropriate geometrical considerations we may obtain compactness of the boundary-to-boundary multilayer on $\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$ for every $p \in (1,\infty)$. This compactness result then leads to the Fredholm solvability of the Dirichlet problem for Δ^3 on bounded infinitesimally flat AR domains with boundary data in $\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$ for every $p \in (1,\infty)$.

Theorem 6.3.5. *Fix $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded infinitesimally flat AR domain with surface measure σ and outward unit normal $\nu = (\nu_1, \ldots, \nu_n)$. Consider the polyharmonic operator Δ^3 and the coefficient tensor $A(3,3)$ given by*

$$A_{\alpha\beta} = A_{\beta\alpha} \quad \textit{for every } \alpha, \beta \in \mathbb{N}_0^n,\ |\alpha| = |\beta| = 3. \tag{6.87}$$

For every $r, s, t \in \{1, \ldots, n\}$*, let*

$$A_{(3e_r)(3e_r)} = 1, \tag{6.88}$$

and if the entries in (r, s) *are distinct, then let*

$$A_{(3e_r)(e_r+2e_s)} = -3 \text{ and } A_{(2e_r+e_s)(2e_r+e_s)} = 9. \tag{6.89}$$

If the entries in the triplet (r, s, t) *are distinct, then let*

$$A_{(2e_r+e_s)(e_s+2e_t)} = -3 \text{ and } A_{(e_r+e_s+e_t)(e_r+e_s+e_t)} = 24. \tag{6.90}$$

Otherwise set

$$A_{\alpha\beta} = 0, \quad \alpha, \beta \in \mathbb{N}_0^n, \text{ with } |\alpha| = |\beta| = 3. \tag{6.91}$$

Then the boundary-to-boundary multilayer associated with $A(3,3)$ is a compact operator on $\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$ for every $p\in(1,\infty)$. Consequently, for every integrability exponent $p\in(1,\infty)$, the Dirichlet problem for Δ^3 in Ω with prescribed boundary data $\dot f=\{f_\gamma\}_{\gamma\in\mathbb{N}_0^n:|\gamma|\leq 2}\in\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$

$$(\textit{Dir}_{\Delta^3,p})\begin{cases} u\in C^6(\Omega),\ \ \Delta^3 u=0\ \ \textit{in }\Omega, \\ \\ \displaystyle\sum_{|\gamma|\leq 2}\mathcal{N}_\kappa(\partial^\gamma u)\in L^p(\partial\Omega,\sigma), \\ \\ \mathrm{Tr}_2^{\kappa-nt}u=\dot f\in\mathrm{WA}_2[L^p(\partial\Omega,\sigma)], \end{cases} \tag{6.92}$$

is Fredholm solvable.

Proof. Fix an arbitrary integrability exponent $p\in(1,\infty)$. Note that for every array $\dot f\in\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$, the array $\{(\dot K_A\dot f)_\gamma\}_{|\gamma|\leq 2}$ is a Lebesgue-Whitney array, hence by the compatibility conditions CC in (2.40)

$$\partial_{\tau_{jk}}\big[(\dot K_A\dot f)_\gamma\big]:=\nu_j(\dot K_A\dot f)_{\gamma+e_k}-\nu_k(\dot K_A\dot f)_{\gamma+e_j}\ \ \text{for }|\gamma|\leq 1, \tag{6.93}$$

and for every $j,k\in\{1,\dots,n\}$. It follows from Theorem 6.3.4 that when $|\gamma|=1$, all the terms

$$(\dot K_A\dot f)_{\gamma+e_k}\ \text{ are chord-dot-normal singular integral operators.} \tag{6.94}$$

Hence by Theorem 6.2.4 they are compact operators on $L^p(\partial\Omega,\sigma)$. Since multiplication by functions in $L^\infty(\partial\Omega,\sigma)$ defines a continuous linear operator on $L^p(\partial\Omega,\sigma)$, the conclusion is that the right-hand side of (6.93) defines a compact operator on $L^p(\partial\Omega,\sigma)$, hence for $|\gamma|=1$ and for $j,k\in\{1,\ldots,n\}$, the operator on the left-hand side of (6.93) is compact from $\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$ to $L^p(\partial\Omega,\sigma)$.

A similar argument for the case $\gamma=0$ allow us to conclude that if $\Omega\subset\mathbb{R}^n$ is an infinitesimally flat AR domain, then the compactness property at the top-level of the operator $\dot{K}_A$ propagates to the arrays' lower levels. Therefore

$$\dot{K}_A:\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]\to\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]\ \text{ is compact for every } p\in(1,\infty). \tag{6.95}$$

Since $\dot{K}_A$ is a compact operator on $\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$, it follows that

$$\frac{1}{2}I+\dot{K}_A\ \text{ is a Fredholm operator of index zero for every } p\in(1,\infty). \tag{6.96}$$

In particular, the range of the operator

$$\frac{1}{2}I+\dot{K}_A:\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]\to\mathrm{WA}_2[L^p(\partial\Omega,\sigma)] \tag{6.97}$$

is a closed subspace of finite codimension in $\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$.

Now suppose $\dot{f} \in \mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$ is the boundary data for the Dirichlet problem $(\mathrm{Dir}_{\Delta^3,p})$. By Definition 6.3.1 and Theorem 6.3.2 it follows that for any $\dot{g} \in \mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$ the function $\dot{\mathscr{D}}_A\dot{g}$ is such that

$$\dot{\mathscr{D}}_A\dot{g} \in C^\infty(\Omega), \ \ \Delta^3(\dot{\mathscr{D}}_A\dot{g}) = 0 \text{ in } \Omega, \ \text{ and } \sum_{|\gamma|\leq 2} \mathcal{N}_\kappa(\partial^\gamma \dot{\mathscr{D}}_A\dot{g}) \in L^p(\partial\Omega,\sigma). \tag{6.98}$$

Moreover, it follows from the jump formula in Theorem 6.3.2 that

$$\mathrm{Tr}_2^{\kappa-nt}(\dot{\mathscr{D}}_A\dot{g}) = \Big(\frac{1}{2}I + \dot{K}_A\Big)\dot{g} \ \ \sigma\text{-a.e. on } \partial\Omega. \tag{6.99}$$

By the Fredholmness stated in (6.96), the equation

$$\Big(\frac{1}{2}I + \dot{K}_A\Big)\dot{g} = \dot{f} \tag{6.100}$$

is solvable up to finite dimensional subspaces of $\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$, and this implies the Fredholm solvability of the Dirichlet problem for Δ^3 $(\mathrm{Dir}_{\Delta^3,p})$ on bounded infinitesimally flat AR domains and $p \in (1,\infty)$. □

The above result holds in all dimensions of Euclidean space, and it represents the first application of Fredholm Theory to the triharmonic operator Δ^3 in dimensions three and higher with boundary data in L^p-based function spaces. In the next section, we outline further research directions within the same programmatic framework.

6.4 Further Directions

The following conjectures arise as consequences of the results obtained thus far and are aligned with the broader program of investigating the most general geometric setting in which boundary layer potentials can be employed to establish well-posedness results for higher-order elliptic boundary value problems. These conjectures extend the applicability of the Fredholm Theory and provide a foundation for further developments in the study of higher-order operators in L^p-based function spaces.

Fix $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded infinitesimally flat AR domain. It is conjectured that the singular integral operator associated with the Dirichlet problem for the triharmonic operator Δ^3 and coefficient tensor $A(3,3)$, namely, the operator

$$\frac{1}{2}I + \dot{K}_A \tag{6.101}$$

where $\dot{K}_A$ is the boundary-to-boundary multilayer potential operator, is an invertible operator on $\mathrm{WA}_2[L^p(\partial\Omega,\sigma)]$ for every $p \in (1,\infty)$. If this conjecture holds, then the Dirichlet problem $(\mathrm{Dir}_{\Delta^3,p})$ in Ω has a solution for all integrability exponents $p \in (1,\infty)$. Furthermore, it is conjectured that the Dirichlet problem $(\mathrm{Dir}_{\Delta^3,p})$ in Ω is well-posed for all $p \in (1,\infty)$.

It is also conjectured that for every $m \in \mathbb{N}$ the polyharmonic operator Δ^m admits a coefficient tensor which would imply the validity of the Fredholm Theory approach for the Dirichlet problem for the polyharmonic operator in Ω with prescribed boundary data in $\mathrm{WA}_{m-1}[L^p(\partial\Omega, \sigma)]$. More broadly, it is conjectured that the Dirichlet problem for Δ^m is well-posed for every $p \in (1, \infty)$.

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