CONCRETE FADING AND ITS EFFECT ON STUDENTS' ALGEBRAIC PROBLEM SOLVING & COMPUTATIONAL SKILLS

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ABSTRACT

Algebra I encompasses several topics that serve as a basis for students' subsequent mathematics courses as they progress in school. Some of the key topics that students struggle with is solving linear equations and algebraic word problems. There are several factors that may contribute to this ongoing struggle for students such as the structure of the textbooks, the teacher instruction and misconceptions of components of algebraic equations. A promising solution to the potential contributing factors is concrete fading. In this study, concreteness fading refers to an instructional technique that represents topics in a particular sequence from a concrete, real-world representation to a semi-concrete diagram (e.g., tape diagram) to an abstract representation (e.g., algebraic equations). The current study aims to investigate the influence concrete fading has on student learning while studying concrete fading in two ninth grade Algebra I general education classes at an urban high school. In particular, the study aims to answer the following: 1) What are some ways that students who received concrete fading think differently than the control group? 2) How do these differences seem to be related to the intervention? Both classes were taught by the same teacher. One class was assigned to the treatment group that received the concrete fading lessons and the other class was assigned to the control group that was taught as business as usual by the teacher. The study was intended to be quasi-experimental study, but due to challenges, it was primarily qualitative in nature focusing on eight students where the analysis included analyzing student work and student interviews responses along with quantitative analysis of the pre and two post-tests. Results revealed that the treatment group does think differently than the control group based on student work and the interview responses.

Keywords: concrete fading, instructional technique, Algebra I, tape diagram, semi-concrete diagram, word problems, algebraic equations, real-world problems, equation solving

To my dad Jonathan, my mom Jenny, Cheryl, Kristen, Jessica and Brendon as well as my grandparents: Gong Gong Po, Paw Paw Won, Grandpa Rickett, Grandma Adele and Grandma Lois. Grandpa Rickett and Grandma Adele, from one doctor to another.

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TABLE OF CONTENTS

Page

LIST OF TABLES

Table Page

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Algebra I is a gatekeeper course that prefigures student success in secondary math. However, students continue to struggle with various topics in Algebra I, especially when they are learning algebraic notation including equations (Koedinger, Alibali & Nathan, 2008; Ottmar & Landy, 2017). Results from a 2011 National Assessment of Educational Progress showed that students continue to struggle with mathematics (National Center for Education Statistics, 2011; Ottmar & Landy, 2017). Even though less than half of the students were below basic, less than half of the students scored in the proficient or advanced categories (Ottmar & Landy, 2017; National Center for Education Statistics, 2011). Students who fall below basic struggle with many topics and also have very little understanding of the basics of Algebra such as solving equations (Koedinger et al., 2008; Ottmar & Landy, 2017). In this study, students' understanding in mathematics is defined as what students know, which is referred to as the ability to 'respond' (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1997, p. 5) by applying what they learned to other mathematical situations.

Algebra I encompasses several topics such as proportions, the number systems including fractions, expressions, inequalities, algebraic equations, and functions (Bush & Karp, 2013) that serve as the foundation for students' subsequent mathematics courses. Solving algebraic linear equations and algebraic word problem solving are two of the most challenging concepts that students in this course struggle with (Koedinger & Nathan, 2004; National Mathematics Advisory Panel, 2008; Ottmar & Landy, 2017). But what is contributing to this struggle?

There are several factors contributing to this persistent problem as to why students continue to struggle with solving linear equations and algebraic problem solving. As Nathan and Koedinger (2000a, 2000b) discussed, teachers' beliefs of how students learn and the structure of textbooks can impact the way topics are taught in mathematics classes. Teachers at the secondary level, especially high school teachers, as well as the textbooks firmly support the symbolic-precedence view (Nathan & Koedinger, 2000a). Per Nathan and Koedinger (2000a, 2000b), students are typically first taught the concept in abstract symbols using equations and expressions. Then, perhaps, conclude the lesson with a verbal story problem.

A verbal story problem provides context so that the student can implement any strategy to solve the problem (Nathan & Koedinger, 2000b). However, teachers structuring a lesson as such by having students learn with symbols first does not align with the math education field's beliefs about what is helpful for students' long-term understanding (Nathan & Koedinger, 2000a, 2000b). Working in symbolic equations and expressions first can be too ambiguous and abstract for students, whereas working with verbal word problems initially will allow them to have context, see the real-world connections (Nathan & Koedinger, 2000a, 2000b). In addition, verbal problems can provide a context to help students make sense of abstract concepts.

This study explored the influence of an instructional approach referred to as "concrete fading" on students' computational skills of learning the process of solving one-variable (e.g., solving equations) and algebraic problem-solving skills (e.g., solving word problems) in two general education classes. The definition of concrete fading proposed in this study is a three-stage progression that first begins with representing the concept in a relatable, concrete context described in a word problem. Then the concept is illustrated in a semi-concrete tape diagram to support students in generating an equation, and finally the concept is represented by an abstract equation including variables and symbols. In particular, the study aimed to answer the following research questions: 1.) What are some ways that students who received concrete fading think differently than the control group? 2.) How do those differences seem to be related to the intervention?

In the next sections, I first discuss the theoretical framework in which concrete fading originated. Second, I provide a literature review in chapter 2 on concrete fading including studies that discuss word-problem solving and algebraic equations including instructional and student misconceptions, studies that support concrete fading and studies that apply the concrete fading technique. The literature has served as a base for the current study, which will be reported in Chapters 3 and 4. I then discuss my findings in Chapter 5.

CHAPTER 2

REVIEW OF LITERATURE

Theoretical Framework

I will review the organization of this notion of concrete fading, which is aligned with Bruner's three modes of representation by first discussing the idea behind going from concrete to abstract representations, the theory of concrete fading and then the connections between such representations. I will then review an alternative form of concreteness fading, which still follows concrete, semi-concrete, and abstract that is more appropriate for my study. I will (a) discuss the importance of going from concrete to abstract representations; (b) Nathan and Fyfe's (2018) model along with Bruner's three stages; (c) an alternative model supported by Theory of Realistic Mathematics Education, RME; and (d) compare both models with a table and highlight that both follow concrete, semi-concrete and abstract. I will be using this alternative model of concrete fading: concrete, semi-concrete and abstract for my study.

Importance of Sequence from Concrete to Abstract Representations

When students learn in the sequence concrete to abstract, they learn with concrete representations and can then apply what they have learned into more formal and abstract representations (Ding & Li, 2014; Freudenthal, 1991; Gravemeijer, 1994). For example, the progression of the three stages shown below in Figure 1 can support students in going from working with two apples to just working with the numeral 2. The goal of the apple example was for students to move from a physical object to working with numbers. Formal representations are more generalized and provide less concrete context for students (Barnes, 2005; Heuvel-Panhuizen & Drijvers, 2014) such as abstract representations including using numbers, operations and equations (Fyfe & Nathan, 2018). Formal representations involve students using 'mathematical language' to solve abstract equations with techniques that stemmed from their experiences in the previous stages where they grounded their understanding in a concrete context (Barnes, 2005, p. 50; Fyfe & Nathan, 2018). Being able to understand and work in formal as well as abstract representations is the objective, especially in Algebra I as this serves as the basis for every other subsequent mathematics course that students will take.

The second stage is when students will see a sketch of two apples based on the example of concrete fading discussed by Fyfe and Nathan (2018). The concrete and real-world objects are being phased out and students are slowly being introduced to the abstract. Through this stage, students are beginning to see how the number 2 can be represented without tangible objects. For example, the apples drawn on paper, in the example previously discussed, no longer provide students the actual physical object; thus, slowly becoming more abstract. In the same vein, word problems situations can be faded into linear diagrams (e.g., tape diagrams, Ding, Chen & Hassler, 2018) because tape diagrams only visualize the quantitative relationships with the distractive contextual information removed. In the final stage, still using the example from Nathan and Fyfe's study (2018), students will work in the abstract by seeing the numbers and/or equations. Now they no longer have objects; they are solely working with numerals, which is the abstract representation. If students were confused at the final stage just working strictly with numbers, then they may refer to the physical objects and/or the picture from the previous two stages to help them. The two apples and the picture of the two apples that they have seen in the first and second stages will have been stored in their storage of knowledge (Fyfe et al., 2014) that they have built as they progressed onto the final stage. In terms of the 'relativity' of concreteness among the three stages, the first stage with the two physical apples in comparison with stage 2 of the two-dimensional sketch of the apples is less concrete. By having two physical objects show what the numeral 2 is as a quantity, that actual object can help students envision that numeral. Additionally, as students progress through the three stages transitioning from concrete to abstract, the distractions and irrelevance from the concrete representations fade away (Fyfe et al., 2015; Fyfe & Nathan, 2018; Harp & Mayer, 1998). Irrelevance is described as directing students' focus away from the concept that the representations are illustrating (Fyfe & Nathan, 2018). The apple becomes irrelevant as students reach the symbolic stage because the underlying concept is the numeral 2, not the apple. Using this particular sequence of representations of concrete to abstract is what is known as concrete fading, which is supported by Bruner's (1966) modes of representation.

Modes of Representation & Concreteness Fading

Concrete fading is supported by Bruner's (1966; Nathan & Fyfe, 2018) modes of representations that consist of three progressive stages: enactive, iconic and symbolic. In fact, Nathan and Fyfe (2018) argued that concreteness fading itself is an instructional theory. The goal of the modes of representation and concrete fading is for students to first work in concrete representations and then graduate to working in the abstract and symbolic representations. When using concrete fading, even though the representations may differ in appearance, the same underlying concept is being illustrated across all representations. The present study applied a variation of concrete fading different from

Nathan and Fyfe (2018) described below, which aligns with the current studies discussed in the literature review (e.g., Ding & Li, 2014).

The modes of representation consist of three stages: enactive, iconic and symbolic. The enactive stage is concrete and hands-on where students physically embody the concept using objects such as physical manipulatives. Through this stage, students work with physical objects i.e., blocks for counting, ‘balance scales’ and ‘base-ten blocks’ (Nathan & Fyfe, 2018, p. 7) and are physically interacting with the actual objects that are representing the concept (Fyfe & Nathan, 2018). By having ‘physical interactions’ (Nathan & Fyfe, 2018) with the actual object, students ‘know how’ (Curry & Bruner, 1972) the concept works.

An example of concrete fading is provided by Nathan and Fyfe (2018) who have argued for concrete fading as a theory of instruction (see Figure 1) where students can use their prior knowledge, apply what they learned from the less idealized representations and apply it to more idealized representations (Nathan, 2012).

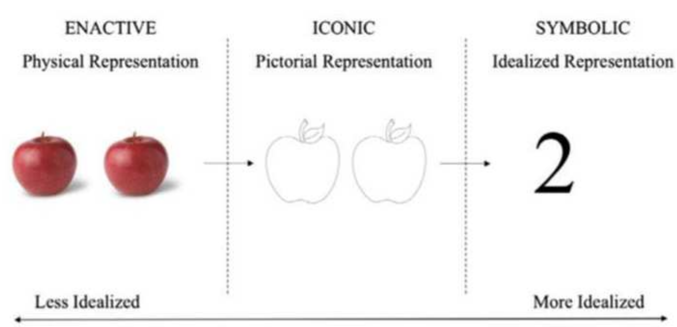


Figure 1. Bruner’s Three-stage Progression. Illustrated by Nathan & Fyfe, 2018, p. 2.

The example involves students learning the numeral ‘2’, where they first physically interact with two, three-dimensional physical apples. It is important to note that the apples that students are physically interacting with are actual apples rather than another object. Students learn how the number 2 is being used in real life, i.e., they are given two apples and see the quantity of 2 using real-life objects that they have been exposed to prior. By working with apples, students can relate to it because they may be more familiar with apples versus looking at the numeral 2. In addition, their past experiences and prior knowledge may activate and they could think of other instances they have seen collections of two objects. However, concrete representations in the first stage may come in several different forms such as 'perceptually' (e.g., using hand-drawn dollar bills that are drawn on green paper with a face and the value written on all four corners; McNeil, Uttal, Jarvin & Sternberg, 2009) or 'conceptually' (e.g., word problems; Ding & Li, 2014; Fyfe & Nathan, 2018, p. 7-9). In particular, information that is conceptual is information that students can draw from their own prior knowledge and experiences (Nathan & Fyfe, 2018). In other words, when a word problem refers to a concept, students may draw on their prior knowledge to bring up a mental image of that concept. For example, if students read a word problem that used the word 'pizza', they may think of instances where they have had pizza, which could either relate or not relate to the problem presented at all. Similarly, conceptual information may also be represented in a word problem in the form of a story and provides context unlike a word equation with no context that simply has numbers and operations (Koedinger & Nathan, 2004; Nathan & Fyfe, 2018).

Next, the iconic stage is where the concreteness is phased away and gradually moved into abstraction such as numbers, operation symbols and equations (Nathan & Fyfe, 2018). This stage involves using images such as 'graphic or pictorial models' (Nathan & Fyfe, 2018, p. 3). This is different from the perceptual, concrete representation in the first stage because even representations such as the hand-drawn bills in that stage are made so that it replicates an actual bill (Nathan & Fyfe, 2018). By having two-dimensional apples drawn on paper, students no longer physically interact with them, which makes this stage semi-concrete. The final stage is symbolic where students now work with the concept in the most abstract representation which involves equations with numbers, variables and operations. Referring back to the apple example, the two apples drawn on paper then become abstract using just numbers. The last example would just show the numeral "2". Figure 1, provided by Nathan and Fyfe (2018, p. 2) shows the progression of the stages with the example of the two apples.

Connections Among Representations

With the three modes of representations and three stages, there lies a connection among all of them that depict the underlying concept. That is, the purpose of having three stages is that through the first two stages, students are able to ground their understanding of the concept. When students ground their understanding, it suggests that they are able to visualize the concept in a concrete, real-world, meaningful context (Baranes, Perry & Stigler, 1989; Fyfe et al., 2015; Kotovsky, Hayes & Simon, 1985) and are able to apply it to more idealized and abstract representations (Fyfe & Nathan, 2018). Similarly, when students establish a foundation of the fundamentals, they are creating a 'storage of knowledge' that they may always refer to when progressing into abstract representations (Fyfe, McNeil, Son & Goldstone, 2014). However, it is important to note that students will not progress onto the subsequent stages involving higher, more advanced skills if the current stage of learning the fundamental skills is not yet mastered, which Bruner (1983) refers to as 'readiness'. If they continue to progress into the abstract representations without having fully acquired and mastered the skills needed, then the misunderstanding can no longer be addressed as students will not see the value in continuing to try to understand (Bruner, 1983).

However, through these three stages, Fyfe et al. (2014) also stress that students see one representation at a time in the beginning. After students see one at a time, then all of the representations in that particular sequence will be presented together so that students can discuss the link between the concrete and abstract representations (Fyfe et al., 2014). The summary of the instructional framework used in this current study is a variation of the definition of concrete fading described above by Nathan and Fyfe (2018).

Verbal reasoning. Connections among representations are then supported by students' abilities to verbally reason why and how they are connected. When students can justify how all of the representations are related, then that suggests that they have deepened their understanding. A good justification is when students are able to connect new and old ideas (Bruner, 2012) by relating all of the mathematical concepts they learned. It is essential for students to know that the same underlying concept is represented in all of the representations and that the concrete and abstract representations are 'mutual referents'—they are not completely different, unrelated representations from one another (Fyfe & Nathan, 2018, p. 9). When students can justify that all of the representations are related and that they are not separate entities, they are conveying how the concept is represented in multiple, different ways as well as the reason why each example was shown. If students are not made aware of the relationship across the concrete and abstract representations, they will not fully understand the topic. Hence, teachers need to ask deep questions to direct students' attention to the connections and to elicit students' explanations. Students' answers should focus on the relationship of the representations and how they connect to each other. The modes of representations along with verbal reasoning (Bruner, 1966) form the basis for what is known as concrete fading; however, this current study applies an alternative form of concrete fading that is supported by both Nathan and Fyfe (2018) and Bruner's (1966) modes of representations.

Alternative Form of Concrete Fading

The modes of representation first begin with concrete, physical objects that students can act on. However, rather than students using concrete objects in the first stage, the present study will begin with a different kind of concrete contextual representation provided by a word problem, then fade into tape diagrams and finally into abstract, symbolic equations. Using real-world, concrete contexts in this stage rather than using physical manipulatives may be more suitable for the present study as it involves high school mathematics (Nathan & Koedinger, 2000a). As Nathan and Koedinger (2000a) argued, algebra should support students in enhancing their algebraic reasoning concerning variables so that they can build on their understanding of variables and relations through several different types of representations: 'equations, tables, diagrams, and verbal relations'. Koedinger et al. (2008) have also argued that algebraic reasoning is key for students in high school because in their previous study (Nathan & Koedinger, 2004), students solved had more success while solving word problems than equations. Thus, this current study has the potential to further support students with the variation of concrete fading that is being defined based on the results of the past aforementioned studies. Therefore, the concrete context provided in this first stage would be a word problem where students are asked to find out what the problem is asking and what information they are being given. The concrete context will be a meaningful word problem that is relatable for students based on their experiences (Ding & Li, 2014) that aligns with the Theory of Realistic Mathematics Education, RME as discussed below.

Theory of Realistic Mathematics Education (RME)

In addition to Bruner's (1966) modes of representation, this study is also supported by the Theory of Realistic Mathematics Education, RME (Heuvel-Panhuizen & Drijvers, 2014), which derived from Edu Wijdeveld, Fred Goffree and Adri Teffer in 1968 (Heuvel-Panhuizen & Drijvers, 2014) in the first stage. During the enactive stage, the emphasis on 'physical objects' or 'manipulatives' is shifted to 'real-world context' in the current study. A real-world context is defined as daily activities and 'practical tasks' that are purposeful, meaningful and something that the students can connect to (Barnes, 2005, p. 43; Daniels & Anghileri, 1995, p. 23). Therefore, the concreteness may also come from the reality and its purpose of the concept provided by the meaningful problem that students can relate to (Barnes, 2005; Gravemeijer, 1994). As Gravemeijer (1994; Barnes, 2005) argued, students should learn from real-world contexts and activities as opposed to the traditional learning methods of the generalized, ambiguous mathematical notation. Just like when they are working with physical manipulatives such as the apple example when their prior knowledge is activated (Fyfe & Nathan, 2018), when students read a real-world problem from a concrete context, their prior knowledge (i.e., about what they may know about the context), situational context (i.e., how they relate to the situation that is being described through the real-world problem) as well as their ability to relate to the problem from their experiences can be activated (Koedinger & Nathan, 2004). In addition, problems that align with RME can also come from a 'fantasy world' that is not considered a real-world context, but is considered 'real' to the student (Heuvel-Panhuizen & Drijvers, 2014, p. 521). Thus, a real-world problem, fictional or mathematical, has the potential to support students in their understanding of abstract concepts because the problem can provide more concreteness to the concept even if it may not be realistic.

Because all mathematical concepts have originated from 'actual life', it is important to note that a word problem itself is not being described as concrete, rather, the real-world situations or contexts are when they are described by a word problem (Aleksandrov, Kolmogorov & Lavrent'Ev, 1999, p. 2). The real-world situations or contexts from a word problem have the potential to help students connect to the problem based on their own real-world experiences (Ding & Li, 2014). As students progress through the stages of concrete fading and then work in the abstract, they are 'removed from reality' (Mason, 1989, p. 2), i.e., they are no longer working in representations that are 'meaningful' (Mason, 1989, p. 2) to them. Thus, further supporting that word problem contexts are relatively more concrete than semi-concrete representations, such as tape diagrams and abstract equations used in this study, which is further discussed.

Below is a table that compares the concreteness fading that is supported by Bruner (1966), but described by Nathan and Fyfe (2019), Ng and Lee (2009) and the alternative model used in the current study. As shown in Table 1, after using a real-world context supported by RME (Heuvel-Panhuizen & Drijvers, 2014) in the first stage, the next stage would be solving a word problem by generating an equation that represents the situation that uses linear quantity models that are semi-concrete images to help solve as described by Ding et al. (2019). In this study, Ding et al. (2019) explored how these

Table 1

Comparison of Models

Study	Stage 1: Concrete	Stage 2: Semi-concrete	Stage 3: Abstract
Fyfe and Nathan, 2018	Real object	Sketch	Number
Ng & Lee, 2009	Text	Pictorial	Symbolic
Current study	Real-world context supported by RME (Heuvel-Panhuizen & Drijvers, 2014)	Tape diagram	Equation

models were used in Chinese and US elementary classrooms. Linear quantity models include 'pre-tapes, tape diagrams, and number line diagrams' (Ding et al., 2019, p. 105). Using such visual models are beneficial because they allow students to see all of the information in one diagram. By representing the information using a linear quantity model, students' problem-solving skills and the ability to inference can improve (Ding et al., 2019; Larkin & Simon, 1987). These types of models are less concrete than real-world, concrete contexts and more abstract (Ding et al., 2019). Specifically, the tape diagram that will represent the information from the story problem is now represented in a tape diagram with rectangles (Ding et al., 2019; Murata, 2008). Another example provided by Pashler et al. (2007) is when some teachers teach students about counting, place value, and/or the subset of real numbers including integers, rational numbers by using a number line, the concepts begin to lose the concrete contexts and gradually phase the abstract symbolism in because students are working with only numbers as opposed to the context of a word problem. It is important to note that the number line diagram has similar natures as a tape diagram where number line diagram is more abstract than a tape diagram (Ding et al., 2019). As evidenced by both examples, they only show the critical information without the distracting details provided by the contextual information that do not illustrate the underlying concept (e.g., shopping for clothes can direct students' attention away from the concept that is being illustrated when finding the total cost of the outfit) when used for problem solving. This type of progression is similar to Ng and Lee's (2009) transition of "text, pictorial and symbolic" (Ng & Lee, 2009, p. 284), which is later discussed in the literature review with the additional studies provided by Murata (2008), Ding et al. (2019) and Ding (2021).

The final stage is when students are solving an equation that uses solely abstract, symbolic expressions and equations. The purpose of concrete fading in this study is to allow students to understand the mathematical concept in abstract symbols and equations as well as being able to understand the relation when solving the problem symbolically. Students will first see the concept in the concrete form by seeing the real-world application provided by the story problem, then slowly phasing out the irrelevant details that take students' focus away from the underlying concept using semi-concrete tape diagrams and finally graduating into abstract representations involving equations, variables and operation symbols. The common feature is to fade the concreteness into relatively less concrete mode (semi-concrete) and eventually an abstract one. A more specific example is provided in the design section in Chapter 4: Methods.

Literature Review

This literature review comprises five sections. First, I discuss student difficulties of solving linear, algebraic word problems. Second, I focus on studies that discuss the importance of student misconceptions of solving linear equations. Third, I discuss studies that support the components of concrete fading including sense-making and multiple representations. Finally, I discuss studies that apply concrete fading as well as reviews of the instructional technique.

Algebraic Word Problems

Even though word problems can help students with meaning-making and make sense of the abstract concepts, prior studies reported various student struggles while generating an equation that helps represent the quantitative relationship represented in the word problem. I will first provide the classical 'students and professors' problem that students often struggle with. The student challenges presented in this problem and other word problems suggest that students struggle with using variables, simplifying expressions and representing quantitative relationships, which are key to setting up algebraic equations in order to solve word problems. I will then discuss tools that can benefit students with generating equations from word problems. Please refer to Appendix A that summarizes the major studies discussed in this paper that includes the design of the study, content of the study, the instructional activity, the level of the students, and the p-values.

Clement, Lochhead and Monk (1981) discussed the challenges that students face when solving the "Students-and-Professors" problem. Students continue to struggle with translating 'words, data tables or pictures' (Clement et al., 1981, p. 289) into an equation, which is evidenced in the following example when they are asked to write an equation that represents the situation: "There are six times as many students as professors at this university" (Clement et al., 1981, p. 288). Students often write the equation as 6S=P, where 'S' represents the number of students and 'P' represents the number of professors. The transcripts of students' interviews revealed that students applied the "word ordered matching" (Clement et al., 1981; Paige & Simon, 1966) where students take the positioning of the words and put them into an equation, i.e., 6S=P because 'students' were stated first before 'professors.' Another finding from the transcripts revealed that students drew a picture of six individual circles labeled 'S' and one circle labeled 'P' to show that there are six students for every one professor. However, students still could not provide the correct answer and still wrote: 6S=P because there were six circles labeled 'S' to one circle labeled 'P'. This may indicate that they lack an understanding of variables as well as quantitative equivalence as described in the story situation. In addition, Rosnick (1981; Fisher, 1988) revealed that students are so convinced that there are six students to one professor, that when students saw the correct equation: S=6P, they thought that 'P' represented the number of students and 'S' represented the number of professors.

Students' thinking here is related to the misconceptions discussed later in this paper by Bush and Karp (2013) and the results provided by Fyfe et al., (2017) where students were confused about the role of variables when they performed better when 'mnemonic letters' were not used (Fyfe et al., 2017, p. 96). However, a potential solution to having students be successful in this type of translating is to use concrete fading as defined in this study by using a tape diagram as opposed to the circles drawn. Students can learn how to generate an equation and understand the quantitative relationship through a semi-concrete diagram, e.g., a tape diagram. Tape diagrams, discussed later in this review can potentially benefit students in depicting as well as visualizing the quantitative relationships from a story problem (Murata, 2008). In addition, the tape diagram helps students address the important and necessary information provided by the problem in order to solve it including generating and solving an equation that represents the situation (Murata, 2008). For example, if there are six times as many students as professors, the problem states that there are more students than professors. Teachers should first draw one, short tape diagram that represents professors. Then they should ask students how to draw the tape diagram that represents the number of students in the problem and allow students the opportunity to draw the tape diagram. With the assistance of the tape diagram, students can be supported to see that they need six ‘P’ strips so that it is as large as the ‘S’ strip in order to obtain the same length as the tape that represents the number of students, S, thus, S=6P. Alternatively, the student’s tape needs to be shrunk into $\frac{1}{6}$ in order to have the same length as the professor’s tape diagram, i.e., $P = \frac{1}{6}S$. By asking students questions, it is essential as tape diagrams and other representations can create a space for discussion.

In a more recent study conducted by Ng and Lee (2009) who further argued how students can benefit from the tape diagram, examined both primary school teachers and students in primary grades learning arithmetic involving multi-step equations. The instrument used in both of these studies was 10 questions including arithmetic, story problems and problems that involved students using several concepts to solve. One example of a story problem used was: “A cow weighs 150 kg more than a dog. A goat weighs 130 kg less than the cow. Altogether, the three animals weigh 410 kg. What is the mass of the cow?” (Ng & Lee, 2009, p. 286).

The first study focused on the teachers where they answered the questions and were interviewed. The interviews revealed that teachers should draw the diagrams in a way that provides meaning (Ng & Lee, 2009). For example, if the length of rectangle A is 90 and the length of rectangle B is 45, then rectangle A should be about double of rectangle B. The second study focused on the students where they answered the same 10 questions that the teachers did, but they were not interviewed. Findings revealed students who used the tape diagram, also called Singapore model method, were very detailed and generated correct 'arithmetic equations' while drawing their diagrams (Ng & Lee, 2009, p. 310). The model method, which is very similar to the tape diagrams used in this study, helps students illustrate what is going on in the algebraic word problem and has them working with three types of representations: "text, pictorial and symbolic" (Ng & Lee, 2009, p. 284). The three types of representation are consistent with the definition of concrete fading in this study as shown in Table 1. In the 'text' phase, students read the words. In the 'pictorial' phase, students construct a model that represents what is provided in the text. In the 'symbolic phase', students try to generate a series of equations in a particular sequence to solve the problem (Ng & Lee, 2009). Additionally, the model method progresses into more abstraction as students get older, thus suggesting that the "text, pictorial and symbolic" representations (Ng & Lee, 2009, p. 284) are more appropriate for high school students. This is similar to Koedinger et al. (2008) as problem become more difficult, it is sometimes more effective for students to learn the concept abstractly before moving onto concrete. In the earlier grades, students use "objects, pictures, and symbols" then graduate to rectangles, which is similar to Bruner's (1966) model and the study conducted by Fyfe et al. (2015) with elementary level students. The rectangles represent a known value if it is an arithmetic word problem whereas in an algebraic word problem, the rectangles represent values that are unknown. Based on findings revealed by Ding et al. (2019), students in the US are taught similarly to the model method when being introduced to finding missing numbers. However, sometimes students were asked to draw a bar model, such as ‘cubes or bar models’, or generate an equation that represented the situation (Ding et al., 2019, p. 111). In addition, both studies revealed that the model method is helpful for some arithmetic word problems (Ng & Lee, 2009).

Similarly, in a later study conducted by Ding et al. (2019), US and Chinese elementary school teachers’ methods of how they used ‘linear quantity models’ that included tape diagrams in the classroom were examined (Ding et al., 2019, p. 105). The grades teachers taught in the US varied from first to fourth grade whereas the grades teachers taught in China varied from first through third grade. The topics teachers were teaching were additive and multiplicative inverses (Ding et al., 2019). Findings revealed that Chinese teachers used tape diagrams to represent the quantitative relationship embedded in the story problems whereas US teachers sometimes used tape diagrams as a method to find the computational answer. Very rarely would US teachers use the tape diagram to show the relationship among quantities (Ding et al., 2019). Another finding suggests that ‘deep questioning’ (Ding et al., 2019, p. 127) can help engage students and allow for a better understanding of the concepts and relationships. Similar to Ng and Lee (2009), deep discussion and conversation were key factors in helping students deepen their understanding. The results of this study (Ding et al., 2019) suggest the need for the current study because the tape diagrams used in this study were to support students in understanding quantitative relationships.

Additionally, tape diagrams can promote understanding of the abstract symbolic representations because moving from a diagram with minimal understanding to abstract symbols and equations can be difficult for students (Ding et al., 2019; Murata & Kattubadi, 2012). Just because students are able to 'draw' a diagram does not suggest that students can really see the structure of the diagram (Ding et al., 2019; Gravemeijer, 1999). In other words, they may only see the surface features, i.e., that tape diagram is split into a few unequal pieces. Hence, the teacher's questioning may be able to support students in directing their attention to the underlying concept, meaning, relationship and structures (Ding et al., 2019). In addition, tape diagrams are considered to be 'powerful tools' because of the potential benefit it has for students in helping them understand quantitative relationships (Ding et al., 2019, p. 105). In particular, students' abilities to inference and problem solve may also benefit from these diagrams (Ding et al., 2019; Larkin & Simon, 1983). Another support these diagrams are beneficial is because they are widely used in countries such as Singapore (Beckman, 2004; Cai & Moyer, 2008; Ding et al., 2019; Ng & Lee, 2009), Japan (Ding et al., 2019; Murata, 2008), and China (Ding et al., 2019; Ding & Li, 2014) which all have performed very highly in mathematics.

Across-study Analysis

The results of the studies conducted by Clement et al. (1981), Ng and Lee (2009) and Ding et al. (2019) suggest how students can benefit from the 'fading' stage of concrete fading with semi-concrete diagrams such as a tape diagram. Students struggled with trying to generate and solve the equation for the students-and-professor problem; however, a tape diagram, such as the one used in this study, may have helped students visualize the quantitative relationships embedded in that story problem. Not only will a tape diagram benefit students' learning, but also discussion as Murata (2008) and Ding et al. (2019) argue. Students have the potential to learn more if they are involved in conversation and describing how they would construct a tape diagram that represents the situation. Ding et al. (2019) revealed how US teachers primarily used the tape diagrams to solve the computational problems unlike Chinese teachers who used them to depict the quantitative relationships represented in the word problems. Ng and Lee (2009) argued earlier in their study as students who used the model method (i.e., tape diagrams) were more successful. However, the studies conducted by Ding et al. (2019) and Ng and Lee (2009) were for students in primary and middle grades whereas the current study was for students at the high school level. It is worthwhile for students to try as they struggle with solving equations as detailed in the following subsections. It is important to note that these studies used word problems. While word problems can help students make sense of the abstract mathematical concepts and procedures, tape diagrams can help them understand the quantitative relationships embedded in the problem, which is one of the sources of student learning difficulty with algebraic word problems as opposed to finding the computational answers.

Algebraic Equation Solving

Concrete fading can also be expected to support solving algebraic equations because the story situation and diagrams can help with making sense of those computational procedures as well. Having reviewed student difficulties with algebraic word problem solving and tools to help students, this subsection of the literature review includes studies that focus on teaching techniques that contribute to students' struggles with various topics in Algebra I. Students' computational skills and their ability to solve computational problems is key to the current study as students were asked to solve word problems and equations. I first discuss how conceptual knowledge and understanding play a role in student learning of solving equations. I then discuss possible sources that contribute to student difficulties. Finally, I discuss studies that focus on student difficulties and misconceptions of specific topics in Algebra I, in particular, solving linear equations. Please refer to Appendix A that summarizes the major studies discussed in this paper that includes the design of the study, content of the study, the instructional activity, the level of the students, and the p-values.

Conceptual Knowledge of Solving Equations

Booth and Koedinger (2008) conducted a study that offered more insight into students solving equations with high school students. This study examined students' abilities and procedures they used when solving linear equations. Students used a program, Algebra Cognitive Tutor, which used multiple representations. The multiple representations included 'tables, symbols, and graphs' (Booth & Koedinger, 2008, p. 572). The data collection involved looking at the students' 'procedural performance and learning' with the errors and solutions they provided on their pre- and post-tests (Booth & Koedinger, 2008, p. 572). The assessments tested for students' conceptual and procedural knowledge of the equal sign as well as the negative sign.

Booth and Koedinger (2008) analyzed the data by examining students' abilities to correctly solve an equation based on their conceptual knowledge of the equal sign. Results revealed that if students have deeper conceptual knowledge, then they will have a significantly higher number of correct solutions with the equal sign (Booth & Koedinger, 2008). Students with lower conceptual knowledge made significantly more errors with equality such as only changing one side of the equation by adding, subtracting, multiplying or dividing by a number or just completely ‘dropping the equals sign’ (Booth & Koedinger, 2008, p. 573).

The results of this study indicate that students struggle with correctly solving equations because they either lack a conceptual understanding or do not have the correct conceptual understanding of the equals sign, which is related to the students not having the correct techniques to solve such problems (Booth & Koedinger, 2008). With the concrete fading technique, students are supported in gaining an understanding of the fundamentals of the equals sign by using a concrete, real-world, relatable example such as the example provided above by Ding and Li (2014). Students can then build on that concrete understanding of the concept and reference it at any time as they move onto semi-concrete and abstract representations.

In another study conducted by Booth and Davenport (2013) focused on students ranging in grades 7-9 who have never taken Algebra I. This study measured: ‘feature encoding’, ‘feature knowledge’ and ‘equation solving’ (Booth & Davenport, 2013, p. 418). The encoding of the features involved students being shown a problem on the board for a certain amount of time before it disappeared from the board. They then had to write down the equations they saw from memory. There was a total of six equations that consisted of at most 10 parts including the operation signs, equal signs and variables (Booth & Davenport, 2013). Feature knowledge involved students answering questions that assessed what they knew about various topics such as the equals sign and variables (Booth & Davenport, 2013). Equation solving had students solving one- or two-step linear equations. Findings revealed that there was a significant difference in the encoding errors with the negative sign, equals sign and variables. Over half of the students solved the equations correctly. In addition, there was a significant difference for students' feature knowledge involving variables (Booth & Davenport, 2013).

The results of this study suggest the need for the current study because of how mathematical information is presented to students and their conceptual knowledge. Conceptual knowledge is defined as students' abilities to understand the underlying concept and how it connects to other concepts within one subject (Booth & Davenport, 2013; Hiebert & Wearne, 1996). Students may not understand or 'see' (Booth & Davenport, 2013, p. 422) the most important components of the problem when being shown the information. The conceptual knowledge of students learning solving equations is most pertinent, however (Booth & Davenport, 2013). If students' conceptual knowledge of this topic expands and the way information is presented to them, then their understanding can only grow which may be related to using the tape diagram because it has potential to help students visualize the quantitative relationships and problem structures, which is what the current study assessed.

Teacher Instruction

Not only is it important how students are presented information, but teacher instruction is very important as well when it comes to students' understanding of a concept. Nathan and Koedinger (2000a) conducted a study that speaks to possible sources of difficulties such as teacher beliefs when teaching mathematics as stated in the previous sections that continue to be problematic today. Teachers at the three levels: elementary, middle and high school were given a set of problems to rank from what they thought would be the easiest to hardest for students to solve. Middle school teachers ranked in the reverse order of the elementary and high school teachers who ranked symbolic equations as the simplest for students to solve while ranking the verbal word problems the hardest to solve. Similarly, there was a significant difference for high school teachers who strongly believe that teaching with abstract symbols are necessary skills to have prior to learning verbal story problems (Nathan & Koedinger, 2000a). The results of this study further demonstrate that high school teachers believe and continue to teach in a symbolic-precedence view even though students perform better on verbal word problems (Nathan & Koedinger, 2000b). A symbolic-precedence view is defined as students who work with symbolic equations first and then phase into word problems whereas a verbal-precedence view is where students work with word problems and then phase into symbolic equations and expressions (Nathan & Koedinger, 2000a).

The results of this study could be attributed to how verbal word problems provide context and the 'familiarity property' where students can activate their prior knowledge to help them relate as well as to make sense of the problem (Koedinger & Nathan, 2004; Koedinger et al., 2008). However, teaching in a symbolic-precedence view does not align with the field's beliefs about what is beneficial for students as evidenced in a similar study conducted by Nathan and Koedinger (2000b). In this study (Nathan & Koedinger, 2000b), a group of secondary teachers was asked to rank a series of problems from their school district textbooks on what they thought students would have the least difficulty to the most difficulty solving. The researchers ranked a set of six problems in the reverse order. The problems were either in the form start-unknown or result-unknown and set up in one of the three ways: word equation, word problem and symbolic equation (Nathan & Koedinger, 2000b). The data analysis involved a two-way ANOVA between the problem type (start-unknown and result-unknown) and the setup of the problem (word equation, word problem, and symbolic equation).

Results revealed that teachers ranked symbolic equations in either problem type as easiest for students to solve while ranking the word equation and word problem as most difficult to solve (Nathan & Koedinger, 2000b). Contrary to teachers' beliefs, students' performance on verbal word problems was higher than their performance on symbolic equations based on Nathan and Koedinger's (1998) study where students were significantly more likely to correctly solve problems in the form of a story problem or word equation as opposed to an equation (Nathan & Koedinger, 2000a, 2000b). The results of the performance are further supported by later studies conducted by Nathan and Koedinger (2004) and Koedinger et al. (2008) where verbal word problems provide context and allow students to relate to the story that can activate prior knowledge for simpler algebraic problems. The findings of these two studies (Nathan & Koedinger, 2000a, 2000b) also suggest that students' verbal skills are developed before their symbolic skills even though the structure of textbooks and the beliefs of teachers support the reverse of the development of students' skills. However, it is important to note these results will not always hold true for every topic in mathematics. The study conducted by Koedinger et al. (2008) discussed later in this paper, may indicate a better support of students' learning more challenging algebraic story problems by teaching with equations first and then ending with verbal problems.

These studies (Nathan & Koedinger, 2000a, 2000b) focused on teacher thinking and did not try to apply or test concrete fading, but they do align with concrete fading because they go from concrete representations to abstract. However, the intermediate stage differs from the current study, i.e., the ‘fading’ stage that could be a semi-concrete diagram such as a tape diagram (Murata, 2008), which is explored in the current study. Hence, concrete fading, as defined in this study, can offer a potential solution for students based on the results of these two studies (Nathan & Koedinger, 2000a, 2000b) because they will first learn with a real-world, concrete context represented in a story problem rather than a symbolic equation. Students are further supported in their ability to visualize the math concept in something that they are familiar with rather than in abstract equations. Then, once they are able to visualize the concept, they can then apply that to a tape diagram that can allow students to see the quantitative relationship embedded in the story problem. Finally, students will progress onto working in abstract equations. This sequence has the potential to lead to more success in solving symbolic equations for students.

Student Misconceptions

Teacher instruction may also play a part in student misconceptions based on studies Bush and Karp (2013) and Fyfe et al. (2017). Bush and Karp (2013) highlight the most common misconceptions in Algebra for students in middle school that could be factors for students who struggle with solving equations. For the purpose of this study, the review of this article will focus solely on the misconceptions students have regarding solving algebraic equations, which is also applicable to high school students. The first and most common misconception students have when solving equations is the purpose of the equals sign. While the role of the equals sign is to compare two quantities, students treat it as an operator meaning that they define the equals sign as ‘the answer is’ (Baroudi, 2006; Bush & Karp, 2013, p. 620). For example, students would solve

3*(22+34) as: 22+34=56*3=168 (Bush & Karp, 2013). The first part is correct, 22+34=56; however, when multiplying by 3 after 56, no longer makes the equation true. When students treat as such, they assume that the equals sign is supposed to show the answer on the other side, which is problematic as students take more advanced math courses in their academic career.

Understanding the role of the equals sign is one of the basic fundamentals needed for solving algebraic equations and should be addressed before moving on to more abstract variables (Bush & Karp, 2013). Equality is a symbol for relation, which shows that both sides of the equation are of the same 'total value' (Bush & Karp, 2013, p. 620) and should be treated as 'the same as' (Bush & Karp, 2013, p. 621; Stacey & McGregor, 1997) rather than the 'answer is' (Baroudi, 2006; Bush & Karp, 2013, p. 620). If students continue with this misconception, then they may become confused on how they arrived at that answer when they are substituting a value in for a variable or that the equals sign is just shorthand with no significance.

The second misconception is simplifying algebraic expressions. For example, students may think that 10x-3=7x (Bush & Karp, 2013; Kieran 1992). Students who simplify as such may indicate a lack of conceptual understanding of the meaning of the variable and also may not understand how the answer can be an expression, i.e., 10-3x, the answer would be 10-3x rather than a number (Booth 1986; Booth & Watson, 1990; Bush & Karp, 2013; Kilpatrick, Swafford & Findell, 2001; Stacey & MacGregor, 1997). If students understand how to correctly simplify, then they would know that 10 and 3x are not like-terms because 10 does not have an x, or, that 3 has an x, so the answer would be 10-3x. Also, students need to understand that 3x is multiplication: 3*x. These misconceptions could affect their answer. For example, if they have to substitute that incorrect answer into another variable their whole answer is completely off. Likewise, when they have gone into more advanced equations where they have to combine like-terms, they may simplify incorrectly because that is what they did in their earlier math courses.

The third misconception is the notation and symbolism used in Algebra (Bush & Karp, 2013). Teachers do not realize that students struggle with notation such as variables and assume that they understand the reason and role of the variables despite students' tests where their struggles with variables were evident (Asquith, Stephens, Knuth & Alibali, 2007; Bush & Karp, 2013). The reason for this struggle is because students have many misconceptions when it comes to variables because they can take on many roles such as 'labels, constants, unknowns, generalized numbers, varying quantities, parameters and abstract symbols' (Bush & Karp, 2013, p. 622; Philipp, 1992).

With the various roles, comes some of these misconceptions for students that include: students see variables as 'labels' (Ascquith et al., 2007; Bush & Karp, 2013, p. 622; Clement, 1982; Stacey & MacGregor, 1997; Usiskin, 1988); students assume that multiple variables cannot represent identical values (Bush & Karp, 2013; Stephens, 2005; Swan, 2000); a value for a variable is related to where it is in the alphabet (i.e., if there are two variables 'a' and 'b', the value of 'a' should be lower than the value of 'b' because 'a' precedes 'b' in the alphabet); Asquith et al., 2007; Bush & Karp, 2013; Herscovics & Kieran, 1980; MacGregor & Stacey, 1997); and how variables represent a quantity that varies versus a missing value (Asquith et al., 2007; Bush & Karp, 2013; Stacy & MacGregor, 2000; Stephens, 2005; Usiskin, 1998). If students are unsure of the purpose of the variable, they could just switch the value of the variables thinking that x=9, a=1 rather than x=1, a=9. Similarly, the misconception could cause confusion and lack of success in solving symbolic equations as evidenced by Nathan & Koedinger (2000b) where students performed worse on symbolic equations than story problems and word equations. However, as students get older, their ability to work with variables improves as shown in Koedinger et al. (2008) where students benefited from abstract to concrete representations when learning more advanced concepts in math (Bush & Karp, 2013; MacGregor & Stacey, 1997).

In addition to misconceptions, Bush and Karp (2013) discuss an area of difficulty that is important to the current study. The difficulty is when students go from solving one-step and two-step equations to solving equations that have variables on both sides, it involves not only using arithmetic, but also applying various algebraic properties of equality and the distributive property (Bush & Karp, 2013: Herscovics & Linchevski, 1994; Kieran, 1980). Students may continue to struggle because they may not yet understand the basics of solving equations. If students do not understand the meaning of equivalence (Bush & Karp, 2013) or the meaning of variables, then to apply more algebraic properties with variables on both sides of the equation may cause more confusion for students. If students continue with this misconception, they will struggle with solving more abstract and advanced equations. This is why concreteness fading can be expected to provide meaning for these abstract symbols and support the computation process.

Concrete fading is a potential solution to aid students who are experiencing challenges described by Bush and Karp (2013). Even though students are being taught with some degree of concrete fading by being introduced to a variable, solving one-step equations, then solving two-step equations to finally solving an equation with variables on both sides with each example becoming more abstract, the definition this study is proposing is different. The definition of concrete fading that is defined in this study is that students begin with a meaningful, concrete context in the form of a story problem. Then students learn the concept with a semi-concrete diagram. Finally, students learn the concept in an abstract representation such as an equation. This sequence will enable students to build a 'store of images' (Bruner, 1966; Fyfe & McNeil, 2012, p. 446) that students may refer to if they are confused in the later stages.

A concrete context can also allow students to better simplify expressions such as 25x+5 so they can see how 25x and 5 are different terms, and not be combined to 30x. A real-world, concrete example of how these two quantities cannot be combined can help students see how these two terms are not alike, e.g., $25 per sweater and a $5 flat fee for shipping cannot be combined because one represents a sweater and one represents the shipping fee. In addition, a concrete context can help students understand the role of variables because students can see how the variable could represent an unknown, a quantity that varies, labels or constant (Bush & Karp, 2013, p. 622; Philipp, 1992) is used in the real world. From there, teachers will phase out the concrete context by using semi-concrete diagrams such as tape diagrams (Murata, 2008) and phase them into abstract symbols. As students understand how these concepts are used in the real world and are given concrete examples, they can then apply it to tape diagrams and to abstract equations.

Many studies have focused on the misconceptions discussed by Bush and Karp (2013). Fyfe et al. (2017) conducted a study with middle school students that focused on the misconception of the equal sign. The focus of this study (Fyfe et al., 2017) was to investigate if there was a connection between the 'construct map of knowledge' and with what students knew of math equivalence; if there was a difference between the level of classes, specifically pre-algebra and algebra students; and if there was a relationship between students' abilities to formally reason when working with algebraic expressions and equality (Fyfe et al., 2017, p. 89). The data collection included two assessments that students completed. The assessment had two different forms with slight differences. Form 1 used letters 'c' and 'b' to represent the price of a cake and brownie, respectively, whereas form 2 used letters 'x' and 'y' to represent the cake and brownie, respectively (Fyfe et al., 2017). The questions on the assessments were of three different formats: 1) solve an equation using fill-in-the-blanks; 2) questions that assessed students' ability to successfully indicate if the left side was equal to the right side regardless of how the equation was written; and 3) questions written in the abstract to assess if students knew how to define the equals sign symbol.

Results revealed that even though both classes performed similarly on most problems, students in Algebra performed better on some problems than the students in pre-Algebra. Algebra students were better able to identify the properties of equality, i.e., when adding the same number to both sides to an equation, the two sides are still equivalent. In addition, Algebra students were better able to explain the role of the equals sign when given '1 quarter = 25 pennies' (Fyfe et al., 2017, p. 96). These students responded that 1 quarter is 'the same as' 25 pennies whereas pre-algebra students said 1 quarter is ‘equal to’ 25 pennies (Fyfe et al, 2017, p. 96). Another finding of this study revealed that what students know about equivalence as well as variables is significantly associated with students’ conceptual understanding of algebraic expression (Fyfe et al., 2017). In other words, students were confused about what the variables represented. If the variable chosen was to represent the quantity rather than using ‘x’ and ‘y’, e.g., using ‘a’ for apples rather than using ‘x’ for apples, students struggled understanding the expressions (Fyfe et al., 2017). This finding is similar to Clement et al. (1981) where students were confused about what the variables represented because they were so convinced that ‘P’ represented the number of students and ‘S’ represented the number of professors after seeing the correct equation that represented that word problem.

Concrete fading has the potential to address these student struggles because this technique can help students build a grounded understanding of solving equations and the role of the equal sign. When they can visualize equality in a concrete way, teachers can then use the concrete context to help students make sense of the quantitative relationship presented in the story problem through a tape diagram and, then, make meaning of the abstract symbols and equations. If students are not making sense of the equality sign in that these two quantities are the same rather than one side being the answer, students will not be able to correctly identify if two quantities are equivalent.

Across-study Analysis

Students struggle with the notion of equality and have a misconception that the equals sign is supposed to show the answer or ‘perform an operation’ (Bush & Karp, 2013). There are several studies that demonstrate this at the secondary level while solving linear equations. Fyfe et al., (2017) show how pre-algebra students struggled with the role of the variable and what it means for two quantities to be equal. Booth and Koedinger (2008) discuss how students' lack of conceptual understanding of the equals sign can affect students' errors and their ability to correctly solve a linear equation. Similarly, Booth and Davenport (2013) revealed how information is presented to students may possibly affect their learning. Another factor that can contribute to students' struggles of solving equations is the instruction students receive. Teachers continue to follow the symbolic-precedence view teaching students in symbols first whereas a verbal-precedence view may be more beneficial since it begins with a real-world problem that could provide context as well as activate prior knowledge (Nathan & Koedinger, 2000a, 2000b, 2004; Koedinger et al., 2008). Concrete fading, as defined in this study, has the potential to address all the issues as it enables students to build the foundation based on their personal experience as well as prior knowledge, to understand the quantitative relationships represented in story problems and to understand how to solve abstract equations. As the material becomes confusing, they can use the previous stages as references. If they are still confused, they can then ask the teacher as opposed to asking the teacher immediately.

Sense-making & Multiple Representations

Sense making and multiple representations also possess key components of the definition of concrete fading that this study applied that students can benefit from. Though these studies in this subsection do not necessarily apply concrete fading, they do support the components of the instructional technique. The current study defines concrete fading begins with a representation that is a real-world, concrete context described by a story problem that then fades into a semi-concrete tape diagram representation in which students use to generate a mathematical equation that represents that situation and then conclude by working with equations that include numbers and variables. This definition was shown earlier in Table 1. The goal is for students to accept the quantitative relationships illustrated in the word problems, which is described by Ding (2016) and Ding and Li (2014). Additionally, concrete fading focuses on students' abilities to make sense of mathematical concepts (Ding, 2016; Ding & Li, 2014; Murata, 2008) as well as the students' exposure to making connections between concrete (e.g., familiar, meaningful representations that students can relate to; Koedinger et al., 2008) to abstract representations (i.e., symbolic equations; Koedinger & Nathan, 2004). However, the sequence of concrete to abstract representations may not always align with the field for what is beneficial for students, especially for more challenging problems as discussed by Koedinger et al. (2008). Hence, the use of word problems does not mean that they are easy problems to solve, rather, the quantitative relationships embedded in the context ought to be understood by students in order to solve them. In Koedinger et al. (2008), one possible reason that story problems challenge students may be due to their lack of 'sense-making' of the embedded quantitative relationships, but that is not to say that this would be a situation where sensible reasoning is not plausible. This calls for instructional support for these students. Below I review various studies focusing on sense-making and multiple representations. Appendix A provides a brief summary of the major studies discussed in this paper that includes the design of the study, content of the study, the instructional activity, the level of the students, and the p-values.

Ding (2016) conducted a study that focuses on sense-making and the structure of US and Chinese textbooks when teaching inverse relationships. Inverse relationships are one of the fundamental, key, basic concepts of learning mathematics (Ding, 2016). The purpose of this study was to investigate and compare US and Chinese textbooks. In addition, the study focused on how the textbooks are structured for instructional purposes. These purposes refer to the methods that are implemented when teaching students and the effect it has on their learning (Ding, 2016; Lizzio, Wilson, & Simmons, 2002). There were two US textbooks used: *Houghton Mifflin* and *Everyday Mathematics* for students in kindergarten through sixth grade. These textbooks were compared to the Chinese textbook: *Jiang Su Educational Press textbook* for students in the first through sixth grades (Ding, 2016; Greenes et al., 2005; EM; HM; JSEP, Su & Wang 2005; University of Chicago School Mathematics Project, 2007). This study focused on the role and effect of worked examples in supporting students' learning about inverse relationships (i.e., operations that are 'opposite' of each other such as addition and subtraction or multiplication and division) and how many concrete and abstract representations are used in the text to teach inverse relationships (Ding, 2016).

Ding (2016) investigated the number of times the inverse relationship was in the textbook. Every instance of the inverse relation that was present in any of the textbooks was categorized and then identified as either a concrete or an abstract representation (Ding, 2016). Concrete representations included using 'physical objects' such as manipulatives to represent the mathematical concept or the concept represented in a 'real-world situation' (Ding, 2016, p. 48; Ding & Li, 2014). Abstract representations included using symbols to represent concepts in mathematics, which takes away the concreteness of the concept as well as unnecessary distractions (Ding, 2016; Ding & Li, 2014; Fyfe & McNeil, 2012).

Results showed that the Chinese textbooks aligned with the concrete fading technique, supported more meaningful connections and focused on sense-making as opposed to the US textbooks (Ding, 2016). The textbooks used in the US consisted of many problems that were of the same format and did not vary in presentation (Ding, 2016). The exercises presented in the textbooks also did not have many concrete problems, but there were several abstract problems such as symbolic equations and expressions. This observation is consistent with teachers and the way they teach with the symbolic-precedence view, which goes from abstract to concrete representations (Nathan & Koedinger, 2000a, 2000b). This view was discussed earlier. Since the structure of the textbooks begins the lessons with abstract equations and expressions, teachers think that students benefit more from learning with symbols first before moving onto verbal word problems, which contradicts the findings of the studies conducted by Nathan and Koedinger (2000a, 2000b). When students learn symbols first, there may be too much ambiguity and abstraction for them. Learning in the abstract first can make it difficult for students to grasp the concept, thus, making it possible that they could build one misconception on top of another.

However, Chinese textbooks focus on students building the foundation of their skills by starting off with an example that has a real-world application. Then the examples fade into abstract representations, which is discussed in the next study by Ding and Li (2014; Ding, 2016). Concrete representations first allow students' prior knowledge and contextual knowledge to be applied for sense-making (Ding, 2016; Resnick & Omanson, 1987). Sense-making refers to students being able to 'make sense' of mathematical concepts, e.g., having students develop their mathematical instincts of inverse relations (Ding, 2016, p. 57). An example provided by Ding (2016) is an initial unknown problem that asks students to find how many books were there initially after combining the number of books lent out and the number of books leftover (Ding, 2016). The purpose of this problem was for students to make sense of the quantitative relationships and why the addition operation was used in the example. If an individual returns the books that were lent out, then the number of books leftover can be added to find the initial number of books. Thus, the equation used is: book lent-out + leftover books = initial number of books. Students are now supported in applying this foundational knowledge as the representations become more idealized. As the examples get more abstract, students are able to reference the real-world, concrete contexts if they are confused, which is later discussed in Koedinger and Nathan's (2004) study. Even though this study (Ding, 2016) is about an elementary school textbook, sense-making is a universal expectation for students in K-12. Thus, concrete fading can be a potential solution that can help students make connections across concepts, visualize a concept in a real-world, concrete context then another example that is a bridge between the concrete and abstract representations before moving onto strictly abstract representations that dominate US textbooks. A further analysis of concrete fading in Chinese textbooks is discussed in the next study by Ding and Li (2014).

Real-world Situations & Sense-making

This subsection is taking a deeper look into the previous subsection of sense-making and multiple representations. Ding and Li (2014) conducted a study that demonstrates how concrete fading promotes sense-making when students are learning the distributive property through a textbook analysis of lesson examples. The purpose of this study was to investigate how the examples in a Chinese textbook are structured from concrete to abstract representations. These representations take on the same definitions as stated in the study conducted by Ding (2016; Ding & Li, 2014) discussed earlier. The sequence of representations investigated was: within a worked example and from a worked example to the practice problems students were asked to solve within one concept (Ding & Li, 2014). The textbook used in this study was *Jiang Su Educational Press textbook* (Ding & Li, 2014; Su & Wang 2005) covering grades 1-6.

The results for fading within a worked example revealed that several of the worked examples were concrete representations in the form of word problems (Ding & Li, 2014). An example of the worked example is provided by Ding and Li (2014). The first example includes pictures that contain a real-world, concrete context of two friends buying clothes: 5 suits (jeans and jackets). The prices were labeled on the clothes: 65 Yen for a jacket and 45 Yen for a pair of pants. The question is asking how much is the total cost? There are two solutions provided that show how students can solve for the answer: 1) first find out the cost for 5 pants and 5 jackets, respectively, and then find the total cost or 2) first find the cost for each suit (1 pair of pants + 1 jacket) and then find the total cost. The above two reasoning processes led to the following two solutions below. Please refer to Table 2 below (Ding & Li, 2014, p. 111).

Table 2

Comparison of Methods

Method 1	Method 2
$65 \times 5 + 45 \times 5$ $= 325 + 225$ $= 550$	$(65 + 45) \times 5$ $= 110 \times 5$ $= 550$

A comparison of these two solutions generated an instance of the distributive property: $65 \times 5 + 45 \times 5 = (65 + 45) \times 5$. The final example is generalized by showing the distributive property using variables and symbols: $(a + b) \times c = a \times c + b \times c$ (Ding & Li, 2014, p. 111). Through this example, students are able to understand the topic in a concrete context and picture, i.e., buying clothes. This problem-solving process enables students to discover the instance of the distributive property. Moreover, this word problem context enables students to make sense of why the two number sentences: $65 \times 5 + 45 \times 5$ and $(65 + 45) \times 5$ are equivalent. After, students use solely numbers and thought bubbles without pictures or a word problem that represents the quantitative relationship embedded in the story problem. Finally, they work in the abstract with variables.

Tape Diagram & Sense-making

In addition to concrete, real-world, concrete contexts that help with sense-making, semi-concrete diagrams such as tape diagrams can also help students better understand and help them learn abstract representations as discussed by Murata (2008) and Ding (2021). Murata (2008) highlights the features of the tape diagram for grades 2 and 3. A tape diagram is used to help students see the 'mathematical relationship' while solving problems (Murata, 2008, p. 391). Please refer to Figure 2 below for an example of a tape diagram provided by Ding et al. (2019, p. 106; Murata, 2008).

Students are able to apprehend what the problem is asking, what information are they given and how it relates to the problem. Similarly, this type of diagram can act as a tool for students who are not yet able to understand when given a story problem (Murata, 2008). In addition, tape diagrams are beneficial because they assist with 'meaning-

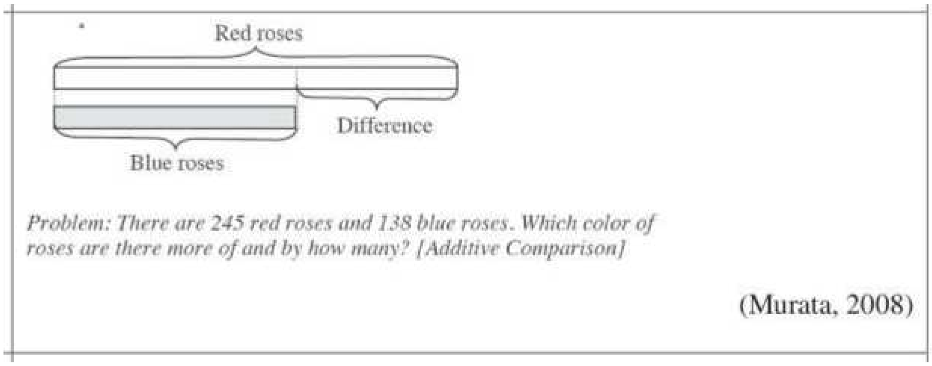


Figure 2. Tape Diagram Example. Ding et al., 2018, p. 106; Murata, 2008

making' (Murata, 2008, p. 396). Meaning-making allows students to go beyond the mathematical calculation and understand what is the math behind the equation. This is recognized by giving students 'time and space' to ponder about the representations when going from verbal word representations to abstract representations including numbers (Murata, 2008, p. 399). As Lubienski (2000) argued, providing students the time to think and understand the initial problem before progressing onto the abstract representation is vital. Also, a tape diagram can support students in their understanding of what operations need to be performed to solve the problem and understand the relationship among them (Murata, 2008).

In an example provided by Murata (2008), students were given a word problem that was aided with a tape diagram asking how many total paper rings were there. They were then asked to write an equation that was represented by the tape diagram. After, students were asked how many total paper rings there were with supplemental pictures of adding the hundreds, tens and ones places (Murata, 2008). The sequence of this example supports concrete fading because students work in a semi-concrete diagram that fades into an equation with symbols and numbers (Murata, 2008). Similar to Ding and Li (2014), the progression of the representations allows students to understand the abstract equation involving numbers when they first began working with the concept concretely (Murata, 2008).

In another study conducted by Ding (2021), the sequence of going from concrete to abstract was shown using a concrete, semi-concrete using a tape diagram and then an abstract representation. Like Ding and Li (2014), when students in grade 3 were trying to solve the price of a clothing outfit. First, students looked what types of quantities were given and what they were asked to solve for. Second, they drew a tape diagram, which is the semi-concrete representation. After constructing the tape diagram, students generated an equation from that tape diagram. Finally, students were asked to compare the equation with the tape diagram (Ding, 2021). This type of progression of representations from concrete, semi-concrete to abstract aligns with the definition of concrete fading used in the current study.

Sequence of Learning

Real-world situations and tape diagrams align with the sequence of learning as revealed by Koedinger and Nathan (2004) and Koedinger et al. (2008). Koedinger and Nathan's (2004) study explored how students may benefit from learning in a particular sequence: verbal problems to symbolic equations. Student learning and performance depend heavily on the representations that they are given. This study focused on three different types of phases: symbolic facilitation, situation facilitation and verbal facilitation (Koedinger & Nathan, 2004). Symbolic facilitation is the belief that symbolic representations are thought to be easier for students to solve because verbal word problems require reading comprehension that may cause confusion. Situation facilitation is the belief that story problems provide context for students, which can prompt them to use any strategy to solve story problems, word equations and symbolic equations (Baranes et al., 1989; T.N. Carraher, D.W., Carraher, & Schliemann, 1987; Koedinger & Nathan, 2004). Verbal facilitation is the belief that students' verbal skills are developed prior to their symbolic skills (Koedinger & Nathan, 2004).

Koedinger and Nathan's (2004) study consisted of two studies involving high school students who took a difficulty factors assessment, DFA: Difficulty Factors Assessment 1, DFA1, for study 1 and Difficulty Factors Assessment 2, DFA2, for study 2 (Koedinger & Nathan, 2004). The DFA included different types of problems that contained 'factors' that would have an effect on the difficulty of the problem (Koedinger & Nathan, 2004, p. 139). Study 1 consisted of high school Algebra I and Geometry students. Study 2 consisted of high school students enrolled in Algebra I. The problems in the assessment were in the format of: the problem (story problem, word equation, symbolic equation); positioning of the unknown (result-unknown and start-unknown); numbers that included positive decimals and whole numbers; and operations that were manipulated (addition and multiplication versus division and subtraction). Story problems provide a story or scenario to a math problem (e.g., situation facilitation) whereas word equations state what operations to use in words. Within story problems, there are story-implicit operators that do not directly state what operations to use, but use words that imply the operation such as 'lost' or 'gained'. Story-explicit operators use words that clearly state which operation to use such as 'multiply' or 'divide' (Koedinger at al., 2008). A symbolic equation uses solely numbers, variables and operations (e.g., symbolic facilitation). Table 3 is provided below to show the differences among the three formats.

Table 3

Math Problem Format

Story Problem	Word Equation	Symbolic Equation
Megan received $80 for her birthday, but had to give $15 to her brother for lunch. She then divided her money by 5 to see how much extra money she should have for the next five days. How much extra money would she have per day?	Starting off with 80, if I subtract 15 and then divide by 5, what number do I get?	$\frac{80-15}{5}=x$

The strategies students used to solve start-unknown were analyzed as well as their errors they made. The errors were placed into one of the three groups: if there was no response; errors with arithmetic; and other errors that suggested lack of conceptual understanding (Koedinger & Nathan, 2004). A three-factor ANOVA was conducted for both studies that compared the factor of difficulty (representation type, the positioning of the unknown and the type of number). Study 1 also ran three different one-factor repeated measures ANOVAs for every type of representation (Koedinger & Nathan, 2004).

Findings revealed that there was a significant difference between how students performed on story problems and symbolic equations as well as word equations and symbolic equations. Even though they performed slightly better on story problems, story problems and word equations did not differ significantly (Koedinger & Nathan, 2004). Study 2 yielded similar results. However, there was a significant difference between the number type and the problem representation. Students had no difficulty when using whole numbers for story problems and word equations but struggled using decimals on word equations. Although Koedinger and Nathan (2004) did not use the term ‘concrete fading’, the results of this study support concrete fading (from concrete to abstract) and suggest that the technique has the potential to benefit students because students performed better on story problems and word equations than they did on symbolic equations. Through these types of representations, students are provided context from story problems and word equations that will allow them to relate to the material to help better understand the concept when they move on to abstract problems (Koedinger & Nathan, 2004).

In a later study conducted by Koedinger et al. (2008), which was follow up to the Koedinger and Nathan (2004), was a two-experimental study that examined the trade-off between the benefits of concrete and abstract representations with undergraduate students. The students were enrolled in either an Algebra course similar to Algebra I in high school or a more difficult course that was similar to Algebra I and II in high school. The purpose of this study was to investigate if transitioning from concrete to abstract was more beneficial for students learning less complicated algebraic problems while learning from abstract to concrete is more beneficial for students learning more advanced mathematical problems (Koedinger et al., 2008).

In experiment 1, students completed a ‘six-item ‘difficulty factors assessment’’ (Koedinger et al., 2008, p. 373). The six categories were single-reference and double-reference problems in one of the three forms: story-implicit operators, story-explicit operators and equations (Koedinger et al., 2008, p. 374). Single-reference problems are when the unknown is only referenced one time in a problem as opposed to double-reference problems when the story refers to the unknown twice in a problem (Koedinger et al., 2008). Findings revealed that students performed better on the single-reference problems where the story problems were more grounded and concrete than they did on the single-reference abstract equations, which was hypothesized for the current study, but did not happen. However, students performed significantly better on the abstract equations when it was a double-reference problem rather than single reference (Koedinger et al., 2008).

In experiment 2, students from another university also completed a difficulty factors assessment that assessed solving equations and solving story problems (Koedinger et al., 2008). Similar to results in experiment 1, students performed significantly better on single-reference verbal problems than on single-reference equations. However, students performed better on double-reference symbolic equations than on double-reference verbal problems (Koedinger et al., 2008). The results from both experiments, however, may be interpreted in two ways based on what the authors suggest and what is interpreted based on the broader literature. First, the results suggest that there is an advantage to using symbols first when teaching more difficult algebraic problems since it involves formal mathematical notation such as formulas that do not require as much manipulation whereas informal strategies (i.e., 'unwind strategy'; Koedinger et al., 2008, p. 370) may require a lot of mathematical manipulation and calculation of transforming equations to try to solve causing more errors (Koedinger et al., 2008; Mayer, 1982). For example, when students attempted to solve a double-reference story problem using informal strategies, some students were using the incorrect arithmetic operations and applying the unwind strategy incorrectly (Koedinger et al., 2008). Thus, incorrectly generating the correct equations from the story problems.

The results also indicate that students perform better on simpler algebraic problems when the problems are initially presented in a verbal format such as a story problem. Students' success on these problems could be attributed to the 'familiarity property' (Koedinger et al., 2008, p. 385) where students can activate their prior knowledge to help them understand and solve the problem. Verbal word problems also provide contextual and situational knowledge for students (Koedinger & Nathan, 2004).

Based on the broader literature, these findings also indicate the need of support from instruction for students to understand quantitative relationships that are depicted in story problems because informal strategies do not always help as seen in this study (Koedinger & Nathan, 2004). Students may not understand the story situation described in harder word problems, which may demand a tool to help illustrate the embedded relationships. In other words, the advantage of concrete situation was not fully taken by students. As such, a tool like a tape diagram may be effective in supporting students (Murata, 2008). Students may not automatically transfer from complex word problems to abstract tasks because in both experiments, students performed better on double-reference story problems when they used formal strategies than informal strategies (Koedinger et al., 2008). Perhaps if semi-concrete diagrams such as tape diagrams (Murata, 2008) were used, students can better understand the quantitative relationships, which were not used in the study conducted by Koedinger et al. (2008), but was considered in the present study.

Across-study Analysis

As evidenced by Ding (2016), Ding and Li (2014), Murata (2008), Ding (2021) and Koedinger and Nathan (2004), studies that align with this study's definition of concrete fading that include problems that focus on components such as meaning-making, sense-making, and different types of representations greatly benefit students. Ding (2016) explored how US textbooks lack meaningful representations and do not focus on sense-making unlike Chinese textbooks. If students are presented with representations that are not familiar to them, then they may not even try to answer the problem (Koedinger et al., 2008). Using real-world examples provides context as Ding and Li (2014), Koedinger and Nathan (2004) and Koedinger et al., (2008) conveyed in their studies. When the examples begin with a concrete context then fade the pictures away and introduce algebraic solutions using numbers and variables (Ding & Li, 2014) with tape diagrams (Ding, 2021), which Murata (2008) discussed, can benefit students greatly. Tape diagrams are very important as they can enable students to see the mathematical relationship and make sense of the mathematical process of the equations they are using to solve. Once they see the relationship through the tape diagram, the teachers can begin to have students work in the abstract. This transition described going from concrete to semi-concrete to abstract can be beneficial for students as evidenced in the previous subsection. However, Koedinger et al. (2008), suggest that more complex algebraic problems may not benefit students from a sequence starting with concrete representations because there is a trade-off where the order of going from concrete to abstract representations will not always be beneficial depending on the complexity of the concept. Future studies need to explore whether concrete situations molded with diagram will contribute to problem solutions. In other words, whether concreteness fading approach will be helpful for high school students.

Concrete Fading Support Student Learning

This subsection will discuss reviews and studies that focus on concrete fading. I will review four experimental studies below, followed by a comparison across these studies. Please refer to Appendix A for a table that summarizes the major studies discussed in the paper that includes the design of the study, content of the study, the instructional activity, the level of the students, and the p-values.

Concrete fading has been proven to be successful for students in the classroom and offers plenty of insight as discussed in this subsection. Though the technique has been used in many studies, most studies are conducted at the elementary (Ching & Wu, 2019; Fyfe et al., 2015), middle school (Ottmar & Landy, 2017) and undergraduate levels (Fyfe & McNeil, 2012), thus, there is a need to explore concrete fading at the high school level. The studies discussed indicate that the instructional technique suggests a promising direction for students and their improvement in the classroom for mathematics at the high school level; however, it has yet to be studied and investigated. Studies that investigated algebraic learning within high school students (e.g., Koedinger et al., 2004, 2008) concur with the hypothesis that starting with story problems can be beneficial; however, the ‘concreteness fading’ approach has not yet been investigated in the prior studies primarily because there is no ‘fading’ stage such as the current study which goes from concrete to semi-concrete (i.e., fading stage) to abstract.

Fyfe et al. (2014) reviewed concrete fading in a mathematics and science class and discussed the necessary components for successful implementation of concrete fading in the classroom. First, for successful implementation, concrete fading must focus on students grounding their understanding with concrete representations that will help them understand the concept before moving on to the abstract representations (Fyfe et al., 2014). An example was provided in detail earlier by Ding and Li (2014) where students learned the distributive property. Students first begin with a concrete context that was illustrated with pictures and words where a couple of girls were buying clothes. This problem task was then modeled by tape diagrams, leading to two different numerical solutions that together indicated the distributive property. The progression of these examples can support students in seeing the quantitative relationships embedded in the story problem.

Second, students are exposed to a concrete context provided by a word problem so that they are able to visualize how the property is used in real life from the word problem when the girls are buying jackets and jeans as discussed earlier by Ding and Li (2014). They need to find the total amount it would cost by using the distributive property. Finally, the transition from concrete to less concrete to abstract through concrete fading has the potential to support students in understanding the mathematical concepts because they are able to cement their understanding by connecting the concepts they are expected to establish in the earlier stages (Fyfe et al., 2014). Several studies discussed below focus on this instructional technique as an intervention that offers great benefits for students.

The first experimental study is a study conducted by Fyfe and McNeil (2012) involving 80 undergraduates learning the commutative and associative properties along with identity and inverses through a learning phase and two transfer phases. The definitions of these properties are provided by Fyfe and McNeil (2012, p. 442) shown below in Table 4. Students were assigned to one of three conditions: abstract (generic),

Table 4

Rules of a Commutative Group (Fyfe & McNeil, 2012, p. 442)

Rules	Definition
Associativity	For any elements x, y, z: $[(x + y) + z = [x + (y + z)]$
Commutativity	For any elements x, y: $x + y = y + x$
Identity	There is an element, $\mathbf{I}$, such that for any element x: $x + \mathbf{I} = x$
Inverses	For any element, x, there exists another element y, such that $x + y = \mathbf{I}$

concrete (concrete-only) and concrete fading (fading; Fyfe & McNeil, 2012). The study investigated the effectiveness of concrete fading; the initial benefits of students' learning when the sequence of representations is abstract to concrete (going from a representation that is abstract to something that is recognizable to them); and how students may benefit later when the sequence of the representations reverses from concrete to abstract (Fyfe & McNeil, 2012). Students took a series of post-assessments: immediate and two delayed post-tests. The questions on the assessments varied across each condition. The questions for the concrete-only group did not use real-world contexts, rather used strictly images such as pictures of measuring cups that would have various portions of the cup shaded and students were asked to answer what was remaining after being shown the identity of the cup (Fyfe & McNeil, 2012). The abstract-only group answered questions that were solely in abstract symbols including circles and diamonds. A combination of these two shapes was added and resulted in either a circle or a diamond. Students were asked to find what was remaining after being shown the identity of the shapes. The fading group answered questions that were a combination of the previous two groups as well as Roman Numerals: I, II, and III. A combination of these Roman Numerals was added and then students were asked to find what was remaining after being shown the identity. These questions were in a particular order that aligned with concrete fading: the measuring cups (concrete), Roman Numerals (fading), and then abstract symbols (generic; Fyfe & McNeil, 2012).

Findings revealed that students in the fading condition outperformed students in the other two conditions in all of the test phases, leading to better transfer knowledge. Transfer knowledge is defined as students' ability to apply that knowledge elsewhere other than in that particular context (Fyfe et al., 2015; Fyfe & McNeil, 2012). In the fading condition, students were first taught using representations that they were able to recognize. After, the concreteness was slowly fading away by taking away the irrelevance from the concrete objects and then phased into abstract symbols (Fyfe & McNeil, 2012). Findings also revealed that even though students in the generic group initially performed significantly better than the students in the concrete-only group for transfer, as time progressed, the concrete-only group made a huge improvement. Thus, the difference between the two groups' transfer was not significant (Fyfe & McNeil, 2012). Students in the concrete condition were able to learn about the properties using representations that were relatable which allowed them to have a 'store of images', which students in the generic condition did not have (Bruner, 1966; Fyfe & McNeil, 2012, p. 446). In the generic condition, students did not have a 'store of images' (Bruner, 1966; Fyfe & McNeil, 2012, p. 446) that they were able to refer to if they got confused unlike the concrete and fading conditions where that storage was established (Fyfe & McNeil, 2012). The results revealed by Fyfe and McNeil (2012) suggest that concrete fading is very beneficial for students because it helps with their transfer knowledge. Concrete fading could also help students because the transition from concrete to fading to abstract allows students to create a ‘store of images’ (Bruner, 1966; Fyfe & McNeil, 2012, p. 446) that they can refer to when the abstract representations become too difficult.

Not only has concrete fading helped undergraduates, but it also proved to benefit students at the middle school and elementary levels. Ottmar and Landy (2017) conducted a study involving seventh-grade students’ learning of the commutative property as well as simplifying and manipulating mathematical expressions. Students completed a pre-test, an immediate post-test after the first day, a one-day delayed post-test after the second day and a one-month delayed post-test (Ottmar & Landy, 2017). They were assigned to one of two groups: concrete fading and concrete introduction. Concrete fading is defined as representations that are concrete and phase into abstract representations (Bruner, 1966; Fyfe et al., 2014; Goldstone & Son, 2005; Fyfe & McNeil, 2012; Ottmar & Landy, 2017). Concrete introduction is defined as representations that are abstract at first and then phase into concrete representations, which is used more frequently with students learning algebra (Nathan, 2012; Ottmar & Landy, 2017).

The students in the concrete fading group first learned the dynamic lesson followed by the static lesson whereas the students in the concrete introduction group learned in the reverse order (Ottmar & Landy, 2017). The dynamic lesson involved students using a program application, PS, Pushing Symbols (Ottmar & Landy, 2017). In the lesson, they were asked to change the expressions they were given by moving the tiles. The tiles were color-coded according to the like-terms, i.e., all of the terms with x were of the same color; all of the constants were of the same color; all of the terms with y were of the same color. In the static lesson, students learned through worked examples and solutions that were handwritten (Ottmar & Landy, 2017). Students would solve the example on the iPad which was a two-column set up. The program they used to solve the example would have the students' answers on the left-hand side while the correct process and solution were on the right-hand side. Results revealed that even though neither group differed significantly for the pre-test or the immediate post-test, students' prior knowledge significantly affected how they performed on the assessments after the first day (Ottmar & Landy, 2017). After the second day, students' scores differed significantly after the one-day delayed post-test. Thus, indicating that students may be more successful when learning the dynamic lesson followed by the static lesson (Ottmar & Landy, 2017). However, neither group differed significantly at the one-month delayed post-test.

The results of the study conducted by Ottmar and Landy (2017) demonstrate that concrete fading is a potential solution for student learning. Even though students in the concrete fading showed a significant difference only a few days after instruction, they still outperformed students in the concrete introduction condition. However, the benefits of retention are unclear as there was no significant difference one month after instruction. The results could differ if different types of examples such as concrete contexts and tape diagrams were used such as the examples that the present study used. Students in this study were using color-coded tiles that involved rearranging them to manipulate and simplify the expression. Perhaps a concrete context that students can relate to as opposed to the tiles used in the study could help students gain a better understanding before moving onto simplifying symbolic expressions.

Fyfe et al. (2015) conducted a later study consisting of three experiments at the elementary level with second and third grade students studying math equivalence.

Students were assigned to four different groups: abstract; concrete; concrete fading (concrete to abstract); and concrete introduction (abstract to concrete; Fyfe et al., 2015). In the first experiment, students learned the lesson and then completed a test which assessed students' transfer knowledge (Fyfe et al., 2015). The concept of the lesson was the same across each condition; however, the presentation of the problems differed for each of the conditions. The concrete group used animals with stickers as well as a balancing scale using objects whereas the abstract group used symbolic math equations such as 5+3=5+__? (Fyfe et al., 2015). The concrete fading group used concrete materials from the concrete group, then a 'fading' worksheet with animals, and concluded with symbolic math equations from the abstract group whereas the concrete introduction group learned in the reverse order (Fyfe et al., 2015, p. 107). The second experiment focused on first and second grade students that were split into two conditions: abstract and 'play-to-abstract' where the first two stages closely replicated the lessons of the concrete and fading groups, but without any instruction to try to engage students, and then were taught with equations in the abstract representation (Fyfe et al., 2015, p. 109). The third experiment focused on students with higher prior knowledge where the problems and the procedures were more difficult for each condition (Fyfe et al., 2015).

The findings of this study (Fyfe et al., 2015) revealed that, regardless of the level of prior knowledge, students in the concrete fading condition outperformed students in all of the other conditions for all three experiments as well as on the assessments (Fyfe et al., 2015). In addition, the transfer scores in experiment 3 were not as low as in the previous two experiments. The results were attributed to the fact that students in the fading condition were more likely to implement the more challenging strategies to solve and solve the challenging problems correctly compared to any other condition (Fyfe et al., 2015). As evidenced in this study (Fyfe et al., 2015), concrete fading is a potential instructional technique that could help students because they have the opportunity to see the concept represented in a concrete, familiar representation (Koedinger et al. 2008). The concrete materials are something that students can relate to and are realistic for them that can help them to correctly solve the problem when the concept begins to lose its concreteness in the fading stage and abstract symbols in the final stage (Fyfe et al., 2015).

Ching and Wu (2019) conducted a similar, more recent study using concrete fading with younger students who were in kindergarten. They were randomly assigned to one of the five groups: the control group, concrete, abstract, concrete fading (concrete to abstract) and concrete introduction (abstract to concrete). Students completed a pre-test, learned the lesson, completed the post-test, and a delayed post-test. The lessons differed depending on the condition. In the concrete condition, students worked with Unifix cubes, where students were able to see how the total number of cubes changed as cubes were added or taken away (Ching & Wu, 2019). In the abstract condition, students were asked to provide the answer after being given an expression with numbers and operations, which was similar to the control group. The concrete fading lesson began with using Unifix cubes, then they worked with pictures of cubes on paper being added and subtracted and then ended their lessons with working with numbers and operation symbols (Ching & Wu, 2019). The lesson for the concrete introduction condition was the reverse of the fading lesson. Students then completed their post-tests, which assessed students' understanding and knowledge of inverse relations; their transfer; and their ability to complete problems that did not explicitly apply the inverse relations (Ching &

Wu, 2019). Chi-squared tests and a one-way ANOVA comparing the student's group (control, concrete, abstract, concrete fading and concrete introduction) to the three different types of problems on the tests were conducted.

Results revealed that students with lower prior knowledge in the concrete fading condition benefited the most as their post-test results had a larger improvement over the students' post-test scores in the abstract-only condition (Ching & Wu, 2019). In this study, students benefited because concrete representations were more helpful to them than abstract representations as concrete representations allow them to see the concept in a relatable, concrete context and then transfer that knowledge (Ching & Wu, 2019; Fyfe & Nathan, 2018). There was also no significant difference for the students with higher prior knowledge across the conditions. Thus, the results of this study support how concrete fading benefits younger students with lower prior knowledge whereas the benefit for students with higher prior knowledge was inconclusive unlike the results revealed by Fyfe et al. (2015; Ching & Wu, 2019).

Though concrete fading only had a significant effect on students with lower prior knowledge, the results may differ if this study was conducted at the secondary level. As evidenced by studies discussed earlier in this section, students at the high school level continue to struggle and build one misconception on top of another possibly due to their lack of grounded understanding, which could affect students of both high and low levels of prior knowledge. The findings of this study (Ching & Wu, 2019) further argue that concrete fading has a profound effect on younger students with lower prior knowledge and can help these students significantly.

Across-study Analysis

In all of these studies, there were two commonalities found. First, students in the concrete fading outperformed students in the other conditions (Ching & Wu, 2019; Fyfe et al., 2014; Fyfe & McNeil, 2012; Ottmar & Landy, 2017). Thus, further supporting that concrete fading has the potential to benefit students greatly. Students were able to ground their understanding using concrete, real-world representations of various concepts (Fyfe & McNeil, 2012). These types of representations convey to students that there is a concreteness and reality to the mathematical concept. Similarly, it will also allow them to make connections with their own experiences and knowledge of the concept (Fyfe et al., 2014). As evidenced by Fyfe and McNeil (2012), students in the fading condition outperformed students in any other condition learning the various mathematical properties, possibly, because they began learning with something that was recognizable to them, i.e., measuring cups. They then were able to create a 'store of images' (Bruner, 1966; Fyfe & McNeil, 2012) in their minds to refer to when abstract representations were too challenging. Second, concrete fading helped younger students with lower prior knowledge significantly with inverse relationships (Ching & Wu, 2019). Similarly, students exposed to concrete fading were more likely to correctly implement a more advanced strategy and answer correctly when solving a problem assessing math equivalence (Fyfe et al., 2015) compared to other instructional styles. However, the effect of concrete fading is not yet known for students at the high school level, only at the elementary, middle and undergraduate levels. Concrete fading has the potential to help students tremendously who are continuing to struggle at this level.

Despite the previous studies that shed light on current research, there are a few shortcomings that the current study addressed. First, this current study included school students that investigated their performance on two of the most crucial, fundamental topics of Algebra: solving linear equations and generating equations from word problems. Second, the initial stage did not use physical objects or images, but real-world, concrete contexts provided by a word problem instead, which contained concrete examples compared to studies conducted by Fyfe and McNeil (2012), Ottmar and Landy (2017) and Ching and Wu (2019). By using real-world, concrete contexts, high school students are more appropriately supported in trying to visualize the abstract, mathematical concept in a more meaningful, concrete, realistic way that they can relate to (Nathan & Koedinger, 2000a). The current study addressed these gaps in the research. This is further discussed in the design section.

Summary of Review

Combining all of the limitations and unresolved issues discussed in the literature above suggests the need of the current study. The concrete fading studies conducted by prior studies (Ching & Wu, 2019; Fyfe & McNeil, 2012; Fyfe et al., 2015; Ottmar & Landy, 2017) did not focus on high school students nor did they focus on solving linear equations. In addition, unlike the initial stage of lessons or design of the studies conducted by Ottmar and Landy (2017) and Booth and Koedinger (2008) where students used physical objects or realistic images, the current study introduced students to a story problem in the initial stage. Students were given a concrete, real-world context that aimed to help them better visualize the abstract mathematical concept as well as aimed see the quantitative relationship embedded in the story problems using tape diagrams.

After students attempted to make sense of the relationship, teachers progressed into abstract representations. However, other studies conducted by Nathan and Koedinger (2000a, 2000b, 2004) and Koedinger et al. (2008) did not explicitly address the 'fading' stage where high school students transition from concrete to abstract, but did align with concrete fading by going from concrete to abstract representations. The 'fading' stage could be potentially beneficial for students as this stage uses tape diagrams that could help students better generate linear equations as evidenced by Clement et al. (1981). The current study included a fading stage that involved using tape diagrams that aimed to help students better visualize, make sense and meaning of the quantitative relationships of the equations represented in word problems (Ding, 2016; Ding & Li, 2014; Ding et al., 2019; Murata, 2008; Ng and Lee, 2009).

Current Study

The purpose of this study was to investigate how concrete fading makes an influence on student learning with two Algebra I classes while students are solving linear equations and word problems taught by one teacher. In particular, this study explored two research questions below:

> RQ1: What are some ways that students who received concrete fading think differently than the control group?
>
> RQ2: How do those differences seem to be related to the intervention?

It is expected that students who have learned the concrete fading technique will have another method that they can use correctly to solve word problems over students in the control group who received the traditional instruction. The teacher's traditional instructions aligned with the given textbook, *Algebra 1: A common core curriculum* (Larson & Boswell, 2019), which was different from concreteness fading in that the control group used the textbook with the sequence that began with abstract examples and got more abstract from there.

CHAPTER 3

METHODS

This chapter discusses the overall design and the type of analysis conducted that provided the findings for this study. First, I discuss the design of the study. Second, I discuss the intervention materials along with participants involved in the study. Third, I discuss teacher training as well as data collection and procedure. Within data collection also includes a description of the assessments used.

Overall Design

The design of this study intended to be quasi-experimental, but it was revised and modified because of the COVID-19 pandemic. The initial design of this study was to have 40-60 students as the sample size and to rely on the quantitative results, but consent was challenging to get because of the pandemic. There were also challenges of receiving permission to conduct the study at various school districts. Due to these challenges, the study changed to primarily qualitative analysis with eight students.

The two classes were split into treatment group and control group.[1] Students took a pre-test before the three lessons, an immediate post-test and a four-week post-test. Eight students were interviewed based on the consent provided. These eight students represented both groups: the treatment and the control. Please refer to Figure 3 below for the design of the study.

Figure 3 shows how concrete fading occurs in two different parts of the lesson for the treatment group. There is concrete fading within a word problem and then throughout

[1] Author was not teacher for either of the classes

	Treatment group	Control group
Part 1 of lesson	Word problem 1. Problem situation 2. Tape diagram 3. Algebraic equation	Equation
Part 2 of lesson	Equation	Word problem 1. Problem situation 2. Algebraic equation

Figure 3. Overall Design of Study.

the lesson. The concrete fading within a word problem goes from the problem situation, to tape diagram where students are supported in visualizing the quantitative relationship embedded in the word problem and then generating an equation from the tape diagram in the previous stage. Similarly, the concrete fading throughout the lesson transitions from concrete to abstract by going from a word problem to an equation. Prior to the interventions, it is important to note that students were likely taught in the more traditional style presented by their textbook. In other words, students were mostly taught using abstract representations with minimal concrete examples. This is discussed further in the Discussion chapter.

Participants & Research Setting

The participants were one Algebra I teacher and eight ninth grade students from two Algebra I classes based on consent provided. One class was assigned the treatment group in which the students received instruction through the concrete fading technique and the other class was the control group where the students received traditional instruction taught by the same teacher. Both classes were general education classes. The high school was located in the northeastern United States on the East coast. Based on the school's website reporting of 2021 data, which is not explicitly referenced to preserve the anonymity of the school and teacher, over 90% of the students racially identify as non-white and less than 10% of the students qualify for reduced and/or free lunch.

Consent forms were given to every student and the teacher. The consent forms were given back when the author came back to observe and to interview the students. One class had three students consent to participate and the other class had five students consent to participate. The classroom breakdown of the participants for each class was provided by the teacher and shown below in Table 5.

Table 5

Class Breakdown

Class	Percentage of Male	Percentage of Female
Control (n=3)	66%	33%
Treatment (n=5)	40%	60%

Lesson Content & Ideas

The learning goals for the students were to help students improve their word problem skills and their computational skills with equation solving. In other words, at the end of the three-lesson sequence, students will be able correctly generate an equation from the word problem and correctly solve an equation as well as being able to represent the quantitative relationship embedded in the word problems. In addition, I expected students' conceptual understanding of equation solving and quantitative relationships represented in word problems to improve.

In the current study, I use concrete fading within a lesson and within an example as shown in Figure 3. This study first presents a problem situation that is then represented using a tape diagram. A tape diagram is used because of its semi-concreteness that will help fade into abstract. I will call the word problem concrete, the tape-diagram semi-concrete and the equation abstract. Though it may be difficult for students, it is still powerful enough to represent the linear quantitative relationship (Ding et al., 2019). The tape diagrams are new to both the students and teacher because students never saw it in early grades unlike students in East Asia who fade into learning with tape diagrams in grade 2 (Murata, 2008). Since the tape diagram is new, the teacher provided instructional scaffolding in order to help students reach stage 3 of generating an abstract equation, which was generating an equation from the tape diagram. In addition, the overall lesson design uses concrete fading as the students begin with a word problem and then fade into symbolic equations.

Each problem was analyzed by looking at how the problem was solved, the main ideas and concepts. Then the problems were solved using the tape diagram. Some of the examples, however, did not show how effective the tape diagram can be, so practice tasks were then analyzed to try to find problems that would show how effective the tape diagram could be. Once the problems were found, the lessons were created. Because more examples were added to the treatment lessons, those same examples were added to the control group. These examples were the same problem but taught differently, i.e., without a tape diagram and no explicit discussion that asked students to connect the story problem with the generated equations. Necessary modifications were made and the control and treatment lessons were designed.

Students in the control group learned with an emphasis on symbolic, abstract equations where the teacher primarily lectured that involved students learning with equations first and discussion was not part of the lesson. The teacher taught students how to solve word problems and linear equations and what strategies to use. The lesson plan consisted of presenting the word problem and using an equation to get the answer. In contrast, students in the treatment group had more choice of how to solve word problems at the beginning of the lesson where the sequence of these lessons was concrete fading within an example. These examples were more student-driven in that students first discussed what the problem was asking, then used a diagram-based strategy with a semi-concrete diagram in the second example and the teacher did not lecture as much.

Content Validity

The lesson content and ideas in this current study align with the content and the textbook students were using at the school. The material was shown to the author's defense committee, administration and the teacher involved in the study. The material was created using the textbook used at the high school, *Algebra 1: A common core curriculum* (Larson & Boswell, 2019). Every problem that was used for the lesson was either an example or practice task that was provided in the textbook. The final versions are provided in Appendix B.

Control Unit & Lesson Design

There were three lessons for each condition. The lessons contained the same word problems and abstract equations, but differed in their sequence of the examples and techniques used to solve them. In particular, the treatment received three concrete fading lessons and the control group received lessons aligned with the textbook in that the teacher presented the content in the manner that the textbook did. The control group lessons were developed such that where the teacher's notes and examples were based on the textbook, *Algebra 1: A common core curriculum* (Larson & Boswell, 2019). The textbook lessons first began with defining terms and properties of equality. Then, there were three to five examples, but all of the examples were abstract representations using variables, numbers, operations symbols and equations. Finally, the lessons concluded with three to five word problems. The control group also had additional word problems that were the same word problems used in the treatment group of the corresponding lesson; however, these word problems were taught using symbolic equations rather than tape diagrams. The word problems in the control lesson were taught at the end but in the treatment group, they were taught at the beginning. All of the lesson examples contained the solved solution.

Enactment of the Control Lesson

The control group was taught in a reverse sequence of the treatment group. The problems were also represented in a different way than the treatment group. Figure 4 highlights the main differences between the control and treatment groups, such as the discussion portion. In the control group, the teacher first taught students how to solve symbolic equations, then he taught them how to solve the word problems provided by *Algebra 1: A common core curriculum* (Larson & Boswell, 2019) using symbolic equations unlike the treatment group that used tape diagrams to help them generate an equation, which is shown below in Figure 5. The control group was asked just to answer the word problem with no discussion before, during or after the example.

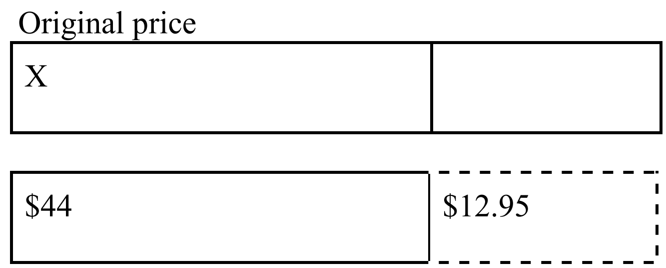
Treatment Unit & Lesson Design

The lessons of the treatment group applied the concrete fading approach as defined in the theoretical framework section in this study. The treatment lessons suggested how many days the teacher should spend on the lessons and how to split them up over a certain number of days; however, the number of days spent on the lesson was up to the teacher. While *Algebra 1: A common core curriculum* (Larson & Boswell, 2019) used symbolic equations, the treatment lessons consisted of relatable, concrete contexts provided by story problems used in the textbook to solve equations as opposed to starting off the concept in an abstract representation with a symbolic equation. In this way, students were supported in grounding their understanding in a concrete manner with concrete contexts before applying what they have learned to semi-concrete and abstract representations.

In this paragraph, I describe an example of one of the treatment lessons. Students first began with a concrete context described by a verbal word problem that was from *Algebra 1: A common core curriculum* (Larson & Boswell, 2019), which provided concrete context. Students were first asked to examine what the problem was asking and what information they were given unlike in the control group where students were not asked to examine the information given in the word problem. Then the teacher worked with that same example using algebraic equations along with a semi-concrete tape diagram, as shown below in the treatment lesson, section 1.1 (Larson & Boswell, 2019, p. 12): *'A discounted amusement park ticket costs $12.95 less than the original price x. Write and solve an equation to find the original price.'* The discounted price of $44 was shown in the diagram provided by the textbook. The semi-concrete diagram was two tape diagrams. One tape diagram represented the original cost of the ticket, x, while the other tape diagram represented the discounted price, $44, along with the discount, $12.95, in a dotted box to indicate $12.95 was being subtracted. The tape diagram was used to show the mathematical relationship in case students were not able to visualize it in the word problems (Murata, 2008) because they were trying to solve what is the original price of the ticket that would equal to the discounted ticket price, $44, plus the discount, $12.95. Please refer to Figure 4 below for a side-by-side comparison of the treatment and control lessons for the first example of section 1.1.

Along with the semi-concrete diagram, the teacher showed how to solve for the price using variables and numbers. In this way, students were working in less concrete representations while phasing into the abstract representation such as variables, operation symbols and numbers. Students then used the inverse operations to solve for 'x' as instructed by the teacher. After, the class formally discussed the solution with a tape diagram. Specifically, why was the tape diagram designed in this way; how did it help students solve; how did the tape diagram relate to the generated equation? This type of questioning had the potential to enable students for a better understanding of generating an equation as compared to traditional instruction in the control group where the teacher taught closely in line with the textbook.

The final example was an equation that asked students to solve for the variable given an abstract, symbolic equation. As students studied sections in chapter 1, the concepts became more abstract. Thus, students may establish a grounded understanding if they were in the treatment group because they were taught with word problems then tape diagrams and, finally, abstract equations. There were no differences with the abstract tasks between the two groups. Please refer to Appendix B for the lesson plans of sections 1.1-1.3 for the control and treatment groups. The teacher's lesson plans (sections 1.1-1.3) for the control group provided were the lesson sections and were based on *Algebra 1: A common core curriculum* (Larson & Boswell, 2019).

Control	**Treatment**
Practice task #15: (Presented at the end of lesson) A discounted amusement park ticket costs \$12.95 less than the original price *x*. Write and solve an equation to find the original price. (Please note that the discounted price \$44 was shown in the picture provided by the textbook) $x - 12.95 = 44$ $x = 44 + 12.95$ $x = 56.95$	Practice task #15: (Presented at the beginning of the lesson) A discounted amusement park ticket costs \$12.95 less than the original price *x*. Write and solve an equation to find the original price. (Please note that the discounted price \$44 was shown in the picture provided by the textbook) ***Ask students what the problem is asking and what information they are given before using tape diagram*** *For the teacher: Next, guide the students to draw tape diagram and introduce how to use equations to solve it. Please have students construct this diagram as they may not be familiar with it. Steps below to follow. Please also be cognizant of the scaling as this is very important in the tape diagram.* *For the teacher: Start by drawing blue brace such as the one above and write '\$44' above the brace. Ask students how to represent the price of the ticket provided that the ticket costs \$12.95 less than the original price. Then, discuss with students how to draw one tape diagram to represent the price of the ticket. The tape diagram will be split into 2 unequal pieces. The part that is the same as \$44 and the part that shows the difference of \$12.95. Ask students to generate equation.* 

	After the diagrams are drawn, ask a couple of questions about the diagrams so students will clearly understand what each tape (or part of the tape or both tapes) represent and why both tapes have the same length.
	
	● *What does the tape diagram in the example represent? Which two parts does the top tape diagram contain?*
	    ○ *How does it relate to the equation below?*
	        ■ *Separate sections of unequal length, combine all together to get total of the ticket*
	            ● *Ask students and emphasize why they have to be unequal*
	            ● *If we take away the discount, 12.95 from the original prices, x, we can obtain the current price, $44. If we add back the 'discount' of $12.95 to the current price, $44, we can find out the original price, x.*
	        ■ *Also shows that when combine the ticket total, a total of $44 is the cost of the ticket*
	            ● *Also showing role of equality and variables*
	● *How does the tape diagram help us to solve the problem?*
	    ○ *Separate sections of unequal length, combine all together to get the original price?*
	● *Why did we draw the tape diagram like that? What does each part of the tape diagram mean?*
	    ○ *See above*

	● *How do all of the tape diagram and equation examples relate to each other?* ○ *Still solving for the missing value on both sides* ○ *Using equations and any of the operations* ■ *Always get the variable on one side* ● *Real-world examples to abstract equations* *Then teacher writes the equation* x-12.95=44 *For the teacher: Before solving the equation, again ask questions about the left and right side of the equation (44+12.95, x) OR (x-12.95, 44) referring to the diagram.* *x=44+12.95* *x=$56.95*

Figure 4. An Example of Word Problem in Both Conditions. *Algebra 1: A common core curriculum* (Larson & Boswell, 2019).

Enactment of the Treatment Lesson

For each of the three lessons, students were first presented with a real-world context that was described by a word problem and asked to look at what information they were given, e.g., practice task #15: '*A discounted amusement park ticket costs $12.95 less than the original price p. Write and solve an equation to find the original price'*. The discounted ticket price of $44 was provided in the picture of the textbook. By using a concrete context described by a word problem, the students were able to imagine the situation. Then, the teacher helped the students construct a tape diagram that represented the situation described by the word problem. When the teacher was observed, he constructed part of the diagram and the students were asked to draw the rest of the diagram that is shown above in Figure 4. By being asked to see what information was given at the first stage, students were provided 'time and space' to ponder about going from the concrete context to the abstract equation (Murata, 2008, p. 399). Having students have this ‘space’ gave them time to think and understand the initial problem before generating an abstract representation, e.g., an equation, which is very important as Lubienski (2000) argued.

After students created the diagram with the teacher’s scaffold, they then generated equations that related to the tape diagram, which was the final stage where variables and operational symbols were used. Below are the equations and the process the teacher used in Figure 5 to show students how to solve the problem.

$$x - 12.95 = 44$$
$$x = 44 + 12.95$$
$$x = \$56.95$$

Figure 5. Example #1 for the Treatment Group. *Algebra 1: A common core curriculum* (Larson & Boswell, 2019).

After, the teacher asked follow-up questions to facilitate a class discussion. The follow-up questions focused on asking why one section of the tape diagram was bigger or smaller than the other part; how did the equation generated relate to the tape diagram; and what did both sides of the equation represent? When the teacher was observed, the discussion was not that long; however, the students seemed eager to answer and answered the questions correctly. Discussion was included because it was to help students engage and allow for a better understanding of the concepts and relationships between the tape diagram and the generated equations. Ding et al., (2019, p. 127) and Ng and Lee (2009) also emphasized how ‘deep questioning’ and conversation were key factors in helping students deepen their understanding. All of the lesson examples contained the solved solutions with the tape diagram.

Teacher Training

Prior to the training, the author first visited the school to speak to the students and teacher regarding the study and to clarify any questions students may have had. The teacher was also provided hard copies of the instruments and all of the tests. The training for the teacher involved meeting with the author of this study. These trainings occurred over the author's school district Zoom account rather than in person due to the COVID-19 pandemic. The training lasted for 1.5 hours. During the training, the author walked through each example, modeled two lessons and then had the teacher model two treatment lessons of his choosing to the author. The teacher was provided the lesson plans that were based on the lessons from *Algebra 1: A common core curriculum* (Larson & Boswell, 2019). He was asked to look and examine the treatment lesson plans to ensure content validity. The teacher was able to draw during the training session, collect artifacts from the session and asked clarifying questions to achieve understanding of the concrete fading lessons. Through the training, he was able to understand how to draw the diagrams; how to involve students to draw a part of it; what questions should be asked to ensure students' understanding of the meaning of the tape diagram; connections between the diagram and generated equations; how to use diagrams to help students make sense of the computation process (i.e., when they add the same number on both sides and what it looks like on tape diagrams) for all the students in the treatment groups unlike the control group where students were asked to generate an equation right after reading the word problem.

The teacher was then observed by the author once throughout the sections (1.1-1.3) to ensure that the treatment was appropriately being implemented with fidelity in the classroom. The teacher covered everything that was discussed in the training session because he asked the students how the tape diagram related to the generated equation, what did the tape diagram represent in relation to the problem and why was the tape diagram split into unequal portions. The lesson flowed smoothly and the teacher executed everything that was covered in the Zoom training. Even though the teacher did not have formal lesson plans for either group besides the ones given to him, he faithfully implemented the lessons for the control group.

Measures & Variables

The study was designed initially as mixed-methods, but due to challenges in obtaining students' permission, this study analyzed the responses of eight students (three from control and five from treatment) who gave consent. With this limitation in mind, the analysis was more qualitative in nature. The measures were student work on all of the pre- and post-assessments and students' explanations of their thinking that were provided by the interviews. All of these tests were used to examine the extent to which the influence concrete fading had or did not have on student learning as well as how students thought differently in both the treatment and control group.

Test Items

The pre- and post-assessments slightly differed from each other with a total of three different versions and were created by the author. All assessments were the same across both classes and had a duration of a portion of the block (25-30 minutes) with each block being 87 minutes. All tests consisted of 12 questions that were aligned with the three lessons. The delayed post-test also included a question that asked whether students used technology, e.g., calculator. The questions consisted of five word problems and the remaining seven problems were algebraic equations. Please refer to Appendix C for the pre and post-tests. The word problems did not ask students to solve the word problem, rather, they asked for students to generate the equation that modeled the situation. They were also provided a hint that drawing may be helpful so that students may want to draw the tape diagram. Since the control group did not use a tape diagram, the directions could not ask students to draw a tape diagram to generate an equation from the word problem. Students were also asked to generate equations because of the struggle they may have had with comprehending word problems such as the ones they encountered with the students-and-professors problem while trying to generate an equation (Clement et al., 1981; Fisher, 1988). One way of student thinking that led to the challenge in the study conducted by Clement et al. (1981) was that students would place the variables and numbers in the exact order in which they were read in the problem. Additionally, the algebraic equations at the end of the assessment of the current study involved students solving for the variable.

Interview Items

The interview questions for students focused on student thinking, how compelled they felt to use one technique over another, what did the students find valuable when solving the problems and what happened if they got stuck on a problem. Please refer to Appendix D for the complete interview items. The purpose of this study was to investigate whether students in the treatment group thought differently and their differences while they solved linear equations and algebraic word problems. Thus, the interview questions asked students how did they solve certain word problems and symbolic equations; how were they thinking to solve the word problems when they first read it; why did they choose that particular method to solve the problems and/or generate an equation. These prompts explored student thinking because the questions asked when students saw certain problems and why did they solve the problem the way they did. This is further discussed in the Results chapter.

Data Collection & Procedures

The data collection included a pre-test, immediate post-test and a four-week delayed post-test. The pre-test assessed what students knew prior to the lesson. The immediate post-test assessed how well they knew the material right after the lesson. The delayed post-test assessed students' skill proficiency (i.e., their skill of solving equations including inverse operations, combining like-terms) and how much information was retained after learning more topics in class.

All of the pre- and post-tests used had the same structure and concepts, but differed slightly with a few changes with numbers and contexts. The story problems were of the same format on all of the tests and as in the examples shown in class, but the quantities changed such as number of students, the tip, the total bill. In other words, the mathematical content of items on the tests were aligned with examples from the class. Also, all of the test questions were modeled after the examples done in class where the only difference was the examples in class asked for an answer and the assessments simply asked for a generated equation for the word problems.

Pre-test

The pre-test was taken prior to the lessons to assess students' current level of solving linear equations and algebraic word problems as well as their prior knowledge.

Immediate & Delayed Post-tests

The immediate post-test was taken immediately following the lessons (elaborated upon in Results). This post-test assessed understanding of solving equations and generating equations represented by a story problem. The four-week delayed post-test assessed skill proficiency (e.g., combining like-terms; applying the correct inverse operations). The types of wrong answers that students provided on tests were evaluated, not just scored as 'incorrect', as a means of exploring differences. The final post-test was very important as this was when students would have begun a whole new unit and revealed the impact concrete fading had on students' skill proficiency over a longer period of time. However, concrete fading is aimed towards enabling students in the treatment groups to a better, stronger and more understanding with concrete contexts and then apply that understanding to a tape diagram, which is further discussed.

Student Interviews

In addition to the assessments, three to five students from each condition agreed to be interviewed, with a total of eight students. These students who were interviewed agreed to participate. These students were from both conditions: treatment group and control group. The purpose of these interviews was to find out: why students chose to solve problems the way they did (i.e., what was their thinking behind trying to solve the problem using that technique); did the techniques in class make them think of the problems differently when they were solving them from pre to post; and do they think differently on the post-test than on the pre-test?

All of the student interviews were audio recordings conducted through the author's school district Zoom and Gmail accounts (camera was disabled) and the cloud recording and transcription features on Zoom were used. That audio was then downloaded to view the transcript. Subjects' names and faces were not shown and remained confidential. The interviews took place during their class period in the hallway right outside of their classroom. The author's computer was used as the student was interviewed in the hallway. The students were a little distracted when their classmates walked in and out of the classroom. Please refer below for the student interview questions. Appendix D provides all of the students' responses to all of the questions.

1. What were you thinking while solving #'s 2, 3, and 7?
2. What did you find helpful when you got stuck solving a problem?
3. When you were first saw the word problem (any from #'s 1-5), how were you thinking of how to generate an equation that models the situation?
4. Why did you feel compelled to use the techniques taught? If you did not use the techniques taught, then why did you feel compelled to not use them?
5. What's changed about your understanding? What caused the changes?

Data Analysis

Quantitative Analysis of Student Tests

This data included analysis of the percentage correct and incorrect of the pre- and post-assessments. Students' work was also looked at and partial credit was awarded when possible, i.e., arithmetic errors and algebraic errors. Arithmetic errors were defined as students who have the correct work, but made an error in simplifying, i.e., 8/2=5 (e.g., student 4's work shown in Figure 6); however, full credit was awarded if they got the correct answer with no work (e.g., student 8's work shown in Figure 7) because the directions did not state to show work. Algebraic errors were defined as students who applied the correct inverse operation but attached a variable to it, i.e., divide by 3x or applied the correct inverse operation, but combined like-terms wrong (e.g., student 6's work). These types of errors in the student work suggests that the student know how to solve the problem, but perhaps made a mistake; thus, students were awarded partial credit depending on the error, e.g., algebraic or arithmetic. Some of the students specified that they had scrap paper, but none of the students turned it in. Below is Figure 6 which shows student 4's error on his pre-test where he received partial credit of .25 points because he did solve it correctly with the exception of his arithmetic error.

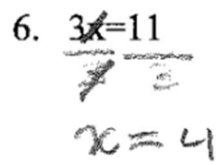


Figure 6. Student 4's Work on Pre-test; Arithmetic Error.

In addition, student 8's work is shown below in Figure 7 from her immediate post-test where she got most of the answers correct. Even though her work was not shown and specified in the interview that her work was on scrap paper, she was given full credit on the second round of grading because the directions did not specify to show work.

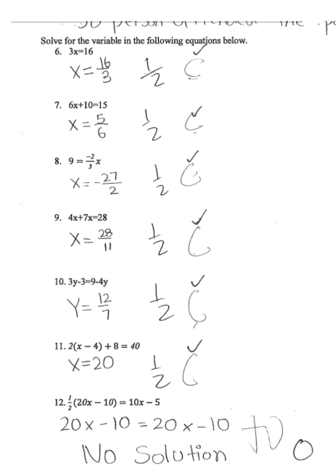


Figure 7. Student 8's Immediate Post-test.

Grading & Reliability

The author graded each student's test twice and also had a colleague grade. The first and second round of grades looked at by the author changed for some students because some answers were scored incorrect when the answer was correct or vice-versa. Also, the tests did not ask to show work, so when students had the correct answer, but no work, they were initially awarded only half credit on the first round, but then given full credit on the second round of grading after looking at the directions. After the author graded twice, then her colleague graded the tests. The names and student id's (if provided) were covered with a post-it and written on to ensure confidentiality. There were a few discrepancies where he awarded .25 or .5 points when the author gave 0 points. The second grader awarded partial credit if students had some of the numbers in the generated equation correct even though the rest of the equation was incorrect. The only partial credit awarded by the author was for arithmetic and algebraic errors. If students made arithmetic errors they earned .25 points partial credit and algebraic errors earned them .24 partial credit. These numbers were chosen so that when the students' scores are viewed, the audience can tell if it's 2.24/12, then a student got two problems completely correct with an algebraic error. The students either answered the question or left it blank; therefore, there was not an instance where students made four algebraic errors that would equal up to 1 point. If there are whole numbers, that means the student got the whole problem correct. Partial credit of these two errors was only awarded if the work related to the problem, e.g., the inverse operation was correct, but the final answer was wrong; or the inverse operation was correct, but attached a variable onto it.

Qualitative Analysis of Interview & Test Data

The qualitative analysis included an analysis of the interviews of the eight students from each class as well as analysis of student work on all of the tests. The analysis for the interviews involved transcribing the audio recordings of the interviews. Students' interview responses and test responses were then categorized into: errors (e.g., dropping the equals sign or negative sign, arithmetic errors); strategy choice (e.g., formal, informal, tape diagrams); skills used while solving the problem (e.g., correctly applying the inverse operations, combining like-terms, simplifying fractions); and if students referenced the real-world, concrete word problem as they progressed into semi-concrete and abstract problems. The work on students' tests suggested misconceptions where there were multiple errors of the same kind. The sources for the analysis were the same, but they were analyzed differently. Similarly, students' work on the word problems and interview responses were used to analyze RQ1 to investigate how both conditions think whereas student interview responses were used to analyze RQ2 to see how the differences seemed to be related to the intervention. Not all of the categories were used to analyze the interviews as most of the data mainly offered insights on their strategy choice and skills. Below I elaborate each type of category qualitatively analyzed. There are some connections among the subcategories; however, they will be separated into different sections because there are slight differences among the four.

Errors & Misconceptions

Errors were placed into categories such as the ones discussed by Booth and Koedinger (2008) and Booth and Davenport (2013) as detailed above in the literature review where students dropped the equals or negative signs as well as arithmetic errors such as writing 8/2=5. As detailed in Results, many students made arithmetic and algebraic errors. Student 4, for example, wrote that 11/3=4 as shown above, which fell into arithmetic errors. Since this error only occurred once with this student, I called it an error. If the error happened multiple times, I tagged it as potentially indicating a misconception and attempted to confirm that via interview data. If I was unable to, I do not make a determination, but will describe possibilities in the discussion section.

If multiple errors were made, they categorized into the type of potential misconceptions if their work suggested any (e.g., equality sign, simplifying expressions; Bush & Karp, 2013) along with examining if their work suggested if they understood the quantitative relationship behind the equations for the word problems. Specifically, did the students' work suggest: they treated the equals sign as an operator (e.g., 3+4=7*9=63); they switched the value of the variables suggesting that they thought one variable should be the greater number; they simplified expressions correctly and combine like-terms (e.g., if the expression was 3+4x, did they write 7x instead of 3+4x; i.e., thinking that 3 and 4x are like-terms to combine?)? Student 2's error, for example, suggested a misconception with the equals and negative sign. Another misconception that was suggested throughout the tests was simplifying expressions.

Strategy Choice

Additionally, strategy choice was categorized to investigate if students in the treatment condition adapted the tape diagram to their solution strategies or did they use informal strategy, specifically with word problems. Informal strategies involve solutions that do not involve using 'algebraic symbols' such as the 'unwind strategy' where students work 'backwards' to solve a problem by starting with the information they were given or guess-and-check where students plug in values and try to find what number will give them the answer (Koedinger et al., 2008, p. 370) that was more effective to use than a formal strategy (e.g., ‘algebra equations’; Koedinger et al., 2008, p. 370) or did they use formal strategies of solving? I wanted to see the progression of strategy choice from pre to immediate to delayed post, especially for the treatment group to see if their thinking changed. Also, the interview responses were also able to provide insight into strategy choice.

Skills

The skills students applied were then analyzed for improvement on all of the tests, i.e., did they incorrectly apply the skill on the pre-test, but corrected it on the post-tests or vice-versa? This analysis involved looking if students were able to solve linear equations, e.g., correctly applying the inverse operation, distributing correctly, combining like-terms. Analyzing skills were important because it suggested how students’ skills in the treatment group were affected by concrete fading. I also wanted to see how students improved or declined in their ability to apply inverse operations, distributing and combining like-terms across all of their tests for both conditions. As detailed in Results, most of the students, e.g., students 1, 2, 4, 7, and 8, were able to correctly apply skills when solving linear equations on the immediate post-test, but then some students, especially 1 and 2, incorrectly applied the skill on the delayed post-test.

Real-world Reference

To determine if students referenced the story problem as they progressed onto less concrete and more abstract problems, they had to explicitly state that they referred back to the story problem or relate the subsequent problems to the initial story problem, which no student stated.

Connecting Tests & Interviews

The analyses of the interviewers were then compared to students' work on their assessments to see if there was a connection between their responses and work as detailed in Results.

CHAPTER 4

RESULTS

In particular, this study addresses the two questions: What are some ways that students who received concrete fading think differently than the control group? How do those differences seem to be related to the intervention? I first present an overall picture based on student data and then report detailed findings to both research questions.

Results of Grading of the Tests

Table 6 below displays the students' test scores. A dashed line indicates that that student did not take that particular test. The scores are separated into treatment and control group showing the scores that each student received on each test. Table 6 also displays each student's score on both tasks: the word problem tasks and the solving equations task. This table indicates that students scored higher on the equation solving and received very little to no credit on word problems in both the treatment and control groups. Additionally, Table 6 indicates how students in the control group made more arithmetic errors than the treatment group that made more algebraic errors. There were five word problems and seven equations on all of the tests.

In addition, table 7 below summarizes the types of errors and misconceptions that the tests suggested provided by the students' tests as well as their strategy choices. If the cell is left blank, then there was not enough information provided by the student test. For instance, there was no evidence that the students referenced real-world problems based on their interview responses.

Table 6

Students' Test Scores

Student	Pre- Test	Immediate post-test	Delayed post-test
Control Group			
Total Scores			
1.	3.75/12	3.25/12	1.25/12
2.	1/12	5.49/12	1/12
3.	0/12	-----	0/12
Word Problems			
1.	1/5	2/5	1/5
2.	1/5	.25/5	1/5
3.	0/5	-----	0/5
Equation Solving			
1.	2.75/5	1.25/5	.25/5
2.	0/5	5.24/5	0/5
3.	0/5	-----	0/5
Treatment Group			
Total Scores			
4.	4.73/12	4.72/12	7/12
5.	.24/12	0/12	7/12
6.	----	.48/12	.24/12
7.	----	6/12	6/12
8.	6/12	7/12	6/12
Word Problems			
4.	1/5	1/5	0/5
5.	0/5	0/5	0/5
6.	----	0/5	0/5
7.	----	0/5	0/5
8.	0/5	1/5	0/5
Solving Equations			
4.	3.73/7	3.72/7	7/7
5.	0/7	0/7	7/7
6.	----	.48/7	.24/7
7.	----	6/7	6/7
8.	6/7	6/7	6/7

As indicated by Table 7, students in the control group made more arithmetic errors while the treatment group made more algebraic errors and struggled with combining like-terms. Also indicated by Table 7, many of the students in both conditions used formal methods as their preferred strategy choice. Some students chose to use the

Table 7

Types of Errors and Strategy Choice

Student	Error with equation solving: (Dropping equals sign/negative sign; arithmetic error)	Strategy choice: (formal, informal, tape diagram)	Skills: (Inverse operation, combining like-terms, simplifying)	Reference real-world problems
		Control		
1.	Arithmetic error: Pre, Immediate, post on #6: performed correct inverse operation 11/3=20 on pre-test 16/3=3.3 on immediate 10/3=1.1 on delayed	Formal	Did not distribute	
2.	4x+7x=22 4+7=11*2=22 on pretest 3-9=6 on pre 9-30=21 on delayed	Formal	Combined like-terms incorrectly 3x/11 on pre	
3.			Multiple skills incorrect	
		Treatment		
4.	-8+8=16 on #11 on immediate post 11/3=4 on #6 pre-test	Formal	Combined like-terms incorrectly (combined variables with constants; -(6x-10=4x)	

Table 7

Continued

Student	Error with equation solving: (Dropping equals sign/negative sign; arithmetic error)	Strategy choice: (formal, informal, tape diagram)	Skills: (Inverse operation, combining like-terms, simplifying)	Reference real-world problems
		Treatment		
5.	Dropped variable when writing equation for #2 on pre-test	Formal and tape diagram on immediate post-test (used incorrectly)	Combined like-terms incorrectly	
6.	Divide by 3x and not 3; when subtracting, would combine constants and variables	Formal and tape diagram on immediate post-test (used incorrectly)		
7.		Formal and tape diagram on immediate post-test (but used incorrectly)		
8.		Used broken tape diagram on immediate post correctly for #3 Used constructed circles on pre and immediate for #1	Did not distribute correctly on #12 for any of the tests	Does not speak English well, so she struggled with real-world problems

tape diagram and the interviews allowed me to see why and how they implemented the tape diagram to show how the conditions think differently.

Word Problems

Generally, the students did not generate the equations for the word problems as the prompt requested and showed little to no improvement over time. They typically wrote equations in an attempt to find a numerical solution where seven of the eight students wrote an equation that had no obvious relationship to the word problem. The researcher could not see a way that the equation might represent or model the situation described in the word problem. For example, student 3, began to write absolute value equations that were unrelated to solving the type of equations necessary to solve the word problems on the delayed test. That notation could have been part of the current or recent topics she was learning.

The interviews provided some explanation for these findings. The interviews suggest that students struggled with word problems and believed they were to determine a solution, not just write an equation to model the situation. Student 1 said the word problems were difficult to understand, but he was trying to figure out ways to solve the problem. Student 4 felt that the word problems were difficult to solve even though he underlined key words and numbers to try to help him solve and the prompt did not even ask to solve, just to generate an equation.

Equation Solving

Through analyzing student work, the errors students made were categorized into either arithmetic and algebra as described above in Table 7. The control group did not make as many algebraic errors as the treatment group with one student making two arithmetic errors. The students in the treatment group, however, made more algebraic errors with three students making atleast two of these errors. The following subsections further elaborate every student's work as well as the differences between the two conditions.

RQ #1: What are Some Ways That Students Who Received Concrete Fading Think Differently Than the Control Group?

Students in the treatment group either attempted to use or claimed that the tape diagrams were helpful in their attempts to complete word problems. Students in the control group did not reference any informal, concrete, or semi-concrete ways of reasoning. In the following section, I will report this research question in two separate subheadings: word problems and equation solving. These subheadings are then reported by condition. In each condition, I discuss students' pre-tests, immediate post-test and delayed post-test. Students' work on the pre-tests are included to show the change from or differences from before the intervention to after the intervention.

Word Problems

Overall, students' work along with their interview responses showed differences in students' thinking because when trying to solve word problems, some of the students in the treatment group drew tape diagrams or a similar drawing unlike students in the control group. Students in the control group did not draw any type of diagram. They used numerical methods in an attempt to get an answer. I define numerical methods as applying operations with numbers, i.e., mathematical calculations. Even though all of the students' implementation of the diagram was incorrect and most stated that they preferred their teacher's way of teaching with equations rather than the tape diagram to solve word problems, their work demonstrated that the control and treatment group do think differently.

Control Group (Students 1-3)

Student 1 attempted to produce an answer via numerical methods, but did not provide the correct answer nor the correct operation. However, he was able to answer #2 correctly on all of his three tests (see Table 8). He mentioned in his interview that the word problems were difficult to understand and he was trying to solve the problem. He also said that he forgot a lot because the school went remote right after winter break because of the pandemic. Student 1's work from pre to the post-tests did not show much change in his thinking other than getting #2 correct on all of his tests.

Similarly, student 2 attempted to produce an answer using numerical methods, but they were not the correct answer nor were the correct operations used. He was able to generate the correct equation for #2 on his pre-test. On #3 of his immediate post-test, he wrote x=0.04x+48 when the daily grams of fiber were 38 (see Table 10 for the task). On his delayed post-test, he only got #2 correct. In the interviews, he mentioned in the interview that he would use the numbers for the word problems, but forgot the equation, which is suggested on his work with calculations that appeared to be random: "I was trying to remember, like, physically the…like, sometimes, I'll forget equations and like, it's just something that happens". He did not specify if #3 on his immediate post-test was a careless mistake when asked how to explain that problem. Student 2's work did show change from his pre to immediate post for #3, which is shown below in Table 8.

Student 3 wrote numbers in absolute value bars on her pre-test and then produced an answer using numerical methods on her delayed post-test for every problem, which were both wrong, in an attempt to generate an equation on her delayed post-test like with students 1 and 2. Student 3 did not complete an immediate post-test. In her interview, she stated:

> I just basically, like, tried my best and, like, used the techniques we learned during math class, and, I'm going to be honest, I'm not really good at math so, I just tried one of the techniques to get my answer.

and when she got stuck, "I, like, studied a little bit from my notes". Thus, student 3's work did not change from her pre-test to her delayed post-test.

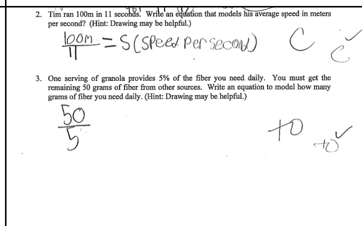
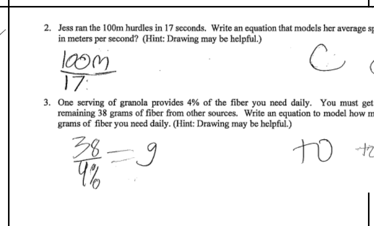
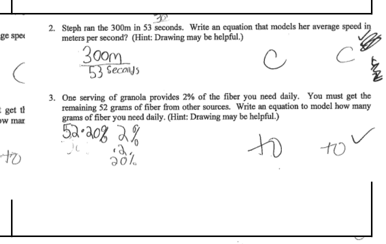
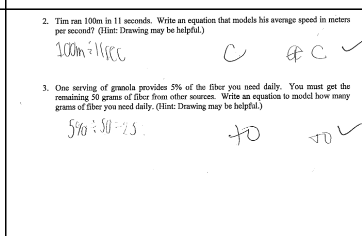
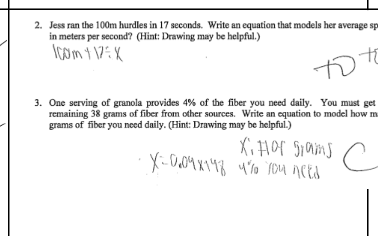
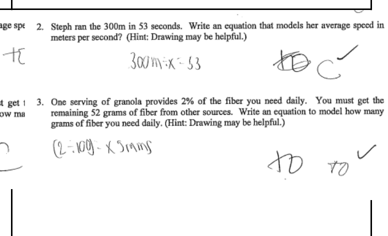
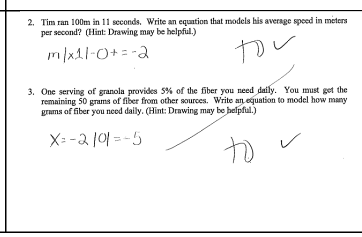
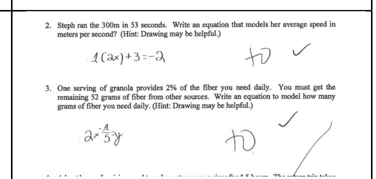
The similarities among these three students (students 1-3) in the control group is that they struggled with trying to generate an equation from a word problem. Students 1 and 2 forgot how to solve the word problems while student 3 tried to use the techniques shown in class. The techniques refer to how the teacher typically taught word problems with linear equations, that are in line with the textbook where the teacher would read the problem and generate an equation based on the problem read. Student work in the control group of the of #'s 2-3 of each test is provided in Table 8 below.

Treatment Group (Students 4-8)

Student 4 used numerical methods in an attempt to produce an equation, but only got #2 correct on the pre and immediate post-test. His delayed post-test contained one number for each question but with no work. In his interview, he stated that "three weeks ago, I would have not understood that at all, but with the new technique that I've learned, this is easy for me now." The new technique referred to the concrete fading and tape diagram because the tape diagrams are new to both the student and teacher as detailed in the next chapter. Student 4's preference for the tape diagram affirms what Ding et al. (2019) found where the 'new technique' can be related to the 'power' of the tape diagram. Even though he stated that the new technique allowed him to solve equations

Table 8

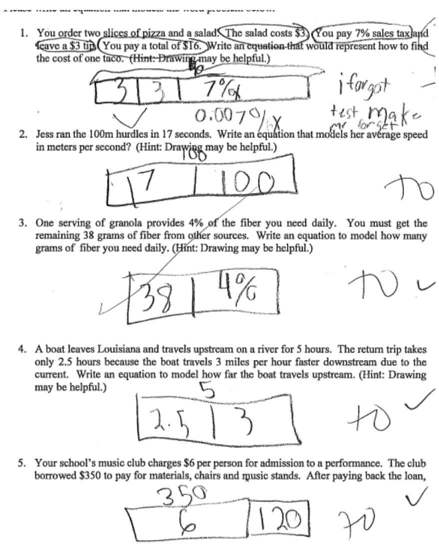
Control Group Work of #2 and #3 of All Tests

Student	Pre-test	Immediate Post-test	Delayed Post-test
Student 1			
Student 2			
Student 3			

more easily, he did not use it on either of the post-tests and still preferred to use the teacher's traditional way of solving using word problems. Student 4 also stated that he would underline key words to try to help him the word problems, specifically for problem #1 by looking at tip and tax. Underlining the key words gave him an idea of what he needed to solve. The teacher did not explicitly say to underline keywords, but the first question he did ask was what information was given and what is the problem asking students to solve. Student 4's work from pre to immediate did not show change, but from pre to delayed post did show change where he did not get #2 correct.

Student 5 also used numerical methods on her pre-test to try generate an equation, but was unsuccessful. On her immediate post-test, the only question she made an attempt to answer was #3 by constructing a tape diagram (see Table 9). #3 on the immediate post-test asked: *"One serving of granola provides 4% of the fiber you need daily. You must get the remaining 38 grams of fiber from other sources. Write an equation to model how many grams of fiber you need daily"*. However, to construct a correct equation, students have two options. The first option is that they can use two tape diagrams: one to represent the total (i.e., one side of the equation) and the other tape diagram to represent the fiber needed along with the other fiber sources (i.e., the other side of the equation). The second option is that they can draw one tape diagram that contains two parts: fiber from granola and from the other resource with the total being labeled as 'x' so that it would read: (1-4%)x=38. The issue here seems to be that she was not clear about the quantitative relationship, i.e., total= from granola + from the other. In addition, she does not know that the whole tape represents the quantity of total, 'x', and she does not know how to present two parts of fiber, i.e., 4%x and (1-4%)x, which is similar to students 6 and 7 below. On student 5's post-delayed test, she wrote one number down for each word problem with no work supporting how she got those numbers. In her interview, she did not really remember how she got her answers, but she said for this particular problem she was thinking of percentages since the problem included percentages. Also, she stated when she got stuck on a word problem, she would use the boxes to help her answer the question because drawing something out helped her more than just looking at the problem. In particular, when she was trying to explain how to generate an equation for #3, she said that 38 should be bigger than 4%, which coincidentally was correct for this problem; thus, her knowledge with expression (e.g., using 4% x to represent the daily fiber needed) hindered her use of tape diagrams. Student 5's work did show change from pre to immediate where she attempted a tape diagram as shown in Table 9, but from pre to delayed post, there was no change.

Student 6 did not complete a pre-test, but on his immediate post-test, he constructed tape diagrams for every word problem. His tape diagrams were incorrect because he did not include everything in the tape diagram for #1. For #2, he included the distance and time in one box. The time is not being multiplied to a variable either. He is missing the variable and how the tape diagram should be split into 17 equal sections. On #3, he constructed his tape diagram incorrectly, just like with student 5. He could have constructed two as described above or one tape diagram. For #4, his tape diagram includes the hours (e.g., 2.5) and miles per hour (e.g., 3) in one tape diagram with the hours it would take going upstream on top (e.g., 5), but did not include a variable with these quantities because it was 3 miles per hour faster. Just like with #3, he could have constructed two tape diagrams or one. On #5, he put the price for one person (i.e., 6) with the profit (i.e., 120) on one tape diagram, which the profit and the amount borrowed (i.e., 350) should have been two sections on the tape diagram with 6x on top representing $6 per person. He was correct in that there should be two sections in the tape diagram and a number on top, but the numbers were misplaced and 6 needed a variable with it, i.e., 6x, where 'x' represents per person. Figure 9 below shows all of the tape diagrams student 6 drew on his immediate post-test. On student 6's delayed post-test, however, was blank and no attempt of the answer. When asked how the problems in the interview, he replied by looking at the numbers given and putting them in the box. The box referred to the tape diagrams as he pointed to them during the interview. For #3, he wanted to get the numbers out first because the words confused him. When he was trying to explain


1. You order two slices of pizza and a salad. The salad costs $3. You pay 7% sales tax and leave a $3 tip. You pay a total of $16. Write an equation that would represent how to find the cost of one taco. (Hint: Drawing may be helpful.)

2. Jess ran the 100m hurdles in 17 seconds. Write an equation that models her average speed in meters per second? (Hint: Drawing may be helpful.)

3. One serving of granola provides 4% of the fiber you need daily. You must get the remaining 38 grams of fiber from other sources. Write an equation to model how many grams of fiber you need daily. (Hint: Drawing may be helpful.)

4. A boat leaves Louisiana and travels upstream on a river for 5 hours. The return trip takes only 2.5 hours because the boat travels 3 miles per hour faster downstream due to the current. Write an equation to model how far the boat travels upstream. (Hint: Drawing may be helpful.)

5. Your school's music club charges $6 per person for admission to a performance. The club borrowed $350 to pay for materials, chairs and music stands. After paying back the loan,

Figure 6. Student 6's Tape Diagrams.

how he went about trying to answer the question, he said that one portion of the tape diagram should be bigger if one value is greater than another; however, his work and interview responses indicate that he did not understand that 4% is not the same as 4. Thus, his knowledge with expression (e.g., using 4% x to the daily amount of fiber needed) hindered his use of tape diagrams like with student 5 as well as an understanding of percents and ratio. He also stated in his interview that graphing it out, he was referring to the tape diagram, made it easier for him to see in terms of solving the problem. He said the tape diagram helped him solve problems more quickly because it used to take him about 20 minutes. Student 6's work from immediate to delayed post did show change because he drew tape diagrams on his immediate post, but then did not write anything on his delayed post-test.

Student 7 also did not take a pre-test, but attempted the tape diagram on #'s 1, 3 and 4, and then wrote a number without work for #2 and an incorrect equation on #5 on the immediate post-test. Her tape diagrams were incorrect because she did not include everything in the tape diagram for #1. #3, she did not include an 'x' with 4%, she just wrote 4% just like with students 5 and 6, but she did write the total, 'x', above. For #4, her tape diagram is constructed incorrectly because she does not have a variable being multiplied to the number like student 6. Her delayed post-test was very similar to students 4 and 5 in that she wrote one number without any work to support how she came up with those numbers for the word problems. In her interview, however, student 7 preferred to use what her teacher taught while solving abstract equations to solve word problems because she:

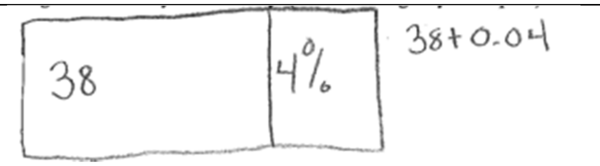
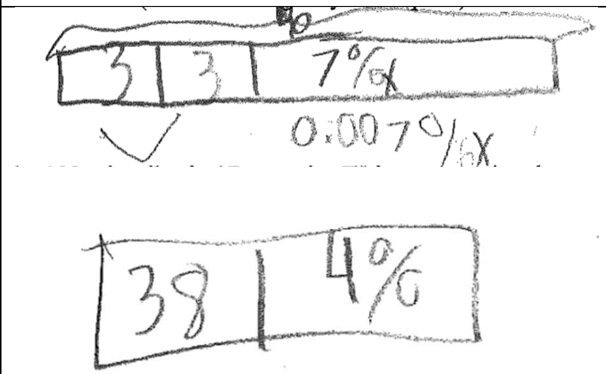
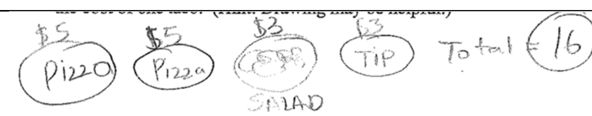
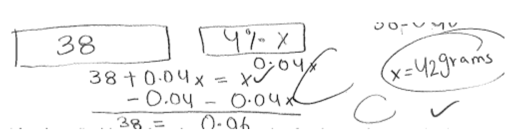
> felt it was more easier if [she] just used what [she's] better at, so [she] just used the things [she] was good at because if [she] used it as another way or another equation…another method [she] didn't think [she] was going to get it done correctly, so [she] just used the one method [she] was good at…the first method [they] learned.

Thus, she did not explain her thinking of why she used the tape diagram to help her generate an equation. Due to using a tape diagram, though, her thinking differed still from the control group. Student 7's work from immediate to delayed post differed because she constructed a few tape diagrams on the immediate-post-test, but then did not construct any on the delayed post-test. Please refer to Table 9 below for the tape diagrams of students in the treatment group.

Student 8 had a unique answer to #1 on her pre and immediate post-test when it asked to generate an equation: "*You order two slices of pizza and a salad. The salad costs $3. You pay 7% sales tax and leave a $3 tip. You pay a total of $16."* There was no difference with the circles on either test. She had implemented drawn circles that were similar to the tape diagram, but did not fall into the definition for the tape diagram that this study proposes (see Table 9 below, the bottom row). The circles were a similar idea; however, the way she drew it, did not show the quantitative relationship correctly.

Table 9

Students' Tape Diagrams

Student 5 #3 on Immediate post-test	
Student 6 #1 on Immediate post-test #3 on Immediate post-test	
Student 7 #'s 1, 3, 4 on Immediate post-test	Used tape diagram on #'s 1, 3, 4 immediate post-test, but used incorrectly 1. 2. 
Student 8 #1 on Immediate post #3 on Immediate post-test	

In particular, she incorrectly assigned numerical values for the pizza because she did not include the tax, though, she showed that the total was $16. Similar to the results in

Clement et al. (1981), having circles rather than a tape diagram may not show the quantitative relationship, thus, possibly leading her to the incorrect equation. This is further discussed in the discussion section.

The rest of her questions on her pre-test had numbers without any work or numerical methods in an attempt to produce an answer. On #3 her immediate post-test, she used, what appears to be a broken tape diagram and generated the correct equation. She correctly constructed where one portion of the tape diagram was 38 and the other portion was 4%x unlike her constructed circles where incorrect values were assigned to some of the food. The only part that was missing was the tape diagram that was labeled 'x' to show the total number of grams. So it is unclear if she knows which part of the tape diagram denotes 'x'. The other problems she either wrote a number without any work to support it or 'I don't know!'. On her delayed post-test, she wrote out the information for #1, but some of the numerical methods did not apply to the problem. In her interview, she stated that she does not understand English very well, so trying to come up with an equation may have been difficult for her. Student 8's work differed, however, from pre to immediate because she did not draw a tape diagram on the pre-test whereas her immediate post-test she used a broken tape diagram for #3. On her delayed post-test, she did not draw anything.

Comparison to control & treatment groups. As discussed above, the differences between these two conditions' thinking are that students from control groups did not even try to complete the problems or attempt to explain what they wrote because they said they forgot or just did not know what to do whereas students in the treatment group attempted to explain what they wrote and answered as well as the methods they used from the immediate to delayed post-test. The treatment group attempted to answer more questions on the immediate post-test than the control group. Four students used a tape diagram or what seemed to be a broken tape diagram even though three of them preferred to use equations to solve word problems. Students used the tape diagram to try get the equation for the word problem, e.g., students 5 and 6. Similarly, some of the students in the treatment group had positive thoughts towards the tape diagram saying that it helped them see what was going on in the problem. The control group attempted to answer using numerical methods to try to provide an answer. They were not able to answer how why they wrote the numbers they did or explain how they got the correct equations that are represented in the word problem. However, the results of the delayed post-test showed students in the control group used numerical methods in an attempt to produce an answer whereas some students in the treatment group wrote a number without any work as the answer for the word problem. Thus, the thinking between the two conditions does differ at different stages. In particular, at the immediate post-test where the treatment group generally used the tape diagram or another method that was similar to the tape diagram whereas the control group used numerical methods in an attempt to answer the question.

The work that students did suggests that their understanding goes deeper than the fundamentals of solving equations. As seen through students 5 and 6, their work and interview responses indicate that they struggled with percentages, ratios and representing expressions. With these struggles, it could have made it more challenging for them to implement the tape diagram because of its semi-concrete features, which is further discussed in the discussion chapter.

Equation Solving

The equation solving results are reported below and separated by condition.

Control Group

Student 1 was able to answer the first three equations correctly and he applied the correct inverse operations to solve for the variable on his pre-test; however, made an arithmetic error on #6 where he divided 11 by 3, but the rest of his work his wrong. He also had the correct answer for #11, but his work was wrong. His work does suggest he may have used the guess and check method (Koedinger et al., 2008), but did not confirm nor was he asked in the interview. His immediate post-test, he was able to answer two correctly. On #6, he applied the correct inverse operation by dividing 16 by 3, but his answer was 3.3 rather than the correct answer of 5.3 repeating. On his delayed post-test, he applied the correct inverse operation again for #6, but the final answer was incorrect because he said the answer was 1.1, when the correct answer was 3.3 repeating. He mentioned in his interview that the pre-test was hard, but the lessons helped him. However, after going remote, he said forgot everything when it came to the delayed post-test. Student 1's work over the duration of the study changed in that he got more answers incorrect from pre to immediate to delayed post-test.

Student 2 did not answer any of the equations correctly on the pre-test. Based on his work, he seemed to struggle with the equal sign and subtracting numbers where the difference should be a negative number. Like with the word problems, he used numerical methods in an attempt to get an answer. On #6 he was supposed to divide 11 by 3, but he wrote 3x/11 instead. For #9 the problem was 4x+7x=22 and he wrote on the next line: 4+7=11*2=22. He got the correct value for 'x,' but his work and process were incorrect.

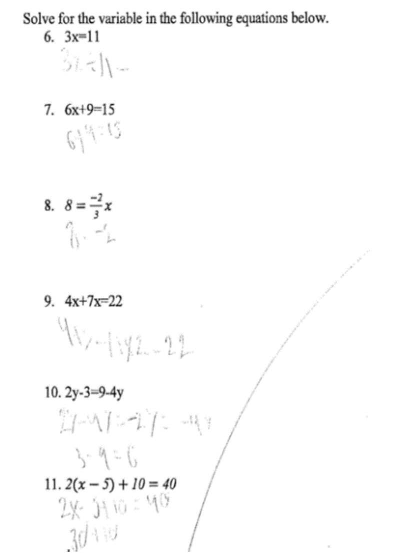
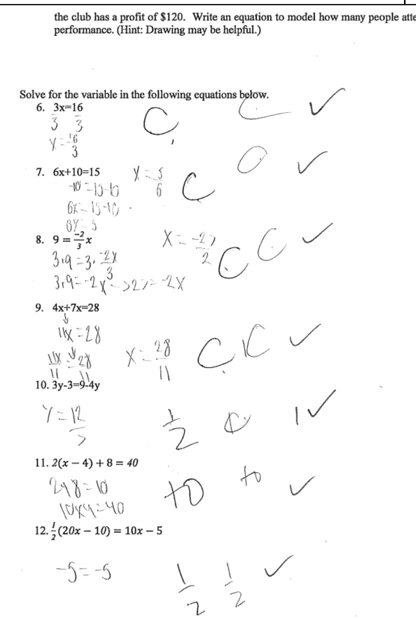
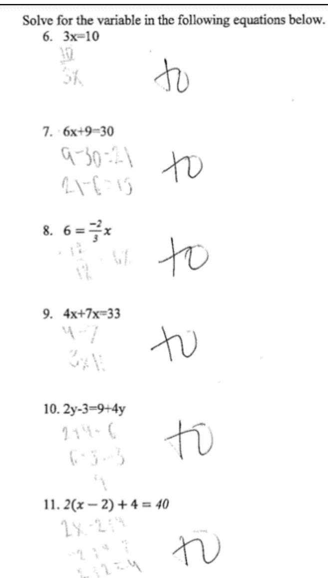
On #10, he was using numerical methods, but wrote 3-9=6. On his immediate post-test, he answered all but two questions correct. He combined like-terms and simplified correctly. He did not solve #11 (#11: "*2(x-4)+8=40"*) correctly because he used numerical methods when he wrote 2+8=10 and 10*4=40. However, he was in the right direction with his algebraic error on #12 when he wrote -5=-5, which is the equation to show that the solution is 'infinitely many solutions', but did not write that. On his delayed post-test, he used more numerical methods to try to solve like he did in his pre-test. Notably on #6, he divided by 3x rather than 3. Also on #7, he did not have a negative sign when he wrote 9-30=21. In his interview, he stated he found a number to write, but forgot the equations. Student 2's work did show change because from pre to immediate post-test shows that he made huge improvements with equation solving, but then did not get any of the equations correct on his delayed post-test.

Students 1 and 2 performed significantly better from pre-to immediate, but then performed the worst on the delayed post, which I illustrate with student 2's progression of tests in Table 10 below. An example of his work is on #9, when he rewrote the problem to 4-7 when it read 4x+7x on the delayed post-test. As shown in the Table 10 below, he answered the majority of the abstract equations correct on the immediate post, but then really declined on the delayed post.

Student 3 did not write anything on her equation solving portion on her pre-test. Her delayed post-test consisted of numerical methods to try to solve for the variable. She said in her interview that she tried her best to solve, but is not good at math. Student 3's work did change because her pre-test was blank whereas her delayed post-test, there was an attempt to write an answer.

Table 10

Progression of Student 2's Tests

Student 2's (control) Pre-test	Student 2's (control) Immediate post-test	Student 2's (control) Delayed Post test
 Solve for the variable in the following equations below. 6. 3x=11 7. 6x+9=15 8. $8 = \frac{-2}{3}x$ 9. 4x+7x=22 10. 2y-3=9-4y 11. $2(x-5)+10=40$	 the club has a profit of $120. Write an equation to model how many people atte performance. (Hint: Drawing may be helpful.) Solve for the variable in the following equations below. 6. 3x=16 7. 6x+10=15 8. $9 = \frac{-2}{3}x$ 9. 4x+7x=28 10. 3y-3=9-4y 11. $2(x-4)+8=40$ 12. $\frac{1}{2}(20x-10)=10x-5$	 Solve for the variable in the following equations below. 6. 3x=10 7. 6x+9=30 8. $6 = \frac{-2}{3}x$ 9. 4x+7x=33 10. 2y-3=9+4y 11. $2(x-2)+4=40$

The challenges for students in the control group were algebraic errors and negative signs for equation solving. Students 2 and 3 generally struggled with where to start to solve equations, inverse operations and combining like-terms whereas student 1 struggled with simplifying when dividing. Though some got some of the problems correct, they were inconsistent which problems were correct across all tests.

Treatment Group

Student 4's work suggested he struggled more with algebraic errors that differed a lot from each other, though, he did make a couple of arithmetic errors. On his pre-test, he correctly applied the inverse operation and simplified on three equations. On #6, he was correct when dividing by 3, but then simplified incorrectly. For #8, he did not multiply by 3 on the other side, only to one side of the equation. On #11, he distributed correctly, but combined like-terms wrong. His immediate post-test, he solved three equations correctly just like on the pre-test, but on #'s 11-12 he distributed correctly, but then combined like-terms incorrectly. His delayed post-test, he answered all of the equations correctly. As he stated in his interview, "I pretty much used the simple technique for Algebra I, which is just to solve it: 6- on each side and solve the solution." The simple technique refers to the traditional way of learning mathematics by using equations. In other words, student 4's response suggests that students think about equation solving as a process that may be reinforced by the teacher, not as a relationship among quantities. So, he may have learned that equation solving should be the primary method to solve problems. Student 4's work changed from pre to post where he made more algebraic errors on his immediate post than pre, but then got all of the equations correct on the delayed post.

Similarly, student 5's work suggested that she made algebraic errors when she attempted to answer #9 and #10 on her pre-test. On #9, she applied the correct inverse operation, but then applied it to every term on the same side of the equation. On #10, she applied the correct inverse operation, but combined -4y-2y=-2y. For this particular problem, she did not subtract 2y to both terms on the same side of the equation like she did for #9. She did not answer the rest of the equations. Her immediate post-test was similar where she attempted four problems and the rest were left blank. She attempted to apply the inverse operations, distributed and combined like-terms, but did not do any correctly. On her post-delayed test, she got all of the problems correct. In her interview she said she would multiply 6 and 9 and “stuff”. She did not provide any more insight into her sudden and significant improvement. Similarly, her work does not change from pre to immediate as much as it did from immediate to post-delayed. Her work on her pre-test showed an algebraic error where her work on her immediate post did not show either type of error.

Student 6’s work also showed he struggled with algebraic errors when he attempted to answer three equations on his immediate post-test. On #6, he divided by 3x rather than 3; however, on #8, he divided by 3 on one side but not the other, which was not the correct inverse operation to apply. His work also suggests that it could be that the operation was applied incorrectly rather than the incorrect inverse operation was applied. On #9 he applied the correct inverse operation but then combined like-terms incorrectly when he wrote 28-7x=21. He then divided by 4x rather than 4 like he did with #6. For #9, he could have combined like-terms first and then the inverse operation, but the inverse operation was still correct. The problem was 4x+7x=28, and he subtracted 7x first rather than writing 11x=28. He did not attempt to answer the other four equations. On his delayed post-test, he wrote ‘x=’ a number but did not have work to support his answers. On #9, he did the same thing he did on #9 on his immediate post and got the answer wrong. The problem read: “*4x+7x=33*”. He tried to subtract 4x from both sides of the equation, rather than combine 4x+7x. Then he incorrectly combined and wrote: 33-4x=29. Interestingly, on his next line he wrote 7x=29, he divided by 7 and not 7x this time. In his interview, he stated that he divided by 3x, not 3, but it was unclear if he was describing what he wrote or if he really thought to divide with the variable. He also stated for #7 on his immediate post-test, he would flip 6x to -6x and divide by 9 if he could, when asked how would he solve that since he did not solve it on his test. For #9, he said he would try to get x and since there were two x's, there could only be one. He went on to say he could either do "4-7 or 7-4". He did 7x-28 and tried to get the answer 4x=21 and divide by 4x on each side to get the answer. He said for more than one problem to divide by a number and its variable. Student 6's work showed changes from his immediate to delayed in that he made more algebraic errors than he did on his delayed post.

Student 7 got all of the problems correct on her immediate post-test with the exception of #12. #12 involved distributing as well as identifying if the answer was 'no solution' or 'infinitely many solutions.' On her immediate post-test, she wrote that the answer was x=0 and on her delayed post, she wrote the infinity sign as her answer. Her delayed post-test results were identical to her immediate post-test because she received the same score and got the same problem wrong. When asked how she solved #7 in her interview she stated that she would try to get x by itself, she should subtract 6 first and then "proceed as it goes". Student 7's work did not change much from immediate to delayed post where she got everything correct except #12 on both tests.

Student 8 got everything correct on her pre-test except #12 where she changed her subtraction sign to a division sign then changed it to an addition as shown in Figure 8.

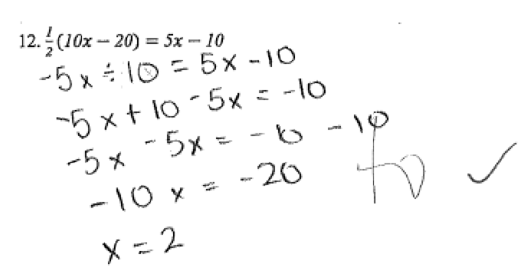


Figure 8. Student 8's Equation Solving for #12.

On her immediate post-test, she got everything correct except for #12 again where she wrote ‘no solution’. She specified in her interview that she had scrap paper, but it was not turned in. Her delayed post-test shows she got everything correct except for #12 where she wrote ‘no solution’ again. Student 8’s work from pre to immediate does not show much change because she only got #12 wrong on both as well as on the delayed post-test.

The challenges in the treatment group were more focused on algebraic errors rather than arithmetic errors in the control group. Many of the students in the treatment group were able to apply the correct inverse operation, but struggled with combining like-terms. Students’ work on their all of their tests showed the types of errors they made before, during and after the intervention.

Comparison to control & treatment groups. Overall, solving equations did not show much of a difference in thinking between the two groups other than students in the treatment group generally performed better or the same than the control group across all three tests. This suggests two interpretations. First, the intervention did not transfer from word problems to equation solving. This makes sense as students did not fully grasp the intervention with word problems. Second, students who performed higher at the beginning may not have needed the intervention as much or that the intervention allowed them to perform the same throughout. Also, the treatment group was able to explain in more detail how they solved equations unlike the control group based on the interview responses; however, based on their work, students of both conditions still struggled with combining like-terms, applying inverse operations and equations that resulted in ‘no solution’ or ‘infinitely many solutions’.

RQ #2: How Do Those Differences Seem to be Related to the Intervention?

The most pronounced differences between students in the treatment and control groups was in terms of their solving of word problems. The students' responses among the students suggest differences relating to the intervention. Below I mainly report how students' performance in the treatment group seemed to be related to the intervention.

The lessons in the treatment group focused on concrete fading in two different areas: the lessons (i.e., word problem to equation) and within word problem (i.e., problem situation to tape diagram to equation). The teacher executed everything that was covered in the training lesson very well in the classroom; however, students still struggled with word problems, in particular, with generating an equation to represent the situation.

Students Who Preferred Tape Diagram

Students in the treatment group who performed lower (e.g., students 5-7) on the word problems by not getting any problems correct preferred the tape diagram. As stated above, these students' responses indicated that the tape diagram helped them visualize and understand the concepts more rather than solving the problems in their head. The semi-concrete representation helped student 6 see what was going on and student 5's responses suggested that drawing something out was more helpful to her rather than just looking at the problem. Even though they implemented the tape diagram incorrectly, they still attempted to use it to solve and had positive thoughts about it. Their work for #3 was able to show that one portion of the tape diagram should be bigger than another portion as shown in Table 9, which was emphasized in the lesson. The teacher would ask during and after the tape diagram was constructed, why one portion is bigger than the other? The main difference is students' responses suggest that they not only saw it as another method, but also as a way to visualize (e.g., student 6) and draw out what was going on in the problem (e.g., student 5).

Student 7 also attempted to use the tape diagram on her immediate post-test; however, she said that she preferred to use traditional methods to solve word problems. As evidenced in her work, she did not resist using the tape diagram despite her preference for traditional techniques. Although these students' tape diagrams were constructed incorrectly, they were still willing to use it. However, during the interview, student 7 did not state how she used the tape diagram or why she drew it to help her generate an equation. She seemed much more focus on the equations rather than the tape diagram.

Students Who Used Other Methods

Students 4 and 8 differed in their responses related to the intervention in that they either did not use a tape diagram or their tape diagram varied from the ones presented in class. Both of these students got one word problem correct on at least one of their tests. Student 4's generated equation was correct even though he did not use the tape diagram on both the pre and immediate post-tests. In his interview, he stated that he preferred to use traditional methods to solve the word problem rather than use the tape diagram, which he did. His preference was unaffected by the intervention because, unlike student 7, he did not even attempt to use the tape diagram. All of the word problems in the lesson involved constructing a tape diagram. In contrast, student 8 got #3 by using, what appeared to be, a broken tape diagram, which was different from the tape diagram shown in the intervention lessons. Her broken tape diagram helped her generate the correct equation that was represented in the word problem.

Key Features of the Intervention

The work students showed on their tests including their attempts of the tape diagram suggest that they were trying to connect the tape diagram to the generated equation, which is one of the key features of the intervention. The tests were the same contexts except the numbers were changed provided in Appendix C.

Another key feature of the intervention was to have discussion in class with the tape diagram where the teacher asked questions that included: how does the tape diagram relate to the problem; how does it relate to the generated equation; why is one portion bigger or smaller than the other; why are the portions not equal; what does the whole tape diagram represent; what does each portion of the tape diagram represent? Students 5-7 attempted to answer these questions on their tests and explain in their interviews through the construction of their tape diagrams. Students 5 and 6 drew a tape diagram by constructing two boxes, where one portion was bigger than the other and described why they drew it. Similarly, student 7 drew tape diagrams to try to help her generate equation, but never described it in the interview. Student 8, though she did not use a full tape diagram, was successful in drawing a broken tape diagram where the portions were not connected, but was able to generate the correct equation.

Thus, the results from both of the tests and the interview responses suggested that students in the treatment group have another technique to solve (e.g., the tape diagram) and their differences in thinking were influenced because some thought it was a better technique to use. As revealed in the interviews, the tape diagrams helped them visualize what is going on rather than just looking that problem.

As seen through this study, students were trying to use the tape diagram and a couple did think it was useful, which shows the potential of the tape diagram in supporting these students. Their tendency and interests in using the tape diagram was attributed to the intervention as this was a totally new technique for them. However, their performance shows that they did not implement the tape diagram correctly, which did not contribute to their scores. This calls for further exploration in terms of how this tape diagram can be successfully taught, which is further discussed in the Discussion chapter.

CHAPTER 5

DISCUSSION, LIMITATIONS & FUTURE DIRECTIONS

Discussion

In this study, I aimed to answer the following questions: 1) What are some ways that students who received concrete fading think differently than the control group?; 2) How do those differences seem to be related to the intervention? The study suggests that concrete fading and tape diagrams can help students think more clearly about solving word problems despite that not translating to higher test scores. In terms of solution process, four students intended to use the concrete fading technique with tape diagrams to assist with word problem solving. Students in the treatment group scored much more consistently and higher on the delayed post-test than students in the control group. The majority of students in both groups also stated that they preferred solve word problems using traditional methods (i.e., equations) that the teacher used that were presented in the textbook rather than the intervention lessons, though, some did prefer to draw tape diagrams based on their interview responses detailed in the Results. Two of the students' responses suggested that it is because they are used to solving word problems by generating an equation, not using a tape diagram. Student 7 said that she was good at using equations and did not want to mess up her solution using another method. These results contradict what Nathan and Koedinger (2000a; 2000b) discussed where verbal problems would be less difficult for students to solve. A reason could be is that students did not fully understand the tape diagram (e.g., how to construct them, the purpose of them, how to use them) prior to moving onto abstract equations perhaps because of the duration of the study, their willingness to use the tape diagram on their own and how the students were not encouraged to use the tape diagram after the lesson. For various reasons, discussed below, students did not receive much credit for the word problems unlike the abstract equations.

The design of this study had conceptual replications to the studies conducted by Ching and Wu (2019), Ottmar and Landy (2017) and Fyfe and McNeil (2012) where there were two conditions: control and concrete fading, the sequence of the examples and the progression of the examples from concrete to abstract. The control group learned how to solve equations and word problems without tape diagrams and the sequence was the reverse of the treatment group, which had tape diagrams. The treatment group learned through a lesson that was in a different sequence, but the content covered and the problems were the same as the control group.

The purpose of this study was to investigate students' thinking depending on the type of instructional technique that they received. There were differences between the conditions and differences within the treatment group for the word problems. The majority of the students in the treatment group attempted the tape diagram and showed changes from pre to immediate post-test. Students in the control group differed from the treatment group, but not much from pre to immediate post-test within their own group. Below, I first discuss findings related to the research questions, followed by more discussions with concreteness fading and problem solving.

Differences In Thinking Between Conditions

Between the two conditions, students' test responses were different. Four students in the treatment group used the tape diagram and students in the control group used numerical methods to solve. The results obtained could suggest that students did not grasp the intervention, so it was hard for them to equation solve as well as the equations did not really promote the use of the tape diagram. Through students' work and interview responses of both groups, it was inferred students may have issues with both variables and expressions. Even though both conditions had the same conceptual difficulties, four students in the treatment group tried to use the tape diagram to visualize the problem situations. However, their prior knowledge with variables and expressions involving percents hindered their ability to use the tape diagrams correctly like with students 5 and 6. Student 6's work, in particular, suggested that he struggled with percentage and ratios as evidenced by his work discussed in the Results chapter. In addition, another issue seemed to be that they did not know how to represent the distance based on the quantitative relationship: time*speed=distance for #2. Similarly, they did not seem to know how to use variables to represent the quantities such as for #4 going upstream and downstream, where the distance for upstream is 2.5(x+3) and distance for downstream is 5x. It seems that the students remain in arithmetic thinking and cannot use variables and expressions to represent key quantities, which is the key to generating an equation. They did draw the diagrams, but they did not know how to show understanding of what each part of the diagram represents or how to represent them algebraically.

Thus, these differences seem to be related to the potential effectiveness of the tape diagram that Ding et al. (2019) discussed. Student 5 said that the tape diagram helped more because it provided a visual, which helps students better understand quantitative relationships (Ding et al., 2019). Even though students 5, 6, and 7 implemented the tape diagram incorrectly and student 8 did not draw the tape full diagram, what they all attempted to draw showed their attempts to inference and problem solve, which was likely inspired by the intervention that using the tape diagram to facilitate concreteness fading (Ding et al., 2019; Larkin & Simon, 1983).

Based on the results of the current study, it would potentially be more difficult for students to see the benefit of using the tape diagram, e.g., the mathematical relationship and making meaning of the math behind the equations. This could have played a part in their inability to see the 'mathematical relationship' (Murata, 2008, p. 391) while using the tape diagram. Even though the teacher was very excited and enthusiastic to teach solving equations and word problems in a different way, students were not able to see the relationship in word problems as well as the relationship between tape diagrams and equation solving as evidenced by their interview responses and work on their tests.

Another important finding to discuss is students' performance on the word problems within the treatment group seem to be related to the differences highlighted by Ching and Wu (2019) within the fading condition where lower-performing students benefitted greatly from concrete fading, but the benefit was unknown for higher-performing students. Because none of the students in the treatment group was high performing, I separated students into ones who received credit on word problems (e.g., students 4 and 8) using different methods other than tape diagrams and students who did not receive credit (e.g., students 5, 6, and 7) who used tape diagrams. Student 4 who received full credit on one word problem preferred the way the teacher normally taught, i.e., traditional methods including using equations rather than tape diagrams. Student 8 did get the problem correct using a broken tape diagram for #3 on her immediate post-test. Even though all of the students were considered low performing on the word problems, the ones who received no credit preferred the tape diagram. Student 8 also came up with a different way to try to help her solve #1, though her equation was wrong. Her work and responses indicate that she did not understand the relationships in the word problems as she said she did not understand and was confused. Her teacher did not provide much assistance and said to try her best. Therefore, it is possible she excelled in the abstract equations because there were no words to confuse her unless she already knew how to solve them.

This study also confirmed that concrete fading does provide students another way to think and four students seemed to favor this instructional technique, three of which did not get any of the word problems correct. In this study, the results did not appear as positive with the tape diagram as I predicted. There are several possible reasons. First, the top-performing students in this study may not have tried because perhaps they were confident and did not need to use the diagram to help. It turned out that they did not know how to solve the questions. Thus, students who were more confident solving algebraic equations preferred to solve word problems without drawings, but that could also be attributed to what their teacher expected on his tests in class. However, in reality, these students struggled just as much with word problems as evidenced by their low scores on the word problems. Thus, concrete fading will likely be a benefit for all students when solving word problems, which, of course, demands better ways to teach it such as focusing more on teacher training and other reasons that I will discuss later in detail. Second, because this study only covered three lessons, the number of lessons may have been insufficient for the lower-level students to become proficient with tape diagrams. It is important to note that the students likely did not grasp the tape diagram, which may have affected the 'concreteness fading' approach, perhaps because of the number of lessons. When the teacher was observed, he assigned problems for students to solve, but did not encourage them to use the tape diagram, nor did he allow time for students to try it on their own suggesting that students reverted to the way that are used to solving, i.e., with no tape diagram. Again, the concreteness fading approach does not mean a simple representational sequence where students can go from one stage to another without understanding the current stage. Rather, the concepts need to be understood correctly at one stage through a particular sequence, i.e., concrete to abstract for the current study. As Ding et al. (2019) discussed in their study, tape diagrams are semi-concrete in nature. Students should be engaged in co-constructing of tape diagrams. In fact, just because students are able to construct a tape diagram, does not imply that students really understand the quantitative relationship involved in the word problem or the mathematical relationship being described. In particular to this study, students were able to answer the pieces of questions that the teacher asked; however, it did not mean that students understood the material or were able to draw the tape diagram independently. As this lesson provided instructional scaffolding for the students since the tape diagram was new to both the students and teacher, students struggled when trying to construct the tape diagram independently. Another factor that could have affected this is because students were taught very closely in line with the textbook prior to the intervention, meaning the teacher primarily lectured about a mathematical concept where students were mostly taught using abstract representations with very minimal concrete examples. Thus, their strategy of choice to solve word problems could be to choose equations immediately. The changes from pre to immediate post-tests for students 5-8, however, did show that immediately after the intervention, students did not revert to equations unlike student 4 or the control group on the immediate post-test. This study also showed some similarities with the literature as discussed below in more detail.

Concrete Fading

The results of the study did yield some of the expected results and provided insight into the literature and implications for further research. Based on the literature, similar results revealed by Fyfe et al. (2017), students in the treatment group outperformed students in the control group, (Fyfe et al., 2017). In Fyfe et al.'s study (2017), students who received concrete fading outperformed all other conditions on all of the tests, which align with delayed post-test results of the current study. Students in the control group scored more or less the same on the immediate post-test, but then their scores decreased significantly on the delayed post-test unlike students in the treatment group. Most of the students in the treatment group outperformed students in the control group for the delayed post-test, similar to results discussed by Fyfe and McNeil (2012). In the study conducted by Fyfe and McNeil (2012), the concrete fading condition had the most correct on the three-week post-test compared to other two tests and conditions, but did not take a pre-test. Their transfer was also the highest at the three-week post-test.

Findings in this study were also closely related to the results of Ching and Wu (2019) where the lower-performing students made more an attempt to apply the tape diagram and one was successful using a broken tape diagram. Another student in the treatment group who was successful on one problem, but did not use tape diagrams. Similarly, another interesting finding of this study was that the results of this study were different from the study conducted by Ottmar and Landy (2017) where students' scores in the concrete fading condition differed significantly than the concrete introduction condition only a few days after instruction, but there was no difference one month after instruction. However, the enactment of concrete fading in this study differed slightly than the enactment used in Ottmar and Landy (2017) because the tape diagram was used for the 'fading' stage. This is important because students in the treatment performed similarly to students in the control group on the immediate post-test, but outperformed students in the control group on the delayed post-test on abstract equations in the current study.

Word Problems

None of the students employed informal methods that Koedinger et al. (2008) shared to solve (i.e., unwind strategy, guess-and-check). Since students' prior knowledge was low, I was surprised to see that students did not try another way to solve that they would think of on their own. Though, student 1's work did suggest guess and check, he did not state that in his interview. As detailed in the results and methods sections, student 8 did use a different type of method that was not considered formal, but may be considered informal. She drew circles to represent various quantities. Again, her diagram did not constitute a tape diagram because it did not show the quantitative relationship, which is similar to what was discussed by Clement et al. (1981). However, she did draw a broken tape diagram for another problem to answer a problem correctly.

In contrast many of the students' work and responses suggested some misconceptions identified by Bush and Karp (2013), including the equals and negative sign. If the error was made once, I called it an error whereas if the error was made more than once, I tagged the errors as possibly suggesting a misconception. Student 2 dropped variables, then plugged in the correct value for 'x', but his work was not correct. In addition, student 2 also made errors when combining like-terms, equals sign and negative sign as detailed in Results. His error with the equals sign suggests that he treated the equals sign as an operator and only made this error once. His error with the negative sign suggests that he struggled with solving equations at different points of the study i.e., pre-test and delayed post-test. Student 5 applied an inverse operation, but then applied it to every term on one side of the equation, but this was an error because she only made the error once. Student 6 made an algebraic error twice by dividing by 3x and 4x rather than 3 and 4, respectively, as well as applying the inverse operation to every term on one side of the equation like student 5. Since student 6 made that type of algebraic error twice (dividing with the variable) along with combining like terms (as detailed in Results), his errors suggested a misconception. Student 4 made an arithmetic error once on #6, so I called that an error with no suggestion of a misconception.

In addition, some students did state that the word problems were challenging for them, which is also evidenced by their work on the tests. Not many students in the control group attempted to answer whereas some students in the treatment group attempted to answer, especially when completing the immediate post-test. In general, students demonstrated a lack of success with word problems (some used key word strategies, some had no clue and just looked at the numbers at a surface level) because their interview responses indicated that they were confused when it came to solving word problems. The analysis suggests that there seems to be a lack of understanding of expressions (e.g., 4%x), which seems to be one of the reasons that hindered their use of tape diagrams to represent these quantities. There also could be a lack of understanding percent and/or proportion for the students involved in the study.

Challenges of Implementation

The other challenges of implementing the tape diagram were, as previously stated were students' opportunities of constructing a tape diagram independently and their willingness to use it to solve word problems. Even though the current study did not yield some of the desired results, it did show how a short intervention could still reveal how students who received the treatment were able to think differently. Similarly, students were very open to using the tape diagram, especially students who were lower performing. They had very positive things to say with the tape diagram. There were also six students who felt that they did not need to draw and preferred to just use equations to solve because that is what they are used to, which is similar to Nathan and Koedinger (2000a, 2000b). Students seemed to want to have revert back to what they are used to, which is using equations to solve word problems. Their method of going back to what they are used to may suggest to them that that is the 'correct' way to solve problems. All of these challenges show that future directions should focus on training the teacher beyond what is needed for the lesson, the duration of the lesson, and students' prior knowledge.

Limitations & Future Directions

The limitations of this study were the sample size, how concrete fading was treated in a holistic way, not all tests were taken, the number of lessons, students' opportunities to work on the tape diagram independently and the trade-off. First, eight students were the sample size based on who provided consent to participate in the study. The reason there were only eight students was because there was a challenge in recruiting students. Students were going from in-person classes to remote back to in-person because of the pandemic. If the study were to be expanded, results may differ. Second, the intervention used in this study treated concrete fading in a holistic way. It contained different components such as teacher and student discussion of the tape diagram. Thus, it was not clear if concrete fading (i.e., concrete to abstract) or the tape diagram itself would make an impact on student learning or its implementation (how the tape diagram was discussed and how much students have learned) would make a difference in students' understanding. Similarly, the current study was changing two types of instruction in the classroom: lecture to less lecture and symbolic first to concrete fading. Hence, based on the difference in outcomes, it remains unclear whether it was the difference in the amount of lecture or that it was using symbols versus a story problem first. In other words, did students in the treatment group perform differently because the lessons were not as teacher-driven or was it because the sequence of the lessons was rearranged from the control group (i.e., word problems to symbolic equations or symbolic equations to word problems) or was it including discussion into the classroom? Third, some students missed a pre-test or immediate or delayed post-test, so the results could have revealed a little bit more had those scores and tests been included in the analysis. Fourth, this study only involved three lessons. If there were more lessons, examples and practice problems that allowed students to apply the tape diagram, they may be more successful in using it. If they use it more, they may have a better understanding of solving linear equations. Fifth, students were not asked to draw diagrams independently in a lesson, which could have been served as an assessment of students' understanding of the intervention.

The final limitation refers to the study conducted by Koedinger et al. (2008), which states that at some point, there is a 'trade-off' in the sequence of representations in mathematics courses. In other words, would it be more beneficial for students to learn using abstract equations first and then word problems or vice-versa? As discussed by Koedinger et al. (2008) in the literature review, some mathematical concepts are more beneficial for students when teachers align with the concrete fading approach by going from concrete to abstract whereas some other concepts are more beneficial for students when teachers align with concrete introduction by going from abstract to concrete (Koedinger et al., 2008). Students performed better on the double-reference (i.e., the variable is referenced twice rather than once) when they were given abstract equations rather than story problems whereas the reverse happened when they solved single-reference problems. However, students in the study conducted by Koedinger et al. (2008) were not taught with concrete fading as defined in this study nor were they taught how to use tape diagrams to understand quantitative relationships with more difficult story problems. Though, the present study tried to bring concrete fading with the support of a 'powerful' (Ding et al., 2019, p. 105) representation that was expected to be useful as discussed in the literature review.

In conclusion, this study underestimated the challenges for students to learn and grasp the tape diagram. This is a powerful tool as stated by Ding et al. (2019), but not that easy to grasp by just looking at how others draw them. The results of the current study do reveal how teacher training should go beyond the lesson so that the teacher can become comfortable in using the intervention materials to encourage students to feel comfortable in applying the new methods as no student used it after the immediate post-test. Also, the results of this study reveal that students who performed lower may benefit from a much longer intervention that lasts more than three lessons. In addition, the tape diagram should be used in early school just as with students in East Asia (Murata, 2008) as a way to show the quantitative relationship, not just as a tool for illustrating a computational process. This study suggests that using it as a tool for computation does not provide benefits to students in traditional skills of solving equations. This way, students will be more comfortable using tape diagrams to represent easier concepts that are necessary for prior knowledge of the later concepts because of the lack of familiarity as well as the semi-concreteness may make it difficult for students to master. Additionally, there should be studies that focus on the effect of the tape diagram. In other words, both the control and treatment group study word problems, but one group has the tape diagram while the other group does not. Finally, qualitative studies involving students' learning of the tape diagram should be conducted so that the students' learning difficulties with the tape diagram can be better understood based on the results of the current study.

Conclusion

This study examined the influence concrete fading had in two Algebra I classes taught by the same teacher while learning how to solve linear equations and algebraic word problems. Though various studies that have applied or supported concrete fading, the focus at the high school level is lacking. As Nathan and Koedinger (2000a, 2000b) revealed more than two decades ago, teachers continue to teach in the abstract first even when students continue to lack the fundamentals of the concept. By learning in the abstract first, students' challenges and struggles of solving equations as well as word problems are greatly affected as evidenced by the misconceptions discussed by Bush and Karp (2013), the errors discussed by Booth and Koedinger (2008) and the word problem discussed by Clement et al. (1981). The errors that students made and methods used when solving equations indicate that students had a lower conceptual understanding of the equals sign (Booth & Koedinger, 2008; Bush & Karp, 2013). Similarly, students continued to struggle with generating an equation that represented the number of students and professors, suggesting that students struggled with what the variables represented and quantitative equivalence (Clement et al., 1981; Fisher, 1988; Rosnick, 1981). Thus, both potential factors contributing to students' standardized test scores where less than 50% of the students scored into the proficient or advanced range (National Center for Education Statistics, 2011; Ottmar & Landy, 2017). However, these studies were conducted more than seven years ago. Thus, the question remains: how may we better address these difficulties through instruction in current classroom settings?

Concrete fading is a promising approach to the problem of students' struggles of solving linear equations and word problems. Concrete fading, as defined in this study, is a three-stage progression supported by Bruner's (1966) modes of representations that begin with concrete representations provided by a story problem, then progress into a semi-concrete diagram and, finally, morph into abstract representations. Through this instructional approach, the concrete representations along with the real-world application had the potential to enable students to ground their understanding of the mathematical concepts, but they continued to struggle. The results of this current study suggest that further research should focus on students' abilities to understand word problems that could help them visualize the quantitative relationship embedded in the story problems with tape diagrams as they graduate into abstract representations more effectively. As students revealed in their interview they struggled with the words and preferred to solve using techniques presented by the textbook rather than another method as discussed in the Discussion chapter. Nevertheless, findings in this study did not address a few factors. A reason could be that this study did not really show the benefit is because concrete fading is not just a simple representational sequence. Rather, the concepts need to be understood correctly through different stages, which may not have been established within three lessons. If concrete fading encompasses all of these factors (e.g., translating word problems more closely, constructing a tape diagram, opportunities to construct independently and generating equations) that may allow students the opportunity to build a foundation of the concepts so that they may have the ability to build their own storage of knowledge (Fyfe et al., 2014) which can deepen their understanding.

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APPENDIX A

MAJOR STUDIES

The following table summarizes the major studies discussed in this paper that includes the design of the study, content of the study, the instructional activity, the level of the students, and the p-values.

Study	Design	Instructional Activity	Content	Grade Level	P-Value
Concrete Fading					
Fyfe et al, 2014	Review of concrete fading	Reviewed concrete fading in a mathematics and science class that discusses what is needed for successful implementation of concrete fading in the classroom.	Components of concrete fading such as concrete to abstract	Various	N/A
Fyfe & McNeil, 2012	Split into three conditions: Concrete-only Abstract-	Students were taught examples then completed a few practice problems. Students then completed an immediate, one-week delayed and three-week post test.	Commutative Property Associative Property Identity Inverse	Undergraduates	There was a significant difference between students in the fading condition and the other two conditions: $p < .001$. Students in the fading condition outperformed students in the concrete-only and abstract-only conditions. Scores of the fading condition after various transfer post-tests, which remained significant throughout:

	only (Generic) Concrete fading	Lessons varied depending on the condition listed below. Concrete-only group: strictly images such as pictures of measuring cups Abstract-only group: solely in abstract symbols including circles and diamonds. Fading group: measuring cups, Roman Numerals and abstract symbols			Immediate: $p < .001$ One-week post: $p = .009$ Three-week post: $p < .01$ The generic and concrete groups differed significantly at the immediate post-test with $p = .03$; however, the significance no longer exists at the one-week with $p = .11$ or the three-week with $p = .43$.

Ottmar & Landy, 2017	Split into two conditions: Concrete Fading Concrete Introduction	Students completed a pre-test, an immediate post-test after the first day, a one-day delayed post-test after the second day and a one-month delayed post-test (Ottmar & Landy, 2017). Concrete Fading: dynamic (using PS) then static Concrete Introduction: static then dynamic	Commutative property; simplifying algebraic expressions	7th grade	Results revealed that neither group differed significantly for the pre-test (p=.44) or the immediate post-test (p=.61). However, students' prior knowledge significantly affected how they performed on the assessments (p<.01; Ottmar & Landy, 2017). After the second day, students' scores differed significantly after the one-day delayed post-test, p<.01. However, neither group differed significantly at the one-month delayed post-test, p=.48.

Fyfe et al., 2015	Split into four conditions: Concrete-only Abstract-only Concrete fading Concrete Introduction Experiment 2: Abstract Play-to-abstract	Study had three experiments. Students received 'one-on-one' instruction (Fyfe et al., 2015, p. 106). Students then completed a transfer test. Lessons varied depending on the condition listed below. Concrete: animals with stickers as well as a balancing scale using objects Abstract: symbolic math equations such as 5+3=5+__? (Fyfe et al., 2015). Concrete fading group: animals with stickers as well as a balancing scale using objects, 'fading' worksheet with animals, and	Math Equivalence	2nd-3rd Grades	Students in the concrete fading condition outperformed students on the transfer test across all conditions with $p = .001$. Students in the fading condition also had a higher probability of getting the correct answer with $p = .01$ compared to the other conditions. Students in the fading condition did not use the different strategy that the other three conditions with used and was significantly difference with $p = .06$

		symbolic math equations Concrete introduction group: symbolic math equations, ‘fading’ worksheet with animals, and animals with stickers as well as a balancing scale using object ‘Play-to-abstract’: first two stages closely replicated the lessons of the concrete and fading groups, but without any instruction to try to engage students, and then abstract representations			
Ching & Wu, 2019	Split into five conditions	Students completed a pre-test, learned two lessons, and then	Inverse Relationship	Kindergarten	Students in the fading condition outperformed the control group for both post-tests with $p < .001$. The higher prior knowledge students did not have

	: Control Concrete-only Abstract-only Concrete fading Concrete Introduction	completed an immediate post-test, a delayed post-test, a delayed post-test eight weeks later. Concrete: Unifix cubes Abstract: expression with numbers and operations Control: similar to abstract group Concrete fading: began with using Unifix cubes, then they worked with pictures of cubes on paper being added and subtracted and then numbers and operation symbols Concrete introduction: began with numbers and operation symbols, then they worked with pictures of cubes on paper being			any significant differences with any other condition as $= 1$. Students with lower prior knowledge in the fading condition performed significantly better than students in the abstract-only group for both post-tests with $p < .001$ as well as with the concrete introduction group for the delayed post-test with $p = .02$. Students with lower prior knowledge in the fading condition did not have a significance difference with concrete-only (immediate post-test, $p = .1$; post-delayed, $p = .07$) or the concrete introduction at the immediate post-test with $p = .23$.

		added and subtracted and then Unifix cubes			
Sense-Making & Multiple Representations					
Ding, 2016	Investigated the inverse relationship of US and Chinese textbooks		Inverse Relationship	Kindergarten-6th grade	Chinese textbooks had more examples than US textbooks whereas US textbooks had more practice problems than Chinese textbooks.
Ding & Li, 2014	Investigated the distributive property in a Chinese textbook	The sequence of representations investigated occurred in three different areas: from a worked example; from a worked example to the practice problems students were asked to solve within one concept; and how the sequence is structured across several topics across all grades (Ding & Li, 2014).	Distributive Property	1st-6th grades	Results for the first area: fading within a worked example, revealed that several of the worked examples were concrete representations in the form of word problems and the practice problems consisted of both concrete and abstract representations (Ding & Li, 2014) Results for the second area: the sequence of representations from a worked example to the practice problems within one topic, revealed that concrete fading was evident. Several of the concrete representations were word problems while the abstract representations focused on computation or fill-in-the-blank with a number or sign (Ding & Li, 2014). Results for the third area: the sequence of representations across all grades had sub-areas.

Murata, 2008	Investigated benefits of tape diagram	Discussed how tape diagrams can allow students to see the mathematical relationships and help with problem solving (Murata, 2008).	Addition and subtraction	2nd-3rd grades	Tape diagrams are beneficial because they assist with 'meaning-making' (Murata, 2008, p. 396). Meaning-making allows students to go beyond the mathematical calculation and understand what is the math behind it. Also by using a tape diagram, students can see what operations need to be performed to solve the problem and see the relationship. An example provided by Murata (2008), students are given a word problem that is aided with a tape diagram asking how many total paper rings are there. They are then asked to write an equation that is represented by the tape diagram. After, students are asked how many total paper rings there are with supplemental pictures of adding the hundreds, tens and ones place (Murata, 2008). The sequence of this example supports concrete fading because students work in a semi-concrete diagram that fades into an equation with symbols and numbers (Murata, 2008).
Ding, 2021 (p. 157-162 only)	Sequence of concrete to semi-concrete to abstract	Discussed the importance of concrete fading using the tape diagram and comparing the diagram to the generated equation.	Distributive property	3rd grade	Discussed the importance of concrete fading using the tape diagram and comparing the diagram to the generated equation.

Koedinger & Nathan, 2004	Study 1 High school Algebra I and Geometry students as well as students in eighth grade in Algebra I. Study 2 High school students in Algebra I	Students took a difficulty factors assessment, DFA: Difficulty Factors Assessment 1, DFA1, for study 1 and Difficulty Factors Assessment 2, DFA2, for study 2 (Koedinger & Nathan, 2004). The problems in the assessment were in the format of the type of problem (story problem, word equation, symbolic equation); positioning of the unknown (result-unknown and start-unknown); numbers that included positive decimals and whole numbers; and operations that were manipulated (addition and multiplication versus division and subtraction).	Solving for the variable with linear equations in various forms for both experiments.	Eighth grade and high school	DFA1 Students did significantly better on story problems and word equations ($p < .001$) than they did on symbolic equations (Koedinger et al., 2008). DFA2 Students also did significantly better on story problems and word equations ($p < .001$) than they did on symbolic equations (Koedinger et al., 2008).

Koedinger et al., 2008	Experiment 1 Students who participated were either enrolled in a course that was similar to Algebra I in high school or another course that was enrolled in a course that taught Algebra I and II topics in high school. Experiment 2 Students from another university	Experiment 1 Students were given a 'six-item 'difficulty factors assessment'' to complete (Koedinger et al., 2008, p. 373). The problems were single and double reference problems in one of the formats: story-implicit operators, story-explicit operators and equations. Experiment 2 Students also completed a difficulty factors assessment that had students solve linear equations in the form of an equation and story problem (Koedinger et al., 2008).	Solving for the variable with linear equations in various forms for both experiments.	College level for both experiments	Experiment 1 Students performed significantly better ($p < .0001$) on the abstract equations when it was a double-reference problem rather than single reference (Koedinger et al., 2008). The difficulty of the problem was also a significant difference, $p < .0001$ (Koedinger et al., 2008) Experiment 2 Students performed significantly better, $p < .02$ on single-reference verbal problems than on single-reference equations.

Algebraic Word Problems					
Clement et al., 1981	N/A	"There are six times as many students as professors at this university" (Clement et al., 1981, p. 288).	Generating equation from word problem	Secondary	The transcripts of students' interviews revealed that students applied the "word ordered matching" (Clement et al., 2981; Paige & Simon, 1966) where students take the positioning of the words and put them into an equation, i.e., 6S=P because 'students' were stated first before 'professors.' Another finding from the transcripts revealed that students drew a picture of six individual circles labeled 'S' and one circle labeled 'P' to show that there are six students for every one professor. However, students still could not provide the correct answer and still wrote: 6S=P because there were six circles labeled 'S' to one circle labeled 'P'.
Fisher, 1988	N/A	"'There are six times as many students as professors at this university'" (Clement et al., 1981, p. 288).	Generating equation from word problem	Secondary	In addition, Rosnick (1981; Fisher, 1988) revealed that students are so convinced that there are six students for every one professor, that when students saw the correct equation: S=6P, they thought that 'P' represented the number of students and 'S' represented the number of professors.
Ng & Lee, 2009	Two studies Study 1 Teachers Study 2	The instrument used in both of these studies was 10 questions including an arithmetic, story problems and problems that	Arithmetic	Primary grades	Findings revealed students who used the model method were very detailed and generated correct 'arithmetic equations' while drawing their diagrams (Ng & Lee, 2009, p. 310). In addition, both studies revealed that the model method is helpful for some arithmetic word

	Students	involved students using several concepts to solve that students learn at the end of the year of Primary 4.			problems, but not all (i.e., when equations have two unknowns), which is very similar to the findings stated by Koedinger et al. (2008; Ng & Lee, 2009).
Ding et al., 2019	Comparison of US versus Chinese teachers	Teachers teaching with tape diagrams	Additive and Multiplicative Inverses	1st-4th grade teachers	Findings revealed that Chinese teachers used tape diagrams to represent the quantitative relationship embedded in the story problems whereas US teachers primarily used tape diagrams as a method to show how to solve computational problems. Another finding suggests that 'deep questioning' (Ding et al., 2019, p. 127) can help engage students and allow for a better understanding of the concepts and relationships.
Algebraic Equation Solving					
Bush & Karp, 2013	Review of misconceptions	Bush & Karp, 2013, highlight the most common misconceptions for middle school students taking Algebra.	Equals sign, simplifying algebraic expressions, notation and symbolism used in Algebra, role of variables, solving equation with variable on both sides	Middle school Algebra	N/A

Fyfe et al., 2017	Pre-algebra and algebra students from two different schools	The questions on the assessments were of three different formats: 1) solve an equation using fill-in-the-blanks; 2) questions that assessed students' ability to successfully indicate if the left side was equal to the right side regardless of how the equation was written; and 3) questions written in the abstract to assess if students truly understood the equals sign. Two forms of assessment were used Form 1 Used letters c and b to represent the price of cake and brownie, respectively. Form 2	Equals sign	Middle school	Algebra performed better on some problems than the students in pre-Algebra. Algebra students were better able to identify the properties of equality, i.e., when adding the same number to both sides to an equation, the two sides are still equivalent. In addition, algebra students were better able to explain the role of the equals sign when given 1 quarter = 25 pennies (Fyfe et al., 2017, p. 96). These students responded that 1 quarter is 'the same as' 25 pennies whereas pre-algebra students said 1 quarter is 'equal to' 25 pennies (Fyfe et al, 2017, p. 96). In addition, how the variables are labeled is a significant, $p = .006$. Students performed significantly better using variables such x and y to represent quantities rather than 'mnemonic letters' (Fyfe et al., 2017, p. 96).

		Used letters x and y to represent the cake and brownie.			
Nathan & Koedinger, 2000a	Teachers at the three levels: elementary, middle and high school	Teachers ranked a set of problems from what they thought would be the easiest to hardest for students to solve	Various topics	Elementary, middle and high school teachers	High school teachers' belief that teaching in symbols is what grounds students' understanding to apply to verbal word problems was significantly different, $p < .0005$. Both elementary and high school teachers ranked the problems symbolic equations as the simplest for students to solve while ranking the verbal word problems the hardest to solve. However, middle school teachers ranked in the reverse order of the other teachers (Nathan & Koedinger, 2000a)
Nathan & Koedinger, 2000b	Teachers who teach various grades ranging from 7th to 12th grade.	Teachers ranked a series of problems from their school district textbooks on what they thought students would have the least difficulty to the most difficulty solving. The researchers ranked a set of six problems in the reverse order.	Problems were start- or result-unknown in any of the formats: story problem, word equation and symbolic equation (Nathan & Koedinger, 2000b).	Teachers who taught from 7th to 12th grade.	The ranking of the problems was statistically different, $p < .05$. Teachers ranked symbolic equations in either problem type as easiest for students to solve while ranking the word equation and word problem as most difficult to solve (Nathan & Koedinger, 2000b).
Booth &	High school	Students used a program, Algebra	Linear equations	High school	The more conceptual knowledge students have, they will have a higher number of correct solutions

Koedinger, 2008	students using program, Algebra Cognitive Tutor	Cognitive Tutor, which used multiple representations. The data collection involved looking at the students' 'procedural performance and learning' with the errors and solutions they provided on their pre- and post-tests (Booth & Koedinger, 2008, p. 572). The assessments tested for students' conceptual and procedural knowledge of the equals sign as well as the negative sign.			with the equals sign ($p < .01$) and the negative sign ($p < .01$) (Booth & Koedinger, 2008). Students with lower conceptual knowledge made significantly more errors ($p < .05$) with equality. However, the errors made with the negative sign were not significantly different, $p < .10$.
Booth & Davenport, 2013	Feature encoding, feature knowledge and equation solving (Booth & Davenport, 2013, p.	Students' errors on encoding equations after seeing them on the board for and writing them down after they disappeared from the board. Assessed students on what they knew about	Solving one- and two-step equations. Equals sign, negative sign and variables	7th-9th grades	Findings revealed that there was a significant difference in the encoding errors with the negative sign, equals sign and variables, $p < .01$ s. In addition, there was a significant difference for students' feature knowledge involving variables, $p < .05$.

	418).	solving one- and two-step equations along with what they know about the equals sign and variables			

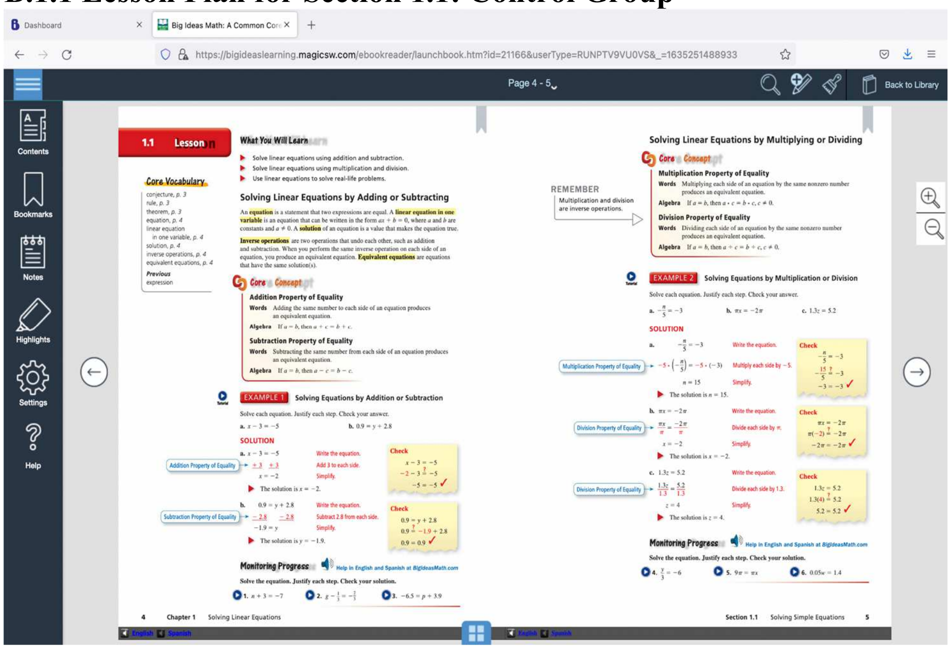
APPENDIX B

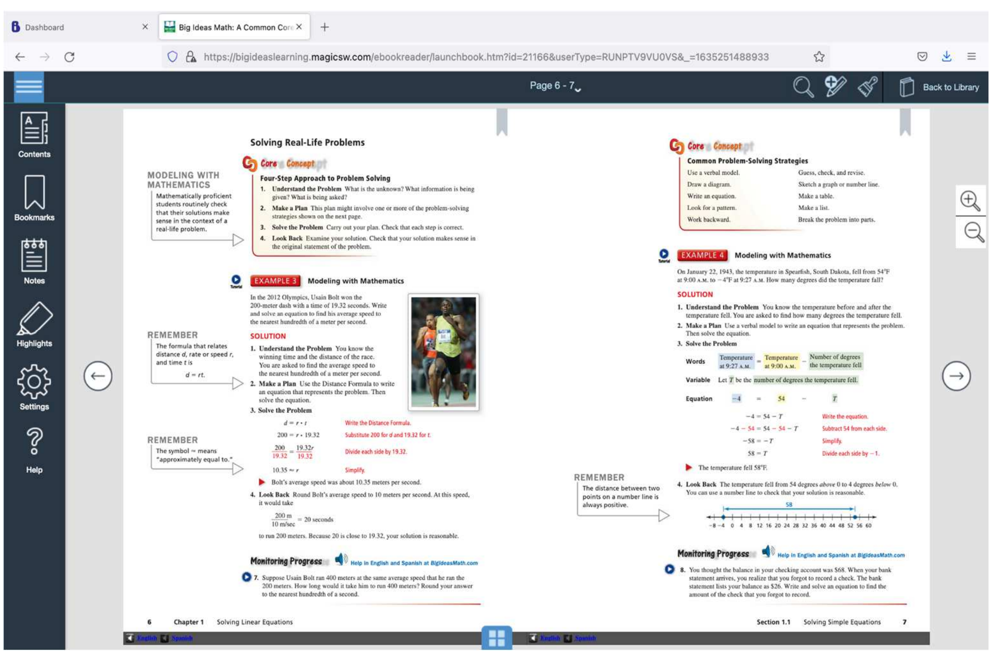
LESSON PLANS & TEXTBOOK PAGES

B.1 Lesson Plans for Control Group

The textbook pages that align with the lesson plans (sections 1.1-1.3) for the control group are based on *Algebra 1: A common core curriculum* (Larson & Boswell, 2019).

B.1.1 Lesson Plan for Section 1.1: Control Group





Teach abstract equations provided by the book and then conclude with word problem for 1.1:

1. Practice task #15: A discounted amusement park ticket costs $12.95 less than the original price *p*. Write and solve an equation to find the original price.

   $x - 12.95 = 44$

   $+12.95 \quad + 12.95$

   $x = 44 + 12.95$

   $x = 56.95$

2. Before we go into that problem, suppose Bolt ran the 200-meter dash in 20 seconds.

   Let r→ the rate at which he runs and draw the tape diagram below

   $20r = 200$

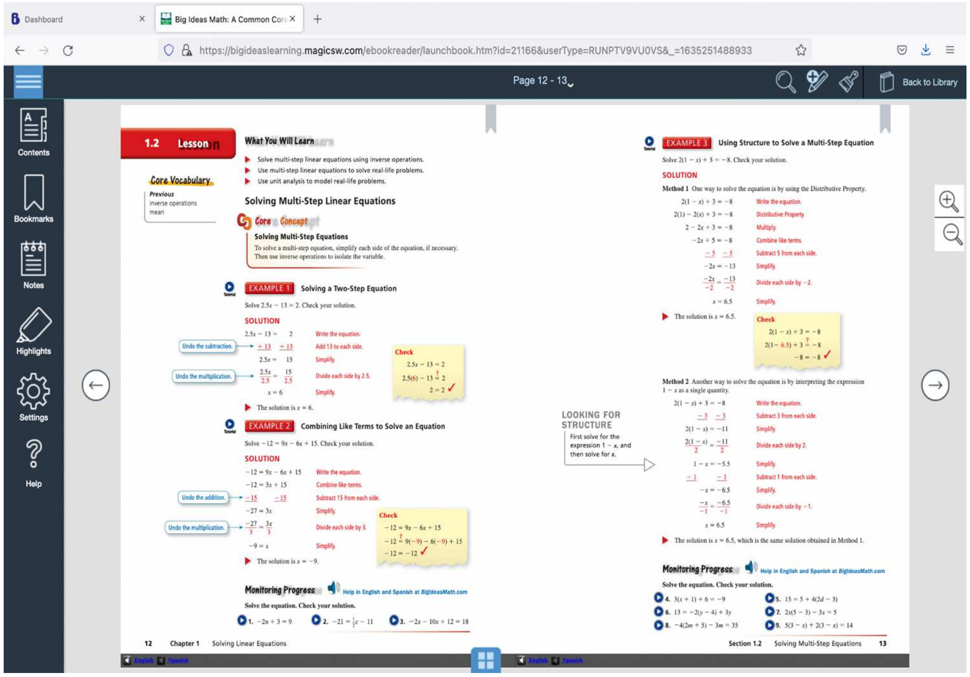
   $\frac{20r}{20} = \frac{200}{20}$

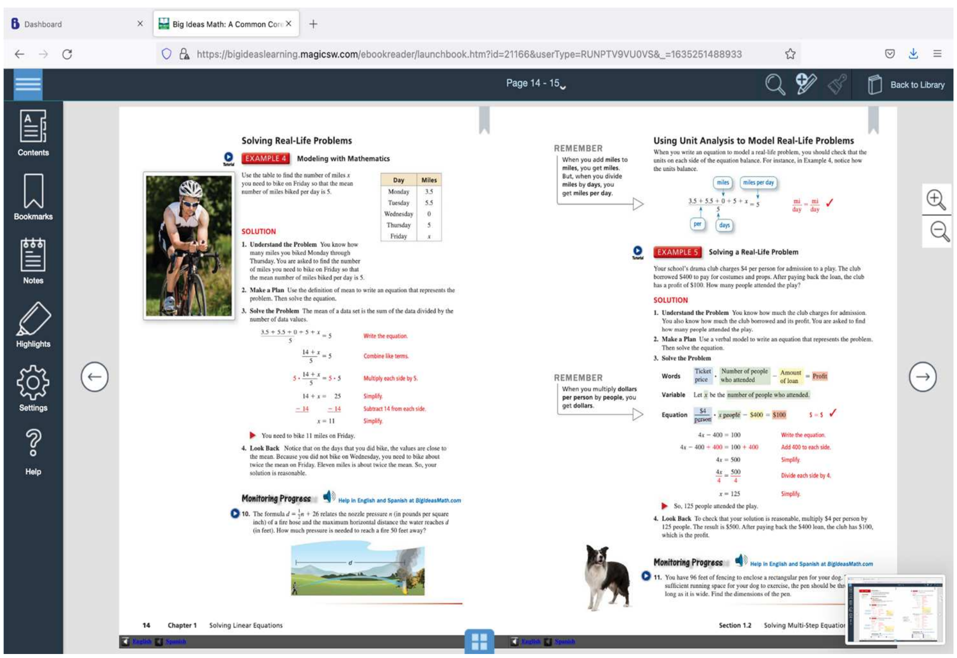
   $r = 10$

   Bolt's rate would be 10 m/sec

3. Now change 20 seconds from #2 to 19.32 seconds and refer to the problem in the textbook (example 3).

B.1.2 Lesson Plan for Section 1.2: Control Group





Teach abstract equations provided by the book and then conclude with word problem for 1.2:

1. Practice problem #37: You order two tacos and a salad. The salad costs $2.50. You pay 8% sales tax and leave a $3 tip. You pay a total of $13.80. How much does one taco cost?

x→ the number of tacos

8% sales tax is: $13.80(.08) = 1.1$

$$1.1 + 3 + 2.50 + 2x = 13.80$$

$$6.5 + 2x = 13.80$$

$$-6.5 \qquad -6.5$$

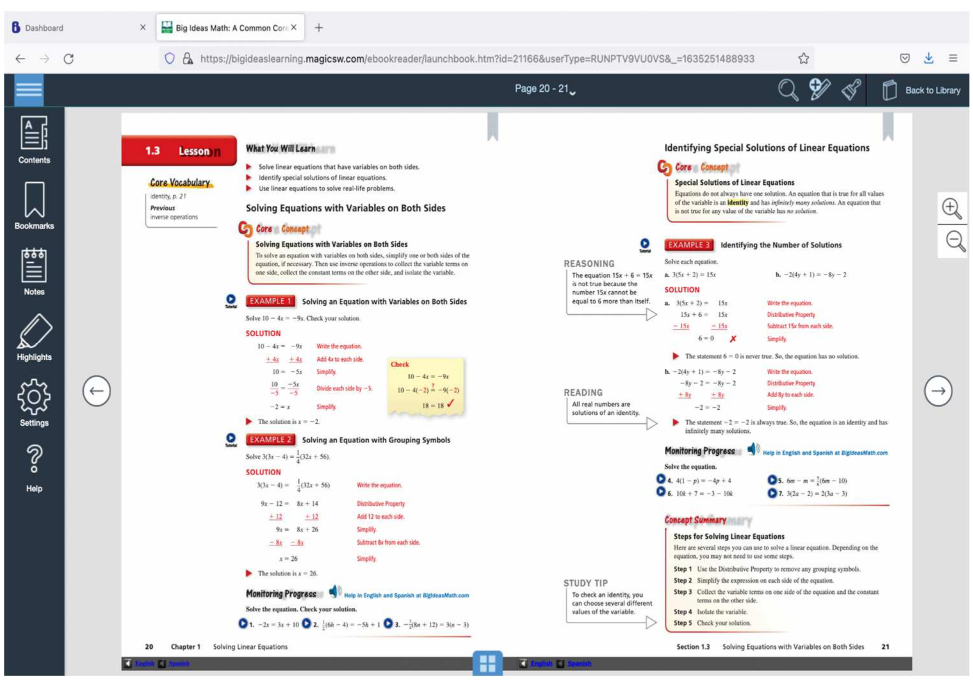
$$2x = 7.3$$

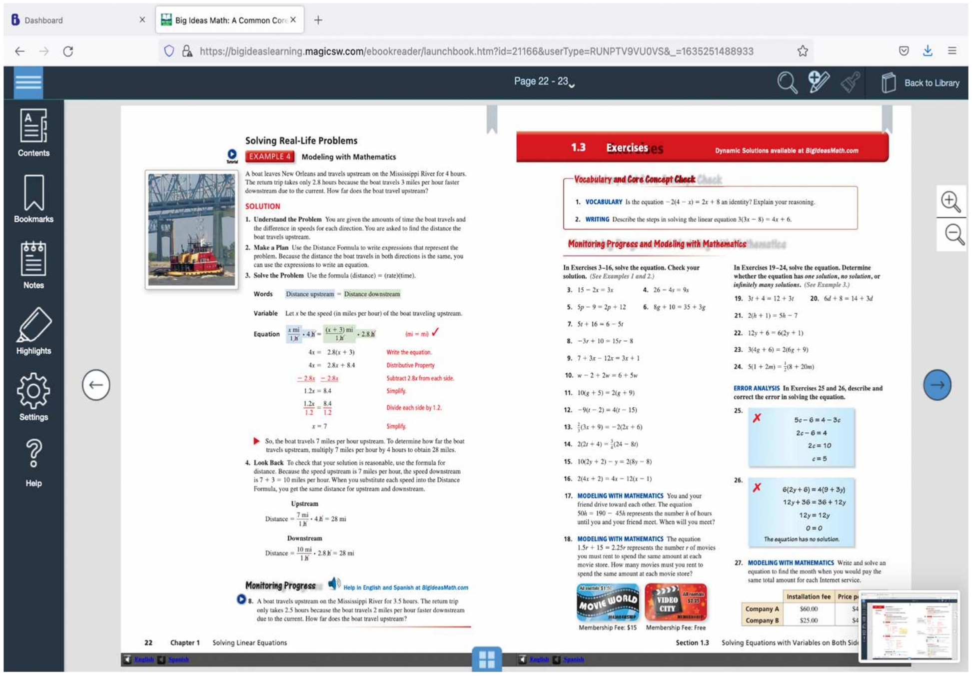
$$\frac{2x}{2} = \frac{7.3}{2}$$

$$x = 3.65$$

Each taco costs $3.65.

B.1.3 Lesson Plan for Section 1.3: Control Group





Teach abstract equations provided by the book and then conclude with word problem for 1.3:

1. One serving of granola provides 4% of the protein you need daily. You must get the remaining 48 grams of protein from other sources. How many grams of protein do you need daily?

   x→ # of grams of protein needed daily

   $$x = .04x + 48$$

   $$-.04x \quad -.04x$$

   $$.96x = 48$$

   $$\frac{.96x}{.96} = \frac{48}{.96}$$

   $$x = 50$$

   You need 50 grams of protein daily.

B.2 Lesson plans for Treatment group

Example problems are based on:

Hong, K. T., Mei, Y. S., & Lim, J. (2009). *Singapore Model Method for Learning Mathematics*. EPB Pan Pacific.

The remaining problems (sections 1.1-1.3) for the treatment group are the example problems from *Algebra 1: A common core curriculum* (Larson & Boswell, 2019).

B.2.1 Lesson Plan for Section 1.1: Treatment Group

Title: Solving Simple Equations

(Suggested 1 day)

Part 1. Additive Task

1. Practice task #15: A discounted amusement park ticket costs $12.95 less than the original price p. Write and solve an equation to find the original price.

Ask students what the problem is asking and what information they are given before using tape diagram

For the teacher: Next, guide the students to draw tape diagram and introduce how to use equations to solve it. Please have students construct this diagram as they may not be familiar with it. Steps below to follow. Please also be cognizant of the scaling as this is very important in the tape diagram.

For the teacher: Start by drawing blue brace such as the one above and write '$44' above the brace. Ask students how to represent the price of the ticket provided that the ticket costs $12.95 less than the original price. Then, discuss with students how to draw one tape diagram to represent the price of the ticket. The tape diagram will be split into 2 unequal pieces. Ask students to generate equation.

After the diagrams are drawn, ask a couple of questions about the diagrams so students will clearly understand what each tape (or part of the tape or both tapes) represent and why both tapes have the same length. (Questions below)

$$x - 12.95 = 44$$

For the teacher: Before solving the equation, again ask questions about the left and right side of the equation (x-12.95, 44) referring to the diagram.

$$x = 44 + 12.95$$

$$x = \$56.95$$

- *How do all of these examples relate to each other?*
    - *Still solving for the missing value on both sides*
    - *Using equations and any of the operations*
        - *Always get the variable on one side*
- *Real-world examples to abstract equations*

- *What does the tape diagram in the example represent?*
    - *How does it relate to the equation below?*
        - *Separate sections of unequal length, combine all together to get total of the ticket*
            - *Ask students and emphasize why they have to be unequal*
        - *Also shows that when combine the lunch total, a total of $44 is the cost of the ticket*
            - *Also showing role of equality and variables*
- *How does the tape diagram help us to solve the problem?*
    - *Separate sections of unequal length, combine all together to get total of the ticket*
- *Why did we draw the tape diagram like that?*
    - *See above*

2. On January 22, 1943, the temperature in Spearfish, South Dakota, fell from 54°F at 9:00AM to -4°F at 9:27 AM. How many degrees did the temperature fall?

   $-4 = 54 - T$; T is the number of degrees that dropped

   $-54 \quad -54$

   $-58 = -T$

   $\frac{-58}{-1} = \frac{-T}{-1}$

   $58 = T$

3. Solve the following: $x - 3 = -5$.

   Teach in the same way the textbook suggests

4. Solve the following: $.9 = y + 2.8$.

Teach in the same way the textbook suggests

Part II: Multiplicative Tasks

1. Before this part of the lesson, review what we multiply a fraction by to get 1.

   $\frac{2}{3}(\frac{3}{2}) =?$

   $\frac{1}{3}x = 1$

   $\frac{3}{4}x = 1$

   $\frac{-4}{5}x = 1$

   $3x = 1$

   $x(\frac{6}{7}) = 1$

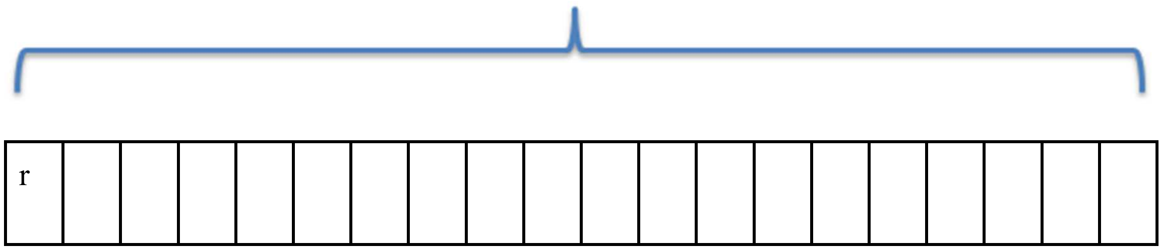
2. In the 2012 Olympics, Usain Bolt won the 200-meter dash in 19.32 seconds. Write and solve an equation to find his average speed to the nearest hundredth of a meter per second.

Ask students what the problem is asking and what information they are given before using tape diagram

Before we go into that problem, suppose Bolt ran the 200-meter dash in 20 seconds.

Let r→ the rate at which he runs and draw the tape diagram below

For the teacher: Draw a box and label 'r' inside. Next, guide the students to draw tape diagram and introduce how to use equations to solve it. Please have students construct this diagram as they may not be familiar with it. Steps below to follow. Please also be cognizant of the scaling as this is very important in the tape diagram.



For the teacher: Start by drawing blue brace such as the one above and write '200m' above the brace. Ask students how to represent the rate given that the time is 20s. Then, discuss with students how to draw one tape diagram to represent the distance. The tape diagram will be split into 20 equal pieces. Ask students to generate equation.

After the diagrams are drawn, ask a couple of questions about the diagrams so students will clearly understand what each tape (or part of the tape or both tapes) represent and why both tapes have the same length. (Questions below)

$$20r = 200$$

For the teacher: Before solving the equation, again ask questions about the left and right side of the equation (20r, 200) referring to the diagram.

$$\frac{20r}{20} = \frac{200}{20}$$

Even for solving the equation, you may still use the tape to help students visualize what happens when one "$\div 20$*" on both sides of equation. It shows the correspondence between 20r and 200.*

$$r = 10$$

Bolt's rate would be 10 m/sec

- *How do all of these examples relate to each other?*

        - *Still solving for the missing value on both sides*
        - *Using equations and any of the operations*
            - *Always get the variable on one side*
- *Real-world examples to abstract equations*
- *What does the tape diagram in the example represent?*
    - *How does it relate to the equation below?*
        - *Shows that when you multiply the time and the rate, it will multiply to the distance: 200m.*
        - *Equal number because 'r' represents each block*
            - *Ask students and emphasize why the blocks are equal*
        - *Also shows equality and role of variables*
    - *How does the tape diagram help us to solve the problem?*
        - *See above*
- *Why did we draw the tape diagram like that?*
    - *See above*

Now to move onto the actual solution:

$$d = rt$$

$$200 = r(19.32)$$

$$\frac{200}{19.32} = \frac{r(19.32)}{19.32}$$

$$10.35 \approx r$$

$Bolt\ ran$ 10*.35 m/sec.*

3. Solve: $-\frac{n}{5} = -3$.

   Teach in the same way the textbook suggests

4. Solve: $\pi x = -2\pi$

    Teach in the same way the textbook suggests

5. Solve $1.3z = 5.2$

    Teach in the same way the textbook suggests

B.2.2. Lesson Plan for Section 1.2: Treatment Group

Title: Solving Multi-Step Equations

(Suggested: 1 day)

1. Use the table to find the number of miles you need to bike on Friday so that the mean number of miles biked per day is 5.

$$\frac{3.5 + 5.5 + 0 + 5 + x}{5} = 5$$

$$\frac{14 + x}{5} = 5$$

$$5(\frac{14 + x}{5}) = 5(5)$$

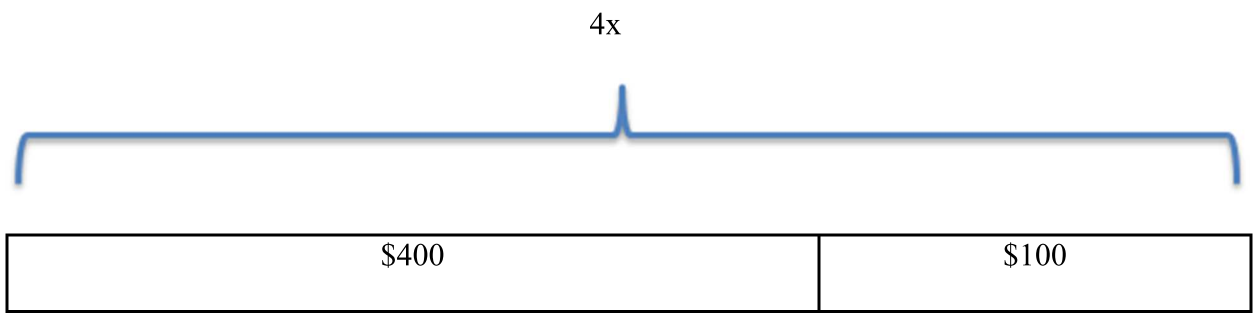
$$14 + x = 25$$

$$-14 \quad -14$$

$$x = 11$$

2. Your school's drama club charges \$4 per person for admission to a play. The club borrowed \$400 to pay for costumes and props. After paying back the loan, the club has a profit of \$100. How many people attended the play?

Ask students what the problem is asking and what information they are given before using tape diagram



For the teacher: Next, guide the students to draw tape diagram and introduce how to use equations to solve it. Please have students construct this diagram as they may not be familiar with it. Steps below to follow. Please also be cognizant of the scaling as this is very important in the tape diagram.

For the teacher: Start by drawing a blue brace and label it '4x'. Then, discuss with students how to draw the tape diagram given that there is a profit of $100 and the club borrowed $400. The two portions in the second tape diagram should be split into two unequal pieces. Ask students to generate equation.

After the diagrams are drawn, ask a couple of questions about the diagrams so students will clearly understand what each tape (or part of the tape or both tapes) represent and why both tapes have the same length. (Questions below)

x→ the number of people who attended

$4x = 400 + 100$ OR

$4x - 400 = 100$

For the teacher: Before solving the equation, again ask questions about the left and right side of the equation (4x, 400+100) OR (4x-400, 100) referring to the diagram.

$$4x - 400 = 100$$

$$+400 \quad + 400$$

Even for solving the equation, you may still use the tape to help students visualize what happens when one "+400" on both sides of equation. It shows the correspondence between 4x and 400+100 OR 4x-400 and 100.

$$4x = 500$$

$$\frac{4x}{4} = \frac{500}{4}$$

$$x = 125$$

125 people attended

- *How do all of these examples relate to each other?*
    - *Still solving for the missing value on both sides*
    - *Using equations and any of the operations*
        - *Always get the variable on one side*
- *Real-world examples to abstract equations*
- *What does the tape diagram in the example represent?*
    - *How does it relate to the equation below?*
        - *To show that when you add what was borrowed the profit, you get the cost of the cost of what the people paid OR that if you subtract the loan from the cost, you get the profit*
            - *The blocks are of unequal length because they cannot be the same value, we cannot assume that so we can split the tape diagram into two equal pieces.*

        - *Ask students and emphasize why we cannot assume that the pieces will be of equal length*
    - *Also showing role of equality and variables*
- *How does the tape diagram help us to solve the problem?*
    - *See above*
- *Why did we draw the tape diagram like that?*
    - *See above*

3. Practice problem #37: You order two tacos and a salad. The salad costs $2.50. You pay 8% sales tax and leave a $3 tip. You pay a total of $13.80. How much does one taco cost?

Ask students what the problem is asking and what information they are given before using tape diagram

$13.80

Taco	Taco	Salad	Tax	Tip
		$2.50	8% ×13.80	$3

For the teacher: Next, guide the students to draw tape diagram and introduce how to use equations to solve it. Draw a blue bracket with $13.80 to represent total cost and the tape diagram split into five pieces- two pieces equal while the remaining three pieces are not equal to them. Please have students construct this diagram as they may not be familiar *with it. Steps below to follow. Please also be cognizant of the scaling as this is very important in the tape diagram.*

For the teacher: Start by drawing a blue brace and label it '13.80'. Then, discuss with students how to draw the tape diagram given that you paid 8% sales tax, $3 tip, a salad is $2.50 and you bought two tacos. The blocks representing the tacos are equal while the block representing the salad will not be equal to them. Ask students to generate equation.

After the diagrams are drawn, ask a couple of questions about the diagrams so students will clearly understand what each tape (or part of the tape or both tapes) represent and why both tapes have the same length. (Questions below)

x→ the number of tacos

8% sales tax is: $13.80(.08) = 1.1$

$$1.1 + 3 + 2.50 + 2x = 13.80$$

$$6.5 + 2x = 13.80$$

For the teacher: Before solving the equation, again ask questions about the left and right side of the equation (1.1+3+2.5+2x, 13.80) referring to the diagram.

$$6.5 + 2x = 13.80$$

$$-6.5 \qquad -6.5$$

Even for solving the equation, you may still use the tape to help students visualize what happens when one "-6.5" on both sides of equation. It shows the correspondence between 1.1+3+2.5+2x and 13.80.

$$2x = 7.3$$

$$\frac{2x}{2} = \frac{7.3}{2}$$

$$x = 3.65$$

Each taco costs $3.65.

- *How do all of these examples relate to each other?*
    - *Still solving for the missing value on both sides*
    - *Using equations and any of the operations*
        - *Always get the variable on one side*

Real-world examples to abstract equations

- *What does the tape diagram in the example represent?*
- *How does it relate to the equation below?*
    - *To show that when you add including the salad, two tacos, tip and tax will get you the total of the bill*
        - *The blocks are of unequal length because they cannot be the same value, we cannot assume that so we can split the tape diagram into equal pieces, except for the tacos.*
        - *Ask students and emphasize why we cannot assume that the pieces will be of equal length*
    - *Also showing role of equality and variables*
- *How does the tape diagram help us to solve the problem?*
    - *See above*
- *Why did we draw the tape diagram like that?*
    - *See above*

4. Solve: $2.5x - 13 = 2$

Teach in the same way the textbook suggests

5. Solve: $-12 = 9x - 6x + 15$

   Teach in the same way the textbook suggests

6. Solve $2(1 - x) + 3 = -8$

   Teach in the same way the textbook suggests

B.2.3 Lesson Plan for Section 1.3: Treatment Group

Title: Solving Equations with Variables on Both Sides

(Suggested 1 day)

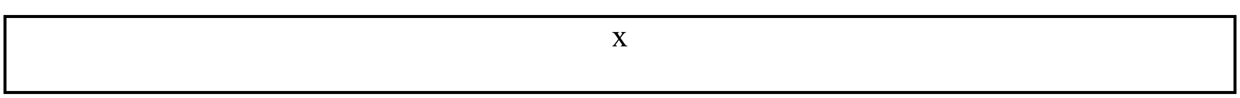
1. One serving of granola provides 4% of the protein you need daily. You must get the remaining 48 grams of protein from other sources. How many grams of protein do you need daily?

Ask students what the problem is asking and what information they are given before using tape diagram

For the teacher: Next, guide the students to draw tape diagram and introduce how to use equations to solve it. Please have students construct this diagram as they may not be familiar with it. Steps below to follow. Please also be cognizant of the scaling as this is very important in the tape diagram.

x→ # of grams of protein needed daily

Total # of grams of protein



Daily grams of protein

.4x	48

For the teacher: Start with drawing a tape diagram to the total number of grams of protein. And, ask students how to represent the daily grams of protein based on the information given. The teacher can discuss with students how to draw the tape diagram for the daily grams of protein.

After the diagrams are drawn, ask a couple of questions about the diagrams so students will clearly understand what each tape (or part of the tape or both tapes) represent and why both tapes have the same length. Ask students to generate equation. (Questions below)

$$x = .04x + 48$$

For the teacher: Before solving the equation, again ask questions about the left and right side of the equation (x, .04x+48) referring to the diagram.

$$x = .04x + 48$$

$$-.04x \quad -.04x$$

Even for solving the equation, you may still use the tape to help students visualize what happens when one "-0.04x" on both sides of equation. It shows the correspondence between x and .04x+48).

$$.96x \ = 48$$

$$\frac{.96x}{.96} = \frac{48}{.96}$$

$$x = 50$$

You need 50 grams of protein daily.

- *How do all of these examples relate to each other?*
    - *Still solving for the missing value on both sides*
    - *Using equations and any of the operations*
        - *Always get the variable on one side*

  Real-world examples to abstract equations

- *What does the tape diagram in example to represent? How does it relate to the equation below?*
    - *To show that the daily number of grams of protein is equal to the total number of grams of protein.*
    - *Also to show that when you plug in the daily number of grams of protein, the lengths of the tape diagram will be the same*
    - *Also emphasize that the daily grams of protein cannot be split into two equal pieces- we cannot assume that they are equal*
    - *Also showing role of equality and variables*
- *How does the tape diagram help us to solve the problem?*
    - *See above*
    - *And to help generate equation*
- *Why did we draw the tape diagram like that?*
    - *Also to compare quantities and to show relationships between the two equations*

2. A boat leaves New Orleans and travels upstream on the Mississippi River for 4 hours. The return trip takes only 2.8 hours because the boat travels 3 miles per hour faster downstream due to the current. How far does the boat travel upstream?

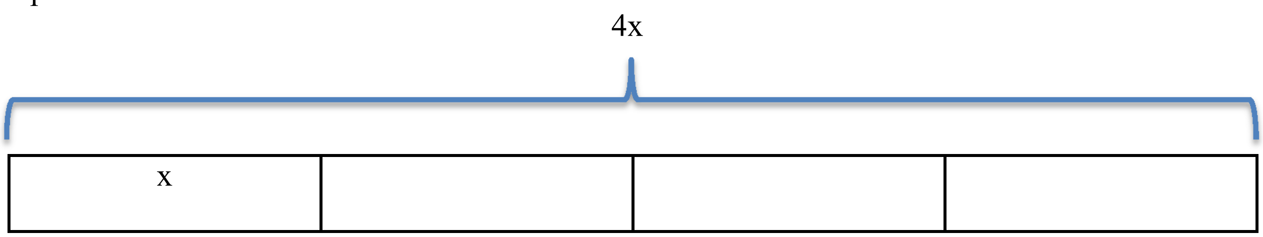
Ask students what the problem is asking and what information they are given before using tape diagram

For the teacher: Next, guide the students to draw tape diagram and introduce how to use equations to solve it. Please have students construct this diagram as they may not be familiar with it. Steps below to follow. Please also be cognizant of the scaling as this is very important in the tape diagram.

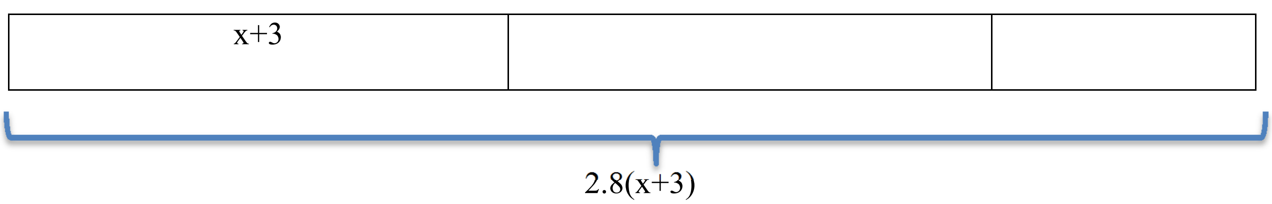
x→ the speed in mph of the boat going upstream

Upstream speed is x mph; downstream is (x+3) mph

Upstream



Downstream



For the teacher: Start with drawing a tape diagram to represent the upstream speed. And, ask students how to represent the downstream speed? The teacher can discuss with students how to draw the tape diagram for the downstream speed.

After the diagrams are drawn, ask a couple of questions about the diagrams so students will clearly understand what each tape (or part of the tape or both tapes) represent and why both tapes have the same length. Ask students to generate equation. (Questions below)

$$4x = 2.8(x + 3)$$

For the teacher: Before solving the equation, again ask questions about the left and right side of the equation (4x, 2.8(x+3)) referring to the diagram.

$$4x = 2.8x + 8.4$$

$$-2.8\,x \;-2.8x$$

Even for solving the equation, you may still use the tape to help students visualize what happens when one "-2.8x" on both sides of equation. It shows the correspondence between 4x and 2.8(x+3).

$$1.2x \;= 8.4$$

$$\frac{1.2x}{1.2} = \frac{8.4}{1.2}$$

$$x = 7$$

The boat travels 7 mph upstream.

- *How do all of these examples relate to each other?*
    - *Still solving for the missing value on both sides*
    - *Using equations and any of the operations*
        - *Always get the variable on one side*

  Real-world examples to abstract equations

- *What does the tape diagram in example to represent? How does it relate to the equation below?*
    - *To show the distance upstream is the same as the distance downstream; hence, the tape diagrams must be of the same length*
    - *Also to show that when you plug in the rate, the lengths of the tape diagram will be the same*

- *Also showing role of equality and variables*

- *How does the tape diagram help us to solve the problem?*
  - *See above*
  - *And to help generate equation*
- *Why did we draw the tape diagram like that?*
  - *Also to compare quantities and to show relationships between the two equations*

3. Solve: $10 - 4x = -9x$

   Teach in the same way the textbook suggests

4. Solve: $3(3x - 4) = \frac{1}{4}(32x + 56)$

   Teach in the same way the textbook suggests

5. Solve $3(5x + 2) = 15x$

   Teach in the same way the textbook suggests

6. Solve $-2(4y + 1) = -8y - 2$

   Teach in the same way the textbook suggests

APPENDIX C

ASSESSMENTS

Pre-Test Assessment

Name: Period: Date: Student ID #:

Algebra IA

Please write an equation that models the word problem below.

1. You order three tacos and a salad. The salad costs \$3.50. You pay 6% sales tax and leave a \$4 tip. You pay a total of \$17. Write an equation that would represent how to find the cost of one taco. (Hint: Drawing may be helpful.)

2. Tim ran 100m in 11 seconds. Write an equation that models his average speed in meters per second? (Hint: Drawing may be helpful.)

3. One serving of granola provides 5% of the fiber you need daily. You must get the remaining 50 grams of fiber from other sources. Write an equation to model how many grams of fiber you need daily. (Hint: Drawing may be helpful.)

4. A boat leaves Louisiana and travels upstream on a river for 6 hours. The return trip takes only 3 hours because the boat travels 4 miles per hour faster downstream due to the current. Write an equation to model how far the boat travels upstream. (Hint: Drawing may be helpful.)

5. Your school's music club charges \$5 per person for admission to a performance. The club borrowed \$300 to pay for materials, chairs and music stands. After paying back the loan, the club has a profit of \$100. Write an equation to model how many people attended the performance. (Hint: Drawing may be helpful.)

Solve for the variable in the following equations below.

6. 3x=11
7. 6x+9=15
8. $8 = \frac{-2}{3}x$
9. 4x+7x=22
10. 2y-3=9-4y
11. $2(x - 5) + 10 = 40$
12. $\frac{1}{2}(10x - 20) = 5x - 10$

Immediate Post-Test Assessment

Name: Period: Date: Student ID #:

Algebra IA

Please write an equation that models the word problem below.

1. You order two slices of pizza and a salad. The salad costs $3. You pay 7% sales tax and leave a $3 tip. You pay a total of $16. Write an equation that would represent how to find the cost of one pizza. (Hint: Drawing may be helpful.)

2. Jess ran the 100m hurdles in 17 seconds. Write an equation that models her average speed in meters per second? (Hint: Drawing may be helpful.)

3. One serving of granola provides 4% of the fiber you need daily. You must get the remaining 38 grams of fiber from other sources. Write an equation to model how many grams of fiber you need daily. (Hint: Drawing may be helpful.)

4. A boat leaves Louisiana and travels upstream on a river for 5 hours. The return trip takes only 2.5 hours because the boat travels 3 miles per hour faster downstream due to the current. Write an equation to model how far the boat travels upstream. (Hint: Drawing may be helpful.)

5. Your school's music club charges $6 per person for admission to a performance. The club borrowed $350 to pay for materials, chairs and music stands. After paying

back the loan, the club has a profit of $120. Write an equation to model how many people attended the performance. (Hint: Drawing may be helpful.)

Solve for the variable in the following equations below.

6. 3x=16
7. 6x+10=15
8. $9 = \frac{-2}{3}x$
9. 4x+7x=28
10. 3y-3=9-4y
11. $2(x-4)+8=40$
12. $\frac{1}{2}(20x-10)=10x-5$

Three or Four-week Delayed Post-Test Assessment

Name: Period: Date: Student ID #:

Algebra IA

Please write an equation that models the word problem below.

1. You order two tacos and two salads. The salad costs $6. You pay 6% sales tax and leave a $5 tip. You pay a total of $20. Write an equation that would represent how to find the cost of one taco. (Hint: Drawing may be helpful.)

2. Steph ran the 300m in 53 seconds. Write an equation that models her average speed in meters per second? (Hint: Drawing may be helpful.)

3. One serving of granola provides 2% of the fiber you need daily. You must get the remaining 52 grams of fiber from other sources. Write an equation to model how many grams of fiber you need daily. (Hint: Drawing may be helpful.)

4. A boat leaves Louisiana and travels upstream on a river for 4.5 hours. The return trip takes only 2 hours because the boat travels 3.5 miles per hour faster downstream due to the current. Write an equation to model how far the boat travels upstream. (Hint: Drawing may be helpful.)

5. Your school's music club charges $4 per person for admission to a performance. The club borrowed $400 to pay for materials, chairs and music stands. After paying

back the loan, the club has a profit of \$150. Write an equation to model how many people attended the performance. (Hint: Drawing may be helpful.)

Solve for the variable in the following equations below.

6. 3x=10
7. 6x+9=30
8. $6 = \frac{-2}{3}x$
9. 4x+7x=33
10. 2y-3=9+4y
11. $2(x - 2) + 4 = 40$
12. $\frac{1}{3}(30x - 18) = 10x - 6$

APPENDIX D

INTERVIEW RESPONSES

#1 What were you thinking while solving #’s 2, 3, and #7?

Student	**Notes**
	Control
1.	Pre-test was hard, then lessons help, then forgot because of going remote for delayed post-test
2.	Found a number to write, but sometimes he’ll forget equations
3.	Tried her best, used the techniques they learned in class. Not good at math and tried to use the techniques
	Treatment
4.	He understood a lot with the help of the teacher and notes. Relied on the solving rather than word equations. Hard for him to solve the word equations, but the rest was easy (the equations)
5.	Doesn’t really remember #2 Thinking about multiplying to get answer #3 Thinking of percentages since they had it for the fiber #7 Multiplied 6 and 9 and ‘stuff’
6.	#2 Looking at whatever numbers and put them in the box May have forgotten #3 wanted to get the numbers out of the way because words are confusing to him #7 Flip 6x to -6x, divide by 9 if he could
7.	#2 solve by finding the meters per second using the numbers given #3 grams and percent to write equation #7 tried to get x by itself first- subtract 6 first and then proceed as it goes
8.	#2 She did not get it right 100 m and 11 sec, so she needed the speed m/sec- doesn’t know how to do this #3 Doesn’t understand Author’s thought: how did you get the 59/50- she didn’t know #7 Used her notes and solved

#2 What did you find helpful when you got stuck solving a problem?

Student	Notes
Control	
1.	Lessons help; tried to problem-solve in head
2.	Used notes that he had
3.	Studied a little from her notes
Treatment	
4.	Relying on the notes Teacher gave key notes on when to subtract
5.	To use the method of the boxes when she got stuck Original method was helpful- what the teacher taught
6.	Double checking Try a lot of techniques best he could (i.e., my technique and the one Mr. Rios taught)
7.	Kept re-reading the question so she knows it better On the back- kept checking work and seeing if she was doing it correctly
8.	Used her notes For #'s 6-12, she used her notes For #'s 1-5, she doesn't understand and was confused. Teacher said try your best, so he couldn't provide much help

#3 When you were first saw the word problem (any from #'s 1-5), how were you thinking of how to generate an equation that models the situation?

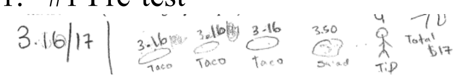
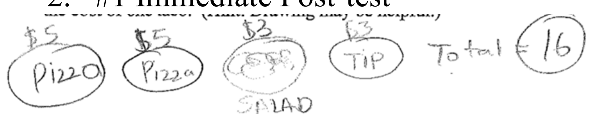
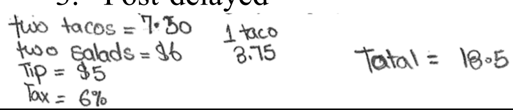
Student	Notes
Control	
1.	Difficult because didn't understand and trying to find ways to solve the problems
2.	Based on numbers on the problem and the way they go through (Lisa's thought: perhaps the order the numbers were shown)
3.	Using the equations
Treatment	
4.	Underlined key points in the word problems for the tip, tax- got the idea he needs to make the equation and do something with the percentage to find the solution
5.	Boxes- make 38 bigger than 4% for #3
6.	Get the numbers out first

	Read best you can such as #1 with salad, tax, the 6 is the total, and just find the total; 3+3 then divide by 7 or something like that
7.	Pull out the numbers first- go through each one and put the numbers into the problem and equation
8.	#1 solved using a tape-like diagram drew circles for salad, tax and total of $16- she needed to find taco Lisa: struggled to answer her questions because she doesn't speak English

Additional Notes from recordings (Author had to rephrase this question a lot)

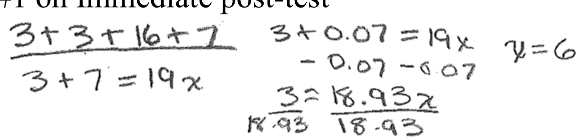
#4 Why did you feel compelled to use the techniques taught? If you did not use the techniques taught, then why did you feel compelled to not use them?

Student	**Notes**
	Control
1.	Used because helpful for him and prove to him that it helped- gave him firm understanding
2.	Good techniques- same, but a little different
3.	It helped her more
	Treatment
4.	Equations were hard, so relying on the techniques was useful. #2, he knew division was needed because of the key points and techniques taught so it helped him On the back- he used the techniques taught (simple technique for Algebra I)
5.	Techniques are easier than using mental math Easier to draw something out than just to look
6.	Easiest way because with a lot of addition, multiplication and division- makes it easier to graph and to see Try to get x, so there are two x's for #9, there can only be one, so 4-7 or 7-4. Did 7x-28 and try to get the answer: 4x=21 and divide by 4x on each side to get the answer.
7.	Felt easier to use what she was better at- things she was good at. If she used another method with equation, she didn't think she was going to get it correctly First method they learned was easier like the method she used on #'s 6-12
8.	She solved it the way she did because her drawing was helpful on the front Used her notes and paper to solve problems #'s 6-12

	1. #1 Pre-test  2. #1 Immediate Post-test  3. Post-delayed 

Additional Notes from recordings (Author had to rephrase the question a few times)

#5 What’s changed about your understanding? What caused the changes?

Student	Notes
Control	
1.	Had hard time answering- lessons helped him
2.	Just a different way of writing it
3.	Her understanding changed because it helped her more and gave her more knowledge
Treatment	
4.	3 weeks ago, he wouldn’t have understood, but the new technique he learned is easy for him now. -Lisa: unclear what he meant because he never used a tape diagram #1 on Immediate post-test  #2 on the immediate post-test

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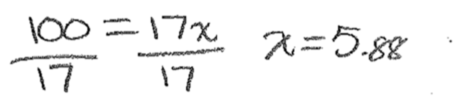
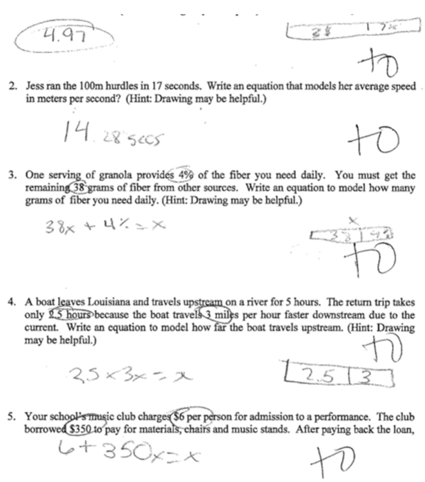
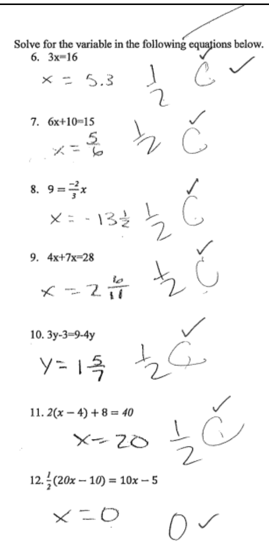
<tr><td></td><td>
</td></tr>
<tr><td>5.</td><td>Not good just saying something, she needs visual help to understand rather than writing numbers and just do this and that.</td></tr>
<tr><td>6.</td><td>Used to take him about 20min to get the answer, but now tape diagram makes it quicker- he sees everything and divides</td></tr>
<tr><td>7.</td><td>Nothing really changed- different way of doing it, but she prefers equation style
Easier to do equation than showing her work out

Work on her immediate post
Front: everything incorrect


4.97
2. Jess ran the 100m hurdles in 17 seconds. Write an equation that models her average speed in meters per second? (Hint: Drawing may be helpful.)
14.28 secs
3. One serving of granola provides 4% of the fiber you need daily. You must get the remaining 38 grams of fiber from other sources. Write an equation to model how many grams of fiber you need daily. (Hint: Drawing may be helpful.)
38x + 4% = x
4. A boat leaves Louisiana and travels upstream on a river for 5 hours. The return trip takes only 2.5 hours because the boat travels 3 miles per hour faster downstream due to the current. Write an equation to model how far the boat travels upstream. (Hint: Drawing may be helpful.)
2.5 × 3x = x
5. Your school's music club charges $6 per person for admission to a performance. The club borrowed $350 to pay for materials, chairs and music stands. After paying back the loan,
6 + 350x = x

Back: partial credit- no scrap paper handed in, she said she had it</td></tr>
</table>

	 Solve for the variable in the following equations below. 6. 3x=16 — x = 5.3 ½ C ✓ 7. 6x+10=15 — $x = \frac{5}{6}$ ½ C ✓ 8. $9 = \frac{-2}{3}x$ — $x = -13\frac{1}{2}$ ½ C ✓ 9. 4x+7x=28 — $x = 2\frac{6}{11}$ ½ C ✓ 10. 3y-3=9-4y — $y = 1\frac{5}{7}$ ½ C ✓ 11. $2(x-4)+8=40$ — x = 20 ½ C ✓ 12. $\frac{1}{2}(20x-10) = 10x-5$ — x = 0 0 ✓
8.	Author: had to rephrase this question She solved it how she normally does because the teacher does it in the class.