THE GEOMETRY OF END-PERIODIC MAPPING TORI

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ABSTRACT

Let S be a boundaryless infinite-type surface with finitely many ends and consider an end-periodic homeomorphism f of S. The end-periodicity of f ensures that M_f, its associated mapping torus, has a compactification as a 3-manifold with boundary; further, if f is atoroidal, then M_f admits a hyperbolic metric. Such maps admit invariant *positive and negative Handel-Miller laminations*, Λ^+, Λ^-, whose leaves naturally project to the arc and curve complex of a given compact subsurface $Y \subset S$.

As an end-periodic analogy to work of Minsky in the finite-type setting, we show that for every $\epsilon > 0$ there exists $K > 0$ (depending only on ϵ and the *capacity* of f) for which $d_Y(\Lambda^+, \Lambda^-) \geq K$ implies $\inf_{\sigma \in \mathrm{AH}(M_f)}\{\ell_\sigma(\partial Y)\} \leq \epsilon$. Here $\ell_\sigma(\partial Y)$ denotes the total geodesic length of ∂Y in (M_f, σ), and the infimum is taken over all hyperbolic structures on M_f.

This work produces the following: given a closed surface Σ, we provide a family of closed, fibered hyperbolic manifolds in which Σ is totally geodesically embedded, (almost) transverse to the pseudo-Anosov flow, with arbitrarily small systole.

This thesis is dedicated to my parents, Cassandra and Gregory. The lessons I’ve learned from you outweigh those from all my years in the classroom.

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CHAPTER 1

INTRODUCTION

The interplay between laminations, mapping classes of surfaces, and hyperbolic 3-manifolds is central to the study of surfaces and the geometry of their mapping tori. The geometry of a closed, fibered, hyperbolic 3-manifold may be further understood by analyzing the combinatorial data of its monodromy. While the connections between such objects are well-studied in the finite-type setting, this work examines fibered hyperbolic 3-manifolds constructed via end-periodic homeomorphisms of infinite-type surfaces.

Let S be a boundaryless infinite-type surface with finitely many ends, all non-planar. We say a homeomorphism f of S is *end-periodic* if each of its ends is either *attracting* or *repelling* under f, and that is it *atoroidal* if it admits no periodic curve up to isotopy. Field-Kim-Leininger-Loving [10] provide plentiful examples of such maps as the composition of commuting handle shifts with sufficiently large powers of a pseudo-Anosov map on a compact subsurface of S. In recent work, Field-Kent-Leininger-Loving [9] introduce a description of the complexity of an end-periodic map by the pair $(\chi(f), \xi(f))$; intuitively, $\xi(f)$ measures the amount of shifting in and out of the ends, and $\chi(f)$ is the complexity of a compact subsurface known as a *minimal core* of the map f. As a joint notion of complexity, we define the *capacity genus of f* by $\eth(f) = \chi(f) + \xi(f)^2$. See Section 2.6.3. for precise definitions of these terms.

The *mapping torus* of f is the fibered, boundaryless 3-manifold with monodromy f defined by the quotient

$$M_f := S \times [0,1]/(x,1) \sim (f(x),0).$$

Interestingly, the end-periodic nature of the homeomorphism allows its mapping torus to admit a compactification $\overline{M}_f$ as a 3-manifold with boundary (see e.g. Section 2.8.). The structure of the homeomorphism's action on curves determine topological properties of the manifold. In particular, if f is atoroidal, then M_f is atoroidal, and by a further application of Thurston's hyperbolization theorem for manifolds with boundary, we have that M_f is hyperbolizable (see [10] as stated for irreducible end-periodic maps). In general, when an irreducible manifold with incompressible boundary M satisfies $\chi(\partial M) < 0$, if its interior admits a hyperbolic structure, then it admits a plethora of hyperbolic structures parameterized by the deformation space AH(M) (see [13, Theorem 8.4]).

In unpublished work during the 1990s, Handel and Miller developed a lamination theory for end-periodic maps introducing the *negative* and *positive Handel-Miller laminations* Λ^- and Λ^+ of S. As one would hope, there are many parallels between the Handel-Miller laminations of an end-periodic homeomorphism and the stable and unstable measured laminations λ^+, λ^- of a pseudo-Anosov homeomorphism of a closed surface (see [16, Section 8], [5, Section 4]). From the perspective of a compact subsurface $Y \subset S$, the leaves of the positive (resp. negative) Handel-Miller lamination project to $\pi_Y(\Lambda^{\pm})$, a finite (possibly empty), disjoint collection arcs in the arc and curve complex $\mathcal{AC}(Y)$ of Y. From there we may consider the quantity $d_Y(\Lambda^+, \Lambda^-)$ which measures the distance between the respective subsurface projections $\pi_Y(\Lambda^+)$ and $\pi_Y(\Lambda^-)$ in $\mathcal{AC}(Y)$.

Given a subsurface $Y \subset S \subset M_f$ and a hyperbolic structure $s \in \mathrm{AH}(M_f)$, we consider $\ell_s(\partial Y)$, the summed length of the geodesic representatives of each component of ∂Y in (M_f, s).

Theorem A. *For any $D, \varepsilon > 0$, there exists $K = K(D, \epsilon)$ such that for any atoroidal end-periodic homeomorphism $f : S \to S$ with capacity genus $\eth(f) \leq D$, there exists $s \in AH(M_f)$ satisfying the following:*

For any connected, compact subsurface $Y \subset S \subset M_f$, we have:

$$d_Y(\Lambda^+, \Lambda^-) \geq K \implies \ell_s(\partial Y) \leq \varepsilon.$$

We note that our proof begins with an arbitrary atoroidal end-periodic homeomorphism f and produces a hyperbolic metric $s \in \mathrm{AH}(M_f)$ satisfying the implication of Theorem A. In particular, we emphasize that the metric s is chosen independently of D and ε. We hope such a statement, with constants controlled by the capacity genus of the homeomorphism, holds for any hyperbolic metric in $\mathrm{AH}(M_f)$.

When $\overline{M}_f$ is acylindrical, equivalently when f is *strongly irreducible* (see Theorem 2.20), there exists a unique metric $s^* \in \mathrm{AH}(M_f)$ whose convex core has totally geodesic boundary [18, Section 4]. For a strongly irreducible homeomorphism f, let s^* denote this unique structure. We wish to strengthen our main theorem so that s^* realizes the bounds of Theorem A in the following way:

Conjecture 1.1. *For any $D, \varepsilon > 0$, there exists $K = K(D, \varepsilon)$ such that for any strongly irreducible end-periodic homeomorphism $f : S \to S$ with capacity genus $\eth(f) \leq D$, and any connected, compact subsurface $Y \subset S \subset M_f$, we have:*

$$d_Y(\Lambda^+, \Lambda^-) \geq K \implies \ell_{s^*}(\partial Y) \leq \varepsilon.$$

Such a result would gain control over the totally geodesic surfaces which arise as boundaries of compact, irreducible, acylindrical hyperbolic manifolds with boundary. The next theorem confirms the conjecture when ∂M is restricted to consist solely of genus-2 surfaces.

Theorem B. *For any $D, \varepsilon > 0$, there exists K satisfying the following: suppose that $f : S \to S$ is a strongly irreducible end-periodic homeomorphism whose compactified boundary ∂M consists solely of genus-2 surfaces, and that $\eth(f) \leq D$; then for any compact essential subsurface $Y \subset S \subset M_f$, we have:*

$$d_Y(\Lambda^+, \Lambda^-) \geq K \implies \ell_{s^*}(\partial Y) \leq \varepsilon.$$

Results in Chapter 8 confirm that this phenomenon is not restricted to the genus-2 setting. For each genus $g \geq 2$, Example 8.7 provides a family of strongly irreducible homeomorphisms f_n, whose boundary consists of genus-g surfaces with arbitrarily thin totally geodesic boundary metrics.

1.1. Totally Geodesic Boundary Structures

Let Σ be a closed surface, and let $\mathcal{M}(\Sigma)$ denote its moduli space: the unmarked space of hyperbolic structures on Σ. Let $\mathcal{S}(\Sigma)$ denote the collection of all strongly irreducible homeomorphisms f on any boundary-less infinite-type surfaces with finitely many, all non-planar ends with $\Sigma \subset \partial\overline{M}_f$.

When f is strongly irreducible, we let $(\overline{M}_f, s_f^*)$ denote its acylindrical mapping torus adorned with its unique totally geodesic boundary structure. Let $\partial\mathfrak{E}(\Sigma) \subset \mathcal{M}(\Sigma)$ denote the collection of unmarked totally geodesic boundary metrics recovered over all $f \in \mathcal{S}(\Sigma)$, i.e.:

$$\partial\mathfrak{E}(\Sigma) := \{(\Sigma, s_f^*) \mid f \in \mathcal{S}(\Sigma)\}.$$

Question 1.2. *Suppose $\mathcal{S}(\Sigma)$ is non-empty. Is $\partial\mathfrak{E}(\Sigma)$ dense in $\mathcal{M}(\Sigma)$?*

Example 8.7 provides a sliver of hope toward a positive answer to Question 1.2. Namely, we show for $g \geq 2$, $\partial\mathfrak{E}(\Sigma_g)$ escapes each ϵ-thick part of $\mathcal{M}(\Sigma)$. We note that this construction can be extended so that $\partial\overline{M}_f$ is the union of an even number of connected genus-g surfaces $\Sigma^1 \sqcup \ldots \sqcup \Sigma^{2n}$.

Our work in the world of infinite-type surfaces culminates in a result about totally geodesic closed surfaces of closed, fibered hyperbolic 3-manifolds. Given a hyperbolic surface X, we let $\ell(X)$ denote the length of its systole, i.e. its shortest closed geodesic.

Theorem C. *For each $g \geq 2, \varepsilon > 0$, there exists a closed, fibered, hyperbolic manifold $N = N(g, \varepsilon)$ satisfying the following. N admits a totally geodesic embedding of a genus-g hyperbolic surface Σ_g with $\ell(\Sigma_g) \leq \varepsilon$. Further, Σ_g is (almost) transverse to a circular pseudo-Anosov flow on N.*

With a doubling construction of Landry-Minsky-Taylor [16], a positive answer to Question 1.2 would show that up to arbitrarily small deformations, every hyperbolic surface appears as a totally geodesic subsurface of some *fibered* hyperbolic 3-manifold.

1.2. Motivation from the Finite-type Setting

Question 1.2 is inspired by work of Fujii-Soma [11] who show that $\mathcal{R}(\Sigma)$, the space of totally geodesic boundary structures represented by acylindrical manifolds with boundary homeomorphic to Σ, is dense in $\mathcal{M}(\Sigma)$. We also take inspiration from work of Kent [14] who conjectures that such a statement is true for the collection of knot complements with totally geodesic boundary.

Theorems A and B are directly inspired by [19, Theorem B] of Minsky which formed one half of his *Bounded Geometry Theorem*. The theorem was developed surrounding the program to resolve the *Ending Lamination Conjecture* (now the *Ending Lamination Theorem* [4]). The Bounded Geometry Theorem establishes a combinatorial condition using the manifold's ending laminations to detect when a Kleinian surface group has bounded injectivity radius, a setting in which Ending Lamination Conjecture was proven [22].

As reflected in the statement of Theorem A, Minsky's theorem relates the subsurface projection distance $d_Y(\lambda^+, \lambda^-)$ between the stable and unstable laminations of a pseudo-Anosov homeomorphism to the length of the geodesic representative of ∂Y in the hyperbolic mapping torus of the pseudo-Anosov homeomorphism. In this setting, λ^+, λ^- are precisely the ending laminations of the cover corresponding to the given fiber of the hyperbolic manifold. We rephrase the theorem in the following setting [1]:

Theorem 1.3 (Minsky [19]). *Given F a closed, connected genus-g surface and $\varepsilon > 0$, there exists $K = K(g, \epsilon) > 0$ so that if N is a hyperbolic manifold fibered by F with pseudo-Anosov monodromy ϕ and its unstable and stable laminations λ^+, λ^- and Y is a proper, essential subsurface of F, then:*

$$d_Y(\lambda^+, \lambda^-) \geq K \implies \ell_N(\partial Y) \leq \varepsilon.$$

[1] This follows from Theorem B of [19] by considering the $(\widetilde{N}_\infty, s)$, the infinite-cyclic cover of N_ϕ corresponding to the fiber, with its hyperbolic metric s induced by the cover which isometrically immerses onto N_ϕ. In this setting, the ending laminations $\nu_\pm$ of $(\widetilde{M}_\infty, s)$ are precisely the stable and unstable laminations $\lambda^\pm$. By an observation of Minsky, there exists an $L_0 > 0$ for which the collection of curves $\pi_Y(C_0(s, L_0)) \subset \mathcal{N}_1(\pi_Y(\lambda^+) \cup \pi_Y(\lambda^-))$. In other words, $d_Y(\lambda^+, \lambda^-) \asymp \mathrm{diam}_Y(C_0(s, L_0))$, up to a negligent constant.

We use this theorem both as motivation and a tool to extend its result to the setting of infinite-type surfaces and their atoroidal end-periodic homemomorphisms. The constant K in Theorem 1.3 depends only on ε and the topological complexity of F; i.e. for a fixed ε and fixed genus, the constant is independent of the choice of pseudo-Anosov monodromy, and hence the hyperbolic metric on N. Rather than depending on any notion of "complexity" of the surface S, the constant K in Theorem A, and Theorem B depends on the aforementioned capacity genus of f.

CHAPTER 2

BACKGROUND

2.1. Surfaces and Subsurfaces

Throughout this work, we'll consider surfaces with finitely many ends, all non-planar, and compact boundary. All surfaces in this work are orientable. We say a surface S is finite-type if its fundamental group $\pi_1(S)$ is finitely generated, and that it is infinite-type otherwise. Finite-type surfaces are classified by their number of genus $g(X)$ and number of boundary components $b(X)$. Let $\chi(X) = 2 - 2g(X) - b(X)$ be its Euler characteristic, and $\xi(X) = 3g(X) + 3 + b(X)$ its complexity. When the surface is disconnected, its Euler characteristic and complexity are simply the sums of the Euler characteristics and complexities of each component. All finite-type surfaces within this work satisfy $\chi(X) \leq 0$, so that for any subsurface $Y \subset X$, $|\chi(Y)| \leq |\chi(X)|$.

More generally, Kerekjarto–Richards prove that all surfaces are classified by the homemorphism type of the pair formed by their space of ends and non-planar ends $(\mathrm{Ends}(S), \mathrm{Ends}_{np}(S))$ [25]. We refer the reader to the expository work of Aramayona-Vlamis [1] for further exposition of infinite-type surfaces and ends spaces.

2.2. The Mapping Class Group

Given a surface X, its *mapping class group* is defined by

$$\mathrm{Map}(X) = \mathrm{Homeo}^+(X, \partial X)/\,\mathrm{Homeo}_0(X, \partial X),$$

where $\mathrm{Homeo}^+(X, \partial X)$ denotes the group of orientation preseving homeomorphisms of X which restrict to the identity on ∂X, and $\mathrm{Homeo}_0(X, \partial X)$ denotes the normal subgroup of such homeomorphisms which are isotopic to the identity map.

When F is of finite-type, the mapping class group is finitely generated, and work of Nielsen and Thurston provide a classification of elements of the mapping class group.

Theorem 2.1 (The classification of the mapping class group [6])**.** *For F with $\chi(F) < 0$, an element $f \in \mathrm{Map}(F)$ satisfies exactly one of the following:*

1. *(periodic) f is finite-order;*
2. *(infinite-order reducible) f is infinite order and preserves some multicurve α;*
3. *(pseudo-Anosov) there exists a pseudo-Anosov representative of f which includes a pair of transverse measured foliations known as the unstable and stable foliations $(\mathcal{F}^u, \mu^u)$ and $(\mathcal{F}^s, \mu^s)$ and a number $\lambda > 1$ so that the representative preserves the foliations and the pushforward measures satisfy $f \cdot \mu^u = \lambda \mu^u$ and $f \cdot \mu^s = \frac{1}{\lambda}\mu^s$.*

The classification gives an equivalent condition to the otherwise wordy definition of a pseudo-Anosov homeomorphism: f is pseudo-Anosov mapping class if and only if it is infinite-order and admits no periodic curve.

By a process of "straightening" leaves of the unstable and stable foliations, we may also consider the unstable and stable *geodesic laminations* λ^+, λ^- of f (see **??**).

2.3. The Arc and Curve Complex

We consider isotopy classes of essential, simple, closed loops and proper isotopy classes of essential, properly embedded, simple paths on X. Recall that an path or loop is *simple* if it is an embedding of the circle or closed interval, and *essential* if it is non-peripheral and non-nullhomotopic in the surface. We use the terms *arcs* and *curves* to refer the proper isotopy classes of essential paths and loops, and reserve the terms *paths* and *loops* for particular embeddings of the closed interval or circle. We refer to a disjoint union of loops as a *multiloop*, and a disjoint union of distinct curves as a *multicurve*.

Let $\mathcal{AC}(X)$ denote the *arc and curve graph* of X, the simplicial graph whose vertices consist of arcs and curves of X, with edges between vertices whose representatives may be realized disjointly on the surface. The graph $\mathcal{AC}(X)$ is a metric space endowed with the path metric $d_{\mathcal{AC}(X)}$. The *curve graph* $\mathcal{C}(X)$ is the induced subgraph of $\mathcal{AC}(X)$ whose vertices are all curves of X. When X is of finite-type both $\mathcal{AC}(X)$ and $\mathcal{C}(X)$ have infinite-diameter, and are quasi-isometric to one another (see [15]). We refer the reader to [21], [26] for more information about this graph and its (coarse) geometry.

Given a compact annulus Y, the definition of $\mathcal{AC}(Y)$ requires more care. We define $\mathcal{AC}(Y)$ as the graph whose vertices are essential arcs from one end of Y to the other, modulo isotopies which *fix the endpoints* in ∂Y, again with edges between vertices which admit disjoint representatives.

We say a subset $Y \subset X$ is an *essential subsurface* if it is a connected, compact surface, not homeomorphic to a pair of pants, and further require component of ∂Y to represent an essential curve. Mapping classes of Y are represented by elements of $\mathrm{Homeo}(Y, \partial Y)$, the set of homeomorphisms which restrict to the identity on ∂Y. We naturally embed $\mathrm{Map}(Y) \leq \mathrm{Map}(X)$ by extending a representative of mapping class of Y to be the identity on $X - Y$, yielding a homeomorphism of X supported on Y. We will refer to a mapping class $g \in \mathrm{Map}(Y)$ as *fully supported* if g acts loxodromically on $\mathcal{AC}(Y)$. When Y is an annulus, each nontrivial element (i.e. some power of a Dehn twist) of $\mathrm{Map}(Y)$ is fully supported. For non-annular surfaces, fully supported mapping classes are precisely the pseudo-Anosov elements:

Theorem 2.2 (Minimal translation length [20]). *Let Y be a compact, connected surface with $\chi(Y) \leq 0$. Then, there exists a constant $c = c(Y)$, depending only on the topological type of Y, so that for any fully supported $g \in \mathrm{Map}(Y)$ we have:*

$$c|n| \leq d_{\mathcal{AC}(Y)}(\alpha, g^n(\alpha)).$$

Let $\tau_Y(g)$ denote the *stable translation length* of g on $\mathcal{AC}(Y)$, i.e.:

$$\tau_Y(g) := \lim_{n\to\infty} \frac{d_{\mathcal{AC}(Y)}(\alpha, g^n(\alpha))}{n}$$

It follows from the theorem above that when $g \in \mathrm{Map}(Y)$ is fully supported, $\tau_Y(g) \geq c$. We also note that $\tau_Y(g) \leq d_Y(\alpha, g(\alpha))$ for all $\alpha \in \mathcal{AC}(Y)$.

2.4. Subsurface Projections

Following [21], given an essential $Y \subset X$, we define the *subsurface projection map* to $\mathcal{P}(\mathcal{AC}(Y))$, the power set of $\mathcal{AC}(Y)$,

$$\pi_Y : \mathcal{AC}(X) \to \mathcal{P}(\mathcal{AC}(Y))$$

by the following:

Consider the cover $p : \widetilde{X}_Y \to X$ corresponding to the subgroup $\pi_1(Y) \leq \pi_1(X)$ and its compactification X_Y given by the action of $\pi_1(Y)$ on $\overline{\mathbb{H}}^2$. Here, X_Y is the quotient of $\mathbb{H}^2 \sqcup \Omega(\pi_1(Y))$ by the action of $\pi_1(Y)$, where $\Omega(\pi_1(Y))$ denotes the domain of discontinuity of the group action. Given $\alpha \in \mathcal{AC}(X)$, $p^{-1}(\alpha) \subset Y$ lifts to a disjoint collection of paths and loops, and we define $\pi_Y(\alpha)$ to be the collection of components of $p^{-1}(\alpha)$ which represent (essential) arcs and curves of X_Y. Whenever the projection is non-empty, $\mathrm{diam}_{\mathcal{AC}(Y)}(\pi(\alpha)) \leq 1$, providing a coarsely well-defined map from $\mathcal{AC}(X) \to \mathcal{AC}(Y) \cup \{\varnothing\}$. Note that we may realize α and ∂Y in minimal position by taking the geodesic representatives of the homotopy classes of α and ∂Y (see Figure 1). When Y is a non-annular subsurface of X, this produces a subsurface Y^* with totally geodesic boundary ∂Y^*. In this setting, $\pi_Y(\alpha)$ is the set of vertices in $\mathcal{AC}(Y)$ comprising of each component of $Y^* \cap \alpha^*$ which represents an essential curve or arc in Y^*.

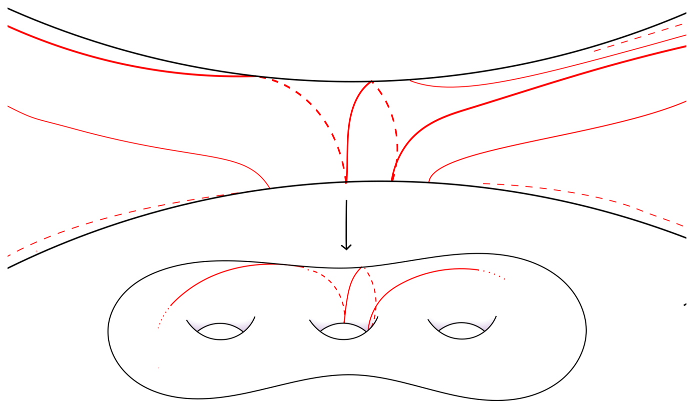

Figure 1. The lift of a path of X to an annular cover.

We will say that a subset $A \subset \mathcal{AC}(S)$ is *disjoint* from Y if $\pi_Y(A) = \varnothing$, and that it *meets* Y (equivalently, that Y meets A) otherwise. If A is disjoint from Y we will typically choose representative loops and paths which realize it as so.

Given a subsurface and subsets A, $B \subset \mathcal{AC}(S)$, both meeting Y, we define their *subsurface projection distance to* Y:

$$d_Y(A, B) := \mathrm{diam}_{\mathcal{AC}(Y)}(\pi_Y(A) \cup \pi_Y(B)).$$

We'll list a few observations about subsurface projection distances which will be helpful for calculations.

Remark 2.3. *Let A, A_i be subsets of $\mathcal{AC}(Y)$ with $\pi_Y(A), \pi_Y(A_i) \neq \varnothing$.*

1. $d_Y(A_1, A_n) \leq \sum_{1 \leq i \leq n} d_Y(A_i, A_{i+1})$

2. *For any* $h \in \mathrm{Map}(S)$*,* $d_Y(A_1, A_2) = d_{h(Y)}(h(A_1), h(A_2))$*. So, if* h *is supported on* Y*, then* $d_Y(A_1, A_2) = d_Y(h(A_1), h(A_2))$*.*

3. *For any subset* $A \subset \mathcal{AC}(Y)$*,*

$$\tau_Y(h) \leq d_Y(A, h(A))$$

4. *For subsets* $A'_i \subset A_i$*,*

$$d_Y(A'_1, A'_2) \leq d_Y(A_1, A_2)$$

5. *If there exists a subsurface* W *for which* $\pi_W(A)$*,* $\pi_W(\partial Y) \neq \varnothing$ *and* $d_W(A, \partial Y) \geq 2$*, then* $\pi_Y(A) \neq \varnothing$*.*

It's important to note that $\pi_Y(\alpha) \neq \varnothing$ does not imply $\pi_Y(h(\alpha)) \neq \varnothing$. In light of this, we will apply (5) to ensure the projections are non-empty during such calculations.

2.5. Foliations and Laminations of Surfaces

A *geodesic lamination* Λ of X is a closed subset of X which is the union of complete, simple geodesics known as *leaves* of the geodesic lamination. We are mostly concerned with the underlying topology of Λ rather than any local structure, so we will simply refer to a closed subset as a lamination if it is topologically equivalent to a geodesic lamination.

Given $\mathcal{G}$, a disjoint union of complete geodesics on X, we let $\overline{\mathcal{G}}$ denote its closure as a subset in X; we note that $\overline{\mathcal{G}}$ is (the support of) a geodesic lamination of X (see [6, Lemma 3.2]).

Since a given leaf of a lamination is not necessarily a proper embedding of $\mathbb{R} \hookrightarrow X$, it may not be appropriate to denote it as a vertex of the arc and curve graph of the surface. However, we may still define $\pi_Y(\Lambda^{\pm})$ by lifting the lamination to $\widetilde{\Lambda}_Y \subset X_Y$, so that $\pi_Y(\Lambda)$ denotes the (possibly empty) collection of vertices of $\mathcal{AC}(Y)$ which represent (essential) arcs and curves of $\widetilde{\Lambda}_Y$.

A singular foliation $\mathcal{F}$ of a surface is a decomposition of X into 1-manifolds known as *leaves* and a finite collection $\mathcal{P}$ of *singularities* so that the collection $\mathcal{F} - \mathcal{P}$ forms a foliation of $X - \mathcal{P}$. A leaf which is disjoint from the singular collection $\mathcal{P}$ is a *regular* leaf, while one which meets $\mathcal{P}$ is *singular*. For a singular foliation, a *leaf-line* of $\mathcal{F}$ is either a regular leaf of $\mathcal{F}$, or bi-infinite line formed by a union of at least two singular leaves so that any pair of leaves which meet at singularity are adjacent.

A closed subset Λ of $\mathcal{F}$ formed by the union of leaf-lines of $\mathcal{F}$ is known as a *(singular) sublamination* of $\mathcal{F}$. We use the same terminology of leaf, leaf-line, and singular leaf to refer to the leaves of the sublamination.

A singular, Reeb-less foliation on a hyperbolic surface X has a corresponding lamination, $\Lambda(\mathcal{F}) \subset X$, obtained by the following process of "straightening" leaf-lines of $\mathcal{F}$ [17, Theorem 1]. The lifts of each leaf-line of $\mathcal{F}$ form a disjoint collection of bi-infinite quasi-geodesics of $\overline{\mathbb{H}^2}$ which defines a subset $\partial^2(\mathcal{F})$ of $\partial^2(\mathbb{H}^2)$, the space of unoriented geodesics of $\mathbb{H}^2$. For each point $(z, w) \in \partial^2(\mathcal{F})$, consider the unique bi-infinite geodesic representative between (z, w). The union of such geodesics is $\pi_1(X)$-invariant lamination of $\overline{\mathbb{H}^2}$ which descends to a lamination $\Lambda(\mathcal{F})$ of X.

For a compact, essential subsurface $Y \subset X$, we define $\pi_Y(\mathcal{F}) := \pi_Y(\Lambda(\mathcal{F}))$. Similarly, if Λ is a (singular) sublamination of $\mathcal{F}$, we define $\pi_Y(\Lambda)$ to be the union of $\pi_Y(\ell)$ over all leaf-lines ℓ of Λ.

We will later refer to singular 2-dimensional foliations of 3-manifolds, but only those obtained as the suspension of a singular foliation of a surface. For such singular foliations, the 1-dimensional leaves of the surface foliation suspend to 2-dimensional leaves, and the finite set of singularities suspend to 1-dimensional singular closed orbits.

2.6. End-periodic Homeomorphisms

From this section onward, let S be a boundaryless infinite-type surface with finitely many ends. Recall that $\mathrm{Ends}(S)$ denotes its finite space of ends.

A homeomorphism f of S is *end-periodic* if for each end $\epsilon \in \mathrm{Ends}(S)$ there exists $p \in \mathbb{Z}_{\geq 1}$ and a neighborhood U_ϵ of ϵ so that the end ϵ is either *attracting* or *repelling* in the following sense:

- (ϵ is an *attracting* end): $f^p(U_\epsilon) \subset U_\epsilon$ is a proper inclusion, and the collection $\{f^{pn}(U_\epsilon)\}_{n\in\mathbb{Z}}$ forms a neighborhood basis of ϵ.

- (ϵ is a *repelling* end): $f^{-p}(U_\epsilon) \subset U_\epsilon$ is a proper inclusion, and the collection

  $\{f^{pn}(U_\epsilon)\}_{n\in\mathbb{Z}}$ forms a neighborhood basis of ϵ.

Following terminology of Field-Kim-Leininger-Loving [10], we refer to a neighborhood U_ϵ which realizes ϵ as an attracting or repelling end as a *nesting neighborhood* of ϵ. The map f naturally acts on $\mathrm{Ends}(S)$, partitioning the finite set of ends so that $\mathrm{Ends}(S) = \mathrm{Ends}^+(S) \sqcup \mathrm{Ends}^-(S)$. $\mathrm{Ends}^\pm(S)$ further decomposes into a collection of end-orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The period of an end ϵ is the cardinality of its end-orbit; or equivalently, it is the minimal p for which $f^p(\epsilon) = \epsilon$. We will use $+, -$ to denote an attracting (respectively repelling) end, one of its neighborhoods, or a curve in a such a neighborhood. We will typically order the ends in an attracting orbit so that $\epsilon_i = f^i(\epsilon_0)$ and in a repelling orbit so that $\epsilon_i = f^{-i}(\epsilon_0)$.

As in [8], we organize nesting neighborhoods of $\mathrm{Ends}^\pm(S)$ and define a positive (resp. negative) *ladder* of f as a neighborhood of $\mathrm{Ends}^\pm(S)$ of the form

$$U_+ = \bigcup_{\epsilon\in\mathrm{Ends}^+(S)} U_\epsilon \qquad U_- = \bigcup_{\epsilon\in\mathrm{Ends}^-(S)} U_\epsilon$$

where $\{U_\epsilon\}_{\epsilon \in \mathrm{Ends}^{\pm}(S)}$ are pairwise disjoint, and each U_ϵ is a nesting neighborhood of ϵ. We say the ladder $U_\pm$ is *tight* if $f^{\pm 1}(U_\pm) \subset U_\pm$ is a proper inclusion. We use tight ladders to define the *positive escaping set* and *negative escaping set* of f respectively:

$$\mathcal{U}_+ = \bigcup_{n \in \mathbb{Z}} f^n(U_+) \qquad \mathcal{U}_- = \bigcup_{n \in \mathbb{Z}} f^n(U_-)$$

We will also refer to the *escaping set of an end* ϵ of period p by:

$$\mathcal{U}_\epsilon = \bigcup_{n \in \mathbb{Z}} f^{pn}(U_\epsilon),$$

where U_ϵ is a connected nesting neighborhood of ϵ.

From the definition of end-periodicity, one can verify that these definitions are independent of the choice of $U_\pm$ (or U_ϵ). Observe that each end-periodic map admits a tight ladder—for example, take the positive (negative) ladder consisting of the forward (backward) f-orbits of each nesting neighborhood for each attracting (repelling) end. As the name implies, the escaping sets form the collection of points whose positive (resp. negative) iterates under f eventually land in an attracting (resp. repelling) nesting neighborhood, hence "escape" the end.

We say that a loop is *positively* (resp. *negatively*) *escaping* under f if it is contained in the positive (resp. negative) escaping set of f. The escaping sets $\mathcal{U}_\pm$ themselves are surfaces of infinite type, thus there are uncountably many loops (representing uncountably many distinct curves) which escape into the positive or negative ends under iteration of the map f.

We recall work of Fenley, exposited in [10], which describes the "end behavior" of the end-periodic homeomorphism:

Lemma 2.4. *(Fenley [8], Field-Leininger-Loving-Kim [10]). Let f be an end-periodic homeomorphism. Then f acts freely, properly discontinuously, and cocompactly on $\mathcal{U}_\pm$. The quotient of the action $\mathcal{U}_\pm/\langle f\rangle$ defines the following surfaces*

$$\Sigma_+ := \mathcal{U}_+/\langle f\rangle \qquad \Sigma_- := \mathcal{U}_-/\langle f\rangle$$

Further, the surfaces satisfy $\chi(\Sigma_+) = \chi(\Sigma_-)$.

Observe that the components of $\Sigma = \Sigma_+ \cup \Sigma_-$ are in bijection with the end-orbits of f.

A loop or line α is *periodic* under f if $\exists\, m \neq 0$ s.t. $f^m(\alpha)$ is properly isotopic to α. We say a loop or line α is *reducing* $\exists\, m, n \in \mathbb{Z}$ s.t. $f^n(\alpha) \subset \mathcal{U}_+$, and $f^m(\alpha) \subset \mathcal{U}_-$, in other words, if α is both positive and negative escaping. We say a curve or arc is periodic or reducing if one of its representatives is.

We say an end-periodic homeomorphism is *atoroidal* if it admits no periodic curves. As the name implies, an atoroidal end-periodic homeomorphism yields an atoroidal mapping torus M_f (see Theorem 2.20). Field-Kim-Leininger-Loving introduce *strongly irreducible* end-periodic homeomorphisms, which are atoroidal end-perioidic homeomorphisms which admit no periodic arcs or reducing curves.

We direct the reader to Chapter 8 for more examples of end-periodic homeomorphisms, including Field-Kim-Leininger-Loving's construction of strongly irreducible end-periodic homeomorphisms.

We say that an essential multiloop of $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) is a positive (resp. negative) *juncture* if it is the boundary of a tight positive (resp. negative) ladder. It's worth noting that a multiloop does not necessarily bound a tight ladder of a given homeomorphism, even if one of its friends in its isotopy class does. In light of this, we emphasize that we are typically concerned with the genuine representative loop rather than its isotopy class.

Fix j_+ and j_-, a positive and negative juncture of f. We define the *positive juncture orbit* J^+, and *negative juncture orbit* J^- (with respect to j_+ and j_-, respectively) as the multiloops

$$J^+ = \bigcup_{k\in\mathbb{Z}} f^k(j_+) \qquad J^- = \bigcup_{k\in\mathbb{Z}} f^k(j_-).$$

To verify that these unions are indeed multiloops, consider the tight ladders $U_\pm$ bounded by $j_\pm$. It follows that $\forall k \in \mathbb{Z}_{\geq 0}$, $f^k(U_\pm) \subset U_\pm$ is a proper inclusion, which implies that each component of $f^i(j_\pm)$ is either disjoint from, or coincides with some component of $f^k(j_\pm)$ for all $i, k \in \mathbb{Z}$.

Remark 2.5. *Consider the partial juncture orbits*

$$J^+(k,\infty) = \bigcup_{i \geq k} f^i(j_+) \qquad J^+(-\infty,k) = \bigcup_{i \leq k} f^i(j_+)$$

$$J^-(-\infty,k) = \bigcup_{i \leq k} f^i(j_-) \qquad J^-(k,\infty) = \bigcup_{i \geq k} f^i(j_-).$$

The partial juncture orbits $J^+(k,\infty)$ and $J^-(-\infty,k)$ are both closed subsets of S.

While the juncture orbits are somewhat uninteresting in the ends of S, they may have more interesting behavior when considering their closure in the entire surface.

2.6.1. Handel-Miller Laminations

Fix a hyperbolic metric X on S, and let J_+, J_- be a pair of positive and negative juncture orbits for f. Let $\mathcal{J}_\pm$ be the union of tightened geodesics for each curve in $J_\pm$. Since $J_\pm$ is an multiloop, $\mathcal{J}^\pm$ is foliated by geodesics and defines a geodesic lamination $\overline{\mathcal{J}^\pm}$. The *Handel-Miller laminations* [5, Section 4] are the geodesic laminations defined as

$$\Lambda^+ = \overline{\mathcal{J}^-} - \mathcal{J}^-$$

$$\Lambda^- = \overline{\mathcal{J}^+} - \mathcal{J}^+.$$

It will be helpful to consider the Hausdorff distance between the junctures and the respective laminations. Let $\mathcal{J}_k^+$ denote the geodesic representative of the multiloop $J^+(-\infty, k)$, and $\mathcal{J}_k^-$ denote the geodesic representative of $J^-(k, \infty)$. Then for every $\varepsilon > 0$, there exists K for which: $\mathcal{J}_k^+ \subset N_\varepsilon(\Lambda^-)$ and $\Lambda^- \subset N_\varepsilon(\mathcal{J}_k^-)$ for all $k \leq K$ (likewise, $\mathcal{J}_k^- \subset N_\epsilon(\Lambda^+)$ and $\Lambda^+ \subset N_\varepsilon(\mathcal{J}_k^-)$ for all $k \geq K$). In this sense, the negative iterates of a positive juncture limit to the negative Handel-Miller lamination, and the positive iterates of a negative juncture orbit limits to the positive Handel-Miller lamination.

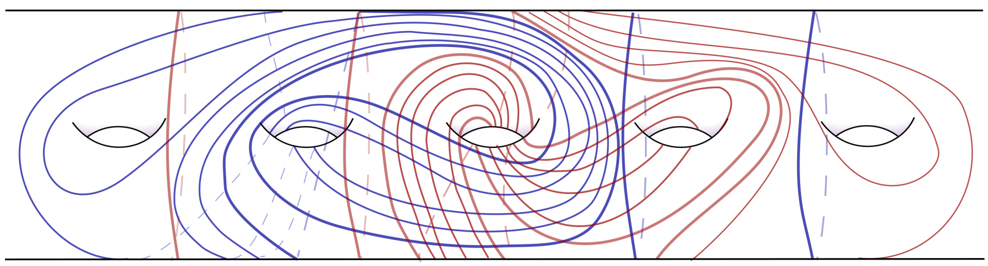

Figure 2. A few components of the juncture multicurves.

Observation 2.6. *For any positive juncture orbit, its geodesic multicurve $\mathcal{J}^+$ is disjoint from the negative Handel-Miller lamination Λ^-; likewise, any for negative juncture orbit, its geodesic multicurve $\mathcal{J}^+$ is disjoint from the positive Handel-Miller lamination Λ^+.*

Observation 2.7 (Laminations avoid ladders)**.** *Let j_+ be a juncture, and U_+^* be the neighborhood of $\mathrm{Ends}^+(S)$ bounded by the geodesic representative of j_+. Then $\Lambda^- \cap U_+^* = \varnothing$. Similarly, for any negative juncture j_-, the corresponding neighborhood U_-^* is disjoint from Λ^+.*

After fixing a hyperbolic metric X of S, the Handel-Miller geodesic laminations are independent of the isotopy class of f or the choice of juncture orbits. Any two choices of juncture orbit pairs (J_1^+, J_1^-) and (J_2^+, J_2^-) yield the same Handel-Miller laminations $\Lambda^{\mp}$ [5, Corollary 4.72]. Note that the Λ^+, Λ^- may be empty if $\mathcal{J}^+, \mathcal{J}^+$ themselves are closed—Cantwell-Conlon-Fenley show that this is the case precisely when the map is a *translation*, i.e. there exists a connected neighborhood U_e of an end e so that $\mathcal{U}_e = \bigcup_{n \in \mathbb{Z}} f^n(U_e) = S$ [5, Corollary 4.78].

The Handel-Miller laminations detect escaping curves in the following sense:

Proposition 2.8. *Let γ be a loop of S. The following are equivalent:*

1. *γ is positively escaping*
2. *γ meets only finitely many components of J_+*
3. *γ is disjoint from the negative Handel-Miller lamination Λ^-*

Similarly, γ is negatively escaping iff it meets only finitely many components of J_+ iff the homotopy class can be made disjoint from the positive Handel-Miller lamination Λ^+.

Proof. We'll prove the statement for a positively escaping loop; the proof for the negatively escaping is analogous. Suppose $\gamma \subset \mathcal{U}_+$, and let $J_+ = \bigcup f^k(j_+)$ be any positive juncture orbit of f. Given that J_+ is closed in $\mathcal{U}_+$, and by the compactness of γ, there are only finitely many i for which $f^i(j_+)$ meets γ. In other words, γ is disjoint from $J^+(-\infty, k)$ for sufficiently large k. Since $\Lambda^- \subset \overline{J^+(-\infty, k)}$ for all k, the converse of Remark 5.1 implies that γ does not meet Λ^- transversely. This shows $(i) \implies (ii) \implies (iii)$.

We'll show that $(iii) \implies (ii) \implies (i)$, assuming Λ^- is non-empty (otherwise, the statement is vacuously true). If γ can be made disjoint from Λ^-, then there exists $\varepsilon > 0$ for which γ is disjoint from $N_\varepsilon(\Lambda^-)$. Since the negative iterates of j_+ converge to the negative lamination, for sufficiently large k, $J^+(-\infty, k) \subset N_\epsilon(\Lambda^-)$, and is therefore disjoint from γ. Again, since $J^+(k, \infty)$ is closed, γ intersects at most finitely many of its components. Thus, γ meets only finitely many components of J_+. Since the junctures separate S, it must be that γ is contained in the positive ladder defined by $f^k(j_+)$. This completes the proof. □

Remark 2.9. *For any compact, essential subsurface, Y,*

1. $\pi_Y(\Lambda^\pm) \in \mathcal{P}(\mathcal{AC}(S))$ *is either empty or a diameter-1 subset of $\mathcal{AC}(S)$.*

2. *If $\pi_Y(\Lambda^\pm) = \varnothing$, then $\Lambda^\pm$ may be realized disjointly from the subsurface Y.*

3. *The following are equivalent:*

   (a) *Both laminations Λ^+ and Λ^- meet Y*

   (b) *Both laminations Λ^+ and Λ^- meet ∂Y*

   (c) *Y is not contained in any positive or negative ladder of f.*

Proof. The first bullet follows immediately. The second would be immediate if the components of $\Lambda^{\pm} \cap Y$ were apriori proper paths. Indeed, a corollary of Cantwell-Conlon-Fenley [5, Corollary 4.47] states that no leaf of $\Lambda^{\pm}$ is contained in a bounded region of S. In fact, they show that for each leaf of the positive (resp. negative) lamination, each of its ends escape into attracting (resp. repelling) ends of S. Thus, the components of the intersection $\Lambda^{\pm} \cap Y$ are never half leaves, hence are compact arcs of Y which necessarily meet ∂Y. This discussion shows $(i) \iff (ii)$.

We'll prove $(i) \iff (iii)$ by proving the contrapositive. If Y were contained any positive (resp. negative) ladder, then it meets at most finitely many components of J_{-}. By Proposition 2.8, it is disjoint from Λ^{+}, a contradiction. Similarly, if Y misses some lamination, then Proposition 2.8 shows that it meets only finitely many components of $J_{\pm}$, which means up to an isotopy of the subsurface, it is contained in some ladder. $\square$

Corollary 2.10. *Suppose f is (strongly) irreducible. For any curve $\gamma \subset S$, either γ meets Λ^{+}, γ meets Λ^{-}, or both. Similarly, for each essential subsurface Y of S, either Y meets Λ^{+}, Y meets Λ^{-} or both laminations.*

In other words, each loop or essential subsurface is either positively escaping, negatively escaping, or "trapped" by both laminations.

Proof. If γ is disjoint from Λ^{+}, then it is negatively escaping and is contained in $\mathcal{U}_{+}$, similarly if it is disjoint from Λ^{-}, it is positively escaping and is contained in $\mathcal{U}_{-}$. This implies that γ is a reducing curve which is ruled out by the assumptions of a (strongly) irreducible end-periodic homeomorphism. The proof for a subsurface is identical. $\square$

2.6.1.1. Pseudo-invariance

For any hyperbolic structure X, there is a representative homeomorphism $f^\sharp$ known as a *Handel-Miller representative* of f which is isotopic to f and preserves the geodesic Handel-Miller laminations (and a choice of geodesic juncture orbits). We point the reader to work of Fenley [8], and Cantwell-Colon-Fenley [5] for further exposition on the laminations and its Handel-Miller representatives, but end this section with a few observations. Subsurface projection distance gives a explicit sense in which our original end-periodic homeomorphism f preserves the Handel-Miller geodesic laminations:

Lemma 2.11 (Pseudo f-invariance)**.** *For any essential subsurface* Y*,* $\pi_Y(f(\Lambda^\pm)) = \pi_Y(\Lambda^\pm)$ *and* $d_Y(\Lambda^+, \Lambda^-) = d_{f^n(Y)}(\Lambda^+, \Lambda^-)$.

Proof. Let Λ^+, Λ^- denote the Handel-Miller geodesic laminations, and let $f^\sharp$ denote a Handel-Miller representative homeomorphism isotopic to f so that $f^\sharp(\Lambda^\pm) = \Lambda^\pm$. We'll assume $\pi_Y(\Lambda^-), \pi_Y(\Lambda^+) \neq \varnothing$. Lift $\Lambda^\pm$ and $f(\Lambda^\pm)$, to laminations $\widetilde{\Lambda}^\pm$ and $\widetilde{f\Lambda}^\pm$ of the cover $\widetilde{X}_Y$ corresponding to Y. The isotopy between the lifts $\widetilde{f}$ and $\widetilde{f^\sharp}$ produces a bijection between any properly embedded leaf $\widetilde{f^\sharp\lambda}$ of $\widetilde{\Lambda}^\pm$ to the leaf $\widetilde{f(\lambda)}$ of $\widetilde{f\Lambda}^\pm$. It follows that $\pi_Y(f^n(\Lambda^\pm)) = \pi_Y(\Lambda^\pm)$ so that

$$d_{f^nY}(\Lambda^+, \Lambda^-) = \mathrm{diam}_{\mathcal{AC}(f^nY)}(\pi_Y(f^n(\Lambda^+) \cup \pi_Y(f^n(\Lambda^-)) = d_Y(\Lambda^+, \Lambda^-).$$

□

2.6.2. Cores

A disjoint pair of positive and negative tight ladders, U_+, U_-, defines a compact surface $C = S - (U_+ \cup U_-)$ called a *core*. The boundary of the core consists of positive and negative junctures denoted respectively by $\partial_\pm C = \partial C \cap \overline{U_\pm}$. A core C' is a *subcore* of C if $C' \subset C$, and each component of $\partial C'$ is an essential loop of C. A core C is *minimal* if its Euler chracteristic realizes the quantity

$$\min\{|\chi(C')| : C' \text{ is a core of } S\}.$$

Since cores bound positive and negative tight ladders, we may expand a core by the action of f. We define a core C to be f*-extended* if it contains a subcore C' so that

$$\partial_+ C' \subset \bigcup_{k>0} f^{-k}(\partial_+ C), \quad \text{and} \quad \partial_- C' \subset \bigcup_{k>0} f^{k}(\partial_- C).$$

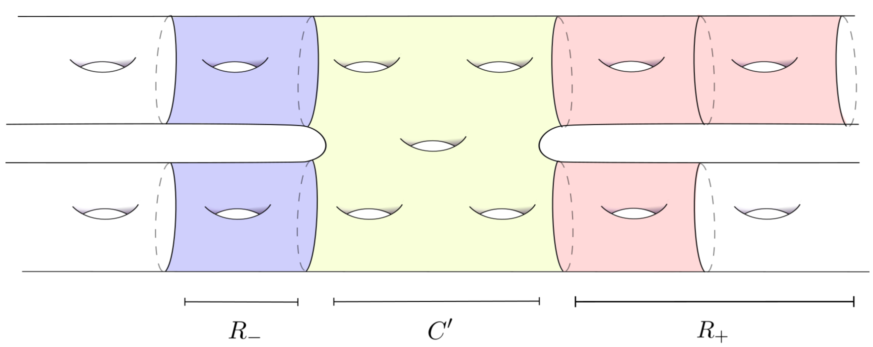

Figure 3. An f-extended core

We note to the reader that neither cores nor components of ladders $U_+ \cap \mathcal{U}_e$ in a particular end e are necessarily connected. The following remark notes that each end of S has a unique unbounded component of $S - C$, which is connected.

Remark 2.12. *For any core $C = S - (U_+ \cup U_-)$, the collection $U_1, \dots U_n$ of unbounded components of $U_+ \cup U_-$ are in bijection with $\mathrm{Ends}(S)$.*

Proof. Consider K_0, the union of the closure of all pre-compact components of $U_+ \cup U_-$, and let $K = C \cup K_0$. Then, K is compact and the components of $S - K$ define distinct end-neighborhoods, all of whom are necessarily the non-precompact, i.e. unbounded, components of $S - (U_+ \cup U_-)$. □

2.6.3. Complexity of end-periodic maps

The cores and shifting behavior of an end-periodic map f provide a notion of complexity for the map. Field-Kent-Leininger-Loving [9] define the *core characteristic* $\chi(f)$ of f as

$$\chi(f) := \min\{|\chi(C)| : C \text{ is a core of } S\}$$

and [10] define the *end complexity* $\xi(f)$ of f as

$$\xi(f) := \xi(\Sigma_+ \cup \Sigma_-).$$

We direct the reader's attention to Lemma 2.4 which states that $\partial\overline{M}_f$ is homeomorphic to the compact subsurface $\Sigma_+ \cup \Sigma_-$. As a joint notion of complexity, we define the *capacity genus* $\eth(f)$ of f by the following:

$$\eth(f) = \chi(f) + \xi(f)^2.$$

2.7. Mapping Tori

We'll give a brief overview of fibered 3-manifolds with finite-type fibers. Let F be a finite-type surface with $\chi(F) < 0$, and let ϕ be a homeomorphism of F. The mapping torus of ϕ, denoted by M_ϕ is obtained by $F \times [0,1]/(x,1) \sim (\phi(x), 0)$.

Theorem 2.13 (Thurston [29])**.** *A closed, fibered* 3*-manifold* M *admits a hyperbolic structure if and only if it is homotopically atoroidal.*

It then follows by the Nielsen-Thurston classification (Theorem 2.1), and by observing that essential tori of M_ϕ correspond to periodic curves of ϕ, that M_ϕ admits a hyperbolic structure if and only if ϕ is pseudo-Anosov. In the next section, we'll discuss the analogous case in the end-periodic setting.

2.8. The Compactified End-periodic Mapping Torus

Note that M_f is obtained by the quotient map

$$\pi : S \times \mathbb{R} \to S \times \mathbb{R}/\langle F \rangle$$

where $F(x,t) = (f(x), t-1)$. This induces a 2-dimensional foliation given by the fibers of the form $S_t = S \times \{t\}$.

We recall Field-Kim-Leininger-Loving's discussion of the compactification of M_f which was introduced by Fenley [8, Section 3]. Let $\widetilde{M_\infty}$ be the partial compactification of $S \times \mathbb{R}$ given by $\widetilde{M_\infty} = S \times \mathbb{R} \sqcup \mathcal{U}_{+\infty} \sqcup \mathcal{U}_{-\infty}$, where $\mathcal{U}_{\pm\infty} = \mathcal{U}_\pm \times \{\pm\infty\}$. Consider the foliation $\widetilde{\mathcal{F}}_\infty$ of $\widetilde{M_\infty}$ whose leaves consist of the pre-fibers $S_t = S \times \{t\}$ along with $\mathcal{U}_{+\infty}$ and $\mathcal{U}_{-\infty}$. The action of F on $S \times \mathbb{R}$ extends to $\widetilde{M_\infty}$ so that $F(x, \pm\infty) = (f(x), \pm\infty)$ preserving the leaves of $\widetilde{\mathcal{F}}$. Let $\mathcal{F}$ denote the 2-dimensional depth-one foliation of $\overline{M}_f$ obtained by the quotient of $\widetilde{\mathcal{F}}_\infty$. Each component Σ of ∂M is a depth-zero leaf of $\mathcal{F}$, while the S_t prefibers are depth-one leaves.

Theorem 2.14. *(Field-Kim-Leininger-Loving) $\overline{M}_f := \widetilde{M_\infty}/\langle F \rangle$ is a compact 3-manifold with boundary whose interior is homeomorphic to M_f. Further, $\partial \overline{M}_f = \partial_+ \overline{M}_f \sqcup \partial_- \overline{M}_f$ where $\partial_\pm \overline{M}_f = \Sigma_\pm$.*

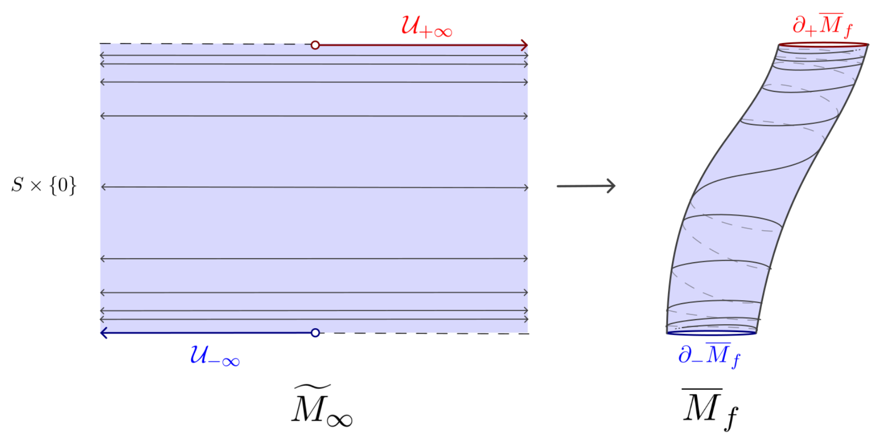

Figure 4. A schematic of A schematic of the mapping torus compactification via the quotient.

The flow on $\widetilde{M}_\infty = S \times \mathbb{R}$ given by $\hat{\varphi}((x,s),t) = (x, s+t)$ extends to a semi-flow on $\widetilde{M}_\infty$ by adding endpoints $(x, \pm\infty)$ to each flow line $\ell_x(t) = (x,t)$ of $x \in \mathcal{U}_\pm$. This vertical flow and semi-flow are both F-equivariant, thus they descend to a flow and semi-flow on M_f, known as the *suspension (semi-)flow* φ_M on $\overline{M}_f$. We will recycle notation and use φ_M to refer to both the flow on M_f and the semi-flow on $\overline{M}_f$. Let $\mathcal{L}$ be the 1-dimensional oriented foliation induced from the suspension semi-flow lines of φ_M on $\overline{M}_f$.

Proposition 2.15 (Product structure at the boundary)**.** *In a product neighborhood $U \cong \partial_+ M \times (k,\infty]$ of $\partial_+ M$, the foliation $\mathcal{L}$ is modeled by the vertical semi-flow given by:*

$$\varphi_M : (\partial_+ M \times (k,\infty]) \times \mathbb{R} \to (\partial_+ M \times (k,\infty])$$

$$((x,t),s) \mapsto (x, t+s)$$

The same statement holds in a product neighborhood $\partial_- M \times [-\infty, k)$ of $\partial_- M$.

Proof. Consider the quotient from the partially compactified space $\pi : \widetilde{M_\infty} \to M$. We work in the positive "end" of the manifold, but note that the proof is analogous for the negative "end". Choose a representative multi-loop j_+ of S which bounds a tight ladder U_+ in S. Consider the open subsurface $R_+ \subset S_0$ bounded by j_+ and $f(j_+)$.

Since the ladder is tight, it follows that no two points of $R_+ \times (\mathbb{R} \sqcup \{\infty\}) \subset S_0 \times (\mathbb{R} \sqcup \{\infty\})$ are identified under the action of $\langle F \rangle$. Thus, $\pi|_{R_+ \times \mathbb{R}}$ is an embedding, and the product structure of the flow on $R_+ \times (\mathbb{R} \cup \{\infty\})$ is preserved in its image in M. By patching together a second "chart" of the form $A_0 \times \mathbb{R} \sqcup \{\infty\}$, where A_0 is an open annulus of j_+, it follows that $\mathcal{L}$ is a foliation by vertical lines in this collar neighborhood of $\partial_+ M$. $\square$

Keeping in mind the product structure of $\mathcal{L}$ in the boundary, a *spiraling neighborhood* of a component $\Sigma \subset \partial M$ is a collar neighborhood U_Σ of Σ so that:

(i) U_Σ is a product foliated by arcs of $\mathcal{L}$, and

(ii) $\partial U_\Sigma = \Sigma \sqcup \Sigma_U$, where $\Sigma_U \cong \Sigma$ is transverse to $\mathcal{L}$ and the foliation $\mathcal{F}$.

We refer to a spiraling neighborhood of ∂M (or $\partial_+ M$ or $\partial_- M$) as a union of disjoint spiraling neighborhoods over each of its components (see e.g. [16, Section 3.1]).

Lemma 2.16. *Given a positive juncture $j_+ = \partial U_+$ there exists a spiraling neighborhood U^+ of $\partial_+ M$ so that $U_+ \subset S \cap U^+$. If f is pure, i.e f preserves each end of S, then there exists a spiraling neighborhood so that $U^+ = S \cap U^+$.*

Note that the analogous statement holds for a negative juncture.

Proof. If f is pure, consider the surface $R_0 = U_+ - f(U_+) \subset S_0$. Use φ_M to carry R_0 to a surface R which is transverse to the flow, and bounded by the multiloops j_0 and $F(j_0)$. Observe that under the quotient map, R is a closed surface which is transverse to the flow, homeomorphic to $\partial_+ M$, and $M \setminus\!\setminus R$, denoting M "cut along" R, has a product component foliated by $\mathcal{L}$ and bounded by $\partial_+ M$ and R, hence this component is a spiraling neighborhood.

Now, more generally, Let $\mathcal{O} = \epsilon_0, \dots \epsilon_{p-1}$ be the end-orbit of period p corresponding to Σ, with corresponding escaping sets $\mathcal{U}_i$. Consider the multiloop $J_0 = \bigcup f^{p-i}(j_i)$ of U_0. Let $j_{\mathcal{O}}$ be a separating multiloop of $\mathcal{U}_0$ which is to the "left" of J_0, i.e. the nesting neighborhood defined by $j_{\mathcal{O}}$ contains each of the nesting neighborhoods defined by $f^{p-i}(j_i)$.

We repeat the same process above with $j_{\mathcal{O}}$, finding a surface R of $\widetilde{M}_\infty$ which is transverse to $\mathcal{L}$ and bounded by $j_{\mathcal{O}}$ and $F^p(j_{\mathcal{O}})$. Again, project R to its quotient in M so that there is product component of $M \setminus\!\setminus R$ which forms a spiraling neighborhood U^+. Now, since $j_{\mathcal{O}}$ was chosen to the "left" of J_0, we have $U_+ \subset U^+$. □

2.8.1. Junctures at infinity

Given a juncture $j \subset S_0$ which satisfies the juncture distinction property, we define its *juncture annulus* A_j as the union of compact π_1-injective, embedded annuli:

$$A_j = \begin{cases} \{\varphi_M(j,t) \,|\, t \in [0,\infty]\}, \text{when } j \text{ is a positive juncture} \\ \{\varphi_M(j,t) \,|\, t \in [-\infty,t)\}, \text{when } j \text{ is a negative juncture} \end{cases}$$

We let j_∞ denote $A_j \cap \partial M$, the boundary of the juncture annulus "at" infinity. For a core C, we let $\partial_\infty C$ denote the multiloop of ∂M formed by the union

$$\partial_\infty C = (\partial_+ C)_\infty \cup (\partial_- C)_\infty.$$

Using Proposition 2.15, the next observation notes that we can produce a juncture annulus and juncture orbit from particular loops of Σ.

Observation 2.17. *Suppose that $j_\infty \subset \partial M$ is the boundary of a juncture annulus. Let γ be a simple closed loop of $\Sigma - j_\infty \subset \partial M$. The flow lines terminating in γ form an π_1-injective annulus $A_\gamma \cong \gamma \times (-\infty, \infty]$ (or $\gamma \times [-\infty, \infty)$) which meets S_0 in a multi-loop J_γ which is the orbit of an escaping loop of f.*

2.8.2. Properly embedded cores

We say a positive or negative juncture j satisfies the *juncture distinction property* if for all $k \in \mathbb{Z}_{\neq 0}$ we have $f^k(j) \cap j = \varnothing$. We note that for a *pure* mapping class, i.e. one which acts by the identity on $\mathrm{Ends}(S)$, each juncture satisfies the juncture distinction property. This is more subtle when an end has period ≥ 2.

Note that this is an adjustment of [5]'s notion of the juncture intersection property– their definition requires any pair of components of $\{f^k(j)\}$ to be either disjoint or coincide, while our definition disallows the latter. We emphasize that we consider junctures as loops, i.e. subsets, and encourage the reader to temporarily break the habit of reducing a loop to its isotopy class.

Corollary 2.18. *Suppose that C_0 is a core whose junctures $\partial_+ C_0$ and $\partial_- C_0$ both satisfy the juncture intersection property. Then there is a isotopy along $\mathcal{L}$ which carries C_0 to a properly embedded subsurface $(\overline{C}_0, \partial\overline{C}_0) \hookrightarrow (M, \partial M)$ which is transverse to $\mathcal{L}$.*

Proof. Notice that since the boundary of C_0 satisfies the juncture distinction property, we may consider the compact cylinders $A^+ = \partial_- C \times [0, \infty]$ and $A^- = \partial_- C \times [-\infty, 0]$ are each disjoint from one another and intersect C_0 only at its boundary. Consider the homotopy equivalence of M obtained by collapsing the cylinder A^+ by the constant map $(x,t) \mapsto (x, \infty)$ and A^- by the map $(x,t) \mapsto (x, -\infty)$. The image of C_0 under this homotopy equivalence is a properly embedded subsurface, and since it was obtained by collapsing flowlines of φ_M, the properly embedded surface remains transverse to $\mathcal{L}$.

Observe that if the junctures of a core do not satisfy the juncture distinction property, then this process of collapsing juncture annuli may not yield an embedded surface. □

More generally, we'll say a properly embedded surface is a *properly embedded core*, or more specifically an *$\mathcal{L}$-embedding* of a core C_0 if is positively transverse to $\mathcal{L}$, and there is an (free) isotopy along the flowlines of $\mathcal{L}$ which carries the surface to C_0.

We'll end this section with an equivalent condition for a core to meet each flow line of $\mathcal{L}$. This ensures that the $\mathcal{L}$-embedding of the core is a *prefiber* of $\mathcal{L}$, i.e. a properly embedded surface whose interior meets each flow line of $\mathcal{L}$:

Proposition 2.19. *A core C_0 meets each flow line of $\mathcal{L}$ if and only if the orbit $\{f^k(C_0)\}_{k \in \mathbb{Z}}$ covers S_0.*

Proof. Let $x \in S_0$ and let ℓ_x denote the flowline which meets x. Note that $\ell_x \cap S_0 = \{f^k(x) : k \in \mathbb{Z}\}$ and $\ell_x = \ell_{f^k(x)}$ for all $x \in S_0$ and $k \in \mathbb{Z}$. Now assume each leaf ℓ of $\mathcal{L}$ meets C_0, then $\ell_x \cap C_0 \subseteq \{f^k(x)\}_{k\in\mathbb{Z}}$, implying that $x \in f^k(C_0)$ for some power k. Thus, $S \subset \bigcup_{k\in\mathbb{Z}} f^k(C_0)$.

Similarly, assume that $\bigcup_{k\in\mathbb{Z}} f^k(C_0) = S_0$ and let $\ell = \ell_x$ be a leaf of $\mathcal{L}$. Since there exists a power k so that $f^k(x) \in C_0$, we have $\ell_{f^k(x)} \cap C_0 = \ell \cap C_0 \neq \varnothing$, meaning each flow line meets C_0. □

2.9. Geometrization of the Mapping Torus

For M, a compact, irreducible 3-manifold with incompressible boundary, we say a compact subsurface Y is *essential* in M if it is π_1-injective, properly embedded, and the inclusion $i : (Y, \partial Y) \hookrightarrow (M, \partial M)$ is not isotopic (rel. boundary) into ∂M. M is *acylindrical* if it does not admit an essential annulus, and is *atoroidal* if it does not admit an embedded, π_1-injective torus.

Theorem 2.20 (Field-Kim-Leininger-Loving [10])**.** *If f is an atoroidal*[1] *end-periodic homeomorphism, then $\overline{M}_f$ is a compact, irreducible, 3-manifold with incompressible boundary. Further, if f is strongly irreducible, then $\overline{M}_f$ is acylindrical.*

[1]Their theorem is stated in terms of an irreducible end-periodic homeomorphism, but the more general atoroidal setting carries through nonetheless.

We'll briefly discuss the geometrization of M_f. We consider the space of hyperbolic structures as the space of discrete, faithful representations $\rho : \pi_1(M_f) \to \mathrm{Isom}(\mathbb{H}^3)$, up to conjugation, and denote this deformation space by AH(M_f). It follows from Theorem 2.20 and Thurston's Geometrization of Haken Manifolds [28] (see [13]) that M_f admits a hyperbolic structure. Any geometrically finite structure $s \in AH(M_f)$ yields a convex-hyperbolic structure on $\overline{M}_f$ as its compact convex core. As discussed in the introduction, when $\overline{M}_f$ is acylindrical, we let $s_f^* \in$ AH(M_f) denote the unique metric for which the boundary of its convex core admits a totally geodesic structure [18, Section 4]. We'll summarize this discussion as follows:

Corollary 2.21. *If f is an atoroidal end-periodic homeomorphism, then M_f admits a non-empty deformation space AH(M_f) of hyperbolic structures. When f is strongly irreducible, there exists a unique metric $s_f^* \in$ AH(M_f) whose convex core admits a totally geodesic boundary structure.*

By the Ahlfors-Bers' isomorphism theorem (c.f. [13][Theorem 8.4]), the interior of AH(M_f), represented by the geometrically finite hyperbolic structures on M_f, is parameterized by $\mathcal{T}(\partial M)$.

CHAPTER 3

SETUP AND OUTLINE

Here we fix notation for use in Sections 3-7. Let f be an atoroidal end-periodic homeomorphism of S, a boundary-less surface of infinite type with finitely many ends, all non-planar. We assume that f is not a translation. We fix our favorite hyperbolic structure X on S and let Λ^+, Λ^- denote the positive and negative Handel-Miller laminations of f. We let C_{min} denote a fixed choice of minimal core of f. A subsurface $Y \subset S$ will always denote a compact, connected essential subsurface which meets both Λ^+ and Λ^-.

Let M_f be its (boundary-less) mapping torus, and M be the compactification as a 3-manifold with boundary as in Theorem 2.14. We consider $S = S_0$ as a fixed leaf of the depth-one foliation $\mathcal{F}$ defined in the previous section. We let $\mathcal{L}$ denote the 1-dimensional foliation of M by flow lines of the suspension semi-flow φ_M.

When f is strongly irreducible, we let (M_f, s^*) denote the unique structure whose convex core has totally geodesic boundary.

3.1. Overview

Throughout the paper we will examine M_f through various perspectives. We are interested in M_f as a fibered manifold, and although any fiber of M_f must be of infinite type, we uncover the geometric structure of the manifold by appealing to the finite-type setting. Our primary tools come from recent work of Landry-Minsky-Taylor [16] who embed M into a closed, fibered hyperbolic manifold via their h-*double* construction. They produce a closed manifold $N = N(M, h)$ by identifying two copies of the compact manifold with boundary M via a component-wise homeomorphism $h : \partial M \to \partial M$. The authors show that for the right choice of h, the resulting h-double $N = N(M, h)$ is fibered and hyperbolic. This allows for the use of tools for closed, fibered, hyperbolic manifolds to apply back to M_f.

In the language of Landry-Minsky-Taylor, the h-double construction provides a *spun pseudo-Anosov package* $(N, \mathcal{F}, \phi, M)$: a closed hyperbolic fibered 3-manifold N; a depth-one foliation $\mathcal{F}$ of N which is transverse to φ, a (dynamic blown up of) a circular pseudo-Anosov flow on N; and a compact manifold with boundary M obtained from N by cutting along the depth-zero leaves of $\mathcal{F}$.

Explicitly, S lives as a leaf of $\mathcal{F}$, and the *spun pseudo-Anosov* representative $f^\flat$, determined by the spun pseudo-Anosov package, is the first-return map on S under the circular flow φ which is isotopic to the original end-periodic homeomorphism.

A significant chunk of this paper is devoted to proving the following lengthy proposition. In brief, this proposition facilitates a careful application of Minsky's Theorem 1.3 to N so that we may coherently translate from the pseudo-Anosov lamination data and the geometry of N to the end-periodic Handel-Miller lamination data and geometry of M.

Proposition 3.1 (A core and gluing package)**.** *Let C_0 be a core of f with $|\chi(C_0)| \leq \eth(f)$ and $h : \partial M \to \partial M$ be a gluing homeomorphism so that:*

1. *C_0 is an f-extended core and meets each flow line of $\mathcal{L}$,*

2. *C_0 admits an $\mathcal{L}$-embedding $(C, \partial C) \hookrightarrow (M, \partial M)$,*

3. *h preserves and reverses the orientation of each component of ∂C, and*

4. $N = N(M,h)$ *is atoroidal.*

Then:

(i) The closed surface $\hat{F} = \hat{F}(C,h)$ *obtained as the* h*-double of* C *is isotopic to a fiber* F *of* N *of genus* $g(f) \leq \eth(F)$*.*

(ii) Let λ^+, λ^- *denote the unstable and stable laminations of the pseudo-Anosov monodromy on* F*. Then, given a subsurface* Y *with* $d_Y(\Lambda^+, \Lambda^-) \geq 3$ *there exists a subsurface* $W \subset F$*, isotopic to* Y *in* M*, so that*

$$d_Y(\Lambda^+, \Lambda^-) \leq d_W(\lambda^+, \lambda^-) + 2.$$

Here is a heuristic of the proof of Theorem A, assuming we've proven Proposition 3.1, and have provided a core C and $h : \partial C \to \partial C$ which satisfies its hypotheses. We use Proposition 3.1(i) to ensure that the topology of the fiber of N is controlled by the $\eth(f)$. With Proposition 3.1(ii), we may directly compare the subsurface projection data $d_Y(\Lambda^+, \Lambda^-)$ and $d_W(\lambda^+, \lambda^-)$ between a subsurface $Y \subset S$ and a subsurface $W \subset F$ isotopic to Y in N so that $\ell_N(\partial Y) = \ell_N(\partial W)$. After proving Proposition 3.1, we apply Minsky's Theorem 1.3 to N and conclude the argument by noting there exists a hyperbolic structure $s \in \text{AH}(M_f)$ so that $(M_f, s) \hookrightarrow N$ is an isometric immersion. This provides a bound $\ell_s(\partial Y) \leq \ell_N(\partial Y)$, and the proof of the theorem follows.

3.2. Plan

In Chapters 4 and 5 we record the proof of Proposition 3.1 (i) and (ii), assuming we have an abstract core C and gluing involution which satisfies the hypotheses of the proposition. We use tools of Landry-Minsky-Taylor throughout. The proof of Proposition 3.1 (i) is a fairly direct application of Landry-Minsky-Taylor, while that of Proposition 3.1 (ii) requires more tinkering before we may apply their work. Essentially, we'll show that any subsurface which realizes sufficiently large distance $d_Y(\Lambda^+, \Lambda^-)$ can be translated under f to be contained in our choice of core, and therefore, by Proposition 3.1(i) in the constructed fiber of N. Finally, we will use Theorem 5.5 of Landry-Minsky-Taylor which directly compares the Handel-Miller laminations with the *positive and negative invariant singular laminations* of the spun pseudo-Anosov representative $f^\flat$ of f.

The bulk of the work is in Chapter 6, where we make an explicit choice of core C and gluing homeomorphism h which satisfy the hypotheses of Proposition 3.1. We briefly discuss the complications that arise when doing this. To build an atoroidal $N(M, h)$, we must carefully choose h so that no two curves which bound essential properly embedded annuli of M are identified under the gluing. However, in order to produce a closed surface, we must balance this task with ensuring that h acts preserves the boundary of our chosen core, as apriori, such cores may bound essential annuli.

If f is strongly irreducible, then for certain homemorphisms, as when ∂M consists of genus-2 surfaces in Theorem B, or the homeomorphisms constructed in Example 8.5 and Example 8.7, we show that there is a h for which $(\overline{M}_f, s_f^*) \hookrightarrow N(M,h)$ isometrically embeds into $N(M,h)$. It is for these homeomorphisms that Conjecture 1.1 holds.

We end the paper with Chapter 8 which describes a few examples of families of end-periodic and strongly irreducible end-periodic homeomorphisms, both to serve as a reference to the reader, and to use in application of Theorem B which produces a proof of Theorem C. We'll provide plenty of examples which one may apply Theorem B to get short curves in totally geodesic structures. Namely, for a fixed core C of a handle shift ρ on the ladder surface, we construct families of strongly irreducible end-periodic homeomorphisms $\{f_n\}$ with uniformly bounded capacity so that

$$\lim_{n\to\infty} d_C(\Lambda_n^+, \Lambda_n^-) \to \infty,$$

hence $\ell_{s_n^*}(\partial C) \to 0$ in the respective totally geodesic boundary metrics.

CHAPTER 4

THE DOUBLING CONSTRUCTION

We begin this section by describing the h-double construction of Landry-Minsky-Taylor which produces a closed, fibered, 3-manifold N in which M embeds. The homotopy equivalence of surfaces obtained from Section 2.8.2. is described by [16, Section 3] in terms of spiraling neighborhoods and *juncture classes* of ∂M. It will be helpful to compare their terminology with ours. The fibration $M_f \to S^1$ induces a cohomology class $[j] \in H^1(\partial M)$ known as the *juncture class* of f, which is the pullback of the fibration class $H^1(M) \to \mathbb{Z}$ under the inclusion map $\partial M \hookrightarrow M$. Indeed, for any properly embedded core $C \subset S$, the first cohomology class $[\partial C] \in H^1(\partial M)$ corresponding to the algebraic intersection form is precisely the juncture class $[j]$ of f.

4.1. The Construction

Let the tuple $(M, \mathcal{F}, \mathcal{L})$ denote M, the oriented 3-manifold with boundary; $\mathcal{F}$, the co-oriented 2-dimensional foliation; and $\mathcal{L}$, the oriented 1-dimensional foliation induced by the suspension flow of f. Let $(M^{\downarrow}, \mathcal{F}^{\downarrow}, \mathcal{L}^{\downarrow})$, denote the same with the opposite (co-)orientations in each factor. Let $\sigma : M \to M^{\downarrow}$ denote the identity map, which by construction reverses the (co-)orientations of M, $\mathcal{F}$ and $\mathcal{L}$.

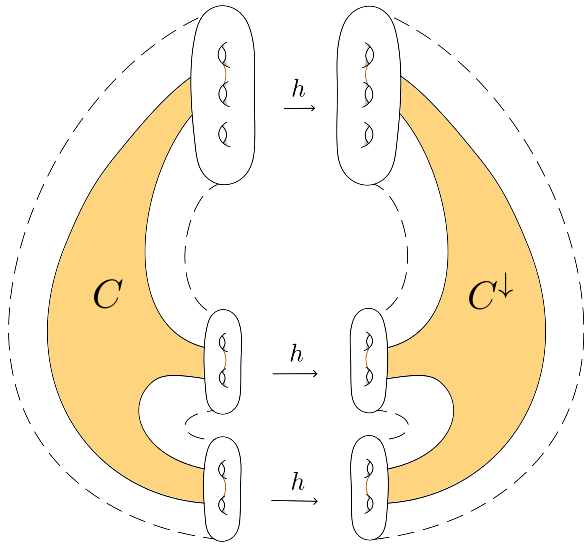

Figure 5. A cartoon of the h-double of M.

Given a component-wise orientation-preserving homeomorphism $h : \partial M \to \partial M$, the *$h$-double of M* is the closed manifold defined by $N(M,h) = M \cup_{\sigma\circ h} M^{\downarrow}$, i.e. by gluing ∂M and $\partial M^{\downarrow}$ by the composition $(\sigma \circ h)|_{\partial M}$. Note that $N(M,h)$ inherits its orientation from those of M and $M^{\downarrow}$, and admits a 1-dimensional foliation obtained from the union $\mathcal{L} \cup \mathcal{L}^{\downarrow}$, and depth-one foliation obtained by $\mathcal{F} \cup \mathcal{F}^{\downarrow}$. We will denote the extensions of $\mathcal{L}$ and $\mathcal{F}$ by the same notation in the context of the closed manifold. Suppose W is a properly embedded surface $(W, \partial W) \hookrightarrow (M, \partial M)$, and $h(\partial W) = \partial W$. Then for $W^{\downarrow} = \sigma(W)$, the union $W \cup W^{\downarrow}$ is a closed surface of $N(M,h)$ which we will denote by $\hat{F} = \hat{F}(W,h)$.

We rephrase a proposition of Landry-Minsky-Taylor below. Their proposition is stated in much higher generality; they only require h to preserve the juncture class $[j]$, and do not specify the topology of the fiber.

Corollary 4.1. *(Landry-Minsky-Taylor [16, Prop. 3.9]) Let C be a properly embedded core which meets each flow line of $\mathcal{L}$. Let $h : \partial M \to \partial M$ be a homeomorphism which preserves ∂C, while reversing the orientation of component. Then the surface $\hat{F} = \hat{F}(C, h)$ is isotopic to a fiber of N.*

With this proposition, we may prove Proposition 3.1(i). The proposition follows directly from the above proposition of Landry-Minsky-Taylor, whose argument we summarize here.

Proof of Proposition 3.1(i). We assume we have a core C_0 which meets each flow line and that $(C, \partial C) \hookrightarrow (M, \partial M)$ is the $\mathcal{L}$-embedding of C_0. Further, we assume h preserves ∂C while reversing the orientation of each component, and that $N = N(M, h)$ is atoroidal. We wish to show that the doubled surface is isotopic to a cross-section of $\mathcal{L}$, i.e. a fiber of N.

As properly embedded cores, the subsurfaces C and $C^{\downarrow}$ are positively transverse to $\mathcal{L}$ and $\mathcal{L}^{\downarrow}$ and meet each flow line of the respective flows. The boundaries of the co-oriented subsurfaces $C \subset M$ and $C^{\downarrow} \subset M^{\downarrow}$ are identified under the h-double gluing to form a new subsurface $\hat{F} = \hat{F}(C, h) = C \cup C^{\downarrow}$. We observe that in the closed manifold N, the closed surface $\hat{F}$ meets each flow line of the identified flow $\mathcal{L}$.

While the open subsurface int$(C) \cup$int$(C^{\downarrow})$ is positively transverse to the flow, we need to verify that F is consistently positively transverse to this flow at the source of the gluing. It suffices to show that the induced co-orientations of C and $C^{\downarrow}$ are consistent in the doubled manifold. Since C is positively transverse to $\mathcal{L}$, it induces a co-orientation of $\partial C \subset \partial M$, represented by $[\partial C] \in H^1(\partial M)$. Likewise, the positive co-orientation of $C^{\downarrow}$ induces a co-orientation of $\partial C^{\downarrow} \subset \partial M^{\downarrow}$, represented by $[\partial C^{\downarrow}] \in H^1(\partial M^{\downarrow})$. We check that this induces a consistent co-orientation of $\partial C \subset \partial M \subset N$. Indeed, since $\sigma^*[\partial C] = -[\partial C^{\downarrow}]$, it follows that $\sigma^* h^*([\partial C]) = \sigma(-[\partial C]) = [\partial C^{\downarrow}]$.

Now, since ∂C is contained in a product neighborhood foliated by product fibers of $\mathcal{L}$, we may isotope $\hat{F}$ along flow lines to a closed surface F which is transverse to $\mathcal{L}$ at each point. Thus, F meets each flow line of $\mathcal{L}$, and the first return map represents the monodromy of the fibration producing an atoroidal manifold.

Finally, it follows from a straight-forward calculation that after doubling C, a compact surface with g genera, b boundary components, that the resulting closed surface has genus $2g + b - 1 = |\chi(C)| - 1 \leq |\chi(C)| \leq \eth(f)$. $\square$

The following remark follows from the construction of Landry-Minsky-Taylor, and Thurston's Hyperbolization Theorem for fibered manifolds [29].

Remark 4.2. *Any C and h as in Proposition 3.1 produces a spun pseudo-Anosov package $(N, \mathcal{F}, \varphi, M)$ for f.*

(i) $N = N(M, h)$ is a fibered, hyperbolic manifold with fiber F isotopic to $\hat{F} = \hat{F}(C, h)$.

(ii) There is a pseudo-Anosov homeomorphism homeomorphism $\phi : F \to F$ with associated pseudo-Anosov flow $\varphi^{\flat}$ on N_{ϕ}, and a homeomorphism $\iota : N \to N_{\phi}$ so that $\iota(S)$ is transverse to a dynamic blow up φ of $\varphi^{\flat}$.

(iii) Let $f^\flat$ denote the first return map of S (conjugated by ι) under φ; this map is called a spun pseudo-Anosov representative of f, and is isotopic to f.

We note that the first return map of $\mathcal{L}$ for the fiber F is isotopic to a pseudo-Anosov homeomorphism. We require extra tools to compare the end-periodic suspension flow φ_M with the pseudo-Anosov suspension flow $\varphi^\flat$ (or its blown up flow φ).

CHAPTER 5

JUNCTURES, LAMINATIONS AND SUBSURFACES

As a reminder to the casual reader, a subsurface $Y \subset S$ will always denote a compact, connected essential subsurface which meets both Λ^+ and Λ^-– we call such a subsurface *suitable* to f. By the (pseudo) f-invariance of the Handel-Miller laminations Lemma 2.11, $\Lambda^\pm$ meets Y if and only if $\Lambda^\pm$ meets $f^n(Y)$ for all $n \in \mathbb{Z}$, and $d_Y(\Lambda^+, \Lambda^-) = d_{f^n(Y)}(\Lambda^+, \Lambda^-)$.

We will apply the following remark to $\mathcal{J}^\pm$ or one of its partial juncture orbits $\mathcal{J}_k^\pm$. In either case, the closure of the union of geodesics is the negative (positive) Handel-Miller lamination.

Remark 5.1. *Let $\mathcal{G}$ be an union of complete geodesics of a hyperbolic surface X, and $\Lambda = \overline{\mathcal{G}}$. If Λ intersects a geodesic γ transversely, then so must $\mathcal{G}$.*

Proof. Let $\widetilde{\Lambda}$, $\widetilde{\mathcal{G}}$, $\widetilde{\gamma}$ be the respective lifts of Λ, $\mathcal{G}$ and γ to the hyperbolic plane via the universal covering map. The endpoints of each of the lifts define subspaces $\partial^2(\Lambda)$, $\partial^2(\mathcal{G})$ in the double boundary $\partial^2(\mathbb{H}^2)$. Let $\widetilde{\ell} \subset \widetilde{\Lambda}$ be a leaf which meets a component $\widetilde{\gamma}_0 \subset \widetilde{\gamma}$. Since $\widetilde{\ell}$ represents a limit point of $\partial^2(\mathcal{G})$, there exists a leaf $\ell' \subset \mathcal{G}$ so that $\ell' \subset U \times V$, the neighborhood formed by the components U and V of $\partial\mathbb{H}^2 - \{\gamma_0\}$. Thus, ℓ' and γ must intersect. □

5.1. Subsurface Projection Distances

Remark 5.2. *Suppose that $Y \subset S$ is suitable to f. Then, for any pair of positive and negative juncture orbits J^+, J^-, we have:*

(i) J_+ *and* J_- *meet* Y.

(ii) $d_Y(J_\pm, \Lambda_\mp) \leq 1$

(iii) $|d_Y(J_+, J_-) - d_Y(\Lambda^+, \Lambda^-)| \leq 2$

Proof. By Remark 2.9, if Λ^+ and Λ^- meet Y, they must also meet ∂Y. It follows by applying Remark 5.1 to $\mathcal{G} = \mathcal{J}^\pm, \overline{\mathcal{G}} = \Lambda^\mp$ and $\gamma = \partial Y$, that ∂Y must also meet the juncture orbits J^- and J^+ respectively. Recall that the projection map π_Y is defined by lifting arcs and curves to the cover $\widetilde{X}_Y$ corresponding to Y. By Observation 2.6, which states that $\mathcal{J}^\pm$ and $\Lambda^\mp$ are disjoint in X, the corresponding lifts $\widetilde{\mathcal{J}}^\pm$ and $\widetilde{\Lambda}^\mp$ are disjoint in $\widetilde{X}_Y$, so the respective projections satisfy $\pi_Y(J^\pm) \subset N_1(\Lambda^\mp)$ and $\pi_Y(\Lambda^\pm) \subset N_1(J^\mp)$. Finally, for any $j^\pm \in J^\pm$ with $d_Y(j^-, j^+) = d_Y(J^+, J^-)$, and any $\lambda^\pm \in \pi_Y(\Lambda^\pm)$ $d_Y(\lambda^+, \lambda^-) = d_Y(\Lambda^+, \Lambda^-)$, it follows that

$$d_Y(\lambda^+, \lambda^-) - 2 \leq d_Y(j^-, j^+) \leq d_Y(\lambda^+, \lambda^-) + 2.$$

□

We note it is not true that a subsurface which meets $J_\mp$ also meets its lamination counterpart $\Lambda^\pm$. For example, one can imagine a subsurface contained in a negative ladder U_- which meets J_-, but it must be disjoint from Λ^+ (see Observation 2.7).

Lemma 5.3. *Let Y be a suitable subsurface. Then for any juncture orbit $J_+ = \{f^k(j_+)\}_{k\in\mathbb{Z}}$ there exists a minimal $K \in \mathbb{Z}$ for which $\pi_Y(f^K(j_+)) = \varnothing$.*

One obtains the analogous statement, i.e. the existence of a maximal K for negative juncture orbits by swapping $+$ for $-$ in the argument.

Proof. Fix a juncture $j_+ \subset J_+$ and let $\kappa = \{k_n\}$ be an enumeration of $\{k \in \mathbb{Z} : \pi_Y(f^k(j_+)) = \varnothing\}$, a non-empty set by Remark 5.2. Consider our favorite hyperbolic metric X on S and pull ∂Y and $J_\kappa = \bigcup(f^{k_n}(j_+))$ tight to geodesic multicurves ∂Y^* and $\mathcal{J}_\kappa$ so that $\partial Y^* \cap \mathcal{J}_\kappa = \varnothing$ (see Remark 2.9).

Assume by contradiction that κ is an arbitrarily negative sequence, and observe that closure of the geodesic multicurve $\overline{\mathcal{J}_\kappa}$ contains Λ^- as a sublamination. It follows from Remark 5.1 that the lamination J_κ meets ∂Y, a contradiction. Thus, the sequence $\{k_n\}$ is bounded below and has a minimal element K. □

Remark 5.2 shows that with a small adjustment of ± 2, our results about the subsurface projection distances for juncture orbits also apply to the subsurface projection distance between the Handel-Miller laminations. A bound on the quantity $d_Y(\Lambda^+, \Lambda^-)$ restricts the position of Y with respect to junctures of f:

Lemma 5.4. *Let C be an f-extended core of S. Then given any surface $Y \subset S$ with $d_Y(\Lambda^+, \Lambda^-) \geq 4$, there exists $n \in \mathbb{Z}$ so that $f^n(Y) \subset C$.*

Proof. Consider the juncture orbits $J_+ = \{f^{-k}(\partial_+ C)\}_{k\in\mathbb{Z}}$ and $J_- = \{f^k(\partial_- C)\}_{k\in\mathbb{Z}}$. Let C' be an essential subcore of C whose boundary is contained in $J_+ \cup J_-$. Recall that we require each component of $\partial C'$ to be essential in C. Let Y be an essential subsurface of S with $d_Y(\Lambda^+, \Lambda^-) \geq 4$.

Let K be the minimal power (from Lemma 5.3) for which $f^K(\partial_+ C)$ is disjoint from Y. Set $W = f^{-K}(Y)$. While W is disjoint from $\partial_+ C$, we observe that W must meet $\partial_+ C'$ by the minimality of K and the fact that Y is suitable.

Now, W is in a favorable position with respect to $\partial_+ C$, and we look to $\partial_- C'$. Either $\pi_W(\partial_- C') = \varnothing$, or $\pi_W(\partial_- C') \neq \varnothing$. In the former case, W is a connected surface of $S - (\partial_+ C \sqcup \partial_- C')$, and not contained in any ladder (see Remark 2.9), so it must be contained in the core C. In the latter case, the disjoint multicurves $\partial_- C'$ *and* $\partial_+ C'$ both meet W so that $d_W(\Lambda^+, \Lambda^-) \leq 3$, a contradiction. Thus, $W = f^{-K}(Y)$ is a subsurface of C. □

5.2. Compatibility of Laminations

We now require tools to compare the Handel-Miller laminations of the spun-pseudo-Anosov representative of the end-periodic map to the pseudo-Anosov laminations from the fiber F of N. We will survey the main characters of the story from the spun pseudo-Anosov package $(N, \mathcal{F}, \varphi, M)$ obtained from Proposition 3.1(i) and Remark 4.2, and refer the reader to Landry-Minsky-Taylor [16, Section 4] for specifics.

Since F is a cross-section of the foliation $\mathcal{L}$ which produces an atoroidal manifold, the first return map $r : F \to F$ of $\mathcal{L}$ is isotopic to a pseudo-Anosov homeomorphism ϕ on F [27]. The pseudo-Anosov homeomorphism induces a circular, pseudo-Anosov suspension flow $\varphi^\flat$ on N_ϕ. Naturally, we identify F with the associated fiber of N_ϕ; the product of the isotopy between r and ϕ induces such a homeomorphism $N_r \to N_\phi$.

Our goal is to compare the foliation $\mathcal{L}$, obtained by the suspension of r, with the flow lines of $\varphi^\flat$. Recall that ϕ is equipped with a pair of unstable and stable foliations $\mathcal{W}_\flat^+$ and $\mathcal{W}_\flat^-$ of F, and corresponding unstable and stable laminations λ^+, λ^- of ϕ (see Section 2.5.). The circular flow φ is obtained from $\varphi^\flat$ by a *dynamic blow up*, an operation on φ which alters the flow at its singular orbits; we refer the reader to [16, Section 2.2] for specifics on this blow up procedure. Under this process, the fiber F remains transverse to φ and still meets each flow line of φ. More crucially, up to an isotopy of N, the depth-one foliation $\mathcal{F}$ of N is positively transverse to φ. We let $\iota : N_r \to N_\phi$ denote the homeomorphism taking F to its copy in N_ϕ, and $\mathcal{F}$ to $\iota(\mathcal{F})$ which is tranvserse to φ. With this, we make no distinction between S (or F) and its image under ι.

The circular flow φ induces unstable and stable 2-dimensional singular foliations of N denoted by W^u and W^s, and by intersecting the leaves of each with S, we obtain stable and unstable singular foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ of S.

Landry-Minsky-Taylor introduce the *positive and negative invariant (singular) (sub)-laminations* of a spun pseudo-Anosov map, which we will denote by $\Lambda_\flat^+$, $\Lambda_\flat^-$ for the spA representative $f^\flat$. In brief, the positive lamination is obtained by removing negatively escaping leaves and contracting half-leaves from the unstable foliation $\mathcal{W}^u$, and the negative lamination is obtained by removing positively escaping leaves and expanding half-leaves from the stable foliation $\mathcal{W}^s$ [16, Proposition 5.8]. In fact, Landry-Minksy-Taylor provide a direct comparision between the Handel-Miller laminations $\Lambda^\pm$ and $\Lambda_\flat^\pm$ by lifting them to the universal cover. Given a singular sublamination Λ of a singular foliation of $\mathbb{H}^2$, recall that $\partial^2\big(\Lambda\big) \subset \partial^2(\mathbb{H}^2)$ denotes the end points of leaf-lines of Λ in the double boundary of $\mathbb{H}^2$.

Theorem 5.5. *Landry-Minksy-Taylor [16, Theorem 8.4]*

$$\partial^2\left(\widetilde{\Lambda}_\flat^\pm\right) = \partial^2\left(\widetilde{\Lambda}^\pm\right).$$

We're now ready to prove Proposition 3.1(ii).

Proof of Proposition 3.1(ii). Let Y be a subsurface of $S \subset \mathcal{F}$ satisfying $d_Y(\Lambda+, \Lambda^-) \geq 4$. We wish to show that there is a subsurface $W \subset F$, isotopic to $Y \subset S$ in the 3-manifold so that the projection distances $d_W(\lambda^+, \lambda^-)$, $d_Y(\Lambda^+, \Lambda^-)$ are comparable. We assume ∂Y avoids the singular orbits of φ. Since C (satisfying the hypotheses of Proposition 3.1) is the f-extension of another core, by Lemma 5.4 there is a power for which the subsurface

$f^n(Y)$ is contained in the core C. There is an isotopy along flowlines of $\mathcal{L}$ which takes $W' = f^n(Y)$ to a subsurface W of the fiber F which defines a (fixed) homeomorphism $W \to W'$ to pass from curves and proper arcs of W' to curves and proper arcs of W. With this, we identify $\mathcal{AC}(W')$ with $\mathcal{AC}(W)$ and $\mathcal{AC}(Y)$.

For brevity, we will just discuss the positive/unstable (singular sub)-laminations and foliations; by replacing $+$ with $-$, our arguments hold for the negative/stable versions of the laminations and foliations.

The lifts $\widetilde{\Lambda}^+$ and $\widetilde{\Lambda}^+_\flat$ to $\mathbb{H}^2$ are $\pi_1(X)$-invariant singular laminations whose endpoints of leaf-lines agree by Theorem 5.5. Note that the leaf-lines of the projections of $\widetilde{\Lambda}$ and $\widetilde{\Lambda}^+_\flat$ to the cover $\widetilde{X}_Y$ still agree. It follows that $\pi_Y(\Lambda^+) = \pi_Y(\Lambda^+_\flat)$. Recall that $\Lambda^+_\flat$ is obtained from $\mathcal{W}^u$ by removing leaves and half-leaves, so that $\pi_Y(\Lambda^+_\flat) \subset \pi_Y(\mathcal{W}^u)$ and $d_Y(W^u, \Lambda^+_\flat) \leq 1$.

Finally, we need to compare the projections $\pi_Y(\mathcal{W}^+)$ and $\pi_Y(\mathcal{W}^+_\flat)$ corresponding to φ and the genuine pseudo-Anosov circular flow $\varphi^\flat$, respectively. Since the flow φ is obtained from $\varphi^\flat$ by introducing *blown annuli*, i.e. mapping tori of finite-trees at singular orbits of $\varphi^\flat$, the singular foliation $\mathcal{W}^+$ is obtained from the singular foliation $\mathcal{W}^+_\flat$ by collapsing the finite-trees which produce the blown annuli. This is a homotopy equivalence of the surface which takes leaf lines of $\mathcal{W}^+$ to leaf lines of $\mathcal{W}^+_\flat$, so that $\pi_Y(\mathcal{W}^u) = \pi_Y(\mathcal{W}^u_\flat)$.

Thus, we have $d_Y(\Lambda^+, \Lambda^-) = d_W(\Lambda^+, \Lambda^-)$ and:

$$d_W(\Lambda^+, \Lambda^-) = d_W(\Lambda^+_\flat, \Lambda^-_\flat) \leq d_W(\mathcal{W}^+, \mathcal{W}^-) + 2$$

$$= d_W(\mathcal{W}_\flat^+, \mathcal{W}_\flat^-) + 2 = d_W(\lambda^+, \lambda^-) + 2$$

$\square$

This completes the proof of Proposition 3.1(ii), and the proof of Proposition 3.1.

CHAPTER 6

A CORE AND GLUING PACKAGE

Our proof of Theorems A and B will refer to a specific choice of a "good" gluing homeomorphism h and "nice" core C, depending on our fixed atoroidal map f. We will use this section to specify the gluing map and core, and show that they satisfy the hypotheses of Proposition 3.1. The separate cases are as follows:

Type I: f is strongly irreducible, and ∂M consists of genus-2 surfaces

Type II: f is strongly irreducible, and ∂M does not consist of genus-2 surfaces

Type III: Otherwise, f is atoroidal, and not strongly irreducible

Game Plan. Given our fixed end-periodic homeomorphism f and minimal core C_{min}, we will first designate a multicurve $\Gamma_\infty \subset \partial M$ to represent the boundary of the nice core. With an application of Observation 2.17, the multicurve Γ_∞ will be used to define our nice core $C_0 \subset S_0$ as an f-extension of C_{min}. We note that the multiloop $\partial_\infty C_0$ consists of possibly multiple parallel components, but each is isotopic into Γ_∞. We will then use Γ_∞ to nail down our particular gluing homeomorphism h. Finally, the homeomorphism provides a particular multiloop Γ^* (isotopic to $\partial_\infty C_0$) which is fixed by h, and we will find an $\mathcal{L}$-embedding of C_0 which is bounded by Γ^*. We nudge the reader to remind them of the subtleties of dealing with (multi)loops as opposed to (multi)curves.

Although C_{min} is a natural candidate for our nice core, we'll take a moment to describe how the good core is selected and used in the proof of our main theorems, and why C_{min} doesn't quite suit our purposes. We seek a homeomorphism of Σ which preserves Γ_∞ while reversing its orientation. Our favorite contenders for this behavior are hyperelliptic involutions which we will explicitly choose for **Type I** and **II**, and utilize for **Type III**. The multicurve $\partial_\infty C_{\text{min}}$ has possibly many components on Σ—it's unclear if there necessarily exists a hyperelliptic involution which preserves it. Further, the boundary of a minimal core may contain reducing curves which bound essential annuli in M. It could be possible that our gluing homemorphism has preserved the boundary components of an essential annulus, creating an essential torus in the doubled manifold N— the sole obstruction to the hyperbolicity of N in this setting.

6.1. Setup

We will fix some notation and choices for use in the rest of this section. We will first choose Γ_∞ so that for each component $\Sigma \subset \partial M$, $\Gamma_\infty \cap \Sigma$ consists of a sole curve which does not bound an annulus of M.

Fix a minimal core $C_{\text{min}} = S - (U_+ \cup U_-)$, and let $R = R_+ \cup R_-$, where $R_+ = \overline{U_+ - f(U_+)}$ and $\overline{R_- = U_- - f^{-1}(U_-)}$. For a component $\Sigma \subset \partial M$, consider its associated end-orbit $\mathcal{O} = \epsilon_0, \ldots, \epsilon_{p-1}$ of period $p = p_\Sigma$, labeled so that $f^{\pm(n)}(\epsilon_0) = \epsilon_n$ (depending on if $\mathcal{O}$ is attracting or repelling). Let $\mathcal{U}_i$ denote the escaping set of ϵ_i. We'll describe a fixed choice of curve $\gamma(\Sigma) \subset \Sigma$ for each component Σ.

Observation 6.1. *There is a (smooth) separating loop of $\mathcal{U}_0$ which is contained in R and is not reducing under f.*

Proof. We'll describe a process to find infinitely many. By Remark 8.4, we need only choose a separating curve γ_0 of $\mathcal{U}_0$ contained in R with $\pi_Y(\gamma_0, \Lambda^+) \geq 2$ for some subsurface Y. For example, one may take γ_0 to be any separating curve homologous to the boundary of U_0, the unique infinite component of $\mathcal{U}_0 \cap U_\pm$ (see Remark 2.12). By applying enough powers of say, a pseudo-Anosov map on the component $R_0 \subset R$ containing the separating curve and replacing it with its image under the map, one obtains $\pi_{R_0}(\gamma_0, \Lambda^+) \geq 2$. □

For each end-orbit, we'll choose γ' to be a smooth separating loop of $\mathcal{U}_0$ contained in R_0 as in the observation. We let $\gamma(\Sigma) = (\gamma_0)_\infty$, i.e., the boundary of the juncture annulus of γ_0 which meets Σ. Doing so for each end-orbit provides a single curve for each component of ∂M, and we let Γ_∞ denote the union of the curves $\gamma(\Sigma)$ over all components of ∂M.

6.2. Good Homeomorphisms

To find a homeomorphism h for which $N(M, h)$ is atoroidal, it suffices to verify that h does not identify the boundaries of any two essential annuli of M. For each component $\Sigma \subset \partial M$, we let $\mathcal{A}_\Sigma \subset \mathcal{C}^{(0)}(\Sigma)$ denote the collection of curves which comprise of the boundaries of essential annuli of M which meet Σ. Naturally, when M is acylindrical $\mathcal{A}_\Sigma$ is empty.

Claim 6.2. *Let h be a homeomorphism for which $h|_\Sigma(\mathcal{A}_\Sigma) \cap \mathcal{A}_\Sigma = \varnothing$ in $\mathcal{C}(\Sigma)$ in each component $\Sigma \subset \partial M$. Then, $N(M, h)$ is atoroidal.*

Proof of the claim. Assume T is an essential torus of $N(M, h)$. Since M is atoroidal, the torus must meet ∂M. By a strengthening of Roussarie-Thurston's theorem (see [12, Theorem 2.9]), we may isotope T so that the torus is transverse to $\mathcal{F} \subset N$. By a further isotopy, we can remove any disks in $T - \partial M$, and assume T meets ∂M in an essential 1-manifold $\Gamma = T \cap \partial M$. Namely $T \setminus\!\!\setminus \Gamma$, the torus cut along the multicurve, is the disjoint union of compact essential annuli.

If f is strongly irreducible, its mapping torus is acylindrical, thus, no such annulus exists and $N(M, h)$ is atoroidal. Otherwise, let α be a component of Γ, and consider the annuli A and $A^{\downarrow} \subset T \setminus\!\!\setminus \Gamma$ whose boundaries meet at α. After cutting $N \setminus\!\!\setminus \partial M$, this produces curves $\alpha \subset M$, and $\alpha^{\downarrow} \subset M^{\downarrow}$, both bounding the respective annuli, which were identified by $h(\alpha) = (\alpha^{\downarrow})$. This contradicts the stated assumption. □

The JSJ decomposition for manifolds with boundary provides A, a (possibly empty) disjoint collection of essential annuli, so that any essential annulus of M is homotopic into a component of $M - A$ [2, Theorem 3.8, 3.9]. In other words, we have either $\mathcal{A}_{\Sigma} = \varnothing$, or $\mathrm{diam}_{\mathcal{C}(\Sigma)}(\mathcal{A}_{\Sigma}) \leq 2$.

6.3. Choice of Homeomorphism

Our homeomorphism will be chosen with respect to the multicurve $\Gamma_\infty = \bigcup \gamma(\Sigma)$ describe aboved. For each component Σ, choose a hyperelliptic involution mapping class about $\gamma(\Sigma)$ so that some representative homeomorphism h'_Σ fixes a representative loop $\gamma'(\Sigma) : S^1 \to \Sigma$ of $\gamma(\Sigma)$ and reverses its orientation; we write $h'(\gamma'(\Sigma)) = -\gamma'(\Sigma)$ to denote this behavior. Let $h' : \partial M \to \partial M$ denote the component-wise homeomorphism formed by the composition of h'_Σ for each component of ∂M. After selecting h, we will describe a multiloop Γ_h which satisfies $h(\Gamma_h) = -\Gamma_h$.

Type I (An isometry of the totally geodesic boundary)**.** *f is strongly irreducible and ∂M consists of totally geodesic surfaces:*

Recall that on the genus-2 surface any hyperelliptic involution mapping class preserves each isotopy class of curve, thus it follows from the $9g-9$ Theorem that h_Σ acts trivially on the Teichmüller space of the surface (see [7, Section 12.1]). So, we let h_Σ be the isometry of (Σ, s^*) which is isotopic to h'_Σ and preserves the geodesic representative $\gamma^*(\Sigma)$ of $\gamma'(\Sigma)$. We let h be the composition of all such isometries over each component of ∂M, so that $\Gamma_h = \bigcup \gamma^*(\Sigma)$, again over all components of ∂M.

Type II (An involution of the boundary)**.** *f is strongly irreducible and ∂M is not homeomorphic to the disjoint union of genus-2 surfaces:*

We let $h = h'$, the homeomorphism above which satisfies $h(\gamma'(\Sigma)) = -\gamma'(\Sigma)$ so that $\Gamma_h = \bigcup \gamma'(\Sigma)$, over all components of ∂M.

Type III (A homeomorphism which mixes annuli:). *f is atoroidal, but not strongly irreducible:*

It follows that M is not acylindrical, i.e. it contains an essential annulus.

Let W_Σ be an open annulus containing $\gamma'(\Sigma)$. We construct a map ϕ_Σ which is supported on $\Sigma_0 = \Sigma - W_\Sigma \cong \Sigma \setminus\setminus \gamma(\Sigma)$. If $\mathcal{A} = \mathcal{A}_\Sigma = \varnothing$, then we will let $\phi_\Sigma = \mathrm{id}_\Sigma$. Otherwise, if $\mathcal{A}$ is non-empty, we will construct it as follows.

Observe that for the homeomorphism h' above, $\mathcal{A}$, $h'(\mathcal{A})$ both meet Σ_0 since $\gamma(\Sigma)$ is a connected non-separating curve of Σ. Critically, $\gamma(\Sigma) \notin \mathcal{A}$ since it was chosen from a curve which is not reducing in the discussion above. We will analyze $\mathcal{A}$, its image $h'(\mathcal{A})$, and the projections $\pi_{\Sigma_0}(\mathcal{A})$ and $\pi_{\Sigma_0}(h'(\mathcal{A}))$. Let $D = \max\{d_{\Sigma_0}(\mathcal{A}, h'(\mathcal{A})), 6\}$, and choose a pseudo-Anosov representative $\phi_\Sigma \in \mathrm{Homeo}(\Sigma_0, \partial\Sigma_0)$ with $\tau_{\mathcal{AC}(\Sigma_0)}(\phi_\Sigma) > D + 3$. We let

$$h = \prod_{\Sigma \subset \partial M} \phi_\Sigma \circ h'.$$

Since $h|_{W_\Sigma} = h'$, we let the preserved multiloop $\Gamma_h = \bigcup \gamma'(\Sigma)$.

For $\alpha \in \pi_{\Sigma_0}(\mathcal{A})$, we have:

$$d_{\Sigma_0}(h(\alpha), \alpha) = d_{\Sigma_0}(\phi_\Sigma(h'(\alpha)), \alpha)$$

$$\geq d_{\Sigma_0}(\phi_\Sigma(h'(\alpha)), h'(\alpha)) - d_{\Sigma_0}(\alpha, h'(\alpha))$$

$$\geq (D+3) - D = 3$$

As above, $\operatorname{diam}(\pi_{\Sigma_0}(\mathcal{A})) \leq 2$. So, it follows that $h(\alpha) \notin \mathcal{A}$ and $\mathcal{A} \cap h(\mathcal{A}) = \varnothing$. Thus, h satisfies Claim 6.2.

6.4. Nice Cores

Our method of selection of core C_0 will again depend on the multicurve Γ_∞ and the homeomorphism chosen above. We'll detail this more general construction carefully.

6.4.1. Multiloop core extension

Let Γ be an arbitrary multiloop contained in $\partial M - \partial_\infty C_{\text{min}}$ which satisfies the following: for each component Σ, $\Gamma_\Sigma = \Gamma \cap \Sigma$ is a multiloop with p_Σ parallel components, where p_Σ denotes the period of the end-orbit $\mathcal{O}$ corresponding to Σ. Given such a multiloop Γ, we will construct a core C_Γ which is f-extended, contains C_{min} and has $\partial_\infty(C_\Gamma) = \Gamma$. We will refer to C_Γ as the Γ-*extension of* C_{min}, and note that as a subset of S it is determined by C_{min}, the multiloop Γ, the labels of each end-orbit, and the labels on each component of Γ_Σ.

We will work in one component $\Sigma \subset \partial M$ at a time letting $p = p_\Sigma$. Let $\Gamma_\Sigma = \gamma_0 \sqcup \gamma_1 \sqcup \ldots \gamma_{p-1}$. By Observation 2.17, each γ_i determines an annulus and loop orbit J_{γ_i}. Now, after flowing along the annuli, consider the parallel components of $\bigcup J_{\gamma_i}$ contained in R_0. With reference to **??**, we will keep the same notation to refer to $\gamma_i \subset S_0$ as the unique component of J_{γ_i} contained in R_0.

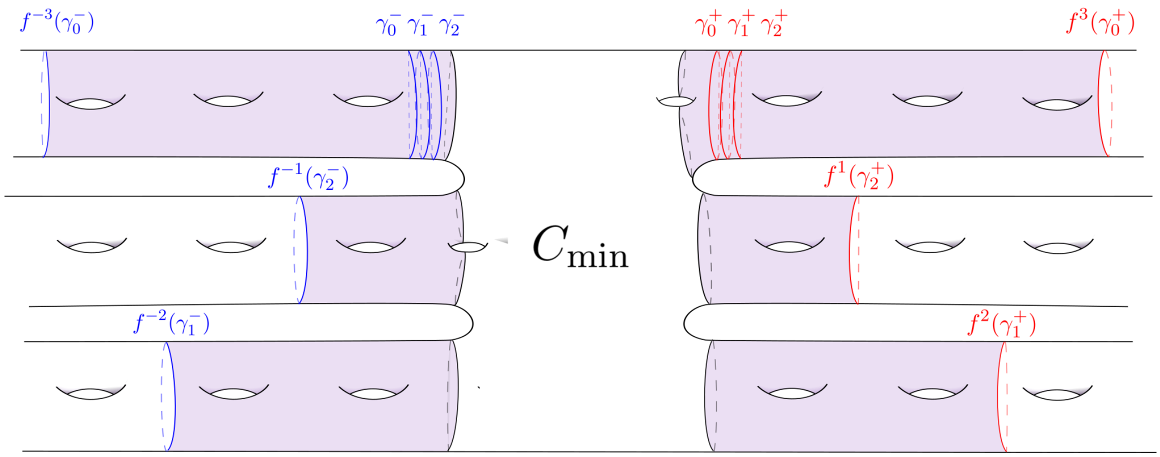

Figure 6. A multiloop core extension of a minimal core.

Let $\Gamma_\mathcal{O}$ denote the following multiloop, depending on if the end-orbit is positive or negative.

$$\Gamma_\mathcal{O} = \begin{cases} \bigcup_{i=0}^{p-1} f^{p-i}(\gamma_i) & \text{if } \mathcal{O} \text{ is attracting.} \\ \bigcup_{i=0}^{p-1} f^{-(p-i)}(\gamma_i) & \text{if } \mathcal{O} \text{ is repelling} \end{cases}$$

For each juncture orbit, the multiloop $\Gamma_\mathcal{O}$ forms a partial juncture of $\mathcal{O}$ so that the union of $\Gamma_\mathcal{O}$ over all end-orbits forms a disjoint union of a positive and negative juncture of f. The Γ-extension C_Γ is the core bounded by these junctures. Observe that it is an f-extended core, and was carefully constructed to satisfy the juncture distinction property.

6.5. Choice of Properly Embedded Core

Claim 6.3. *Let h be a hyperelliptic involution homemorphism on a closed, connected surface Σ, and let γ be an oriented loop so that $h(\gamma) = -\gamma$. Then for each p, there exists a smooth oriented multi loop, $\Gamma = \gamma_0 \cup \gamma_1, \ldots \cup \gamma_{p-1}$ with each component isotopic to γ, so that $h(\Gamma) = -\Gamma$, i.e. h preserves Γ and reverses the orientation of the image of each component of* Γ.

Given the claim above, we are ready to describe our choice of core. For each component Σ, let Γ_Σ be an oriented multiloopconsisting of p_Σ parallel components about $\gamma(\Sigma)$ so that $h(\Gamma_\Sigma)) = -\Gamma_\Sigma$. Finally, we will let Γ^* denote the union of such multiloops over all components of ∂M.

We isotope Γ^* to a multiloop $\Gamma' \subset \partial M - \partial_\infty C_{\text{min}}$; let C_0 be the Γ'-extension of C_{min} as defined in the previous subsection Section 6.4.1.. Now, C_0 is our good core and we will isotope the manifold to get the right $\mathcal{L}$-embedding $(C, \partial C) \hookrightarrow (M, \partial M)$ of C_0 so that $\partial C = \Gamma^*$.

First, we'll observe that both $\partial_\infty C_0$ and Γ^* are smooth, isotopic curves of ∂M, so by the Isotopy Extension Theorem [23, Section 5], there exists a smooth ambient isotopy $\psi_t : \partial M \to \partial M$ so that $\psi_0 = \text{id}_{\partial M}$ and $\psi_1(\partial_\infty C_0) = \Gamma_*$. Let $U = U^+ \sqcup U^-$ be a spiraling neighborhood of ∂M, which we will parametrize by $U^+ = \partial_+ M \times [0, 1]$ and $U^- = \partial_- M \times [-1, 0]$ so that $\partial M \times \{0\}$ is in the interior of the manifold. Now, let Ψ be the diffeomorphism of M given by the identity on $M - U$, $\Psi(x, t) := (\psi_t(x), t)$ on

U^+ and $\Psi(x,t) := (\psi_{-t}(x), t)$ on U^-. Observe that Ψ is isotopic to id_M. Let $\overline{C}_0$ be the $\mathcal{L}$-embedding of C_0 obtained by collapsing juncture annuli, as in Section 2.8.2.. Since $\overline{C}_0$ is transverse to $\mathcal{L}$, $\Psi(\overline{C}_0)$ is transverse to $\Psi(\mathcal{L})$. Now, $\Psi(\overline{C}_0)$ is a properly embedded core bounded by Γ^*.

We'll use Ψ to remark M, considering the tuple $(M, \mathcal{F}, \mathcal{L})$ as $(\Psi(M), \Psi(\mathcal{F}), \Psi(\mathcal{L}))$. We construct the h-double as $N(M,h) = \Psi(M) \cup_h \Psi(M)^{\downarrow}$ so that the union $\Psi(\overline{C}_0) \cup_{\sigma \circ h} \Psi(\overline{C}_0)^{\downarrow}$ is a closed surface $\hat{F}$.

The next lemma shows that our choice of C_0, C and h satisfy the hypotheses of Proposition 3.1.

Lemma 6.4. *In Type I, II, III as above, the core C_0 and homeomorphism h satisfy:*

(1) C_0 is an f-extension, and meets each flow line of $\mathcal{L}$.

(2) $\chi(C_0) \leq \eth(f)$

(3) There is an $\mathcal{L}$-embedding $(C, \partial C) \hookrightarrow (M, \partial M)$ of C_0.

(4) h preserves ∂C, and reverses the orientation on each image of each component of ∂C.

(5) $N = N(M,h)$ is atoroidal.

Proof of (1). The core C_0 is constructed as an f-extension, containing the core bounded by $\bigcup f^{p-i}(\gamma_i)$ ($\bigcup f^{-(p-i)}(\gamma_i)$ if $\mathcal{O}$ is repelling). We'll prove that it meets each flow line of $\mathcal{L}$. For each end-orbit $\mathcal{O}$ of period p, let $L_{\mathcal{O}}$ denote compact, connected subsurface:

$$L_{\mathcal{O}} = \begin{cases} \overline{U_{\gamma_0}} - f^p(U_{\gamma_0}) & \text{if } \mathcal{O} \text{ is attracting.} \\ \overline{U_{\gamma_0}} - f^{-p}(U_{\gamma_0}) & \text{if } \mathcal{O} \text{ is repelling.} \end{cases}$$

Let L_+ be the union of $L_{\mathcal{O}}$ over all positive end-orbits, and similarly, let L_- be the union of $L_{\mathcal{O}}$ over all negative end orbits. By our construction, $L_+, L_- \subset C$. Observe that the compact regions $\{f^k(L_\pm)\}$ cover $U_\pm$. Thus,

$$S = C \cup \bigcup_{k \in \mathbb{Z}} f^k(L_+ \sqcup L_-) \subset \bigcup_{k \in \mathbb{Z}} f^k(C).$$

It follows from Proposition 2.19 that C meets each flow line of $\mathcal{L}$.

□

Proof of (2). We compute the $\chi(C_0) = \chi(C)$. Let Z denote the surface $C_0 - \text{int}(C_{\text{min}})$, and let Z_+ (resp. Z_-) represent the components of Z contained in U_+ (resp. U_-). Recall that by properties of the Euler characteristic, $\chi(C_0) = \chi(C_{\text{min}}) + \chi(Z_+) + \chi(Z_-)$. We'll start by computing the Euler characteristic of Z_+, working in a positive end orbit $\mathcal{O}$ at a time. We let $Z_{\mathcal{O}}$ denote the components of Z_+ contained in $\mathcal{U}_{\mathcal{O}}$, and Z_i denote the component of $Z_{\mathcal{O}}$ which meets $\mathcal{U}_i$. The surface Z_i is bounded by $\partial C_{\text{min}} \cap \mathcal{U}_i$ and $f^i(\gamma_{p-i})$.

Then, it follows by our multiloop core construction that $f^{p-i}(Z_i) \hookrightarrow L_+$. Further, since L_+ is a fundamental domain for the action of f on $\mathcal{U}_+$, L_+ embeds into $\partial_+ M$ so that $\chi(\partial_+ M) = \frac{1}{2}\chi(\partial M) = \frac{1}{3}\xi(f)$. Finally, observe that the sum $\sum p_{\mathcal{O}}$ over all end-orbits is bounded above by $\xi(f)$, since the complexity measures the cardinality of a maximal multicurve of ∂M.

Thus,

$$\chi(Z_{\mathcal{O}}) \leq \sum_{i=0}^{p_{\mathcal{O}}-1} \chi(Z_i) \leq p_{\mathcal{O}} \cdot \chi(L_+) \leq p_{\mathcal{O}} \cdot \chi(\partial_+ M) = \frac{1}{2} p_{\mathcal{O}} \cdot \chi(\partial M) = \frac{1}{3} p_{\mathcal{O}} \cdot \xi(f)$$

and

$$\chi(Z_+ \sqcup Z_-) = \sum_{\mathcal{O}} \chi(Z_{\mathcal{O}}) \leq \sum_{\mathcal{O}} \frac{1}{3} p_{\mathcal{O}} \cdot \xi(f) \leq \frac{1}{3}\xi(f)^2 \leq \xi(f)^2$$

so that

$$\chi(C) \leq \chi(f) + \xi(f)^2 = \eth(f)$$

□

Observe that by Proposition 3.1(i), the genus of F, the fiber of $N(M, h)$ isotopic to $\hat{F}(C, h)$, is bounded by $\eth(f)$.

Proofs of (3)-(5). Since the junctures of C_0 satisfy the juncture distinction property, the core is carried to the properly embedded surface $C = \overline{C}_0$ as in Section 2.8.2.. We use the diffeomorphism Ψ to remark the manifold so that $\partial C = \Gamma$. The homeomorphism h was chosen so that h preserves ∂C while reversing the orientation of each image of each component. Finally, recall that the gluing homeomorphisms for Type I, and Type II are chosen in the strongly irreducible setting where $\mathcal{A} = \varnothing$, while that of Type III was cooked up to satisfy $h(\mathcal{A}_\Sigma) \cap \mathcal{A}_\Sigma \neq \varnothing$ for each component. In any case, $N(M, h)$ is atoroidal by Claim 6.2. This completes the proof of the lemma. $\square$

CHAPTER 7

ISOMETRIC IMMERSIONS

Before proving Theorems A and B, we record some general facts about the compactified mapping torus M and its h-double. These results are generally true for a compact, irreducible atoroidal hyperbolic 3-manifold with boundary, but we'll just state it for our compactified mapping tori.

Lemma 7.1. *Let $h : \partial M \to \partial M$ be a homeomorphism so that $N(M, h)$ is hyperbolic. Then, there exists a hyperbolic metric $s \in AH(M_f)$ and an isometric immersion $\iota : (M_f, s) \to N$ so that for each simple closed curve $\alpha \subset M$, $\ell_s(\alpha) = \ell_N(\iota(\alpha))$.*

Proof. Consider the inclusion $i : M \hookrightarrow N$, and let $\pi : N_{\pi_1(M)} \to N$ be the cover of N corresponding to $i_*(\pi_1(M)) \leq \pi_1(N) \leq \mathrm{Isom}^+(\mathbb{H}^3)$. Naturally, $N_{\pi_1(M)}$ is homeomorphic to M_f. Let s be the metric corresponding to the pullback $\pi : M_f \to N$. This naturally produces an isometric immersion $\pi : (M_f, s) \to N$. For each curve $\alpha \subset N$, and each component $\widetilde{\alpha}_0 \subset \widetilde{\alpha} = \pi^{-1}(\alpha)$, the restriction $\pi|_{\widetilde{\alpha}_0} : \widetilde{\alpha}_0 \to \alpha$ is degree-one. Thus, $\ell_s(\widetilde{\alpha}_0) = \ell_N(\alpha)$. □

Lemma 7.2. *Suppose that f is strongly irreducible, and let (M, s^*) be the totally geodesic boundary structure of its acylindrical compactified mapping torus. Let $h : (\partial M, s^*) \to (\partial M, s^*)$ be an isometry, and let $N = N(M, h)$ be the h-double of M equipped with the hyperbolic metric induced by the gluing.*

Then, $(M, s^*) \hookrightarrow N$ isometrically embeds, and there is an isometric immersion $\iota : (M_f, s^*) \to N$ so that for each simple closed curve $\alpha \subset M$, $\ell_{s^*}(\alpha) = \ell_N(\iota(\alpha))$.

Proof. The first claim is immediate by our assumptions. As above, we consider the cover $\pi : (M_f, s) \to N$ corresponding to $\pi_1(M) \subset \mathrm{Isom}(\mathbb{H}^3)$, where s is the pullback metric from the hyperbolic structure on N induced by the gluing. For concreteness, we check that $(M_f, s) = (M_f, s^*)$. Since $(M, s^*) \subset N$ is a convex, totally geodesic submanifold, its lift $\widetilde{M} \subset \mathbb{H}^3$ is a $\pi_1(M)$-invariant convex subset of $\mathbb{H}^3$ whose boundary consists of totally geodesic embeddings of $\mathbb{H}^2 \hookrightarrow \mathbb{H}^3$. The convex core of (M_f, s) is $\widetilde{M}/\pi_1(M)$, a hyperbolic manifold with totally geodesic boundary. By the uniqueness of such a structure, $s = s^*$. □

7.1. Proofs of Theorem A and Theorem B

We are ready to prove the main theorems.

Proof of Theorem A. Fix $D, \varepsilon > 0$ and let f be an atoroidal end-periodic homeomorphism with $\eth(f) \leq D$. Let C_0 and h be the choice of core and gluing construction from Chapter 6. By Lemma 6.4, C_0 and h satisfy the hypotheses of Proposition 3.1. Further, by Lemma 7.1, there is a hyperbolic structure $s \in \mathrm{AH}(M_f)$ for which (M_f, s) isometrically immerses into the closed, fibered hyperbolic manifold $N = N(M, h)$. It follows from Proposition 3.1(i) and Remark 4.2 that N is fibered by a closed surface F isotopic to $\hat{F}(C, h)$ whose genus is bounded above by $\eth(f)$.

Let $K' = K'(D, \varepsilon)$ be the constant from Theorem 1.3 coming from a surface of genus D and let $K = \max\{K', 4\}+2$. Let Y be a subsurface of $S \subset M$ satisfying $d_Y(\Lambda^+, \Lambda^-) \geq K$. Then by Proposition 3.1(ii), Y is isotopic to $W \subset F$ so that $d_W(\lambda^+, \lambda^-) \geq K'$. We apply Minsky's Theorem 1.3 so that after pulling the multicurve ∂W tight to a geodesic, $\ell_N(\partial W) \leq \varepsilon$. It follows by Lemma 7.1, that $\ell_s(\partial Y) = \ell_s(\partial W) = \ell_N(\partial W) \leq \varepsilon$, completing the proof of Theorem A. □

Proof of Theorem B. This theorem is a special case of Theorem A, we repeat a nearly identical argument to that above. Fix $D, \varepsilon > 0$ and let f be a strongly irreducible end-periodic homeomorphism whose ∂M consists of genus-2 surfaces. Let C and h be the properly embedded surface and gluing homeomorphism chosen from Type I. Recall that both satisfy the hypotheses of Proposition 3.1, and that h is an isometry of (Σ, s^*). By Lemma 7.2, (M_f, s^*) isometrically immerses into N so that $\ell_{s^*}(\alpha) = \ell_N(\alpha)$ for all simple closed curves α of M_f.

Let $K' = K'(D, \varepsilon)$ be the constant from Theorem 1.3 coming from ε and a surface of genus D. Exactly as the proof above, by applying Proposition 3.1 and Theorem 1.3 for $K = \max\{K', 4\} + 2$, it follows that if $d_Y(\Lambda^+, \Lambda^-) \geq K$, then $\ell_{s^*}(\partial Y) = \ell_N(\partial Y) \leq \varepsilon$. This completes the proof of Theorem B.

□

CHAPTER 8

EXAMPLES

We'll detail constructions of end-periodic homeomorphisms before providing a few applications of our main theorems. We note to the reader that notation has been reset.

8.1. Handle Shift Mapping Classes

We will explicitly describe a somewhat natural occurrence of a end-periodic translation on a ladder surface.

Let $\Sigma = \Sigma_g$ be a closed, connected surface of genus g. Let β be a non-separating curve on Σ, let $R = \Sigma \setminus\!\!\setminus \beta$, i.e, the surface with two boundary components cut along β, and consider the following homeomorphism

$$\tau_\beta : \pi_1(\Sigma) \to H_1(\Sigma) \xrightarrow{\iota_\beta} \mathbb{Z}$$

which is the composition of the abelianization map on $\pi_1(\Sigma)$ and the algebraic intersection form with respect to β defined by $\iota_\beta([\alpha]) = \langle [\alpha], [\beta] \rangle$. We observe that $H = \ker \tau_\beta = \ll \pi_1(R) \gg$, the normal closure of $\pi_1(R)$. The cover $\widetilde{S}_H \to \Sigma$, corresponding to H has a cyclic deck transformation group generated by $\rho : \widetilde{S}_H \to \widetilde{S}_H$.

Example 8.1 (A translation map on the ladder)**.** *Given Σ, and β, and H as above:*

(i) *$\widetilde{\Sigma}_H$ is homeomorphic to the ladder surface L.*

(ii) *ρ is a an end-periodic translation map.*

Proof. First, note that $H = \pi_1(\Sigma_H)$ is an infinite-index subgroup of $\pi_1(\Sigma)$. Observe that R lifts homeomorphically to $\widetilde{\Sigma}_H$, and that a given component of $p^{-1}(R)$ is a fundamental domain of the infinite order deck transformation action– this implies that Σ_H is a boundaryless surface of infinite-type. Since $\langle T \rangle$ acts freely and cocompactly on the surface, by the Svarc-Milnor lemma (see [3, Prop. 8.19]), Σ_H is quasi-isometric to $\mathbb{Z}$, thus is two-ended. It follows that Σ_H is homeomorphic to the ladder surface, as it is the unique boundaryless, two-ended surface of infinite-type.

Fixing a basepoint $x \in R$ and lifted basepoint $\widetilde{x}_0 \in p^{-1}(x)$, we label a lift of R by $\widetilde{R}_i$ if it contains the lifted basepoint $T^i(x_0)$. For each i, we have the neighborhood $\overline{U}_+ = \bigcup_{k \geq i} T^i(\widetilde{R}_i)$ so that $T(U_+) \subset U_+$. We may similarly define $\overline{U}_- = \bigcup_{k \leq i} T^i(\widetilde{R}_i)$, and $T^{-1}(U_-) \subset U_-$. Note that the interiors of the respective sets, after taking iterates of the map T, form neighborhood bases for the respective ends, and $\bigcup_{\mathbb{Z}} T^n(U_+) = \bigcup_{\mathbb{Z}} T^n(U_-) = L$. Thus, T is an end-periodic translation map. □

We note that any such map above is isotopic to a handle shift homeomorphism as defined by Patel-Vlamis[24]. Let $\overline{L}$ be the surface whose boundary is homeomorphic to $\mathbb{R} \sqcup \mathbb{R}$ and interior homeomorphic to the Loch Ness Monster surface (the unique boundaryless infinite-type surface with one end). Informally, a handle shift homeomorphism is an embedding of a translation map of $\overline{L}$ onto a S, and the support of the homemorphism is the image of the embedding. We refer the reader to [24, Section 6] for their precise definition of a handle shift homeomorphism. It is worth noting that handle shift homeomorphisms themselves are not necessarily end-periodic, nor isotopic to an end-periodic homeomorphism.

8.2. Constructions of Strongly Irreducible End-periodic Homeomorphisms

The following proposition of Field-Kim-Leininger-Loving uses handle shifts and compactly supported mapping classes to produce a myriad of examples of strongly irreducible homeomorphisms. The authors specify ρ to be the product of commuting handle shift homeomorphisms with disjoint support as in [10, Construction 2.10]. We rephrase their result below as their arguments work for a more general class of end-periodic homeomorphisms.

Proposition 8.2. *(Field-Kim-Leininger-Loving [10, Prop. 6.3]) Given ρ an end-periodic homeomorphism, C a connected core of ρ, and $g \in \mathrm{Homeo}(C, \partial C)$, if:*

(i) ∂C separates each end, i.e. the components of $S - C$ are in bijection with $\mathrm{Ends}(S)$,

(ii) $\rho(\partial_- C), \rho^{-1}(\partial_+ C) \subset C$ with $i(\rho(\partial_- C), \rho^{-1}(\partial_+ C)) = 0$,

(iii) and $d_C(\rho(\partial_- C), g(\rho(\partial_- C)) > 9$,

then $f = \rho g$ is isotopic to a strongly irreducible end-periodic homeomorphism.

Given a choice of end-periodic map ρ, the above conditions are satisfied with ease by extending the core C and taking sufficiently large powers of g as needed.

Consider a family of atoroidal end-periodic homeomorphisms constructed by the form $f_n = \rho g^n$ $n \neq 0$ with $g \in \mathrm{Homeo}(C, \partial C)$ a representative of a pseudo-Anosov mapping class. It follows that each f_n has the same end-capacity, and since C remains a core of each, its core capacity is bounded above by $\chi(C)$. Let $\Lambda^+ n$, Λ^-_n denote the Handel-Miller laminations of f_n. The following proposition shows that $d_C(\Lambda^+_n, \Lambda^-_n) \to 0$.

Proposition 8.3. *Let ρ be an end-periodic homeomorphism with juncture orbits P_+, P_-. Let $g \in \mathrm{Homeo}(Y, \partial Y)$ be so that $g|_{Y_i}$ represents a fully supported mapping class on each connected component $Y_i \subset Y$. Suppose that:*

1. *the homeomorphism $f = \rho g$ is not a translation,*
2. *Y_i meets Λ^+, Λ^-, P^+ and P^-.*

Then,

$$\tau_{Y_i}(g) - d_{Y_i}(P^+, P^-) - 2 \leq d_{Y_i}(\Lambda^+, \Lambda^-).$$

Proof. Let α, η be a pair of positive and negative junctures in P^+,P^- respectively so that $\rho^{-1}(\alpha)$ and $\rho(\eta)$ meet Y_i, but for $k \geq 0$, $\rho^k(\alpha)$ and $\rho^{-k}(\eta)$ are disjoint from Y_i (this exists by the compactness of Y, see the related Lemma 5.3). It follows that α and η define juncture orbits $J^+ = \bigcup_{k\in\mathbb{Z}} f^k(\alpha)$ and $J^- = \bigcup_{k\in\mathbb{Z}} f^k(\eta)$ of f. Recall from Remark 5.2 that $d_C(J^+, J^-) - 2 \leq d_C(\Lambda^+, \Lambda^-)$.

$$\begin{aligned}\tau_{Y_i}(g) = \tau_{Y_i}(g^{-1}) &\leq d_{Y_i}(g^{-1}p^{-1}(\alpha), p^{-1}(\alpha)) \\ &\leq d_{Y_i}(g^{-1}p^{-1}(\alpha), \rho(\eta)) + d_{Y_i}(\rho(\eta), \alpha)\end{aligned}$$

We observe that $\rho(\eta) = \rho g(\eta) = f(\eta)$.

So, continuing the calculation from the previous line, we have:

$$= d_{Y_i}(f^{-1}(\alpha), f(\eta)) + d_{Y_i}(\rho(\eta), \alpha)$$

$$\leq d_{Y_i}(J^+, J^-) + d_{Y_i}(P^+, P^-)$$

$$\leq d_C(\Lambda^+, \Lambda^-) + 2 + d_C(P^+, P^-)$$

□

Recall that $\tau(g)$ is the stable translation length of g in $\mathcal{AC}(C)$, and that any fully supported has positive translation length, and $\tau(g^n) = n\tau(g)$.

8.3. Setup

For the next few examples, we'll fix the following setup: Let L be the ladder surface, and as from Example 8.1, we let ρ be a translation map corresponding to Σ_2 and β as in Figure 7. We label each component of $\widetilde{\beta}$ by the integers as pictured above, and let C_i denote the core bounded by the separating curves $\widetilde{\beta}_i$ and $\widetilde{\beta}_{i+1}$.

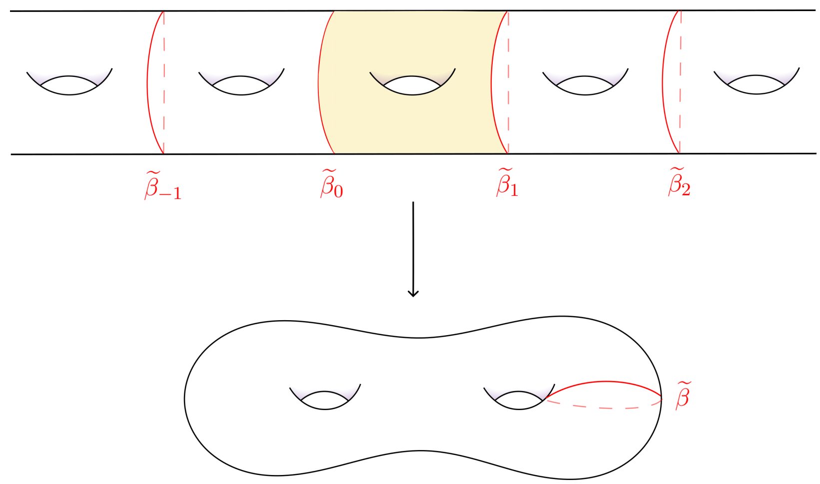

Figure 7. A translation homeomorphism on the ladder surface induced by a cover.

8.4. Short Non-peripheral Curves

The Handel-Miller laminations and juncture orbits give an obstruction to a curve being peripheral in the manifold. Recall that a loop $\gamma \subset S$ is a positively (negatively) escaping with respect to f if and only if it is contained in $\mathcal{U}_+$ ($\mathcal{U}_-$), and that any escaping loop of f is peripheral in M_f.

Remark 8.4. *If $\gamma \subset S$ is a positively escaping loop with respect to f, then for any essential connected subsurface Y, either: $\pi_Y(\Lambda^-) = \varnothing$, $\pi_Y(\gamma) = \varnothing$, or $d_Y(\gamma, \Lambda^-) \leq 1$. Similarly, if γ is negatively escaping, for any essential connected subsurface Y either: $\pi_Y(\Lambda^+) = \varnothing$, $\pi_Y(\gamma) = \varnothing$, or $d_Y(\gamma, \Lambda^+) \leq 1$.*

The remark follows from Proposition 2.8, which states that each positively (resp. negatively) escaping loop γ is disjoint from the negative (resp. positive) lamination.

Example 8.5 (Short non-peripheral curves)**.** *Let L be the ladder surface, and suppose Y, a compact subsurface, is the union of essential connected components $Y_1, \dots, Y_n$. Then, for every $\varepsilon > 0$, there exists $f = f_\varepsilon$, a strongly irreducible end-periodic homeomorphism, and associated mapping torus M_f so that:*

1. *No component of ∂Y is peripheral in $\overline{M}_f$, and*
2. *s^*, the corresponding structure on $\overline{M}_f$ with totally geodesic boundary satisfies*

$$\ell_{s^*}(\partial Y) \leq \varepsilon.$$

Proof. We will first choose an auxiliary end-periodic map f_0 for which $\Gamma = \partial Y$ is non-escaping. Let ρ be the translation depicted in Figure 7. We direct the reader to Remark 2.3 to remind them of a few fun facts which we will use for subsurface projection calculations throughout this argument and the rest of the section.

Let C' be a subsurface formed as the union of consecutive $C_i, \dots, C_{i+k}$ so that Y is an essential subsurface of C'. We'll let $C = C_{i-1} \cup C' \cup C_{i+k+1}$ so that $\pi_C(\rho(\Gamma))$, $\pi_C(\rho^{-1}(\Gamma)) \neq \varnothing$ and $\rho(\partial_- C)$, $\rho^{-1}(\partial_+ C)$, and Γ are mutually disjoint multicurves contained in C.

By considering sufficiently large powers of an arbitrary pseudo-Anosov mapping class on C, we may choose $h \in \mathrm{Homeo}(C, \partial C)$ so that:

(i) $d_C(\rho(\partial_- C), h(\rho(\partial_- C)) \geq 9$

(ii) $d_C(h\rho(\partial_- C), \rho^{-1}(\Gamma)) \geq 5$

(iii) $d_C(h^{-1}(\Gamma), \Gamma) \geq 5$

From the conditions above and Proposition 8.2, it follows that $f_0 = \rho h$ is strongly irreducible. Observe that C must meet both Λ_0^+ and Λ_0^-, the Handel-Miller laminations of f_0 by Corollary 2.10; otherwise produces a reducing loop in ∂C. With a direct application of Remark 8.4, we see that Γ is neither negatively nor positively escaping under the strongly irreducible map f_0, since its Handel-Miller laminations $\Lambda_0^\pm$ satisfy:

$$\begin{aligned} d_{\rho C}(\Lambda_0^-, \Gamma) &\geq d_{\rho C}(f_0^2(\partial_- C), \Gamma) - 2 \\ &= d_{\rho C}(\rho h \rho(\partial_- C), \Gamma) - 2 \\ &\geq 3 \end{aligned} \tag{8.1}$$

$$\begin{aligned} d_C(\Lambda_0^+, \Gamma) &\geq d_C(f_0^{-1}(\partial_+ C), \Gamma) - 1 \\ &\geq d_C(\Gamma, h^{-1}(\Gamma)) - d_C(h^{-1}\rho^{-1}(\partial_+ C), h^{-1}(\Gamma)) - 1 \\ &\geq 3. \end{aligned} \tag{8.2}$$

For each component Y_i of Y, let $g_i \in \mathrm{Homeo}(Y_i, \partial Y_i)$ each be representatives of a fully supported mapping class. We let $g = \prod g_i$ be the composition over all components. Consider $f_n = \rho h g^n = f_0 g^n$ with its associated Handel-Miller laminations Λ_n^+, Λ_n^-. Recall that each f_n has bounded capacity genus, and that g preserves $\partial_\pm C$. It follows that $f_n = \rho h g^n$ satisfies the conditions of Proposition 8.2, hence is a strongly irreducible homeomorphism. Our goal is to apply Proposition 8.3 to f_0 and $f_0 g^n$ to show that $d_{Y_i}(\Lambda_n^+, \Lambda_n^-)$ is arbitrarily large for each $Y_i \subset Y$.

Let $\eta = \partial_- C$ and $\alpha = \partial_+ C$. Note that α, η are positive and negative junctures of f_n for all n.

We'll first check that $f_0^2(\eta)$, $f_0^{-1}(\alpha)$ meet each Y_i. This will verify that

$$P^+ = \bigcup_{k\in\mathbb{Z}} f_0^k(\alpha), \quad P^- = \bigcup_{k\in\mathbb{Z}} f_0^k(\eta),$$

the f_0-juncture orbits for α and η, both meet Y_i. We showed in eq. (8.1) and eq. (8.2) that projection distance between Γ and $f_0^{-1}(\alpha)$, and Γ and $f_0^2(\eta)$ is sufficiently large in $\mathcal{AC}(C)$; the respective multicurves meet each component of Γ, and therefore each component of Y (see Remark 2.3). We will use the same reasoning to verify that Λ_n^+ and Λ_n^- meet Y_i. This calculation also verifies that Γ is neither positively nor negatively escaping. We start by using Remark 5.2 to jump from the laminations to a particular juncture.

$$\begin{aligned} d_C(\Lambda_n^+, \Gamma) &\geq d_C(f_n^{-1}(\alpha), \Gamma) - 1 \\ &\geq d_C(g^{-n}h^{-1}\rho^{-1}(\alpha), \Gamma) - 1 \\ &\geq d_C(f_0^{-1}(\alpha), \Gamma) - 1 \\ &\geq 4. \end{aligned} \tag{8.3}$$

For this calculation, let $W = \rho h(C)$. Recall that g and h are supported on Y and C respectively, and that $\rho(\eta)$ and $\rho^{-1}(\Gamma)$ are both disjoint from Γ.

$$d_W(\Lambda_n^-, \Gamma) \geq d_W(f_n^2(\eta), \Gamma) - 1$$

$$
\begin{aligned}
&= d_W(\rho h g^n \rho h g^n(\eta), \Gamma) - 1 = d_W(\rho h g^n \rho(\eta), \Gamma) - 1 \\
&= d_C(g^n \rho(\eta), h^{-1}\rho^{-1}(\Gamma)) - 1 \\
&\geq -1 - d_C(g^n\rho(\eta), g^n(\Gamma)) + d_C(g^n(\Gamma), h^{-1}(\Gamma)) - d_C(h^{-1}(\Gamma), h^{-1}\rho^{-1}(\Gamma)) \\
&= d_C(\Gamma, h^{-1}(\Gamma)) - 3 \\
&\geq 2.
\end{aligned}
$$

So, f_0 is an end-periodic homeomorphism whose juncture orbits P^+, P^- meet Y_i for each component $Y_i \subset Y$. And for each n, the Handel-Miller laminations Λ_n^+, Λ_n^- both meet Y_i. By Proposition 8.3, we have:

$$
\tau_{Y_i}(g^n) - d_{Y_i}(P^+, P^-) - 2 \leq d_{Y_i}(\Lambda^+, \Lambda^-).
$$

Now, since $g|_{Y_i} \in \mathrm{Map}(Y_i)$ is fully supported, it acts loxodromically on $\mathcal{AC}(Y)$. Let c be the constant from Theorem 2.2 depending only on Y so that $c|n| < \tau_{Y_i}(g^n)$. Let $B = \max\{d_{Y_i}(P^+, P^-) + 2\}$.

Finally, let $\varepsilon > 0$, $D = \eth(f_0)$, and $K = K(D, \varepsilon)$ be the constant Theorem B. Above we verified that $f_n = \rho g^n h$ is a strongly irreducible homeomorphism for all n with $\partial\overline{M}_{f_n} \cong \Sigma_2^+ \sqcup \Sigma_2^-$. Let $N \in \mathbb{Z}$ be so that $K \leq c|N| - B$. Let $M_N = (\overline{M}_{f_N}, s^*)$ be the corresponding compactified mapping torus with totally geodesic boundary. Since $d_Y(\Lambda_N^+, \Lambda_N^-) \geq K$, by Theorem B we have $\ell_{s^*}(\Gamma) = \ell_{s^*}(\partial Y) \leq \varepsilon$ as required. □

8.5. Thin Totally Geodesic Structures

The following example answers a question posed by Kent, and mirrors her main result in [14] which produces arbitrarily short curves of totally geodesic boundaries of knot complements.

Let Σ_+ and Σ_- each be two copies of a closed surface and let $\Sigma = \Sigma_g^+ \cup \Sigma_g^-$. Suppose that $(M, s^*) \in \mathfrak{F}(\Sigma)$, i.e. M is an acylindrical compactified mapping torus of an strongly irreducible end-periodic map.

We let $(\Sigma_\pm, s^*)$ denote the totally geodesic metric on the boundary surface. We let $\ell_{s^*}(\Sigma)$ denote the length of shortest closed (simple) geodesic of (Σ, s^*), i.e. the length of its systole.

Setup. We will use $C = C_{-1} \cup C_0 \cup C_1$ be the core of ρ on the ladder surface L as in Figure 7. Let $P^+ = P^- = \bigcup_{i\in\mathbb{Z}} \rho^i(\partial_- C_0)$. Let $\phi \in \mathrm{Homeo}(C, \partial C)$ be a representative of a pseudo-Anosov mapping class of C.

Remark 8.6 (Some bookkeeping)**.** *Let $\varepsilon > 0$, $D = \chi(C) + 4\xi(\Sigma_2)^2$ and $K = K(D, \varepsilon)$ the corresponding constant from Theorem B.*

There is a power N so that for $n \geq N$, $\rho\phi^n$ is strongly irreducible with $\tau(\phi^n) \geq K(D, \varepsilon) + 3$. Considering all powers of such maps, let $f(n, g) = (\rho\phi^n)^{g-1}$ for $g, n \in \mathbb{Z}$ and let $\Lambda^+(n, g), \Lambda^-(n, g)$ denote the corresponding positive and negative Handel-Miller laminations of the homeomorphism.

Observe that for all $n \geq 1, g \geq 2$, we have $\xi(f(n, g)) = 2\xi(\Sigma_g)$, and $\chi(f(n, g)) \leq \chi(C)$, so that $\eth(f(n, 2)) \leq D$.

Example 8.7 (Thin totally geodesic structures)**.** *For every $g \geq 2$, $\varepsilon > 0$, there exists $N > 0$ so that for all $n \geq N$, the composition $f(n, g) = (\rho\phi^n)^{g-1}$ satisfies the following:*

(i) Its corresponding compactified mapping torus $M(n, g)$ is an acylindrical hyperbolic manifold.

(ii) $\partial M(n, g) \cong \Sigma_g^+ \sqcup \Sigma_g^-$, where each component is a closed genus-g surface.

(iii) The components of ∂C satisfy $\ell_{s^*}(\partial C) \leq \varepsilon$, where s^* denotes the unique totally geodesic boundary structure of $M(n, g)$

In other words, this example is an explicit topological construction of 3-manifolds with totally geodesic boundary structures of any genus ≥ 2 and arbitrarily short systole. To make the notation easier on the eyes, we'll drop the * decoration so that any s denotes the totally geodesic structure of the associated 3-manifold.

Proof. We will first prove the statement when ∂M is the disjoint union of closed genus-2 surfaces. We will construct a sequence of strongly irreducible homeomorphisms with bounded capacity genus with the aim of applying Theorem B to the appropriate subsurface of the infinite-type surface.

Fix $\varepsilon > 0$, let $D = \chi(C) + 4\xi(\Sigma_2)^2$ and $f_2 = f(N, 2) = \rho\phi^N$ be the strongly irreducible homeomorphism as in Remark 8.6.

With Proposition 8.3 and since $d_C(P^+, P^-) = 1$, we have $d_C(\Lambda^+, \Lambda^-) \geq K$. A direct application of Theorem B has that $\ell_s(\partial C) \leq \varepsilon$ in the corresponding totally geodesic structure s for f_2. This concludes the genus-2 case.

To generalize to the genus-g setting, we will consider covers of $M = \overline{M}_{f_2}$ corresponding to the compactified mapping tori of $f_g = (f(N,2))^{g-1}$. Let $\widetilde{M} = \overline{M}_{f_g}$, and observe that $\partial\widetilde{M} \cong \Sigma_g^+ \sqcup \Sigma_g^-$. Indeed, the ladders $U_\pm$ defined by $S - C = U_+ \cup U_-$ satisfy $f_g|_{U_\pm} = \rho|_{U_\pm}$. By directly analyzing ρ, we see that $U_\pm / \langle f_g \rangle$ is a genus g surface.

The covering map from $\mathring{p} : M_{f_g} \to M_{f_2}$ extends to the compactifications $p : \widetilde{M} \to M$. Let $(\widetilde{M}, \widetilde{s})$ denote the metric obtained by the pullback on (M, s) via the covering map p. Observe that $(\partial\widetilde{M}, \widetilde{s}) \hookrightarrow (\widetilde{M}, \widetilde{s})$ is totally geodesic, hence $\widetilde{s}$ is the unique such structure.

Let $\alpha = \partial_+ C$ and $\eta = \partial_- C$. For each component $\widetilde{\alpha}_0$ of $p^{-1}(\alpha)$, the restricted cover $p|_{\widetilde{\alpha}_0}$, is degree one. Thus, $\ell_{\widetilde{s}}(\widetilde{\alpha}_0) = \ell_s(\alpha) \leq \varepsilon$, as shown in the genus-2 case. An identical argument shows $\ell_{\widetilde{s}}(\widetilde{\eta}_0) \leq \varepsilon$ for (any) component $\widetilde{\eta}_0$ of $\widetilde{\eta}$. Thus, for the strongly irreducible map $(\rho\phi^N)^{g-1}$, we have $\ell_{\widetilde{s}}(\Sigma_+), \ell_{\widetilde{s}}(\Sigma_-) \leq \varepsilon$. □

We conclude with a proof of Theorem C.

Proof of Theorem C. We'll reintroduce the same characters from the previous example. Fix $\varepsilon > 0$, $g \geq 2$, and let D, K and N be the constants from Remark 8.6 and Theorem 1.3, and consider the map $f = \rho\phi^N$, and its compactified mapping torus (M, s) with totally geodesic boundary. Once again, we consider f^{g-1} and $p : (\widetilde{M}, \widetilde{s}) \to (M, s)$, the associated $g-1$-fold cover where $(\widetilde{M}, \widetilde{s})$ denotes the metric corresponding to the pullback of s under the map $p : \widetilde{M} \to (M, s)$.

We choose a core and gluing homeomorphism h just as Type I from Chapter 6. In particular, recall that $h : (\partial M, s) \to (\partial M, s)$ is an isometry, and the choices satisfy the conditions of Proposition 3.1. It follows from Remark 4.2 that the h-double $N = N(M, h)$ is hyperbolic and fibered by a closed surface F. Observe that $(\partial M, s)$ is totally geodesic in both (M, s) and N, as it retains its totally geodesic structure from the gluing.

Claim 8.8. *The isometry h lifts to an isometry $\widetilde{h} : (\partial\widetilde{M}, \widetilde{s}) \to (\partial\widetilde{M}, \widetilde{s})$.*

Proof of claim. Notice that for each component of $\widetilde{\Sigma}_0 \subset \widetilde{\Sigma} = p^{-1}(\Sigma)$, the covering map $\widetilde{\Sigma}_0$ to Σ is obtained by the kernel of the following map:

$$\pi_1(\Sigma) \to H_1(\Sigma, \mathbb{Z}) \xrightarrow{\iota_\beta} \mathbb{Z} \xrightarrow{\text{mod } g-1} \mathbb{Z}_{g-1}$$

And since h acts by -1 on $H_1(\Sigma)$, it preserves the corresponding kernel subgroup, thus by the covering criterion, it lifts to a homeomorphism $\widetilde{h}$. Further, $\widetilde{h}$ is a local isometry of $\widetilde{s}$, and since it is a diffeomorphism, it is a global isometry on each component of Σ.

□

Let $\widetilde{N}$ be the closed, hyperbolic $\widetilde{h}$-double of $\widetilde{M}$. The cover $p : \widetilde{M} \to M$ and gluing map $\widetilde{h} : \partial\widetilde{M} \to \partial\widetilde{M}$ extend to a cover $\pi : \widetilde{N} \to N$ by mapping the copies of $\widetilde{M}$ to M via p on each component of $\widetilde{N} \setminus \partial\widetilde{M}$; one should observe this is compatible with the gluing. Again, we see that $(\partial\widetilde{M}, \widetilde{s})$ is totally geodesic in both $(\widetilde{M}, \widetilde{s})$ and $\widetilde{N}$. The 1-dimensional foliation $\mathcal{L}$ of N lifts to a 1-dimensional foliation $\widetilde{\mathcal{L}}$ of $\widetilde{N}$ for which $\widetilde{F} = \pi^{-1}(F)$ is a cross section. Further, the embeddings $\partial M \hookrightarrow M \hookrightarrow N$ lift:

$$\begin{array}{ccccc} (\Sigma_g^{\pm}, \widetilde{s}) & \hookrightarrow & (\widetilde{M}, \widetilde{s}) & \hookrightarrow & \widetilde{N} \\ \downarrow & & \downarrow & & \downarrow \\ (\Sigma_2^{\pm}, s) & \hookrightarrow & (M, s) & \hookrightarrow & N. \end{array}$$

Observe that each embedding is isometric. Example 8.7 shows that each component $\widetilde{\gamma} \subset \partial\widetilde{C} \subset \partial\widetilde{M}$ has $\ell_{\widetilde{s}}(\widetilde{\gamma}) \leq \ell_{\widetilde{s}}(\partial C) \leq \varepsilon$, thus $\ell_{\widetilde{s}}(\Sigma_g) \leq \varepsilon$. In summary, $\widetilde{N}$ is a closed, fibered hyperbolic manifold with a totally geodesic embedding of (two copies of) Σ_g, a hyperbolic genus-g surface with $\ell(\Sigma_g) \leq \varepsilon$. $\square$

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