TIME SERIES BLOCK BOOTSTRAP APPLICATION AND EFFECT OF AGGREGATION AND SYSTEMATIC SAMPLING

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ABSTRACT

In this dissertation, we review the basic properties of the bootstrap and time series application. Then we apply parametric bootstrap on three simulated normal i.i.d. samples and nonparametric bootstrap on four real life financial returns. Among the time series bootstrap methods, we look into the specific method called block bootstrap and investigate the block length consideration to properly select a suitable block size for AR(1) model. We propose a new rule of blocking named as Combinatorially-Augmented Block Bootstrap(CABB). We compare the existing block bootstrap and CABB method using the simulated i.i.d. samples, AR(1) time series, and the real life examples. Both methods perform equally well in estimating AR(1) coefficients. CABB produces a smaller standard deviation based on our simulated and empirical studies. We study two procedures of collecting time series, (i) aggrergation of a flow variable and (ii) systematic sampling of a stock variable. In these two procedures, we derive theorems that calculate exact equations for m aggregated and m^{th} systematically sampled series of the original AR(1) model. We evaluate the performance of block bootstrap estimation of the parameters of ARMA(1,1) and AR(1) model using aggregated and systematically sampled series. Simulation and real data analyses show that in some cases, the performance of the estimation based on the block bootstrap method for the MA(1) parameter of the ARMA(1,1) model in aggregated series is better than the one without using bootstrap. In an extreme case of stock price movement, which is close to a random walk, the block bootstrap estimate using systematically sampled series is closer to the true parameter, defined as the parameter calculated by the theorem. Specifically, the block bootstrap estimate of the parameter of AR(1) model using the systematically sampled series is closer to ϕ^n than that based on the MLE for the AR(1) model. Future reserch problmes include theoretical investigation of CABB, effectiveness of block bootstrap in other time series analysess such as nonlinear or VAR.

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TABLE OF CONTENTS

ABSTRACT **ii**

ACKNOWLEDGEMENTS **iii**

TABLE OF CONTENTS **iv**

LIST OF FIGURES **vi**

LIST OF TABLES **vii**

1 INTRODUCTION **1**

2 LITERATURE REVIEW **5**

2.1 Approximating bootstrap distribution 5

2.2 Approximating parametric bootstrap confidence interval 6

2.3 Edgeworth expansion and bootstrap relationship 7

2.4 Bootstrap simulation methods 8

2.5 Inconsistency of bootstrap distribution estimators 9

2.6 Bootstrap methods of time series 9

2.7 Block bootstrap methods: Time series application 11

2.8 Moving-blocks bootstrap 12

3 BOOTSTRAP SIMULATION **13**

3.1 Parametric bootstrap on i.i.d. 13

3.2 Nonparametric bootstrap on time series 19

4 APPLICATION OF BLOCK BOOTSTRAP TO TIME SERIES DATA **22**

4.1 Block length consideration 22

4.2 Simulation study on block length or the number of blocks 23

4.3 Combinatorially-Augmented Block Bootstrap-Proposed Method 26
4.4 Comparison of two methods in independent sample and time series . . 27
4.4.1 Simulation study of independent sample 27
4.4.2 Simulation study of time series 28
4.4.3 Empirical study: U.S. New One-Family Houses sold - Time Series 30

5 AGGREGATION EFFECT ON BLOCK BOOTSTRAP 32
5.1 Introduction 32
5.2 Simulation study 35
5.3 Empirical study 36

6 SYSTEMATIC SAMPLING EFFECT ON BOOTSTRAP 39
6.1 Introduction 39
6.2 Simulation study 40
6.3 Empirical study 42

7 MORE ON AR(1) AND TWO DIMENSIONAL VAR(1) BLOCK BOOTSTRAP 44

8 CONCLUSION AND FUTURE RESEARCH 47

REFERENCES 48

LIST OF FIGURES

4.1 Simulated AR(1) for the investigation of block length and the number of replications 24
4.2 U.S. New One-Family Houses sold 31

5.1 AR(1) simulation and temporal aggregation 35
5.2 Weekly Initial Claims by US workers from 1/7/1967 to 8/6/1994 Seasonally Adjusted (Source: Federal Reserve Bank of St. Louis) 38

6.1 AR(1) simulation and systematic sampling 41
6.2 S&P index and systematic sampling (Source: Bloomberg) 43

7.1 The AR(1) and two dimensional VAR(1) block bootstrap estimation) . . 46

LIST OF TABLES

2.1 Three methods of parametric bootstrap confidence intervals for θ 6

3.1 100,000 normal data, $X \sim N(0,1),\ Y \sim N(1,1),\ Z \sim N(2,1)$ 15

3.2 Bootstrap replications of the normal samples 16

3.3 Bias, Standard Error and Root Mean Square Error of the ratio estimation 18

3.4 Roughly 10 years of US and Korean stock and bond indeces daily return 20

3.5 Bootstrap replications of samples drawn from daily returns 20

4.1 Simulated AR(1) with various block length and the number of replications 25

4.2 Comparison of two block bootstrap methods on independent sample . . 28

4.3 Comparison of two block bootstrap methods on various AR(1) coefficient $\sigma_a^2 = 16$ 29

4.4 Comparison of two block bootstrap methods on U.S. New One-Family House sold using AR(1) 31

5.1 Parameter estimation of AR(1) model with $\phi = 0.8$, $\sigma_a^2 \sim N(0,1)$, $n = 1440$, and its aggregate model ARMA(1,1) with $m = 3$ and $m = 12$ (200 replications) 35

5.2 Model estimation of ARMA(1,1) model for aggregate Weekly Initial Claims series with 200 replications using two block bootstrap methods 38

6.1 Parameter estimation of AR(1) model with $\phi = 0.8$, $\sigma_a^2 \sim N(0,1)$, $n = 1440$, systematic sampling model of AR(1) $m = 3$ and $m = 12$ 41

6.2 Model estimation of AR(1) model for quarterly and yearly S&P index using two block bootstrap methods $\phi = 0.99$, $\sigma_a^2 = 217.42$ 43

7.1 Estimation of AR(1) coefficient with different block size 44

7.2 The estimation of two dimensional VAR(1) coefficient with a different block size (each 2x2 matrix entry $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}$ according to matrix convention) 45

CHAPTER 1

INTRODUCTION

The bootstrap is an at least two centuries old idea (Efron and Tibshirani (1994, p 10)) that was theoretically analyzed and brought to application in modern settings by Efron (1979). Basically, the bootstrap is claiming that many theoretically unattainable statistical quantities can be obtained via Monte Carlo "resampling" which means that many samples can be drawn repeatedly from the original sample, typically with replacement, and the bootstrap statistic can be approximated by taking an average of an appropriate function of these numbers.

Let random sample data $x_1, \cdots, x_n = X$ whose probability density function (PDF) and cumulative distribution function (CDF) are f and F respectively. We are interested in making inference about population parameter θ . For example, mean, variance, bias, standard errors and likely values under a certain null hypothesis and calculation of confidence limits for θ are interests of bootstrap analysis. In nonparametric bootstrap, for each sample value x_j, there is equal probability of n^{-1}, and the corresponding estimate F is the empirical distribution function (EDF) $\hat{F}$. The bootstrap data set $(x_1^*, \cdots, x_n^*) = X^*$ consists of members of the original data set $x_1^*, \cdots, x_n^*$, some appearing zero times, some once, twice, and so on. Therefore, a bootstrap data set X^* leads to a bootstrap replication of $\hat{\theta}^*$ of the estimate $\hat{\theta} = s(X)$, where θ might be, for example, mean or standard deviation. Then $\hat{\theta}^* = s(X^*)$ is the result of applying the same function $s(\cdot)$ to X^*. The bootstrap estimate of the standard error of a statistic $\hat{\theta}^*$, which is the estimate from the empirical distribution function $\hat{F}$, measures the standard error of the bootstrap estimate.

Efron and Tibshirani (1994, p 47) suggested an algorithm to estimate in nonparametric bootstrap by selecting J number of independent bootstrap samples $X^{*1},\ X^{*2}, \cdots, X^{*J}$, each containing n data values drawn with replacement from the original data set $x_1, \cdots x_n$. They suggested the typical number J will be in the range of 25–200. Using the bootstrap replication corresponding to each bootstrap sample $\hat{\theta}^*(j) = s\left(X^{*j}\right)$ where $j = 1, 2, \cdots J$, the standard error estimate $se_{\hat{F}}\left(\hat{\theta}^*\right)$ can be calculated by the sample standard deviation of the J replications.

$$\text{bias}_{\hat{F}}\left(\hat{\theta}^*\right) = \left\{\sum_{j=i}^{J}\left[\hat{\theta}^*(j) - \left(\Sigma_{j=1}^{J}\hat{\theta}^*(j)/J\right)\right]^2/(J-1)\right\} \tag{1.1}$$

This is the standard error of nonparametric bootstrap. For a parametric bootstrap, instead of sampling with replacement from the data, we draw J samples of size n from the parametric estimate of the population $\hat{F}_{\text{par}}$. This means there is no replacement of the data in terms of the sampling. The rest of the procedure is the same as for nonparametric bootstrap for standard error calculation. Though typically the most popular method of bootstrap is nonparametric as it does not require any knowledge of the existing data's distribution, parametric bootstrap is sometimes useful if there is some knowledge about the form of the underlying population. In addition, as we know the textbook formula of the standard error for a parametric model, we can compare the bootstrap result easily with the formula in the parametric case. A natural extension after the estimate of the standard error of the bootstrap is to estimate bias of the bootstrap. The bootstrap estimate of bias can be defined as

$$\text{bias}_{\hat{F}}\left(\hat{\theta}^*\right) = \sum_{j=1}^{J}\hat{\theta}^*(j)/J - \hat{\theta}. \tag{1.2}$$

where $\hat{\theta} = s(X)$ is an original random sample estimate such as mean or standard deviation.

As a rule of thumb, a bias of less than .25 of standard error can be ignored unless a careful confidence interval calculation is attempted. (Efron and Tibshirani 1994, p 128) Finally, there is the root mean square error which takes into account both bias and standard error.

$$\text{RMSE} = \left(se_{\hat{F}}(\hat{\theta}^*)^2 + \text{bias}^2_{\hat{F}}\right)^{1/2} \tag{1.3}$$

Using the rule of thumb described above, if an absolute value of bias over standard error is less than .25, the RMSE is no more than 3.1% greater than standard error. Efron(1979) proposed a better method than (1.2) to approximate $\mathrm{bias}_{\hat{F}}$ from J bootstrap replications. This method introduced a concept of resampling vector. Let $P_j^* = \#\{x_i^* = x_j\}/n,\ j = 1, 2, \cdots, n$ then resampling vector

$$\mathbf{P}^* = (P_1^*, P_2^*, \cdots, P_n^*) \tag{1.4}$$

Basically, P_j^* designates the probability of its showing up in the bootstrapped sample. Therefore, the J bootstrap samples $X^{*1}, X^{*2}, \cdots, X^{*J}$ assign $\mathbf{P}^{*1}, \mathbf{P}^{*2}, \cdots \mathbf{P}^{*J}$, each vector is a form of (1.4). Let $\bar{\mathbf{P}}^*$ be the average of these vectors, $\mathbf{P}^{*1}, \mathbf{P}^{*2}, \cdots \mathbf{P}^{*J}$ then the better bootstrap bias estimate is

$$\overline{\mathrm{bias}}_{\hat{F}} = E_{\hat{F}}\left[s(X^*)\right] - T(\bar{\mathbf{P}}^*). \tag{1.5}$$

According to Efron and Tibshirani (1994, p 125), $t(\hat{F})$ is the plug-in estimate of θ, which may differ from $\hat{\theta} = s(X)$ and $\mathrm{bias}_{\hat{F}}$ is the plug-in estimate of bias_F, whether or not $\hat{\theta}$ is the plug-in estimate of θ. Note that $T\left(\mathbf{P}^0\right) = \hat{\theta}^* = t(\hat{F})$, where $\mathbf{P}^0 = (1/n, 1/n, \cdots, 1/n)$. Hall (1992) showed that bias (1.5) converges much faster than bias (1.2).

It is worthwhile to compare the jackknife estimation with the bootstrap method. The jackknife, which was proposed by Maurice Quenouille in the mid 1950's, allows us to estimate biases and standard errors. Let data sample $X = (x_1, x_2, \cdots x_n)$, the ith jackknife sample $X_{(i)}$, is defined as X with the ith data point removed,

$$X_{(i)} = (x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \tag{1.6}$$

Then, for $i = 1, 2, \cdots, n$, the ith jackknife replication $\hat{\theta}_{(i)}$ of the statistic $\hat{\theta} = s(X)$ is

$$\hat{\theta}_{(i)} = s\left(X_{(i)}\right) \text{ for } i = 1, 2, \cdots, n. \tag{1.7}$$

Finally, the jackknife estimate of bias is defined as

$$\mathrm{bias}_{\text{jack}} = (n-1)\left\{\Sigma_{i=1}^{n}\hat{\theta}_{(i)/n} - \hat{\theta}\right\} \tag{1.8}$$

It is shown that $\text{bias}_{\text{jack}}$ is a quadratic Taylor series approximation to the plug-in estimate of bias in (1.2). (Efron and Tibshirani 1994, p 292).

Our motivation in this paper is to investigate the effect of bootstrap in various settings including parametric i.i.d. and nonparametric real life example. After literature review, we look at time series application of block bootstrap and show that block bootstrap procedure does not violate the existing correlation structure of the original time series. Then we propose new method of blocking rule, called Combinatorially-Augmented Block Bootstrap(CABB), which can be applied to any block bootstrap method. We compare the CABB result with regular method. We look into the effect of bootstrap in time series aggregation and systematic sampling settings. Our finding shows that CABB lowers standard error compared to the regular block bootstrap method. The accuracy of parameter estimation with CABB is mixed. However, it is not an indication against our new procedure as many times in time series where we don't know the true parameter. In general, the effect of temporal aggregation of a pure form of an AR(p) model is to produce some mixture of an AR and an MA with same or lower order. Systematic sampling effect of a pure form of an AR(p) model becomes same AR(p) with different parameter. In both temporal aggregation and systematic sampling, block bootstrappings helps to recover some information loss caused by aggregation and systematic sampling, which the simple best fitted model estimation could not recover. We showed it using extensive simulation method. Intuitively, it can be explained by smaller confidence interval using CABB since standard error is lower. However, higher bias can be introduced in CABB. Also, CABB does not violate the existing correlation structure of original time series as each pseudo-block is created by simple sequence of arithmetic average. In addition, we investigate the length of block bootstrap. Carlstein (1986) suggested block length formulae to minimize the bias in AR(1) setting. Hall (1995) simply proposed $n^{1/3}$. Unlike previous research by Carlstein (1986) and Hall (1995), our finding shows that minimum length of block bootstrap changes depending on sign of AR(1) parameter. We showed it by extensive simulation method rather than theoretical proof. In all our research, simulated time series as well as real life time series were presented to vindicate our findings.

CHAPTER 2

LITERATURE REVIEW

2.1 Approximating bootstrap distribution

Efron (1979) did ground breaking work on popularizing the bootstrap by proposing three methods of approximating bootstrap distribution: (a) direct theoretical calculation (b) Monte Carlo approximation to the bootstrap distribution by repeated realization of X^* and (c) Taylor series expansion methods to obtain the approximate mean and variance of the bootstrap distribution. This third method turns out to be the same as an infinitesimal jackknife. From Efron's (1990) equation shown in (1.5), better approximation of the bias, by the properties of the multinomial distribution, $\mathbf{P}^*$ has mean vector and covariance matrix $E\mathbf{P}^* = (1/n)e,\ \mathrm{Cov}\,\mathbf{P}^* = (1/n^2)I - (1/n^3)e'e$ where I is identity matrix and $e = (1, 1, \cdots, 1)'$. Given the bootstrap realization of R corresponding to $\mathbf{P}^*$, a Taylor series about the value $\mathbf{P}^* = (1/n)e$ can be expressed as

$$R(\mathbf{P}^*) \doteq R(e/n) + (\mathbf{P}^* - e/n)U' + 0.5(\mathbf{P}^* - e/n)V(\mathbf{P}^* - e/n)' \tag{2.1}$$

$$\text{where } U = \begin{bmatrix} \vdots \\ \frac{\partial R(\mathbf{P}^*)}{\partial P_i^*} \\ \vdots \end{bmatrix}_{P^*=(1/n)e}, \quad V = \begin{bmatrix} \cdots & & \\ \cdots & \frac{\partial^2 R(\mathbf{P}^*)}{\partial P_i^* \partial P_j^*} & \cdots \\ & \vdots & \end{bmatrix}_{P^*=(1/n)e}$$

Then the bootstrap mean and variance can be approximated as

$$E\ R(\mathbf{P}^*) \doteq R(e/n) + 1/2n\,(\Sigma_{i=1}^n V_{ii}/n)\,, \quad \mathrm{Var}\,R(\mathbf{P}^*) \doteq \Sigma_{i=1}^n U_i^2/n^2 \tag{2.2}$$

The bootstrap bias is

$$\mathrm{bias}_F\,\theta(\hat{F}) \approx 1/2n\,(\Sigma_{i=1}^n V_{ii}/n) \tag{2.3}$$

All these exactly agree with those given by Jaeckel's infinitesimal jackknife (1972). Efron and Tibshirani (1985) extended the bootstrap application to more complicated data structure such as autoregressive time series model and Box and Cox's (1964) estimation of the transformation of response in a regression. The bootstrap application shows similar parameter estimations as the original model suggested in both cases.

2.2 Approximating parametric bootstrap confidence interval

In terms of confidence interval of the parametric bootstrap, Efron(1982) introduced a simple method called the percentile method. The standard interval is defined as $\theta \in \hat{\theta} \pm \hat{\sigma} z^{\alpha}$ where z^{α} is the 100α percentile point of a standard normal variate. This interval approximates coverage probability $1 - 2\alpha$. Define $\hat{G}(s)$ to be the parametric bootstrap cdf of $\hat{\theta}^*$,

$$\hat{G}(s) = \Pr_* \left\{ \hat{\theta}^* < s \right\} \tag{2.4}$$

He proposed three different methods that use percentiles of $\hat{G}$ to find the confidence interval. These are shown in Table 2.1.

TABLE 2.1: Three methods of parametric bootstrap confidence intervals for θ

Method	**Abbreviation**	**α level endpoint**	**Correct if**
Percentile	$\theta_P[\alpha]$	$\hat{G}^{-1}(\alpha)$	$\hat{\phi} \sim N(\phi, \tau^2)$ τ constant
Bias Corrected	$\theta_{BC}[\alpha]$	$\hat{G}^{-1}(\Phi\{2z_0 + z^{(\alpha)}\})$	$\hat{\phi} \sim N(\phi - z_0\tau, \tau^2)$ $z_0,\ \tau$ constant
BC_a	$\theta_{BC_a}[\alpha]$	$\hat{G}^{-1}(\Phi\{z_0 + \frac{(z_0+z^{(\alpha)})}{1-a(z_0+z^{(\alpha)})}\})$	$\hat{\phi} \sim N(\phi - z_0\tau_\phi, \tau_\phi^2)$ where $\tau_\phi = 1 + a\phi$ $z_0,\ a$ constant

The percentile method interval is just the interval between the 100α and $100(1-\alpha)$ percentile of the bootstrap distribution of $\hat{\theta}^*$. The second method is to incorporate possible bias in $\hat{\theta}^*$. This is called the bias-corrected percentile method and its interval for θ is exactly correct if $\hat{\phi} \sim N(\phi - z_0\tau,\ \tau^2)$ for some monotone transformation $\hat{\phi} = g(\hat{\theta}),\ \phi = g(\theta)$ and some constant z_0. The third method is to take care of the non-monotone case of parametric family like a chi-square distribution. For nonparametric bootstrap, which is more plausible in real world data application, there are no exact nonparametric confidence intervals for most parameters (see Bahadur and Savage, 1956).

2.3 Edgeworth expansion and bootstrap relationship

Hall (1983) developed the theory of the bootstrap as a first-order inversion of an Edgeworth expansion. Hall (1986a) later derived an explicit formula for the first term in an unconditional Edgeworth-type expansion of coverage probability for the nonparametric bootstrap technique applied to a very broad class of "Studentized" statistics. Condition on the sample x, and let $X_1, \cdots, X_n$ be independent and identically distributed with the n-point distribution $P(X = X_r \mid x) = n^{-1},\ 1 \leq r \leq n$, Let $\bar{X} = n^{-1}\Sigma_{r-1}^{n} X_r$. Define

$$t_\alpha = t_\alpha \equiv \inf\left\{t : P[n^{1/2}g(\bar{X}^0|\bar{X}) \leq |x|] \geq \alpha\right\},\ 0 < \alpha < 1 \tag{2.5}$$

Then t_α is the nonparametric bootstrap approximation to the upper $(1-\alpha)$-level critical point of $n^{1/2}g(\bar{X}^0|\mu)$ and can be used for the construction of confidence intervals or hypothesis tests. With (2.5), an important theorem is developed for actual approximation using an Edgeworth expansion

$$P\{n^{1/2}g(\bar{X}^0|\mu) \leq t_\alpha(x)\} = \alpha + n^{-1}\Psi\{z(\alpha)\}\phi\{z(\alpha)\} + o(n^{-1}) \tag{2.6}$$

where $0 < \alpha < 1$, $z(\alpha)$ is the solution of $\Phi(z) = \alpha$, Φ, ϕ is the standard normal distribution and density functions, and Ψ is the empirical characteristic functions respectively.

$$\begin{aligned} &\Psi(z) = \frac{1}{6}\left(\frac{3}{2}\lambda_3^2 - \lambda_4\right) z(1 + 2z^2) \text{ where } \lambda_i \text{ is the standardized } i^{th}\text{cumulant} \\ &\lambda_3 = E(X-\mu)^3\sigma^{-3},\ \lambda_4 = E(X-\mu)^4\sigma^{-4} - 3 \end{aligned} \tag{2.7}$$

For a very lengthy and technical proof of (2.6), refer to Hall (1986a). Subsequently, Hall (1986b) showed the theory of the number of simulation required to construct the confidence interval. Define unconditional coverage probability as

$$a(v, H) \equiv \sum_{j=0}^{v} \binom{H}{j} \int_0^1 u^j (1-u)^{H-j} dP(p \leq u) \tag{2.8}$$

J is the number of simulation and $0 < v < H - 1$. If there is an infinite amount of simulation, then v is close to αH as $H \to \infty$. Basically,

$$\lim_{H \to \infty} a(v, H) = P(p < \alpha) \tag{2.9}$$

Using the Edgeworth expansion, (2.8) becomes

$$\begin{aligned} a(v, H) &= \alpha' + n^{-1} Q_1(\alpha') + n^{-3/2} Q_2(\alpha') + O(n^{-1} H^{-1} + n^{-2}) \\ \alpha' &= (v+1)(H+1)^{-1},\ Q_i(\alpha) = \Psi_i(z_a)\phi(z_a) \end{aligned} \tag{2.10}$$

He concluded that if H is chosen so that nominal coverage probability equals α, then $a(v, H)$ and $P(p \leq \alpha)$ agree to order $n^{-3/2}$ if H is of larger order than the square root of the sample size.

2.4 Bootstrap simulation methods

Reducing the bias and the standard error are the main objective of efficient bootstrap simulation. Earlier methods by Efron (1982, 1983) include jackknife and cross-validation techniques. Davison, Hinkley and Schechtman (1986) proposed a balanced bootstrap simulation. Let $\bar{X} = n^{-1} \Sigma X_j$ be the simple bootstrapping average of a random sample $X_1, \cdots, X_n$. The bias and the standard error of ordinary bootstrap simulation are respectively (1.3) and (1.2). If each datum X_j occurs equally often in the aggregate of whole J sample, then the standard error can be slightly reduced. To achieve that goal: concatenate J copies of $X_1, \cdots, X_n$ in a string of length nJ, randomly permute this string and then read off the J bootstrap samples as successive blocks of length n in the permuted string.

This operation forms a sample from the hypergeometric distribution with row sums

n and column sums J which makes the bias (1.3) zero. Davison, Hinkley and Schechtman (1986) also suggested explicit use of linear approximations. If the quantity being simulated has an approximation with known moments, they suggested that it is only necessary to simulate the residual from that approximation. That means the moments $E(X)$ and so forth, can be estimated by the linear approximation, then the remains can be simulated. Hall (1989) later proved that Efron (1990)'s centering method and the two methods by Davison, Hinkley, and Schechtman (1986) are asymptotically equivalent. Efron's (1990) centering method was initially introduced in a 1988 Stanford Technical Report.

2.5 Inconsistency of bootstrap distribution estimators

Athreya (1987), Knight (1989), and Hall (1991) showed that consistent bootstrap estimation of sample mean depends on the existence of finite variance. Hall (1993) generalized the inconsistency of bootstrap distribution estimators. Let $\theta_1, \cdots, \theta_p$ denote unknown parameter values, and let $\hat{\theta}_1, \cdots, \hat{\theta}_p$ represent root-n estimators, where n is sample size. Hall showed that the usual bootstrap estimator is consistent if and only if there are no ties for the value of r^{th} largest θ_i. In the event of a tie, the bootstrap estimator does not converge in probability to a constant though it does converge weakly to a distribution. In other words, the bootstrap can produce poor estimators in small to moderate samples, when there are no ties but when two or more close values of θ_i are competing for the rank of r^{th} largest. A possible remedy suggested by Hall (1993) is to find empirical evidence of ties for max θ_i. For example, an *ad hoc* equality test among the larger θ_i's could be applied. White (2000) suggested not to overuse the double bootstrap method as it might abuse data mining and end up producing meaningless forecasting model.

2.6 Bootstrap methods of time series

There are two general bootstrap methods in time series. One is to assume a certain parametric fitting on existing time series and then apply the bootstrap technique on the residual. This method was investigated by David (1977), Freedman (1984), Efron and Tibshirani (1986) and Bose (1988). Though this approach results in good performance, it depends on the existence of a plausible and tractable structural model. The nonparametric method of time series is not assuming any structural fitting of the time series, which is nonparametric and less restrictive. The most famous method of nonparametric time series bootstrapping is the block bootstrapping method, which is discussed in the next section.

The parametric method of time series bootstrapping in general has three approaches to tackle. The first is the most intuitive and easiest to implement and was suggested by Efron (1979). It simply fits an existing time series to an appropriate model such as AR, MA or ARMA, then bootstrap residuals to reconstruct bootstrapped sample time series.

The second approach is to implement linear approximation to the existing time series. Buhlmann (1997,1998), Kreiss (1988,1992), and Paparoditis (1996) proposed an autoregressive-sieve bootstrap method. That is similar to the residual bootstrapping method. Instead of choosing a sample with replacement among the residuals of a fitted model, the methods restricts bootstrap to linear predictors. The method creates a pseudo-time series by selecting a set of p starting values $X_1^*, X_2^*, \cdots, X_p^*$ and, given the past $X_1^*, X_2^*, \cdots, X_j^*,\ j \geq p$, generates the next observation X_{j+1}^* using an estimated version of the best linear predictor $\hat{X}_{j+1} = \sum_{s=1}^{p} a_s(p) X_{j+1-s}^*$ plus an error term which is selected randomly from the set of estimated prediction errors. Basically, it fits the p^{th} order autoregressive model to $X_1, X_2, \cdots, X_n$ where $p < n$. Then, after fitting, there are residuals of $p+1,\ p+2,\ \cdots, n$. Finally, the bootstrap version of the pseudo-time series is generated by model fitted time series plus randomly picked residuals with replacement. In other words, the best fitted p^{th} order autoregressive model is selected, where $p < n$, then $n-p$ amount of residuals are calculated using p^{th} order autoregressive model. Eventually, $n-p$ residuals are bootstrapped randomly and added into existing model's $n-p$ portion. For more theoretical treatments and other statistical properties of the AR-sieve model, refer to Kreiss and Lahiri (2012).

The third approach is bootstrap Markov chains, proposed by Rajarshi (1990). This method estimates Markov transition density nonparametrically using existing time series. For example, a bilinear model, such as $X_n - aX_{n-1} + bX_{n-1}\epsilon_n + \epsilon_n$ or a nonlinear model belongs to the Markov model. Bootstrap samples are generated by the stochastic process implied by transition density.

2.7 Block bootstrap methods: Time series application

Block bootstrap method emerged for nonparametric and less restrictive cases and is based on "blocking" arguments. These blocks are resampled rather than individual data values. Hall (1985) suggested the block approach in the context of spatial data. Kunsch (1989) also used a blocking argument in the bootstrap method. Carlstein (1986) proposed a blocking scheme of non-overlapping bootstrap for time series data. A standard block bootstrap data setup follows. Let the observed series $H = (X_1, \cdots, X_n)$, and b, l denote intgers such that $n = bl$. The standard non-overlapping block bootstrap rule proposed by Carlstein states that H be divided into b disjoint blocks, the k^{th} being $B_k = (X_{(k-1)l+1}, \cdots, X_{kl})$ for $1 \leq k \leq b$. Then we choose blocks $B_1^*, \cdots, B_b^*$ by resampling randomly with replacement, from among $B_1, \cdots, B_N$. If $B_i^* = (X_{i1}^*, \cdots, X_{il}^*)$, the boot strap version of H :

$$H^* = (X_1^*, \cdots, X_n^*) = (X_{11}^*, \cdots, X_{1l}^*, X_{21}^*, \cdots, X_{2l}^*, \cdots X_{b1}^*, \cdots, X_{bl}^*) \tag{2.11}$$

Hall, Horowitz and Jing (1995) showed that the optimal block size for variance or bias estimation, estimation of a one-sided distribution function, and estimation of a two-sided distribution function, equal $n^{1/5}$, $n^{1/4}$ and $n^{1/3}$, respectively. This makes sense intuitively as confidence interval estimation requires more data set than variance or bias estimation. It is worthwhile to mention that though the block bootstrap method is to preserve some correlation within the block, a correlation structure between two blocks, especially the end of one block and the beginning of the next block is not well preserved due to the random picking nature of the bootstrap. It can be seen later section of VAR(1) analysis that the random interaction between two blocks causes the original correlation structure between the two blocks to be broken. Also note that throughout our report, we assume strict stationarity in our data set and analysis. Block bootstrap can distort serial correlation structure of time series in many instances. However, the convenience and intuitive application of blocking argument can overcome the distortion of the serial correlation. Efron and Tibshirani (1994) suggested moving-blocks bootstrap method to preserve the original correlation structure and independence among adjacent blocks.

2.8 Moving-blocks bootstrap

To obtain a bootstrap realization of the time series, Efron and Tibshirani (1994) sample with replacement from all possible blocks and paste them together to form the bootstrap time series. Then, k blocks are chosen to make $\lambda = n/k$. With the moving-blocks bootstrap, the idea is to choose a block size λ large enough so that observations more than λ time units apart will be nearly independent. By sampling the blocks of length λ, they obtain the correlation present in observations less than λ units apart. By making block this way, they ensure that all adjacent blocks in bootstrapped series preserve the original correlation structure and independence among adjacent blocks. Moving-blocks bootstrap has the advantage of being less "model-dependent" than the bootstrapping of residuals approach. The choice of block length λ can be quite important. Our research of block length consideration in chapter 4 can be one of the solution for the selection of λ.

CHAPTER 3

BOOTSTRAP SIMULATION

3.1 Parametric bootstrap on i.i.d.

In this section, we investigate the heuristic approach of determining the necessity of bootstrap method. Basically, we run an exhaustive and thorough bootstrap on both a simulated parametric data set and a real nonparametric data set to see whether the bootstrap results in various scenarios are better than the sample themselves. For example, in parametric bootstrapping, we draw samples without replacement. We examine a few different sample sizes to compare parameter estimation with bootstrapping and without bootstrapping. We investigate simple parameter estimations, such as mean and standard deviation among various data sets. Then we compare the result with parameter estimation of bootstrapping. We generate i.i.d. parametric samples where we know what mean and standard deviation should be. Next, we use real-life nonparametric time series to investigate the effect of bootstrap. Though we estimate same estimators, two results of parametric and nonparametric bootstrappings are not related directly as we don't know true parameters in nonparametric bootstrapping. The parametric setting is as follows.

1. Three sets of 100,000 normal data each are generated: $X \sim N(0,1)$, $Y \sim N(1,1)$ and $Z \sim N(2,1)$. These data are sufficient to represent mean and standard deviation to population level accuracy.

2. From each data set, 10, 20, 50, 100 and 200 random samples are selected without replacement. There are a total of 15 of them with X, Y and Z. These sample data represent random samples for the normal population, X, Y and Z.

3. Then, we run bootstrap replications for each sample to estimate mean, standard error of $X,\ Y$ and Z, and the bias and root mean square error of an estimator $\hat{\theta}$ for θ, where $\theta = (E(X) - E(Y))/(E(Y) - E(Z))$, which is ratio estimator. For example, bioequivalence in biostatistics is one of ratio estimator. Historically, statisticians have worried a lot about the possible biases in ratio estimator. Therefore, we are investigating the bias and the root mean square error of a ratio estimator, Efron and Tibshirani (1994, p 126). Efron and Tibshirani (1994, p 47) suggested obtaining 25 – 200 bootstrap replications to estimate the statistical quantities above. We choose 25, 50, 100 and 200 for the number of bootstrap replications.

Note that the only difference between the parametric and nonparametric bootstrap replication generation process is that the parametric bootstrap pulls its sample from a parametric estimate of the population without replacement while the nonparametric bootstrap pulls empirical unknown distribution with replacement.
We have a true distribution of the normal case for $X,\ Y$ and Z and random samples from the 100,000 population, then run bootstrap replications to compare whether they are good representations of the true population or in fact just random samples themselves that are good enough to explain the population statistics. If there is not much difference between the sample and the bootstrap replication, bootstrap replication is not necessary as it does not provide any additional information about the true population.
In Table 3.1 and Table 3.2, we show samples from the population to demonstrate how the sampled selection is close to the population. Obviously, when the sample size is as small as 10, it does not look anything like normal and bootstrap replication done with estimates from this small sample would distort the true parameters of the population. However, as the sample size approaches 200, it looks more normal. In fact, when the sample size is 200, bootstrapping is quite unnecessary as the sample itself represents the population well enough. In a normal case, without prior assumption, if there is a reasonable conviction that the data are close to normal and the number of data points is more than 200, it can be seen from our simulation experiment that bootstrapping does not do any better than the sample itself. Therefore, in this case, bootstrapping is unnecessary. Another interesting observation is that when the sample size is small, for

example 10, its basic statistics like mean and standard error do not improve much as the number of bootstrap replication increases. However, as the sample size increases to over 100, we notice that as the number of bootstrap replication increases, the standard error decreases. It may be the case that when there is a large enough sample, increasing the number of replications makes sense in terms of decreasing the standard error. However, when the sample size is small, even as the number of replications increases, the standard error only increases or at least is not decreased. Here, we demonstrate that parametric bootstrap, small sample size bootstrap does not generate intended parametric distribution. The root mean square error, which takes care of both bias and standard error, it can be shown in Efron and Tibshirani (1994, p128) that the root mean square equals

$$\begin{aligned}\sqrt{E_F[(\hat{\theta}-\theta)^2]} &= \sqrt{E_F[(se_F(\hat{\theta})^2+\text{bias}_F(\hat{\theta},\theta)^2]}\\ &= se_F(\hat{\theta}).\sqrt{1+\left(\frac{\text{bias}_F}{se_F}\right)} \doteq se_F(\hat{\theta}).\left[1+0.5\left(\frac{\text{bias}_F}{se_F}\right)^2\right]\end{aligned} \tag{3.1}$$

Typically, the bias can be ignored if the root mean square error is no more than about 3.1% greater than the standard error. In our case, the biases are all small and can be ignored. However, in a large sample case with a high number of bootstrap replications, standard error tends to get quite big. This vindicates our previous allegation that when the sample size is 200, bootstrapping does not do any good in terms of parameter estimations. It is better to just use the sample of 200 itself. Table 3.3 shows the summary of the bias, standard error and RMSE of the ratio statistics.

TABLE 3.1: 100,000 normal data, $X \sim N(0,1),\ Y \sim N(1,1),\ Z \sim N(2,1)$

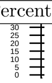
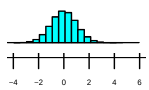
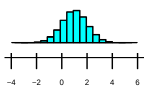
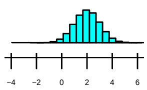
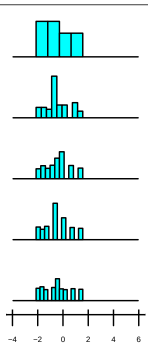
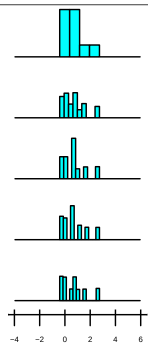
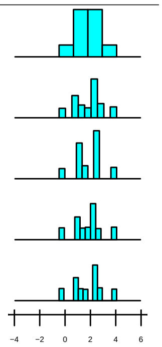
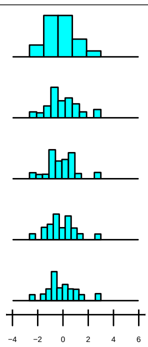
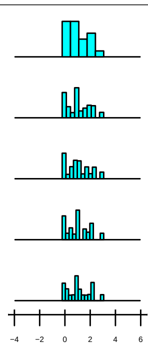
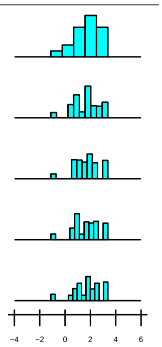
Normal		**x**		**y**		**z**	
	Percent	m/sd	hist	m/sd	hist	m/sd	hist
Population Level		0.001 1.000		1.005 1.004		1.999 1.000	

TABLE 3.2: Bootstrap replications of the normal samples

Sample Size 10	x		y		z	
	m/sd	hist	m/sd	hist	m/sd	hist
sample	-0.448 1.048		0.689 0.911		1.808 1.164	
bootstrap rep 25	-0.447 0.299		0.699 0.237		1.871 0.365	
bootstrap rep 50	-0.418 0.329		0.708 0.273		1.833 0.376	
bootstrap rep 100	-0.468 0.323		0.721 0.291		1.826 0.330	
bootstrap rep 200	-0.481 0.326		0.687 0.266		1.809 0.315	

Sample Size 20	x		y		z	
	m/sd	hist	m/sd	hist	m/sd	hist
sample	-0.174 1.164		1.078 0.865		1.688 1.019	
bootstrap rep 25	-0.108 0.224		1.066 0.174		1.679 0.222	
bootstrap rep 50	-0.122 0.250		1.068 0.172		1.692 0.212	
bootstrap rep 100	-0.180 0.264		1.078 0.164		1.700 0.231	
bootstrap rep 200	-0.160 0.283		1.086 0.187		1.692 0.228	

Sample Size 50	x		y		z	
	m/sd	hist	m/sd	hist	m/sd	hist
sample	-0.106 0.870		1.174 0.996		1.934 0.914	
bootstrap rep 25	-0.101 0.130		1.205 0.140		1.978 0.153	
bootstrap rep 50	-0.110 0.125		1.178 0.156		1.965 0.134	
bootstrap rep 100	-0.118 0.121		1.135 0.138		1.925 0.131	
bootstrap rep 200	-0.090 0.130		1.186 0.147		1.941 0.127	

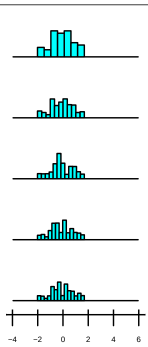
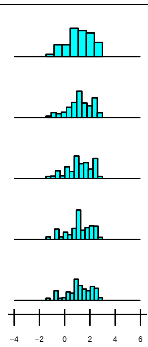
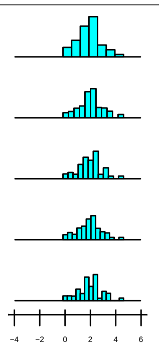
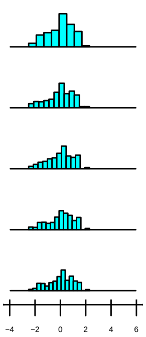
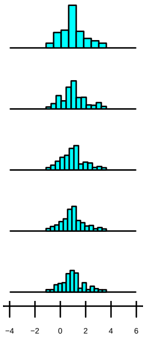
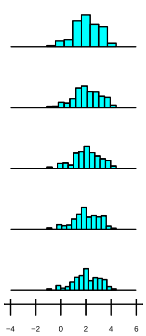




TABLE 3.2, continued: Bootstrap replications of the normal samples

Sample Size 100		x		y		z	
	Percent	m/sd	hist	m/sd	hist	m/sd	hist
sample		0.015 0.990		1.023 0.957		2.043 1.048	
bootstrap rep 25		-0.018 0.102		1.039 0.092		2.035 0.100	
bootstrap rep 50		-0.005 0.095		1.029 0.084		2.045 0.116	
bootstrap rep 100		0.015 0.105		1.011 0.096		2.053 0.105	
bootstrap rep 200		0.015 0.092		1.012 0.097		2.044 0.097	

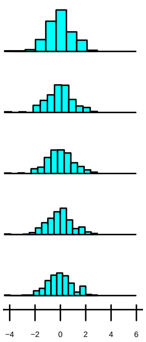
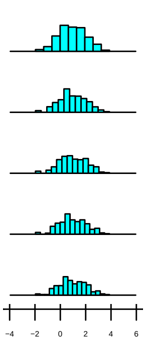
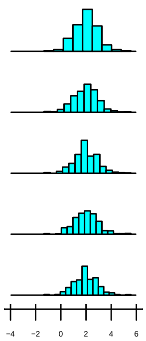
Sample Size 200		x		y		z	
	Percent	m/sd	hist	m/sd	hist	m/sd	hist
sample		0.050 1.042		0.987 1.052		1.997 0.949	
bootstrap rep 25		-0.056 0.084		0.979 0.055		1.981 0.060	
bootstrap rep 50		-0.035 0.076		0.993 0.074		1.985 0.068	
bootstrap rep 100		-0.050 0.064		0.995 0.076		2.001 0.064	
bootstrap rep 200		-0.052 0.073		0.983 0.069		1.994 0.098	

TABLE 3.3: Bias, Standard Error and Root Mean Square Error of the ratio estimation

	$(x-y)/(y-z)$ **ratio**	**# of bootstrap replication**			
sample size		25	50	100	200
10	Bias	-0.1180	-0.1106	-0.0726	0.0127
	S.E.	1.3075	1.3361	1.6665	1.6191
	R.M.S.E.	1.3128	1.3407	1.6681	1.6191
20	Bias	0.0998	0.0464	-0.0063	0.0022
	S.E.	12.3612	9.9055	9.0133	7.3208
	R.M.S.E.	12.3616	9.9056	9.0133	7.3208
50	Bias	-0.2513	-0.2152	-0.1841	-0.2000
	S.E.	8.0181	7.5800	8.6119	20.1186
	R.M.S.E.	8.0221	7.5831	8.6139	20.1196
100	Bias	0.0168	0.0027	-0.0051	0.0032
	S.E.	7.7727	7.2148	8.2253	7.9894
	R.M.S.E.	7.7727	7.2148	8.2253	7.9894
200	Bias	0.0294	0.0351	0.0387	0.0372
	S.E.	3.7998	13.6873	12.2749	63.2823
	R.M.S.E.	3.7999	13.6874	12.2750	63.2824

3.2 Nonparametric bootstrap on time series

In nonparametric bootstrap case, sample size does not matter as we don't know true distribution. This result is summarized in Table 3.4 and Table 3.5. The nonparametric setup follows.

1. Four time series of roughly 10 years of daily return on the US stock index (USS), US bond index(USB), Korean stock index(KRS), and Korean bond index(KRB) are collected. Each return comprises of 3,352 data points. (Source: Bloomberg)

2. Assuming 3,352 data points are the population, 10, 20, 50, 100 and 200 samples are drawn from each series.

3. In each sample, the bootstrap replication of 25, 50, 100 and 200 are generated to calculate mean, standard error of USS, USB, KRS and KRB.

A real-life example also shows that there might be some advantage to running bootstrap replication in small sample sizes like 10, 20, or 50. However, as the sample size gets larger, bootstrap replication might not necessarily show better statistical result than the sample itself. (see Tables 3.4, and 3.5) Bootstrapping in certain circumstances can perform well and provide meaningful and useful information in terms of inference to population statistics. However, simulation results from our normal parametric experiment and real-life example from the financial data set, in the larger sample case (for normal case, 0.2% of population size (100,000 vs 200)) show that the sample itself sometimes provides just as good statistical inference as extensive bootstrap replication. Though bootstrap provides very useful information when it is properly used, researchers still need to pay closer attention when and how to use bootstrap.

TABLE 3.4: Roughly 10 years of US and Korean stock and bond indeces daily return

stock,bond return 13 years daily	Percent	**stock** m/sd	**stock** hist	**bond** m/sd	**bond** hist
US		0.016 1.131		0.017 0.378	
Korean		0.013 1.126		0.018 0.236	

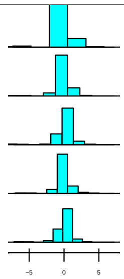
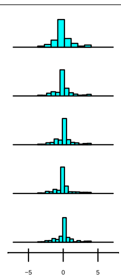
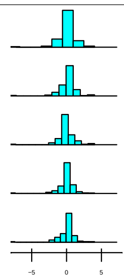
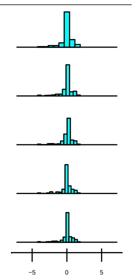
TABLE 3.5: Bootstrap replications of samples drawn from daily returns

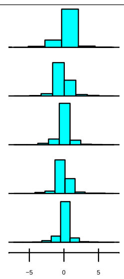
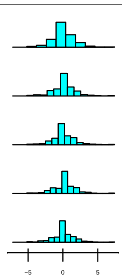
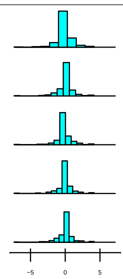
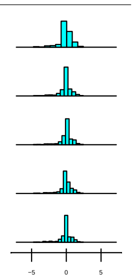
Sample size 10	Percent	**US stock** m/sd	hist	**US bond** m/sd	hist	**KR stock** m/sd	hist	**KR bond** m/sd	hist
sample		0.359 0.765		0.073 0.269		0.079 1.039		0.065 0.117	
bootstrap rep 25		0.279 0.232		0.061 0.058		0.040 0.341		0.056 0.028	
bootstrap rep 50		0.325 0.165		0.063 0.089		0.114 0.311		0.059 0.040	
bootstrap rep 100		0.366 0.248		0.083 0.071		0.037 0.324		0.062 0.031	
bootstrap rep 200		0.340 0.233		0.073 0.077		0.114 0.305		0.058 0.034	

Sample size 20	Percent	**US stock** m/sd	hist	**US bond** m/sd	hist	**KR stock** m/sd	hist	**KR bond** m/sd	hist
sample		0.181 1.635		-0.032 0.343		0.369 0.925		0.110 0.390	
bootstrap rep 25		0.094 0.352		-0.034 0.063		0.362 0.145		0.119 0.080	
bootstrap rep 50		0.151 0.334		-0.039 0.071		0.403 0.227		0.127 0.095	
bootstrap rep 100		0.183 0.337		-0.010 0.069		0.378 0.230		0.116 0.091	
bootstrap rep 200		0.232 0.376		-0.038 0.068		0.377 0.189		0.112 0.084	

Sample size 50	Percent	**US stock** m/sd	hist	**US bond** m/sd	hist	**KR stock** m/sd	hist	**KR bond** m/sd	hist
sample		0.023 0.861		0.082 0.388		0.082 1.451		0.020 0.142	
bootstrap rep 25		-0.004 0.095		0.086 0.055		0.145 0.238		0.023 0.031	
bootstrap rep 50		0.002 0.121		0.080 0.048		0.098 0.218		0.016 0.021	
bootstrap rep 100		0.016 0.110		0.080 0.056		0.067 0.197		0.017 0.019	
bootstrap rep 200		0.011 0.116		0.079 0.055		0.088 0.180		0.020 0.021	

TABLE 3.5, continued: Bootstrap replications of samples drawn from daily returns

Sample size 100	Percent	US stock m/sd	US stock hist	US bond m/sd	US bond hist	KR stock m/sd	KR stock hist	KR bond m/sd	KR bond hist
sample		-0.026 1.834		0.012 0.308		-0.023 1.196		-0.002 0.206	
bootstrap rep 25		-0.048 0.195		0.012 0.027		0.004 0.126		-0.014 0.020	
bootstrap rep 50		-0.040 0.175		0.010 0.026		-0.009 0.111		-0.003 0.019	
bootstrap rep 100		-0.028 0.178		0.015 0.034		-0.023 0.117		-0.005 0.022	
bootstrap rep 200		-0.034 0.182		0.014 0.031		-0.021 0.109		-0.002 0.021	

Sample size 200	Percent	US stock m/sd	US stock hist	US bond m/sd	US bond hist	KR stock m/sd	KR stock hist	KR bond m/sd	KR bond hist
sample		0.030 1.376		0.032 0.390		-0.035 1.092		-0.017 0.231	
bootstrap rep 25		0.046 0.138		0.027 0.032		-0.021 0.078		-0.017 0.018	
bootstrap rep 50		0.025 0.076		0.036 0.028		-0.048 0.077		-0.014 0.014	
bootstrap rep 100		0.055 0.104		0.036 0.027		-0.029 0.074		-0.018 0.017	
bootstrap rep 200		0.041 0.101		0.029 0.025		-0.043 0.076		-0.017 0.017	

CHAPTER 4

APPLICATION OF BLOCK BOOTSTRAP TO TIME SERIES DATA

4.1 Block length consideration

Before we consider the block length, we summarize the justfication of block bootstrap. It is true that independent sampling of blocks does not correspond to the dependence structure of time series. However, Carlstein (1986) showed that strictly stationary process, block bootstrap provides consistent variance estimator under certain conditions. Also, Hall (1995) studied convergence of block bootstrap estimators of certain moments, one-sided and sysmetrical distribution functions. Despite the drawback of inaccurate structural integrity of time series, time series block bootstrap gained popularity due to its intuitive implementation and quite accurate parameter estimation. Moving block bootstrap method mitigates the issue of the dependence structure of time series.

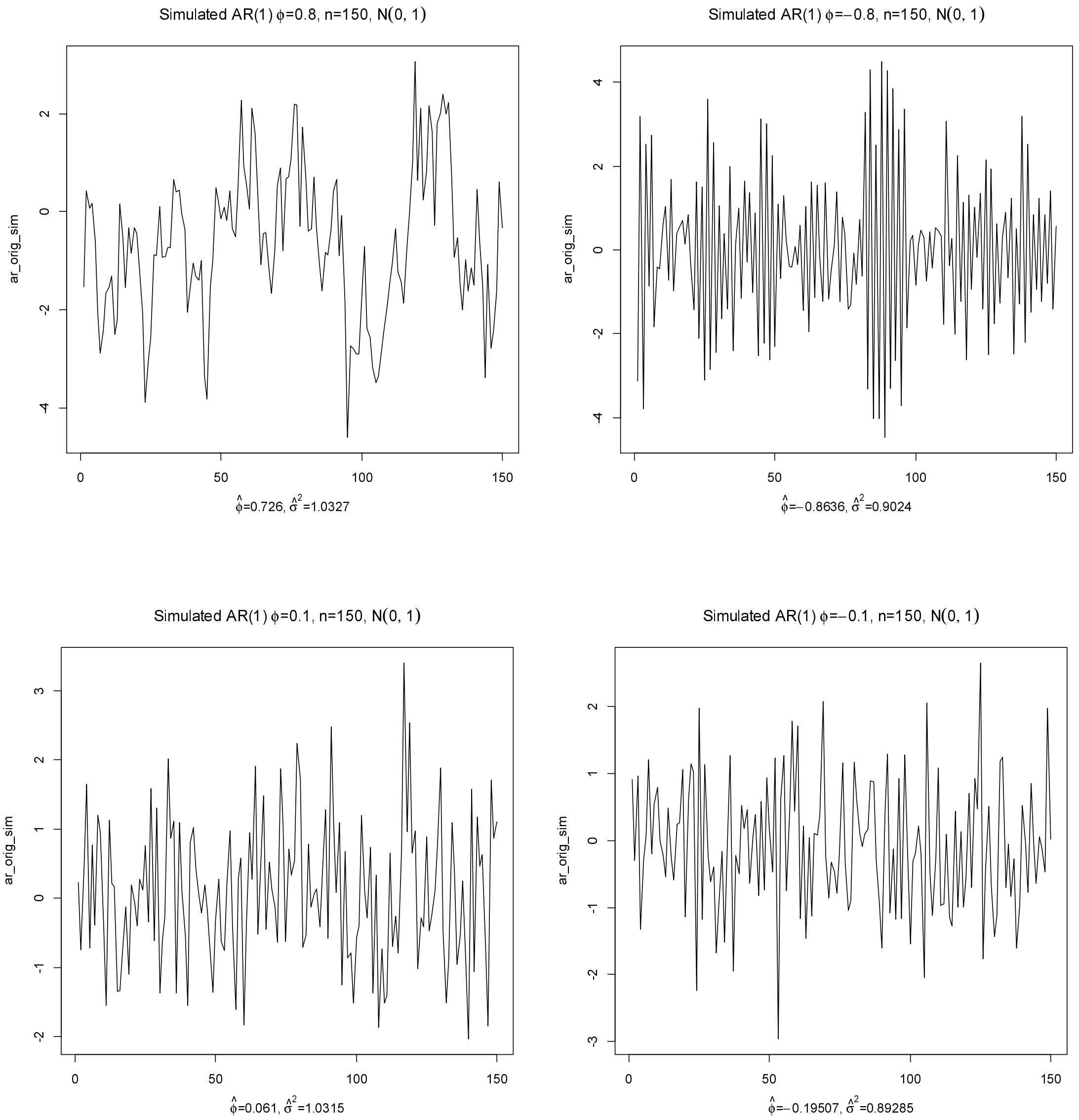
The choice of the optimal length of data has been the subject of many analyses yet there is no general rule for how to choose the proper length of data. Our primary interest in this report is in selecting the block length for an AR(1) analysis of a time series. Hall (1995) showed that $n^{1/3}$ is the optimal block size for a block bootstrap. Box and Tiao (1975) claimed that a minimum of 50 and preferably more than 100 data length is needed to conduct proper time series analysis with an ARIMA model. Carlstein (1986) derived the equations $l_n^* = (2\phi/c)^{2/3}\, n^{1/3}$, $c = (1-\phi)\,(1+\phi)$, where l is the length of the series, to minimize the bias in the AR(1) block bootstrap. However, if $n = 100$, then Hall (1995)'s suggestion makes the block length roughly 5, and Carlstein's formula yields block lengths roughly from 3 ($\phi = \pm 0.2$) to 20 ($\phi = \pm 0.9$). Both results are far short of the minimum 50 that Box and Tiao suggested. This is because Hall and Carlstein considered only the bias and variance aspects, and did not consider the time series correlation structure aspect. Yaffee and McGee (2000) conducted extensive sample size tests on the cross-correlation functions of the intervention model using transfer functions based on Box and Tiao (1975) and concluded that a minimum of 30, (preferably 50) data points are necessary for AR(1) input series with AR(1) and MA(1) transfer functions. We will investigate the various lengths in this study. Based on our findings, we proposed a minimum of 30 data points on each block in block bootstrap, especially for the effect of temporal aggregation and systematic sampling. 50 suggested by Box and Tiao (1975) is necessary for transfer functions. 30 data points are suitable for single time series analysis.

4.2 Simulation study on block length or the number of blocks

We designed extensive and comprehensive block bootstrap procedures to test the performance of various block lengths through an AR(1) setting. Our hypothesis is that Carlstein's AR(1) optimal length is not necessarily the best choice of length for the analysis. Throughout all the following sections, the best fitted parameter estimator is OLS, which is very close to MLE, and bootstrap estimators are simple average and standard deviations of replications. To be able to test the different blocks and different replications, we generated four AR(1) time series innovations as shown in Figure 4.1 $Y_t = \pm 0.8 Y_{t-1} + a_t,\ a_t \sim N(0,1),\ n = 150$, $Y_t = \pm 0.1 Y_{t-1} + a_t,\ a_t \sim N(0,1),\ n = 150$.

Using these innovations, we ran 3, 5, and 10 blocks for $\phi = \pm 0.8$ (50, 30, and 15 block length); 3, 5, and 75 blocks for $\phi = \pm 0.1$; and 10,50 and 100 replications for each block number, for a total of 9 runs for one coefficient. We chose 3 blocks since this is the minimum time series length that Box and Tiao (1975) suggested for any ARIMA model and 5 blocks because it is the minimum block length of 30 we want to investigate. Based on Carlstein's formula to minimize bias, 10 and 75 are the optimal block lengths. We chose 10 to test below recommendation and 50, 100, which are inside Efron's recommendation. Table 4.1 results shows the very interesting outcomes. For $\phi = 0.8$ and $\phi = 0.1$, the number of block 3 with only 10 replications did well in terms of being

FIGURE 4.1: Simulated AR(1) for the investigation of block length and the number of replications



close to the true parameter, while Carlstein's optimal block length did the worst of the three. Therefore, this case proved that in fact Box and Tiao's suggestion did matter and Carlstein's formula failed to produce a proper estimation. However, for $\phi = -0.8$ and $\phi = -0.1$, the results were quite different. Carlstein's optimal block length did the best in estimating the true parameters. Block lengths of 30 and 50 performed worse than Carlstein's length of 15 ($\phi = -0.8$) and 2 ($\phi = -0.1$). In the positive coefficient case, we showed that Carlstein's block length is not necessarily optimal in analyzing an AR(1) block bootstrap. Our hypothesis is upheld.

TABLE 4.1: Simulated AR(1) with various block length and the number of replications

(a) True model : $Y_i = \pm 0.8 Y_{i-1} + a_i$, $a_i \sim N(0,1)$, $n = 150$

Block, rep	3,10		3,50		3,100		5,10		5,50	
ϕ	−0.8	0.8	−0.8	0.8	−0.8	0.8	−0.8	0.8	−0.8	0.8
$\hat{\phi}$	−0.856	0.733	−0.843	0.707	−0.845	0.717	−0.835	0.688	−0.832	0.711
$S_{\hat{\phi}}$	0.017	0.022	0.025	0.034	0.025	0.033	0.022	0.028	0.018	0.035
$\hat{\sigma_a}^2$	0.993	1.076	0.971	1.003	0.973	1.038	1.082	0.977	1.069	1.047
$S_{\hat{\sigma_a}^2}$	0.118	0.085	0.089	0.089	0.121	0.092	0.113	0.068	0.130	0.109
95% Cl $\hat{\phi}$	0.045	0.064	0.089	0.144	0.083	0.146	0.061	0.079	0.063	0.133

Block, rep	5,100		10,10		10,50		10,100	
ϕ	−0.8	0.8	−0.8	0.8	−0.8	0.8	−0.8	0.8
$\hat{\phi}$	−0.835	0.718	−0.829	0.642	−0.803	0.642	−0.811	0.64
$S_{\hat{\phi}}$	0.018	0.032	0.026	0.044	0.030	0.045	0.033	0.045
$\hat{\sigma_a}^2$	1.082	1.067	1.178	1.246	1.192	1.223	1.158	1.208
$S_{\hat{\sigma_a}^2}$	0.133	0.096	0.163	0.148	0.189	0.142	0.159	0.153
95% Cl $\hat{\phi}$	0.065	0.111	0.075	0.139	0.112	0.174	0.127	0.176

(b) True model : $Y_i = \pm 0.1 Y_{i-1} + a_i$, $a_i \sim N(0,1)$, $n = 150$

Block, rep	3,10		3,50		3,100		5,10		5,50	
ϕ	−0.1	0.1	−0.1	0.1	−0.1	0.1	−0.1	0.1	−0.1	0.1
$\hat{\phi}$	−0.184	0.029	−0.207	0.049	−0.197	0.056	−0.183	0.030	−0.180	0.040
$S_{\hat{\phi}}$	0.039	0.046	0.044	0.033	0.046	0.033	0.099	0.065	0.091	0.076
$\hat{\sigma_a}^2$	0.913	0.962	0.882	1.011	0.895	1.032	0.892	0.995	0.880	1.013
$S_{\hat{\sigma_a}^2}$	0.060	0.132	0.067	0.102	0.068	0.105	0.056	0.102	0.070	0.124
95% Cl $\hat{\phi}$	0.106	0.135	0.089	0.128	0.180	0.108	0.280	0.190	0.321	0.304

TABLE 4.1, continued: Simulated AR(1) with various block length and the number of replications

(b) True model : $Y_i = \pm 0.1 Y_{i-1} + a_i$, $a_i \sim N(0,1)$, $n = 150$

Block, rep	5,100		75,10		75,50		75,100	
ϕ	−0.1	0.1	−0.1	0.1	−0.1	0.1	−0.1	0.1
$\hat{\phi}$	−0.189	0.042	−0.117	−0.039	−0.125	−0.008	−0.128	−0.001
$S_{\hat{\phi}}$	0.081	0.084	0.063	0.092	0.079	0.092	0.093	0.089
$\hat{\sigma_a}^2$	0.891	1.022	0.865	1.034	0.920	1.004	0.910	1.021
$S_{\hat{\sigma_a}^2}$	0.064	0.133	0.115	0.056	0.112	0.106	0.115	0.123
95% Cl $\hat{\phi}$	0.302	0.293	0.191	0.270	0.267	0.296	0.353	0.350

4.3 Combinatorially-Augmented Block Bootstrap-Proposed Method

In this section, we propose a new method of creating pseudo bootstrap data using the original data. Motivation of the new method is coming from the fact that we create additional bootstrap data set from existing data. By averaging available blocks according to specified rule, additional data sets are related to original data. The main idea comes from the block bootstrap method. In section 2.7, we described the block bootstrapping method. Equation (2.11) summarizes the final data product of block bootstrapping. We rearrange the block bootstrap data accordingly. From the observed series $H = (X_1, \ldots, X_n)$, and b, l denote integers such that $n = bl$, additional $2^b - 1 - b$ blocks of series are generated according to the specified rule. Therefore, the total number of blocks is $2^b - 1$. From the H, which is divided into b disjoint blocks, let $B_b = \left(X_{(b-1)l+1}, \ldots, X_{bl}\right)$. Additional $2^b - 1 - b$ blocks of series are created by taking the average of all possible combination of $B_1, \ldots, B_b$, being (B_1B_2),…,$(B_{b-1}B_b)$, $(B_1B_2B_3)$,…,$(B_{b-2}B_{b-1}B_b)$,…,$(B_1 \ldots B_b)$. For example, (B_1B_2) is generated by taking the average of each element, where

$$\begin{aligned} (B_1B_2) &= \left(\frac{X_1 + X_{l+1}}{2}, \ldots, \frac{X_l + X_{2l}}{2}\right) \\ (B_1B_2B_3) &= \left(\frac{X_1 + X_{l+1} + X_{2l+1}}{3}, \ldots, \frac{X_l + X_{2l} + X_{3l}}{3}\right) \text{ etc.} \end{aligned} \tag{4.1}$$

Now, we obtain H^{mod}, which includes additional blocks that amount to $2^b - 1$. The total number of non-overlapping blocks we generate from this process is the sum of $\binom{b}{1} + \binom{b}{2} + \ldots + \binom{b}{b} = 2^b - 1$. Then we choose blocks $B_1^*, \ldots, B_b^*$ by resampling randomly with replacement, from among $B_1, \ldots, B_b, (B_1 B_2), \ldots, (B_1 \ldots B_b)$. If $B_i^* = (X_{i1}^*, \ldots, X_{il}^*)$, the bootstrap version of $H^{\mathrm{mod}}$is

$$H^{\mathrm{mod}} = (X_1^*, \ldots, X_n^*) = (X_{11}^*, \ldots, X_{1l}^*, X_{21}^*, \ldots, X_{2l}^*, \ldots, X_{b1}^*, \ldots, X_{bl}^*) \qquad (4.2)$$

Basically, we increase the number of available bootstrappable data by taking the average of all possible combinations of the blocks. This method is applicable not only to time series, but also to any data set that includes an independent sample. One simply needs to separate the data into blocks according to any specified rule to pseudo generate additional available blocks. We call this method augmented block bootstrap.

4.4 Comparison of two methods in independent sample and time series

4.4.1 Simulation study of independent sample

We applied independent sample $n = 100$, $N\left(\mu, \sigma^2\right)$ where $\mu = 0$ and $\sigma^2 = 16$ to the regular block bootstrap and augmented block bootstrap methods. The bootstrap procedure is as follows:

1. Independent sample with $n = 100$, $N\,(0, 16)$ is simulated.

2. A total of 200 replications are performed with each of the regular block bootstrap and the proposed augmented block bootstrap methods. We simply sequentially divide the 100 random sample into 2, 5 and 10 blocks.

3. Then, the means of $\hat{\mu}$ and $S_{\hat{\mu}}$ and the length of 95% CI for $\hat{\mu}$ of 200 replications are recorded.

The bootstrap results shown in Table 4.2 demonstrated that our Combinatorially-Augmented Block Bootstrap has an advantage in reducing the mean of $S_{\hat{\mu}}$ estimation. For 2, 5, and 10 blocks bootstrap replications, the means of $S_{\hat{\mu}}$ from our method were consistently lower than those of the regular method. The estimations of the means of $\hat{\mu}$ had mixed results but were not significantly different among the two methods. The length of 95% CI for the mean was much shorter in our method, especially in the 5 and 10 blocks. We note that there is no general rule about the choice of the number of blocks for independent sample.

TABLE 4.2: Comparison of two block bootstrap methods on independent sample

True value, *i.i.d.*, $N(\mu, \sigma^2)$ $N(0,16)$, 200 replications		Regular block	Augmented block
Block=2	$\hat{\mu}$	0.4208	0.3452
	$S_{\hat{\mu}}$	0.4151	0.3074
	Length of 95% CI for mean	1.1637	1.1637
Block=5	$\hat{\mu}$	0.3942	0.4195
	$S_{\hat{\mu}}$	0.3986	0.2218
	Length of 95% CI for mean	1.4145	0.8634
Block=10	$\hat{\mu}$	0.3669	0.3827
	$S_{\hat{\mu}}$	0.3118	0.1281
	Length of 95% CI for mean	1.2169	0.5079

4.4.2 Simulation study of time series

We designed an extensive and comprehensive block bootstrap procedure to test the performance of our proposed augmented block bootstrap against regular block bootstrap in an AR(1) setting, $Y_t = \phi Y_{t-1} + a_t$, $a_t \sim N(0, 16)$. We note that both methods use nonoverlapping blocks. To test many different cases of time series innovation we simulated AR(1) coefficients, -0.9, -0.8, -0.6, 0.6, 0.8 and 0.9. A procedure follows.

1. 120 points AR(1) with $\phi = -0.9$ time series innovation is simulated.

2. A total of 200 replications are done using both a regular block bootstrap and the proposed augmented block bootstrap methods with a block length of 30 (total 4 blocks).

3. Then, the means of $\hat{\phi}, S_{\hat{\phi}}, \hat{\sigma}_a^2, S_{\hat{\sigma}_a^2}$ and CI of $\hat{\phi}$ for the 200 replications are recorded.

4. 1$\sim$3 are repeated for ϕ =−0.9,−0.8,−0.6,0.6,0.8 and ϕ =0.9.

Results in Table 4.3 showed that in terms of estimation of $\hat{\phi}, S_{\hat{\phi}}, S_{\hat{\sigma}_a^2}$, and the length of 95% CI for the mean, both methods did almost equally well. However, by creating an average of all the available blocks using the rule specified in 4.3, we can clearly see that our Combinatorially-Augmented Block Bootstrap reduced $\hat{\sigma}_a^2$ significantly. Though it might not be desirable to estimate a smaller $\hat{\sigma}_a^2$ than originally planned, our method provides an efficient way of block bootstrapping smaller $\hat{\sigma}_a^2$ without sacrificing parameter accuracy.

TABLE 4.3: Comparison of two block bootstrap methods on various AR(1) coefficient $\sigma_a^2 = 16$

Simulated AR(1) Coefficient	True value −0.9		True value −0.8		True value −0.6	
(200 replications)	Regular	Aug'ed	Regular	Aug'ed	Regular	Aug'ed
$\hat{\phi}$	−0.783	−0.776	−0.789	−0.782	−0.643	−0.654
$S_{\hat{\phi}}$	0.051	0.058	0.048	0.053	0.015	0.044
$\hat{\sigma}_a^2$	29.676	16.040	15.056	8.670	15.720	8.591
$S_{\hat{\sigma}_a^2}$	5.035	5.135	1.992	2.059	2.508	2.769
Length of 95% CI for mean	0.207	0.215	0.190	0.187	0.058	0.176

Simulated AR(1) Coefficient	True value 0.6		True value 0.8		True value 0.9	
(200 replications)	Regular	Aug'ed	Regular	Aug'ed	Regular	Aug'ed
$\hat{\phi}$	0.576	0.640	0.794	0.784	0.828	0.818
$S_{\hat{\phi}}$	0.091	0.077	0.054	0.052	0.072	0.067
$\hat{\sigma}_a^2$	16.447	9.152	22.158	11.424	17.955	10.061
$S_{\hat{\sigma}_a^2}$	2.031	2.559	4.941	4.268	2.874	2.876
Length of 95% CI for mean	0.319	0.294	0.200	0.204	0.248	0.238

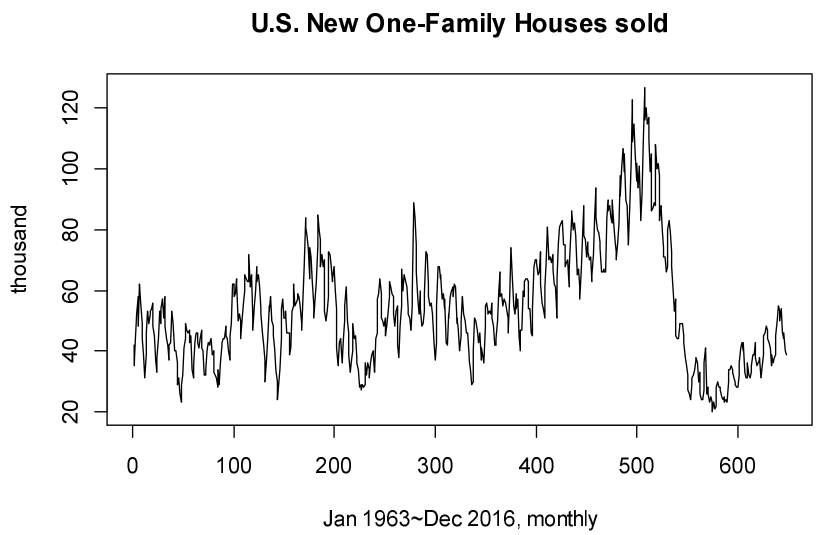
4.4.3 Empirical study: U.S. New One-Family Houses sold - Time Series

We investigated the comparison of two block bootstrapping methods on an empirical time series taken from Federal Reserve Bank of St. Louis website: U.S. new one-family houses sold monthly. The time span for gathering data was Jan 1963 to Dec 2016, and a total of 648 data point. Shown in Figure 4.2, the best fitted model in AR(1) is $Y_t = 0.946Y_{t-1} + a_t$, $a_t \sim N(0, 39.40)$. We ran 200 bootstrap replications on both methods and recorded the mean of $\hat{\phi}, S_{\hat{\phi}}, \hat{\sigma}_a^2$ and $S_{\hat{\sigma}_a^2}$. The number of blocks was chosen according to Carlstein's formula, which is 10 in this case. As this time series had more than 600 data points, 10 blocks met our minimum suggested data length. Table 4.4 summarizes the results of the empirical study of the two methods. A simple estimation of time series yields the AR(1) coefficient of 0.946. The regular block bootstrap method yields 0.85 and our Combinatorially-Augmented Block Bootstrap yields 0.81. Notable performances of our Combinatorially-Augmented Block Bootstrap are smaller $\hat{\sigma}_a^2$ and $S_{\hat{\sigma}_a^2}$. It is due the fact that averaged additional data set creates smaller $\hat{\sigma}_a^2$ and $S_{\hat{\sigma}_a^2}$ by applying close to mean data sets. As in the i.i.d. case and simulated AR(1) time series, our Combinatorially-Augmented Block Bootstrap tends to produce smaller $\hat{\sigma}_a^2$ and $S_{\hat{\sigma}_a^2}$.

TABLE 4.4: Comparison of two block bootstrap methods on U.S. New One-Family House sold using AR(1)

AR(1) Coefficient estimation	Original Series 0.95, $\hat{\sigma}_a^2 = 39.40$	
(based on 200 replication)	Regular	Aug'ed
$\hat{\phi}$	0.8538	0.8080
$S_{\hat{\phi}}$	0.0227	0.0216
$\hat{\sigma}_a^2$	37.073	9.147
$S_{\hat{\sigma}_a^2}$	5.269	1.387

FIGURE 4.2: U.S. New One-Family Houses sold



CHAPTER 5

AGGREGATION EFFECT ON BLOCK BOOTSTRAP

5.1 Introduction

Aggregation issues in time series have been well researched and documented. For example, the analytics of monthly collected data series do not necessarily translate into the same analytics when three months of data are combined into quarterly data and then analyzed as quarterly series. Therefore, one must pay close attention when analyzing aggregated series. In general, aggregation can be separated into two areas of interest. One is a "stock" variable such as stock price, in which the series is observed every m unit by systematic sampling. The other variable is a "flow" variable that occurs by temporal aggregation over the unit of time. Early research on time series aggregation includes Amemiya and Wu (1972) and Brewer (1973). Later Wei (1981), and Stram and Wei (1986) further developed theories and applications in various subjects of time series regarding aggregation and systematic sampling. There has been no research on the time series aggregation effect on block bootstrap method. We will investigate this effect in this paper.

It is reported in various literature including Stram and Wei (1986) that, in general, the temporal aggregation effect of a pure form of an AR(p) model becomes some mixture of an AR and an MA. Wei (2006) shows that the ARMA(p, q) becomes an ARMA(p, r) where $r = (p + 1) + (q - p - 1)/m$. For example, if $m = 2$, the AR(1) becomes an ARMA(1,1). It is worthwhile to note that this general rule does not apply when the distinctive roots of an AR(p) have multiplicity (Stram and Wei (1986)). The

AR(2) with $m = 3$, should yield aggregated series an ARMA(2,2) according to the formula above. However, if there is a multiplicity in two distinctive roots, then it becomes an ARMA(1,1). They show that the AR order of a pure AR(p) series can change under temporal aggregation. In our bootstrap simulation, we chose to apply an AR(1). Therefore, we do not have to be concerned with this multiplicity issue.

Before we investigate the simulation study of the two block bootstrap methods, we summarize a brief theoretical treatment of temporal aggregation and systematic sampling. We focus on the AR(1) case only. According to Stram and Wei (1986) and Amemiya and Wu (1972), for a time series $Y_t = (1 - \phi B)\, Y = a_t$, $a_t \sim N(0,1)$, let $\{Z_l\}$be the series consisting of sums of m non-overlapping points of Y, which is defined as

$$Z_l = \sum_{t=m(l-1)+1}^{ml} \left(1 + B + \ldots + B^{m-1}\right) Y_{ml} \tag{5.1}$$

Using backshift operator B on Z_l

$$BZ_l = Z_{l-1} = \sum_{t=m(l-2)+1}^{m(l-1)} \left(1 + B + \ldots + B^{m-1}\right) Y_{m(l-1)} = B^m \left(1 + B + \ldots + B^{m-1}\right) Y_{ml} \tag{5.2}$$

In short, the backshift operator B applied to Z_l is equivalent to applying B^m to Y_t.

Stram and Wei (1986) define the relation between autocovariance functions of aggregated series Z_l and original series as

$$\gamma_Z(k) = \left(1 + B + \ldots + B^{m-1}\right)^2 \gamma_Y[mk + (m-1)] \tag{5.3}$$

where B is backshift operator, and m is aggregation order. They interpret the autocovariance function of an aggregated series as the linear transformation of the autocovariance of the original series by expanding the polynomial $\left(1 + B + \ldots + B^{m-1}\right)^2$. Amemiya and Wu (1972), Brewer (1973), Stram and Wei (1986), and Wei (2006) only showed model order results for the aggregate on ARMA and ARIMA models. From their results, the aggregate of an AR(1) model becomes ARMA (1,1) mode. However, they did not derive the parameter expressions for the MA term and the white noise variance of the aggregate model. For our purposes, we derive them in Theorem 1.

Theorem 1. *Given the basic AR(1) model, $Y_t = \phi Y_{t-1} + a_t$, where a_t is a normal white noise $N\left(0, \sigma_a^2\right)$, the m^{th} order aggregate $Z_T = \left(1 + B + \ldots + B^{m-1}\right) Y_{mT}$, follows an ARMA(1,1) model, $Z_T = \phi^m Z_{T-1} + A_T - \theta A_{T-1}$ where A_T normal white noise $N\left(0, \sigma_A^2\right)$ such that*

$$\theta = \frac{\left(\phi^{2m} + 1 - 2\alpha\phi^m\right) \pm \sqrt{\left(2\alpha\phi^m - \phi^{2m} - 1\right)^2 - 4\left(\phi^m - \alpha\right)^2}}{2\left(\phi^m - \alpha\right)},$$

where we chose $|\theta| < 1$, and

$$\sigma_A^2 = \frac{\left(1 - \phi^{2m}\right)\left(1 + B + \cdots + B^{m-1}\right)\gamma_Y\left(m-1\right)}{1 + \theta^2 - 2\phi^m\theta}.$$

Proof. Given Y_t, we can compute its covariance functions $\gamma_Y(k)$, for $k = 0, 1, \ldots, (m-1)$ $, \ldots, (2m-1)$. From the ARMA(1,1) of Z_T and the general relation of 3.1, we can compute the variance and first-order autocovariance of Z_T as follows:

$$\gamma_Z(0) = \frac{1 + \theta^2 - 2\phi^m\theta}{1 - \left(\phi^m\right)^2}\sigma_A^2 = \left(1 + B + \ldots + B^{m-1}\right)^2 \gamma_Y\left(m-1\right)$$
$$\gamma_Z(1) = \frac{\left(\phi^m - \phi\right)\left(1 - \phi^m\theta\right)}{1 - \left(\phi^m\right)^2}\sigma_A^2 = \left(1 + B + \ldots + B^{m-1}\right)^2 \gamma_Y\left(2m-1\right)$$

Let

$$\frac{\gamma_Z(1)}{\gamma_Z(0)} = \frac{\left(\phi^m - \phi\right)\left(1 - \phi^m\theta\right)}{1 + \theta^2 - 2\phi^m\theta} = \frac{\left(1 + B + \ldots + B^{m-1}\right)^2 \gamma_Y\left(2m-1\right)}{\left(1 + B + \ldots + B^{m-1}\right)^2 \gamma_Y\left(m-1\right)} = \alpha.$$

We have quadratic equation

$$\left(\phi^m - \alpha\right)\theta^2 - \left(\phi^{2m} - 2\alpha\phi^m + 1\right)\theta + \left(\phi^m - \alpha\right) = 0$$

Hence,

$$\theta = \frac{\left(\phi^{2m} + 1 - 2\alpha\phi^m\right) \pm \sqrt{\left(2\alpha\phi^m - \phi^{2m} - 1\right)^2 - 4\left(\phi^m - \alpha\right)^2}}{2\left(\phi^m - \alpha\right)},$$

where we choose $|\theta| < 1$, and

$$\sigma_A^2 = \frac{\left(1 - \phi^{2m}\right)\left(1 + B + \cdots + B^{m-1}\right)\gamma_Y\left(m-1\right)}{1 + \theta^2 - 2\phi^m\theta}.$$

$\square$

5.2 Simulation study

With an original series AR(1) with $\phi = 0.8$, $n = 1440$, $N \sim (0, 1)$, after temporal aggregation, we ran 200 replications to estimate an ARMA (1,1) on both $m = 3$ and 12, We used a data length of 30 based on our suggestion. Exact solutions for $m = 3$ and 12 from Theorem 1 are $(1 - 0.51B)\,Z_t = (1 + 0.21B)\,a_t$, $\sigma_A^2 = 12.12$ and $(1 - 0.07B)\,a_t$, $\sigma_A^2 = 184.30$ respectively as shown in Figure 5.1.

FIGURE 5.1: AR(1) simulation and temporal aggregation

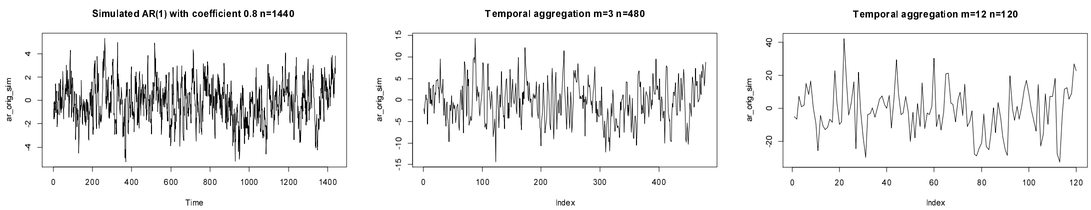


Table 5.1 contains detailed output on our estimation. For $m = 3$, all three estimations, (i.e., the series itself, the regular block bootstrap and our proposed block bootstrap) show not much difference in parameters and their variations. An estimation of $\hat{\theta}$ performed particularly poorly for all of them. For $m = 12$, as both true parameters ϕ, θ are close to 0, an estimation of the series itself does not give any relevant estimation numbers compared to the true parameters. The regular method's $\hat{\sigma}_A^2$is closer to the true parameter value.

TABLE 5.1: Parameter estimation of AR(1) model with $\phi = 0.8$, $\sigma_a^2 \sim N(0, 1)$, $n = 1440$, and its aggregate model ARMA(1,1) with $m = 3$ and $m = 12$ (200 replications)

(a) Estimation of aggregate model for $m = 3$ without bootstrap $(1 - 0.4407B)\,Z_t = (1 - 0.3275B)\,A_t$, $\hat{\sigma}_A^2 = 12$, s.e.$(\hat{\phi}^3) = 0.0623$, s.e.$(\hat{\theta}) = 0.0665$

$m = 3$ ARMA(1,1) true parameter $\phi^3 = 0.51, \theta = -0.21, \sigma_A^2 = 12.12$	Regular(8 blocks)	Augmented(8 blocks)
$\hat{\phi}^3$	0.4173	0.3849
$S_{\hat{\phi}^3}$	0.0545	0.0706
$\hat{\theta}$	-0.3182	-0.3688
$S_{\hat{\theta}}$	0.0778	0.0872
$\hat{\sigma}_A^2$	12.2633	3.2012
$S_{\hat{\sigma}_A^2}$	0.9361	0.7538

TABLE 5.1, continued: Parameter estimation of AR(1) model with $\phi = 0.8$, $\sigma_a^2 \sim N(0,1)$, $n = 1440$, and its aggregate model ARMA(1,1) with $m = 3$ and $m = 12$ (200 replications)

(b) Estimation of aggregate model for $m = 12$ without bootstrap $(1 - 0.2926B)\, Z_t = (1 + 0.0335B)\, A_t$, $\hat{\sigma}_A^2 = 190$, s.e.$(\hat{\phi}^3) = 0.4291$, s.e.$(\hat{\theta}) = 0.4515$

$m = 12$ ARMA(1,1) true parameter $\phi^{12} = 0.07, \theta = -0.19, \sigma_A^2 = 184.30$	Regular(4 blocks)	Augmented(4 blocks)
$\hat{\phi}^{12}$	0.2298	0.2214
$S_{\hat{\phi}^{12}}$	0.4943	0.4447
$\hat{\theta}$	−0.0196	−0.0145
$S_{\hat{\theta}}$	0.4663	0.4252
$\hat{\sigma}_A^2$	186.108	101.914
$S_{\hat{\sigma}_A^2}$	16.9781	29.1099

5.3 Empirical study

For an empirical study of the implication of two block bootstrap methods in temporal aggregation, we used weekly Initial Claims by US workers from 1/7/1967 to 8/6/1994, total 1440 data series, shown in Figure 5.2, taken from Federal Reserve Bank of St. Louis website.(Figure 5.2) With original monthly series of 1440 observations, we fitted a simple AR(1) model and a coefficient turned out to be 0.9775 with $\hat{\sigma}_a^2 = 405.1$. Then we applied both block bootstrap methods to quarterly($m = 3$) and yearly($m = 12$) with an ARMA(1,1) model. Exact solutions for $m = 3$ and 12 using theorem 1 are

$$(1 - 0.93B)\, Z_t = (1 + 0.22B)\, a_t,\ \sigma_A^2 = 7015.8,$$
$$(1 - 0.7610B)\, Z_t = (1 + 0.26B)\, a_t,\ \sigma_A^2 = 343443.7 \text{ respectively.}$$

Results in Table 5.2 showed that the best fitted model of series did not do well on both $m = 3$ and $m = 12$. Both block bootstrap models did well on estimations of ϕ^3 and ϕ^{12} but did poorly on both $\hat{\theta}$s. $\hat{\sigma}_A^2$s were lower on Combinatorially-Augmented Block Bootstrap. Combinatorially-Augmented Block Bootstrap did equally well with conventional method on aggregation model of simulated and empirical cases, especially in case of ϕ^3 and ϕ^{12}. We ensured that the minimum block length is 30 in all bootstrap runs.

FIGURE 5.2: Weekly Initial Claims by US workers from 1/7/1967 to 8/6/1994 Seasonally Adjusted (Source: Federal Reserve Bank of St. Louis)

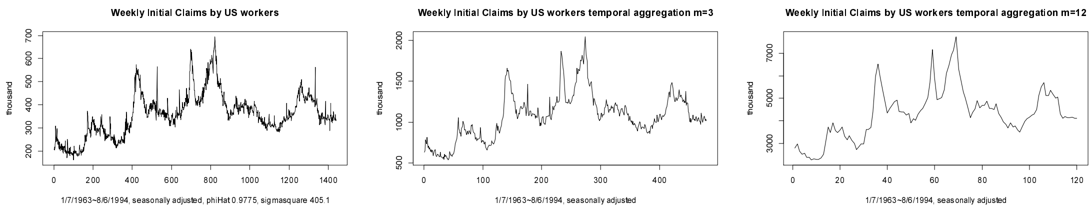


TABLE 5.2: Model estimation of ARMA(1,1) model for aggregate Weekly Initial Claims series with 200 replications using two block bootstrap methods

(a) Estimation of aggregate model for $m = 3$ without bootstrap $(1 - 0.9788B)\, Z_t = (1 - 0.0229B)\, A_t$, $\hat{\sigma}_A^2 = 3263$, s.e.$(\hat{\phi}^3) = 0.0092$, s.e.$(\hat{\theta}) = 0.0446$

$m = 3$ ARMA(1,1) true parameter $\phi^3 = 0.93$, $\theta = -0.22$, $\sigma_A^2 = 7015.8$	Regular (8 blocks)	Augmented (8 blocks)
$\hat{\phi}^3$	0.9290	0.9009
$S_{\hat{\phi}^3}$	0.0270	0.0399
$\hat{\theta}$	−0.0747	−0.0834
$S_{\hat{\theta}}$	0.1085	0.1119
$\hat{\sigma}_A^2$	8498.31	2308.15
$S_{\hat{\sigma}_A^2}$	3012.70	559.17

(b) Estimation of aggregate model for $m = 12$ without bootstrap $(1 - 0.9020B)\, Z_t = (1 - 0.2642B)\, A_t$, $\hat{\sigma}_A^2 = 149778$, s.e.$(\hat{\phi}^{12}) = 0.0401$, s.e.$(\hat{\theta}) = 0.0852$

$m = 12$ ARMA(1,1) true parameter $\phi^{12} = 0.7610$, $\theta = -0.26$, $\sigma_A^2 = 343443.7$	Regular (4 blocks)	Augmented (4 blocks)
$\hat{\phi}^{12}$	0.8389	0.7896
$S_{\hat{\phi}^{12}}$	0.1072	0.1098
$\hat{\theta}$	−0.0749	−0.0510
$S_{\hat{\theta}}$	0.1078	0.0959
$\hat{\sigma}_A^2$	285860.9	115584.7
$S_{\hat{\sigma}_A^2}$	105776.1	77031.48

CHAPTER 6

SYSTEMATIC SAMPLING EFFECT ON BOOTSTRAP

6.1 Introduction

Systematic sampling is simply selecting the m^{th} unit rather than aggregating units over time. It is a stock variable compared to a flow variable in aggregation. Its theoretical detail is well documented in Brewer (1973) and Wei (2006). In summary, the ARMA(p, q) model becomes an ARMA(p, r) where r is less than or equal to the integer part of a $p + (q - p)/m$. Brewer (1973) and Wei (1981) only showed model order results for the systematic sampling on ARMA and ARIMA models. From their results, the systematic sampling of an AR(1) model becomes AR(1) model. However, they did not derive the expression for the white noise variance of the model from systematic sampling. For our purposes, we derive them in Theorem 2.

Theorem 2. *Given the basic AR(1) model, $Y_t = \phi Y_{t-1} + a_t$, where a_t is a normal white noise $N\left(0, \sigma_a^2\right)$, the m^{th} order systematic sampling, $Z_T = Y_{mT}$, follows an AR(1) model, $Z_T = \phi^m Z_{T-1} + A_T$ where A_T is Normal white noise $N\left(0, \sigma_A^2\right)$ and $\sigma_A^2 = \dfrac{1-\phi^{2m}}{1-\phi^2}\sigma_a^2$.*

Proof.

$$
\begin{aligned}
Z_T = Y_{Tm} &= \phi Y_{Tm-1} + a_{Tm} \\
&= \phi\left(\phi Y_{Tm-2} + a_{Tm-1}\right) + a_{Tm} = \phi^2 Y_{Tm-2} + \phi a_{Tm-1} + a_{Tm} \\
&= \phi^2\left(\phi Y_{Tm-3} + a_{Tm-2}\right) + \phi a_{Tm-1} + a_{Tm} = \phi^3 Y_{Tm-3} + \phi^2 Y_{Tm-2} + \phi a_{Tm-1} + a_{Tm} \\
&= \ldots \\
&= \phi^m Y_{Tm-m} + \phi^{m-1} Y_{Tm-m+1} + \phi^{m-2} Y_{Tm-m+2} + \ldots + \phi Y_{Tm-1} + a_{Tm} \\
&= \phi^m Z_{T-1} + A_T
\end{aligned}
$$

where

$$A_T = \phi^{m-1} a_{Tm-m+1} + \phi^{m-2} a_{Tm-m+2} + \ldots + \phi a_{Tm-1} + a_{tm}.$$

Hence,

$$\sigma_A^2 = \text{Var}\left(A_T\right) = \sigma_a^2\left(1 + \phi^2 + \ldots + \phi^{2(m-2)}\right) = \left(\frac{1-\phi^{2m}}{1-\phi^2}\right)\sigma_a^2.$$

□

Wei (1981) also reported that the random walk phenomenon in stock price could be observed due to systematic sampling.

6.2 Simulation study

To be able to compare systematic sampling to temporal aggregation, we used the same simulated AR(1) time series innovation from section 5.2 with $\phi = 0.8$, $\sigma_a^2 \sim N(0,1)$, $n = 1440$. Instead of aggregating by temporal units, we simply selected data points by systematic sampling $m = 3$ and 12, shown in Figure 6.1. Then we applied an AR(1) on both methods of block bootstrapping. Table 6.1 results showed that when $m = 3$, the best fitted model to the series did well on both ϕ^3 and σ_A^2 estimations. Both block bootstrap methods are fairly close to the true value of ϕ^3 as well. $\hat{\sigma}_A^2$ of our method was lower than the true value.

In $m = 12$ case, though the true parameter ϕ^{12} is 0.06872 which is very small, the best fitted model to the series and both block bootstrap methods did fairly well. Again, $\hat{\sigma}_A^2$ of our method was lower than true value. As shown in Table 6.1, exact solutions for systematic sampling cases of $m = 3$ and 12 using Theorem 2 are $(1 - 0.512B)\, Z_t = A_t$, $\sigma_A^2 = 2.0496$ and $(1 - 0.06872B)\, Z_t = A_t$, $\sigma_A^2 = 2.7647$ respectively. (Figure 6.1, Table 6.1)

FIGURE 6.1: AR(1) simulation and systematic sampling

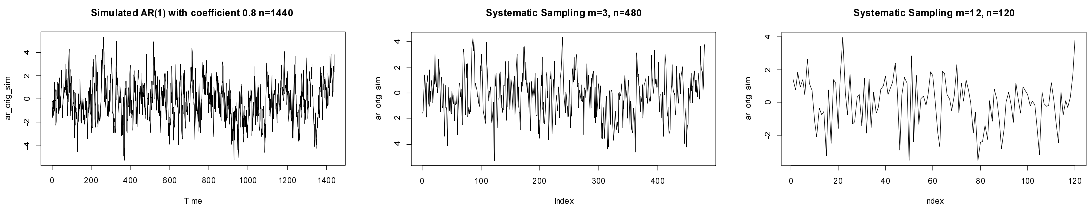


TABLE 6.1: Parameter estimation of AR(1) model with $\phi = 0.8$, $\sigma_a^2 \sim N(0,1)$, $n = 1440$, systematic sampling model of AR(1) $m = 3$ and $m = 12$

(a) Estimation of systematically sampling model for $m = 3$ without bootstrap $(1 - 0.5187B)\, Z_t = A_t$, $\hat{\sigma}_A^2 = 2.0598$

$m = 3$ AR(1) true parameter $\phi^3 = 0.51$, $\sigma_A^2 = 2.0496$	Regular (8 blocks)	Augmented (8 blocks)
$\hat{\phi}^3$	0.4892	0.4680
$S_{\hat{\phi}^3}$	0.0499	0.0458
$\hat{\sigma}_A^2$	2.0819	0.6022
$S_{\hat{\sigma}_A^2}$	0.1110	0.1289

(b) Estimation of systematic sampling model for $m = 12$ without bootstrap $(1 - 0.1107B)\, Z_t = A_t$, $\hat{\sigma}_A^2 = 2.2296$

$m = 12$ AR(1) true parameter $\phi^{12} = 0.06872$, $\sigma_A^2 = 2.7647$	Regular (4 blocks)	Augmented (4 blocks)
$\hat{\phi}^{12}$	0.1091	0.1139
$S_{\hat{\phi}^{12}}$	0.1540	0.1212
$\hat{\sigma}_A^2$	2.1624	1.1303
$S_{\hat{\sigma}_A^2}$	0.2311	0.3159

6.3 Empirical study

For the systematic sampling case for both block bootstrap methods, we collected data from the well-known stock index, S&P 500 stock index from Jan 1 of 2006 to Dec 31 of 2015, shown in Figure 6.2. The overall results presented in Table 6.2 are similar to those of simulated AR(1). However, like the empirical case of temporal aggregation block bootstrap results, Combinatorially-Augmented Block Bootstrap estimated better $\hat{\sigma}_A^2$ than the true σ_A^2. $\hat{\phi}^{12}$ was not a good estimate perhaps due to high $\hat{\sigma}_A^2$. Exact solutions for systematic sampling in the case of $m = 3$ and 12 are $(1 - 9703B)\, Z_t = A_t$, $\sigma_A^2 = 9621.97$ and $(1 - 0.8864B)\, Z_t = A_t$, $\sigma_A^2 = 34873.07$ respectively.

FIGURE 6.2: S&P index and systematic sampling (Source: Bloomberg)

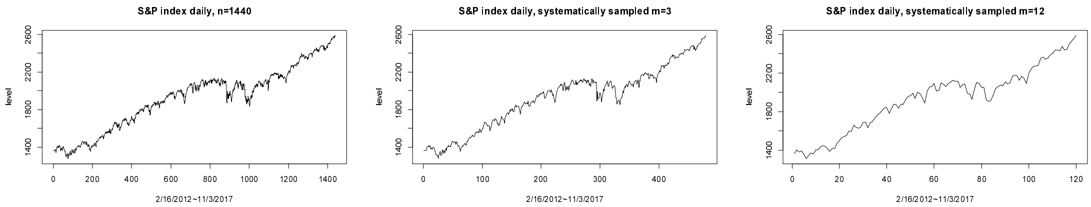


TABLE 6.2: Model estimation of AR(1) model for quarterly and yearly S&P index using two block bootstrap methods $\phi = 0.99$, $\sigma_a^2 = 217.42$

(a) Estimation of systematic sampling model for $m = 3$ without bootstrap $(1 - 0.99B)\,Z_t = A_t$, $\hat{\sigma}_A^2 = 604.6892$

$m = 3$ AR(1) true parameter $\phi^3 = 0.9703$, $\sigma_A^2 = 639.36$	Regular (8 blocks)	Augmented (8 blocks)
$\hat{\phi}^3$	0.9769	0.9682
$S_{\hat{\phi}^3}$	0.0083	0.0103
$\hat{\sigma}_A^2$	4287.25	1192.80
$S_{\hat{\sigma}_A^2}$	1863.03	504.04

(b) Estimation of systematic sampling model for $m = 12$ without bootstrap $(1 - 0.99B)\,Z_t = A_t$, $\hat{\sigma}_A^2 = 1732.984$

$m = 12$ AR(1) true parameter $\phi^{12} = 0.8864$, $\sigma_A^2 = 2341.60$	Regular (4 blocks)	Augmented (4 blocks)
$\hat{\phi}^{12}$	0.9340	0.9163
$S_{\hat{\phi}^{12}}$	0.0348	0.0430
$\hat{\sigma}_A^2$	9178.62	5857.34
$S_{\hat{\sigma}_A^2}$	4267.33	2891.27

CHAPTER 7

MORE ON AR(1) AND TWO DIMENSIONAL VAR(1) BLOCK BOOTSTRAP

Using the block bootstrap data arrangement given in section 4.3, we investigate the performance of a variety of scenarios in AR(1) and two-dimensional VAR(1) simulated time series. Basic setup was done according to Wei (2006). $H = (X_1, \ldots, X_n)$, $n = 1000$ is simulated to replicate AR(1) and, $J = (Y_1, \ldots, Y_n)$, $n = 1000$ is also simulated to replicate two-dimensional VAR(1). With $n = bl$, we will investigate when $b = 2, 4, 5, 10, 20, 25, 40, 50$. The parameters are $\phi = 0.8$ for the AR(1) process and $\phi_{11} = 0.8$, $\phi_{12} = 0.5$, $\phi_{21} = 0.4$, $\phi_{22} = 0.7$ for two-dimensional VAR(1). In VAR(1), both residuals are assumed to be $a_t \sim N(0, 1)$. All simulations in each block are done 500 times. Table 7.1 below shows the summary result.

TABLE 7.1: Estimation of AR(1) coefficient with different block size

# of blocks	2	4	5	10	20	25	40	50
mean	0.819	0.760	0.776	0.808	0.774	0.814	0.750	0.752
std	0.008	0.010	0.017	0.015	0.021	0.013	0.027	0.004

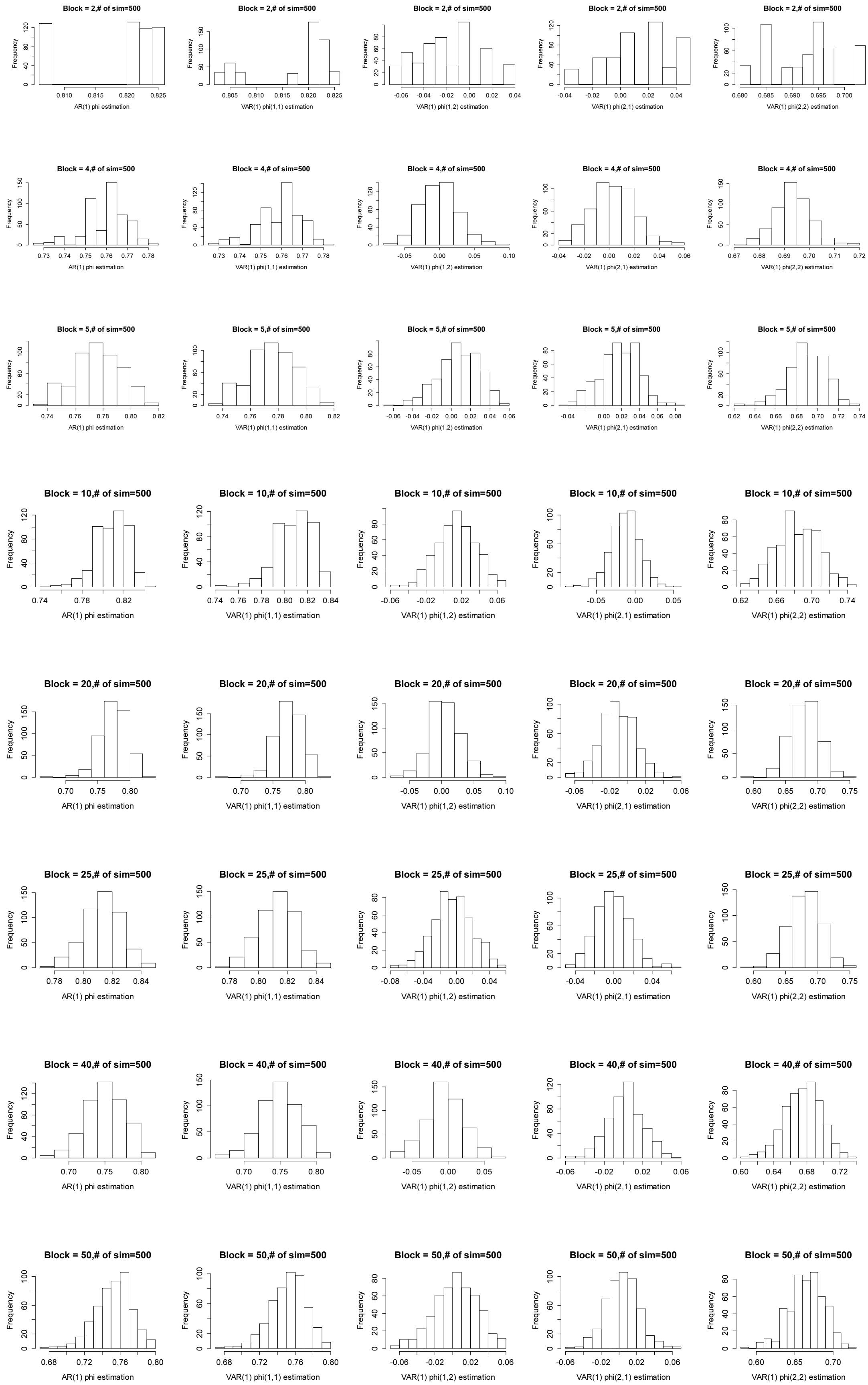
The AR(1) estimation shows that when using 1000 samples data point time series, dividing the sample into 10 blocks results in the best estimation with close to the original parameter of 0.8 with the smallest standard deviation. Carlestein (1986) shows the optimal block length calculation using AR(1). His lengthy calculation of the AR(1) case shows that the optimal subseries length $m_n^* = (2\phi/c)^{2/3}\, n^{1/3}$, $c = (1-\phi)\,(1+\phi)$. This result yields an optimal subseries length of 27.03 in our case which dictates that the number of blocks should be roughly 36. This estimation result is quite similar regardless of the number of blocks, so it is hard to tell which block gives the best result. Our mean is simply the average of 500 estimations of the AR(1) parameter using the block bootstrap method and the standard deviation is based on 500 estimations.

TABLE 7.2: The estimation of two dimensional VAR(1) coefficient with a different block size (each 2x2 matrix entry $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}$ according to matrix convention)

	The number of block			
	2	4	6	8
Mean	$\begin{bmatrix} 0.818 & -0.022 \\ 0.014 & 0.692 \end{bmatrix}$	$\begin{bmatrix} 0.759 & 0.000 \\ 0.017 & 0.694 \end{bmatrix}$	$\begin{bmatrix} 0.776 & 0.008 \\ 0.018 & 0.689 \end{bmatrix}$	$\begin{bmatrix} 0.808 & 0.014 \\ -0.013 & 0.684 \end{bmatrix}$
Std. Error	$\begin{bmatrix} 0.008 & 0.027 \\ 0.023 & 0.007 \end{bmatrix}$	$\begin{bmatrix} 0.01 & 0.026 \\ 0.017 & 0.007 \end{bmatrix}$	$\begin{bmatrix} 0.017 & 0.021 \\ 0.022 & 0.018 \end{bmatrix}$	$\begin{bmatrix} 0.015 & 0.022 \\ 0.019 & 0.024 \end{bmatrix}$
	20	25	40	50
Mean	$\begin{bmatrix} 0.774 & 0.005 \\ 0.011 & 0.679 \end{bmatrix}$	$\begin{bmatrix} 0.813 & 0.005 \\ -0.001 & 0.679 \end{bmatrix}$	$\begin{bmatrix} 0.750 & -0.005 \\ 0.002 & 0.675 \end{bmatrix}$	$\begin{bmatrix} 0.752 & 0.004 \\ 0.004 & 0.665 \end{bmatrix}$
Std. Error	$\begin{bmatrix} 0.021 & 0.024 \\ 0.019 & 0.023 \end{bmatrix}$	$\begin{bmatrix} 0.013 & 0.024 \\ 0.018 & 0.025 \end{bmatrix}$	$\begin{bmatrix} 0.027 & 0.026 \\ 0.018 & 0.022 \end{bmatrix}$	$\begin{bmatrix} 0.020 & 0.025 \\ 0.018 & 0.023 \end{bmatrix}$

The two-dimensional VAR(1) for a sample size of 100 shows that optimal block size is around 10. Unfortunately, other than its own past parameters ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22}, the estimation results were very consistently very poor, showing around 0, compared to the true parameters of 0.5 and 0.4, shown in Table 7.2. Block bootstrap is not suitable for VAR analysis, as it cannot capture correlation structure of the multiple time series.

FIGURE 7.1: The AR(1) and two dimensional VAR(1) block bootstrap estimation)



CHAPTER 8

CONCLUSION AND FUTURE RESEARCH

The bootstrap is a very useful method to estimate the underlying population distribution when the given data set is limited. We investigated the basic properties of bootstrap and block bootstrap. We also investigated the length of the block bootstrap method. We showed that the previously known Carlstein's optimal length is suitable only for negatively correlated series. In various AR(1) coefficient tests, the block bootstrap method did quite well. For the first time in time series research, we investigated block bootstrap methods for two properties of time series data collection procedures, temporal aggregation and systematic sampling. We also explicitly calculated the exact solution of AR(1)'s temporal aggregation to ARMA(1,1) and systematic sampling to other AR(1). We presented the results of both simulated and empirical data, and showed that block bootstrapping can help recover information loss through temporal aggregation and systematic sampling. Our future research can be expanded to explore in which circumstances it makes sense to use block bootstrapping in general ARIMA(p, d, q) and investigate its effects on more complicated data sets, such as VAR or nonlinear time series. Also, theoretical properties of our proposed CABB can be researched to further enhance the effetiveness of the method. Theoretical treatment of information loss recovery using block bootstrapping in aggregation and systematic sampling can be interesting subject to explore.

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