

**Second and Higher Order Elliptic Boundary Value Problems in
Irregular Domains in the Plane**

A Dissertation
Submitted to
the Temple University Graduate Board

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

by
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May, 2024

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May, 2024

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ABSTRACT

The topic of this dissertation lies at the interface between the areas of Harmonic Analysis, Partial Differential Equations, and Geometric Measure Theory, with an emphasis on the study of singular integral operators associated with second and higher order elliptic boundary value problems in non-smooth domains.

The overall aim of this work is to further the development of a systematic treatment of second and higher order elliptic boundary value problems using singular integral operators. This is relevant to the theoretical and numerical treatment of boundary value problems arising in the modeling of physical phenomena such as elasticity, incompressible viscous fluid flow, electromagnetism, anisotropic plate bending, etc., in domains which may exhibit singularities at all boundary locations and all scales. Since physical domains may exhibit asperities and irregularities of a very intricate nature, we wish to develop tools and carry out such an analysis in a very general class of non-smooth domains, which is in the nature of best possible from the geometric measure theoretic point of view.

The dissertation will be focused on three main, interconnected, themes:

- A.** *A systematic study of the poly-Cauchy operator in uniformly rectifiable domains in \mathbb{C} ;*
- B.** *Solvability results for the Neumann problem for the bi-Laplacian in infinite sectors in \mathbb{R}^2 ;*
- C.** *Connections between spectral properties of layer potentials associated with second-order elliptic systems and the underlying tensor of coefficients.*

Theme **A** is based on papers [16, 17, 18] and this work is concerned with the investigation of polyanalytic functions and boundary value problems associated with (integer) powers of the Cauchy-Riemann operator in uniformly rectifiable domains in the complex plane. The goal here is to devise a higher-order analogue of the existing theory for the classical Cauchy operator in which

the salient role of the Cauchy-Riemann operator $\bar{\partial}$ is now played by $\bar{\partial}^m$ for some arbitrary fixed integer $m \in \mathbb{N}$. This analysis includes integral representation formulas, higher-order Fatou theorems, Calderón-Zygmund theory for the poly-Cauchy operators, radiation conditions, and higher-order Hardy spaces.

Theme **B** is based on papers [3, 19] and this regards the Neumann problem for the bi-Laplacian with L^p data in infinite sectors in the plane using Mellin transform techniques, for $p \in (1, \infty)$. We reduce the problem of finding the solvability range of the integrability exponent p for the L^p biharmonic Neumann problem to solving an equation involving quadratic polynomials and trigonometric functions employing the Mellin transform technique. Additionally, we provide the range of the integrability exponent for the existence of a solution to the L^p biharmonic Neumann problem in two-dimensional infinite sectors. The difficulty we are overcoming has to do with the fact that the Mellin symbol involves hypergeometric functions.

Finally regarding theme **C**, based on the ongoing work in [2], the emphasis is the investigation of coefficient tensors associated with second-order elliptic operators in two dimensional infinite sectors and properties of the corresponding singular integral operators, employing Mellin transform. Concretely, we explore the relationship between distinguished coefficient tensors and L^p spectral and Hardy kernel properties of the associated singular integral operators.

ACKNOWLEDGEMENTS

I am grateful to my advisor, Professor Irina Mitrea, whose patience and enthusiasm have shaped my academic journey. Her passionate engagement in our mathematical discussions not only inspired and motivated me but also deepened my understanding of critical problems. I will forever be grateful for her guidance and support.

My sincere thanks also go to Professors Dorina Mitrea and Marius Mitrea for their insightful discussions and contributions to our collaborative projects. I extend my gratitude to Professors Cristian Gutiérrez, Mihaela Ignatova, and Shari Moskow for their invaluable feedback and for serving on my dissertation committee. A special thanks to Professor Gutiérrez for his continued interest in my work and for the opportunity to learn advanced topics in PDEs.

I am thankful to Professor Seick Kim for providing me with an excellent foundation in PDEs during my master's program at Yonsei University, as well as for his generous support.

Appreciation is also due to Professors Yury Grabovsky, Gerardo Mendoza, Matthew Stover, Samuel Taylor, Wei-Shih Yang, and Atilla Yilmaz for broadening my horizons through their engaging and informative courses. I would also like to thank Professor David Futer for his support as an outstanding director of graduate studies at Temple University.

I would also like to acknowledge Professors Maria Lorenz and Jeromy Sivek for their advice and for providing opportunities that improved my teaching skills.

I am incredibly thankful for my family for always being supportive. I also thank all my colleagues and friends for their warm friendship and encouragement throughout my academic career.

My deepest gratitude extends to my wife, Inyeong, whose love, patience, and belief in me have been the greatest source of support in my life. Her constant companionship throughout this journey has been invaluable. Her presence is my constant inspiration, and I dedicate this achievement to her.

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CHAPTER 1

Preliminaries

In this chapter, we introduce notation and review a number of definitions and theorems used in the dissertation.

1.1 Geometric Measure Theory

Throughout, we shall work in $\mathbb{R}^2 \equiv \mathbb{C}$ and we agree that \mathcal{L}^2 stands for the two-dimensional Lebesgue measure, while \mathcal{H}^1 denotes the one-dimensional Hausdorff measure in \mathbb{R}^2 . For $z \in \mathbb{R}^2$ and $r > 0$ we let $B(z, r)$ stand for the open ball in \mathbb{R}^2 centered at z with radius r .

Definition 1.1. *A closed set $\Sigma \subseteq \mathbb{R}^2$ is called an Ahlfors regular set provided there exist constants $c, C \in (0, \infty)$ such that $cr \leq \mathcal{H}^1(B(z, r) \cap \Sigma) \leq Cr$ for all $z \in \Sigma$ and $r \in (0, \text{diam } \Sigma)$.*

Definition 1.2. *For any Lebesgue measurable set $\Omega \subseteq \mathbb{R}^2$, denote by $\partial_*\Omega$ the geometric measure theoretic boundary of Ω , defined as*

$$\partial_*\Omega := \left\{ x \in \partial\Omega : \limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^2(B(x, r) \cap \Omega)}{r^2} > 0, \right. \quad (1.1)$$

$$\left. \limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^2(B(x, r) \setminus \Omega)}{r^2} > 0 \right\}.$$

Definition 1.3. $\Omega \subseteq \mathbb{R}^2$ is said to be a set of locally finite perimeter provided it is Lebesgue measurable and $\mathcal{H}^1(\partial_*\Omega \cap K) < \infty$ for each compact set $K \subseteq \mathbb{R}^2$.

As is apparent from Definition 1.1 and Definition 1.3, any open subset $\Omega \subseteq \mathbb{R}^2$ with an Ahlfors regular topological boundary $\partial\Omega$ is a set of locally finite perimeter. Given a set of locally finite perimeter $\Omega \subseteq \mathbb{R}^2$, a classical result of E. De Giorgi and H. Federer gives that the characteristic function $\mathbf{1}_\Omega$ of Ω satisfies

$$\nabla \mathbf{1}_\Omega = -\nu (\mathcal{H}^1 \llcorner \partial_*\Omega) \text{ in } [\mathcal{D}'(\mathbb{R}^2)]^2, \quad (1.2)$$

where $\nu \in [L^\infty(\partial_*\Omega, \mathcal{H}^1)]^2$ is a vector field, henceforth referred to as the geometric measure theoretic (GMT) outward unit normal to Ω , and \llcorner denotes restriction. Also $\mathcal{D}'(\mathbb{R}^2)$ stands for the space of distributions in \mathbb{R}^2 . Hence, if we abbreviate

$$\sigma := \mathcal{H}^1 \llcorner \partial\Omega \text{ and } \sigma_* := \mathcal{H}^1 \llcorner \partial_*\Omega = \sigma \llcorner \partial_*\Omega, \quad (1.3)$$

we may simply recast (1.2) as $\nabla \mathbf{1}_\Omega = -\nu \sigma_*$ in the sense of vector distributions (cf. [22]) in \mathbb{R}^2 .

The following definition of uniform rectifiability is due to G. David and S. Semmes (cf., e.g., the discussion in [26]).

Definition 1.4. $\Sigma \subseteq \mathbb{R}^2$ is said to be a uniformly rectifiable set (or UR set, for short) if Σ is a closed Ahlfors regular set with the property that there exist constants $\varepsilon, M \in (0, \infty)$ such that for each $z \in \Sigma$ and each $R \in (0, 2 \operatorname{diam}(\Sigma))$ it is possible to find some Lipschitz map $\varphi : [0, R] \rightarrow \mathbb{R}^2$ with $\|\varphi'\|_{L^\infty} \leq M$ such that

$$\mathcal{H}^1(\Sigma \cap B(z, R) \cap \varphi([0, R])) \geq \varepsilon R. \quad (1.4)$$

Also, recall the definition of a UR domain (cf. [11]).

Definition 1.5. An open set $\Omega \subseteq \mathbb{R}^2$ is called a UR domain provided that $\partial\Omega$ is a UR set and

$$\mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0. \quad (1.5)$$

Given a set $\Omega \subseteq \mathbb{R}^2$ let

$$\Omega_+ := \Omega \quad \text{and} \quad \Omega_- := \mathbb{R}^2 \setminus \bar{\Omega}, \quad (1.6)$$

where bar denotes topological closure. It turns out that if Ω is a UR domain with geometric measure theoretic outward unit normal vector ν then Ω_- is also a UR domain such that $\partial_*(\Omega_-) = \partial_*\Omega$, $\partial(\Omega_-) = \partial\Omega$, and whose geometric measure theoretic outward unit normal is $-\nu$.

The following definitions are due to D. Jerison and C. Kenig in [13]

Definition 1.6. *Fix $R \in (0, \infty]$ and $C \in (0, 1)$. An open set $\Omega \subseteq \mathbb{R}^2$ satisfies the corkscrew condition if for each $x \in \partial\Omega$ and $r \in (0, R)$ there exists a point $z \in \Omega$ with the property that $B(z, Cr) \subseteq B(x, r) \cap \Omega$. The point $z \in \Omega$ is called a corkscrew point relative to x and r .*

Next, we recall the Harnack chain condition.

Definition 1.7. *Fix $R \in (0, \infty]$ and $N \in \mathbb{N}$. An open set $\Omega \subseteq \mathbb{R}^2$ is said to satisfy the Harnack chain condition provided whenever $\varepsilon > 0$, $k \in \mathbb{N}$, $z \in \partial\Omega$, $r \in (0, R)$, and $x, y \in B(z, r/4) \cap \Omega$ satisfy $|x - y| \leq 2^k \varepsilon$ and $\min \{\text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega)\} \geq \varepsilon$, there exist open balls B_1, B_2, \dots, B_K in \mathbb{R}^2 with $K \leq Nk$, such that $x \in B_1$, $y \in B_K$, and $B_i \cap B_{i+1} \neq \emptyset$ for every $i \in \{1, \dots, K-1\}$, and*

$$N^{-1} \cdot \text{diam}(B_i) \leq \text{dist}(B_i, \partial\Omega) \leq N \cdot \text{diam}(B_i), \quad (1.7)$$

$$\text{diam}(B_i) \geq N^{-1} \cdot \min \{\text{dist}(x, B_i), \text{dist}(y, B_i)\}, \quad (1.8)$$

for every $i \in \{1, \dots, K\}$.

With Definitions 1.7-1.8 in hand, following [13], recall the notion of NTA domains.

Definition 1.8. *Fix $R \in (0, \infty]$ and $N \in \mathbb{N}$. An open, nonempty, proper subset Ω of \mathbb{R}^2 is said to be a nontangentially accessible domain (or simply an NTA domain) if it satisfies the Harnack chain condition, and if both Ω and $\mathbb{R}^2 \setminus \bar{\Omega}$ satisfy the corkscrew condition, with bar denoting topological closure.*

The following definition introduces the nontangentially accessible boundary.

Definition 1.9. *For any given nonempty open proper set Ω of \mathbb{R}^2 , the nontangentially accessible boundary of Ω is defined as*

$$\partial_{\text{nta}}\Omega := \left\{ x \in \partial\Omega : x \in \overline{\Gamma_\kappa(x)} \text{ for each } \kappa > 0 \right\}. \quad (1.9)$$

1.2 Calderón-Zygmund theory and the Divergence Theorem in UR domains

Fix a UR domain $\Omega \subseteq \mathbb{R}^2$, in the sense of Definition 1.5 and let ν be the geometric measure theoretic outward unit normal vector and σ be the surface measure. Let $\mathcal{C}^1(\mathbb{R}^2)$ denote the space of continuously differentiable functions in \mathbb{R}^2 and define the tangential derivative $\partial_\tau\varphi$ of any given function $\varphi \in \mathcal{C}^1(\mathbb{R}^2)$ as

$$\partial_\tau\varphi := \nu_1(\partial_y\varphi)\Big|_{\partial\Omega} - \nu_2(\partial_x\varphi)\Big|_{\partial\Omega}. \quad (1.10)$$

More generally, we shall say that the tangential derivative of some function $f \in L^p(\partial\Omega, \sigma)$ with $1 < p < \infty$ exists and belongs to the space $L^p(\partial\Omega, \sigma)$ if one can find a function $\partial_\tau f \in L^p(\partial\Omega, \sigma)$ with the property that the following formula (mimicking integration by parts on the boundary) holds:

$$\int_{\partial\Omega} f(\partial_\tau\varphi) d\sigma = - \int_{\partial\Omega} (\partial_\tau f)\varphi d\sigma \text{ for each } \varphi \in \mathcal{C}_c^1(\mathbb{R}^2). \quad (1.11)$$

Above and throughout, $L^p(\partial\Omega, \sigma)$ stands for the Lebesgue space of p integrable functions on $\partial\Omega$ with respect to the measure σ , while $\mathcal{C}_c^1(\mathbb{R}^2)$ stands for the space of continuously differentiable functions with compact support in \mathbb{R}^2 . Finally, for each $p \in (1, \infty)$ define the boundary Sobolev space (cf. [11], [23], [26])

$$L_1^p(\partial\Omega, \sigma) := \{f \in L^p(\partial\Omega, \sigma) : \partial_\tau f \in L^p(\partial\Omega, \sigma)\}, \quad (1.12)$$

and equip it with the natural norm $\|f\|_{L_1^p(\partial\Omega, \sigma)} := \|f\|_{L^p(\partial\Omega, \sigma)} + \|\partial_\tau f\|_{L^p(\partial\Omega, \sigma)}$.

Next, we introduce the notions of the nontangential maximal operator and boundary trace. The nontangential approach regions of aperture $\kappa \in (0, \infty)$ are introduced as

$$\Gamma_\kappa(x_0) := \left\{ x \in \Omega : |x - x_0| < (1 + \kappa) \operatorname{dist}(x, \partial\Omega) \right\} \text{ for each } x_0 \in \partial\Omega. \quad (1.13)$$

Then the nontangential maximal operator \mathcal{N}_κ acts on each \mathcal{L}^2 -measurable function $u : \Omega \rightarrow \mathbb{R}^2$ according to

$$(\mathcal{N}_\kappa u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x), \mathcal{L}^2)} \text{ for each } x \in \partial\Omega, \quad (1.14)$$

where $L^\infty(\Gamma_\kappa(x), \mathcal{L}^2)$ stands for the space of essentially bounded functions on $\Gamma_\kappa(x)$ with respect to \mathcal{L}^2 . If Ω is an exterior domain, that is Ω is the complement of a compact subset of \mathbb{R}^2 , it is assumed that the nontangential maximal operator is truncated. In addition, we agree to denote by $(u|_{\partial\Omega}^{\kappa\text{-n.t.}})(x_0)$ the κ -nontangential trace of a given \mathcal{L}^2 -measurable function $u : \Omega \rightarrow \mathbb{R}^2$ at the point $x_0 \in \partial\Omega$, defined as the vector (which is unique, if it exists) $y \in \mathbb{R}^2$ with the property that

$$\begin{aligned} &\text{for every } \varepsilon > 0 \text{ there exists } r > 0 \text{ such that} \\ &|u(x) - y| < \varepsilon \text{ for } \mathcal{L}^2\text{-a.e. } x \in \Gamma_\kappa(x_0) \cap B(x_0, r), \end{aligned} \quad (1.15)$$

that is,

$$(u|_{\partial\Omega}^{\kappa\text{-n.t.}})(x_0) := \lim_{\Gamma_\kappa(x_0) \ni x \rightarrow x_0} u(x). \quad (1.16)$$

The result below, found in [11], regards Calderón-Zygmund theory properties for singular integral operators of Calderón-Zygmund type in UR domains.

Theorem 1.10. *Assume $\Omega \subseteq \mathbb{R}^2$ is a UR domain with geometric measure theoretic outward unit normal vector ν and surface measure $\sigma = \mathcal{H}^1 \llcorner \partial\Omega$. Fix an aperture parameter $\kappa \in (0, \infty)$ and an integrability exponent $p \in (1, \infty)$. Suppose that $k \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{0\})$ is an odd and positive homogeneous function of degree -1 . Define the singular integral operator*

$$\mathcal{T}f(x) := \int_{\partial\Omega} k(x - y)f(y) d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial\Omega. \quad (1.17)$$

and for each $\varepsilon > 0$ consider the truncated boundary-to-boundary operator T_ε given by

$$T_\varepsilon f(x) := \int_{\partial\Omega \setminus B(x, \varepsilon)} k(x-y)f(y) d\sigma(y), \quad x \in \partial\Omega, \quad (1.18)$$

and the maximal operator T_* given by

$$T_* f(x) := \sup_{\varepsilon > 0} |T_\varepsilon f(x)|, \quad x \in \partial\Omega. \quad (1.19)$$

Then the following properties hold:

(1) For each $f \in L^p(\partial\Omega, \sigma)$ one has

$$\|T_* f\|_{L^p(\partial\Omega, \sigma)} \lesssim \|f\|_{L^p(\partial\Omega, \sigma)}. \quad (1.20)$$

(2) For each $f \in L^p(\partial\Omega, \sigma)$ the limit

$$Tf(x) := \lim_{\varepsilon \rightarrow 0^+} T_\varepsilon f(x) \quad (1.21)$$

exists for σ -a.e. point x on $\partial\Omega$ and the following jump relation

$$\mathcal{T}f \Big|_{\partial\Omega_\pm}^{\kappa\text{-n.t.}}(x) = \mp \frac{\sqrt{-1}}{2} \cdot \widehat{k}(\nu(x)) \cdot f(x) + Tf(x) \quad (1.22)$$

holds for σ -a.e. point on $\partial\Omega$ where $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^2 \setminus \overline{\Omega}$. Here \widehat{k} denotes the Fourier transform of k .

(3) For each $f \in L^p(\partial\Omega, \sigma)$ one has

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{L^p(\partial\Omega, \sigma)} \lesssim \|f\|_{L^p(\partial\Omega, \sigma)}. \quad (1.23)$$

Another very useful tool is the following Divergence Theorem with non-tangential traces. This result can be found in [25] in the more general case of domains in \mathbb{R}^n with $n \geq 2$ and with a lower Ahlfors regular boundary.

Theorem 1.11. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with outward unit normal vector ν and surface measure $\sigma = \mathcal{H}^1 \llcorner \partial\Omega$. Fix an aperture parameter $\kappa \in (0, \infty)$ and*

assume that $\vec{F} = (F_1, F_2) : \Omega \rightarrow \mathbb{C}^2$ is a vector field with Lebesgue measurable components, satisfying

$$\begin{aligned} \vec{F}|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \mathcal{N}_\kappa \vec{F} \text{ belongs to the} \\ \text{space } L^1(\partial\Omega, \sigma), \text{ and } \operatorname{div} \vec{F} := \partial_1 F_1 + \partial_2 F_2 \in L^1(\Omega, \mathcal{L}^2), \end{aligned} \quad (1.24)$$

where all derivatives are considered in the sense of distribution in Ω .

Then there holds

$$\int_{\Omega} \operatorname{div} \vec{F} d\mathcal{L}^2 = \int_{\partial\Omega} \nu \cdot \left(\vec{F}|_{\partial\Omega}^{\kappa\text{-n.t.}} \right) d\sigma \quad (1.25)$$

when either Ω is bounded, or $\partial\Omega$ is unbounded.

1.3 Second-order elliptic systems in the plane

Let $M \in \mathbb{N}$ and consider a collection of complex numbers $A := (a_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}}$. Associated with A , consider the second-order $M \times M$ system in \mathbb{R}^2 , with constant complex coefficients, written as

$$L_A u := \left(\partial_r (a_{rs}^{\alpha\beta} \partial_s u_\beta) \right)_{1 \leq \alpha \leq M} \quad (1.26)$$

when acting on a \mathcal{C}^2 vector valued function $u = (u_\beta)_{1 \leq \beta \leq M}$. Here the Einstein summation convention over repeated indices $1 \leq r, s \leq 2$ and $1 \leq \beta \leq M$ is used.

Given a second-order $M \times M$ system L in \mathbb{R}^2 and a collection of complex numbers $A := (a_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}}$, we shall say that A is associated with L , and write $A \sim L$, provided

$$L = L_A, \quad \text{where } L_A \text{ is as in (1.26)}. \quad (1.27)$$

In such a case, we shall say that A is a **coefficient tensor associated with L** . Denote the collection of all coefficient tensors associated with L by \mathfrak{A}_L , that is

$$\mathfrak{A}_L = \left\{ A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}} : \text{each } a_{rs}^{\alpha\beta} \text{ belongs to } \mathbb{C} \text{ and } L = L_A \right\}. \quad (1.28)$$

Note that there are infinitely many coefficient tensors associated with the second-order elliptic operator L . Indeed, let $B = (b_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}}$ be a collection of complex numbers which is antisymmetric in its lower indices, that is,

$$b_{rs}^{\alpha\beta} = -b_{sr}^{\alpha\beta}, \quad \text{for all } 1 \leq \alpha, \beta \leq M \quad \text{and all } 1 \leq r, s \leq 2. \quad (1.29)$$

Then, for each $A \in \mathfrak{A}_L$ there holds that $A + B := (a_{rs}^{\alpha\beta} + b_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}}$ also satisfies $A + B \in \mathfrak{A}_L$. This is since for each $\alpha \in \{1, \dots, M\}$ there holds that $\partial_r (b_{rs}^{\alpha\beta} \partial_s u_\beta) = -\partial_s (b_{sr}^{\alpha\beta} \partial_r u_\beta)$, implying that $\partial_r (b_{rs}^{\alpha\beta} \partial_s u_\beta) = 0$. Ultimately this gives that $L_{A+B} = L_A = L$, as desired.

Below, let Re denote real part and bar denote complex conjugation.

Definition 1.12. *Assume that $M \in \mathbb{N}$ and fix a second-order $M \times M$ system L in \mathbb{R}^2 . Call the coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}} \in \mathfrak{A}_L$ positive definite provided there exists a real number $\kappa > 0$ such that*

$$\text{Re} \langle A\zeta, \bar{\zeta} \rangle := \text{Re} (a_{rs}^{\alpha\beta} \zeta_s^\beta \bar{\zeta}_\alpha^r) \geq \kappa |\zeta|^2, \quad \forall \zeta = (\zeta_\alpha^r)_{\substack{1 \leq r \leq 2 \\ 1 \leq \alpha \leq M}} \in \mathbb{C}^{2 \times M}. \quad (1.30)$$

Also, call $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}} \in \mathfrak{A}_L$ positive semi-definite if

$$\text{Re} \langle A\zeta, \bar{\zeta} \rangle := \text{Re} (a_{rs}^{\alpha\beta} \zeta_s^\beta \bar{\zeta}_\alpha^r) \geq 0, \quad \forall \zeta = (\zeta_\alpha^r)_{\substack{1 \leq r \leq 2 \\ 1 \leq \alpha \leq M}} \in \mathbb{C}^{2 \times M}. \quad (1.31)$$

For example, consider the 2×2 Lamé system in \mathbb{R}^2 , given by

$$L_{\mu, \lambda} u := \mu \Delta u + (\mu + \lambda) \nabla \text{div} u \quad (1.32)$$

where $u = (u_\beta)_{1 \leq \beta \leq 2}$ is a \mathcal{C}^2 vector valued function, and parameters $\mu > 0$ and $\lambda \in \mathbb{R}$ are fixed such that $\mu + \lambda \geq 0$. Then, for each $r \in \mathbb{C}$ set

$$a_{jk}^{\alpha\beta}(r) := \mu \delta_{jk} \delta_{\alpha\beta} + (\mu + \lambda - r) \delta_{j\alpha} \delta_{k\beta} + r \delta_{j\beta} \delta_{k\alpha}, \quad (1.33)$$

for $1 \leq j, k \leq 2$ and $1 \leq \alpha, \beta \leq 2$, with δ denoting the Kronecker symbol. Then, if

$$A(r) := (a_{jk}^{\alpha\beta}(r))_{\substack{1 \leq j, k \leq 2 \\ 1 \leq \alpha, \beta \leq 2}} \quad \text{where } a_{jk}^{\alpha\beta}(r) \text{ is as in (1.33)}, \quad (1.34)$$

there holds that

$$A(r) \in \mathfrak{A}_{L_{\mu,\lambda}} \quad \text{for each } r \in \mathbb{C}. \quad (1.35)$$

Next, fix $r \in \mathbb{C}$ and $\zeta = (\zeta_\alpha^r)_{\substack{1 \leq r \leq 2 \\ 1 \leq \alpha \leq 2}} \in \mathbb{C}^{2 \times 2}$, and write

$$\begin{aligned} \operatorname{Re}(\langle A(r)\zeta, \bar{\zeta} \rangle) &= \operatorname{Re} \left(a_{jk}^{\alpha\beta}(r) \zeta_k^\beta \bar{\zeta}_\alpha^j \right) \\ &= \mu |\zeta|^2 + \operatorname{Re} \left((\mu + \lambda - r) \delta_{j\alpha} \delta_{k\beta} (\zeta_\alpha^j \bar{\zeta}_\beta^k) \right) + \operatorname{Re} \left(r \delta_{j\beta} \delta_{k\alpha} (\zeta_\alpha^j \bar{\zeta}_\beta^k) \right) \\ &= \mu |\zeta|^2 + \sum_{\alpha,\beta=1}^2 \left[(\mu + \lambda) \operatorname{Re} \left(\zeta_\alpha^\alpha \bar{\zeta}_\beta^\beta \right) - \operatorname{Re} \left(r \zeta_\alpha^\alpha \bar{\zeta}_\beta^\beta \right) + \operatorname{Re} \left(r \zeta_\alpha^\beta \bar{\zeta}_\beta^\alpha \right) \right] \\ &= \mu |\zeta|^2 + (\mu + \lambda - \operatorname{Re}(r)) |\operatorname{Tr}(\zeta)|^2 + \operatorname{Re} (r \cdot \operatorname{Tr} (\zeta \cdot \bar{\zeta})) \\ &\geq \mu |\zeta|^2 - (\mu + \lambda - \operatorname{Re}(r))^- |\operatorname{Tr}(\zeta)|^2 - |r| |\operatorname{Tr} (\zeta \cdot \bar{\zeta})| \end{aligned} \quad (1.36)$$

where $x^- := -\min\{0, x\}$ is the negative part of $x \in \mathbb{R}$, while Tr denotes the trace operator of matrices in $\mathbb{C}^{2 \times 2}$, and \cdot denotes matrix multiplication. Note that

$$|\operatorname{Tr}(\zeta)| = \langle \zeta, I \rangle_F \leq \|\zeta\|_F \|I\|_F = \sqrt{2} |\zeta|, \quad (1.37)$$

where $\langle \cdot, \cdot \rangle_F$ is the Frobenius inner product and $\|\cdot\|_F$ is the Frobenius norm, defined by setting $\langle A, B \rangle_F := \operatorname{Tr}(\overline{A^\top} B)$ and $\|A\|_F := \sqrt{\langle A, A \rangle_F}$, respectively, for all $A, B \in \mathbb{C}^{2 \times 2}$. Here the superscript \top indicates transposition. Combining this with (1.36), we obtain

$$\operatorname{Re}(\langle A(r)\zeta, \bar{\zeta} \rangle) \geq \mu |\zeta|^2 - 2(\mu + \lambda - \operatorname{Re}(r))^- |\zeta|^2 - |r| |\operatorname{Tr} (\zeta \cdot \bar{\zeta})|. \quad (1.38)$$

Moreover, using the triangle inequality and the fact that $2ab \leq a^2 + b^2$ for real numbers a, b ,

$$|\operatorname{Tr} (\zeta \cdot \bar{\zeta})| \leq \sum_{\alpha,\beta=1}^2 |\zeta_\alpha^\beta| |\zeta_\beta^\alpha| \leq \frac{1}{2} \sum_{\alpha,\beta=1}^2 (|\zeta_\alpha^\beta|^2 + |\zeta_\beta^\alpha|^2) = |\zeta|^2. \quad (1.39)$$

Using this and (1.38) we arrive at

$$\operatorname{Re}(\langle A(r)\zeta, \bar{\zeta} \rangle) \geq (\mu - 2(\mu + \lambda - \operatorname{Re}(r))^- - |r|) |\zeta|^2. \quad (1.40)$$

Consequently,

$$\begin{aligned} & \text{the coefficient tensor } A(r) \text{ from (1.33)-(1.34)} \\ & \text{is positive definite whenever } (\mu - 2(\mu + \lambda - \operatorname{Re}(r))^- - |r|) > 0. \end{aligned} \quad (1.41)$$

Note that, (1.41) is satisfied whenever $\lambda + \mu - \operatorname{Re}(r) \geq 0$ and $|r| < \mu$, and that in this case the coefficient tensor $A(r)$ is positive definite with constant $\kappa = \mu - |r|$.

Definition 1.13. *Assume that $M \in \mathbb{N}$ and fix a second-order $M \times M$ system L in \mathbb{R}^2 with constant, complex coefficients. Let $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq r,s \leq 2 \\ 1 \leq \alpha,\beta \leq M}} \in \mathfrak{A}_L$. The coefficient tensor A is called Legendre-Hadamard elliptic provided there exists a real number $\kappa > 0$ such that the following condition is satisfied:*

$$\begin{aligned} & \operatorname{Re}(a_{rs}^{\alpha\beta} \xi_r \xi_s \bar{\eta}_\alpha \eta_\beta) \geq \kappa |\xi|^2 |\eta|^2, \\ & \forall \xi = (\xi_r)_{1 \leq r \leq 2} \in \mathbb{R}^2 \text{ and } \forall \eta = (\eta_\alpha)_{1 \leq \alpha \leq M} \in \mathbb{C}^M. \end{aligned} \quad (1.42)$$

Moreover, the coefficient tensor A is called weakly elliptic provided

$$\det [(a_{rs}^{\alpha\beta} \xi_r \xi_s)_{1 \leq \alpha,\beta \leq M}] \neq 0, \quad \forall \xi = (\xi_r)_{1 \leq r \leq 2} \in \mathbb{R}^2 \setminus \{0\}. \quad (1.43)$$

Note that the positive definiteness condition implies the Legendre-Hadamard ellipticity condition. This is readily seen by observing that, given any vectors $\xi = (\xi_r)_{1 \leq r \leq 2} \in \mathbb{R}^2$ and $\eta = (\eta_\alpha)_{1 \leq \alpha \leq M} \in \mathbb{C}^M$, if we consider a complex-valued matrix $\zeta = (\zeta_\alpha^r)_{\substack{1 \leq r \leq 2 \\ 1 \leq \alpha \leq M}} \in \mathbb{C}^{2 \times M}$ with components $\zeta_\alpha^r := \xi_r \eta_\alpha$ then $|\zeta| = |\xi| |\eta|$.

Next let us also notice that the Legendre-Hadamard ellipticity implies weakly ellipticity. Indeed, if a coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq r,s \leq 2 \\ 1 \leq \alpha,\beta \leq M}}$ is Legendre-Hadamard elliptic, then (1.42) implies that for each $\xi = (\xi_r)_{1 \leq r \leq 2} \in \mathbb{R}^2 \setminus \{0\}$, the characteristic matrix $L(\xi) := (a_{rs}^{\alpha\beta} \xi_r \xi_s)_{1 \leq \alpha,\beta \leq M}$ satisfies

$$\operatorname{Re}(\langle L(\xi)\eta, \bar{\eta} \rangle) \geq \kappa |\xi|^2 |\eta|^2, \quad \forall \eta = (\eta_\alpha)_{1 \leq \alpha \leq M} \in \mathbb{C}^M. \quad (1.44)$$

This forces for each $\xi \in \mathbb{R}^2 \setminus \{0\}$, $L(\xi)$ is positive definite matrix which further implies that A is weakly elliptic. In summary, for every $A \in \mathfrak{A}_L$ we have:

$$\begin{aligned} A \text{ is positive definite} & \implies A \text{ is Legendre-Hadamard elliptic} \\ & \implies A \text{ is weakly elliptic.} \end{aligned} \quad (1.45)$$

Example 1.14. Let $r \in \mathbb{C}$ and $A(r)$ be as in (1.34), the coefficient tensor of the Lamé system $L_{\mu,\lambda}$ from (1.32). Then, straightforward calculations yield that for each $\xi = (\xi_r)_{1 \leq r \leq 2} \in \mathbb{R}^2$ and each $\eta = (\eta_\alpha)_{1 \leq \alpha \leq 2} \in \mathbb{C}^2$, there holds

$$\begin{aligned} a_{jk}^{\alpha\beta}(r)\xi_j\xi_k\overline{\eta_\alpha}\eta_\beta &= \mu|\xi|^2|\eta|^2 + (\mu + \lambda)\langle\xi, \eta\rangle\langle\xi, \overline{\eta}\rangle \\ &= \mu|\xi|^2|\eta|^2 + (\mu + \lambda)|\langle\xi, \eta\rangle|^2. \end{aligned} \quad (1.46)$$

In particular $a_{jk}^{\alpha\beta}(r)\xi_j\xi_k\overline{\eta_\alpha}\eta_\beta \in \mathbb{R}$ and thus

$$\operatorname{Re}(a_{jk}^{\alpha\beta}(r)\xi_j\xi_k\overline{\eta_\alpha}\eta_\beta) = a_{jk}^{\alpha\beta}(r)\xi_j\xi_k\overline{\eta_\alpha}\eta_\beta \geq \mu|\xi|^2|\eta|^2. \quad (1.47)$$

This implies that the coefficient tensor $A(r)$ from (1.34) satisfies the Legendre-Hadamard ellipticity condition for all $r \in \mathbb{C}$. From (1.45), $A(r)$ is also weakly elliptic for all $r \in \mathbb{C}$.

The theorem below states the properties of a special fundamental solution associated with a second-order weakly elliptic differential operator L in \mathbb{R}^2 . These properties can be found in [30] (cf. also [20]) in \mathbb{R}^n for $n \geq 2$.

Theorem 1.15. *Assume that L is an $M \times M$ weakly elliptic, second-order system in \mathbb{R}^2 , with complex constant coefficients as in (1.26). Then there exists a matrix $E = (E_{\alpha\beta})_{1 \leq \alpha, \beta \leq M}$ whose entries are tempered distributions in \mathbb{R}^2 and such that the following properties hold:*

- (a) *For each $\alpha, \beta \in \{1, \dots, M\}$, there holds that $E_{\alpha\beta} \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{0\})$ and $E_{\alpha\beta}(-x) = E_{\alpha\beta}(x)$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.*
- (b) *If δ_y stands for Dirac's delta distribution with mass at y then for each indices $\alpha, \beta \in \{1, \dots, M\}$, and every $x, y \in \mathbb{R}^2$,*

$$\partial_{x_r} a_{rs}^{\alpha\gamma} \partial_{x_s} [E_{\gamma\beta}(x - y)] = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \delta_y(x) & \text{if } \alpha = \beta. \end{cases} \quad (1.48)$$

- (c) *For each $\alpha, \beta \in \{1, \dots, M\}$, one has*

$$E_{\alpha\beta}(x) = \Phi_{\alpha\beta}(x) + c_{\alpha\beta} \ln |x|, \quad \forall x \in \mathbb{R}^2 \setminus \{0\}, \quad (1.49)$$

where $(c_{\alpha\beta})_{1 \leq \alpha, \beta \leq M} \in \mathbb{C}^{M \times M}$ and $\Phi_{\alpha\beta} \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{0\})$ is a homogeneous function of degree 0.

(d) For each $\gamma \in \mathbb{N}_0^2$ there exists a finite constant $C_\gamma > 0$ such that for each $x \in \mathbb{R}^2 \setminus \{0\}$ there holds

$$|\partial^\gamma E(x)| \leq \begin{cases} C_\gamma \cdot |x|^{-|\gamma|} & \text{if } |\gamma| > 0, \\ C_0(1 + |\ln|x||) & \text{if } |\gamma| = 0. \end{cases} \quad (1.50)$$

(e) When restricted to $\mathbb{R}^2 \setminus \{0\}$, the (matrix-valued) distribution \widehat{E} is a \mathcal{C}^∞ function and, with “hat” denoting the Fourier transform in \mathbb{R}^2 ,

$$\widehat{E}(\xi) = - \left[(\xi_r \xi_s a_{rs}^{\alpha\beta})_{1 \leq \alpha, \beta \leq M} \right]^{-1} \quad \text{for each } \xi \in \mathbb{R}^2 \setminus \{0\}. \quad (1.51)$$

(f) One can assign to each elliptic differential operator L as in (1.26) a fundamental solution E_L which satisfies (a)-(e) above and, in addition, $(E_L)^\top = E_{L^\top}$, where the superscript \top denotes transposition.

(g) In the particular case $M = 1$, i.e., in the situation when $L = \operatorname{div} A \nabla$ for some matrix $A = (a_{rs})_{1 \leq r, s \leq 2} \in \mathbb{C}^{2 \times 2}$, an explicit formula for the fundamental solution E of L is

$$E(x) = \frac{1}{4\pi \sqrt{\det(A_{\text{sym}})}} \log(\langle (A_{\text{sym}})^{-1} x, x \rangle) \quad (1.52)$$

for $x \in \mathbb{R}^2 \setminus \{0\}$. Here, $A_{\text{sym}} := \frac{1}{2}(A + A^\top)$ stands for the symmetric part of A and \log denotes the principal branch of the complex logarithm function (defined by the requirement that $z^t = e^{t \log z}$ holds for every $z \in \mathbb{C} \setminus (-\infty, 0]$ and every $t \in \mathbb{R}$).

The following result from [30], describes equivalent properties for coefficient tensors of a given second-order weakly elliptic system. Throughout $\mathcal{S}'(\mathbb{R}^2)$ will stand for the space of tempered distributions in \mathbb{R}^2 .

Proposition 1.16. *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 , and consider the inverse of the*

characteristic matrix of L , i.e., introduce the matrix-valued function defined for each $\xi \in \mathbb{R}^2 \setminus \{0\}$ as

$$\mathbf{E}(\xi) := (\mathbf{E}_{\gamma\beta}(\xi))_{1 \leq \gamma, \beta \leq M} := [L(\xi)]^{-1} \in \mathbb{C}^{M \times M} \quad (1.53)$$

Also, denote by $E = (E_{\alpha\beta})_{1 \leq \alpha, \beta \leq M}$ the fundamental solution associated with the given system L . Then for each coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}} \in \mathfrak{A}_L$ the following conditions are equivalent:

(a) For each $s, s' \in \{1, 2\}$ and each $\alpha, \gamma \in \{1, \dots, M\}$ there holds

$$(x_{s'} a_{rs}^{\beta\alpha} - x_s a_{rs'}^{\beta\alpha})(\partial_r E_{\gamma\beta})(x) = 0 \text{ for all } x = (x_j)_{1 \leq j \leq 2} \in \mathbb{R}^2 \setminus \{0\}. \quad (1.54)$$

(b) For each $s, s' \in \{1, 2\}$ and each $\alpha, \gamma \in \{1, \dots, M\}$ there holds

$$(x_{s'} a_{rs}^{\beta\alpha} - x_s a_{rs'}^{\beta\alpha})(\partial_r E_{\gamma\beta})(x) = 0 \text{ in } \mathcal{S}'(\mathbb{R}^2). \quad (1.55)$$

(c) For each $s, s' \in \{1, 2\}$ and each $\alpha, \gamma \in \{1, \dots, M\}$ one has

$$\left[a_{rs}^{\beta\alpha} \partial_{\xi_{s'}} - a_{rs'}^{\beta\alpha} \partial_{\xi_s} \right] [\xi_r \mathbf{E}_{\gamma\beta}(\xi)] = 0 \text{ in } \mathcal{S}'(\mathbb{R}^2). \quad (1.56)$$

(d) For each $s, s' \in \{1, 2\}$ and each $\alpha, \gamma \in \{1, \dots, M\}$ one has

$$(a_{s's}^{\beta\alpha} - a_{ss'}^{\beta\alpha} + \xi_r a_{rs}^{\beta\alpha} \partial_{\xi_{s'}} - \xi_r a_{rs'}^{\beta\alpha} \partial_{\xi_s}) \mathbf{E}_{\gamma\beta}(\xi) = 0 \text{ for all } \xi \in \mathbb{R}^2 \setminus \{0\} \quad (1.57)$$

and also

$$\int_{S^1} (a_{rs}^{\beta\alpha} \xi_{s'} - a_{rs'}^{\beta\alpha} \xi_s) \xi_r \mathbf{E}_{\gamma\beta}(\xi) d\mathcal{H}^1(\xi) = 0. \quad (1.58)$$

(e) One has

$$\xi_r \xi_s \left[a_{rs'}^{\beta\alpha} (a_{sj}^{\lambda\mu} + a_{js}^{\lambda\mu}) - a_{rs}^{\beta\alpha} (a_{s'j}^{\lambda\mu} + a_{js'}^{\lambda\mu}) \right] \mathbf{E}_{\mu\beta}(\xi) + a_{ss'}^{\lambda\alpha} - a_{s's}^{\lambda\alpha} = 0 \quad (1.59)$$

for all $\xi \in S^1$, all $s, s' \in \{1, 2\}$, and all $\alpha, \lambda \in \{1, \dots, M\}$,

with the cancellation condition

$$\int_{S^1} (a_{rs}^{\beta\alpha} \xi_{s'} - a_{rs'}^{\beta\alpha} \xi_s) \xi_r \mathbf{E}_{\lambda\beta}(\xi) d\mathcal{H}^1(\xi) = 0 \quad (1.60)$$

for all $s, s' \in \{1, 2\}$ and $\alpha, \lambda \in \{1, \dots, M\}$.

(f) For each $\xi \in S^1$ and each $\alpha, \lambda \in \{1, \dots, M\}$,

$$\begin{aligned} & \text{the expression } \left(a_{sj}^{\lambda\mu} + a_{js}^{\lambda\mu} \right) \mathbf{E}_{\mu\beta}(\xi) \xi_s \xi_r a_{rs'}^{\beta\alpha} - a_{s's}^{\lambda\alpha} \\ & \text{is symmetric in the indices } s, s' \in \{1, 2\}, \end{aligned} \quad (1.61)$$

with the condition that for each $\alpha, \lambda \in \{1, \dots, M\}$

$$\begin{aligned} & \text{the expression } \int_{S^1} a_{rs}^{\beta\alpha} \xi_{s'} \xi_r \mathbf{E}_{\lambda\beta}(\xi) d\mathcal{H}^1(\xi) \\ & \text{is symmetric in the indices } s, s' \in \{1, 2\}. \end{aligned} \quad (1.62)$$

(g) There exists a matrix-valued function

$$k = \{k_{\gamma\alpha}\}_{1 \leq \gamma, \alpha \leq M} : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{C}^{M \times M} \quad (1.63)$$

with the property that for each $\gamma, \alpha \in \{1, \dots, M\}$ and $s \in \{1, 2\}$ one has

$$a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x) = x_s k_{\gamma\alpha}(x) \text{ for all } x \in \mathbb{R}^2 \setminus \{0\}. \quad (1.64)$$

(h) For each indices $\alpha, \gamma \in \{1, \dots, M\}$ and each vector $\xi = (\xi_s)_{1 \leq s \leq 2} \in \mathbb{R}^2$ one has

$$a_{rs}^{\beta\alpha} \xi_s (\partial_r E_{\gamma\beta})(x) = 0 \text{ for each } x \in \langle \xi \rangle^T \setminus \{0\}. \quad (1.65)$$

We also recall the following definition from [30],

Definition 1.17. Given a second-order, weakly elliptic, homogeneous, $M \times M$ system L in \mathbb{R}^2 , with constant complex coefficients, call

$$A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}} \in \mathfrak{A}_L \quad (1.66)$$

a distinguished coefficient tensor for the system L provided any of the condition (a)-(h) holds. Also, denote by $\mathfrak{A}_L^{\text{dis}}$ the family of such distinguished coefficient tensors for L . That is

$$\begin{aligned} \mathfrak{A}_L^{\text{dis}} := & \left\{ A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq r, s \leq 2 \\ 1 \leq \alpha, \beta \leq M}} \in \mathfrak{A}_L : \text{conditions (1.59)-(1.60) hold} \right. \\ & \left. \text{for each } s, s' \in \{1, 2\} \text{ and } \alpha, \lambda \in \{1, \dots, M\} \right\}. \end{aligned} \quad (1.67)$$

The following two propositions, again from [30], address the existence and uniqueness of distinguished coefficient tensors under different type of ellipticity assumptions.

Proposition 1.18. *Let $M \in \mathbb{N}$ and consider a homogeneous, second-order, constant complex coefficient, $M \times M$ system L in \mathbb{R}^2 which satisfies the Legendre-Hadamard ellipticity condition. Then $\#\mathfrak{A}_L^{dis} \leq 1$, i.e. \mathfrak{A}_L^{dis} is either empty or a singleton.*

Proposition 1.19. *Let $M \in \mathbb{N}$ and consider a weakly elliptic, homogeneous, second-order, constant complex coefficient, $M \times M$ system L in \mathbb{R}^2 . Then, with \top denoting transposition,*

$$\mathfrak{A}_{L^\top}^{dis} \neq \emptyset \implies \#\mathfrak{A}_L^{dis} \leq 1. \quad (1.68)$$

Let us consider the case of the Laplacian in \mathbb{R}^2 . We may write

$$\Delta = a_{ij}(r)\partial_i\partial_j, \quad (1.69)$$

where for any $r \in \mathbb{C}$, the coefficient tensor $A_\Delta(r) := (a_{ij}(r))_{1 \leq i, j \leq 2}$ is given by

$$A_\Delta(r) := \begin{pmatrix} 1 & r \\ -r & 1 \end{pmatrix}. \quad (1.70)$$

Then $A_\Delta(r)$ is positive definite for any $r \in \mathbb{C}$. Based on direct calculations, for each $k, k' \in \{1, 2\}$ and for each $x = (x_j)_{1 \leq j \leq 2} \in \mathbb{R}^2 \setminus \{0\}$ there holds

$$\begin{aligned} & (x_{k'}a_{jk}(r) - x_k a_{jk'}(r))(\partial_r E)(x) \\ &= (x_2 - rx_1)\frac{1}{2\pi}\frac{x_1}{|x|^2} + (-rx_2 - x_1)\frac{1}{2\pi}\frac{x_2}{|x|^2} \\ &= -\frac{r}{2\pi}, \end{aligned} \quad (1.71)$$

where E is the fundamental solution for Δ in \mathbb{R}^2 . According to (1.54),

$$A_\Delta(r) \text{ is a distinguished coefficient tensor for } \Delta \text{ if and only if } r = 0. \quad (1.72)$$

As another example, consider the weakly elliptic complex Lamé system in \mathbb{R}^2 with Lamé moduli $\mu, \lambda \in \mathbb{C}$, i.e.

$$L_{\mu,\lambda} := \mu\Delta + (\mu + \lambda)\nabla \operatorname{div} \quad \text{with } \mu \neq 0 \text{ and } 2\mu + \lambda \neq 0. \quad (1.73)$$

As mentioned in (1.33), the coefficient tensor $A_{L_{\mu,\lambda}}(r) = (a_{jk}^{\alpha\beta}(r))_{\substack{1 \leq j,k \leq 2 \\ 1 \leq \alpha,\beta \leq 2}}$ is given by

$$a_{jk}^{\alpha\beta}(r) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu + \lambda - r)\delta_{j\alpha}\delta_{k\beta} + r\delta_{j\beta}\delta_{k\alpha}, \quad (1.74)$$

for any $r \in \mathbb{C}$. According to [30],

$$\mathfrak{A}_{L_{\mu,\lambda}}^{\operatorname{dis}} \neq \emptyset \iff 3\mu + \lambda \neq 0. \quad (1.75)$$

Moreover, if $3\mu + \lambda \neq 0$, then

$$A_{L_{\mu,\lambda}}(r) \in \mathfrak{A}_{L_{\mu,\lambda}}^{\operatorname{dis}} \iff r = \frac{\mu(\mu + \lambda)}{3\mu + \lambda}. \quad (1.76)$$

CHAPTER 2

The Poly-Cauchy Operator in UR Domains in the Complex Plane

In this chapter we establish a higher-order analogue of the existing theory for the classical Cauchy operator, in which the salient role of the Cauchy-Riemann operator $\bar{\partial}$ is now played by natural powers of this operator. This analysis is carried out in a very general class of domains and a central role will be played by integral representation formulas, jump relations and higher-order Fatou-type theorems.

2.1 The classical Cauchy operator

In this section, we recall the classical Cauchy singular integral operator and review its important properties which will be then generalized in the higher-order setting throughout this dissertation. We shall continue to work in $\mathbb{R}^2 \equiv \mathbb{C}$ in the class of UR domains introduced in Definition 1.5.

The first order of business is to recall the complex arc-length on the geometric measure theoretic boundary of a set $\Omega \subseteq \mathbb{R}^2$ of locally finite perimeter,

defined as

$$d\zeta := i\nu d\sigma_* = i\nu d\sigma|_{\partial_*\Omega}. \quad (2.1)$$

In addition, bring in the operators $\bar{\partial}$ and ∂ which are, respectively, the Cauchy-Riemann operator and its complex conjugate, given by

$$\partial_{\bar{z}} = \bar{\partial}_z = \bar{\partial} := \frac{1}{2}(\partial_x + i\partial_y) \quad \text{and} \quad \partial_z = \partial := \frac{1}{2}(\partial_x - i\partial_y). \quad (2.2)$$

For an open set of locally finite perimeter $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$, define the action of the boundary-to-domain Cauchy operator on any function $f \in L^1\left(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\right)$ at any point $z \in \Omega$ as

$$(\mathcal{C}f)(z) := \frac{1}{2\pi i} \int_{\partial_*\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi} \int_{\partial_*\Omega} \frac{f(\zeta)}{\zeta - z} \nu(\zeta) d\sigma(\zeta), \quad (2.3)$$

where ν denotes the GMT outward unit normal to Ω . In the same geometric setting, define the action of the boundary-to-boundary Cauchy operator on each function $f \in L^1\left(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\right)$ as

$$(Cf)(z) := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial_*\Omega \\ |\zeta - z| > \varepsilon}} \frac{f(\zeta)}{\zeta - z} d\zeta = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{\substack{\zeta \in \partial_*\Omega \\ |\zeta - z| > \varepsilon}} \frac{f(\zeta)}{\zeta - z} \nu(\zeta) d\sigma(\zeta), \quad (2.4)$$

for σ -a.e. point $z \in \partial_*\Omega$. Note that the integrals in (2.3) are absolutely convergent, so $\mathcal{C}f$ is well defined for each function $f \in L^1\left(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\right)$ and, in fact, $\bar{\partial}(\mathcal{C}f) \equiv 0$ in Ω . Thus, if $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions in Ω , we have

$$\mathcal{C}f \in \mathcal{O}(\Omega) \quad \text{for each } f \in L^1\left(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\right). \quad (2.5)$$

Since $\partial_*\Omega$ is countably rectifiable (according to a classical result of De Giorgi-Federer), it turns out that the boundary-to-boundary Cauchy operator is also well defined when acting on functions from the weighted Lebesgue space $L^1\left(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\right)$. See [28] for details, where more information on the size, regularity, and boundary behavior of those objects may be found.

The result below, found in [28] (cf. also [25]), deals with the nontangential behavior of the Cauchy operator and related matters.

Theorem 2.1. *Assume $\Omega \subseteq \mathbb{R}^2$ is a UR domain. Fix an aperture parameter $\kappa \in (0, \infty)$ and an integrability exponent $p \in (1, \infty)$. Then the following properties hold:*

- (1) [Fatou Theorem and Integral Representation Formula] *For each holomorphic function $u \in \mathcal{O}(\Omega)$ with $\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma)$ the κ -nontangential trace $\left(u\Big|_{\partial\Omega}^{\kappa\text{-n.t.}}\right)(z)$ is meaningfully defined at σ -a.e. point $z \in \partial\Omega$, the function $u\Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ belongs to $L^p(\partial\Omega, \sigma)$, and (also assuming $u(z) = o(1)$ as $z \rightarrow \infty$ if Ω is an exterior domain) one has*

$$u = \mathcal{C}\left(u\Big|_{\partial\Omega}^{\kappa\text{-n.t.}}\right) \text{ in } \Omega. \quad (2.6)$$

- (2) [Jump Relation] *If I denotes the identity operator then, for $f \in L^p(\partial\Omega, \sigma)$,*

$$\left(\mathcal{C}f\right)\Big|_{\partial\Omega}^{\kappa\text{-n.t.}} = \left(\frac{1}{2}I + C\right)f \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \quad (2.7)$$

- (3) [Size Estimate] *For each $f \in L^p(\partial\Omega, \sigma)$ one has*

$$\|\mathcal{N}_\kappa(\mathcal{C}f)\|_{L^p(\partial\Omega, \sigma)} \lesssim \|f\|_{L^p(\partial\Omega, \sigma)}. \quad (2.8)$$

- (4) [Boundedness and Involution Property] *The operator C is bounded from the space $L^p(\partial\Omega, \sigma)$ into itself, that is, for any $f \in L^p(\partial\Omega, \sigma)$ one has $Cf \in L^p(\partial\Omega, \sigma)$ and*

$$\|Cf\|_{L^p(\partial\Omega, \sigma)} \lesssim \|f\|_{L^p(\partial\Omega, \sigma)}. \quad (2.9)$$

In addition,

$$C^2 = \frac{1}{4}I \text{ as operators on } L^p(\partial\Omega, \sigma). \quad (2.10)$$

- (5) [Regularity] *For each $f \in L_1^p(\partial\Omega, \sigma)$ one has*

$$\|\mathcal{N}_\kappa(\nabla\mathcal{C}f)\|_{L^p(\partial\Omega, \sigma)} \lesssim \|f\|_{L_1^p(\partial\Omega, \sigma)}. \quad (2.11)$$

Also, the boundary-to-boundary Cauchy operator C is continuous from the boundary Sobolev space $L_1^p(\partial\Omega, \sigma)$ into itself, i.e., for each function $f \in L_1^p(\partial\Omega, \sigma)$ one has $Cf \in L_1^p(\partial\Omega, \sigma)$ and

$$\|Cf\|_{L_1^p(\partial\Omega, \sigma)} \lesssim \|f\|_{L_1^p(\partial\Omega, \sigma)}. \quad (2.12)$$

Next, we turn our attention to holomorphic Hardy spaces and the behavior of the Cauchy operators in this context. Assume $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is a UR domain. In particular, (1.5) implies that ν , the geometric measure theoretic outward unit normal vector to Ω , is well-defined σ -a.e. on $\partial\Omega$.

Given a UR domain $\Omega \subseteq \mathbb{C}$, recall Ω_- from (1.6). Fix $\kappa \in (0, \infty)$ and $p \in (1, \infty)$ and recall the domain Hardy spaces defined by setting

$$\mathcal{H}^p(\Omega_{\pm}) := \left\{ u \in \mathcal{O}(\Omega_{\pm}) : \mathcal{N}_{\kappa} u \in L^p(\partial\Omega, \sigma), \text{ and also } u(z) = O(|z|^{-1}) \right. \\ \left. \text{as } |z| \rightarrow \infty \text{ in the case when } \Omega_{\pm} \text{ is an exterior domain} \right\} \quad (2.13)$$

equipped with the norm $\|\mathcal{N}_{\kappa} u\|_{L^p(\partial\Omega, \sigma)}$. Note that (cf. [28]) the definition of $\mathcal{H}^p(\Omega_{\pm})$ does not depend on the aperture parameter $\kappa \in (0, \infty)$.

We recall next the Fatou Theorem from [28].

Theorem 2.2. *Let $\Omega \subseteq \mathbb{C}$ be a UR domain and fix $p \in (1, \infty)$. For each $u \in \mathcal{H}^p(\Omega_{\pm})$ there exists $u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ at σ -a.e. point on $\partial\Omega$ and $u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \in L^p(\partial\Omega, \sigma)$.*

Going further, define the boundary Hardy spaces by setting

$$\mathcal{H}_{\pm}^p(\partial\Omega, \sigma) := \left\{ f \in L^p(\partial\Omega, \sigma) : f = u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ for some } u \in \mathcal{H}^p(\Omega_{\pm}) \right\} \quad (2.14)$$

Thanks to Theorem 2.2, it follows that the boundary Hardy spaces $\mathcal{H}_{\pm}^p(\partial\Omega, \sigma)$ are well-defined subspaces of $L^p(\partial\Omega, \sigma)$. According to the Calderón Decomposition Theorem (see [29]), $\mathcal{H}_{\pm}^p(\partial\Omega, \sigma)$ are closed subspaces of $L^p(\partial\Omega, \sigma)$ and

$$L^p(\partial\Omega, \sigma) = \mathcal{H}_{+}^p(\partial\Omega, \sigma) \oplus \mathcal{H}_{-}^p(\partial\Omega, \sigma). \quad (2.15)$$

Introduce next

$$P_{\pm} := \frac{1}{2}I \pm C, \quad (2.16)$$

and note that since $C^2 = \frac{1}{4}I$ on $L^p(\partial\Omega, \sigma)$, there holds that

$$P_{+} + P_{-} = I \quad \text{and} \quad P_{\pm}^2 = P_{\pm} \quad \text{on} \quad L^p(\partial\Omega, \sigma), \quad (2.17)$$

i.e., P_{\pm} are complementary projections of $L^p(\partial\Omega, \sigma)$. Moreover,

$$\text{Im } P_{\pm} = \mathcal{H}_{\pm}^p(\partial\Omega, \sigma) \quad \text{and} \quad \text{Ker } P_{\pm} = \mathcal{H}_{\mp}^p(\partial\Omega, \sigma). \quad (2.18)$$

In addition, $\mathcal{C}_\pm : L^p(\partial\Omega, \sigma) \rightarrow \mathcal{H}^p(\Omega_\pm)$ are well-defined, linear, bounded, and surjective where \mathcal{C}_\pm are versions of \mathcal{C} relative to Ω_\pm . Moreover, the boundary-to-domain Cauchy operators $\mathcal{C} : \mathcal{H}_\pm^p(\partial\Omega, \sigma) \rightarrow \mathcal{H}^p(\Omega_\pm)$ from the boundary Hardy spaces into the domain Hardy spaces are isomorphisms.

We next introduce holomorphic regular Hardy spaces (with integrability exponent $p \in (1, \infty)$ and parameter $\kappa \in (0, \infty)$ fixed, as at the beginning of this discussion)

$$\mathcal{H}_1^p(\Omega_\pm) := \left\{ u \in \mathcal{O}(\Omega) : \mathcal{N}_\kappa u, \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma) \text{ and also } \right. \\ \left. u(z) = O(|z|^{-1}) \text{ as } |z| \rightarrow \infty \text{ when } \Omega_\pm \text{ is an exterior domain} \right\} \quad (2.19)$$

equipped with the norm $\|\mathcal{N}_\kappa\|_{L^p(\partial\Omega, \sigma)} + \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega, \sigma)}$. Also, the regular analogue of boundary holomorphic Hardy space is defined by

$$\mathcal{H}_{1,\pm}^p(\partial\Omega, \sigma) := \left\{ f \in L_1^p(\partial\Omega, \sigma) : f = u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ for some } u \in \mathcal{H}_1^p(\Omega_\pm) \right\}. \quad (2.20)$$

As stated in [29], the spaces $\mathcal{H}_{1,\pm}^p(\partial\Omega, \sigma)$ are well-defined closed subspaces of $L_1^p(\partial\Omega, \sigma)$. Then this brand of Hardy spaces satisfies similar properties of those enjoyed by the usual Hardy spaces.

2.2 The fundamental solution of $\bar{\partial}^m$

Throughout this section fix $m \in \mathbb{N}$. Let us identify each $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$ with the pair $(x, y) \in \mathbb{R}^2$. Then, using (2.2) we may write

$$\Delta = 4\bar{\partial}\partial = 4\partial\bar{\partial} \quad \text{and} \quad \Delta^m = 4^m \bar{\partial}^m \partial^m = 4^m \partial^m \bar{\partial}^m. \quad (2.21)$$

Let Γ_{Δ^m} be the canonical, radial fundamental solution for Δ^m in $\mathbb{R}^2 \equiv \mathbb{C}$ from [[22], Theorem 7.28], for the choice $n = 2$. That is, for each point $z \in \mathbb{C} \setminus \{0\}$

$$\Gamma_{\Delta^m}(z) := \frac{1}{2\pi \cdot 4^{m-1} [(m-1)!]^2} \cdot |z|^{2m-2} \cdot \ln |z|. \quad (2.22)$$

Note that $|z|^{2m-2} \cdot \ln |z|$ can be written as $(\bar{z})^{m-1} z^{m-1} \cdot \frac{1}{2} \ln(z \cdot \bar{z})$. If we define $E_m : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ by setting

$$E_m(z) := 4^m \partial^m \Gamma_{\Delta^m}(z), \quad \forall z \in \mathbb{C} \setminus \{0\}. \quad (2.23)$$

Then, in the sense of distributions in \mathbb{R}^2 we may write

$$\bar{\partial}^m E_m = 4^m \bar{\partial}^m \partial^m \Gamma_{\Delta^m} = \Delta^m \Gamma_{\Delta^m} = \delta, \quad (2.24)$$

where δ stands for the Dirac delta distribution with mass at the origin, showing that that

$$E_m \text{ in (2.23) is a fundamental solution of the operator } \bar{\partial}^m \text{ in } \mathbb{C}. \quad (2.25)$$

Theorem 2.3. *Fix $m \in \mathbb{N}$. Then $E_m : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ defined in (2.23) satisfies*

$$E_m(z) = \frac{1}{\pi(m-1)!} \frac{(\bar{z})^{m-1}}{z}, \quad \text{for } z \in \mathbb{C} \setminus \{0\}. \quad (2.26)$$

Proof. Fix $m \in \mathbb{N}$, $z \in \mathbb{C} \setminus \{0\}$, and introduce the constant

$$c_m := \frac{2}{\pi[(m-1)!]^2}. \quad (2.27)$$

Since $\partial \bar{z} = 0$, according to (2.22) and the remark below it, we may write

$$4^m \partial \Gamma_{\Delta^m}(z) = c_m (\bar{z})^{m-1} z^{m-2} \left\{ (m-1) \ln |z| + \frac{1}{2} \right\}. \quad (2.28)$$

In particular, if $m = 1$ there holds $c_1 = 2/\pi$ and

$$E_1(z) = 4 \partial \Gamma_{\Delta}(z) = \frac{1}{\pi \cdot z}, \quad (2.29)$$

proving (2.26) when $m = 1$.

If $m \geq 2$, applying the operator ∂^{m-1} to both sides of (2.29) gives

$$E_m(z) = 4^m \partial^m \Gamma_{\Delta^m}(z) = c_m (\bar{z})^{m-1} (m-1)! \frac{1}{2z}, \quad (2.30)$$

which, together with (2.27), implies (2.26), completing the proof of Theorem 2.3. \square

Combining (2.25) and Theorem 2.3 we obtain that

$$\mathbb{C} \setminus \{0\} \ni z \mapsto \frac{1}{\pi(m-1)!} \frac{(\bar{z})^{m-1}}{z} \quad (2.31)$$

is a radial fundamental solution of the operator $\bar{\partial}^m$ in \mathbb{C} .

Remark. An alternative proof of (2.31) is by induction on m , starting with the fundamental solution of the Cauchy-Riemann operator $z \mapsto \frac{1}{\pi z}$ in \mathbb{C} (see e.g. [[22], Theorem 7.43]). Thus

$$\bar{\partial} \left(\frac{1}{\pi z} \right) = \delta \quad (2.32)$$

in the sense of distribution. This further implies

$$\bar{\partial}^2 \left(\bar{z} \cdot \frac{1}{\pi z} \right) = \bar{\partial} \left(\frac{1}{\pi z} + \bar{z} \cdot \delta \right) = \delta, \quad (2.33)$$

showing that $\mathbb{C} \setminus \{0\} \ni z \mapsto \frac{\bar{z}}{\pi z}$ is a fundamental solution for $\bar{\partial}^2$ in \mathbb{C} . Going further,

$$\bar{\partial}^3 \left(\frac{\bar{z}^2}{2} \cdot \frac{1}{\pi z} \right) = \bar{\partial}^2 \left(\bar{z} \cdot \frac{1}{\pi z} + \frac{\bar{z}^2}{2} \cdot \delta \right) = \delta, \quad (2.34)$$

etc.

2.3 The poly-Cauchy operator

Throughout this section we shall fix $m \in \mathbb{N}$, and we shall assume that the domain $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is an Ahlfors regular domain with compact boundary and we shall denote its geometric measure theoretic outward unit normal by $\nu = (\nu_1, \nu_2)$ and the surface measure $\mathcal{H}^1 \llcorner \partial\Omega$ by σ . Fix $z_0 \in \Omega$ and observe that for each function $u \in \mathcal{C}^\infty(\bar{\Omega})$ which is polyanalytic of order m in Ω (i.e. $\bar{\partial}^m u = 0$ in Ω) there holds

$$\mathcal{E}'(\Omega) \langle E_m(\cdot - z_0), \bar{\partial}^m u \rangle_{\mathcal{E}(\Omega)} = 0, \quad (2.35)$$

where E_m is the fundamental solution of $\bar{\partial}^m$ from (2.26), and \mathcal{L}^2 is the Lebesgue measure in \mathbb{R}^2 .

Starting from (2.35) and using the integration by parts formula from [25] (cf. also [26]) we may write

$$\begin{aligned}
0 &= {}_{\mathcal{E}'(\Omega)}\langle E_m(\cdot - z_0), \bar{\partial}^m u \rangle_{\mathcal{E}(\Omega)} \tag{2.36} \\
&= -{}_{\mathcal{E}'(\Omega)}\langle (\bar{\partial} E_m)(\cdot - z_0), \bar{\partial}^{m-1} u \rangle_{\mathcal{E}(\Omega)} + \frac{1}{2} \int_{\partial\Omega} E_m(\cdot - z_0) (\bar{\partial}^{m-1} u) \nu \, d\sigma \\
&= -{}_{\mathcal{E}'(\Omega)}\langle (\bar{\partial} E_m)(\cdot - z_0), \bar{\partial}^{m-1} u \rangle_{\mathcal{E}(\Omega)} + \frac{1}{2i} \int_{\partial\Omega} E_m(\cdot - z_0) (\bar{\partial}^{m-1} u) \, d\zeta,
\end{aligned}$$

where $d\zeta := i\nu \, d\sigma$ is the complex arclength on $\partial\Omega$. Integrating by parts again in the first term on the last line of (2.36), appealing again to [25], we obtain

$$\begin{aligned}
0 &= (-1)^2 {}_{\mathcal{E}'(\Omega)}\langle (\bar{\partial}^2 E_m)(\cdot - z_0), \bar{\partial}^{m-2} u \rangle_{\mathcal{E}(\Omega)} \tag{2.37} \\
&\quad + \frac{1}{2i} \int_{\partial\Omega} E_m(\cdot - z_0) (\bar{\partial}^{m-1} u) \, d\zeta - \frac{1}{2i} \int_{\partial\Omega} (\bar{\partial} E_m)(\cdot - z_0) (\bar{\partial}^{m-2} u) \, d\zeta.
\end{aligned}$$

Repeating the integration by parts process in this manner, we eventually end up with

$$\begin{aligned}
0 &= (-1)^m {}_{\mathcal{E}'(\Omega)}\langle (\bar{\partial}^m E_m)(\cdot - z_0) u \rangle_{\mathcal{E}(\Omega)} \\
&\quad + \sum_{k=0}^{m-1} \frac{(-1)^k}{2i} \int_{\partial\Omega} (\bar{\partial}^k E_m)(\zeta - z_0) (\bar{\partial}^{m-1-k} u)(\zeta) \, d\zeta. \tag{2.38}
\end{aligned}$$

Since E_m is the fundamental solution for $\bar{\partial}^m$,

$${}_{\mathcal{E}'(\Omega)}\langle (\bar{\partial}^2 E_m)(\cdot - z_0), \bar{\partial}^{m-2} u \rangle_{\mathcal{E}(\Omega)} = u(z_0), \tag{2.39}$$

and substituting this back into (2.38), we get

$$u(z_0) = \sum_{k=0}^{m-1} \frac{(-1)^{m+1+k}}{2i} \int_{\partial\Omega} (\bar{\partial}^k E_m)(\zeta - z_0) (\bar{\partial}^{m-1-k} u)(\zeta) \, d\zeta, \tag{2.40}$$

or equivalently

$$u(z_0) = \sum_{k=0}^{m-1} \frac{(-1)^k}{2i} \int_{\partial\Omega} (\bar{\partial}^{m-1-k} E_m)(\zeta - z_0) (\bar{\partial}^k u)(\zeta) \, d\zeta. \tag{2.41}$$

Differentiating in (2.26), for each $k \in \{0, \dots, m-1\}$ there holds

$$(\bar{\partial}^{m-1-k} E_m)(z) = \frac{\bar{z}^k}{k!} \cdot \frac{1}{\pi z} \quad \forall z \in \mathbb{C} \setminus \{0\}. \tag{2.42}$$

Combining this with (2.41), we obtain that

$$u(z_0) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z_0 - \zeta)}^k}{k!(\zeta - z_0)} (\bar{\partial}^k u)(\zeta) d\zeta. \quad (2.43)$$

In summary, formula (2.43) shows that one is able to recover the polyanalytic function $u \in \mathcal{C}^\infty(\bar{\Omega})$ from the restriction to the boundary of the partial derivatives $\bar{\partial}^k u$ for $k \in \{0, \dots, m-1\}$ in the geometric context of bounded Ahlfors regular domains in \mathbb{C} .

Our next order of business is to define Lebesgue and Sobolev based spaces of complex Whitney arrays. Recall from (1.10) that the tangential derivative $\partial_\tau \varphi$ of any given function $\varphi \in \mathcal{C}^1(\mathbb{R}^2)$ is defined as

$$\partial_\tau \varphi := \nu_1 (\partial_y \varphi) \Big|_{\partial\Omega} - \nu_2 (\partial_x \varphi) \Big|_{\partial\Omega}, \quad (2.44)$$

where

$$\begin{aligned} \nu_1 = \operatorname{Re} \nu &= \frac{1}{2}(\nu + \bar{\nu}), \quad \nu_2 = \operatorname{Im} \nu = \frac{1}{2i}(\nu - \bar{\nu}), \\ (\partial_x \varphi) \Big|_{\partial\Omega} &= (\bar{\partial} \varphi) \Big|_{\partial\Omega} + (\partial \varphi) \Big|_{\partial\Omega}, \quad (\partial_y \varphi) \Big|_{\partial\Omega} = \frac{1}{i} \left((\bar{\partial} \varphi) \Big|_{\partial\Omega} - (\partial \varphi) \Big|_{\partial\Omega} \right). \end{aligned} \quad (2.45)$$

Substituting (2.45) into (2.44), we obtain

$$\partial_\tau \varphi = i\nu (\partial \varphi) \Big|_{\partial\Omega} - i\bar{\nu} (\bar{\partial} \varphi) \Big|_{\partial\Omega}. \quad (2.46)$$

Given a function $g \in \mathcal{C}^\infty(\bar{\Omega})$, define

$$g_{(a,b)} := (\partial^a \bar{\partial}^b g) \Big|_{\partial\Omega}, \quad \forall a, b \in \mathbb{N}_0. \quad (2.47)$$

Then

$$\begin{aligned} \partial_\tau g_{(a,b)} &= i \left[\nu (\partial^{a+1} \bar{\partial}^b g) \Big|_{\partial\Omega} - \bar{\nu} (\partial^a \bar{\partial}^{b+1} g) \Big|_{\partial\Omega} \right] \\ &= i\nu g_{(a+1,b)} - i\bar{\nu} g_{(a,b+1)}. \end{aligned} \quad (2.48)$$

Hereafter, keeping in mind (2.48), given $p \in [1, \infty]$, we say that a family of L^p functions on $\partial\Omega$

$$\dot{g} := \{g_{(a,b)} : a, b \in \mathbb{N}_0 \text{ with } a + b \leq m - 1\} \quad (2.49)$$

satisfies compatibility conditions in the complex plane, henceforth abbreviated as $CC_{\mathbb{C}}$ provided that

$$\dot{g} \in CC_{\mathbb{C}} \iff \begin{cases} \partial_{\tau} g_{(a,b)} = i\nu g_{(a+1,b)} - i\bar{\nu} g_{(a,b+1)} \quad \sigma\text{-a.e. on } \partial\Omega \\ \text{whenever } a+b \leq m-2 \text{ and } a, b \in \mathbb{N}_0. \end{cases} \quad (2.50)$$

Definition 2.4. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain and $p \in [1, \infty]$. Define the L^p -based complex Whitney array space by setting

$$\begin{aligned} \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] &:= \left\{ \dot{g} = \{g_{(a,b)}\}_{\substack{a+b \leq m-1 \\ a,b \in \mathbb{N}_0}} : \dot{g} \in CC_{\mathbb{C}} \right. \\ &\quad \left. \text{and } g_{(a,b)} \in L^p(\partial\Omega, \sigma) \text{ if } a+b \leq m-1 \right\}, \end{aligned} \quad (2.51)$$

equipped with the norm

$$\|\dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]} := \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \|g_{(a,b)}\|_{L^p(\partial\Omega, \sigma)}. \quad (2.52)$$

Similarly, define the complex Sobolev-Whitney array space as follows.

Definition 2.5. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain and $p \in [1, \infty]$. Define the complex Sobolev-Whitney array space by setting

$$\begin{aligned} \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)] &:= \left\{ \dot{g} = \{g_{(a,b)}\}_{\substack{a+b \leq m-1 \\ a,b \in \mathbb{N}_0}} : \dot{g} \in CC_{\mathbb{C}} \right. \\ &\quad \left. \text{and } g_{(a,b)} \in L_1^p(\partial\Omega, \sigma) \text{ if } a+b \leq m-1 \right\}, \end{aligned} \quad (2.53)$$

equipped with the norm

$$\|\dot{g}\|_{\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]} := \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \|g_{(a,b)}\|_{L_1^p(\partial\Omega, \sigma)}. \quad (2.54)$$

Inspired by (2.43) we define the poly-Cauchy operator, acting on complex Whitney array spaces, as follows.

Definition 2.6. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary, and fix an arbitrary integer $m \in \mathbb{N}$ along with an integrability index $p \in (1, \infty)$. Define

the action of the boundary-to-domain poly-Cauchy operator on each Lebesgue based complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ as

$$(\mathcal{C}_{m-1}\dot{g})(z) := \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(0,k)}(\zeta) d\zeta \text{ for all } z \in \Omega. \quad (2.55)$$

The boundary-to-domain poly-Cauchy operator introduced in Definition 2.6 satisfies a number of remarkable properties, on which we wish to elaborate. First, we have the following Fatou type result and integral representation formula.

Theorem 2.7. *Let $\Omega \subseteq \mathbb{R}^2$ be an arbitrary UR domain, and fix an arbitrary integer $m \in \mathbb{N}$. Pick some aperture parameter $\kappa \in (0, \infty)$ along with some integrability exponent $p \in (1, \infty)$. Let u be a polyanalytic function of order m in Ω , i.e., a function $u \in \mathcal{C}^\infty(\Omega)$ satisfying $\bar{\partial}^m u = 0$ in Ω . Associate with this polyanalytic function the family of auxiliary functions $\{u_j\}_{0 \leq j \leq m-1}$ defined, for each $j \in \{0, 1, \dots, m-1\}$, as*

$$u_j(z) := \sum_{\ell=0}^{m-1-j} \frac{(-1)^\ell \bar{z}^\ell}{\ell!} (\bar{\partial}^{j+\ell} u)(z) \text{ for each } z \in \Omega. \quad (2.56)$$

Make the assumption that

$$\mathcal{N}_\kappa u_j \in L^p(\partial\Omega, \sigma) \text{ for each } j \in \{0, 1, \dots, m-1\}, \quad (2.57)$$

and if Ω is an exterior domain also assume that

$$u_j(z) = o(1) \text{ as } z \rightarrow \infty, \text{ for each } j \in \{0, 1, \dots, m-1\}. \quad (2.58)$$

Then for each $\ell \in \{0, 1, \dots, m-1\}$ it follows that the κ -nontangential trace

$$(\bar{\partial}^\ell u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \quad (2.59)$$

and for each $z \in \Omega$ one has

$$u(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \sum_{k=0}^{m-1} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} (\bar{\partial}^k u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta. \quad (2.60)$$

Note that Theorem 2.7 is a most natural higher-order generalization of item (1) in Theorem 2.1 (to which this reduces when $m := 1$). Here is the proof of Theorem 2.7.

Proof. We proceed by induction on m . The case $m = 1$ is contained in Theorem 2.1. To carry out the induction step, assume $m \geq 2$ and that all claims in the statement are valid for polyanalytic functions of order $m - 1$. Decompose the given u as

$$\begin{aligned} u(z) &= w(z) + \bar{z}^{m-1}\omega(z) \text{ for each } z \in \Omega, \\ \text{where } w(z) &:= u(z) - \frac{\bar{z}^{m-1}}{(m-1)!}(\bar{\partial}^{m-1}u)(z) \text{ and} \\ \omega(z) &:= \frac{1}{(m-1)!}(\bar{\partial}^{m-1}u)(z) \text{ for each } z \in \Omega. \end{aligned} \tag{2.61}$$

Then $\omega \in \mathcal{O}(\Omega)$ and since $\omega = \frac{1}{(m-1)!}u_{m-1}$, we conclude from (2.57)-(2.58) (corresponding to $j := m - 1$) that $\mathcal{N}_\kappa\omega \in L^p(\partial\Omega, \sigma)$ and ω vanishes at infinity if Ω is an exterior domain. As such, all conclusions in item (1) of Theorem 2.1 apply to ω . In terms of u , these imply that

$$(\bar{\partial}^{m-1}u)\Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \tag{2.62}$$

and for each $z \in \Omega$ we have

$$(\bar{\partial}^{m-1}u)(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{\zeta - z} (\bar{\partial}^{m-1}u)\Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta. \tag{2.63}$$

In addition, (2.61) implies that $w \in \mathcal{C}^\infty(\Omega)$ satisfies $\bar{\partial}^{m-1}w = 0$ (hence w is a polyanalytic function of order $m - 1$ in Ω) and, for each $j, \ell \in \mathbb{N}_0$ with $j + \ell \leq m - 2$,

$$(\bar{\partial}^{j+\ell}w)(z) = (\bar{\partial}^{j+\ell}u)(z) - \frac{\bar{z}^{m-1-j-\ell}}{(m-j-\ell-1)!}(\bar{\partial}^{m-1}u)(z) \text{ for each } z \in \Omega. \tag{2.64}$$

This may be used to compute the auxiliary functions associated with w as in

(2.56). Specifically, for each $j \in \{0, 1, \dots, m-2\}$ we have

$$\begin{aligned}
w_j(z) &= \sum_{\ell=0}^{m-2-j} \frac{(-1)^\ell \bar{z}^\ell}{\ell!} (\bar{\partial}^{j+\ell} w)(z) \\
&= \sum_{\ell=0}^{m-2-j} \frac{(-1)^\ell \bar{z}^\ell}{\ell!} (\bar{\partial}^{j+\ell} u)(z) \\
&\quad - \left(\sum_{\ell=0}^{m-2-j} \frac{(-1)^\ell}{\ell!(m-j-\ell-1)!} \right) \bar{z}^{m-1-j} (\bar{\partial}^{m-1} u)(z) \\
&= \sum_{\ell=0}^{m-2-j} \frac{(-1)^\ell \bar{z}^\ell}{\ell!} (\bar{\partial}^{j+\ell} u)(z) - \left(\frac{1}{(m-1-j)!} \cdot ((-1) + 1)^{m-1-j} \right. \\
&\quad \left. - \frac{(-1)^{m-1-j}}{(m-1-j)!} \right) \bar{z}^{m-1-j} (\bar{\partial}^{m-1} u)(z) \\
&= \sum_{\ell=0}^{m-1-j} \frac{(-1)^\ell \bar{z}^\ell}{\ell!} (\bar{\partial}^{j+\ell} u)(z) = u_j(z) \text{ for each } z \in \Omega. \tag{2.65}
\end{aligned}$$

In concert with assumptions (2.57)-(2.58) this guarantees that, for each index $j \in \{0, 1, \dots, m-2\}$, we have $\mathcal{N}_\kappa w_j \in L^p(\partial\Omega, \sigma)$ and w_j vanishes at infinity if Ω is an exterior domain. As such, by the induction hypothesis, (2.62), and (2.64) with $j := 0$ it follows that for each $\ell \in \{0, 1, \dots, m-2\}$ the κ -nontangential trace

$$(\bar{\partial}^\ell u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \tag{2.66}$$

(which, together with (2.62), takes care of (2.59)), and for each $z \in \Omega$ we have

$$\begin{aligned}
w(z) &= \frac{1}{2\pi i} \int_{\partial\Omega} \sum_{k=0}^{m-2} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} (\bar{\partial}^k w) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta \\
&= \frac{1}{2\pi i} \int_{\partial\Omega} \sum_{k=0}^{m-2} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left[(\bar{\partial}^k u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) - \frac{\bar{\zeta}^{m-1-k}}{(m-k-1)!} (\bar{\partial}^{m-1} u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) \right] d\zeta. \tag{2.67}
\end{aligned}$$

Hence, using (2.61), (2.63), and (2.67), we may write

$$\begin{aligned}
u(z) &= w(z) + \frac{\bar{z}^{m-1}}{(m-1)!} (\bar{\partial}^{m-1} u)(z) \\
&= \frac{1}{2\pi i} \int_{\partial\Omega} \sum_{k=0}^{m-2} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left[(\bar{\partial}^k u)|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) - \frac{\bar{\zeta}^{m-1-k}}{(m-k-1)!} (\bar{\partial}^{m-1} u)|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) \right] d\zeta \\
&\quad + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\bar{z}^{m-1}}{(m-1)!(\zeta-z)} (\bar{\partial}^{m-1} u)|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta, \tag{2.68}
\end{aligned}$$

for each $z \in \Omega$. Observe that for each z and ζ we have

$$\frac{\bar{z}^{m-1}}{(m-1)!} = \frac{[(z-\zeta) + \bar{\zeta}]^{m-1}}{(m-1)!} = \sum_{k=0}^{m-1} \frac{\overline{(z-\zeta)^k} \bar{\zeta}^{m-1-k}}{k!(m-1-k)!}. \tag{2.69}$$

Plugging this back in (2.68) and canceling like-terms yields (2.60). This completes the proof of Theorem 2.7. \square

Proposition 2.8. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary and fix $p \in (1, \infty)$. For any $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, and any order $r \in \{0, 1, \dots, m-1\}$, the poly-Cauchy operator $\dot{\mathcal{C}}_{m-1}\dot{g} \in C^\infty(\Omega)$ satisfies*

$$\bar{\partial}^r (\dot{\mathcal{C}}_{m-1}\dot{g})(z) = \sum_{k=r}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^{k-r}}}{(k-r)!(\zeta-z)} g_{(0,k)}(\zeta) d\zeta, \quad \text{for all } z \in \Omega, \tag{2.70}$$

and

$$\bar{\partial}^m (\dot{\mathcal{C}}_{m-1}\dot{g}) = 0 \quad \text{in } \Omega. \tag{2.71}$$

Proof. Fix $m \in \mathbb{N}$, a parameter $r \in \{0, 1, \dots, m\}$, and a complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Then, using (2.55) and the Lebesgue Dominated Convergence Theorem,

$$\bar{\partial}^r (\dot{\mathcal{C}}_{m-1}\dot{g})(z) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\bar{z}}^r \left[\frac{(\bar{z}-\bar{\zeta})^k}{k!(\zeta-z)} \right] g_{(0,k)}(\zeta) d\zeta. \tag{2.72}$$

However, by direct calculation, we get

$$\partial_{\bar{z}}^r \left[\frac{(\bar{z} - \bar{\zeta})^k}{k!(\zeta - z)} \right] = \begin{cases} \frac{k \cdot (k-1) \cdots (k-r+1)(\bar{z} - \bar{\zeta})^{k-r}}{k!(\zeta - z)} & \text{if } k \geq r, \\ 0 & \text{if } k < r. \end{cases} \quad (2.73)$$

This, combined with (2.72) proves (2.70) and (2.71). \square

In the next proposition we study well-posedness and boundedness properties for shift like operators acting on complex Whitney array spaces

Proposition 2.9. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary and fix an index $p \in [1, \infty]$. For any parameter $s \in \{1, \dots, m-1\}$ and any complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, define*

$$\Theta_s^I(\dot{g}) := \{g_{(a+s,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1-s}}, \quad (2.74)$$

and

$$\Theta_s^{II}(\dot{g}) := \{g_{(a,b+s)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1-s}}. \quad (2.75)$$

Then

$$\Theta_s^I : \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{CWA}_{m-1-s}[L^p(\partial\Omega, \sigma)], \quad (2.76)$$

and

$$\Theta_s^{II} : \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{CWA}_{m-1-s}[L^p(\partial\Omega, \sigma)], \quad (2.77)$$

are well defined, linear, and bounded.

Proof. We shall proceed to first prove that the operator in (2.76) is well defined, linear and bounded. To this end, select $s \in \{1, \dots, m-1\}$ and Lebesgue based complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$.

For $a, b \in \mathbb{N}_0$ with $a+b \leq m-2-s$, since $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in CC_{\mathbb{C}}$, we have

$$\partial_{\tau} g_{(a+s,b)} = i\nu g_{(a+s+1,b)} - i\bar{\nu} g_{(a+s,b+1)}, \quad (2.78)$$

implying that $\Theta_s^I(\dot{g}) \in CC_{\mathbb{C}}$, and ultimately that

$$\Theta_s^I(\dot{g}) \in \text{CWA}_{m-1-s}[L^p(\partial\Omega, \sigma)], \quad (2.79)$$

as the components of the array \dot{g} belong to $L^p(\partial\Omega, \sigma)$. This proves that the operator Θ_s^I in (2.76) is well defined, and linearity is immediate. In addition,

$$\begin{aligned} \|\Theta_s^I(\dot{g})\|_{\text{CWA}_{m-1-s}[L^p(\partial\Omega, \sigma)]} &= \sum_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1-s}} \|g_{(a+s, b)}\|_{L^p(\partial\Omega, \sigma)} \\ &\leq \|\dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]}, \end{aligned} \quad (2.80)$$

where in the inequality above we have used that $a + s + b \leq m - 1$. This yields

$$\|\Theta_s^I(\dot{g})\|_{\text{CWA}_{m-1-s}[L^p(\partial\Omega, \sigma)]} \leq \|\dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]}, \quad (2.81)$$

proving the boundedness of the operator Θ_s^I from (2.76). Finally, the properties of the operator Θ_s^{II} from (2.77) are established in a similar fashion. This finishes the proof of the proposition. \square

Remark. Note that (2.70) can be equivalently expressed as

$$\bar{\partial}^r(\mathcal{E}_{m-1}\dot{g})(z) = \sum_{k=0}^{m-1-r} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(0, k+r)}(\zeta) d\zeta, \quad (2.82)$$

and observe that the integral expression on the right hand side in (2.82) is $\mathcal{E}_{m-1-r}(\Theta_r^{II}(\dot{g}))(z)$ where $\Theta_r^{II}(\dot{g}) := \{g_{(a, b+r)}\}_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1-r}}$. That is, for each $\dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, with $p \in (1, \infty)$, there holds

$$\bar{\partial}^r(\mathcal{E}_{m-1}\dot{g}) = \mathcal{E}_{m-1-r}(\Theta_r^{II}(\dot{g})) \quad \text{in } \Omega. \quad (2.83)$$

Proposition 2.10. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary and fix $m \in \mathbb{N}$ and an integrability index $p \in (1, \infty)$. For any Lebesgue based complex Whitney array $\dot{g} = \{g_{(a, b)}\}_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ there holds*

$$\begin{aligned} \partial(\mathcal{E}_{m-1}\dot{g})(z) &= \left(\mathcal{E}_{m-2}(\Theta_1^I(\dot{g})) \right)(z) \\ &\quad - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-1}}}{(m-1)!(\zeta-z)} \right] g_{(0, m-1)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.84)$$

for each $z \in \Omega$. More generally, for all $s \in \{0, 1, \dots, m-1\}$,

$$\begin{aligned} \partial^s \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) &= \left(\dot{\mathcal{C}}_{m-1-s} \left(\Theta_s^I(\dot{g}) \right) \right) (z) \\ &= - \sum_{j=0}^{s-1} \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \partial_z^j \left[\frac{\overline{(z-\zeta)^{m-s+j}}}{(m-s+j)!(\zeta-z)} \right] g_{(s-1-j, m-s+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.85)$$

for each $z \in \Omega$.

Proof. Using that $d\zeta = i\nu d\sigma$ on $\partial\Omega$, for each $z \in \Omega$ we have

$$\partial(\dot{\mathcal{C}}_{m-1} \dot{g})(z) = \sum_{k=0}^{m-1} \frac{i}{2\pi i} \int_{\partial\Omega} \nu(\zeta) \partial_z \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right] g_{(0,k)}(\zeta) d\sigma(\zeta). \quad (2.86)$$

However, the chain rule yields

$$\partial_z \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right] = -\partial_\zeta \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right], \quad (2.87)$$

while

$$-\nu(\zeta) \partial_\zeta = (\bar{\nu}(\zeta) \bar{\partial}_\zeta - \nu(\zeta) \partial_\zeta) - \bar{\nu}(\zeta) \bar{\partial}_\zeta = i\partial_{\tau(\zeta)} - \bar{\nu}(\zeta) \bar{\partial}_\zeta. \quad (2.88)$$

Using (2.87) and (2.88) in (2.86) we obtain that

$$\begin{aligned} \partial(\dot{\mathcal{C}}_{m-1} \dot{g})(z) &= \sum_{k=0}^{m-1} \frac{-1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right] g_{(0,k)}(\zeta) d\sigma(\zeta) \\ &\quad - \sum_{k=0}^{m-1} \frac{1}{2\pi} \int_{\partial\Omega} \bar{\partial}_\zeta \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right] g_{(0,k)}(\zeta) \bar{\nu}(\zeta) d\sigma(\zeta) \\ &= \sum_{k=0}^{m-2} \frac{-1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right] g_{(0,k)}(\zeta) d\sigma(\zeta) \\ &\quad + \left(\frac{-1}{2\pi i} \right) \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-1}}}{(m-1)!(\zeta-z)} \right] g_{(0,m-1)}(\zeta) d\sigma(\zeta) \\ &\quad - \sum_{k=1}^{m-1} \frac{1}{2\pi} \int_{\partial\Omega} \frac{(-1)k \overline{(z-\zeta)^{k-1}}}{k!(\zeta-z)} g_{(0,k)}(\zeta) \bar{\nu}(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.89)$$

Now, let

$$\begin{aligned}
& \sum_{k=0}^{m-2} \frac{-1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right] g_{(0,k)}(\zeta) d\sigma(\zeta) \\
& + \left(\frac{-1}{2\pi i} \right) \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-1}}}{(m-1)!(\zeta-z)} \right] g_{(0,m-1)}(\zeta) d\sigma(\zeta) \\
& - \sum_{k=1}^{m-1} \frac{1}{2\pi} \int_{\partial\Omega} \frac{(-1)k\overline{(z-\zeta)^{k-1}}}{k!(\zeta-z)} g_{(0,k)}(\zeta) \bar{\nu}(\zeta) d\sigma(\zeta) \\
& := I + II + III
\end{aligned} \tag{2.90}$$

Since for any parameter $k \in \{0, \dots, m-2\}$, one can write $(\partial_{\tau} g_{(0,k)})(\zeta)$ as $i\nu(\zeta)g_{(1,k)}(\zeta) - i\bar{\nu}(\zeta)g_{(0,k+1)}(\zeta)$, and $i\nu d\sigma = d\zeta$, one obtains

$$\begin{aligned}
I &= \sum_{k=0}^{m-2} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(1,k)}(\zeta) d\zeta \\
& + \sum_{k=0}^{m-2} \frac{-1}{2\pi} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(0,k+1)}(\zeta) \bar{\nu}(\zeta) d\sigma(\zeta) \\
& = \sum_{k=0}^{m-2} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(1,k)}(\zeta) d\zeta - III.
\end{aligned} \tag{2.91}$$

This and (2.90) imply that

$$\begin{aligned}
\partial(\mathcal{E}_{m-1}\dot{g})(z) &= \sum_{k=0}^{m-2} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(1,k)}(\zeta) d\zeta \\
& - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-1}}}{(m-1)!(\zeta-z)} \right] g_{(0,m-1)}(\zeta) d\sigma(\zeta),
\end{aligned} \tag{2.92}$$

for $z \in \Omega$. This recurrent formula can be iterated. For example, for each $z \in \Omega$

$$\begin{aligned}
\partial^2(\mathcal{E}_{m-1}\dot{g})(z) &= \partial \left(\mathcal{E}_{m-2}(\Theta_1^I(\dot{g})) \right) (z) \\
& - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \partial_z \left[\frac{\overline{(z-\zeta)^{m-1}}}{(m-1)!(\zeta-z)} \right] g_{(0,m-1)}(\zeta) d\sigma(\zeta)
\end{aligned} \tag{2.93}$$

which yields

$$\begin{aligned}
\partial^2(\mathcal{C}_{m-1}\dot{g})(z) &= \left(\mathcal{C}_{m-3} \left((\Theta_1^I(\Theta_1^I(\dot{g}))) \right) \right) (z) \\
&\quad - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-2}}}{(m-2)!(\zeta-z)} \right] (\Theta_1^I(\dot{g}))_{(0,m-2)}(\zeta) d\sigma(\zeta) \\
&\quad - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \partial_z \left[\frac{\overline{(z-\zeta)^{m-1}}}{(m-1)!(\zeta-z)} \right] g_{(0,m-1)}(\zeta) d\sigma(\zeta) \\
&= \left(\mathcal{C}_{m-3} (\Theta_2^I(\dot{g})) \right) (z) \\
&\quad - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-2}}}{(m-2)!(\zeta-z)} \right] g_{(1,m-2)}(\zeta)(\zeta) d\sigma(\zeta) \\
&\quad - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \partial_z \left[\frac{\overline{(z-\zeta)^{m-1}}}{(m-1)!(\zeta-z)} \right] g_{(0,m-1)}(\zeta) d\sigma(\zeta),
\end{aligned} \tag{2.94}$$

More generally, if we assume (2.85) for some $s \in \{0, 1, \dots, m-2\}$, one has

$$\begin{aligned}
&\partial^{s+1} \left(\mathcal{C}_{m-1}\dot{g} \right) (z) \\
&= \left(\mathcal{C}_{m-1-(s+1)} (\Theta_{s+1}^I(\dot{g})) \right) (z) \\
&\quad - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-1-s}}}{(m-1-s)!(\zeta-z)} \right] (\Theta_s^I(\dot{g}))_{(0,m-1-s)}(\zeta) d\sigma(\zeta) \\
&\quad - \sum_{j=1}^s \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \partial_z^j \left[\frac{\overline{(z-\zeta)^{m-s-1+j}}}{(m-s-1+j)!(\zeta-z)} \right] g_{(s-j,m-s-1+j)}(\zeta) d\sigma(\zeta) \\
&= \left(\mathcal{C}_{m-1-(s+1)} (\Theta_{s+1}^I(\dot{g})) \right) (z) \\
&\quad - \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-1-s}}}{(m-1-s)!(\zeta-z)} \right] (\Theta_s^I(\dot{g}))_{(0,m-1-s)}(\zeta) d\sigma(\zeta) \\
&\quad - \sum_{j=1}^s \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \partial_z^j \left[\frac{\overline{(z-\zeta)^{m-1-s+j}}}{(m-1-s+j)!(\zeta-z)} \right] g_{(s-j,m-1-s+j)}(\zeta) d\sigma(\zeta).
\end{aligned} \tag{2.95}$$

Consequently, we get

$$\begin{aligned} \partial^{s+1} \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) &= \left(\dot{\mathcal{C}}_{m-1-(s+1)} \left(\Theta_{s+1}^I(\dot{g}) \right) \right) (z) \\ &\quad - \frac{1}{2\pi i} \sum_{j=0}^s \int_{\partial\Omega} \partial_{\tau(\zeta)} \partial_z^j \left[\frac{\overline{(z-\zeta)^{m-1-s+j}}}{(m-1-s+j)!(\zeta-z)} \right] \times \\ &\quad \times g_{(s-j, m-1-s+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.96)$$

which finishes the proof by induction of (2.85). \square

Corollary 2.11. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary and fix $p \in (1, \infty)$. For any $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, $\forall r, s \in \mathbb{N}_0$ with $r+s \leq m-1$ we have*

$$\begin{aligned} \partial^s \bar{\partial}^r \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) &= \sum_{k=0}^{m-1-s-r} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(s, k+r)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{s-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{j!}{(m-s-r+j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-s-r+j}}}{(\zeta-z)^{j+1}} \right] g_{(s-1-j, m-s+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.97)$$

for each $z \in \Omega$. As a corollary of this and the integration by parts on the boundary formula in (1.11) it follows that whenever $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}}$ actually belongs to the Sobolev-based complex Whitney array space $\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$ then for any pair of numbers $r, s \in \mathbb{N}_0$ with $r+s \leq m-1$ any point $z \in \Omega$ one has

$$\begin{aligned} \partial^s \bar{\partial}^r \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) &= \sum_{k=0}^{m-1-s-r} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(s, k+r)}(\zeta) d\zeta \\ &\quad + \sum_{j=0}^{s-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{j!}{(m-s-r+j)!} \times \\ &\quad \times \frac{\overline{(z-\zeta)^{m-s-r+j}}}{(\zeta-z)^{j+1}} (\partial_{\tau} g_{(s-1-j, m-s+j)})(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.98)$$

Proof. Together, (2.83) and (2.85) give $\forall r, s \in \mathbb{N}_0$ with $r + s \leq m - 1$,

$$\begin{aligned}
\partial^s \bar{\partial}^r \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) &= \left(\dot{\mathcal{C}}_{m-1-s-r} \left(\left(\Theta_r^{II} \left(\Theta_s^I(\dot{g}) \right) \right) \right) \right) (z) \\
&\quad - \sum_{j=0}^{s-1} \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \bar{\partial}_z^r \partial_z^j \left[\frac{\overline{(z-\zeta)^{m-s+j}}}{(m-s+j)! (\zeta-z)} \right] \times \\
&\quad \quad \quad \times g_{(s-1-j, m-s+j)}(\zeta) d\sigma(\zeta) \\
&= \left(\dot{\mathcal{C}}_{m-1-s-r} \left(\Theta_r^{II} \left(\Theta_s^I(\dot{g}) \right) \right) \right) (z) \\
&\quad - \sum_{j=0}^{s-1} \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \partial_z^j \left[\frac{1}{(m-s-r+j)!} \frac{\overline{(z-\zeta)^{m-s-r+j}}}{(\zeta-z)} \right] \times \\
&\quad \quad \quad \times g_{(s-1-j, m-s+j)}(\zeta) d\sigma(\zeta),
\end{aligned} \tag{2.99}$$

where $\Theta_r^{II}(\Theta_s^I(\dot{g})) = \{g_{(a+s, b+r)}\}_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1-s-r}}$. Since $\partial_z^j \left[\frac{1}{\zeta-z} \right] = \frac{j!}{(\zeta-z)^{j+1}}$, this finishes the proof of the Corollary 2.11. \square

The point of the next theorem is that our boundary-to-domain poly-Cauchy operator acting on complex Whitney array spaces can absorb $m - 1$ derivatives without becoming hyper-singular.

Theorem 2.12. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Fix an arbitrary integer $m \in \mathbb{N}$ and select an integrability exponent $p \in (1, \infty)$. Then for any array $\dot{g} = \{g_{(a,b)}\}_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ it follows that*

$$\dot{\mathcal{C}}_{m-1} \dot{g} \in \mathcal{C}^\infty(\Omega) \quad \text{and} \quad \bar{\partial}^m (\dot{\mathcal{C}}_{m-1} \dot{g}) = 0 \quad \text{in } \Omega, \tag{2.100}$$

hence $\dot{\mathcal{C}}_{m-1} \dot{g}$ is a well-defined polyanalytic function of order m in Ω .

2.4 Space of polyanalytic functions

In this section, we investigate a space of polyanalytic functions of order $m \in \mathbb{N}$ and the radiation condition of the poly-Cauchy operator. We also provide a couple of counterexamples to the classical Fatou theorem and the maximum principle for polyanalytic functions of order m with $m > 1$. Consider

a nonempty open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$, and $m \in \mathbb{N}$. Define a space of polyanalytic functions of order m in Ω , denoted by $\text{PA}_m(\Omega)$, i.e.,

$$\text{PA}_m(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{C} : u \in \mathcal{C}^\infty(\Omega) \text{ and } \bar{\partial}^m u \equiv 0 \text{ in } \Omega \right\}. \quad (2.101)$$

Lemma 2.13. *Let $m \in \mathbb{N}$ and $\text{PA}_m(\Omega)$ is the space of polyanalytic functions of order m in Ω as in (2.101). Then we have the following properties about the polyanalytic functions in Ω .*

1. *Let $a, b \in \mathbb{N}$. For $u \in \text{PA}_a(\Omega)$, $v \in \text{PA}_b(\Omega)$, $u \cdot v \in \text{PA}_{a+b-1}(\Omega)$.*
2. *In particular, if $u \in \text{PA}_m(\Omega)$, $v \in \mathcal{O}(\Omega)$ then $u \cdot v \in \text{PA}_m(\Omega)$. In other words, $\text{PA}_m(\Omega)$ is a module over $\mathcal{O}(\Omega)$.*
3. *Let $m \in \mathbb{N}$. Since $1, \bar{z}, \bar{z}^2, \dots, \bar{z}^{m-1} \in \text{PA}_m(\Omega)$, for holomorphic functions $h_0, h_1, \dots, h_{m-1} \in \mathcal{O}(\Omega)$,*

$$\sum_{j=0}^{m-1} \bar{z}^j h_j(z) \in \text{PA}_m(\Omega). \quad (2.102)$$

Proof. Fix $a, b \in \mathbb{N}$ and $u \in \text{PA}_a(\Omega)$, $v \in \text{PA}_b(\Omega)$. Using $\bar{\partial}^\ell u = 0$ for $\ell \geq a$ and Leibniz rule, we have

$$\begin{aligned} \bar{\partial}^{a+b-1}(u \cdot v) &= \sum_{j=0}^{a+b-1} \frac{(a+b-1)!}{j!(a+b-1-j)!} (\bar{\partial}^j u) (\bar{\partial}^{a+b-1-j} v) \\ &= \sum_{j=0}^{a-1} \frac{(a+b-1)!}{j!(a+b-1-j)!} (\bar{\partial}^j u) (\bar{\partial}^{a+b-1-j} v). \end{aligned} \quad (2.103)$$

Observe that if $0 \leq j \leq a-1$ then $-a+1 \leq -j \leq 0$ which further implies $b \leq a+b-1-j \leq a+b-1$. Combining (2.103) with $\bar{\partial}^k v = 0$ for $k \geq b$, we conclude that

$$\bar{\partial}^{a+b-1}(u \cdot v) = 0. \quad (2.104)$$

This proves the first item. Moving on, consider $u \in \text{PA}_m(\Omega)$, $v \in \mathcal{O}(\Omega)$. Applying the Leibniz rule again and $\bar{\partial}^j v = 0$ for all $j \geq 1$ from $v \in \mathcal{O}(\Omega)$, we

obtain

$$\begin{aligned}
\bar{\partial}^m(u \cdot v) &= \sum_{j=0}^m \frac{m!}{j!(m-j)!} (\bar{\partial}^{m-j} u) (\bar{\partial}^j v) \\
&= \bar{\partial}^m u \\
&= 0.
\end{aligned} \tag{2.105}$$

This shows the second item in the lemma. Finally, the last item automatically comes from the second item which finishes the proof. \square

Let $u \in \text{PA}_m(\Omega)$. Define the family of auxiliary functions $\{u_j\}_{0 \leq j \leq m-1}$ associated with given polyanalytic function u of order m in Ω where for each index $j \in \{0, 1, \dots, m-1\}$,

$$u_j(z) := \sum_{\ell=0}^{m-1-j} \frac{(-1)^\ell \bar{z}^\ell}{\ell!} \left(\bar{\partial}^{j+\ell} u \right) (z) \text{ for each } z \in \Omega. \tag{2.106}$$

Definition 2.14. Let $\Omega \subseteq \mathbb{R}^2$ be an exterior domain. Let $m \in \mathbb{N}$ and $u \in \text{PA}_m(\Omega)$. Then u radiates at ∞ provided that each auxiliary function u_j vanishes at ∞ for all $j \in \{0, 1, \dots, m-1\}$, i.e.,

$$u_j(z) = o(1) \text{ as } z \rightarrow \infty. \tag{2.107}$$

Lemma 2.15. Let $\Omega \subseteq \mathbb{R}^2$ be an exterior UR domain and $m \in \mathbb{N}$, $p \in (1, \infty)$. If $\dot{f} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, then $\mathcal{E}_{m-1}\dot{f} \in \text{PA}_m(\Omega)$ satisfies the radiation condition as in the definition 2.14.

Proof. According to (2.82), for $r \in \{0, 1, \dots, m-1\}$

$$\bar{\partial}^r(\mathcal{E}_{m-1}\dot{f})(z) = \sum_{k=0}^{m-1-r} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} f_{(0,k+r)}(\zeta) d\zeta. \tag{2.108}$$

This forces for $j \in \{0, 1, \dots, m-1\}$, $\ell \in \{0, 1, \dots, m-1-j\}$, $z \in \Omega$

$$\bar{\partial}^{j+\ell}(\mathcal{E}_{m-1}\dot{f})(z) = \sum_{k=0}^{m-1-j-\ell} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} f_{(0,k+j+\ell)}(\zeta) d\zeta. \tag{2.109}$$

In particular, using (2.106) and (2.109), for $z \in \Omega$

$$\begin{aligned}
& \left(\mathcal{E}_{m-1} \dot{f} \right)_j (z) \\
&= \sum_{\ell=0}^{m-1-j} \frac{(-1)^\ell \bar{z}^\ell}{\ell!} \sum_{k=0}^{m-1-j-\ell} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)}^k}{k!(\zeta-z)} f_{(0,k+j+\ell)}(\zeta) d\zeta \\
&= \frac{1}{2\pi i} \sum_{\ell=0}^{m-1-j} \sum_{k=0}^{m-1-j-\ell} \int_{\partial\Omega} \frac{(-1)^\ell \bar{z}^\ell \overline{(z-\zeta)}^k}{\ell! k!(\zeta-z)} f_{(0,k+j+\ell)}(\zeta) d\zeta.
\end{aligned} \tag{2.110}$$

Going further,

$$\begin{aligned}
& \sum_{\ell=0}^{m-1-j} \sum_{k=0}^{m-1-j-\ell} \int_{\partial\Omega} \frac{(-1)^\ell \bar{z}^\ell \overline{(z-\zeta)}^k}{\ell! k!(\zeta-z)} f_{(0,k+j+\ell)}(\zeta) d\zeta \\
&= \sum_{\substack{0 \leq \ell \leq m-1-j \\ 0 \leq k \leq m-1-j-\ell}} \int_{\partial\Omega} \frac{(-1)^\ell \bar{z}^\ell \overline{(z-\zeta)}^k}{\ell! k!(\zeta-z)} f_{(0,k+j+\ell)}(\zeta) d\zeta \\
&= \sum_{r=0}^{m-1-j} \sum_{\ell=0}^r \int_{\partial\Omega} \frac{(-1)^\ell}{\ell!(r-\ell)!} (\bar{z} - \bar{\zeta})^{r-\ell} \bar{z}^\ell \frac{1}{(\zeta-z)} f_{(0,r+j)}(\zeta) d\zeta,
\end{aligned} \tag{2.111}$$

where the last equality follows from substituting $k + \ell = r$. Applying the binomial theorem, we observe that for $r \in \{0, 1, \dots, m-1-j\}$

$$\begin{aligned}
& \sum_{\ell=0}^r \frac{(-1)^\ell}{\ell!(r-\ell)!} (\bar{z} - \bar{\zeta})^{r-\ell} \bar{z}^\ell \\
&= \frac{1}{r!} \sum_{\ell=0}^r \frac{r!}{\ell!(r-\ell)!} (\bar{z} - \bar{\zeta})^{r-\ell} (-\bar{z})^\ell \\
&= \frac{1}{r!} (-\bar{\zeta})^r.
\end{aligned} \tag{2.112}$$

Combining this together with (2.110) and (2.111), we conclude that for each index $j \in \{0, 1, \dots, m-1\}$, $z \in \Omega$

$$\left(\mathcal{E}_{m-1} \dot{f} \right)_j (z) = \sum_{r=0}^{m-1-j} \frac{1}{2\pi i \cdot r!} \int_{\partial\Omega} \frac{(-\bar{\zeta})^r}{(\zeta-z)} f_{(0,r+j)}(\zeta) d\zeta. \tag{2.113}$$

This implies for $z \in \Omega$ with sufficiently large modulus $|z|$,

$$\begin{aligned}
\left| \left(\mathcal{E}_{m-1} \dot{f} \right)_j (z) \right| &\leq C(\partial\Omega, m) \sum_{r=0}^{m-1-j} \int_{\partial\Omega} \frac{1}{|\zeta - z|} |f_{(0,r+j)}(\zeta)| |d\zeta| \\
&\leq \frac{C(\partial\Omega, m)}{|z|} \sum_{r=0}^{m-1-j} \int_{\partial\Omega} |f_{(0,r+j)}(\zeta)| d\sigma \\
&\leq \frac{C(\partial\Omega, m, p)}{|z|} \sum_{r=0}^{m-1-j} \|f_{(0,r+j)}\|_{L^p(\partial\Omega, \sigma)} \\
&\leq \frac{C(\partial\Omega, m, p)}{|z|} \|\dot{f}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]}, \tag{2.114}
\end{aligned}$$

which shows for $j \in \{0, 1, \dots, m-1\}$, $\dot{f} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$

$$\left(\mathcal{E}_{m-1} \dot{f} \right)_j (z) = o(1) \text{ as } z \rightarrow \infty. \tag{2.115}$$

This completes the proof of the lemma. \square

Next, we introduce a technical lemma to provide a counterexample to the classical Fatou Theorem for polyanalytic function of order $m \in \mathbb{N}$ with $m > 1$.

Lemma 2.16. *Let $\alpha \in \mathbb{N}$ be even number. For $\theta \in [0, 2\pi)$,*

$$\max(|\cos(\alpha^k \theta)|, |\cos(\alpha^{k+1} \theta)|) > \frac{1}{2\alpha}. \tag{2.116}$$

Proof. First of all, (2.116) is equivalent to the following. For $\theta \in [0, 2\alpha^k \pi)$,

$$\max(|\cos(\theta)|, |\cos(\alpha\theta)|) > \frac{1}{2\alpha}. \tag{2.117}$$

Since the period of $|\cos(\theta)|$ and $|\cos(\alpha\theta)|$ are 2π , it is enough to show that (2.117) holds for $\theta \in [0, 2\pi]$. We first consider when $\theta \in [0, \pi]$. If $|\cos(\theta)| > \frac{1}{2\alpha}$, then we are done. Let us assume that $|\cos(\theta)| \leq \frac{1}{2\alpha}$. Let $\alpha = 2k$ for $k \geq 1$ and $\cos \theta = x$. Since $\theta \in [0, \pi]$, $\cos \theta$ is bijective which forces $\theta = \arccos(x)$. Then $\cos(2k\theta) = \cos(2k \arccos(x))$ for $|x| \leq \frac{1}{4k}$ where $\frac{1}{4k} = \frac{1}{2\alpha}$. Note that the roots of $\cos(2k \arccos(x))$ are

$$x = \cos\left(\frac{t + (1/2)\pi}{2k}\right), \text{ for } t = 0, \dots, 2k-1. \tag{2.118}$$

Moreover,

$$\frac{d}{dx} \cos(2k \arccos(x)) = \sin(2k \arccos(x)) \cdot \frac{2k}{\sqrt{1-x^2}}. \quad (2.119)$$

This implies that the points of the extrema of $\cos(2k \arccos(x))$ are

$$x = \cos\left(\frac{t}{2k}\pi\right), \quad \text{for } t = 0, \dots, 2k. \quad (2.120)$$

Observe that

$$|\cos(2k \arccos(x))| = 1 > \frac{1}{4k} \quad \text{for } x = \cos\left(\frac{t}{2k}\pi\right), \quad (2.121)$$

where $t = 0, \dots, 2k$. Moreover,

$$\frac{1}{4k} < \sin\left(\frac{\pi}{8k}\right) = \cos\left(\frac{k - (1/4)}{2k}\pi\right) < \cos\left(\frac{k - (1/2)}{2k}\pi\right) < \cos\left(\frac{k - 1}{2k}\pi\right), \quad (2.122)$$

where $\cos\left(\frac{k-1}{2k}\pi\right)$ is the first positive extreme point and $\cos\left(\frac{k-(1/2)}{2k}\pi\right)$ is the first positive root of $\cos(2k \arccos(x))$. Combining this together with (2.119)-(2.121), we conclude that

$$\cos(2k \arccos(x)) \text{ is positive and decreasing from } 0 \text{ to } \cos\left(\frac{k - (1/2)}{2k}\pi\right). \quad (2.123)$$

Applying this with (2.122), we have

$$\begin{aligned} \cos\left(2k \arccos\left(\frac{1}{4k}\right)\right) &> \cos\left(2k \arccos\left(\cos\left(\frac{k - (1/4)}{2k}\pi\right)\right)\right) \\ &= \frac{\sqrt{2}}{2} > \frac{1}{4k}. \end{aligned} \quad (2.124)$$

Since $\arccos(-x) = \pi - \arccos(x)$ for $|x| \leq 1$, $\cos(2k \arccos(x))$ is even, thus

$$\cos(2k \arccos(x)) > \frac{1}{4k} \quad \text{for } |x| = \frac{1}{4k}. \quad (2.125)$$

Consequently, $\cos(2k \arccos(x)) > \frac{1}{4k}$ for $|x| \leq \frac{1}{4k}$. Similarly, for the case when $\theta \in (\pi, 2\pi]$ (2.125) holds. This finishes the proof of the lemma. \square

In the following example, we provide a counterexample to the classical Fatou Theorem in unit disk for polyanalytic function of order m which is motivated from the Mazalov's example in [21].

Example 2.17. Let $\alpha \in \mathbb{N}$ with $\alpha \geq 2$ and $m \in \mathbb{N}$ with $m \geq 2$. Consider a function $u_{\alpha,m}(z)$ in unit disk $B(0,1)$ by

$$u_{\alpha,m}(z) := (1 - z\bar{z}) \sum_{k=1}^{\infty} \alpha^k z^{\alpha^k} + \bar{z}^{m-1} \quad (2.126)$$

Then $u_{\alpha,m} \in \text{PA}_m(B(0,1))$. In addition, if $1 - \frac{1}{\alpha^N} \leq |z| \leq 1 - \frac{1}{\alpha^{N+1}}$ for $N \in \mathbb{N}$, then

$$\begin{aligned} |u_{\alpha,m}(z)| &< \frac{2}{\alpha^N} \left(\sum_{k=1}^N \alpha^k + \sum_{k=N+1}^{\infty} \alpha^k \left(1 - \frac{1}{\alpha^{N+1}}\right)^{\alpha^k} \right) + 1 \\ &\leq 4 + 2 \sum_{k=1}^{\infty} \alpha^k e^{-\alpha^{k-1}} + 1 \\ &< 5 + 2\alpha. \end{aligned} \quad (2.127)$$

This shows the function $u_{\alpha,m}$ is bounded in $B(0,1)$. We claim that $u_{\alpha,m}$ does not have radial limits at any point on $\partial B(0,1)$. For more simplicity we denote $(1 - z\bar{z}) \sum_{k=1}^{\infty} \alpha^k z^{\alpha^k}$ by $v_{\alpha}(z)$. Write $z \in B(0,1)$ as $z = re^{i\theta}$ where $r = |z| < 1$ and $\theta = \text{Arg}(z)$. According to [21], for even α sufficiently large, the function v_{α} does not have radial limits at any point on $\partial B(0,1)$. Indeed, the real part of v_{α} , denoted by f_{α} , is

$$f_{\alpha}(z) = \text{Re}(v_{\alpha}(z)) = (1 - r^2) \sum_{k=1}^{\infty} \alpha^k r^{\alpha^k} \cos(\alpha^k \theta). \quad (2.128)$$

Let $\alpha \in \mathbb{N}$ be even number. Fix an arbitrary angle $\theta \in [0, 2\pi)$.

According to Lemma 2.16, we have

$$\max(|\cos(\alpha^k \theta)|, |\cos(\alpha^{k+1} \theta)|) > \frac{1}{2\alpha}. \quad (2.129)$$

Going further, consider the sequences $r_N = 1 - \frac{1}{\alpha^N}$ and $\rho_N = 1 - \frac{\log \alpha}{\alpha^N}$. Suppose that there exists a sequence of indices N such that

$$|\cos(\alpha^N \theta)| > \frac{1}{\sqrt{\alpha}}. \quad (2.130)$$

Then we have

$$\begin{aligned}
& |f_\alpha(r_N e^{i\theta})| \\
&= \left| (1 - r_N^2) \sum_{k=1}^{\infty} \alpha^k (r_N)^{\alpha^k} \cos(\alpha^k \theta) \right| \\
&\geq \frac{1}{\alpha^N} \left(\alpha^N (r_N)^{\alpha^N} |\cos(\alpha^N \theta)| - \sum_{k=1}^{N-1} \alpha^k - \sum_{k=N+1}^{\infty} \alpha^k \left(1 - \frac{1}{\alpha^N}\right)^{\alpha^k} |\cos(\alpha^k \theta)| \right) \\
&> \frac{1}{\alpha^N} \left(\frac{\alpha^N}{\sqrt{\alpha}} \left(1 - \frac{1}{\alpha^N}\right)^{\alpha^N} - 1 - \sum_{k=1}^{\infty} \alpha^k e^{-\alpha^{k-1}} \right) \\
&> \frac{C}{\sqrt{\alpha}} \tag{2.131}
\end{aligned}$$

for sufficiently large α . Similarly,

$$\begin{aligned}
|f_\alpha(\rho_N e^{i\theta})| &\leq (1 - \rho_N^2) \sum_{k=1}^{\infty} \alpha^k (\rho_N)^{\alpha^k} |\cos(\alpha^k \theta)| \\
&\leq \frac{2 \log \alpha}{\alpha^N} \left(\alpha^N \left(1 - \frac{\log \alpha}{\alpha^N}\right)^{\alpha^N} + \sum_{k=1}^{N-1} \alpha^k + \sum_{k=N+1}^{\infty} \alpha^k \left(1 - \frac{\log \alpha}{\alpha^N}\right)^{\alpha^k} \right) \\
&\leq \frac{2 \log \alpha}{\alpha} \left(\alpha \left(1 - \frac{\log \alpha}{\alpha^N}\right)^{\alpha^N} + 1 + \sum_{k=1}^{\infty} \alpha^k e^{-\alpha^k \log \alpha} \right) \\
&< \frac{C \log \alpha}{\alpha} \tag{2.132}
\end{aligned}$$

for all α . Combining this with (2.131), we obtain that for sufficiently large α

$$|f_\alpha(r_N e^{i\theta}) - f_\alpha(\rho_N e^{i\theta})| > \frac{C}{\sqrt{\alpha}}. \tag{2.133}$$

We are left with the case when

$$|\cos(\alpha^k \theta)| \leq \frac{1}{\sqrt{\alpha}} \tag{2.134}$$

for sufficiently large k . Let α be even number. According to (2.116), there exists N such that

$$|\cos(\alpha^{N-1} \theta)| \leq \frac{1}{\sqrt{\alpha}} \tag{2.135}$$

and

$$\frac{1}{2\alpha} < |\cos(\alpha^N \theta)| \leq \frac{1}{\sqrt{\alpha}}. \tag{2.136}$$

This implies that

$$\sum_{k=1}^{N-1} \alpha^k r^{\alpha^k} |\cos(\alpha^k \theta)| < C \alpha^{N-2} \sqrt{\alpha}. \quad (2.137)$$

This forces

$$\begin{aligned} & |f_\alpha(r_N e^{i\theta})| \quad (2.138) \\ &= \left| (1 - r_N^2) \sum_{k=1}^{\infty} \alpha^k (r_N)^{\alpha^k} \cos(\alpha^k \theta) \right| \\ &\geq \frac{1}{\alpha^N} \left(\alpha^N (r_N)^{\alpha^N} |\cos(\alpha^N \theta)| - \sum_{k=1}^{N-1} \alpha^k - \sum_{k=N+1}^{\infty} \alpha^k \left(1 - \frac{1}{\alpha^N}\right)^{\alpha^k} |\cos(\alpha^k \theta)| \right) \\ &> \frac{1}{\alpha} \left(\frac{1}{2} \left(1 - \frac{1}{\alpha^N}\right)^{\alpha^N} - \frac{2}{\alpha} - \sum_{k=1}^{\infty} \alpha^k e^{-\alpha^{k-1}} \right) \\ &> \frac{C}{\alpha} \quad (2.139) \end{aligned}$$

for sufficiently large α and

$$\begin{aligned} & |f_\alpha(\rho_N e^{i\theta})| \quad (2.140) \\ &\leq (1 - \rho_N^2) \sum_{k=1}^{\infty} \alpha^k (\rho_N)^{\alpha^k} |\cos(\alpha^k \theta)| \\ &\leq \frac{2 \log \alpha}{\alpha^N} \left(\sum_{k=1}^{N-1} \alpha^k \left(1 - \frac{\log \alpha}{\alpha^N}\right)^{\alpha^k} |\cos(\alpha^k \theta)| + \sum_{k=N}^{\infty} \alpha^k \left(1 - \frac{\log \alpha}{\alpha^N}\right)^{\alpha^k} \right) \\ &\leq \frac{2 \log \alpha}{\alpha} \left(\frac{C}{\sqrt{\alpha}} + \sum_{k=1}^{\infty} \alpha^k e^{-\alpha^k \log \alpha} \right) \\ &< \frac{C \log \alpha}{\alpha \sqrt{\alpha}} \end{aligned}$$

for all α . In particular, for sufficiently large even number α

$$|f_\alpha(r_N e^{i\theta}) - f_\alpha(\rho_N e^{i\theta})| > \frac{C}{\alpha}. \quad (2.141)$$

This shows that for sufficiently large even number α , the real part of v_α does not have radial limits at any point on $\partial B(0, 1)$. This further implies that there is no radial limit of v_α on $\partial B(0, 1)$. Since $u_{\alpha, m}(z) = v_\alpha(z) + \bar{z}^{m-1}$ and

there exists radial limit of \bar{z}^{m-1} at every point on $\partial B(0, 1)$, thus $u_{\alpha, m}$ does not have radial limits at every point on $\partial B(0, 1)$. This provides an example of a bounded polyanalytic function of order $m \geq 2$ in $B(0, 1)$ which does not have radial limits at any point on $\partial B(0, 1)$.

Next, we provide a counterexample to the classical maximum principle of analytic function in the case of the polyanalytic function of any order $m > 1$.

Example 2.18. Let $m \in \mathbb{N}$ and consider a function $u_m(z) := 1 - (z\bar{z})^{m-1}$. Then $u_m \in \text{PA}_m(B(0, 1))$. However, for $z \in \partial B(0, 1)$,

$$u_m(z) = 0, \quad (2.142)$$

with $\max_{z \in B(0, 1)} |u_m(z)| > 0$ which contradicts to maximum principle.

2.5 Higher-order Fatou Theorems

In this section we study a couple of Fatou theorems and integral representation theorems with different assumption on the domain. The following theorem provides a reproducing formula for polyanalytic functions in bounded open sets with Ahlfors regular boundary.

Theorem 2.19. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a bounded open set with Ahlfors regular boundary. Fix $m \in \mathbb{N}$ and $\kappa \in (0, \infty)$. Abbreviate $\sigma := \mathcal{H}^1|_{\partial\Omega}$. Suppose $u \in \mathcal{C}^\infty(\Omega)$ is such that $\bar{\partial}^m u = 0$ in Ω with the property that for any order $\ell \in \{0, 1, \dots, m-1\}$*

$$\mathcal{N}_\kappa(\bar{\partial}^\ell u) \in L^1(\partial\Omega, \sigma) \quad \text{and} \quad \exists \left(\bar{\partial}^\ell u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \quad (2.143)$$

Then for any $z \in \Omega$ there holds

$$u(z) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial_*\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta. \quad (2.144)$$

Proof. Fix $z \in \Omega$. Let us consider two functions F_1, F_2 in Ω given by

$$\begin{aligned} F_1 &:= \frac{1}{2} \sum_{k=0}^{m-1} (-1)^{m+1+k} (\bar{\partial}^k E_m)(\cdot - z) (\bar{\partial}^{m-1-k} u) \\ F_2 &:= iF_1. \end{aligned} \quad (2.145)$$

Then we have

$$\sum_{k=0}^{m-1} \frac{(-1)^{m+1+k}}{2i} (\bar{\partial}^k E_m)(\cdot - z) (\bar{\partial}^{m-1-k} u) i\nu = \langle \nu, \vec{F} \rangle, \quad (2.146)$$

where $\vec{F} = (F_1, F_2)$. Now, we apply the Divergence Theorem from [25] to the vector field $\vec{F} = (F_1, F_2)$, which leads to

$$(\mathcal{C}_b^\infty(\Omega))^* (\operatorname{div} \vec{F}, 1)_{\mathcal{C}_b^\infty(\Omega)} = \int_{\partial_* \Omega} \langle \nu, \vec{F} \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} \rangle d\sigma, \quad (2.147)$$

where $\mathcal{C}_b^\infty(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : f \text{ bounded in } \Omega\}$. By direct calculation of the divergence of \vec{F} , we get

$$\begin{aligned} \operatorname{div} \vec{F} &= \partial_1 F_1 + \partial_2 F_2 \\ &= \partial_1 F_1 + i\partial_2 F_1 \\ &= 2\bar{\partial} F_1, \end{aligned} \quad (2.148)$$

and

$$\begin{aligned} 2\bar{\partial} F_1 &= 2\bar{\partial} \left[\frac{1}{2} \sum_{k=0}^{m-1} (-1)^{m+1+k} (\bar{\partial}^k E_m)(\cdot - z) (\bar{\partial}^{m-1-k} u) \right] \\ &= \sum_{k=0}^{m-1} (-1)^{m+1+k} (\bar{\partial}^{k+1} E_m)(\cdot - z) (\bar{\partial}^{m-1-k} u) \\ &\quad + \sum_{k=0}^{m-1} (-1)^{m+1+k} (\bar{\partial}^k E_m)(\cdot - z) (\bar{\partial}^{m-k} u) \\ &:= I + II. \end{aligned} \quad (2.149)$$

In particular,

$$\begin{aligned}
I &= \sum_{k=0}^{m-1} (-1)^{m+1+k} (\bar{\partial}^{k+1} E_m)(\cdot - z) (\bar{\partial}^{m-1-k} u) \\
&= \sum_{k=0}^{m-2} (-1)^{m+1+k} (\bar{\partial}^{k+1} E_m)(\cdot - z) (\bar{\partial}^{m-1-k} u) + (\bar{\partial}^m E_m)(\cdot - z) u \\
&= \sum_{k=1}^{m-1} (-1)^{m+k} (\bar{\partial}^k E_m)(\cdot - z) (\bar{\partial}^{m-k} u) + u \delta_z.
\end{aligned} \tag{2.150}$$

Since $\bar{\partial}^m u = 0$ in Ω , we have

$$I = -II + u \delta_z. \tag{2.151}$$

Hence, $\operatorname{div} \vec{F} = u \delta_z \in \mathcal{E}'(\Omega) \hookrightarrow (\mathcal{C}_b^\infty(\Omega))^*$. In addition,

$$\begin{aligned}
u(z) &=_{(\mathcal{C}_b^\infty(\Omega))^*} (u \delta_z, 1)_{(\mathcal{C}_b^\infty(\Omega))} \\
&=_{(\mathcal{C}_b^\infty(\Omega))^*} (\operatorname{div} \vec{F}, 1)_{(\mathcal{C}_b^\infty(\Omega))} \\
&= \int_{\partial_* \Omega} \langle \nu, \vec{F} \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} \rangle d\sigma \\
&= \int_{\partial_* \Omega} \left[\nu_1 \left(F_1 \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} \right) + i \nu_2 \left(F_1 \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} \right) \right] d\sigma \\
&= \frac{1}{i} \int_{\partial_* \Omega} \left(F_1 \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} \right) d\zeta \\
&= \frac{1}{2i} \sum_{k=0}^{m-1} (-1)^{m+1+k} \int_{\partial_* \Omega} (\bar{\partial}^k E_m)(\zeta - z) (\bar{\partial}^{m-1-k} u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta.
\end{aligned} \tag{2.152}$$

In addition, the integral expression in (2.152) can be written as

$$u(z_0) = \sum_{k=0}^{m-1} \frac{(-1)^k}{2i} \int_{\partial_* \Omega} (\bar{\partial}^{m-1-k} E_m)(\zeta - z_0) (\bar{\partial}^k u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta. \tag{2.153}$$

According to (2.42), one has

$$(\bar{\partial}^{m-1-k} E_m)(\zeta) = \frac{\bar{\zeta}^k}{k!} \cdot \frac{1}{\pi \zeta}. \tag{2.154}$$

Substituting this into (2.153) yields

$$u(z_0) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial_* \Omega} \frac{\overline{(z - \zeta)}^k}{k! (\zeta - z)} (\bar{\partial}^k u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta. \tag{2.155}$$

This completes the proof of the theorem. \square

The following theorem presents a new Fatou theorem in arbitrary UR domains and it builds on a new integral representation theorem in bounded UR domains.

Theorem 2.20. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a UR domain. Fix an integrability exponent $p \in (\frac{1}{2}, \infty)$, an aperture parameter $\kappa \in (0, \infty)$ and some power $m \in \mathbb{N}$. Abbreviate $\sigma := \mathcal{H}^1 \llcorner \partial\Omega$. Suppose $u \in \mathcal{C}^\infty(\Omega)$ is such that $\bar{\partial}^m u = 0$ in Ω and for any $\ell \in \{0, 1, \dots, m-1\}$, $\mathcal{N}_\kappa(\bar{\partial}^\ell u) \in L^p(\partial\Omega, \sigma)$. Then*

$$\exists \left(\bar{\partial}^\ell u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega \quad (2.156)$$

for each $\ell \in \{0, 1, \dots, m-1\}$.

In order to prove the new Fatou theorem, we use the following structure theorem.

Theorem 2.21. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be an arbitrary open set, $m \in \mathbb{N}$, $u \in \mathcal{C}^\infty(\Omega)$, $\bar{\partial}^m u = 0$ in Ω . Then $\exists! u_0, u_1, u_2, \dots, u_{m-1} \in \mathcal{O}(\Omega)$ such that*

$$u(z) = u_0(z) + \bar{z}u_1(z) + \dots + \bar{z}^{m-1}u_{m-1}(z), \quad \forall z \in \Omega. \quad (2.157)$$

In fact,

$$\begin{aligned} u_{m-1} &= \frac{1}{(m-1)!} \left(\bar{\partial}^{m-1} u \right) \\ u_{m-2} &= \frac{1}{(m-2)!} \bar{\partial}^{m-2} \left(I - \frac{1}{(m-1)!} \bar{z}^{m-1} \bar{\partial}^{m-1} \right) u \\ &\vdots \\ u_1 &= \bar{\partial} \left(I - \frac{1}{2} \bar{z}^2 \bar{\partial}^2 \right) \dots \left(I - \frac{1}{(m-1)!} \bar{z}^{m-1} \bar{\partial}^{m-1} \right) u \\ u_0 &= (I - \bar{z} \bar{\partial}) \left(I - \frac{1}{2} \bar{z}^2 \bar{\partial}^2 \right) \dots \left(I - \frac{1}{(m-1)!} \bar{z}^{m-1} \bar{\partial}^{m-1} \right) u. \end{aligned} \quad (2.158)$$

Proof. We first claim the following:

$$\text{Ker } \bar{\partial}^m = \text{Ker } \bar{\partial} + \bar{z} \text{Ker } \bar{\partial} + \dots + \bar{z}^{m-1} \text{Ker } \bar{\partial}. \quad (2.159)$$

We prove the claim by induction on m . If $m = 1$, then (2.159) trivially holds. Let us assume (2.159) for $m = k - 1$. It suffices to show that

$$\text{Ker } \bar{\partial}^{k-1} + \bar{z}^{k-1} \text{Ker } \bar{\partial} = \text{Ker } \bar{\partial}^k. \quad (2.160)$$

To justify $\text{Ker } \bar{\partial}^{k-1} + \bar{z}^{k-1} \text{Ker } \bar{\partial} \subseteq \text{Ker } \bar{\partial}^k$, let $f = g + \bar{z}^{k-1}h$ for some $g \in \text{Ker } \bar{\partial}^{k-1}$, $h \in \text{Ker } \bar{\partial}$. Then $\bar{\partial}^k f = \bar{\partial}^k g + \bar{\partial}^k (\bar{z}^{k-1}h) = 0$ which implies that $f \in \text{Ker } \bar{\partial}^k$. For the other direction, let $f \in \text{Ker } \bar{\partial}^k$, then f can be expressed as

$$f = \left(I - \frac{1}{(k-1)!} \bar{z}^{k-1} \bar{\partial}^{k-1} \right) f + \left(\frac{1}{(k-1)!} \bar{z}^{k-1} \bar{\partial}^{k-1} f \right). \quad (2.161)$$

Observe from $f \in \text{Ker } \bar{\partial}^k$ that

$$\begin{aligned} \left(I - \frac{1}{(k-1)!} \bar{z}^{k-1} \bar{\partial}^{k-1} \right) f &\in \text{ker } \bar{\partial}^{k-1} \\ \left(\frac{1}{(k-1)!} \bar{\partial}^{k-1} f \right) &\in \text{ker } \bar{\partial}. \end{aligned} \quad (2.162)$$

This proves the claim (2.159).

In order to complete the proof of the structure theorem, let $u \in \mathcal{C}^\infty(\Omega)$ with $\bar{\partial}^m u = 0$ in Ω . From (2.159), there exist $u_0, u_1, u_2, \dots, u_{m-1} \in \text{Ker } \bar{\partial}$ such that

$$u(z) = u_0(z) + \bar{z}u_1(z) + \dots + \bar{z}^{m-1}u_{m-1}(z), \quad \forall z \in \Omega. \quad (2.163)$$

Since $u \in \text{Ker } \bar{\partial}^m$ and $u_\ell \in \text{Ker } \bar{\partial}$ for $\ell \in \{0, \dots, m-1\}$, taking $\bar{\partial}^{m-1}$ in (2.163) yields

$$\bar{\partial}^{m-1} u = (m-1)! u_{m-1} \quad \text{in } \Omega \quad (2.164)$$

which forces

$$u_{m-1} = \frac{1}{(m-1)!} \left(\bar{\partial}^{m-1} u \right) \quad \text{in } \Omega \quad (2.165)$$

Now, let us substitute (2.165) into (2.163) and take $\bar{\partial}^{m-2}$, then we obtain that

$$\bar{\partial}^{m-2} u = (m-2)! u_{m-2} + \bar{\partial}^{m-2} \left(\frac{1}{(m-1)!} \bar{z}^{m-1} \bar{\partial}^{m-1} u \right) \quad \text{in } \Omega. \quad (2.166)$$

This yields

$$u_{m-2} = \frac{1}{(m-2)!} \bar{\partial}^{m-2} \left(I - \frac{1}{(m-1)!} \bar{z}^{m-1} \bar{\partial}^{m-1} \right) u \text{ in } \Omega. \quad (2.167)$$

By continuing this process, we have (2.158). We are left with justifying the uniqueness. Let $v_0, v_1, \dots, v_{m-1} \in \text{Ker } \bar{\partial}$ such that

$$u(z) = v_0(z) + \bar{z}v_1(z) + \dots + \bar{z}^{m-1}v_{m-1}(z), \quad \forall z \in \Omega. \quad (2.168)$$

In particular,

$$\begin{aligned} & u_0(z) + \bar{z}u_1(z) + \dots + \bar{z}^{m-1}u_{m-1}(z) \\ &= v_0(z) + \bar{z}v_1(z) + \dots + \bar{z}^{m-1}v_{m-1}(z), \end{aligned} \quad (2.169)$$

for $z \in \Omega$. Since $u_\ell, v_\ell \in \text{Ker } \bar{\partial}$ for all $\ell \in \{0, 1, \dots, m-1\}$, taking $\bar{\partial}^{m-1}$ into (2.169) yields

$$(m-1)!u_{m-1}(z) = (m-1)!v_{m-1}(z), \quad \forall z \in \Omega \quad (2.170)$$

which gives $u_{m-1} = v_{m-1}$ in Ω . The equation (2.169) is simplified as

$$\begin{aligned} & u_0(z) + \bar{z}u_1(z) + \dots + \bar{z}^{m-2}u_{m-2}(z) \\ &= v_0(z) + \bar{z}v_1(z) + \dots + \bar{z}^{m-2}v_{m-2}(z), \end{aligned} \quad (2.171)$$

for $z \in \Omega$. We take $\bar{\partial}^{m-2}$ into above, then we have

$$(m-2)!u_{m-2}(z) = (m-2)!v_{m-2}(z), \quad \forall z \in \Omega \quad (2.172)$$

which implies that $u_{m-2} = v_{m-2}$ in Ω . Repeating this process forces $u_\ell = v_\ell$ in Ω for all $\ell \in \{0, 1, \dots, m-1\}$. This finishes the proof of the structure theorem. \square

We now turn to the proof of Theorem 2.20.

Proof of Theorem 2.20. Let u_0, u_1, \dots, u_{m-1} be associated with the given u as

in the Structure Theorem. Since $\bar{\partial}^m u = 0$ in Ω , from (2.158) we have

$$\begin{aligned}
u_{m-1} &\sim \bar{\partial}^{m-1} u \\
u_{m-2} &\sim \bar{\partial}^{m-2} u + \bar{z} \bar{\partial}^{m-1} u \\
&\vdots \\
u_1 &\sim \bar{\partial} u + \bar{z} \bar{\partial}^2 u + \cdots + \bar{z}^{m-2} \bar{\partial}^{m-1} u \\
u_0 &\sim u + \bar{z} \bar{\partial} u + \cdots + \bar{z}^{m-1} \bar{\partial}^{m-1} u.
\end{aligned} \tag{2.173}$$

Recall that we are assuming

$$\mathcal{N}_\kappa(\bar{\partial}^k u) \in L^p(\partial\Omega, \sigma) \text{ for } k \in \{0, 1, \dots, m-1\}. \tag{2.174}$$

Then $\mathcal{N}_\kappa(u_j) \in L^p(\partial\Omega, \sigma)$ and $u_j \in \mathcal{O}(\Omega)$, $\forall j \in \{0, 1, \dots, m-1\}$.

Since Ω is UR domain, from the Fatou Theorem for holomorphic functions in arbitrary UR domain from [28] we have $\exists u_j \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ σ -a.e. on $\partial\Omega$ for $0 \leq j \leq m-1$.

Note that $\forall \ell \in \{0, 1, \dots, m-1\}$, $(\bar{\partial}^\ell u)(z)$ only involves $\bar{z}^j u_k(z)$ for $0 \leq j, k \leq m-1$ since any $\bar{\partial}$ taking on one of the u_0, u_1, \dots, u_{m-1} annihilates it, and $\bar{\partial}^\ell(\bar{z}^k)$ is a constant multiple of a power of $\bar{z}^{k-\ell}$ if $\ell \leq k$, and zero otherwise. That is, for $\ell \in \{0, 1, \dots, m-1\}$

$$(\bar{\partial}^\ell u)(z) = \sum_{0 \leq j, k \leq m-1} a_{jk} \bar{z}^j u_k(z) \tag{2.175}$$

for some constant a_{jk} . This implies that for $\ell \in \{0, 1, \dots, m-1\}$

$$\exists (\bar{\partial}^\ell u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \sigma\text{-a.e. on } \partial\Omega, \tag{2.176}$$

which completes the proof of Theorem 2.20. \square

As a consequence of the New Fatou Theorem in arbitrary UR domains and the Integral Representation Formula in bounded open sets with Ahlfors regular boundary, we obtain a version of the Integral Representation Formula that does not assume the existence of non-tangential boundary traces. Specifically, we present the following new Integral Representation Formula.

Theorem 2.22. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a bounded UR domain. Let $\sigma = \mathcal{H}^1 \lfloor \partial\Omega$. Fix $m \in \mathbb{N}$ and $\kappa \in (0, \infty)$. Assume $u \in \mathcal{C}^\infty(\Omega)$ is such that $\bar{\partial}^m u = 0$ in Ω and $\mathcal{N}_\kappa(\bar{\partial}^\ell u) \in L^1(\partial\Omega, \sigma)$ for each $\ell \in \{0, 1, \dots, m-1\}$. Then, for any order $\ell \in \{0, 1, \dots, m-1\}$,*

$$\exists \left(\bar{\partial}^\ell u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. point on } \partial\Omega \quad (2.177)$$

and for any $z \in \Omega$ there holds

$$u(z) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta. \quad (2.178)$$

Proof. This readily follows from Theorem 2.19 and Theorem 2.20. \square

The new Fatou Theorem 2.20 establishes the existence of non-tangential traces for the differential operator $\bar{\partial}$ up to order $m-1$, under the condition that the non-tangential maximal function of $\bar{\partial}^\ell u$ belongs to $L^p(\partial\Omega, \sigma)$ for each $\ell \in \{0, 1, \dots, m-1\}$ in arbitrary UR domains. We now develop this theorem to include all derivatives up to order $m-1$ in UR domains with compact boundary.

Theorem 2.23. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a UR domain with compact boundary. Fix $m \in \mathbb{N}$, $p \in (1, \infty)$, and $\kappa \in (0, \infty)$. Assume $u \in \mathcal{C}^\infty(\Omega)$ is such that $\bar{\partial}^m u = 0$ in Ω and $\mathcal{N}_\kappa(\nabla^\ell u) \in L^p(\partial\Omega, \sigma)$ for $\ell \in \{0, 1, \dots, m-1\}$. If Ω is an exterior domain, it is assumed that the nontangential maximal operator is truncated and one also asks that the auxiliary functions $\{u_j\}_{0 \leq j \leq m-1}$ associated with u as in (2.56) vanish at infinity. Then, for each $\ell \in \{0, 1, \dots, m-1\}$, there holds*

$$\exists \left(\nabla^\ell u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \quad (2.179)$$

Proof. We first claim that if $\ell \in \mathbb{N}_0$ with $\ell \leq m-2$ is such that

$$\exists \left(\nabla^r u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \forall r \in \{0, 1, \dots, \ell\}, \quad (2.180)$$

then

$$\dot{g} := \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq \ell}} \in \text{CWA}_\ell [L_1^p(\partial\Omega, \sigma)] \quad (2.181)$$

$$\text{where } \forall a, b \in \mathbb{N}_0 \text{ with } a + b \leq \ell \quad g_{(a,b)} := \left(\partial^a \bar{\partial}^b u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}. \quad (2.182)$$

Indeed, according to [28], for Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ if $w \in \mathcal{C}^1(\Omega)$ satisfies

$$\begin{aligned} \mathcal{N}_\kappa w, \mathcal{N}_\kappa(\nabla w) &\in L^p(\partial\Omega, \sigma), \quad \text{for } 1 < p < \infty, \text{ and} \\ &\exists w \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. on } \partial\Omega, \end{aligned} \quad (2.183)$$

then

$$w \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \in L_1^p(\partial\Omega, \sigma). \quad (2.184)$$

In addition, for Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ if $w \in \mathcal{C}^1(\Omega)$ satisfies

$$\begin{aligned} \mathcal{N}_\kappa w, \mathcal{N}_\kappa(\nabla w) &\in L^p(\partial\Omega, \sigma), \quad \text{for } 1 \leq p \leq \infty, \text{ and} \\ &\exists w \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \quad \text{and} \quad \exists (\nabla w) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. on } \partial\Omega, \end{aligned} \quad (2.185)$$

then

$$w \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \in L_1^p(\partial\Omega, \sigma) \quad \text{and} \quad \forall j, k \in \{1, \dots, n\} \quad (2.186)$$

$$\partial_{\tau_{jk}} \left(w \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \right) = \nu_j (\partial_k w) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} - \nu_k (\partial_j w) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}. \quad (2.187)$$

For each $a, b \in \mathbb{N}_0$ with $a + b \leq \ell$ the function $w := \partial^a \bar{\partial}^b u \in \mathcal{C}^\infty(\Omega)$ satisfies

$$\mathcal{N}_\kappa w \in L^p(\partial\Omega, \sigma), \quad \mathcal{N}_\kappa(\nabla w) \in L^p(\partial\Omega, \sigma), \quad (2.188)$$

$$\exists w \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. on } \partial\Omega. \quad (2.189)$$

Indeed, (2.188) follows from the hypotheses that the non-tangential maximal function of the gradient of u belongs to $L^p(\partial\Omega, \sigma)$ up to the order $m - 1$ and (2.189) follows from the assumption on the claim in (2.179) with $r := a + b$. This implies that for $a + b \leq \ell$, the arrays $g_{(a,b)}$ in (2.182) belong $L_1^p(\partial\Omega, \sigma)$. To prove the claim, we are left with showing the compatibility condition CC

for the array $g_{(a,b)}$. Pick $a, b \in \mathbb{N}_0$ with $a + b \leq \ell - 1$, applying (2.187) with $n = 2$ to $w = g_{(a,b)} \in \mathcal{C}^\infty(\Omega)$

$$\begin{aligned} \partial_\tau g_{(a,b)} &= \partial_\tau \left[\left(\partial^a \bar{\partial}^b u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \right] \\ &= \nu_1 \left(\partial_y \left(\partial^a \bar{\partial}^b u \right) \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} - \nu_2 \left(\partial_x \left(\partial^a \bar{\partial}^b u \right) \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}. \end{aligned} \quad (2.190)$$

Combining this with the notation in (2.48)

$$\partial_\tau g_{(a,b)} = i \left(\nu g_{(a+1,b)} - \bar{\nu} g_{(a,b+1)} \right) \quad (2.191)$$

which forces the compatibility condition CC in (2.50) which completes the proof of the claim. With the claim in our hand, we proceed the proof by induction. For $\ell = 0$, we need to show that there exists $u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ at σ -a.e. on $\partial\Omega$. However, this already is conducted in Theorem 2.20. To proceed with inductive step, we assume for $\ell \in \{0, 1, \dots, m-2\}$ is such that

$$\exists (\nabla^r u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. on } \partial\Omega \quad \forall r \in \{0, 1, \dots, \ell\}. \quad (2.192)$$

Then, we claim,

$$\exists (\nabla^{\ell+1} u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. on } \partial\Omega. \quad (2.193)$$

Write (2.60) from Theorem 2.7 as

$$\begin{aligned} u(z) &= \sum_{k=0}^{\ell} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta \\ &\quad + \sum_{k=\ell+1}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta. \end{aligned} \quad (2.194)$$

Introducing $\dot{g} = \{g_{(a,b)}\}_{\substack{a+b \in \mathbb{N}_0 \\ a+b \leq \ell}}$ where $g_{(a,b)} = \left(\partial^a \bar{\partial}^b u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ as in (2.182) and according to the claim in (2.181) we know that $\dot{g} \in \text{CWA}_\ell [L_1^p(\partial\Omega, \sigma)]$. Combining this with the definition of the poly-Cauchy operator in (2.55), we rewrite (2.194) as

$$u(z) = \left(\mathcal{C}_\ell \dot{g} \right) (z) + \sum_{k=\ell+1}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta. \quad (2.195)$$

Recall the Corollary 2.11 written for ℓ in place of $m - 1$, $\forall r, s \in \mathbb{N}_0$ with $r + s \leq \ell$ we have

$$\partial^s \bar{\partial}^r \left(\mathcal{C}_\ell \dot{g} \right) (z) = \sum_{k=0}^{\ell-s-r} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(s,k+r)} d\zeta \quad (2.196)$$

$$\begin{aligned} & - \sum_{j=0}^{s-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{j!}{(\ell+1-s-r+j)!} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{\ell+1-s-r+j}}}{(\zeta-z)^{j+1}} \right] \times \\ & \times g_{(s-1-j, \ell+1-s+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.197)$$

for all $z \in \Omega$. Since for fixed $z \in \Omega$,

$$\varphi(\zeta) := \frac{\overline{(z-\zeta)^{\ell+1-s-r+j}}}{(\zeta-z)^{j+1}} \quad (2.198)$$

is smooth for all ζ near $\partial\Omega$, applying the integration by parts on $\partial\Omega$ to the integral in (2.197) yields

$$\begin{aligned} & \int_{\partial\Omega} \frac{j!}{(\ell+1-s-r+j)!} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{\ell+1-s-r+j}}}{(\zeta-z)^{j+1}} \right] g_{(s-1-j, \ell+1-s+j)}(\zeta) d\sigma(\zeta) \\ & = - \int_{\partial\Omega} \frac{j!}{(\ell+1-s-r+j)!} \frac{\overline{(z-\zeta)^{\ell+1-s-r+j}}}{(\zeta-z)^{j+1}} \times \\ & \quad \times \left(\partial_{\tau} g_{(s-1-j, \ell+1-s+j)} \right) (\zeta) d\sigma(\zeta), \end{aligned} \quad (2.199)$$

where $\partial_{\tau} g_{(s-1-j, \ell+1-s+j)} \in L^p(\partial\Omega, \sigma)$ because $\dot{g} \in \text{CWA}_\ell [L_1^p(\partial\Omega, \sigma)]$. With an eye on the condition we presently seek, assume $r, s \in \mathbb{N}_0$ with $r + s = \ell$ and can write

$$\begin{aligned} \nabla_{x,y} \left(\partial^s \bar{\partial}^r \left(\mathcal{C}_\ell \dot{g} \right) \right) (z) & = \nabla_{x,y} \left(\mathcal{C} g_{(s,r)} \right) (z) \\ & + \sum_{j=0}^{s-1} \nabla_{x,y} \left[\frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{j+1} \frac{\overline{(z-\zeta)^{j+1}}}{(\zeta-z)^{j+1}} \times \right. \\ & \quad \left. \times \left(\partial_{\tau} g_{(s-1-j, \ell+1-s+j)} \right) (\zeta) d\sigma(\zeta) \right], \end{aligned} \quad (2.200)$$

where $g_{(s,r)} \in L_1^p(\partial\Omega, \sigma)$, $\partial_{\tau} g_{(s-1-j, \ell+1-s+j)} \in L^p(\partial\Omega, \sigma)$, and \mathcal{C} is the ordinary boundary-to-domain Cauchy operator as in (2.3). We apply the following two

facts found in [28]. The first fact in [28] that we are applying is that for UR domain $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$, $\sigma := \mathcal{H}^1 \llcorner \partial\Omega$, $f \in L^p_1(\partial\Omega, \sigma)$, $1 < p < \infty$,

$$(\mathcal{C}f)(z) := \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad \forall z \in \Omega. \quad (2.201)$$

Then $\forall \kappa \in (0, \infty)$,

$$\exists [\nabla(\mathcal{C}f)] \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \quad \text{at } \sigma\text{-a.e. on } \partial\Omega. \quad (2.202)$$

The second fact in [28] is a part of the Calderón-Zygmund theory, namely, for UR domain $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$, $\sigma := \mathcal{H}^1 \llcorner \partial\Omega$, $f \in L^p(\partial\Omega, \sigma)$, $1 < p < \infty$,

$$\begin{aligned} k &\in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{0\}), \quad k \text{ odd}, \\ k(\lambda z) &= \lambda^{-1}k(z), \quad \forall z \in \mathbb{C} \setminus \{0\}, \forall \lambda \in (0, \infty), \quad \text{and} \\ (\mathcal{T}f)(z) &:= \int_{\partial\Omega} k(z - \zeta)f(\zeta) d\sigma(\zeta), \quad \forall z \in \Omega. \end{aligned} \quad (2.203)$$

Then $\forall \kappa \in (0, \infty)$,

$$\exists (\mathcal{T}f) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \quad \text{at } \sigma\text{-a.e. on } \partial\Omega. \quad (2.204)$$

Using these two facts found in [28] and (2.200)

$$\exists \left(\nabla^{\ell+1} \left(\mathcal{C} \dot{g} \right) \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \quad \text{at } \sigma\text{-a.e. on } \partial\Omega. \quad (2.205)$$

Back to (2.194), there remains to consider

$$\nabla_{x,y}^{\ell+1} \left[\sum_{k=\ell+1}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z - \zeta)^k}}{k!(\zeta - z)} \left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta \right], \quad (2.206)$$

for $z = x + iy \in \Omega$. Using the fact that $d\zeta = i\nu(\zeta) d\sigma(\zeta)$ as in (2.1), we rewrite (2.206) as

$$\nabla_{x,y}^{\ell+1} \left[\sum_{k=\ell+1}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z - \zeta)^k}}{k!(\zeta - z)} \left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) i\nu(\zeta) d\sigma(\zeta) \right], \quad (2.207)$$

for $z \in \Omega$. According to Theorem 2.20 and the hypotheses of the theorem, one can notice that $\left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\cdot) i\nu(\cdot) \in L^p(\partial\Omega, \sigma)$ for $\ell + 1 \leq k \leq m - 1$.

Observe that the kernel

$$\nabla_{x,y}^{\ell+1} \left[\frac{\overline{(z - \zeta)^k}}{k!(\zeta - z)} \right] \quad (2.208)$$

gives rise of a boundary-to-domain integral operator as in (2.203) if $k = \ell + 1$, and is bounded if $k > \ell + 1$. For $k = \ell + 1$, using the second fact

$$\exists \nabla_{x,y}^{\ell+1} \left[\frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^{\ell+1}}}{(\ell+1)!(\zeta-z)} \left(\bar{\partial}^{\ell+1} u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} i\nu(\zeta) d\sigma(\zeta) \right] \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \quad (2.209)$$

at σ -a.e. on $\partial\Omega$. For $k > \ell + 1$, we use the following result about the weakly singular integral operators found in [28]. For an open set $\Omega \subseteq \mathbb{R}^2$ with an upper Ahlfors regular boundary, $f \in L^1(\partial\Omega, \sigma)$, $\sigma := \mathcal{H}^1 \llcorner \partial\Omega$,

$$(\mathcal{B}f)(x) := \int_{\partial\Omega} b(x,y)f(y)d\sigma(y), \quad \forall x \in \Omega, \quad (2.210)$$

with the property that

$$|b(x,y)| \leq \frac{C}{|x-y|^{1+\alpha}}, \quad \text{for some } \alpha > 0, \text{ constant } C > 0, \quad (2.211)$$

$$b(\cdot, y) \text{ is continuous on } \mathbb{R}^2 \setminus \{y\}. \quad (2.212)$$

Then

$$\exists (\mathcal{B}f) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. on } \partial\Omega. \quad (2.213)$$

Applying this to (2.207) with $k > \ell + 1$, one can conclude that

$$\exists \nabla_{x,y}^{\ell+1} \left[\sum_{k>\ell+1}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left(\bar{\partial}^k u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} (\zeta) i\nu(\zeta) d\sigma(\zeta) \right] \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \quad (2.214)$$

at σ -a.e. on $\partial\Omega$. This finishes the induction which completes proof of the theorem. \square

Turning our attention to a version of the new Integral Representation Formula for exterior domains.

Theorem 2.24. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be exterior with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^1 \llcorner \partial\Omega$ is a doubling measure on $\partial\Omega$. Fix $m \in \mathbb{N}$, $\varepsilon > 0$, and $\kappa \in (0, \infty)$. Suppose $u \in \mathcal{C}^\infty(\Omega)$ is such that $\bar{\partial}^m u = 0$ in Ω with the property that for each $\ell \in \{0, 1, \dots, m-1\}$*

$$\mathcal{N}_\kappa^\varepsilon \left(\bar{\partial}^\ell u \right) \in L^1(\partial\Omega, \sigma) \quad \text{and} \quad \exists \left(\bar{\partial}^\ell u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \text{ at } \sigma\text{-a.e. point on } \partial_{nta}\Omega. \quad (2.215)$$

Then there exists a polynomial P of degree less than or equal to $m - 2$ in \mathbb{R}^2 such that

$$u(z) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial_* \Omega} \frac{\overline{(z - \zeta)^k}}{k!(\zeta - z)} \left(\bar{\partial}^k u \right) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} (\zeta) d\zeta + P(z), \quad (2.216)$$

for each $z \in \Omega$.

Proof. Fix $z \in \Omega$. Let us define a vector field \vec{F}_z by

$$\vec{F}_z(\zeta) := \sum_{k=0}^{m-1} \frac{1}{2\pi i} \frac{\overline{(z - \zeta)^k}}{k!(\zeta - z)} \left(\bar{\partial}^k u \right) (\zeta), \text{ for } \zeta \in \Omega. \quad (2.217)$$

Since we have $u \in \mathcal{C}^\infty(\Omega)$, $\bar{\partial}^m u = 0$ in Ω , and $u(\zeta) = O(|\zeta|^{m-2})$ as $|\zeta| \rightarrow \infty$, this forces

$$\left(\bar{\partial}^k u \right) (\zeta) = O(|\zeta|^{m-2-k}), \forall k \in \{0, 1, \dots, m-1\}. \quad (2.218)$$

This further implies that for $\eta \in \mathbb{N}_0^2$ with $|\eta| \geq m-1$,

$$\begin{aligned} \left(\partial_{x,y}^\eta \vec{F}_z \right) (\zeta) &= C \sum_{k=0}^{m-1} |\zeta|^{k-1-|\eta|} \cdot |\zeta|^{m-2-k} \\ &= o(|\zeta|^{-1}), \end{aligned} \quad (2.219)$$

for some constant $C > 0$. Granted this, we conclude that $\forall \eta \in \mathbb{N}_0^2$ with $|\eta| \geq m-1$,

$$\partial_{x,y}^\eta \left[u - \mathcal{E} \left(\text{Tr}_{m-1}^{\mathbb{C}} u \right) \right] = 0 \text{ in } \Omega. \quad (2.220)$$

In turn, using Taylor Expansion this forces

$$u - \mathcal{E} \left(\text{Tr}_{m-1}^{\mathbb{C}} u \right) := P_0, \quad (2.221)$$

where P_0 is locally polynomial function of degree $\leq m-2$. That is,

$$P_0 \Big|_{\Omega_\infty} \text{ is polynomial of degree } \leq m-2. \quad (2.222)$$

This trivially and uniquely extends to some polynomial P in \mathbb{R}^2 of degree $\leq m-2$. We can therefore apply Divergence Theorem in [26] to obtain (2.216).

This completes the proof of Theorem 2.24. \square

The following is the other Integral Representation result for exterior domains combining with the higher-order Fatou Theorem.

Theorem 2.25. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be UR exterior domain. Set $\sigma := \mathcal{H}^1 \llcorner \partial\Omega$ and fix $m \in \mathbb{N}$ along with $\varepsilon > 0$ and $\kappa \in (0, \infty)$. Assume $u \in \mathcal{C}^\infty(\Omega)$ is such that $\bar{\partial}^m u = 0$ in Ω and $\mathcal{N}_\kappa(\bar{\partial}^\ell u) \in L^1(\partial\Omega, \sigma)$ for each $\ell \in \{0, 1, \dots, m-1\}$. Then $\exists \left(\bar{\partial}^\ell u\right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ at σ -a.e. point on $\partial\Omega$ and there exists a polynomial P of degree less than or equal to $m-2$ in \mathbb{R}^2 such that*

$$u(z) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial_*\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left(\bar{\partial}^k u\right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}(\zeta) d\zeta + P(z), \quad (2.223)$$

for each $z \in \Omega$.

Proof. This is an immediate consequence of Theorem 2.20 and Theorem 2.24. \square

2.6 Higher-order Hardy Spaces

The higher-order Fatou theorem in Theorem 2.23 suggests making the following definition of a couple of higher-order Hardy spaces.

Definition 2.26. *Given a UR domain $\Omega \subseteq \mathbb{R}^2$ with compact boundary, along with an integer $m \in \mathbb{N}$, an integrability exponent $p \in (1, \infty)$, and some aperture parameter $\kappa \in (0, \infty)$, define the higher-order Hardy space $\mathcal{H}^{p,m}(\Omega)$ (of order m in Ω) as the collection of all functions $u \in \mathcal{C}^\infty(\Omega)$ with $\bar{\partial}^m u = 0$ in Ω , satisfying*

$$\mathcal{N}_\kappa(\nabla^\ell u) \in L^p(\partial\Omega, \sigma) \text{ for all } \ell \in \{0, 1, \dots, m-1\}. \quad (2.224)$$

If Ω is an exterior domain, it is assumed that the nontangential maximal operator is truncated and one also asks that the auxiliary functions $\{u_j\}_{0 \leq j \leq m-1}$ associated with u as in (2.56) vanish at infinity. Equip this higher-order Hardy space with

$$\mathcal{H}^{p,m}(\Omega) \ni u \longmapsto \|u\|_{\mathcal{H}^{p,m}(\Omega)} := \sum_{\ell=0}^{m-1} \|\mathcal{N}_\kappa(\nabla^\ell u)\|_{L^p(\partial\Omega, \sigma)}. \quad (2.225)$$

Finally, define the regular higher-order Hardy space $\mathcal{H}_1^{p,m}(\Omega)$ (of order m in Ω) in an analogous fashion, now replacing (2.224) by the stronger condition

$$\mathcal{N}_\kappa(\nabla^\ell u) \in L^p(\partial\Omega, \sigma) \text{ for all } \ell \in \{0, 1, \dots, m\}, \quad (2.226)$$

and equipping $\mathcal{H}_1^{p,m}(\Omega)$ with the norm

$$\mathcal{H}_1^{p,m}(\Omega) \ni u \mapsto \|u\|_{\mathcal{H}_1^{p,m}(\Omega)} := \sum_{\ell=0}^m \|\mathcal{N}_\kappa(\nabla^\ell u)\|_{L^p(\partial\Omega, \sigma)}. \quad (2.227)$$

Note that $\mathcal{P}_{m-2}\Big|_\Omega \subseteq \mathcal{H}^{p,m}(\Omega)$ where $\mathcal{P}_{m-2}\Big|_\Omega$ is a collection of polynomials of degree less than or equal to $m-2$ in Ω .

We may then refine our earlier higher-order Fatou type theorem as follows.

Theorem 2.27. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Fix an integer $m \in \mathbb{N}$, an integrability exponent $p \in (1, \infty)$, and an aperture parameter $\kappa \in (0, \infty)$. Then the higher-order complex nontangential trace operator*

$$\mathrm{Tr}_{m-1}^{\mathbb{C}} : \mathcal{H}^{p,m}(\Omega) \longrightarrow \mathrm{CWA}_{m-1} [L^p(\partial\Omega, \sigma)] \quad (2.228)$$

defined as

$$\mathrm{Tr}_{m-1}^{\mathbb{C}}(u) := \left\{ (\partial^a \bar{\partial}^b u) \Big|_{\partial\Omega} \right\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}}^{\kappa\text{-n.t.}} \text{ for each } u \in \mathcal{H}^{p,m}(\Omega), \quad (2.229)$$

is meaningful, linear, and bounded. In addition,

$$\mathrm{Tr}_{m-1}^{\mathbb{C}} : \mathcal{H}_1^{p,m}(\Omega) \longrightarrow \mathrm{CWA}_{m-1} [L_1^p(\partial\Omega, \sigma)] \quad (2.230)$$

defined as in (2.229) above for each $u \in \mathcal{H}_1^{p,m}(\Omega) \subseteq \mathcal{H}^{p,m}(\Omega)$ is well defined, linear, and bounded. Finally, one has the following poly-Cauchy reproducing formula:

$$u = \mathcal{E}_{m-1}(\mathrm{Tr}_{m-1}^{\mathbb{C}}(u)) \text{ for each } u \in \mathcal{H}^{p,m}(\Omega). \quad (2.231)$$

Proof. Let us first prove that $\mathrm{Tr}_{m-1}^{\mathbb{C}}$ is meaningfully defined, linear, and bounded.

Let $u \in \mathcal{H}^{p,m}(\Omega)$. According to the definition, $\mathrm{Tr}_{m-1}^{\mathbb{C}}$ is linear and $\mathrm{Tr}_{m-1}^{\mathbb{C}}(u)$

satisfies the compatibility condition. In addition, for aperture parameter $\kappa \in (0, \infty)$

$$\begin{aligned} \|\mathrm{Tr}_{m-1}^{\mathbb{C}}(u)\|_{\mathrm{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]} &= \sum_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1}} \left\| \left(\partial^a \bar{\partial}^b u \right) \Big|_{L^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} \right\| \\ &\leq \sum_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1}} \left\| \mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b u \right) \right\|_{L^p(\partial\Omega, \sigma)}. \end{aligned} \quad (2.232)$$

This forces $\mathrm{Tr}_{m-1}^{\mathbb{C}}$ is bounded from $\mathcal{H}^{p,m}(\Omega)$ into $\mathrm{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Similarly, $\mathrm{Tr}_{m-1}^{\mathbb{C}} : \mathcal{H}_1^{p,m}(\Omega) \rightarrow \mathrm{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$ is well defined and linear. Moreover, for $u \in \mathcal{H}_1^{p,m}(\Omega)$

$$\|\mathrm{Tr}_{m-1}^{\mathbb{C}}(u)\|_{\mathrm{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]} = \sum_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1}} \left\| \left(\partial^a \bar{\partial}^b u \right) \Big|_{L_1^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} \right\|. \quad (2.233)$$

For $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$,

$$\begin{aligned} &\left\| \left(\partial^a \bar{\partial}^b u \right) \Big|_{L_1^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} \right\| \\ &= \left\| \left(\partial^a \bar{\partial}^b u \right) \Big|_{L^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} \right\| + \left\| \partial_\tau \left(\partial^a \bar{\partial}^b u \right) \Big|_{L^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} \right\|. \end{aligned} \quad (2.234)$$

Since $\partial_\tau \left(\partial^a \bar{\partial}^b u \right) \Big|_{L^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} = i\nu \left(\partial^{a+1} \bar{\partial}^b u \right) \Big|_{L^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} - i\bar{\nu} \left(\partial^a \bar{\partial}^{b+1} u \right) \Big|_{L^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}}$ on $\partial\Omega$, one can conclude that

$$\sum_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1}} \left\| \left(\partial^a \bar{\partial}^b u \right) \Big|_{L_1^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} \right\| \leq C \sum_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m}} \left\| \left(\partial^a \bar{\partial}^b u \right) \Big|_{L^p(\partial\Omega, \sigma)}^{\kappa\text{-n.t.}} \right\|, \quad (2.235)$$

for some constant $C > 0$. Combining this with (2.233), we obtain that $\mathrm{Tr}_{m-1}^{\mathbb{C}}$ is bounded from $\mathcal{H}_1^{p,m}(\Omega)$ into $\mathrm{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$. Finally, the poly-Cauchy reproducing formula in (2.231) follows from the definitions of $\mathcal{H}^{p,m}(\Omega)$, $\mathrm{Tr}_{m-1}^{\mathbb{C}}$ and the integral representation theorem in Theorem 2.7. \square

The boundary higher-order Hardy spaces are naturally defined from the higher-order Hardy spaces in Definition 2.26 and the higher-order complex nontangential trace operator defined in (2.229) as follows.

Definition 2.28. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Fix an integer $m \in \mathbb{N}$, an integrability exponent $p \in (1, \infty)$, and some aperture parameter $\kappa \in (0, \infty)$, define the boundary higher-order Hardy space $\mathcal{H}^{p,m}(\partial\Omega, \sigma)$ (of order m in Ω) by

$$\left. \begin{aligned} \mathcal{H}^{p,m}(\partial\Omega, \sigma) &:= \left\{ \dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \right. \\ \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] &: \dot{g} = \text{Tr}_{m-1}^{\mathbb{C}}(u) \text{ for some} \\ &\left. u \in \mathcal{H}^{p,m}(\Omega) \right\} \end{aligned} \quad (2.236)$$

In addition, the boundary regular higher-order Hardy space $\mathcal{H}_1^{p,m}(\partial\Omega, \sigma)$ is defined by

$$\left. \begin{aligned} \mathcal{H}_1^{p,m}(\partial\Omega, \sigma) &:= \left\{ \dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \right. \\ \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)] &: \dot{g} = \text{Tr}_{m-1}^{\mathbb{C}}(u) \text{ for some} \\ &\left. u \in \mathcal{H}_1^{p,m}(\Omega) \right\} \end{aligned} \quad (2.237)$$

Along with these definitions of the boundary higher-order Hardy spaces, the higher-order boundary trace operator becomes an isomorphism from each domain Hardy space to the boundary Hardy space.

Proposition 2.29. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Fix an integer $m \in \mathbb{N}$, an integrability exponent $p \in (1, \infty)$, and an aperture parameter $\kappa \in (0, \infty)$. Then the higher-order complex nontangential trace operators

$$\text{Tr}_{m-1}^{\mathbb{C}} : \mathcal{H}^{p,m}(\Omega) \longrightarrow \mathcal{H}^{p,m}(\partial\Omega, \sigma), \quad (2.238)$$

$$\text{Tr}_{m-1}^{\mathbb{C}} : \mathcal{H}_1^{p,m}(\Omega) \longrightarrow \mathcal{H}_1^{p,m}(\partial\Omega, \sigma), \quad (2.239)$$

are isomorphisms.

Proof. Let $u \in \mathcal{H}^{p,m}(\Omega)$. According to the poly-Cauchy reproducing formula in (2.231), one has

$$u = \mathcal{E}_{m-1}(\text{Tr}_{m-1}^{\mathbb{C}}(u)). \quad (2.240)$$

This forces $\text{Tr}_{m-1}^{\mathbb{C}}$ in (2.238) is injective. The fact that $\text{Tr}_{m-1}^{\mathbb{C}}$ is surjective follows from the definition of the boundary higher-order Hardy space as in (2.236) which gives for each array $\dot{g} \in \mathcal{H}^{p,m}(\partial\Omega, \sigma)$ there exists $u \in \mathcal{H}^{p,m}(\Omega)$ such that

$$\text{Tr}_{m-1}^{\mathbb{C}}(u) = \dot{g}. \quad (2.241)$$

The same argument works for the regular higher-order Hardy spaces. This finishes the proof. \square

Proposition 2.30. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Set $\sigma := \mathcal{H}^1[\partial\Omega]$ and fix integrability exponent $p \in (1, \infty)$, then*

$$\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] = \mathcal{H}_+^{p,m}(\partial\Omega, \sigma) + \mathcal{H}_-^{p,m}(\partial\Omega, \sigma), \quad (2.242)$$

and

$$\mathcal{H}_+^{p,m}(\partial\Omega, \sigma) \cap \mathcal{H}_-^{p,m}(\partial\Omega, \sigma) = \text{Tr}_{m-1}^{\mathbb{C}}\mathcal{P}_{m-2}, \quad (2.243)$$

where $\mathcal{H}_{\pm}^{p,m}(\partial\Omega, \sigma)$ are the boundary higher-order Hardy spaces associated with $\Omega_+ \equiv \Omega$ and $\Omega_- \equiv \mathbb{R}^2 \setminus \bar{\Omega}$, respectively.

Proof. Let $\dot{g} \in \mathcal{H}_+^{p,m}(\partial\Omega, \sigma) + \mathcal{H}_-^{p,m}(\partial\Omega, \sigma)$, then

$$\exists u_{\pm} \in \mathcal{H}^{p,m}(\Omega_{\pm}) \text{ such that } \dot{g} = \text{Tr}_{m-1}^{\mathbb{C}}(u_+ + u_-). \quad (2.244)$$

This forces

$$g_{(a,b)} = \left(\partial^a \bar{\partial}^b (u_+ + u_-) \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}, \quad \forall a, b \in \mathbb{N}_0 \text{ with } a + b \leq m - 1. \quad (2.245)$$

This readily implies that $\dot{g} \in CC_{\mathbb{C}}$ which shows that $\dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$.

For any $\dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$,

$$\dot{g} = \left(\frac{1}{2}I + \mathcal{C}_{m-1} \right) \dot{g} - \left(-\frac{1}{2}I + \mathcal{C}_{m-1} \right) \dot{g}, \quad (2.246)$$

where $\left(\frac{1}{2}I + \mathcal{C}_{m-1} \right) \dot{g} \in \mathcal{H}_+^{p,m}(\partial\Omega, \sigma)$ and $\left(-\frac{1}{2}I + \mathcal{C}_{m-1} \right) \dot{g} \in \mathcal{H}_-^{p,m}(\partial\Omega, \sigma)$.

Since $\mathcal{P}_{m-2} \Big|_{\Omega_{\pm}} \subseteq \mathcal{H}^{p,m}(\Omega_{\pm})$, we have $\text{Tr}_{m-1}^{\mathbb{C}}\mathcal{P}_{m-2} \subseteq \mathcal{H}_{\pm}^{p,m}(\partial\Omega, \sigma)$.

There remains to show that

$$\mathcal{H}_+^{p,m}(\partial\Omega, \sigma) \cap \mathcal{H}_-^{p,m}(\partial\Omega, \sigma) \subseteq \text{Tr}_{m-1}^{\mathbb{C}}\mathcal{P}_{m-2}. \quad (2.247)$$

Consider $\dot{g} \in \mathcal{H}_+^{p,m}(\partial\Omega, \sigma) \cap \mathcal{H}_-^{p,m}(\partial\Omega, \sigma)$, then

$$\exists u_{\pm} \in \mathcal{H}^{p,m}(\Omega_{\pm}) \text{ such that } \operatorname{Tr}_{m-1}^{\mathbb{C}} u_+ = \dot{g} = \operatorname{Tr}_{m-1}^{\mathbb{C}} u_- \quad (2.248)$$

which implies that

$$\left(\partial^a \bar{\partial}^b u_+ \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} = g_{(a,b)} = \left(\partial^a \bar{\partial}^b u_- \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}, \quad (2.249)$$

for all $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$. Define $u := \begin{cases} u_+ & \text{in } \Omega_+ \\ u_- & \text{in } \Omega_- \end{cases}$. Observe that $u_{\pm} \in L_{\text{bdd}}^{2p}(\Omega_{\pm}, \mathcal{L}^2)$, we get

$$u \in L_{\text{loc}}^{2p}(\mathbb{R}^2, \mathcal{L}^2) \subseteq L_{\text{loc}}^1(\mathbb{R}^2, \mathcal{L}^2). \quad (2.250)$$

We claim that $\bar{\partial}^m u = 0$ in $\mathbb{R}^2 \equiv \mathbb{C}$, in the sense of distributions. To justify this claim, pick $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ and write

$$\int_{\mathbb{R}^2} u \bar{\partial}^m \varphi d\mathcal{L}^2 = \int_{\Omega_+} u_+ \bar{\partial}^m \varphi d\mathcal{L}^2 + \int_{\Omega_-} u_- \bar{\partial}^m \varphi d\mathcal{L}^2 \quad (2.251)$$

We apply the integration by parts as follows.

$$\int_{\Omega} u \bar{\partial} \varphi d\mathcal{L}^2 = - \int_{\Omega} (\bar{\partial} u) \varphi d\mathcal{L}^2 + \frac{1}{2} \int_{\partial_* \Omega} u \varphi \nu d\sigma. \quad (2.252)$$

Since $\bar{\partial}^m u_{\pm} = 0$ in Ω_{\pm} and $\nu_{\Omega_-} = -\nu_{\Omega_+}$, repeating the integration by parts m times as above forces

$$\begin{aligned} \int_{\Omega_+} u_+ \bar{\partial}^m \varphi d\mathcal{L}^2 + \int_{\Omega_-} u_- \bar{\partial}^m \varphi d\mathcal{L}^2 &= \sum_{k=0}^{m-1} \frac{(-1)^k}{2} \int_{\partial\Omega} \left(\bar{\partial}^k u_+ \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \varphi \nu d\sigma \\ &\quad - \sum_{k=0}^{m-1} \frac{(-1)^k}{2} \int_{\partial\Omega} \left(\bar{\partial}^k u_- \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} \varphi \nu d\sigma. \end{aligned} \quad (2.253)$$

In turn, $\operatorname{Tr}_{m-1}^{\mathbb{C}} u_+ = \operatorname{Tr}_{m-1}^{\mathbb{C}} u_-$, we see that

$$\int_{\Omega_+} u_+ \bar{\partial}^m \varphi d\mathcal{L}^2 + \int_{\Omega_-} u_- \bar{\partial}^m \varphi d\mathcal{L}^2 = 0. \quad (2.254)$$

This shows that $\int_{\mathbb{R}^2} u \bar{\partial}^m \varphi d\mathcal{L}^2 = 0$ in (2.251), and hence $\bar{\partial}^m u = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. In particular, elliptic regularity theory gives us that

$$u \in \mathcal{C}^\infty(\mathbb{R}^2) \text{ and } \bar{\partial}^m u = 0 \text{ in } \mathbb{R}^2 \equiv \mathbb{C}, \text{ in a classical sense.} \quad (2.255)$$

Also, near infinity

$$u = u_- = O(|z|^{m-2}) \text{ as } |z| \rightarrow \infty. \quad (2.256)$$

We claim that (2.255) and (2.256) imply that

$$\nabla^\ell u = 0 \text{ in } \mathbb{R}^2, \forall \ell \geq m - 1. \quad (2.257)$$

which, in turn, force $u \in \mathcal{P}_{m-2}$, as wanted. To prove (2.257), fix an arbitrary point $z_0 \in \mathbb{C}$. For each $R > 0$, write the Cauchy Integral Representation Formula for the function in the bounded smooth domain $B(z_0, R)$,

$$u(z) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial B(z,R)} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \left(\bar{\partial}^k u \right) (\zeta) d\zeta, \forall z \in B(z_0, R). \quad (2.258)$$

Hence, $\forall \ell \geq m - 1$, the above implies

$$(\nabla_{x,y}^\ell u)(z) = \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial B(z,R)} \nabla_{x,y}^\ell \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right] \left(\bar{\partial}^k u \right) (\zeta) d\zeta. \quad (2.259)$$

This shows that

$$|(\nabla^\ell u)(z_0)| \leq C \left(\sum_{k=0}^{m-1} R \cdot R^{k-\ell-1} \right) \sup_{\zeta \in \partial B(z_0, R)} |(\bar{\partial}^k u)(\zeta)|. \quad (2.260)$$

According to (2.256), one has $|(\bar{\partial}^k u)(\zeta)| = O(|\zeta|^{m-2-k})$ as $|\zeta| \rightarrow \infty$. For each $0 \leq k \leq m - 1$ and $\ell \geq m - 1$, we further have

$$|(\nabla^\ell u)(z_0)| \leq O(R^{m-2-\ell}) = o(1) \text{ as } R \rightarrow \infty. \quad (2.261)$$

Passing $R \rightarrow \infty$ proves $(\nabla^\ell u)(z_0) = 0, \forall \ell \geq m - 1$, and since $z_0 \in \mathbb{C}$ was arbitrary this proves (2.257) above. \square

Theorem 2.31. Consider a UR domain $\Omega \subseteq \mathbb{R}^2$ with compact boundary, an integrability exponent $p \in (1, \infty)$, and aperture parameter $\kappa \in (0, \infty)$. The poly-Cauchy operator acting on the Lebesgue based complex Whitney array space

$$\dot{\mathcal{C}}_{m-1} : \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \mathcal{H}^{p,m}(\Omega) \quad (2.262)$$

is well defined, linear, bounded, and surjective where the higher-order Hardy space is defined as in Definition 2.26.

Proof. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Let us consider a complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, and $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$. According to the Corollary 2.11 we have

$$\begin{aligned} \partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) &= \sum_{k=0}^{m-1-a-b} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(a,k+b)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{a-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{j!}{(m-a-b+j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-a-b+j}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.263)$$

for each $z \in \Omega$. Observe that the part of the first sum in (2.263) corresponding to $k = 0$ is Calderón-Zygmund operator providing desired nontangential maximal function estimate. The kernels of the remaining terms of the first sum in (2.263) are weakly singular kernels which directly yield the nontangential maximal estimates. By similar argument, we obtain the nontangential maximal estimates of the second sum in (2.263). This implies that $\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) \right) \in L^p(\partial\Omega, \sigma)$ for all $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$. In addition, from (2.71), $\bar{\partial}^m (\dot{\mathcal{C}}_{m-1} \dot{g}) = 0$ in Ω and according to Lemma 2.15, the auxiliary functions associated with the poly-Cauchy operator vanish at infinity, in the case when Ω is an exterior domain. This shows for any complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, $\dot{\mathcal{C}}_{m-1} \dot{g} \in \mathcal{H}^{p,m}(\Omega)$.

From the definition of the poly-Cauchy operator in (2.55), $\dot{\mathcal{C}}_{m-1}$ is linear operator on $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Moreover, from the nontangential maximal

estimates aforementioned we obtain that for any $\dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ there exists $C > 0$ such that

$$\begin{aligned}
\|\dot{\mathcal{C}}_{m-1}\dot{g}\|_{\mathcal{H}^{p,m}(\Omega)} &= \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \|\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1}\dot{g} \right) \right)\|_{L^p(\partial\Omega, \sigma)} \\
&\leq C \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \left[\sum_{k=0}^{m-1-a-b} \|g_{(a,k+b)}\|_{L^p(\partial\Omega, \sigma)} \right. \\
&\quad \left. + \sum_{j=0}^{a-1} \|g_{(a-1-j, m-a+j)}\|_{L^p(\partial\Omega, \sigma)} \right] \\
&\leq C \|\dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]}, \tag{2.264}
\end{aligned}$$

which proves that $\dot{\mathcal{C}}_{m-1} : \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \mathcal{H}^{p,m}(\Omega)$ is bounded. In order to prove surjectivity, let us fix $u \in \mathcal{H}^{p,m}(\Omega)$. Thanks to the Higher order Fatou theorem in Theorem 2.27, $\partial^a \bar{\partial}^b u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ is well defined in $L^p(\partial\Omega, \sigma)$ for $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$. Let $\dot{g} := \text{Tr}_{m-1}^{\mathbb{C}} u$. According to Theorem 2.27, one can conclude that $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ where for $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$, $g_{(a,b)} = \partial^a \bar{\partial}^b u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ on $\partial\Omega$. Since $u \in \mathcal{H}^{p,m}(\Omega)$, $\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b u \right) \in L^p(\partial\Omega, \sigma)$, for $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$ which implies that $g_{(a,b)} \in L^p(\partial\Omega, \sigma)$. Moreover, $\dot{g} \in CC_{\mathbb{C}}$ follows from the construction of \dot{g} through the derivatives of u which forces $\dot{g} := \text{Tr}_{m-1}^{\mathbb{C}} u \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. According to the integral representation theorem for the poly-Cauchy operator, we get $\dot{\mathcal{C}}_{m-1}\dot{g} = u$ in $\mathcal{H}^{p,m}(\Omega)$. This finishes the proof of the theorem. \square

Corollary 2.32. *Given a UR domain $\Omega \subseteq \mathbb{R}^2$ with compact boundary, an integrability exponent $p \in (1, \infty)$, and aperture parameter $\kappa \in (0, \infty)$, there holds*

$$\dot{\mathcal{C}}_{m-1} : \mathcal{H}^{p,m}(\partial\Omega, \sigma) \rightarrow \mathcal{H}^{p,m}(\Omega) \text{ is an isomorphism.} \tag{2.265}$$

Proof. According to the theorem 2.31, it suffices to show the injectivity because $\mathcal{H}^{p,m}(\partial\Omega)$ is a subset of $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. In order to prove the injectivity let $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \mathcal{H}^{p,m}(\partial\Omega, \sigma)$, then there exists $u \in \mathcal{H}^{p,m}(\Omega)$ such that

$g_{(a,b)} = \left(\partial^a \bar{\partial}^b u \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ on $\partial\Omega$. Applying the reproducing formula in Theorem 2.56, there holds

$$\left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) = u(z). \quad (2.266)$$

for each $z \in \Omega$. This forces

$$\dot{\mathcal{C}}_{m-1} \dot{g} \equiv 0 \implies u \equiv 0 \implies \dot{g} \equiv \dot{0}. \quad (2.267)$$

This shows that $\dot{\mathcal{C}}_{m-1} : \mathcal{H}^{p,m}(\partial\Omega, \sigma) \rightarrow \mathcal{H}^{p,m}(\Omega)$ is injective which completes the proof of the corollary. \square

Theorem 2.33. *Consider a UR domain $\Omega \subseteq \mathbb{R}^2$ with compact boundary, an integrability exponent $p \in (1, \infty)$, and aperture parameter $\kappa \in (0, \infty)$. There holds*

$$\dot{\mathcal{C}}_{m-1} : \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)] \rightarrow \mathcal{H}_1^{p,m}(\Omega) \quad (2.268)$$

is well defined, linear, bounded, and surjective where the regular higher-order Hardy space defined as in Definition 2.26.

Proof. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Fix a sobolev based complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$, and $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$. According to the Corollary 2.11 we have

$$\begin{aligned} \partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) &= \sum_{k=0}^{m-1-a-b} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(a,k+b)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{a-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{j!}{(m-a-b+j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-a-b+j}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.269)$$

for each $z \in \Omega$. Similar to the Theorem 2.31, $\mathcal{N}_\kappa(\partial^\alpha u) \in L^p(\partial\Omega, \sigma)$ whenever

$|\alpha| \leq m - 1$. Let $a + b = m - 1$, then

$$\begin{aligned} \partial^a \bar{\partial}^b \left(\mathcal{E}_{m-1} \dot{g} \right) (z) &= \frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{\zeta - z} g_{(a,b)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{a-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{j+1} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{j+1}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.270)$$

for each $z \in \Omega$. By direct calculation, we have

$$\begin{aligned} &\partial^a \bar{\partial}^{b+1} \left(\mathcal{E}_{m-1} \dot{g} \right) (z) \\ &= - \sum_{j=1}^{a-1} \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^j}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.271)$$

and

$$\begin{aligned} &\partial^{a+1} \bar{\partial}^b \left(\mathcal{E}_{m-1} \dot{g} \right) (z) \\ &= \frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{(\zeta-z)^2} g_{(a,b)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{a-1} \frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{j+1}}}{(\zeta-z)^{j+2}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.272)$$

for each $z \in \Omega$. Since $\dot{g} \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$, from the definition we have $\partial_\tau g_{(a,b)}, \partial_\tau g_{(a-1-j, m-a+j)} \in L^p(\partial\Omega, \sigma)$ for all $j = 0, \dots, a-1$. Applying integration by parts and Calderón-Zygmund theory, we obtain the desired nontangential maximal estimates except the first term in (2.272). Using $d\zeta = i\nu d\sigma$ on $\partial\Omega$, $\forall z \in \Omega$,

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{(\zeta-z)^2} g_{(a,b)}(\zeta) d\zeta = \frac{1}{2\pi i} \int_{\partial\Omega} i\nu(\zeta) \frac{1}{(\zeta-z)^2} g_{(a,b)}(\zeta) d\sigma(\zeta). \quad (2.273)$$

Since $i\nu(\zeta) \frac{1}{(\zeta-z)^2} = -\partial_{\tau(\zeta)} \left[\frac{1}{\zeta-z} \right]$, we get

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{(\zeta-z)^2} g_{(a,b)}(\zeta) d\zeta = -\frac{1}{2\pi i} \int_{\partial\Omega} \partial_{\tau(\zeta)} \left[\frac{1}{\zeta-z} \right] g_{(a,b)}(\zeta) d\sigma(\zeta). \quad (2.274)$$

Since $a + b = m - 1$, $\partial_\tau g_{(a,b)} \in L^p(\partial\Omega, \sigma)$. The nontangential maximal estimate for the first term in (2.272) follows from the integration by parts and Calderón-Zygmund theory again. This shows that $\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) \right) \in L^p(\partial\Omega, \sigma)$ for $a, b \in \mathbb{N}_0$ with $a + b \leq m$. According to (2.71), $\bar{\partial}^m \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) = 0$ in Ω . Additionally, in the case when Ω is an exterior domain, the auxiliary functions associated with the poly-Cauchy operator vanishes according to Lemma 2.15. Consequently, for all $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$, one has $\dot{\mathcal{C}}_{m-1} \dot{g} \in \mathcal{H}_1^{p,m}(\Omega)$.

Moving on, it is clear from definition that $\dot{\mathcal{C}}_{m-1}$ is linear operator. For the boundedness, let $\dot{g} \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$, then

$$\begin{aligned} & \|\dot{\mathcal{C}}_{m-1} \dot{g}\|_{\mathcal{H}_1^{p,m}(\Omega)} \\ &= \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m}} \|\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) \right)\|_{L^p(\partial\Omega, \sigma)} \\ &= \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \|\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) \right)\|_{L^p(\partial\Omega, \sigma)} \\ & \quad + \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b=m}} \|\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) \right)\|_{L^p(\partial\Omega, \sigma)}. \end{aligned} \quad (2.275)$$

From the nontangential estimates mentioned above for $\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right)$ whenever $a + b \leq m$, we obtain that there exists $C > 0$ such that

$$\sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \|\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) \right)\|_{L^p(\partial\Omega, \sigma)} \leq C \|\dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]}, \quad (2.276)$$

and

$$\sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m}} \|\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) \right)\|_{L^p(\partial\Omega, \sigma)} \leq C \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b=m-1}} \|\partial_\tau g_{(a,b)}\|_{L^p(\partial\Omega, \sigma)}. \quad (2.277)$$

In conclusion,

$$\|\dot{\mathcal{C}}_{m-1} \dot{g}\|_{\mathcal{H}_1^{p,m}(\Omega)} \leq C \|\dot{g}\|_{\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]}. \quad (2.278)$$

This proves that $\dot{\mathcal{C}}_{m-1} : \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)] \rightarrow \mathcal{H}_1^{p,m}(\Omega)$ is bounded operator. To this end, let us fix $u \in \mathcal{H}_1^{p,m}(\Omega)$. Thanks to the Higher order Fatou theorem

in Theorem 2.27, $\partial^a \bar{\partial}^b u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ is well defined in $L_1^p(\partial\Omega, \sigma)$ for all natural numbers $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$. Let $\dot{g} := \text{Tr}_{m-1}^{\mathbb{C}} u$. Then for all sobolev based complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$ where each entry $g_{(a,b)} = \partial^a \bar{\partial}^b u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ on $\partial\Omega$. Since $\mathcal{N}_\kappa \left(\partial^a \bar{\partial}^b u \right) \in L^p(\partial\Omega, \sigma)$ for $a, b \in \mathbb{N}_0$ with $a + b \leq m$, we have $g_{(a,b)} \in L^p(\partial\Omega, \sigma)$ for $a + b \leq m$ which implies that $g_{(a,b)} \in L_1^p(\partial\Omega, \sigma)$ for $a + b \leq m - 1$. In particular, $g_{(a,b)}$ consists of derivatives of u which implies that $\dot{g} \in CC_{\mathbb{C}}$. This shows that the array $\dot{g} := \text{Tr}_{m-1}^{\mathbb{C}} u \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$ for $u \in \mathcal{H}_1^{p,m}(\Omega)$. Thanks to the integral representation formula for the poly-Cauchy operator, we have $\dot{\mathcal{C}}_{m-1} \dot{g} = u$ in $\mathcal{H}_1^{p,m}(\Omega)$. This proves the theorem 2.33. \square

Using same argument in Corollary 2.32, the poly-Cauchy operator is an isomorphism from the boundary regular higher-order Hardy space to the regular higher-order Hardy space.

Corollary 2.34. *Given a UR domain $\Omega \subseteq \mathbb{R}^2$ with compact boundary, an integrability exponent $p \in (1, \infty)$, and aperture parameter $\kappa \in (0, \infty)$, there holds*

$$\dot{\mathcal{C}}_{m-1} : \mathcal{H}_1^{p,m}(\partial\Omega, \sigma) \rightarrow \mathcal{H}_1^{p,m}(\Omega) \text{ is an isomorphism.} \quad (2.279)$$

2.7 The boundary-to-boundary poly-Cauchy operator

In this section, we prove the jump relation associated with the poly-Cauchy operator with boundary-to-boundary version of the poly-Cauchy operator which is a most natural higher-order generalization of item (2) in Theorem 2.1. Proceeding forward, define a boundary-to-boundary poly-Cauchy operator as follows.

Definition 2.35. *Given a UR domain $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ with compact boundary and $p \in (1, \infty)$, for $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ define*

boundary-to-boundary poly-Cauchy operator $\dot{C}_{m-1}\dot{g} := \left\{ \left(\dot{C}_{m-1}\dot{g} \right)_{(a,b)} \right\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}}$
 by setting, for $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$,

$$\begin{aligned} \left(\dot{C}_{m-1}\dot{g} \right)_{(a,b)}(z) &:= \sum_{k=0}^{m-1-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(a,k+b)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-a-b+j}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta) \end{aligned} \quad (2.280)$$

at σ -a.e. point $z \in \partial\Omega$.

As indicated in our next theorem this operator acts naturally between our complex Whitney array spaces.

Theorem 2.36. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary, and fix an arbitrary integer $m \in \mathbb{N}$. Also, pick an integrability exponent $p \in (1, \infty)$.*

Then the boundary-to-boundary poly-Cauchy operator \dot{C}_{m-1} yields well-defined, linear, and bounded mappings both on the Lebesgue-based complex Whitney array space $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ and on the Sobolev-based complex Whitney array space $\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$.

As such, for each array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ one has

$$\|\dot{C}_{m-1}\dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]} \lesssim \|\dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]}, \quad (2.281)$$

and for each array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$ one has

$$\|\dot{C}_{m-1}\dot{g}\|_{\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]} \lesssim \|\dot{g}\|_{\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]}. \quad (2.282)$$

Proof. We first show that for $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, the

boundary-to-boundary poly-Cauchy operator $\dot{C}_{m-1}\dot{g} = \left\{ \left(\dot{C}_{m-1}\dot{g} \right)_{(a,b)} \right\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}}$

also satisfies the compatibility condition. Indeed, for any $a, b \in \mathbb{N}_0$ with $a + b \leq m - 2$,

$$\begin{aligned}
\left(\dot{C}_{m-1}\dot{g}\right)_{(a,b)}(z) &= \sum_{k=0}^{m-1-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(a,k+b)}(\zeta) d\zeta \\
&\quad - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\
&\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-a-b+j}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta)
\end{aligned} \tag{2.283}$$

at σ -a.e. point $z \in \partial\Omega$. Taking tangential derivative to (2.283) yields

$$\begin{aligned}
&\left(\partial_{\tau} \left(\dot{C}_{m-1}\dot{g}\right)_{(a,b)}\right)(z) \\
&= - \sum_{k=0}^{m-1-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} \right] g_{(a,k+b)}(\zeta) d\zeta \\
&\quad + \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\
&\quad \times \partial_{\tau(\zeta)}^2 \left[\frac{\overline{(z-\zeta)^{m-a-b+j}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta).
\end{aligned} \tag{2.284}$$

Applying the integration by parts for $0 \leq k \leq m - 2 - a - b$ to the first term

in (2.284) gives

$$\begin{aligned}
& \left(\partial_\tau \left(\dot{C}_{m-1} \dot{g} \right)_{(a,b)} \right) (z) \\
&= - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \partial_{\tau(\zeta)} \left[\frac{\overline{(z - \zeta)^{(m-1-a-b)}}}{(m-1-a-b)! (\zeta - z)} \right] g_{(a,m-1-a)}(\zeta) d\zeta \\
&+ \sum_{k=0}^{m-2-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{\overline{(z - \zeta)^k}}{k! (\zeta - z)} (\partial_\tau g_{(a,k+b)})(\zeta) d\zeta \\
&+ \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\
&\quad \times \partial_{\tau(\zeta)}^2 \left[\frac{\overline{(z - \zeta)^{m-a-b+j}}}{(\zeta - z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta). \tag{2.285}
\end{aligned}$$

Based on the calculation and the compatibility condition of \dot{g} , we obtain

$$\left(\partial_\tau \left(\dot{C}_{m-1} \dot{g} \right)_{(a,b)} \right) (z) = I + i\nu \cdot \left(\dot{C}_{m-1}^1 \dot{g} \right)_{(a,b)} (z) - i\bar{\nu} \cdot \left(\dot{C}_{m-1}^2 \dot{g} \right)_{(a,b)} (z), \tag{2.286}$$

where

$$I = - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \partial_{\tau(\zeta)} \left[\frac{\overline{(z - \zeta)^{(m-1-a-b)}}}{(m-1-a-b)! (\zeta - z)} \right] g_{(a,m-1-a)}(\zeta) d\zeta, \tag{2.287}$$

$$\begin{aligned}
\left(\dot{C}_{m-1}^1 \dot{g} \right)_{(a,b)} (z) &= \sum_{k=0}^{m-2-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{\overline{(z - \zeta)^k}}{k! (\zeta - z)} g_{(a+1, k+b)}(\zeta) d\zeta \\
&- \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{(j+1)!}{(m-a-b+j)!} \times \\
&\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z - \zeta)^{m-a-b+j}}}{(\zeta - z)^{j+2}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \tag{2.288}
\end{aligned}$$

and

$$\begin{aligned}
\left(\dot{C}_{m-1}^2 \dot{g}\right)_{(a,b)}(z) &= \sum_{k=0}^{m-2-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{\overline{(z-\zeta)}^k}{k!(\zeta-z)} g_{(a,k+b+1)}(\zeta) d\zeta \\
&\quad - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{(j+1)!}{(m-1-a-b+j)!} \times \\
&\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)}^{m-1-a-b+j}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j,m-a+j)}(\zeta) d\sigma(\zeta). \tag{2.289}
\end{aligned}$$

Using $d\zeta = i\nu d\sigma(\zeta)$, it can be shown that

$$I + i\nu \cdot \left(\dot{C}_{m-1}^1 \dot{g}\right)_{(a,b)}(z) = \left(\dot{C}_{m-1} \dot{g}\right)_{(a+1,b)}(z). \tag{2.290}$$

From (2.289), let us also point out that

$$\left(\dot{C}_{m-1}^2 \dot{g}\right)_{(a,b)}(z) = \left(\dot{C}_{m-1} \dot{g}\right)_{(a,b+1)}(z). \tag{2.291}$$

Substituting (2.290) and (2.291) into (2.286) gives for each $a, b \in \mathbb{N}_0$ with $a+b \leq m-2$,

$$\left(\partial_{\tau} \left(\dot{C}_{m-1} \dot{g}\right)_{(a,b)}\right)(z) = i\nu \cdot \left(\dot{C}_{m-1} \dot{g}\right)_{(a+1,b)}(z) - i\bar{\nu} \cdot \left(\dot{C}_{m-1} \dot{g}\right)_{(a,b+1)}(z), \tag{2.292}$$

at σ -a.e. point $z \in \partial\Omega$. This verifies that $\dot{C}_{m-1} \dot{g} = \left\{ \left(\dot{C}_{m-1} \dot{g}\right)_{(a,b)} \right\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}}$ also satisfies the compatibility condition provided the Lebesgue based complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. From the definition of the boundary-to-boundary poly-Cauchy operator in (2.280), \dot{C}_{m-1} is a linear operator. In order to prove the boundedness, let us consider a Lebesgue based complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Then,

$$\|\dot{C}_{m-1} \dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]} = \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \left\| \left(\dot{C}_{m-1} \dot{g}\right)_{(a,b)} \right\|_{L^p(\partial\Omega, \sigma)}. \tag{2.293}$$

Let us denote

$$\left(\dot{C}_{m-1}\dot{g}\right)_{(a,b)}(z) := \sum_{k=0}^{m-1-a-b} \left(\dot{C}_{m-1}^{k,1}\dot{g}\right)_{(a,b)}(z) - \sum_{j=0}^{a-1} \left(\dot{C}_{m-1}^{j,2}\dot{g}\right)_{(a,b)}(z) \quad (2.294)$$

where for $0 \leq k \leq m-1-a-b$

$$\left(\dot{C}_{m-1}^{k,1}\dot{g}\right)_{(a,b)}(z) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(a,k+b)}(\zeta) d\zeta, \quad (2.295)$$

and for $0 \leq j \leq a-1$

$$\begin{aligned} \left(\dot{C}_{m-1}^{j,2}\dot{g}\right)_{(a,b)}(z) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-a-b+j}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j,m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.296)$$

at σ -a.e. point $z \in \partial\Omega$. Observe that for $0 < k \leq m-1-a-b$, $\dot{C}_{m-1}^{k,1}$ is weakly singular integral operator which is bounded on $L^p(\partial\Omega, \sigma)$. If $k=0$, then $\dot{C}_{m-1}^{0,1}$ is reduced to the boundary-to-boundary Cauchy operator which is also bounded on $L^p(\partial\Omega, \sigma)$ due to the Calderón-Zygmund theory. Next, we turn to the boundedness of $\dot{C}_{m-1}^{j,2}$. For $a, b \in \mathbb{N}_0$ with $a+b < m-1$, $\dot{C}_{m-1}^{j,2}$ is weakly singular integral operator for any $0 \leq j \leq a-1$ which is bounded on $L^p(\partial\Omega, \sigma)$. If $a+b = m-1$, then $\dot{C}_{m-1}^{j,2}$ becomes the Calderón-Zygmund integral operator which is also bounded on $L^p(\partial\Omega, \sigma)$ according to the Calderón-Zygmund theory. Consequently, for any $0 \leq k \leq m-1-a-b$, $0 \leq j \leq a-1$, we obtain

$$\begin{aligned} \left\| \left(\dot{C}_{m-1}^{k,1}\dot{g}\right)_{(a,b)} \right\|_{L^p(\partial\Omega, \sigma)} &\lesssim \|g_{(a,k+b)}\|_{L^p(\partial\Omega, \sigma)}, \\ \left\| \left(\dot{C}_{m-1}^{j,2}\dot{g}\right)_{(a,b)} \right\|_{L^p(\partial\Omega, \sigma)} &\lesssim \|g_{(a-1-j,m-a+j)}\|_{L^p(\partial\Omega, \sigma)}. \end{aligned} \quad (2.297)$$

Combining this with (2.294), one has

$$\left\| \dot{C}_{m-1}\dot{g} \right\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]} \lesssim \|\dot{g}\|_{\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]}. \quad (2.298)$$

Now, we are left with showing the regularity property in (2.282). Let us consider $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$. According to Definition 2.5, one has

$$\|\dot{C}_{m-1}\dot{g}\|_{\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]} = \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \left\| \left(\dot{C}_{m-1}\dot{g} \right)_{(a,b)} \right\|_{L_1^p(\partial\Omega, \sigma)}. \quad (2.299)$$

For $a, b \in \mathbb{N}_0$ with $a + b \leq m - 2$, using the compatibility condition of $\dot{C}_{m-1}\dot{g}$ and the boundedness as in (2.298), it can be checked that

$$\sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-2}} \left\| \left(\dot{C}_{m-1}\dot{g} \right)_{(a,b)} \right\|_{L_1^p(\partial\Omega, \sigma)} \lesssim \sum_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \|g_{(a,b)}\|_{L^p(\partial\Omega, \sigma)}. \quad (2.300)$$

If $a, b \in \mathbb{N}_0$ with $a + b = m - 1$, then $\dot{C}_{m-1}^{k,1}$ in (2.295) turns out to be the boundary-to-boundary Cauchy operator acting on $g_{(a,b)}$. According to the regularity result in Theorem 2.1, one has

$$\left\| \left(\dot{C}_{m-1}^{k,1}\dot{g} \right)_{(a,b)} \right\|_{L_1^p(\partial\Omega, \sigma)} \lesssim \|g_{(a,b)}\|_{L_1^p(\partial\Omega, \sigma)}. \quad (2.301)$$

Next, we turn our attention to $\dot{C}_{m-1}^{j,2}$. Applying the integration by parts gives for $0 \leq j \leq a - 1$

$$\begin{aligned} \left(\dot{C}_{m-1}^{j,2}\dot{g} \right)_{(a,b)}(z) &= - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{1}{j+1} \times \\ &\quad \times \frac{\overline{(z - \zeta)^{j+1}}}{(\zeta - z)^{j+1}} (\partial_\tau g_{(a-1-j, m-a+j)})(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.302)$$

at σ -a.e. point $z \in \partial\Omega$. Observe that the kernel in (2.302) is weakly singular which forces

$$\left\| \left(\dot{C}_{m-1}^{j,2}\dot{g} \right)_{(a,b)} \right\|_{L^p(\partial\Omega, \sigma)} \lesssim \left\| \partial_\tau g_{(a-1-j, m-a+j)} \right\|_{L^p(\partial\Omega, \sigma)}. \quad (2.303)$$

In addition,

$$\begin{aligned} \left(\partial_\tau \dot{C}_{m-1}^{j,2}\dot{g} \right)_{(a,b)}(z) &= - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{1}{j+1} \times \\ &\quad \times \partial_{\tau(z)} \left[\frac{\overline{(z - \zeta)^{j+1}}}{(\zeta - z)^{j+1}} \right] (\partial_\tau g_{(a-1-j, m-a+j)})(\zeta) d\sigma(\zeta) \end{aligned} \quad (2.304)$$

which is Calderón-Zygmund operator. Applying the Calderón-Zygmund theory provides

$$\left\| \left(\partial_\tau \dot{C}_{m-1}^{j,2} \dot{g} \right)_{(a,b)} \right\|_{L^p(\partial\Omega, \sigma)} \lesssim \left\| \partial_\tau g_{(a-1-j, m-a+j)} \right\|_{L^p(\partial\Omega, \sigma)}. \quad (2.305)$$

In conclusion, for each array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]$ we obtain that

$$\left\| \dot{C}_{m-1} \dot{g} \right\|_{\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]} \lesssim \left\| \dot{g} \right\|_{\text{CWA}_{m-1}[L_1^p(\partial\Omega, \sigma)]}. \quad (2.306)$$

This completes the proof of the theorem. \square

The theorem above also indicates the higher-order generalization of the boundedness and regularity of the classical Cauchy operator in Theorem 2.1. Remarkably, the boundary-to-boundary poly-Cauchy operator \dot{C}_{m-1} is tied up with its boundary-to-domain version $\dot{\mathcal{C}}_{m-1}$ via the jump-formula described below.

Theorem 2.37. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Fix an integer $m \in \mathbb{N}$ along with an integrability exponent $p \in (1, \infty)$. Then for each given array $\dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ one has*

$$\text{Tr}_{m-1}^{\text{C}} \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) = \left(\frac{1}{2} I + \dot{C}_{m-1} \right) \dot{g}. \quad (2.307)$$

Proof. Fix $\kappa > 0$. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Let $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$. From the formula (2.97), for each Lebesgue based complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ we have

$$\begin{aligned} \partial^a \bar{\partial}^b \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) (z) &= \sum_{k=0}^{m-1-a-b} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(a,k+b)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{a-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{j!}{(m-a-b+j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-a-b+j}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.308)$$

$$(2.309)$$

for each $z \in \Omega$. The integral kernel in (2.308) is either weakly singular, or Calderón-Zygmund operator. If $k > 0$, then the integral kernel in (2.308) is weakly singular which does not jump on the boundary. In the case when $k = 0$, we employ the jump relation (2.7) to write

$$\left(\frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{\zeta - z} g_{(a,b)}(\zeta) d\zeta \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} (z) \quad (2.310)$$

$$= ((\mathcal{C}g_{(a,b)})(z)) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} (z) \quad (2.311)$$

$$= \frac{1}{2} g_{(a,b)}(z) + \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{1}{\zeta - z} g_{(a,b)}(\zeta) d\zeta, \quad (2.312)$$

for almost every $z \in \partial\Omega$. If $a + b < m - 1$, then the integral kernel in (2.309) is weakly singular for all $0 \leq j \leq a - 1$ which does not jump on the boundary. In the case when $a, b \in \mathbb{N}_0$ satisfies $a + b = m - 1$. For simplicity, we denote $\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}}$ by $k_0(z - \zeta)$ for all $0 \leq j \leq a - 1$, and we apply the jump relations again to write

$$\begin{aligned} & \left(\int_{\partial\Omega} \partial_{\tau(\zeta)} [k_0(\cdot - \zeta)] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta) \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} (z) \\ &= \frac{1}{2\sqrt{-1}} \left[\nu_1 \nu_2 \widehat{k}_0(\nu) - \nu_2 \nu_1 \widehat{k}_0(\nu) \right] (z) g_{(a-1-j, m-a+j)}(z) \\ & \quad + \lim_{\varepsilon \rightarrow 0^+} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \partial_{\tau(\zeta)} [k_0(z - \zeta)] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta) \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \partial_{\tau(\zeta)} [k_0(z - \zeta)] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta) \end{aligned} \quad (2.313)$$

for almost every $z \in \partial\Omega$. Remark that

$$\begin{aligned} k_0(z - \zeta) &= -\frac{\pi(m - a - b + j)!}{j!} (\partial^j \bar{\partial}^{a+b-1-j} E_m)(z - \zeta), \\ \frac{(z - \zeta)^k}{k!(\zeta - z)} &= -\pi(\bar{\partial}^{m-1-k} E_m)(z - \zeta), \end{aligned} \quad (2.314)$$

where E_m is the fundamental solution for $\bar{\partial}^m$. This forces

$$\begin{aligned}
& \left(\frac{1}{2\pi i} \int_{\partial\Omega} \frac{j!}{(m-a-b+j)!} \times \right. \\
& \quad \left. \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-a-b+j}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta) \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} (z) \\
&= - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \partial_{\tau(\zeta)} \left[\left(\partial^j \bar{\partial}^{a+b-1-j} E_m \right) (z-\zeta) \right] \times \\
& \quad \times g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \tag{2.315}
\end{aligned}$$

and

$$\begin{aligned}
& \left(\sum_{k=0}^{m-1-a-b} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(a, k+b)}(\zeta) d\zeta \right) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} (z) \tag{2.316} \\
&= \frac{1}{2} g_{(a,b)}(z) - \sum_{k=0}^{m-1-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \left(\bar{\partial}^{m-1-k} E_m \right) (z-\zeta) g_{(a, k+b)}(\zeta) d\zeta,
\end{aligned}$$

for almost every $z \in \partial\Omega$. In conclusion, for the boundary-to-boundary poly-Cauchy operator \mathcal{C}_{m-1} in Definition 2.35 there holds

$$\text{Tr}_{m-1}^{\mathbb{C}} \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right) = \left(\frac{1}{2} I + \dot{C}_{m-1} \right) \dot{g}, \quad \text{at } \sigma\text{-almost every point on } \partial\Omega. \tag{2.317}$$

This completes the proof of the theorem 2.37 \square

As a corollary of Theorem 2.27 and (2.307) one obtains the higher-order version of the involution property as in the item (4) in Theorem 2.1.

Corollary 2.38. *Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary. Fix an integer $m \in \mathbb{N}$ along with an integrability exponent $p \in (1, \infty)$. There holds*

$$\left(\dot{C}_{m-1} \right)^2 = \frac{1}{4} I \quad \text{on } \text{CWA}_{m-1} [L^p(\partial\Omega, \sigma)], \tag{2.318}$$

where \dot{C}_{m-1} is the boundary-to-boundary poly-Cauchy operator defined in Definition 2.35.

Proof. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary, and consider array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. According to the Theorem 2.31, the poly-Cauchy operator $\dot{\mathcal{C}}_{m-1}\dot{g} \in \mathcal{H}^{p,m}(\Omega)$. Applying (2.231),

$$\dot{\mathcal{C}}_{m-1}\dot{g} = \dot{\mathcal{C}}_{m-1} \left(\text{Tr}_{m-1}^{\mathbb{C}} \left(\dot{\mathcal{C}}_{m-1}\dot{g} \right) \right) \quad \text{in } \Omega. \quad (2.319)$$

Combining this with taking $\dot{\mathcal{C}}_{m-1}$ to (2.307), we obtain that

$$\dot{\mathcal{C}}_{m-1}\dot{g} = \dot{\mathcal{C}}_{m-1} \left(\left(\frac{1}{2}I + \dot{C}_{m-1} \right) \dot{g} \right) \quad \text{on } \mathcal{H}^{p,m}(\Omega). \quad (2.320)$$

Taking $\text{Tr}_{m-1}^{\mathbb{C}}$ into the identity (2.320), we have

$$\text{Tr}_{m-1}^{\mathbb{C}} \left(\dot{\mathcal{C}}_{m-1}\dot{g} \right) = \text{Tr}_{m-1}^{\mathbb{C}} \left(\dot{\mathcal{C}}_{m-1} \right) \left(\left(\frac{1}{2}I + \dot{C}_{m-1} \right) \dot{g} \right) \quad (2.321)$$

on $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. The jump formula in (2.307) forces

$$\left(\frac{1}{2}I + \dot{C}_{m-1} \right) \dot{g} = \left(\frac{1}{2}I + \dot{C}_{m-1} \right) \left(\left(\frac{1}{2}I + \dot{C}_{m-1} \right) \dot{g} \right) \quad (2.322)$$

on $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, namely,

$$\left(\frac{1}{2}I + \dot{C}_{m-1} \right) \dot{g} = \left(\frac{1}{4}I + \dot{C}_{m-1} + \left(\dot{C}_{m-1} \right)^2 \right) \dot{g}, \quad (2.323)$$

for $\dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. This implies that

$$\left(\dot{C}_{m-1} \right)^2 \dot{g} = \frac{1}{4}\dot{g} \quad \text{on } \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \quad (2.324)$$

This finishes the proof of the Corollary. \square

2.8 Identification of the poly-Cauchy operator as the double multi-layer potential operator

The main goal of this section is to identify that the poly-Cauchy operator is a genuine double multi-layer potential operator associated with polylaplacian under appropriate identification. We first introduce real and complex Whitney array spaces, and identification map between these two array spaces.

Definition 2.39. For $p \in [1, \infty]$, define a real Whitney array space and a complex Whitney array space by

$$\begin{aligned} & \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \\ &= \left\{ \dot{f} = \{f_{(j,k)}\}_{\substack{j,k \in \mathbb{N}_0 \\ j+k \leq m-1}} \in CC_{\mathbb{R}} : f_{(j,k)} \in L^p(\partial\Omega, \sigma) \text{ if } j+k \leq m-1 \right\}, \end{aligned} \quad (2.325)$$

$$\begin{aligned} & \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \\ &= \left\{ \dot{g} = \{g_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in CC_{\mathbb{C}} : g_{(r,s)} \in L^p(\partial\Omega, \sigma) \text{ if } r+s \leq m-1 \right\}, \end{aligned} \quad (2.326)$$

respectively, where

$$\begin{aligned} \dot{f} = \{f_{(j,k)}\}_{\substack{j,k \in \mathbb{N}_0 \\ j+k \leq m-1}} \in CC_{\mathbb{R}} & \iff \\ \left\{ \begin{array}{l} \partial_{\tau} f_{(j,k)} = \nu_1 f_{(j,k+1)} - \nu_2 f_{(j+1,k)} \text{ } \sigma\text{-a.e. on } \partial\Omega \\ \text{whenever } j+k \leq m-2 \text{ and } j,k \in \mathbb{N}_0. \end{array} \right. & , \end{aligned} \quad (2.327)$$

and

$$\begin{aligned} \dot{g} = \{g_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in CC_{\mathbb{C}} & \iff \\ \left\{ \begin{array}{l} \partial_{\tau} g_{(r,s)} = i\nu g_{(r+1,s)} - i\bar{\nu} g_{(r,s+1)} \text{ } \sigma\text{-a.e. on } \partial\Omega \\ \text{whenever } r+s \leq m-2 \text{ and } r,s \in \mathbb{N}_0. \end{array} \right. & . \end{aligned} \quad (2.328)$$

We introduce a couple of useful combinatorial lemmas.

Lemma 2.40. For $r, s \in \mathbb{N}_0$, $0 \leq d \leq r+s$, we have

$$\begin{aligned} & \frac{1}{2^{r+s}} \sum_{\ell=0}^{r+s} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s \\ a+b=\ell}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \\ & \times \sum_{\substack{0 \leq p \leq \ell \\ 0 \leq q \leq r+s-\ell \\ p+q=d}} \frac{\ell!}{p!(\ell-p)!} \frac{(r+s-\ell)!}{q!(r+s-\ell-q)!} (-1)^{r-a-q} = \delta_{dr}, \end{aligned} \quad (2.329)$$

where

$$\delta_{dr} = \begin{cases} 1 & \text{if } d = r \\ 0 & \text{otherwise} \end{cases}. \quad (2.330)$$

Proof. Fix $r, s \in \mathbb{N}_0$. For $x, y \in \mathbb{R}$, recall

$$\begin{aligned} z &= x - iy, \quad \bar{z} = x + iy, \\ x &= \frac{z + \bar{z}}{2}, \quad y = i \frac{z - \bar{z}}{2}. \end{aligned} \quad (2.331)$$

Applying binomial formula combining with (2.331), we have

$$\begin{aligned} z^r \bar{z}^s &= (x - iy)^r (x + iy)^s \quad (2.332) \\ &= \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} i^{r+s-(a+b)} (-1)^{r-a} x^{a+b} y^{r+s-(a+b)} \\ &= \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} i^{r+s-(a+b)} (-1)^{r-a} \left(\frac{z + \bar{z}}{2} \right)^{a+b} \left(i \frac{z - \bar{z}}{2} \right)^{r+s-(a+b)} \\ &= \frac{1}{2^{r+s}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \\ &\quad \times \sum_{\substack{0 \leq p \leq a+b \\ 0 \leq q \leq r+s-(a+b)}} \frac{(a+b)!}{p!(a+b-p)!} \frac{(r+s-a-b)!}{q!(r+s-a-b-q)!} (-1)^{r-a-q} z^{p+q} \bar{z}^{r+s-(p+q)}, \end{aligned}$$

where $c = r + s$. This forces

$$\begin{aligned} z^r \bar{z}^s &= \frac{1}{2^c} \sum_{\ell=0}^c \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s \\ a+b=\ell}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \\ &\quad \times \sum_{d=0}^{r+s} \sum_{\substack{0 \leq p \leq \ell \\ 0 \leq q \leq c-\ell \\ p+q=d}} \frac{\ell!}{p!(\ell-p)!} \frac{(c-\ell)!}{q!(c-\ell-q)!} (-1)^{r-a-q} z^d \bar{z}^{c-d} \\ &= \sum_{d=0}^c \left[\frac{1}{2^c} \sum_{\ell=0}^c \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s \\ a+b=\ell}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \right. \\ &\quad \left. \times \sum_{\substack{0 \leq p \leq \ell \\ 0 \leq q \leq c-\ell \\ p+q=d}} \frac{\ell!}{p!(\ell-p)!} \frac{(c-\ell)!}{q!(c-\ell-q)!} (-1)^{r-a-q} \right] z^d \bar{z}^{c-d} \quad (2.333) \end{aligned}$$

On the other hand,

$$z^r \bar{z}^s = \sum_{d=0}^c \delta_{dr} z^d \bar{z}^{c-d}. \quad (2.334)$$

Comparing coefficients with (2.332) completes the proof. \square

Lemma 2.41. Fix $m \in \mathbb{N}$, $k \in \{0, \dots, m\}$. For $r \in \{0, 1, \dots, m-k\}$, the following holds.

$$\frac{1}{2^{m-k}} \sum_{\ell=0}^{m-k} \sum_{\substack{0 \leq p \leq m-k-\ell \\ 0 \leq q \leq \ell \\ p+q=r}} (-1)^q \cdot \frac{(m-k)!}{p!(m-k-\ell-p)!q!(\ell-q)!} = \delta_{r0}, \quad (2.335)$$

where

$$\delta_{r0} = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases}. \quad (2.336)$$

Proof. Combining (2.331) with the binomial formula yields, for each $z = x - iy$ where $x, y \in \mathbb{R}$,

$$\begin{aligned} \bar{z}^{m-k} &= (x + iy)^{m-k} = \sum_{\ell=0}^{m-k} \frac{(m-k)!}{\ell!(m-k-\ell)!} x^{m-k-\ell} (iy)^\ell \\ &= \sum_{\ell=0}^{m-k} \frac{(m-k)!}{\ell!(m-k-\ell)!} \left(\frac{z + \bar{z}}{2} \right)^{m-k-\ell} \left(\frac{\bar{z} - z}{2} \right)^\ell, \end{aligned} \quad (2.337)$$

which forces

$$\begin{aligned} &\bar{z}^{m-k} \\ &= \frac{1}{2^{m-k}} \sum_{\ell=0}^{m-k} (-1)^\ell \frac{(m-k)!}{\ell!(m-k-\ell)!} \times \\ &\quad \times \sum_{\substack{0 \leq p \leq m-k-\ell \\ 0 \leq q \leq \ell}} (-1)^{\ell-q} \frac{(m-k-\ell)!}{p!(m-k-\ell-p)!} \frac{\ell!}{q!(\ell-q)!} z^{p+q} \bar{z}^{m-k-p-q} \\ &= \frac{1}{2^{m-k}} \sum_{\ell=0}^{m-k} \sum_{\substack{0 \leq p \leq m-k-\ell \\ 0 \leq q \leq \ell}} (-1)^q \frac{(m-k)!}{p!(m-k-\ell-p)!q!(\ell-q)!} z^{p+q} \bar{z}^{m-k-(p+q)} \\ &= \frac{1}{2^{m-k}} \sum_{r=0}^{m-k} \sum_{\ell=0}^{m-k} \sum_{\substack{0 \leq p \leq m-k-\ell \\ 0 \leq q \leq \ell \\ p+q=r}} (-1)^q \frac{(m-k)!}{p!(m-k-\ell-p)!q!(\ell-q)!} z^r \bar{z}^{m-k-r}. \end{aligned} \quad (2.338)$$

Note that

$$\bar{z}^{m-k} = \sum_{r=0}^{m-k} \delta_{r0} z^r \bar{z}^{m-k-r}. \quad (2.339)$$

Combining this with (2.338), this finishes the proof. \square

For $p \in [1, \infty]$, we introduce a map Ψ from the real Whitney array into the complex Whitney array

$$\Psi : \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)], \quad (2.340)$$

defined by for $\dot{f} = \{f_{(j,k)}\}_{\substack{j,k \in \mathbb{N}_0 \\ j+k \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ and $r, s \in \mathbb{N}_0$ such that $r + s \leq m - 1$

$$\begin{aligned} \left(\Psi(\dot{f}) \right)_{(r,s)} &:= \frac{1}{2^{r+s}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \\ &\quad \times (-1)^{r-a} i^{r+s-(a+b)} f_{(a+b, r+s-a-b)}. \end{aligned} \quad (2.341)$$

Lemma 2.42. $\Psi : \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ defined as in (2.341) is well-defined linear map for $m \in \mathbb{N}$, $p \in [1, \infty]$.

Proof. From the construction, Ψ is a linear map and $\Psi(\dot{0}) = \dot{0}$. For any real Whitney array $\dot{f} = \{f_{(j,k)}\}_{\substack{j,k \in \mathbb{N}_0 \\ j+k \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ and $r, s \in \mathbb{N}_0$ such that $r + s \leq m - 1$, we have

$$\begin{aligned} &\left\| \left(\Psi(\dot{f}) \right)_{(r,s)} \right\|_{L^p(\partial\Omega, \sigma)} \\ &= \left\| \frac{1}{2^{r+s}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} f_{(a+b, r+s-a-b)} \right\|_{L^p(\partial\Omega, \sigma)} \\ &\leq C(m) \sum_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \|f_{(a+b, r+s-a-b)}\|_{L^p(\partial\Omega, \sigma)} \\ &\leq C(m) \|\dot{f}\|_{\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]}, \end{aligned} \quad (2.342)$$

which implies for $r, s \in \mathbb{N}_0$ with $r + s \leq m - 1$

$$\left(\Psi(\dot{f}) \right)_{(r,s)} \in L^p(\partial\Omega, \sigma). \quad (2.343)$$

Moreover, for $r, s \in \mathbb{N}_0$ with $r + s \leq m - 2$, applying the following relations

$$\nu_1 = \frac{\nu + \bar{\nu}}{2}, \nu_2 = \frac{\bar{\nu} - \nu}{2}i, \quad (2.344)$$

we obtain that

$$\begin{aligned} \partial_\tau \left(\Psi(\dot{f}) \right)_{(r,s)} &= \frac{1}{2^{r+s}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} \partial_\tau f_{(a+b, r+s-a-b)} \\ &= \frac{1}{2^{r+s}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \\ &\quad \times (-1)^{r-a} i^{r+s-(a+b)} (\nu_1 f_{(a+b, r+s+1-a-b)} - \nu_2 f_{(a+b+1, r+s-a-b)}), \end{aligned} \quad (2.345)$$

which implies

$$\begin{aligned} \partial_\tau \left(\Psi(f) \right)_{(r,s)} &= \frac{1}{2^{r+s}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} \times \\ &\quad \times \left[i\nu \left(\frac{1}{2} \cdot (f_{(a+b+1, r+s-a-b)} - i f_{(a+b, r+s+1-a-b)}) \right) \right. \\ &\quad \left. - i\bar{\nu} \left(\frac{1}{2} \cdot (f_{(a+b+1, r+s-a-b)} + i f_{(a+b, r+s+1-a-b)}) \right) \right]. \end{aligned} \quad (2.346)$$

This forces

$$\begin{aligned} \partial_\tau \left(\Psi(f) \right)_{(r,s)} &= i\nu \cdot \left[\frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} \times \right. \\ &\quad \left. \times (f_{(a+b+1, r+s-a-b)} - i f_{(a+b, r+s+1-a-b)}) \right] \\ &\quad - i\bar{\nu} \cdot \left[\frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} \times \right. \\ &\quad \left. \times (f_{(a+b+1, r+s-a-b)} + i f_{(a+b, r+s+1-a-b)}) \right]. \end{aligned} \quad (2.347)$$

Observe that

$$\begin{aligned}
& \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} f_{(a+b+1, r+s-a-b)} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{1 \leq a \leq r+1 \\ 0 \leq b \leq s}} \frac{r!}{(a-1)!(r+1-a)!} \times \\
& \quad \times \frac{s!}{b!(s-b)!} (-1)^{r+1-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)}, \tag{2.348}
\end{aligned}$$

which forces

$$\begin{aligned}
& \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} f_{(a+b+1, r+s-a-b)} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{1 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{(a-1)!(r+1-a)!} \frac{s!}{b!(s-b)!} \times \\
& \quad \times (-1)^{r+1-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
& \quad + \frac{1}{2^{r+s+1}} \sum_{0 \leq b \leq s} \frac{s!}{b!(s-b)!} i^{s-b} f_{(r+1+b, s-b)}, \tag{2.349}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{-i}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} f_{(a+b, r+s+1-a-b)} \tag{2.350} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r+1-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)},
\end{aligned}$$

which implies

$$\begin{aligned}
& \frac{-i}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} f_{(a+b, r+s+1-a-b)} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{1 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r+1-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
& \quad + \frac{1}{2^{r+s+1}} \sum_{0 \leq b \leq s} \frac{s!}{b!(s-b)!} (-1)^{r+1} i^{r+s+1-b} f_{(b, r+s+1-b)}. \tag{2.351}
\end{aligned}$$

In addition, for $1 \leq a \leq r$

$$\begin{aligned}
& \frac{1}{(a-1)!(r+1-a)!} + \frac{1}{a!(r-a)!} \\
&= \frac{1}{(a-1)!(r-a)!(r+1-a)} + \frac{1}{(a-1)!(r-a)!a} \\
&= \frac{1}{(a-1)!(r-a)!} \left[\frac{1}{r+1-a} + \frac{1}{a} \right] \\
&= \frac{r+1}{a!(r+1-a)!}.
\end{aligned} \tag{2.352}$$

Combining this together with (2.348), (2.350), we conclude that

$$\begin{aligned}
& \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} \times \\
& \times (f_{(a+b+1, r+s-a-b)} - i f_{(a+b, r+s+1-a-b)}) \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{1 \leq a \leq r \\ 0 \leq b \leq s}} \frac{(r+1)!}{a!(r+1-a)!} \frac{s!}{b!(s-b)!} (-1)^{r+1-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
& + \frac{1}{2^{r+s+1}} \sum_{0 \leq b \leq s} \frac{s!}{b!(s-b)!} i^{s-b} f_{(r+1+b, s-b)} \\
& + \frac{1}{2^{r+s+1}} \sum_{0 \leq b \leq s} \frac{s!}{b!(s-b)!} (-1)^{r+1} i^{r+s+1-b} f_{(b, r+s+1-b)} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r+1 \\ 0 \leq b \leq s}} \frac{(r+1)!}{a!(r+1-a)!} \frac{s!}{b!(s-b)!} (-1)^{r+1-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
&= \left(\Psi(\dot{f}) \right)_{(r+1, s)}.
\end{aligned} \tag{2.353}$$

Similarly,

$$\begin{aligned}
& \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} f_{(a+b+1, r+s-a-b)} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 1 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{(b-1)!(s+1-b)!} (-1)^{r-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
& + \frac{1}{2^{r+s+1}} \sum_{0 \leq a \leq r} \frac{r!}{a!(r-a)!} (-1)^{r-a} i^{r-a} f_{(a+s+1, r-a)},
\end{aligned} \tag{2.354}$$

and

$$\begin{aligned}
& \frac{i}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} f_{(a+b, r+s+1-a-b)} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 1 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
&+ \frac{1}{2^{r+s+1}} \sum_{0 \leq a \leq r} \frac{r!}{a!(r-a)!} (-1)^{r-a} i^{r+s+1-a} f_{(a, r+s+1-a)}. \tag{2.355}
\end{aligned}$$

Combining (2.352) for b, s together with (2.354), (2.355), we conclude that

$$\begin{aligned}
& \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} \times \\
& \times (f_{(a+b+1, r+s-a-b)} + i f_{(a+b, r+s+1-a-b)}) \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 1 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{(s+1)!}{b!(s+1-b)!} (-1)^{r-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
&+ \frac{1}{2^{r+s+1}} \sum_{0 \leq a \leq r} \frac{r!}{a!(r-a)!} (-1)^{r-a} i^{r-a} f_{(a+s+1, r-a)} \\
&+ \frac{1}{2^{r+s+1}} \sum_{0 \leq a \leq r} \frac{r!}{a!(r-a)!} (-1)^{r-a} i^{r+s+1-a} f_{(a, r+s+1-a)} \\
&= \frac{1}{2^{r+s+1}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s+1}} \frac{r!}{a!(r-a)!} \frac{(s+1)!}{b!(s+1-b)!} (-1)^{r-a} i^{r+s+1-(a+b)} f_{(a+b, r+s+1-a-b)} \\
&= \left(\Psi(\dot{f}) \right)_{(r, s+1)}. \tag{2.356}
\end{aligned}$$

Applying (2.353) and (2.356) to (2.347), we obtain for any $r, s \in \mathbb{N}_0$ with $r+s \leq m-2$

$$\partial_\tau \left(\Psi(\dot{f}) \right)_{(r, s)} = i\nu \cdot \left(\Psi(\dot{f}) \right)_{(r+1, s)} - i\bar{\nu} \cdot \left(\Psi(\dot{f}) \right)_{(r, s+1)}. \tag{2.357}$$

This with (2.343) shows for $\dot{f} = \{f_{(j,k)}\}_{\substack{j,k \in \mathbb{N}_0 \\ j+k \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$,

$$\Psi(\dot{f}) = \left\{ \left(\Psi(\dot{f}) \right)_{(r, s)} \right\}_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \tag{2.358}$$

Consequently,

$$\Psi : \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \quad (2.359)$$

defined as in (2.341) is well-defined linear map. \square

Next, for $p \in [1, \infty]$, we introduce a map Φ from the complex Whitney array into the real Whitney array

$$\Phi : \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \quad (2.360)$$

defined by for $\dot{g} = \{g_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ and $j, k \in \mathbb{N}_0$ with $j + k \leq m - 1$

$$\left(\Phi(\dot{g})\right)_{(j,k)} := i^k \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b, j+k-a-b)}. \quad (2.361)$$

Lemma 2.43. $\Phi : \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ defined as in (2.361) is well-defined linear map for $m \in \mathbb{N}$, $p \in [1, \infty]$.

Proof. Notice that Φ is a linear map and $\Phi(\dot{0}) = \dot{0}$. Similar to (2.342), for $j, k \in \mathbb{N}_0$ with $j + k \leq m - 1$

$$\left(\Phi(\dot{g})\right)_{(j,k)} \in L^p(\partial\Omega, \sigma). \quad (2.362)$$

Since $\nu = \nu_1 + i\nu_2$, $\bar{\nu} = \nu_1 - i\nu_2$, we obtain

$$\begin{aligned} & \partial_\tau \left(\Phi(\dot{g})\right)_{(j,k)} \\ &= i^k \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} \partial_\tau g_{(a+b, j+k-a-b)} \\ &= i^k \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} (i\nu g_{(a+b+1, j+k-a-b)} - i\bar{\nu} g_{(a+b, j+k+1-a-b)}) \\ &= i^k \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} \times \\ & \quad \times (-1)^{k-b} \left[i\nu_1 (g_{(a+b+1, j+k-a-b)} - g_{(a+b, j+k+1-a-b)}) \right. \\ & \quad \left. - \nu_2 (g_{(a+b+1, j+k-a-b)} + g_{(a+b, j+k+1-a-b)}) \right]. \quad (2.363) \end{aligned}$$

Going further,

$$\begin{aligned}
& \partial_\tau \left(\Phi(\dot{g}) \right)_{(j,k)} \tag{2.364} \\
&= \nu_1 \cdot \left[i^{k+1} \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} \times \right. \\
&\quad \left. \times (-1)^{k-b} (g_{(a+b+1, j+k-a-b)} - g_{(a+b, j+k+1-a-b)}) \right] \\
&\quad - \nu_2 \cdot \left[i^k \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} \times \right. \\
&\quad \left. \times (-1)^{k-b} (g_{(a+b+1, j+k-a-b)} + g_{(a+b, j+k+1-a-b)}) \right].
\end{aligned}$$

In particular,

$$\begin{aligned}
& \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b+1, j+k-a-b)} \\
&= \sum_{\substack{0 \leq a \leq j \\ 1 \leq b \leq k+1}} \frac{j!}{a!(j-a)!} \frac{k!}{(b-1)!(k+1-b)!} (-1)^{k+1-b} g_{(a+b, j+k+1-a-b)} \\
&= \sum_{\substack{0 \leq a \leq j \\ 1 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{(b-1)!(k+1-b)!} (-1)^{k+1-b} g_{(a+b, j+k+1-a-b)} \\
&\quad + \sum_{0 \leq a \leq j} \frac{j!}{a!(j-a)!} g_{(a+k+1, j-a)}, \tag{2.365}
\end{aligned}$$

and

$$\begin{aligned}
& - \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b, j+k+1-a-b)} \\
&= \sum_{\substack{0 \leq a \leq j \\ 1 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k+1-b} g_{(a+b, j+k+1-a-b)} \\
&\quad + \sum_{0 \leq a \leq j} \frac{j!}{a!(j-a)!} (-1)^{k+1} g_{(a, j+k+1-a)}. \tag{2.366}
\end{aligned}$$

Applying (2.352) for b, k , we have

$$\frac{1}{(b-1)!(k+1-b)!} + \frac{1}{b!(k-b)!} = \frac{k+1}{b!(k+1-b)!}. \quad (2.367)$$

Combining this together with (2.365), (2.366), we obtain that

$$\begin{aligned} & i^{k+1} \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} (g_{(a+b+1, j+k-a-b)} - g_{(a+b, j+k+1-a-b)}) \\ &= i^{k+1} \left[\sum_{\substack{0 \leq a \leq j \\ 1 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{(k+1)!}{b!(k+1-b)!} (-1)^{k+1-b} g_{(a+b, j+k+1-a-b)} \right. \\ & \quad \left. + \sum_{0 \leq a \leq j} \frac{j!}{a!(j-a)!} g_{(a+k+1, j-a)} + \sum_{0 \leq a \leq j} \frac{j!}{a!(j-a)!} (-1)^{k+1} g_{(a, j+k+1-a)} \right] \\ &= i^{k+1} \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k+1}} \frac{j!}{a!(j-a)!} \frac{(k+1)!}{b!(k+1-b)!} (-1)^{k+1-b} g_{(a+b, j+k+1-a-b)} \\ &= \left(\Phi(\hat{j}) \right)_{(j, k+1)}. \end{aligned} \quad (2.368)$$

Similarly,

$$\begin{aligned} & \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b+1, j+k-a-b)} \\ &= \sum_{\substack{1 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{(a-1)!(j+1-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b, j+k+1-a-b)} \\ & \quad + \sum_{0 \leq b \leq k} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(j+1+b, k-b)}, \end{aligned} \quad (2.369)$$

and

$$\begin{aligned} & \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b, j+k+1-a-b)} \\ &= \sum_{\substack{1 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b, j+k+1-a-b)} \\ & \quad + \sum_{0 \leq b \leq k} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(b, j+k+1-b)}. \end{aligned} \quad (2.370)$$

Combining (2.352) for a, j together with (2.369) and (2.370), we obtain that

$$\begin{aligned}
& i^k \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} (g_{(a+b+1, j+k-a-b)} + g_{(a+b, j+k+1-a-b)}) \\
&= i^k \left[\sum_{\substack{1 \leq a \leq j \\ 0 \leq b \leq k}} \frac{(j+1)!}{a!(j+1-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b, j+k+1-a-b)} \right. \\
&\quad \left. + \sum_{0 \leq b \leq k} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(j+1+b, k-b)} + \sum_{0 \leq b \leq k} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(b, j+k+1-b)} \right] \\
&= i^k \sum_{\substack{0 \leq a \leq j+1 \\ 0 \leq b \leq k}} \frac{(j+1)!}{a!(j+1-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} g_{(a+b, j+k+1-a-b)} \\
&= \left(\Phi(\dot{g}) \right)_{(j+1, k)}. \tag{2.371}
\end{aligned}$$

Consequently, substituting (2.368) and (2.371) into (2.364) yields

$$\partial_\tau \left(\Phi(\dot{g}) \right)_{(j, k)} = \nu_1 \cdot \left(\Phi(\dot{g}) \right)_{(j, k+1)} - \nu_2 \cdot \left(\Phi(\dot{g}) \right)_{(j+1, k)}. \tag{2.372}$$

This with (2.362) implies that for $\dot{g} = \{g_{(r, s)}\}_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$,

$$\Phi(\dot{g}) = \left\{ \left(\Phi(\dot{g}) \right)_{(j, k)} \right\}_{\substack{j, k \in \mathbb{N}_0 \\ j+k \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \tag{2.373}$$

defined as in (2.361) is well-defined linear map. \square

Proposition 2.44. *Fix $m \in \mathbb{N}$, $p \in [1, \infty]$. The maps Ψ defined in (2.341) and Φ defined in (2.361) are inverse to one another.*

Proof. Let $\dot{g} = \{g_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, then

$$\begin{aligned}
& \left(\Psi(\Phi(\dot{g})) \right)_{(r,s)} \\
&= \frac{1}{2^{r+s}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} \left(\Phi(\dot{g}) \right)_{(a+b, r+s-a-b)} \\
&= \frac{1}{2^{r+s}} \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{r-a} i^{r+s-(a+b)} \times \\
&\quad \times i^{r+s-(a+b)} \sum_{\substack{0 \leq p \leq a+b \\ 0 \leq q \leq r+s-(a+b)}} \frac{(a+b)!}{p!(a+b-p)!} \frac{(r+s-a-b)!}{q!(r+s-a-b-q)!} \times \\
&\quad \times (-1)^{r+s-a-b-q} g_{(p+q, r+s-p-q)}. \tag{2.374}
\end{aligned}$$

Let $c = r + s$, then one can conclude

$$\begin{aligned}
& \left(\Psi(\Phi(\dot{g})) \right)_{(r,s)} \\
&= \frac{1}{2^c} \sum_{\ell=0}^c \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s \\ a+b=\ell}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \\
&\quad \times \sum_{\substack{0 \leq p \leq \ell \\ 0 \leq q \leq c-\ell}} \frac{\ell!}{p!(\ell-p)!} \frac{(c-\ell)!}{q!(c-\ell-q)!} (-1)^{r-a-q} g_{(p+q, c-p-q)} \\
&= \frac{1}{2^c} \sum_{\ell=0}^c \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s \\ a+b=\ell}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \\
&\quad \times \sum_{\substack{d=0 \\ 0 \leq p \leq \ell \\ 0 \leq q \leq c-\ell \\ p+q=d}}^c \sum_{\substack{0 \leq p \leq \ell \\ 0 \leq q \leq c-\ell \\ p+q=d}} \frac{\ell!}{p!(\ell-p)!} \frac{(c-\ell)!}{q!(c-\ell-q)!} (-1)^{r-a-q} g_{(d, c-d)} \\
&= \sum_{d=0}^c g_{(d, c-d)} \left[\frac{1}{2^c} \sum_{\ell=0}^c \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s \\ a+b=\ell}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} \times \right. \\
&\quad \left. \times \sum_{\substack{0 \leq p \leq \ell \\ 0 \leq q \leq c-\ell \\ p+q=d}} \frac{\ell!}{p!(\ell-p)!} \frac{(c-\ell)!}{q!(c-\ell-q)!} (-1)^{r-a-q} \right]. \tag{2.375}
\end{aligned}$$

Combining this with lemma 2.40, we conclude that

$$\left(\Psi(\Phi(\dot{g}))\right)_{(r,s)} = \sum_{d=0}^c g_{(d,c-d)} \delta_{dr}, \quad (2.376)$$

which forces

$$\left(\Psi(\Phi(\dot{g}))\right)_{(r,s)} = g_{(r,s)}. \quad (2.377)$$

Similarly, for $\dot{f} = \{f_{(j,k)}\}_{\substack{j,k \in \mathbb{N}_0 \\ j+k \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$

$$\begin{aligned} & \left(\Phi(\Psi(\dot{f}))\right)_{(j,k)} \\ &= i^k \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} \left(\Psi(\dot{f})\right)_{(a+b, j+k-a-b)} \\ &= i^k \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^{k-b} \frac{1}{2^{j+k}} \sum_{\substack{0 \leq p \leq a+b \\ 0 \leq q \leq j+k-a-b}} \frac{(a+b)!}{p!(a+b-p)!} \times \\ & \quad \times \frac{(j+k-a-b)!}{q!(j+k-a-b-q)!} (-1)^{a+b-p} i^{j+k-p-q} f_{(p+q, j+k-p-q)}. \end{aligned} \quad (2.378)$$

Let $c = j + k$, then

$$\begin{aligned} & \left(\Phi(\Psi(\dot{f}))\right)_{(j,k)} \\ &= \frac{i^j}{2^c} \sum_{\ell=0}^c \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k \\ a+b=\ell}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} (-1)^a \times \\ & \quad \times \sum_{d=0}^c \sum_{\substack{0 \leq p \leq \ell \\ 0 \leq q \leq c-\ell \\ p+q=d}} \frac{\ell!}{p!(\ell-p)!} \frac{(c-\ell)!}{q!(c-\ell-q)!} (-1)^p i^{-d} f_{(d,c-d)} \\ &= i^j \sum_{d=0}^c i^{-d} (-1)^{d-j} f_{(d,c-d)} \left[\frac{1}{2^c} \sum_{\ell=0}^c \sum_{\substack{0 \leq a \leq j \\ 0 \leq b \leq k \\ a+b=\ell}} \frac{j!}{a!(j-a)!} \frac{k!}{b!(k-b)!} \times \right. \\ & \quad \left. \times \sum_{\substack{0 \leq p \leq \ell \\ 0 \leq q \leq c-\ell \\ p+q=d}} \frac{\ell!}{p!(\ell-p)!} \frac{(c-\ell)!}{q!(c-\ell-q)!} (-1)^{j-a-q} \right]. \end{aligned} \quad (2.379)$$

Combining this with lemma 2.40, we obtain

$$\left(\Phi(\Psi(\dot{f}))\right)_{(j,k)} = i^j \sum_{d=0}^c i^{-d} (-1)^{d-j} f_{(d,c-d)} \delta_{dj}, \quad (2.380)$$

which implies

$$\left(\Phi(\Psi(\dot{f}))\right)_{(j,k)} = f_{(j,k)}. \quad (2.381)$$

This completes the proof of the proposition. \square

We recall the definition of double multi-layer potential operator. Let L be a homogeneous differential operator of order $2m$ with constant coefficients which is given by

$$L := \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta} \partial^\beta. \quad (2.382)$$

Definition 2.45. Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain with compact boundary. Let L be a matrix-valued constant coefficient, homogeneous, weakly elliptic differential operator of order $2m$ in \mathbb{R}^n and denote by E a fundamental solution for L in \mathbb{R}^n . Then the double multi-layer potential operator associated with coefficient tensor $A = (A_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ |\alpha|=|\beta|=m}}$ acting on the real Whitney array $\dot{f} = \{f_\delta\}_{|\delta| \leq m-1} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ for $p \in (1, \infty)$ is given by

$$\begin{aligned} \mathcal{D}_L^A \dot{f}(X) := & - \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{k=1}^m \sum_{\substack{|\delta|=m-k \\ |\gamma|=k-1 \\ \gamma+\delta+e_j=\alpha}} \left\{ \frac{\alpha!(m-k)!(k-1)!}{m! \gamma! \delta!} \times \right. \\ & \left. \times \int_{\partial\Omega} \nu_j(Y) (\partial^{\beta+\gamma} E)(X-Y) A_{\beta\alpha} f_\delta(Y) d\sigma(Y) \right\} \end{aligned} \quad (2.383)$$

for $X \in \mathbb{R}^n \setminus \partial\Omega$.

Lemma 2.46. Fix $m \in \mathbb{N}$. Define a tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ by

$$\tilde{A}_{(m-s,s),(m-q,q)} = \frac{(m!)^2}{(m-s)!(m-q)!s!q!} (-1)^q i^{s+q}, \quad \text{for } 0 \leq s, q \leq m. \quad (2.384)$$

Then, $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ as in (2.384) is a coefficient tensor associated with Δ^m in \mathbb{R}^2 .

Proof. Fix $m \in \mathbb{N}$. Consider $\alpha, \beta \in \mathbb{N}_0^2$ with $|\alpha| = |\beta| = m$. We can rewrite α, β as $\alpha = (m - s, s)$, $\beta = (m - q, q)$ for some $0 \leq s, q \leq m$. Then,

$$\begin{aligned}
\sum_{|\alpha|=|\beta|=m} \partial^\alpha \tilde{A}_{\alpha\beta} \partial^\beta &= \sum_{\substack{0 \leq s \leq m \\ 0 \leq q \leq m}} \partial^{(m-s,s)} \tilde{A}_{(m-s,s),(m-q,q)} \partial^{(m-q,q)} & (2.385) \\
&= \sum_{\substack{0 \leq s \leq m \\ 0 \leq q \leq m}} \tilde{A}_{(m-s,s),(m-q,q)} \partial_x^{m-s} \partial_y^s \partial_x^{m-q} \partial_y^q \\
&= \sum_{\substack{0 \leq s \leq m \\ 0 \leq q \leq m}} \frac{(m!)^2}{(m-s)!(m-q)!s!q!} (-1)^q i^{s+q} \partial_x^{m-s} \partial_y^s \partial_x^{m-q} \partial_y^q.
\end{aligned}$$

This forces

$$\begin{aligned}
\sum_{|\alpha|=|\beta|=m} \partial^\alpha \tilde{A}_{\alpha\beta} \partial^\beta &= \sum_{s=0}^m \frac{m!}{(m-s)!s!} \partial_x^{m-s} (i\partial_y)^s \sum_{q=0}^m \frac{m!}{(m-q)!q!} \partial_x^{m-q} (-i\partial_y)^q \\
&= (\partial_x + i\partial_y)^m (\partial_x - i\partial_y)^m \\
&= 4^m \left(\frac{\partial_x + i\partial_y}{2} \right)^m \left(\frac{\partial_x - i\partial_y}{2} \right)^m \\
&= 4^m \bar{\partial}_z^m \partial_z^m. & (2.386)
\end{aligned}$$

Combining this with (2.21), we conclude that

$$\sum_{|\alpha|=|\beta|=m} \partial^\alpha \tilde{A}_{\alpha\beta} \partial^\beta = \Delta^m. \quad (2.387)$$

This finishes the proof of the lemma. \square

In the next theorem, we prove that the boundary-to-domain poly-Cauchy operator of order m is a genuine double multi-layer potential operator associated with polylaplacian of order m and its coefficient tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ under the identification map $\Phi : \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ defined in (2.360).

Theorem 2.47. *Fix $m \in \mathbb{N}$ and $p \in (1, \infty)$. The boundary-to-domain poly-Cauchy operator \mathcal{E}_{m-1} is a double multi-layer potential operator associated*

with Δ^m and the coefficient tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ under appropriate identification. Namely, for any $\dot{g} = \{g_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$

$$\mathcal{C}_{m-1}(\dot{g}) \equiv \dot{\mathcal{D}}_m(\Phi(\dot{g})) \quad \text{in } \Omega, \quad (2.388)$$

where Φ is defined in (2.360) and $\dot{\mathcal{D}}_m = \dot{\mathcal{D}}_{\Delta^m}^{\tilde{A}}$ is the double multi-layer potential associated with the coefficient tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ given in Lemma 2.46.

Proof. Let us first find the double multi-layer potential operator $\dot{\mathcal{D}}_m$ associated with Δ^m and its coefficient tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ using the idea employing bilinear form used in [23]. Consider a bilinear form $\tilde{\mathcal{B}}$ by

$$\tilde{\mathcal{B}}(u, v) = \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}} \int_{\Omega} \langle \tilde{A}_{\alpha\beta} \partial^{\beta} u, \partial^{\alpha} v \rangle dY, \quad (2.389)$$

for reasonable functions u, v and domain Ω . Using the similar argument in the proof of Lemma 2.46, we have

$$\begin{aligned} & \tilde{\mathcal{B}}(u, v) \\ &= \sum_{\substack{0 \leq p \leq m \\ 0 \leq q \leq m}} \int_{\Omega} \left\langle \frac{m!}{(m-p)!p!} \partial_x^{m-p} (i\partial_y)^p u, \frac{m!}{(m-q)!q!} \partial_x^{m-q} (-i\partial_y)^q v \right\rangle dY. \end{aligned} \quad (2.390)$$

Going further, applying binomial formula as in Lemma 2.46,

$$\begin{aligned} \tilde{\mathcal{B}}(u, v) &= 4^m \int_{\Omega} \left\langle \left(\frac{\partial_x + i\partial_y}{2} \right)^m u, \left(\frac{\partial_x - i\partial_y}{2} \right)^m v \right\rangle dY \\ &= 4^m \int_{\Omega} \langle \bar{\partial}_z^m u, \partial_z^m v \rangle dY. \end{aligned} \quad (2.391)$$

Applying integration by parts m times as in (2.36)-(2.38), we may write

$$\begin{aligned} \tilde{\mathcal{B}}(u, v) &= 4^m \sum_{k=0}^{m-1} \frac{(-1)^k}{2} \int_{\partial\Omega} \left(\bar{\partial}_z^k \partial_z^m v \right) \cdot \left(\bar{\partial}_z^{m-k-1} u \right) \nu d\sigma(Y) \\ &\quad + (-1)^m \int_{\Omega} u (\Delta^m v) dY, \end{aligned} \quad (2.392)$$

where $\nu = \nu_1 + i\nu_2$ is the complex outward unit normal vector to Ω . The boundary integral in (2.392) can be written as

$$4^m \sum_{k=0}^{m-1} \frac{(-1)^k}{2} \int_{\partial\Omega} \left(\bar{\partial}_z^k \partial_z^m v \right) \cdot \left(\bar{\partial}_z^{m-k-1} u \right) \nu d\sigma(Y) \quad (2.393)$$

$$= \sum_{k=0}^{m-1} 2^{m+k} (-1)^k \int_{\partial\Omega} \left(\bar{\partial}_z^k \partial_z^m v \right) \cdot (2\bar{\partial}_z)^{m-k-1} u \nu d\sigma(Y). \quad (2.394)$$

Substituting $2\bar{\partial}_z = \partial_x + i\partial_y$ into (2.394), we obtain that

$$\begin{aligned} & \sum_{k=0}^{m-1} 2^{m+k} (-1)^k \int_{\partial\Omega} \left(\bar{\partial}_z^k \partial_z^m v \right) \cdot (2\bar{\partial}_z)^{m-k-1} u \nu d\sigma(Y) \\ &= \sum_{k=0}^{m-1} 2^{m+k} (-1)^k \int_{\partial\Omega} \left(\bar{\partial}_z^k \partial_z^m v \right) \cdot (\partial_x + i\partial_y)^{m-k-1} u \nu d\sigma(Y) \\ &= \sum_{k=0}^{m-1} 2^{m+k} (-1)^k \int_{\partial\Omega} \left(\bar{\partial}_z^k \partial_z^m v \right) \cdot \sum_{\ell=0}^{m-k-1} i^\ell \frac{(m-k-1)!}{\ell!(m-k-1-\ell)!} \partial_x^{m-k-1-\ell} \partial_y^\ell u \nu d\sigma(Y) \\ &= \sum_{k=0}^{m-1} \sum_{\ell=0}^{m-k-1} 2^{m+k} (-1)^k i^\ell \frac{(m-k-1)!}{\ell!(m-k-1-\ell)!} \times \\ & \quad \times \int_{\partial\Omega} \left(\bar{\partial}_z^k \partial_z^m v \right) \cdot \partial_x^{m-k-1-\ell} \partial_y^\ell u \nu d\sigma(Y). \end{aligned} \quad (2.395)$$

Combining this with (2.392), we end up with

$$\begin{aligned} \tilde{\mathcal{B}}(u, v) &= (-1)^m \int_{\Omega} u (\Delta^m v) dY \\ & \quad + \sum_{k=0}^{m-1} \sum_{\ell=0}^{m-k-1} 2^{m+k} (-1)^k i^\ell \frac{(m-k-1)!}{\ell!(m-k-1-\ell)!} \times \\ & \quad \times \int_{\partial\Omega} \left(\bar{\partial}_z^k \partial_z^m v \right) \cdot \partial_x^{m-k-1-\ell} \partial_y^\ell u \nu d\sigma(Y). \end{aligned} \quad (2.396)$$

Substituting $\ell = s$, $m-k-1-\ell = r$ into (2.396), we obtain that

$$\begin{aligned} \tilde{\mathcal{B}}(u, v) &= (-1)^m \int_{\Omega} u (\Delta^m v) dY \\ & \quad + \sum_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} 2^{2m-1-(r+s)} (-1)^{m-1-(r+s)} i^s \frac{(r+s)!}{r!s!} \times \\ & \quad \times \int_{\partial\Omega} \left(\bar{\partial}_z^{m-1-(r+s)} \partial_z^m v \right) \cdot \partial_x^r \partial_y^s u \nu d\sigma(Y). \end{aligned} \quad (2.397)$$

According to [23], the double multi-layer associated with Δ^m and its coefficient tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ is obtained by the boundary integral in (2.396), in which, the function v is replaced by $\mathbb{E}_{\Delta^m}(X - \cdot)$, the family of derivatives $\{\partial_x^r \partial_y^s u\}_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}}$ by the family of Lebesgue based real Whitney arrays $\dot{f} = \{f_{(r,s)}\}_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, and then multiplying by $(-1)^{m-1}$. Consequently, the double multi-layer potential operator $\dot{\mathcal{D}}_m$ associated with Δ^m and its coefficient tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ is given by

$$\begin{aligned} \dot{\mathcal{D}}_m \dot{f}(X) = & - \sum_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} 2^{2m-1-(r+s)} i^s \frac{(r+s)!}{r!s!} \times \\ & \times \int_{\partial\Omega} (\bar{\partial}_z^{m-1-(r+s)} \partial_z^m \mathbb{E}_{\Delta^m})(X - Y) f_{(r,s)}(Y) \nu(Y) d\sigma(Y), \end{aligned} \quad (2.398)$$

where $X \in \Omega$, $\dot{f} = \{f_{(r,s)}\}_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, and $\nu = \nu_1 + i\nu_2$ is the complex outward unit normal vector to Ω .

Moving on, consider $\dot{g} = \{g_{(r,s)}\}_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ and recall $\Phi : \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \rightarrow \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ defined in (2.361). According to (2.398), we have for $z \in \Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$

$$\begin{aligned} \dot{\mathcal{D}}_m(\Phi(\dot{g}))(z) = & - \sum_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} 2^{2m-1-(r+s)} i^s \frac{(r+s)!}{r!s!} \times \\ & \times \int_{\partial\Omega} (\bar{\partial}_z^{m-1-(r+s)} \partial_z^m \mathbb{E}_{\Delta^m})(z - \zeta) \left(\Phi(\dot{g}) \right)_{(r,s)}(\zeta) \nu(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.399)$$

From (2.23), we substitute $4^m \partial_z^m \mathbb{E}_{\Delta^m} = E_m$ into (2.399) where E_m is the fundamental solution for $\bar{\partial}_z^m$ which forces

$$\begin{aligned} \dot{\mathcal{D}}_m(\Phi(\dot{g}))(z) = & - \sum_{\substack{r, s \in \mathbb{N}_0 \\ r+s \leq m-1}} 2^{-(r+s+1)} i^s \frac{(r+s)!}{r!s!} \times \\ & \times \int_{\partial\Omega} (\bar{\partial}_z^{m-1-(r+s)} E_m)(z - \zeta) \left(\Phi(\dot{g}) \right)_{(r,s)}(\zeta) \nu(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.400)$$

Similar to (2.397), substituting $k = m - 1 - (r + s)$, $\ell = s$ into (2.400), we

obtain that

$$\begin{aligned} \dot{\mathcal{D}}_m(\Phi(\dot{g}))(z) &= - \sum_{k=0}^{m-1} \sum_{\ell=0}^{m-1-k} 2^{-(m-k)} i^\ell \frac{(m-1-k)!}{(m-1-(k+\ell))! \ell!} \times \\ &\quad \times \int_{\partial\Omega} (\bar{\partial}_z^k E_m)(z-\zeta) \left(\Phi(\dot{g})\right)_{(m-1-(k+\ell), \ell)}(\zeta) \nu(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.401)$$

Going further, from (2.361), for $r, s \in \mathbb{N}_0$ with $r+s \leq m-1$

$$\left(\Phi(\dot{g})\right)_{(r,s)} := i^s \sum_{\substack{0 \leq a \leq r \\ 0 \leq b \leq s}} \frac{r!}{a!(r-a)!} \frac{s!}{b!(s-b)!} (-1)^{s-b} g_{(a+b, r+s-a-b)}. \quad (2.402)$$

Combining this with (2.401), we obtain

$$\begin{aligned} \dot{\mathcal{D}}_m(\Phi(\dot{g}))(z) &= - \sum_{k=0}^{m-1} \sum_{\ell=0}^{m-1-k} 2^{-(m-k)} i^\ell \frac{(m-1-k)!}{(m-1-(k+\ell))! \ell!} \times \\ &\quad \times \int_{\partial\Omega} \left\{ (\bar{\partial}_z^k E_m)(z-\zeta) i^\ell \sum_{\substack{0 \leq a \leq m-1-k-\ell \\ 0 \leq b \leq \ell}} \frac{(m-1-(k+\ell))!}{a!(m-1-(k+\ell+a))!} \times \right. \\ &\quad \left. \times \frac{\ell!}{b!(\ell-b)!} (-1)^{\ell-b} g_{(a+b, m-1-k-(a+b))}(\zeta) \nu(\zeta) \right\} d\sigma(\zeta), \end{aligned} \quad (2.403)$$

which turns out to be

$$\begin{aligned} \dot{\mathcal{D}}_m(\Phi(\dot{g}))(z) &= - \sum_{k=0}^{m-1} \sum_{\ell=0}^{m-1-k} 2^{-(m-k)} \int_{\partial\Omega} \left\{ (\bar{\partial}_z^k E_m)(z-\zeta) \times \right. \\ &\quad \times \sum_{\substack{0 \leq a \leq m-1-k-\ell \\ 0 \leq b \leq \ell}} (-1)^b \frac{(m-1-k)!}{a!(m-1-k-\ell-a)! b!(\ell-b)!} \times \\ &\quad \left. \times g_{(a+b, m-1-k-(a+b))}(\zeta) \nu(\zeta) \right\} d\sigma(\zeta). \\ &= - \sum_{k=0}^{m-1} \frac{1}{2} \int_{\partial\Omega} \left\{ (\bar{\partial}_z^k E_m)(z-\zeta) \sum_{c=0}^{m-1-k} g_{(c, m-1-k-c)}(\zeta) \nu(\zeta) \times \right. \\ &\quad \left. \times \frac{1}{2^{m-1-k}} \sum_{\ell=0}^{m-1-k} \sum_{\substack{0 \leq a \leq m-1-k-\ell \\ 0 \leq b \leq \ell \\ a+b=c}} (-1)^b \frac{(m-1-k)!}{a!(m-1-k-\ell-a)! b!(\ell-b)!} \right\} d\sigma(\zeta). \end{aligned} \quad (2.404)$$

Applying the combinatorial identity in Lemma 2.41, one can conclude

$$\frac{1}{2^{m-1-k}} \sum_{\ell=0}^{m-1-k} \sum_{\substack{0 \leq a \leq m-1-k-\ell \\ 0 \leq b \leq \ell \\ a+b=c}} (-1)^b \frac{(m-1-k)!}{a!(m-1-k-\ell-a)!b!(\ell-b)!} = \delta_{c0}. \quad (2.405)$$

Substituting this into (2.404), we have

$$\begin{aligned} & \dot{\mathcal{D}}_m(\Phi(\dot{g}))(z) \\ &= - \sum_{k=0}^{m-1} \frac{1}{2} \int_{\partial\Omega} (\bar{\partial}_z^k E_m)(z-\zeta) \sum_{c=0}^{m-1-k} g_{(c,m-1-k-c)}(\zeta) \delta_{c0} \nu(\zeta) d\sigma(\zeta) \\ &= - \sum_{k=0}^{m-1} \frac{1}{2} \int_{\partial\Omega} (\bar{\partial}_z^k E_m)(z-\zeta) g_{(0,m-1-k)}(\zeta) \nu(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.406)$$

Substituting $m-1-k=k$ into the summation in (2.406), we have

$$\begin{aligned} & \dot{\mathcal{D}}_m(\Phi(\dot{g}))(z) \\ &= - \sum_{k=0}^{m-1} \frac{1}{2} \int_{\partial\Omega} (\bar{\partial}_z^{m-1-k} E_m)(z-\zeta) g_{(0,k)}(\zeta) \nu(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.407)$$

According to (2.42), for $k \in \{0, \dots, m-1\}$

$$(\bar{\partial}_z^{m-1-k} E_m)(z-\zeta) = \frac{1}{\pi} \cdot \frac{\overline{(z-\zeta)^k}}{k!(z-\zeta)}. \quad (2.408)$$

Substituting this into (2.407) forces

$$\begin{aligned} & - \sum_{k=0}^{m-1} \frac{1}{2} \int_{\partial\Omega} (\bar{\partial}_z^{m-1-k} E_m)(z-\zeta) g_{(0,k)}(\zeta) \nu(\zeta) d\sigma(\zeta) \\ &= \sum_{k=0}^{m-1} \frac{1}{2\pi} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(0,k)}(\zeta) \nu(\zeta) d\sigma(\zeta) \\ &= \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(0,k)}(\zeta) i \nu(\zeta) d\sigma(\zeta) \\ &= \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(0,k)}(\zeta) d\zeta \\ &= (\mathcal{C}_{m-1} \dot{g})(z), \end{aligned} \quad (2.409)$$

for $z \in \Omega$. Combining this together with (2.407), we eventually end up with

$$\dot{\mathcal{D}}_m(\Phi(\dot{g}))(z) = \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right)(z), \quad (2.410)$$

for $\dot{g} = \left\{ g_{(r,s)} \right\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ and $z \in \Omega$. This finishes the proof of the theorem. \square

Remark. According to [30], by direct calculation based on the definition of the double layer potential operator, coefficient tensor associated with the second-order elliptic operators, a tensor $A' = (A'_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ |\alpha|=|\beta|=1}}$ given by

$$A' = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad (2.411)$$

is a coefficient tensor associated with Δ and the double layer potential operator associated with Δ and $A' = (A'_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ |\alpha|=|\beta|=1}}$ is identically equal to the classical Cauchy operator on $L^p(\partial\Omega, \sigma)$. Namely,

$$\mathcal{C}(g)(z) = \mathcal{D}_\Delta^{A'}(g)(z), \quad (2.412)$$

for $z \in \Omega$ and $g \in L^p(\partial\Omega, \sigma)$. Now, we verify the identity (2.412) using the theorem 2.47. Consider $m = 1$, then the complex Whitney array space and the real Whitney array space are identified as the Lebesgue space on the boundary, namely,

$$\text{CWA}_0[L^p(\partial\Omega, \sigma)] \equiv \text{RWA}_0[L^p(\partial\Omega, \sigma)] \equiv L^p(\partial\Omega, \sigma). \quad (2.413)$$

In addition, the isomorphisms Ψ, Φ defined in (2.341), (2.361) become identity maps on $L^p(\partial\Omega, \sigma)$. A coefficient tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ |\alpha|=|\beta|=1}}$ associated with Δ in lemma 2.46 turns out to be

$$\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ |\alpha|=|\beta|=1}} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad (2.414)$$

which implies that $\tilde{A} = A'$ where A' defined as in (2.411). Consequently, according to the theorem 2.47, one can conclude that for any $g \in L^p(\partial\Omega, \sigma)$

$$\mathcal{C}(g) \equiv \mathcal{D}_\Delta^{A'}(g) \text{ in } \Omega. \quad (2.415)$$

In conclusion, Theorem 2.47 is a higher-order generalization of the identification between the classical Cauchy operator and the double layer potential operator associated with a specific coefficient tensor and Δ .

2.9 The Distinguished Double Multi-layer associated with Δ^m

The theory of layer potential operator and its applications to partial differential equations has been an important area of research for many years. In recent years, there has been a growing interest in the extension of this theory to higher order partial differential equations, such as the polylaplacian operator. The layer potential theory for higher-order elliptic partial differential equations has been developed based on various function spaces appropriate for the higher-order setting (cf. [23]).

The layer potential operators associated with the higher-order elliptic operators are called the multi-layer potential operators and these operators are much more sophisticated than the classical layer potential operators associated with the second-order elliptic operators.

In second-order elliptic partial differential equations, the double layer potential having chord-dot-normal structure play an important role to solve the boundary value problems. The boundary-to-boundary double layer potential operators having the chord-dot-normal structure vanish whenever the underlying domain is a half space. Coefficient tensors leading to chord-dot-normal type double layers are called distinguished. One of the important consequence of the distinguished coefficient tensor is relevant to the Fredholm operator theory to treat boundary value problems through layer potential theory. Specifically, a boundary-to-boundary double layer potential operator associated with a coefficient tensor A is compact on the boundary of any smooth bounded domain if and only if the coefficient tensor is distinguished. The properties of the coefficient tensors and the distinguished coefficient tensors for second-order

elliptic operators has been developed with many key properties which are related to treatment of the boundary value problems (cf. [30]). In the context of higher-order operators, finding the distinguished double multi-layer potential operator associated with the polylaplacian is much more challenging.

In this section, we investigate the relationship between the distinguished double multi-layer potential operator associated with the polylaplacian in two dimensions and the poly-Cauchy operator. We use the method of layer potentials and poly-Cauchy operator to construct the distinguished double multi-layer potential operator and we study the boundary value problems associated with the polylaplacian.

Let us consider the domain $\Omega \subseteq \mathbb{R}^2$ be a UR domain with compact boundary, fix a natural power $m \in \mathbb{N}$, an integrability exponent $p \in (1, \infty)$, and a aperture parameter $\kappa \in (0, \infty)$. The Dirichlet problem for the polylaplacian Δ^m in $\Omega \subseteq \mathbb{R}^2$ with given data $\dot{f} = \{f_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ takes the form

$$\begin{cases} \Delta^m u = 0 & \text{in } \Omega \\ \mathcal{N}_\kappa(\nabla^{m-1} u) \in L^p(\partial\Omega, \sigma) \\ \text{Tr}_{m-1}^{\mathbb{R}^2}(u) = \dot{f} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \end{cases} \quad (2.416)$$

where the two dimensional higher-order boundary trace $\text{Tr}_{m-1}^{\mathbb{R}^2}$ is defined by

$$\text{Tr}_{m-1}^{\mathbb{R}^2}(u) = \left\{ \left(\partial_x^a \partial_y^b u \right) \Big|_{\partial\Omega} \right\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}}^{\kappa\text{-n.t.}}. \quad (2.417)$$

We recall the definition of double multi-layer potential operator. Let L be a homogeneous differential operator of order $2m$ with constant coefficients which is given by

$$L := \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta} \partial^\beta. \quad (2.418)$$

Here, $A = (A_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^n \\ |\alpha|=|\beta|=m}}$ is called a coefficient tensor associated with L . Note that there are infinitely many coefficient tensors associated with L .

According to [12] (cf. also [23]), for any coefficient tensor $A = (A_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ associated with Δ^m in $\Omega \subseteq \mathbb{R}^2$, the double multi-layer potential operator asso-

ciated with $\dot{\mathcal{D}}_{\Delta^m}^A \dot{g}$ acting on $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ satisfies the PDE condition and size condition in (2.416), namely,

$$\begin{aligned} \Delta^m(\dot{\mathcal{D}}_{\Delta^m}^A \dot{g}) &= 0 \quad \text{in } \Omega, \\ \mathcal{N}_\kappa(\nabla^{m-1} \dot{\mathcal{D}}_{\Delta^m}^A \dot{g}) &\in L^p(\partial\Omega, \sigma). \end{aligned} \quad (2.419)$$

In order to solve the Dirichlet problem in (2.416) using the layer potential theory, we are left with the boundary condition, that is,

$$\text{Tr}_{m-1}^{\mathbb{R}^2}(\dot{\mathcal{D}}_{\Delta^m}^A \dot{g}) = \dot{f}. \quad (2.420)$$

According to jump relation in [12], (2.420) is reduced to solving the following integral equation.

$$\left(\frac{1}{2}I + \dot{K}_A\right) \dot{g} = \dot{f} \quad \text{on } \partial\Omega, \quad (2.421)$$

where \dot{K}_A denotes the boundary-to-boundary double multi-layer potential operator. If we assume that \dot{K}_A is compact operator on $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, then this implies $\frac{1}{2}I + \dot{K}_A$ is Fredholm operator of index 0 on $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. This yields

$$\begin{aligned} &\frac{1}{2}I + \dot{K}_A \text{ is invertible on } \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \\ \iff &\frac{1}{2}I + \dot{K}_A \text{ is injective or surjective on } \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \end{aligned} \quad (2.422)$$

Now, it is natural to think of the largest class of the domain for which the operator $\frac{1}{2}I + \dot{K}_A$ is compact on $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Let us start with the basic property between the polyanalytic and the polyharmonic.

Proposition 2.48. *Let $\Omega \subseteq \mathbb{R}^2$ is an open set. Fix an integer $m \in \mathbb{N}$. Let u be a polyanalytic function of order m in Ω , that is, a function $u \in \mathcal{C}^\infty(\Omega)$ satisfying $\bar{\partial}^m u = 0$ in Ω . Then the real part and the imaginary part of u are polyharmonic functions of order m in Ω , i.e., a function $\text{Re } u, \text{Im } u \in \mathcal{C}^\infty(\Omega)$ satisfying $\Delta^m(\text{Re } u) = 0$ $\Delta^m(\text{Im } u) = 0$ in Ω .*

Proof. We proceed the proof by induction. Let $m = 1$ and $u = u_1 + iu_2$ where u_1, u_2 are real-valued

functions which are real part and imaginary part of u , respectively. Since $\bar{\partial}u = 0$ in Ω , one has

$$\partial_x u_1 = \partial_y u_2, \quad (2.423)$$

$$\partial_y u_1 = -\partial_x u_2. \quad (2.424)$$

Taking ∂_x into (2.423) and combining this with (2.424), one can conclude that

$$(\partial_x^2 + \partial_y^2)u_1 = 0 \quad \text{in } \Omega. \quad (2.425)$$

Similarly, taking ∂_y into (2.423) and combining this with (2.424), one has

$$(\partial_x^2 + \partial_y^2)u_2 = 0 \quad \text{in } \Omega. \quad (2.426)$$

This shows u_1, u_2 are harmonic in Ω . To carry out the induction step, assume that $m \geq 2$ and that all claims in the statement are valid for polyanalytic functions of order $m - 1$. Let us assume that $u = u_1 + iu_2 \in \mathcal{C}^\infty(\Omega)$ is a polyanalytic function of order m . We claim that the real part and the imaginary part of u are polyharmonic of order m . First of all, $\bar{\partial}^m u = 0$ in Ω implies $\bar{\partial}^{m-1}(\bar{\partial}u) = 0$ in Ω . This means $\bar{\partial}u$ is a polyanalytic of order $m - 1$. According to the inductive assumption, one has the real part and the imaginary part of $\bar{\partial}u$ are polyharmonic functions of order $m - 1$. Since $\bar{\partial}u = [(\partial_x u_1 - \partial_y u_2) + i(\partial_y u_1 + \partial_x u_2)]/2$, we obtain that

$$\Delta^{m-1} \partial_x u_1 = \Delta^{m-1} \partial_y u_2, \quad (2.427)$$

$$\Delta^{m-1} \partial_y u_1 = -\Delta^{m-1} \partial_x u_2, \quad (2.428)$$

in Ω . To prove the claim, we look at the polylaplacians of u_1 and u_2 which satisfy

$$\Delta^m u_i = \Delta^{m-1}(\Delta u_i) = \Delta^{m-1}(\partial_x^2 u_i + \partial_y^2 u_i), \quad (2.429)$$

for $i = 1, 2$. In particular, for $i = 1, 2$

$$\Delta^m u_i = \partial_x \Delta^{m-1} \partial_x u_i + \partial_y \Delta^{m-1} \partial_y u_i. \quad (2.430)$$

Substituting (2.427) and (2.428) into (2.430) yields

$$\Delta^m u_i = 0 \quad \text{in } \Omega \quad \text{for } i = 1, 2. \quad (2.431)$$

This proves the claim and therefore this finishes the proof. \square

Now, we first look at the classical Cauchy operator and the classical harmonic double layer potential operator. Let $\Omega \subseteq \mathbb{R}^2$ be a UR domain and $p \in (1, \infty)$. Then the boundary-to-domain Cauchy operator on any function $f \in L^p(\partial\Omega, \sigma)$ at any point $z \in \Omega$ is defined as

$$(\mathcal{C}f)(z) := \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \nu(\zeta) d\sigma(\zeta), \quad (2.432)$$

where $\nu = \nu_1 + i\nu_2$ denotes the complex outward unit normal vector to Ω . The classical harmonic double layer potential operator on $f \in L^p(\partial\Omega, \sigma)$ at any point $x \in \Omega$ is defined by

$$(\mathcal{D}f)(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle \nu(y), y - x \rangle}{|x - y|^2} f(y) d\sigma(y), \quad (2.433)$$

where $\nu = (\nu_1, \nu_2)$ denotes the two dimensional outward unit normal vector to Ω . There is a well-known relationship between the Cauchy operator and the harmonic double layer potential operator that the real part of the Cauchy operator turns out to be precisely the harmonic double layer potential. Indeed,

$$\begin{aligned} \operatorname{Re}(\mathcal{C}f)(z) &= \operatorname{Re} \frac{1}{2\pi} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \nu(\zeta) d\sigma(\zeta) \\ &= \frac{1}{2\pi} \operatorname{Re} \int_{\partial\Omega} \frac{(\zeta - z)\overline{\nu(\zeta)}}{|\zeta - z|^2} f(\zeta) d\sigma(\zeta) \\ &= \frac{1}{2\pi} \int_{\partial\Omega} \frac{\operatorname{Re}[\nu(\zeta)\overline{(\zeta - z)}]}{|\zeta - z|^2} f(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.434)$$

Note that for $z_1 = a + ib$, $z_2 = c + id$, $\operatorname{Re}(z_1 \cdot \overline{z_2}) = \langle (a, b), (c, d) \rangle$. This yields,

$$\operatorname{Re}(\mathcal{C}f)(z) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle \nu(\zeta), \zeta - z \rangle}{|z - \zeta|^2} f(\zeta) d\sigma(\zeta). \quad (2.435)$$

Since \mathbb{C} can be identified as \mathbb{R}^2 , we replace z by $x \in \mathbb{R}^2$ and ζ by $y \in \mathbb{R}^2$ which forces for $x \in \Omega$

$$\operatorname{Re}(\mathcal{C}f)(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle \nu(y), y - x \rangle}{|x - y|^2} f(y) d\sigma(y), \quad (2.436)$$

which is precisely the harmonic double layer potential operator. We can understand this property in the sense of the coefficient tensor in two dimensions. According to [30], for fixed positive integer M , a homogeneous, second-order, constant complex-coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^n has infinitely many coefficient tensors which yield infinitely many double layer potential operators. In particular, if we consider a $M \times M$ system written as for $1 \leq r, s \leq n$

$$Lu := \left(\partial_r (a_{rs}^{\alpha\beta} \partial_s u_\beta) \right)_{1 \leq \alpha \leq M} \quad (2.437)$$

acting on a \mathcal{C}^2 vector valued function $u = (u_\beta)_{1 \leq \beta \leq M}$. We assume that L is a homogeneous, second-order, constant complex-coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^n , in the sense that

$$\begin{aligned} \det[A_{\xi\xi}] &= \det \left[(a_{rs}^{\alpha\beta} \xi_r \xi_s)_{1 \leq \alpha, \beta \leq M} \right] \neq 0, \\ \text{for each vector } \xi &= (\xi_r)_{1 \leq r \leq n} \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (2.438)$$

Then, the double layer potential operator \mathcal{D}_A is given by

$$(\mathcal{D}_A g)(x) := \int_{\partial\Omega} \left(\partial_\nu^{A^T} (E_{L^T}(x - \cdot)) \right)^T (y) g(y) d\sigma(y), \quad x \in \Omega \quad (2.439)$$

where $g = (g_1, g_2, \dots, g_M) : \partial\Omega \rightarrow \mathbb{R}^M$, E_L is the fundamental solution with respect to the differential operator L , and ∂_ν^A denotes the conormal derivative associated with the coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq n}}$ which is defined as

$$(\partial_\nu^A u)_\alpha(x) := \nu_r a_{rs}^{\alpha\beta} \partial_s u_\beta \Big|_{\partial\Omega}^{n.t.}(x). \quad (2.440)$$

In particular, for the scalar Laplacian Δ in two dimensions, it can be shown that

$$\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=1}} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad (2.441)$$

is a coefficient tensor associated with Δ in \mathbb{R}^2 . In addition, the double layer potential operator associated with this specific coefficient tensor \tilde{A} becomes the classical Cauchy operator. As we observed in this section earlier, the real part of the classical Cauchy operator is the classical harmonic double layer

potential which is a distinguished double layer potential operator associated with Δ . In other words,

$$\operatorname{Re} \mathcal{D}_{\tilde{A}} \text{ is a distinguished double layer associated with } \Delta \text{ in } \mathbb{R}^2, \quad (2.442)$$

where \tilde{A} is a coefficient tensor associated with Δ such that $\mathcal{D}_{\tilde{A}}$ becomes the Cauchy operator. In particular, if we look at the definition of $\mathcal{D}_{\tilde{A}}$, everything in the kernel are real valued except the entries of the coefficient tensor. This implies that $\operatorname{Re} \mathcal{D}_{\tilde{A}} = \mathcal{D}_{\operatorname{Re} \tilde{A}}$. In fact, $\operatorname{Re} \tilde{A}$ is also a coefficient tensor associated with Δ because for the coefficient tensor $\tilde{A} = \left(\tilde{A}_{\alpha\beta} \right)_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=1}}$

$$\partial_\alpha \left(\tilde{A}_{\alpha\beta} \partial_\beta \right) = \Delta, \quad (2.443)$$

this yields

$$\operatorname{Re} \left[\partial_\alpha \left(\tilde{A}_{\alpha\beta} \partial_\beta \right) \right] = \Delta, \quad (2.444)$$

which further implies

$$\partial_\alpha \left(\operatorname{Re} \left[\tilde{A}_{\alpha\beta} \right] \partial_\beta \right) = \Delta. \quad (2.445)$$

Here, our strategy to tackle the polylaplacian is to use the real part of the coefficient tensor associated with Δ^m such that the double multi-layer potential $\dot{\mathcal{D}}_m = \dot{\mathcal{D}}_{\Delta^m}^{\tilde{A}}$ associated with the coefficient tensor $\tilde{A} = \left(\tilde{A}_{\alpha\beta} \right)_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ is identified as the poly-Cauchy operator of order m . In Theorem 2.47, we were able to identify the poly-Cauchy operator of order m as the double multi-layer associated with Δ^m and the coefficient tensor $\tilde{A} = \left(\tilde{A}_{\alpha\beta} \right)_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ whose entries are given by

$$\tilde{A}_{(m-s,s),(m-q,q)} = \frac{(m!)^2}{(m-s)!(m-q)!s!q!} (-1)^q i^{s+q}, \quad \text{for } 0 \leq s, q \leq m. \quad (2.446)$$

We claim that the double multi-layer potential associated with the real part of this coefficient tensor becomes the distinguished double multi-layer potential operator whose kernel is either weakly singular or having chord-dot-normal structure. In other words, $\operatorname{Re} \tilde{A} = \left(\operatorname{Re} \tilde{A}_{\alpha\beta} \right)_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ is the distinguished coefficient tensor associated with Δ^m . Let $p \in (1, \infty)$. According to (2.400), for

each real Whitney array $\dot{f} = \{f_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$,

$$\begin{aligned} \dot{\mathcal{D}}_m(\dot{f})(z) = & - \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} 2^{-(r+s+1)} i^s \frac{(r+s)!}{r!s!} \times \\ & \times \int_{\partial\Omega} (\bar{\partial}_z^{m-1-(r+s)} E_m)(z-\zeta) f_{(r,s)}(\zeta) \nu(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.447)$$

where $\dot{\mathcal{D}}_m$ denotes the double multi-layer potential operator associated with the coefficient tensor $\tilde{A} = (\tilde{A}_{\alpha\beta})_{\substack{\alpha,\beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ given in (2.446) and E_m is the fundamental solution for $\bar{\partial}_z^m$. According to (2.26) with direct calculations, one has for $r, s \in \mathbb{N}_0$ with $r+s \leq m-1$,

$$(\bar{\partial}_z^{m-1-(r+s)} E_m)(z-\zeta) = \frac{1}{\pi(r+s)!} \frac{\overline{(z-\zeta)^{r+s}}}{z-\zeta}. \quad (2.448)$$

Substituting this into (2.447) yields

$$\begin{aligned} \dot{\mathcal{D}}_m(\dot{f})(z) = & - \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \frac{1}{2^{(r+s+1)\pi}} i^s \frac{1}{r!s!} \times \\ & \times \int_{\partial\Omega} \frac{\overline{(z-\zeta)^{r+s}}}{z-\zeta} f_{(r,s)}(\zeta) \nu(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.449)$$

Moreover, as we verified earlier in this section, the double multi-layer associated with $\text{Re } \tilde{A}$ acting on \dot{f} is going to be $\text{Re } \dot{\mathcal{D}}_m(\dot{f})(z)$. In particular,

$$\begin{aligned} \text{Re } \dot{\mathcal{D}}_m(\dot{f})(z) = & - \text{Re} \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \frac{1}{2^{(r+s+1)\pi}} i^s \frac{1}{r!s!} \times \\ & \times \int_{\partial\Omega} \frac{\overline{(z-\zeta)^{r+s}}}{z-\zeta} f_{(r,s)}(\zeta) \nu(\zeta) d\sigma(\zeta) \\ = & - \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \frac{1}{2^{(r+s+1)\pi} r!s!} \times \\ & \times \int_{\partial\Omega} \frac{\text{Re} \left[i^s \overline{(z-\zeta)^{r+s+1}} \nu(\zeta) \right]}{|\zeta-z|^2} f_{(r,s)}(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.450)$$

This forces

$$\begin{aligned}
& \operatorname{Re} \dot{\mathcal{D}}_m(\dot{f})(z) \\
&= - \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+2s \leq m-1}} \frac{1}{2^{(r+2s+1)} \pi r! (2s)!} \int_{\partial\Omega} \frac{\operatorname{Re} \left[i^{2s} \overline{(z-\zeta)^{r+2s+1}} \nu(\zeta) \right]}{|\zeta-z|^2} f_{(r,2s)}(\zeta) d\sigma(\zeta) \\
&\quad - \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+(2s+1) \leq m-1}} \frac{1}{2^{(r+2s+2)} \pi r! (2s+1)!} \times \\
&\quad \times \int_{\partial\Omega} \frac{\operatorname{Re} \left[i^{2s+1} \overline{(z-\zeta)^{r+2s+2}} \nu(\zeta) \right]}{|\zeta-z|^2} f_{(r,2s+1)}(\zeta) d\sigma(\zeta). \tag{2.451}
\end{aligned}$$

To simplify this expression we use the following basic algebraic identities. For any $j \in \mathbb{N}$ and $z = a + ib \in \Omega$,

$$\begin{aligned}
& \operatorname{Re} [\nu \cdot \overline{z^j}] = \operatorname{Re} [\overline{\nu} \cdot z^j] \\
&= \operatorname{Re} \left[\overline{\nu} \cdot \sum_{0 \leq k \leq j} \frac{j!}{(j-k)! k!} a^{j-k} i^k b^k \right] \\
&= \operatorname{Re} \left[\overline{\nu} \cdot \left(\sum_{\substack{0 \leq k \leq j \\ k=2\ell}} (-1)^\ell \frac{j!}{(j-2\ell)! (2\ell)!} a^{j-2\ell} b^{2\ell} \right. \right. \\
&\quad \left. \left. + i \sum_{\substack{0 \leq k \leq j \\ k=2\ell+1}} (-1)^\ell \frac{j!}{(j-(2\ell+1)! (2\ell+1)!} a^{j-(2\ell+1)} b^{2\ell+1} \right) \right] \\
&= \nu_1 \cdot \sum_{\substack{0 \leq k \leq j \\ k=2\ell}} (-1)^\ell \frac{j!}{(j-2\ell)! (2\ell)!} a^{j-2\ell} b^{2\ell} \\
&\quad + \nu_2 \cdot \sum_{\substack{0 \leq k \leq j \\ k=2\ell+1}} (-1)^\ell \frac{j!}{(j-(2\ell+1)! (2\ell+1)!} a^{j-(2\ell+1)} b^{2\ell+1}. \tag{2.452}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\operatorname{Re} [i\nu \cdot \overline{z^j}] &= -\operatorname{Re} [i\overline{\nu} \cdot z^j] \\
&= -\operatorname{Re} \left[i\overline{\nu} \cdot \left(\sum_{\substack{0 \leq k \leq j \\ k=2\ell}} (-1)^\ell \frac{j!}{(j-2\ell)!(2\ell)!} a^{j-2\ell} b^{2\ell} \right. \right. \\
&\quad \left. \left. + i \sum_{\substack{0 \leq k \leq j \\ k=2\ell+1}} (-1)^\ell \frac{j!}{(j-(2\ell+1)!(2\ell+1)!} a^{j-(2\ell+1)} b^{2\ell+1} \right) \right] \\
&= \nu_1 \cdot \sum_{\substack{0 \leq k \leq j \\ k=2\ell+1}} (-1)^\ell \frac{j!}{(j-(2\ell+1)!(2\ell+1)!} a^{j-(2\ell+1)} b^{2\ell+1} \\
&\quad - \nu_2 \cdot \sum_{\substack{0 \leq k \leq j \\ k=2\ell}} (-1)^\ell \frac{j!}{(j-2\ell)!(2\ell)!} a^{j-2\ell} b^{2\ell}. \tag{2.453}
\end{aligned}$$

For more simplicity, denote for $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned}
\mathfrak{C}(x^j) &:= \left(\sum_{\substack{0 \leq k \leq j \\ k=2\ell}} (-1)^\ell \frac{j!}{(j-2\ell)!(2\ell)!} x_1^{j-2\ell} x_2^{2\ell} \right. \\
&\quad \left. , \sum_{\substack{0 \leq k \leq j \\ k=2\ell+1}} (-1)^\ell \frac{j!}{(j-(2\ell+1)!(2\ell+1)!} x_1^{j-(2\ell+1)} x_2^{2\ell+1} \right). \tag{2.454}
\end{aligned}$$

Combining this with (2.452) and (2.453), for any complex numbers $z = a + ib$ and $\zeta = c + id$, the complex outward unit normal vector $\nu(\zeta) = \nu_1(\zeta) + i\nu_2(\zeta)$, and positive integer $j \in \mathbb{N}$, one has

$$\begin{aligned}
\operatorname{Re} [\nu(\zeta) \cdot \overline{(z - \zeta)^j}] &= \langle \nu(y), \mathfrak{C}((x - y)^j) \rangle, \\
\operatorname{Re} [i\nu \cdot \overline{(z - \zeta)^j}] &= \langle \tau(y), \mathfrak{C}((x - y)^j) \rangle, \tag{2.455}
\end{aligned}$$

where $x = (a, b)$, $y = (c, d)$, $\nu(y) = (\nu_1(y), \nu_2(y))$ is the two dimensional outward unit normal vector, and $\tau(y) = (-\nu_2(y), \nu_1(y))$ is the two dimensional unit tangent vector. Applying the identities in (2.452)-(2.454) into (2.451), replacing $z \in \Omega$ by $x \in \Omega$ and $\zeta \in \partial\Omega$ by $y \in \partial\Omega$ due to the identification

yields

$$\begin{aligned}
& \operatorname{Re} \dot{\mathcal{D}}_m(\dot{f})(x) \\
&= \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+2s \leq m-1}} \frac{(-1)^{s+1}}{2^{(r+2s+1)} \pi r! (2s)!} \int_{\partial\Omega} \frac{\langle \nu(y), \mathfrak{C}((x-y)^{r+2s+1}) \rangle}{|x-y|^2} f_{(r,2s)}(y) d\sigma(y) \\
&+ \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+(2s+1) \leq m-1}} \frac{(-1)^{s+1}}{2^{(r+2s+2)} \pi r! (2s+1)!} \times \\
&\times \int_{\partial\Omega} \frac{\langle \tau(y), \mathfrak{C}((x-y)^{r+2s+2}) \rangle}{|x-y|^2} f_{(r,2s+1)}(y) d\sigma(y). \tag{2.456}
\end{aligned}$$

Observe that the case when $r = s = 0$ for the first term, the integral expression of the first term in (2.456) turns out to be the classical harmonic double layer potential acting on $f_{(0,0)}$ given by

$$\frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle \nu(y), y-x \rangle}{|x-y|^2} f_{(0,0)}(y) d\sigma(y). \tag{2.457}$$

To verify that $\operatorname{Re} \tilde{A}$ is a distinguished coefficient tensor associated with Δ^m , we turn our attention to the boundary-to-boundary double multi-layer potential operator associated with the coefficient tensor $\operatorname{Re} \tilde{A}$. For simplicity, let us denote the coefficient tensor $\operatorname{Re} \tilde{A}$ by $\mathcal{A} = (\mathcal{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ given by for $0 \leq s, q \leq m$

$$\mathcal{A}_{(m-s,s),(m-q,q)} = \begin{cases} \frac{(m!)^2}{(m-s)!(m-q)!s!q!} (-1)^{\frac{s+3q}{2}} & \text{if } s+q \text{ is even} \\ 0 & \text{if } s+q \text{ is odd} \end{cases} \tag{2.458}$$

To treat the boundary value problem, we turn our attention to the boundary-to-boundary double multi-layer potential operator whose kernel has the chord-dot-normal structure. Let us denote the double multi-layer potential operator associated with \mathcal{A} by $\dot{\mathcal{D}}_{\mathcal{A},m}$ which is given by for each Lebesgue based real

Whitney array $\dot{f} = \{f_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$,

$$\begin{aligned}
& \dot{\mathcal{D}}_{\mathcal{A},m}(\dot{f})(x) \\
&= \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+2s \leq m-1}} \frac{(-1)^{s+1}}{2^{(r+2s+1)} \pi r! (2s)!} \int_{\partial\Omega} \frac{\langle \nu(y), \mathfrak{C}((x-y)^{r+2s+1}) \rangle}{|x-y|^2} f_{(r,2s)}(y) d\sigma(y) \\
&+ \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+(2s+1) \leq m-1}} \frac{(-1)^{s+1}}{2^{(r+2s+2)} \pi r! (2s+1)!} \times \\
&\times \int_{\partial\Omega} \frac{\langle \tau(y), \mathfrak{C}((x-y)^{r+2s+2}) \rangle}{|x-y|^2} f_{(r,2s+1)}(y) d\sigma(y). \tag{2.459}
\end{aligned}$$

for $x \in \Omega$. Recall that for $\dot{f} = \{f_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$,

$$\dot{\mathcal{D}}_{\mathcal{A},m}(\dot{f}) = \text{Re} [\dot{\mathcal{C}}_{m-1}(\Psi(\dot{f}))], \tag{2.460}$$

where Ψ is defined in (2.341). According to Proposition 2.44, Φ and Ψ are inverse to one another, then there exists $\dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ such that $\Psi(\dot{f}) = \dot{g}$ and $\dot{f} = \Phi(\dot{g})$. Substituting this into (2.460) yields

$$\dot{\mathcal{D}}_{\mathcal{A},m}(\Phi(\dot{g})) = \text{Re} [\dot{\mathcal{C}}_{m-1}(\dot{g})]. \tag{2.461}$$

The real part can be expressed as a half of the summation with its conjugate expression which forces

$$\dot{\mathcal{D}}_{\mathcal{A},m}(\Phi(\dot{g})) = \frac{1}{2} (\dot{\mathcal{C}}_{m-1} + \overline{\dot{\mathcal{C}}_{m-1}}) \dot{g}. \tag{2.462}$$

Then for $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$,

$$\partial_x^a \partial_y^b (\dot{\mathcal{D}}_{\mathcal{A},m}(\Phi(\dot{g}))) = \frac{1}{2} [\partial_x^a \partial_y^b ((\dot{\mathcal{C}}_{m-1} + \overline{\dot{\mathcal{C}}_{m-1}}) \dot{g})]. \tag{2.463}$$

Using the identities in (2.11), the right-hand side in (2.463) is reduced to

$$\frac{1}{2} \left[i^b \sum_{\substack{0 \leq j \leq a \\ 0 \leq k \leq b}} \frac{(-1)^{b-k} a! b!}{(a-j)! j! (b-k)! k!} \partial_z^{j+k} \overline{\partial_z^{a+b-(j+k)}} ((\dot{\mathcal{C}}_{m-1} + \overline{\dot{\mathcal{C}}_{m-1}}) \dot{g}) \right]. \tag{2.464}$$

Moreover, using the jump relation in Theorem 2.307, one can conclude that

$$\begin{aligned} & \left. \partial_z^{j+k} \overline{\partial_z^{a+b-(j+k)}} (\dot{\mathcal{C}}_{m-1}(\dot{g})) \right|_{\partial\Omega}^{\kappa\text{-n.t.}} \\ &= \frac{1}{2} g_{(j+k, a+b-(j+k))} + \left(\dot{\mathcal{C}}_{m-1} \dot{g} \right)_{(j+k, a+b-(j+k))}. \end{aligned} \quad (2.465)$$

Combining this with (2.463) and (2.464) implies that

$$\begin{aligned} & \left. \partial_x^a \partial_y^b \left(\dot{\mathcal{D}}_{\mathcal{A}, m}(\Phi(\dot{g})) \right) \right|_{\partial\Omega}^{\kappa\text{-n.t.}} \\ &= \frac{1}{2} \left[i^b \sum_{\substack{0 \leq j \leq a \\ 0 \leq k \leq b}} \frac{(-1)^{b-k} a! b!}{(a-j)! j! (b-k)! k!} \left[g_{(j+k, a+b-(j+k))} \right. \right. \\ & \quad \left. \left. + \left((\dot{\mathcal{C}}_{m-1} + \overline{\dot{\mathcal{C}}_{m-1}}) \dot{g} \right)_{(j+k, a+b-(j+k))} \right] \right] \end{aligned} \quad (2.466)$$

$$\begin{aligned} &= \frac{1}{2} i^b \sum_{\substack{0 \leq j \leq a \\ 0 \leq k \leq b}} \frac{(-1)^{b-k} a! b!}{(a-j)! j! (b-k)! k!} g_{(j+k, a+b-(j+k))} \\ & \quad + \frac{1}{2} i^b \sum_{\substack{0 \leq j \leq a \\ 0 \leq k \leq b}} \frac{(-1)^{b-k} a! b!}{(a-j)! j! (b-k)! k!} \left((\dot{\mathcal{C}}_{m-1} + \overline{\dot{\mathcal{C}}_{m-1}}) \dot{g} \right)_{(j+k, a+b-(j+k))}. \end{aligned} \quad (2.467)$$

According to the definition of Φ in (2.361), one has

$$\begin{aligned} & \left. \partial_x^a \partial_y^b \left(\dot{\mathcal{D}}_{\mathcal{A}, m}(\Phi(\dot{g})) \right) \right|_{\partial\Omega}^{\kappa\text{-n.t.}} \\ &= \frac{1}{2} \left(\Phi(\dot{g}) \right)_{(a,b)} + \frac{1}{2} \left(\Phi \left((\dot{\mathcal{C}}_{m-1} + \overline{\dot{\mathcal{C}}_{m-1}}) \dot{g} \right) \right)_{(a,b)}. \end{aligned} \quad (2.468)$$

In other words,

$$\begin{aligned} & \left. \partial_x^a \partial_y^b \left(\dot{\mathcal{D}}_{\mathcal{A}, m}(\dot{f}) \right) \right|_{\partial\Omega}^{\kappa\text{-n.t.}} \\ &= \frac{1}{2} f_{(a,b)} + \frac{1}{2} \left(\Phi \left((\dot{\mathcal{C}}_{m-1} + \overline{\dot{\mathcal{C}}_{m-1}}) \Psi(\dot{f}) \right) \right)_{(a,b)}. \end{aligned} \quad (2.469)$$

Combining this with the jump relation of the double multi-layer potential operator in [23] (cf. also [24]), for $\dot{f} = \{f_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$

$$\left(\dot{K}_{\mathcal{A}, m}(\dot{f}) \right)_{(a,b)} = \frac{1}{2} \left(\Phi \left((\dot{\mathcal{C}}_{m-1} + \overline{\dot{\mathcal{C}}_{m-1}}) \Psi(\dot{f}) \right) \right)_{(a,b)}, \quad (2.470)$$

where $\dot{K}_{\mathcal{A},m}$ is the boundary-to-boundary double multi-layer potential operator associated with the coefficient tensor \mathcal{A} . From the definition of the boundary-to-boundary poly-Cauchy operator in (2.280), for each $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$,

$$\begin{aligned} \left(\dot{C}_{m-1}\dot{g}\right)_{(a,b)}(z) &= \sum_{k=0}^{m-1-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(a,k+b)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{(z-\zeta)^{m-a-b+j}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j,m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.471)$$

and

$$\begin{aligned} \left(\overline{\dot{C}_{m-1}\dot{g}}\right)_{(a,b)}(z) &= \sum_{k=0}^{m-1-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{(z-\zeta)^k}{k!(\bar{\zeta}-\bar{z})} g_{(a,k+b)}(\zeta) d\bar{\zeta} \\ &\quad + \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{(z-\zeta)^{m-a-b+j}}{(\bar{\zeta}-\bar{z})^{j+1}} \right] g_{(a-1-j,m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.472)$$

at σ -a.e. point $z \in \partial\Omega$. For simplicity, let

$$\dot{C}_{m-1} := \dot{C}_{m-1}^1 + \dot{C}_{m-1}^2, \quad (2.473)$$

where for $\dot{g} \in \text{CWA}_{m-1} [L^p(\partial\Omega, \sigma)]$ and $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$

$$\left(\dot{C}_{m-1}^1 \dot{g} \right)_{(a,b)}(z) := \sum_{k=0}^{m-1-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{(z-\zeta)^k}{k!(\zeta-z)} g_{(a,k+b)}(\zeta) d\zeta \quad (2.474)$$

$$\begin{aligned} \left(\dot{C}_{m-1}^2 \dot{g} \right)_{(a,b)}(z) := & - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\ & \times \partial_{\tau(\zeta)} \left[\frac{(z-\zeta)^{m-a-b+j}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.475)$$

To investigate the compactness of $\dot{C}_{m-1} + \overline{\dot{C}_{m-1}}$ on $\text{CWA}_{m-1} [L^p(\partial\Omega, \sigma)]$, we first look at $\dot{C}_{m-1}^1 + \overline{\dot{C}_{m-1}^1}$. Observe that for $k = 0$, we get

$$(C + \overline{C}) g_{(a,b)}. \quad (2.476)$$

Based on the calculation, one has $\overline{C} = -C^*$ where C^* is canonical adjoint operator of C . According to [[11], Theorem 4.6.8],

$$\begin{aligned} (C - C^*) \text{ is compact on } L^p(\partial\Omega, \sigma) \text{ for some } p \in (1, \infty) \\ \iff \Omega \text{ is a regular SKT domain.} \end{aligned} \quad (2.477)$$

This implies that $C + \overline{C}$ is compact for any $p \in (1, \infty)$ if and only if $\Omega \subseteq \mathbb{R}^2$ is a regular SKT domain from (2.477). If $0 < k \leq m - 1 - a - b$, one can observe that the integral expressions in (2.474) for \dot{C}_{m-1} and $\overline{\dot{C}_{m-1}}$ are weakly singular for any $p \in (1, \infty)$ which is compact provided that $\Omega \subseteq \mathbb{R}^2$ is a UR domain with compact boundary. Next, we turn our attention to \dot{C}_{m-1}^2 and $\overline{\dot{C}_{m-1}^2}$. According to (2.475), we have for $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$

$$\begin{aligned} & \left(\left(\dot{C}_{m-1}^2 \dot{g} \right)_{(a,b)} \right) \\ = & - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-a-b+j)!} \times \\ & \times \partial_{\tau(\zeta)} \left[\frac{(z-\zeta)^{m-a-b+j}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.478)$$

For $0 \leq j \leq a - 1$,

$$\begin{aligned} & \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}} \right] \\ &= i \left(\nu \cdot \partial_{\zeta} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}} \right] - \bar{\nu} \cdot \bar{\partial}_{\zeta} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}} \right] \right). \end{aligned} \quad (2.479)$$

In particular,

$$\begin{aligned} \partial_{\zeta} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}} \right] &= - \frac{(z - \zeta)^{m-a-b+j} (j+1) (\zeta - z)^j}{(\zeta - z)^{2(j+1)}} \\ &= (-1)^{j+1} \frac{\overline{(z - \zeta)^{m-a-b+j}} (j+1) (z - \zeta)^j}{(\zeta - z)^{2(j+1)}}. \end{aligned} \quad (2.480)$$

Similarly,

$$\begin{aligned} \bar{\partial}_{\zeta} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}} \right] &= - \frac{(m-a-b+j) (z - \zeta)^{m-a-b+j-1} (\zeta - z)^{j+1}}{(\zeta - z)^{2(j+1)}} \\ &= (-1)^j \frac{(m-a-b+j) \overline{(z - \zeta)^{m-a-b+j-1}} (\zeta - z)^{j+1}}{(\zeta - z)^{2(j+1)}}. \end{aligned} \quad (2.481)$$

Substituting (2.480) and (2.481) into (2.479) yields

$$\begin{aligned} & \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}} \right] \\ &= i (-1)^{j+1} (z - \zeta)^j \overline{(z - \zeta)^j} \times \\ & \quad \times \frac{\overline{(z - \zeta)^{m-a-b-1}}}{(\zeta - z)^{2(j+1)}} \cdot \left((j+1) \nu \cdot \overline{(z - \zeta)} + (m-a-b+j) \bar{\nu} \cdot (z - \zeta) \right). \end{aligned} \quad (2.482)$$

Combining this with (2.478), one can conclude that for $0 \leq a + b < m - 1$, $\left(\left(\dot{C}_{m-1}^2 \right) \dot{g} \right)_{(a,b)}$ is weakly singular integral on $\partial\Omega$. This yields for $a, b \in \mathbb{N}_0$ with $0 \leq a + b < m - 1$, \dot{C}_{m-1}^2 is compact operator on $\text{CWA}_{m-1}[L^p(\partial\Omega)]$ for all $p \in (1, \infty)$ provided that $\Omega \subseteq \mathbb{R}^2$ is a UR domain with compact boundary.

Next we consider the case when $a + b = m - 1$. If $a + b = m - 1$, according to (2.482), we obtain that

$$\begin{aligned} & \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}} \right] \\ &= i(-1)^{j+1} (z - \zeta)^j \overline{(z - \zeta)^j} \times \\ & \times \frac{1}{(\zeta - z)^{2(j+1)}} \cdot \left((j+1)\nu \cdot \overline{(z - \zeta)} + (j+1)\bar{\nu} \cdot (z - \zeta) \right). \end{aligned} \quad (2.483)$$

Going further,

$$\begin{aligned} & \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\zeta - z)^{j+1}} \right] \\ &= i(j+1)(-1)^{j+1} (z - \zeta)^j \overline{(z - \zeta)^j} \times \\ & \times \frac{1}{(\zeta - z)^{2(j+1)}} \cdot \left(\nu \cdot \overline{(z - \zeta)} + \bar{\nu} \cdot (z - \zeta) \right) \\ &= 2i(j+1)(-1)^{j+1} (z - \zeta)^j \overline{(z - \zeta)^j} \times \\ & \times \frac{1}{(\zeta - z)^{2(j+1)}} \cdot \langle z - \zeta, \nu \rangle. \end{aligned} \quad (2.484)$$

Substituting this into (2.478), one has for $a + b = m - 1$ and for $z \in \partial\Omega$

$$\begin{aligned} & \left(\left(\dot{C}_{m-1}^2 \right) \dot{g} \right)_{(a,b)}(z) \\ &= \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{(-1)^j}{\pi} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{|z - \zeta|^{2j}}{(\zeta - z)^{2(j+1)}} \cdot \langle z - \zeta, \nu \rangle g_{(a-1-j, b+1+j)}(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.485)$$

Moreover, according to (2.472) and (2.475), we have for any $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$

$$\begin{aligned} \left(\overline{\dot{C}_{m-1}^2 \dot{g}} \right)_{(a,b)}(z) &= \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{j!}{(m - a - b + j)!} \times \\ & \times \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta), \end{aligned} \quad (2.486)$$

at σ -a.e. point $z \in \partial\Omega$. For $0 \leq j \leq a - 1$,

$$\begin{aligned} \partial_\zeta \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] &= - \frac{(m-a-b+j)(z - \zeta)^{m-a-b+j-1} \overline{(\zeta - z)^{j+1}}}{(\bar{\zeta} - \bar{z})^{2(j+1)}} \\ &= (-1)^j \frac{(m-a-b+j)(z - \zeta)^{m-a-b+j-1} \overline{(z - \zeta)^{j+1}}}{(\bar{\zeta} - \bar{z})^{2(j+1)}}, \end{aligned} \quad (2.487)$$

and

$$\begin{aligned} \bar{\partial}_\zeta \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] &= - \frac{(z - \zeta)^{m-a-b+j} (j+1) \overline{(\zeta - z)^j}}{(\bar{\zeta} - \bar{z})^{2(j+1)}} \\ &= (-1)^{j+1} \frac{(z - \zeta)^{m-a-b+j} (j+1) \overline{(z - \zeta)^j}}{(\bar{\zeta} - \bar{z})^{2(j+1)}}. \end{aligned} \quad (2.488)$$

Substituting (2.488) and (2.487) into the kernel in (2.486), we obtain that

$$\begin{aligned} &\partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] \quad (2.489) \\ &= i \left(\nu(\zeta) \partial_\zeta \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] - \bar{\nu}(\zeta) \bar{\partial}_\zeta \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] \right) \\ &= i (-1)^j (z - \zeta)^j \overline{(z - \zeta)^j} \times \\ &\quad \times \frac{(z - \zeta)^{m-a-b-1}}{(\bar{\zeta} - \bar{z})^{2(j+1)}} \cdot \left((m-a-b+j) \nu(\zeta) \cdot \overline{(z - \zeta)} + (j+1) \bar{\nu}(\zeta) \cdot (z - \zeta) \right). \end{aligned}$$

Combining this with (2.486) yields for $0 \leq a + b < m - 1$, $\left(\overline{\dot{C}_{m-1}^2 \dot{g}} \right)_{(a,b)}$ is weakly singular integral operator on $\partial\Omega$. This yields for $a, b \in \mathbb{N}_0$ with $0 \leq a + b < m - 1$, $\left(\overline{\dot{C}_{m-1}^2 \dot{g}} \right)_{(a,b)}$ is compact operator on $\text{CWA}_{m-1}[L^p(\partial\Omega)]$ for all $p \in (1, \infty)$. Next we consider the remaining case when $a + b = m - 1$. If

$a + b = m - 1$, according to (2.489), we obtain that

$$\begin{aligned}
& \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] \\
&= i \left(\nu(\zeta) \partial_{\zeta} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] - \bar{\nu}(\zeta) \bar{\partial}_{\zeta} \left[\frac{(z - \zeta)^{m-a-b+j}}{(\bar{\zeta} - \bar{z})^{j+1}} \right] \right) \\
&= i(-1)^j (z - \zeta)^j \overline{(z - \zeta)^j} \times \\
& \quad \times \frac{1}{(\bar{\zeta} - \bar{z})^{2(j+1)}} \cdot \left((j+1)\nu(\zeta) \cdot \overline{(z - \zeta)} + (j+1)\bar{\nu}(\zeta) \cdot (z - \zeta) \right) \\
&= i(j+1)(-1)^j (z - \zeta)^j \overline{(z - \zeta)^j} \times \\
& \quad \times \frac{1}{(\bar{\zeta} - \bar{z})^{2(j+1)}} \cdot \left(\nu(\zeta) \cdot \overline{(z - \zeta)} + \bar{\nu}(\zeta) \cdot (z - \zeta) \right) \\
&= 2i(j+1)(-1)^j (z - \zeta)^j \overline{(z - \zeta)^j} \times \\
& \quad \times \frac{1}{(\bar{\zeta} - \bar{z})^{2(j+1)}} \cdot \langle z - \zeta, \nu(\zeta) \rangle. \tag{2.490}
\end{aligned}$$

Substituting this into (2.486), one has for $a + b = m - 1$ and for $z \in \partial\Omega$

$$\begin{aligned}
& \left(\overline{\dot{C}_{m-1}^2 \dot{g}} \right)_{(a,b)}(z) \tag{2.491} \\
&= \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{(-1)^j}{\pi} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{|z - \zeta|^{2j}}{(\bar{\zeta} - \bar{z})^{2(j+1)}} \cdot \langle z - \zeta, \nu \rangle g_{(a-1-j, b+1+j)}(\zeta) d\sigma(\zeta).
\end{aligned}$$

Consequently, from (2.485) and (2.491), for $a, b \in \mathbb{N}_0$ with $a + b = m - 1$, we have

$$\begin{aligned}
& \left(\left(\overline{\dot{C}_{m-1}^2 + \dot{C}_{m-1}^2} \right) \dot{g} \right)_{(a,b)} \\
&= \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} k_j(z - \zeta) \cdot \langle z - \zeta, \nu \rangle g_{(a-1-j, b+1+j)}(\zeta) d\sigma(\zeta), \tag{2.492}
\end{aligned}$$

where for $z \in \mathbb{R}^2 \setminus \{0\}$, $0 \leq j \leq a - 1$

$$k_j(z) := \frac{(-1)^j |z|^{2j}}{\pi} \cdot \left[\frac{1}{z^{2(j+1)}} + \frac{1}{\bar{z}^{2(j+1)}} \right]. \tag{2.493}$$

In particular, for $z \in \mathbb{R}^2 \setminus \{0\}$, $0 \leq j \leq a - 1$

$$k_j(z) = \frac{(-1)^j \cdot [z^{2(j+1)} + \bar{z}^{2(j+1)}]}{\pi |z|^{2(j+2)}}. \quad (2.494)$$

Observe that for $0 \leq j \leq a - 1$, $k_j : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ is smooth, even, and positive homogeneous of degree -2 . According to [11], the singular integral operator in (2.485) having chord-dot-normal structure with the kernel k_j which is smooth, even, and positive homogeneous of degree -2 in $\mathbb{R}^2 \setminus \{0\}$ is compact on $L^p(\partial\Omega, \sigma)$ for all $p \in (1, \infty)$ provided that $\Omega \subseteq \mathbb{R}^2$ is a regular SKT domain. Consequently,

$$\begin{aligned} \dot{C}_{m-1} + \overline{\dot{C}_{m-1}} \text{ is compact on } \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)] \text{ for any } p \in (1, \infty) \\ \iff \Omega \subseteq \mathbb{R}^2 \text{ is a regular SKT domain with compact boundary} \end{aligned} \quad (2.495)$$

In particular, for any complex Whitney array $\dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, and for each $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$, $\left((\dot{C}_{m-1} + \overline{\dot{C}_{m-1}}) \dot{g} \right)_{(a,b)}$ has either weakly singular kernel or chord-dot-normal structure. From the definition of the maps Ψ and Φ as in (2.341) and (2.361), these maps are preserving the order and the chord-dot-normal structure of the kernel. Combining this with (2.470) implies that for $\dot{f} = \{f_{(r,s)}\}_{\substack{r,s \in \mathbb{N}_0 \\ r+s \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$, $\left(\dot{K}_{\mathcal{A},m}(\dot{f}) \right)_{(a,b)}$ has either weakly singular kernel or chord-dot-normal structure. This shows that $\mathcal{A} = (\mathcal{A}_{\alpha\beta})_{\substack{\alpha, \beta \in \mathbb{N}_0^2 \\ |\alpha|=|\beta|=m}}$ given in (2.458) is distinguished coefficient tensor associated with Δ^m . Combining (2.470) with (2.495) we have that if $\Omega \subseteq \mathbb{R}^2$ is bounded regular SKT domain, then $\dot{K}_{\mathcal{A},m}$ is compact on $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. This yields

$$\frac{1}{2}I + \dot{K}_{\mathcal{A},m} \text{ is Fredholm of index 0 on } \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \quad (2.496)$$

Next, we verify that $\frac{1}{2}I + \dot{K}_{\mathcal{A},m}$ is indeed invertible on $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ for any $1 < p < \infty$ using the duality argument on the Whitney array spaces in [23]. In order to proceed the proof of the invertibility, we first provide the jump formula for the single multi-layer potential operators in bounded Lipschitz domains.

Proposition 2.49. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Fix a integer $m \in \mathbb{N}$ and an integrability exponent $p \in (1, \infty)$. Also, $\dot{\mathcal{S}}_{A,m}$ stand for the single multi-layer potential operator associated with Δ^m and the coefficient tensor A . Then*

$$\partial_\nu^A \dot{\mathcal{S}}_{A,m} : (\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^* \rightarrow (\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^* \quad (2.497)$$

is well-defined, linear and bounded operator where ∂_ν^A is the conormal derivative associated with A defined as in (2.440). In particular, there holds

$$\partial_\nu^A \dot{\mathcal{S}}_{A,m} = -\frac{1}{2}I + \dot{K}_{A,m}^* \quad \text{on } (\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^*, \quad (2.498)$$

where $\dot{K}_{A,m}^*$ is the adjoint operator of $\dot{K}_{A,m}$.

Proof. Since $\dot{K}_{A,m}^*$ is well-defined, linear and bounded operator on the dual of Whitney array space $(\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^*$ (cf. Theorem 4.14 in [23]),

$$-\frac{1}{2}I + \dot{K}_{A,m}^* \quad (2.499)$$

is well defined, linear and bounded on $(\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^*$. In addition, $\partial_\nu^A \dot{\mathcal{S}}_{A,m}$ is well-defined, linear and bounded operator with the jump formula as in (2.498) on $(\text{RWA}_{m-1}[B_s^{p,q}(\partial\Omega, \sigma)])^*$ for $1 < p, q < \infty$, $s > 0$ (cf. Proposition 5.27 in [23]). Combining this with the property that $(\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^*$ is embedded into $(\text{RWA}_{m-1}[B_s^{p,q}(\partial\Omega, \sigma)])^*$ completes the proof of the proposition. \square

Next, we provide the invertibility result for the case when $p = 2$.

Proposition 2.50. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded regular SKT domain. Fix an integer $m \in \mathbb{N}$. There holds*

$$\frac{1}{2}I + \dot{K}_{\mathcal{A},m} \quad \text{is invertible on } \text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)], \quad (2.500)$$

where $\dot{K}_{\mathcal{A},m}$ is a distinguished boundary-to-boundary double multi-layer potential operator associated with Δ^m and the distinguished coefficient tensor \mathcal{A} .

Proof. Since $\frac{1}{2}I + \dot{K}_{\mathcal{A},m}^*$ is injective on $(\text{RWA}_{m-1}[B_{1/2}^{2,2}(\partial\Omega, \sigma)])^*$ (cf. Theorem 6.8 in [23]), and $(\text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)])^* \hookrightarrow (\text{RWA}_{m-1}[B_{1/2}^{2,2}(\partial\Omega, \sigma)])^*$, one can conclude that

$$\frac{1}{2}I + \dot{K}_{\mathcal{A},m}^* \text{ is injective on } (\text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)])^*, \quad (2.501)$$

which implies that

$$\frac{1}{2}I + \dot{K}_{\mathcal{A},m} \text{ is surjective on } \text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)]. \quad (2.502)$$

According to (2.496),

$$\frac{1}{2}I + \dot{K}_{\mathcal{A},m} \text{ is Fredholm of index 0 on } \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \quad (2.503)$$

Combining this with (2.502) finishes the proof. \square

In the next theorem, we extend the invertibility result for any $1 < p < \infty$.

Theorem 2.51. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded regular SKT domain. Fix an integer $m \in \mathbb{N}$ and an integrability exponent $1 < p < \infty$. There holds*

$$\frac{1}{2}I + \dot{K}_{\mathcal{A},m} \text{ is invertible on } \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)], \quad (2.504)$$

where $\dot{K}_{\mathcal{A},m}$ is a distinguished boundary-to-boundary double multi-layer potential operator associated with Δ^m and the distinguished coefficient tensor \mathcal{A} .

Proof. According to Proposition 2.50,

$$\frac{1}{2}I + \dot{K}_{\mathcal{A},m} \text{ is invertible on } \text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)], \quad (2.505)$$

which implies that $\frac{1}{2}I + \dot{K}_{\mathcal{A},m}$ is injective on $\text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)]$. Since $\partial\Omega$ is compact, $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ is a subset of $\text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)]$ for $p \geq 2$. This forces $\frac{1}{2}I + \dot{K}_{\mathcal{A},m}$ is injective on $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ for $p \geq 2$. Combining this with the Fredholmness in (2.503) provides $\frac{1}{2}I + \dot{K}_{\mathcal{A},m}$ is invertible on $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ for $p \in [2, \infty)$. For the case when $p \in (1, 2)$, consider the adjoint operator

$$\frac{1}{2}I + \dot{K}_{\mathcal{A},m}^* : (\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^* \rightarrow (\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^*. \quad (2.506)$$

Since $\partial\Omega$ is compact, $(\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^*$ is a subset of $(\text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)])^*$ for $p \in (1, 2)$. In particular, from (2.501)

$$\frac{1}{2}I + \dot{K}_{\mathcal{A}, m}^* \text{ is injective on } (\text{RWA}_{m-1}[L^2(\partial\Omega, \sigma)])^*, \quad (2.507)$$

which further implies $\frac{1}{2}I + \dot{K}_{\mathcal{A}, m}^*$ is injective on $(\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])^*$ for any $p \in (1, 2)$, thus we have $\frac{1}{2}I + \dot{K}_{\mathcal{A}, m}^*$ is surjective on $(\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)])$ for $p \in (1, 2)$. Combining this with the Fredholmness in (2.503) gives that $\frac{1}{2}I + \dot{K}_{\mathcal{A}, m}^*$ is invertible on $\text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ for $p \in (1, 2)$. This completes the proof of the theorem. \square

Finally, we establish the solvability of the Dirichlet problem for Δ^m in bounded regular SKT domain.

Theorem 2.52. *Let $\Omega \subseteq \mathbb{R}^2$ is a bounded regular SKT domain. Fix an integer $m \in \mathbb{N}$ and $p \in (1, \infty)$. Then the higher-order Laplace equation in Ω with Dirichlet boundary data $\dot{f} = \{f_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)]$*

$$\begin{cases} \Delta^m u = 0 & \text{in } \Omega \\ \mathcal{N}_\kappa(\nabla^{m-1} u) \in L^p(\partial\Omega, \sigma) \\ \text{Tr}_{m-1}^{\mathbb{R}^2}(u) = \dot{f} \in \text{RWA}_{m-1}[L^p(\partial\Omega, \sigma)] \end{cases} \quad (2.508)$$

has a solution.

Proof. This follows from (2.419), (2.420), (2.421), and Theorem 2.51. \square

2.10 Riemann-Hilbert problems for polyanalytic functions

The classical Riemann-Hilbert problem is a boundary value problem in complex analysis that seeks an analytic function satisfying size and boundary conditions involving twisting coefficients. The Riemann-Hilbert problem has important applications in many areas of mathematics, including algebraic geometry, representation theory, mathematical physics, and number theory. It

has been extensively studied over the years, and many important results have been obtained. The Riemann-Hilbert problem for polyanalytic functions is a natural generalization of the classical Riemann-Hilbert problem to polyanalytic functions. In recent years, there have been several important developments in the theory of the type of Riemann-Hilbert problems for polyanalytic functions such as the Schwarz problem, the one-sided Riemann-Hilbert problem, and the Riemann-Hilbert problem for polyanalytic functions (cf. [4], [10], [42], [43]). In this section, we investigate the Riemann-Hilbert problem for polyanalytic function in rough domains employing the results of polyanalytic functions and the poly-Cauchy operator that we have constructed in this paper.

For the classical Riemann-Hilbert problem in unit disk $\mathbb{D} \subseteq \mathbb{C}$, let us consider a twisting coefficient $\phi : S^1 \rightarrow \mathbb{R}$ with the property that

$$\begin{aligned} \phi(x) &\neq 0 \quad \forall x \in S^1, \\ \phi &= \psi \Big|_{S^1} \quad \text{for some } \psi \in C_c^\infty(\mathbb{R}^2), \end{aligned} \tag{2.509}$$

where $S^1 = \partial\mathbb{D}$. Let us denote $\mathbb{D}^+ := \mathbb{D}$ and $\mathbb{D}^- = \mathbb{C} \setminus \overline{\mathbb{D}}$. Consider analytic functions $u^\pm \in \mathcal{O}(\mathbb{D}^\pm)$ where u^- decays at ∞ . Fix $\kappa \in (0, \infty)$ and $p \in (1, \infty)$. For given $f \in L^p(\partial\mathbb{D}, \sigma)$, the Riemann-Hilbert problem takes the form

$$\left\{ \begin{array}{l} \bar{\partial}u^\pm = 0 \quad \text{in } \mathbb{D}^\pm, \\ u^- \text{ decays at } \infty, \\ \mathcal{N}_\kappa(u^\pm) \in L^p(\partial\mathbb{D}, \sigma), \\ \phi \cdot u^+ \Big|_{\partial\mathbb{D}^+}^{\kappa\text{-n.t.}} - u^- \Big|_{\partial\mathbb{D}^-}^{\kappa\text{-n.t.}} = f \in L^p(\partial\mathbb{D}, \sigma). \end{array} \right. \tag{2.510}$$

For this problem the classical Cauchy operator \mathcal{C} plays an important role with the compactness property of commutator $[M_\phi, CZ]$ on $L^p(\partial\mathbb{D}, \sigma)$. In this section, we consider the Riemann-Hilbert problems for polyanalytic functions with natural generalization of the multiplication at the level of array. We first need an appropriate definition of the multiplication at the level of array.

Definition 2.53. Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be an open set. Fix $m \in \mathbb{N}$. Consider a function $\phi : \partial\Omega \rightarrow \mathbb{R}$ which is a restriction of some function ψ which is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$, i.e., $\phi = \psi|_{\partial\Omega}$ and an array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}}$ defined on $\partial\Omega$ satisfies the complex compatibility condition as in (2.50). Define a multiplication with array \odot by for each $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$

$$(\phi \odot \dot{g})_{(a,b)} := \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta!\gamma!} (\partial_{\mathbb{C}}^{\beta} \psi)|_{\partial\Omega} \cdot g_{\gamma}, \quad (2.511)$$

where for $\beta = (\beta_1, \beta_2)$

$$\partial_{\mathbb{C}}^{\beta} \psi := \partial_z^{\beta_1} \bar{\partial}_z^{\beta_2} \psi \quad \text{in a neighborhood of } \partial\Omega. \quad (2.512)$$

Remark 2.54. There are some basic observations of the operation \odot defined as in (2.511).

1. If $m = 1$, then \odot simply becomes a multiplication of functions.
2. If $\phi \equiv c$ is a constant function on $\partial\Omega$ for some $c \in \mathbb{R}$, that is, $\psi(z) = c$ in a neighborhood of $\partial\Omega$ where $\phi = \psi|_{\partial\Omega}$, then for each $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$, there holds

$$(c \odot \dot{g})_{(a,b)} = g_{(a,b)}. \quad (2.513)$$

In the following lemma, we verify that this multiplication with an array is a natural in the sense that $\phi \odot \dot{g}$ also satisfies the complex compatibility condition.

Lemma 2.55. Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a UR domain. Fix $m \in \mathbb{N}$. Consider a function $\phi : \partial\Omega \rightarrow \mathbb{R}$ which is a restriction of some function ψ which is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$, i.e., $\phi = \psi|_{\partial\Omega}$ and an array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}}$ defined on $\partial\Omega$ satisfies the complex compatibility condition as in (2.50). Then $\phi \odot \dot{g} \in CC_{\mathbb{C}}$ where \odot is defined in (2.511) and $CC_{\mathbb{C}}$ is defined in (2.50).

Proof. Consider $a, b \in \mathbb{N}_0$ with $a + b \leq m - 2$. Then,

$$\partial_\tau \left((\phi \odot \dot{g})_{(a,b)} \right) = \partial_\tau \left[\sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta! \gamma!} (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma \right], \quad (2.514)$$

which implies

$$\partial_\tau \left((\phi \odot \dot{g})_{(a,b)} \right) = \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta! \gamma!} \partial_\tau \left((\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma \right). \quad (2.515)$$

In particular,

$$\partial_\tau \left((\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma \right) = \partial_\tau \left((\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \right) \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot \partial_\tau (g_\gamma). \quad (2.516)$$

Applying definition of the tangential derivative as in (2.46) for $\partial_\tau \left((\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \right)$ and compatibility condition as in (2.50) for $\partial_\tau (g_\gamma)$, one has

$$\begin{aligned} & \partial_\tau \left((\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma \right) \\ &= i(\nu(\partial_{\mathbb{C}}^{\beta+e_1} \psi)|_{\partial\Omega} - \bar{\nu}(\partial_{\mathbb{C}}^{\beta+e_2} \psi)|_{\partial\Omega}) \cdot g_\gamma + i(\nu g_{\gamma+e_1} - \bar{\nu} g_{\gamma+e_2}) \cdot (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \\ &= i(\nu((\partial_{\mathbb{C}}^{\beta+e_1} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_1}) \\ & \quad - \bar{\nu}((\partial_{\mathbb{C}}^{\beta+e_2} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_2})). \end{aligned} \quad (2.517)$$

Substituting this into (2.515) yields

$$\begin{aligned} \partial_\tau \left((\phi \odot \dot{g})_{(a,b)} \right) &= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta! \gamma!} i \left[\nu((\partial_{\mathbb{C}}^{\beta+e_1} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_1}) \right. \\ & \quad \left. - \bar{\nu}((\partial_{\mathbb{C}}^{\beta+e_2} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_2}) \right]. \end{aligned} \quad (2.518)$$

One can rewrite the expression on the right-hand side in (2.518) as

$$\begin{aligned} & i\nu \cdot \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta! \gamma!} \left((\partial_{\mathbb{C}}^{\beta+e_1} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_1} \right) \\ & - i\bar{\nu} \cdot \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta! \gamma!} \left((\partial_{\mathbb{C}}^{\beta+e_2} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_2} \right). \end{aligned} \quad (2.519)$$

Based on the basic calculation by substituting $\beta + e_i = \beta'$, for $i = 1, 2$, one can conclude that

$$\begin{aligned}
\sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta! \gamma!} (\partial_{\mathbb{C}}^{\beta + e_i} \psi)|_{\partial\Omega} \cdot g_{\gamma} &= \sum_{\substack{\beta', \gamma \in \mathbb{N}_0^2 \\ \beta' + \gamma = (a, b) + e_i \\ \beta'_i \geq 1}} \frac{a!b!}{(\beta' - e_i)! \gamma!} (\partial_{\mathbb{C}}^{\beta'} \psi)|_{\partial\Omega} \cdot g_{\gamma} \\
&= \sum_{\substack{\beta', \gamma \in \mathbb{N}_0^2 \\ \beta' + \gamma = (a, b) + e_i \\ \beta'_i \geq 1}} \frac{a!b! \beta'_i}{\beta'! \gamma!} (\partial_{\mathbb{C}}^{\beta'} \psi)|_{\partial\Omega} \cdot g_{\gamma} \\
&= \sum_{\substack{\beta', \gamma \in \mathbb{N}_0^2 \\ \beta' + \gamma = (a, b) + e_i}} \frac{a!b! \beta'_i}{\beta'! \gamma!} (\partial_{\mathbb{C}}^{\beta'} \psi)|_{\partial\Omega} \cdot g_{\gamma}, \quad (2.520)
\end{aligned}$$

where $\beta' = (\beta'_1, \beta'_2)$. Indeed, (2.520) comes from the fact that if $\beta'_i = 0$, then the expression in the summation turns out to be 0. Replacing β' by β , one has

$$\sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta! \gamma!} (\partial_{\mathbb{C}}^{\beta + e_i} \psi)|_{\partial\Omega} \cdot g_{\gamma} = \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b) + e_i}} \frac{a!b! \beta_i}{\beta! \gamma!} (\partial_{\mathbb{C}}^{\beta} \psi)|_{\partial\Omega} \cdot g_{\gamma}, \quad (2.521)$$

where $\beta = (\beta_1, \beta_2)$. Similarly, by substituting $\gamma + e_i = \gamma'$, for $i = 1, 2$, one has

$$\begin{aligned}
\sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta! \gamma!} (\partial_{\mathbb{C}}^{\beta} \psi)|_{\partial\Omega} \cdot g_{\gamma + e_i} &= \sum_{\substack{\beta, \gamma' \in \mathbb{N}_0^2 \\ \beta + \gamma' = (a, b) + e_i \\ \gamma'_i \geq 1}} \frac{a!b!}{\beta! (\gamma' - e_i)!} (\partial_{\mathbb{C}}^{\beta} \psi)|_{\partial\Omega} \cdot g_{\gamma'} \\
&= \sum_{\substack{\beta, \gamma' \in \mathbb{N}_0^2 \\ \beta + \gamma' = (a, b) + e_i \\ \gamma'_i \geq 1}} \frac{a!b! \gamma'_i}{\beta! \gamma'!} (\partial_{\mathbb{C}}^{\beta} \psi)|_{\partial\Omega} \cdot g_{\gamma'} \\
&= \sum_{\substack{\beta, \gamma' \in \mathbb{N}_0^2 \\ \beta + \gamma' = (a, b) + e_i}} \frac{a!b! \gamma'_i}{\beta! \gamma'!} (\partial_{\mathbb{C}}^{\beta} \psi)|_{\partial\Omega} \cdot g_{\gamma'}, \quad (2.522)
\end{aligned}$$

where $\gamma' = (\gamma'_1, \gamma'_2)$. Replacing γ' by γ , one can conclude that

$$\sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta! \gamma!} (\partial_{\mathbb{C}}^{\beta} \psi)|_{\partial\Omega} \cdot g_{\gamma + e_i} = \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b) + e_i}} \frac{a!b! \gamma_i}{\beta! \gamma!} (\partial_{\mathbb{C}}^{\beta} \psi)|_{\partial\Omega} \cdot g_{\gamma}, \quad (2.523)$$

where $\gamma = (\gamma_1, \gamma_2)$. Combining (2.521) and (2.523) forces, for $i = 1, 2$

$$\begin{aligned}
& \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta!\gamma!} ((\partial_{\mathbb{C}}^{\beta+e_i} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_i}) \\
&= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b) + e_i}} \frac{a!b!}{\beta!\gamma!} (\beta_i + \gamma_i) (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma.
\end{aligned} \tag{2.524}$$

In conclusion, if $i = 1$,

$$\begin{aligned}
& \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta!\gamma!} ((\partial_{\mathbb{C}}^{\beta+e_1} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_1}) \\
&= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a+1, b)}} \frac{a!b!}{\beta!\gamma!} (a+1) (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma \\
&= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a+1, b)}} \frac{(a+1)!b!}{\beta!\gamma!} (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma \\
&= (\phi \odot \dot{g})_{(a+1, b)},
\end{aligned} \tag{2.525}$$

and if $i = 2$,

$$\begin{aligned}
& \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta!\gamma!} ((\partial_{\mathbb{C}}^{\beta+e_2} \psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_{\gamma+e_2}) \\
&= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b+1)}} \frac{a!b!}{\beta!\gamma!} (b+1) (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma \\
&= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b+1)}} \frac{a!(b+1)!}{\beta!\gamma!} (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma \\
&= (\phi \odot \dot{g})_{(a, b+1)}.
\end{aligned} \tag{2.526}$$

Substituting (2.525) and (2.526) into (2.519), one has

$$\begin{aligned}
& i\nu \cdot \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta!\gamma!} ((\partial_{\mathbb{C}}^{\beta+e_1}\psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta\psi)|_{\partial\Omega} \cdot g_{\gamma+e_1}) \\
& - i\bar{\nu} \cdot \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a, b)}} \frac{a!b!}{\beta!\gamma!} ((\partial_{\mathbb{C}}^{\beta+e_2}\psi)|_{\partial\Omega} \cdot g_\gamma + (\partial_{\mathbb{C}}^\beta\psi)|_{\partial\Omega} \cdot g_{\gamma+e_2}) \\
& = i \left(\nu \cdot (\phi \odot \dot{g})_{(a+1, b)} - \bar{\nu} \cdot (\phi \odot \dot{g})_{(a, b+1)} \right). \tag{2.527}
\end{aligned}$$

Combining this with (2.518), we obtain that for any $a, b \in \mathbb{N}_0$ with $a+b \leq m-2$

$$\partial_\tau \left((\phi \odot \dot{g})_{(a, b)} \right) = i \left(\nu \cdot (\phi \odot \dot{g})_{(a+1, b)} - \bar{\nu} \cdot (\phi \odot \dot{g})_{(a, b+1)} \right). \tag{2.528}$$

This finishes the proof. \square

From the lemma 2.55, we naturally get the following corollary.

Corollary 2.56. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a UR domain with compact boundary. Fix $m \in \mathbb{N}$ and an integrability exponent $p \in [1, \infty]$. Consider a function $\phi : \partial\Omega \rightarrow \mathbb{R}$ which is a restriction of some function ψ that is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$. That is, $\phi = \psi|_{\partial\Omega}$ and a complex Whitney array $\dot{g} = \{g_{(a, b)}\}_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, then $\phi \odot \dot{g} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$.*

Proof. This follows from the lemma 2.55 with the basic property that for any function $g \in L^p(\partial\Omega, \sigma)$ and multi-index $\beta \in \mathbb{N}_0^2$, $(\partial_{\mathbb{C}}^\beta\psi)|_{\partial\Omega} \cdot g$ also belongs to $L^p(\partial\Omega, \sigma)$. \square

Proposition 2.57. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a UR domain with compact boundary. Fix $m \in \mathbb{N}$ and $p \in [1, \infty]$. Consider a function $\phi : \partial\Omega \rightarrow \mathbb{R}$ which is a restriction of some function ψ which is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$, i.e., $\phi = \psi|_{\partial\Omega}$. Define an operator \odot_ϕ on $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ as*

$$\odot_\phi \dot{g} := \phi \odot \dot{g}, \tag{2.529}$$

where \odot is defined as in (2.511) and $\dot{g} = \{g_{(a, b)}\}_{\substack{a, b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Then \odot_ϕ is well defined and for any functions $\phi_1, \phi_2 : \partial\Omega \rightarrow \mathbb{R}$ satisfy the same hypotheses as ϕ there holds

$$\odot_{\phi_1} (\odot_{\phi_2}) = \odot_{\phi_1 \cdot \phi_2} \text{ on } \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \tag{2.530}$$

Proof. According to Corollary 2.56, \odot_ϕ is well defined on $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Next, we shall prove the identity (2.530). According to the definition in (2.511), for an array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$, one has

$$\begin{aligned} [\odot_{\phi_1}(\odot_{\phi_2}\dot{g})]_{(a,b)} &= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta!\gamma!} (\partial_{\mathbb{C}}^\beta \psi_1)|_{\partial\Omega} \cdot (\odot_{\phi_2}\dot{g})_{\gamma} r & (2.531) \\ &= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta!\gamma!} (\partial_{\mathbb{C}}^\beta \psi_1)|_{\partial\Omega} \cdot \sum_{\substack{\beta', \gamma' \in \mathbb{N}_0 \\ \beta' + \gamma' = \gamma}} \frac{\gamma!}{(\beta')!(\gamma')!} (\partial_{\mathbb{C}}^{\beta'} \psi_2)|_{\partial\Omega} \cdot g_{\gamma'} \\ &= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0 \\ \beta + \gamma = (a,b)}} \sum_{\substack{\beta', \gamma' \in \mathbb{N}_0 \\ \beta' + \gamma' = \gamma}} \frac{a!b!}{\beta!(\beta')!(\gamma')!} (\partial_{\mathbb{C}}^\beta \psi_1)|_{\partial\Omega} (\partial_{\mathbb{C}}^{\beta'} \psi_2)|_{\partial\Omega} \cdot g_{\gamma'}, \end{aligned}$$

where for $i = 1, 2$, ψ_i is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$ with the property that $\phi_i = \psi_i|_{\partial\Omega}$. Substituting $\beta' + \gamma' = \gamma$ into the first summation indices in (2.531) yields

$$[\odot_{\phi_1}(\odot_{\phi_2}\dot{g})]_{(a,b)} = \sum_{\substack{\beta, \beta', \gamma' \in \mathbb{N}_0 \\ \beta + \beta' + \gamma' = (a,b)}} \frac{a!b!}{\beta!(\beta')!(\gamma')!} (\partial_{\mathbb{C}}^\beta \psi_1)|_{\partial\Omega} (\partial_{\mathbb{C}}^{\beta'} \psi_2)|_{\partial\Omega} \cdot g_{\gamma'}. \quad (2.532)$$

On the other hand,

$$[\odot_{\phi_1 \cdot \phi_2} \dot{g}]_{(a,b)} = \sum_{\substack{\beta, \gamma \in \mathbb{N}_0 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta!\gamma!} (\partial_{\mathbb{C}}^\beta (\psi_1 \psi_2))|_{\partial\Omega} \cdot g_{\gamma}. \quad (2.533)$$

Applying Leibniz formula, one has

$$\begin{aligned} [\odot_{\phi_1 \cdot \phi_2} \dot{g}]_{(a,b)} &= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta!\gamma!} \sum_{\substack{\beta', \gamma' \in \mathbb{N}_0 \\ \beta' + \gamma' = \beta}} \frac{\beta!}{(\beta')!(\gamma')!} (\partial_{\mathbb{C}}^{\beta'} \psi_1)|_{\partial\Omega} (\partial_{\mathbb{C}}^{\gamma'} \psi_2)|_{\partial\Omega} \cdot g_{\gamma} \\ &= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0 \\ \beta + \gamma = (a,b)}} \sum_{\substack{\beta', \gamma' \in \mathbb{N}_0 \\ \beta' + \gamma' = \beta}} \frac{a!b!}{\gamma!(\beta')!(\gamma')!} (\partial_{\mathbb{C}}^{\beta'} \psi_1)|_{\partial\Omega} (\partial_{\mathbb{C}}^{\gamma'} \psi_2)|_{\partial\Omega} \cdot g_{\gamma}. \quad (2.534) \end{aligned}$$

Substituting $\beta' + \gamma' = \beta$ into the first summation indices in (2.534), we obtain that

$$[\odot_{\phi_1 \cdot \phi_2} \dot{g}]_{(a,b)} = \sum_{\substack{\beta', \gamma', \gamma \in \mathbb{N}_0 \\ \beta' + \gamma' + \gamma = (a,b)}} \frac{a!b!}{\gamma!(\beta')!(\gamma')!} (\partial_{\mathbb{C}}^{\beta'} \psi_1)|_{\partial\Omega} (\partial_{\mathbb{C}}^{\gamma'} \psi_2)|_{\partial\Omega} \cdot g_{\gamma}. \quad (2.535)$$

We simply change the indices by $\beta' = \beta$, $\gamma' = \beta'$, $\gamma = \gamma'$, then we have

$$[\odot_{\phi_1 \cdot \phi_2} \dot{g}]_{(a,b)} = \sum_{\substack{\beta, \beta', \gamma' \in \mathbb{N}_0 \\ \beta + \beta' + \gamma' = (a,b)}} \frac{a!b!}{\beta!(\beta')!(\gamma')!} (\partial_{\mathbb{C}}^{\beta} \psi_1)|_{\partial\Omega} (\partial_{\mathbb{C}}^{\beta'} \psi_2)|_{\partial\Omega} \cdot g_{\gamma'}. \quad (2.536)$$

Comparing (2.536) with (2.532), for any $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$, one can conclude that

$$[\odot_{\phi_1} (\odot_{\phi_2} \dot{g})]_{(a,b)} = [\odot_{\phi_1 \cdot \phi_2} \dot{g}]_{(a,b)}, \quad (2.537)$$

which implies that

$$\odot_{\phi_1} (\odot_{\phi_2}) = \odot_{\phi_1 \cdot \phi_2} \quad \text{on } \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \quad (2.538)$$

This finishes the proof. \square

Corollary 2.58. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a UR domain with compact boundary. Fix $m \in \mathbb{N}$ and $p \in [1, \infty]$. Consider a function $\phi : \partial\Omega \rightarrow \mathbb{R}$ which is a restriction of some function ψ which is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$, i.e., $\phi = \psi|_{\partial\Omega}$ and assume that $\psi(z) \neq 0$ for all $z \in \partial\Omega$. Then \odot_{ϕ} is an invertible operator on $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ with an inverse $\odot_{(1/\phi)}$ where \odot_{ϕ} is defined as in (2.529).*

Proof. Since $\psi(z) \neq 0$ for all $z \in \partial\Omega$, $(1/\psi)$ is well defined in a neighborhood of $\partial\Omega$. In addition, $(1/\phi) = (1/\psi)|_{\partial\Omega}$ where $(1/\psi)$ is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$. According to the proposition 2.57, one has

$$\odot_{\phi} (\odot_{(1/\phi)}) = \odot_{(1/\phi)} (\odot_{\phi}) = \odot_1 \quad \text{on } \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \quad (2.539)$$

Combining this with (2.513), one can conclude that

$$\odot_{\phi} (\odot_{(1/\phi)}) = \odot_{(1/\phi)} (\odot_{\phi}) = I \quad \text{on } \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \quad (2.540)$$

This completes the proof. \square

According to [7] by Coifman, Rochberg, and Weiss and [39] by Uchiyama, the commutator of the Calderón-Zygmund operator $[M_{\phi}, CZ]$ is compact on $L^p(\partial\Omega, \sigma)$ for all $p \in (1, \infty)$ under the assumption for ϕ as in Definition 2.53. In

the Riemann-Hilbert problem, this compactness property of the commutator $[M_\phi, CZ]$ plays an important role to employ the Fredholm operator theory. With an appropriate definition of the multiplication at the level of array as in Definition 2.53, our next business is to verify the compactness property of commutator with the boundary-to-boundary poly-Cauchy operator of order m .

Proposition 2.59. *Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a UR domain with compact boundary. Fix $m \in \mathbb{N}$ and $p \in (1, \infty)$. Consider a function $\phi : \partial\Omega \rightarrow \mathbb{R}$ which is a restriction of some function ψ which is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$, i.e., $\phi = \psi|_{\partial\Omega}$ and an array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Define a commutator of the boundary-to-boundary poly-Cauchy operator (of order m) with ϕ acting on \dot{g} by*

$$[M_\phi, \dot{C}_{m-1}]\dot{g} := \phi \odot (\dot{C}_{m-1}\dot{g}) - \dot{C}_{m-1}(\phi \odot \dot{g}), \quad (2.541)$$

where \odot is defined as in (2.511). Then $[M_\phi, \dot{C}_{m-1}]$ is compact operator on the complex Whitney array space $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$

Proof. Let $a, b \in \mathbb{N}_0$ with $a+b \leq m-1$ and consider a complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. According to (2.511) one has

$$\begin{aligned} & \left(\phi \odot (\dot{C}_{m-1}\dot{g}) \right)_{(a,b)} \\ &= \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta!\gamma!} (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot (\dot{C}_{m-1}\dot{g})_\gamma \\ &= \sum_{\substack{\beta_i, \gamma_i \in \mathbb{N}_0 \\ \beta_1 + \gamma_1 = a \\ \beta_2 + \gamma_2 = b}} \frac{a!b!}{\beta_1!\beta_2!\gamma_1!\gamma_2!} (\partial_z^{\beta_1} \bar{\partial}_z^{\beta_2} \psi)|_{\partial\Omega} \cdot (\dot{C}_{m-1}\dot{g})_{(\gamma_1, \gamma_2)} \end{aligned} \quad (2.542)$$

From the definition 2.35, we have

$$\begin{aligned}
\left(\dot{C}_{m-1}\dot{g}\right)_{(\gamma_1,\gamma_2)}(z) &= \sum_{k=0}^{m-1-\gamma_1-\gamma_2} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{\overline{(z-\zeta)^k}}{k!(\zeta-z)} g_{(\gamma_1,k+\gamma_2)}(\zeta) d\zeta \\
&\quad - \sum_{j=0}^{\gamma_1-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{j!}{(m-\gamma_1-\gamma_2+j)!} \times \\
&\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{m-\gamma_1-\gamma_2+j}}}{(\zeta-z)^{j+1}} \right] g_{(\gamma_1-1-j,m-\gamma_1+j)}(\zeta) d\sigma(\zeta)
\end{aligned} \tag{2.543}$$

at σ -a.e. point $z \in \partial\Omega$. Observe that for the first term in (2.543), if $k > 0$, then the kernel becomes weakly singular. For the second term, one can conclude that if $\gamma_1 + \gamma_2 < m - 1$, then the kernel is weakly singular. Now, we look at the remaining cases when $k = 0$, $\gamma_1, \gamma_2 \in \mathbb{N}_0$ with $\gamma_1 \leq a$, $\gamma_2 \leq b$ for the first term which becomes the boundary-to-boundary classical Cauchy operator acting on $g_{(\gamma_1,\gamma_2)}$ and the case when $\gamma_1 + \gamma_2 = m - 1$ which means $a + b = m - 1$, $\beta_i = 0$, $\gamma_1 = a$, $\gamma_2 = b$ for the second term. For more simplicity, denote the second term when $\gamma_1 + \gamma_2 = m - 1$ by

$$\begin{aligned}
\left(\dot{C}'_{m-1}\dot{g}\right)_{(a,b)} &:= - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta-z| > \varepsilon}} \frac{1}{j+1} \times \\
&\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z-\zeta)^{j+1}}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j,m-a+j)}(\zeta) d\sigma(\zeta).
\end{aligned} \tag{2.544}$$

Therefore, the remaining term in (2.542) when the kernel of the singular integral operator is not weakly singular can be expressed as follows.

$$\sum_{\substack{\beta_i, \gamma_i \in \mathbb{N}_0 \\ \beta_1 + \gamma_1 = a \\ \beta_2 + \gamma_2 = b}} \frac{a!b!}{\beta_1!\beta_2!\gamma_1!\gamma_2!} (\partial_z^{\beta_1} \bar{\partial}_z^{\beta_2} \psi)|_{\partial\Omega} \cdot C g_{(\gamma_1,\gamma_2)} + \phi \cdot \left(\dot{C}'_{m-1}\dot{g}\right)_{(a,b)}. \tag{2.545}$$

Using the properties that $L_1^p(\partial\Omega, \sigma)$ is compactly embedded into $L^p(\partial\Omega, \sigma)$,

and C is bounded operator on $L^p(\partial\Omega, \sigma)$, one can conclude that

$$\sum_{\substack{\beta_i, \gamma_i \in \mathbb{N}_0 \\ \beta_1 + \gamma_1 = a \\ \beta_2 + \gamma_2 = b}} \frac{a!b!}{\beta_1!\beta_2!\gamma_1!\gamma_2!} (\partial_z^{\beta_1} \bar{\partial}_z^{\beta_2} \psi)|_{\partial\Omega} \cdot C \text{ is compact operator on } L_1^p(\partial\Omega, \sigma). \quad (2.546)$$

Since $g_{(\gamma_1, \gamma_2)} \in L_1^p(\partial\Omega, \sigma)$ for $\gamma_1, \gamma_2 \in \mathbb{N}_0$ with $\gamma_1 + \gamma_2 < m - 1$, we consider $\gamma_1 + \gamma_2 = m - 1$ over the summation in (2.545). The integral expression in (2.545) is reduced to

$$\phi \cdot Cg_{(a,b)} + \phi \cdot \left(\dot{C}'_{m-1} \dot{g} \right)_{(a,b)}. \quad (2.547)$$

Turning our attention to the second term in (2.541), for any $a, b \in \mathbb{N}_0$ with $a + b \leq m - 1$,

$$\begin{aligned} \left(\dot{C}_{m-1}(\phi \odot \dot{g}) \right)_{(a,b)} &= \sum_{k=0}^{m-1-a-b} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{\overline{(z - \zeta)^k}}{k!(\zeta - z)} (\phi \odot \dot{g})_{(a, k+b)}(\zeta) d\zeta \\ &\quad - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{j!}{(m - a - b + j)!} \times \\ &\quad \times \partial_{\tau(\zeta)} \left[\frac{\overline{(z - \zeta)^{m-a-b+j}}}{(\zeta - z)^{j+1}} \right] (\phi \odot \dot{g})_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta) \end{aligned} \quad (2.548)$$

at σ -a.e. point $z \in \partial\Omega$. As we observed above, if $k > 0$, then the kernel of the first term in (2.548) becomes weakly singular, and if $a + b < m - 1$, then the kernel of the second term in (2.548) becomes also weakly singular. Therefore, the remaining term in (2.548) whose kernel is not weakly singular is

$$\begin{aligned} C(\phi \odot \dot{g})_{(a,b)}, \quad &\text{if } a + b < m - 1, \\ C(\phi \odot \dot{g})_{(a,b)} + \left(\dot{C}'_{m-1}(\phi \odot \dot{g}) \right)_{(a,b)}, \quad &\text{if } a + b = m - 1. \end{aligned} \quad (2.549)$$

According to Corollary 2.56, $\phi \odot \dot{g}$ belongs to $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ which implies if $a + b < m - 1$, then $(\phi \odot \dot{g})_{(a,b)} \in L_1^p(\partial\Omega, \sigma)$. Since $L_1^p(\partial\Omega, \sigma)$ is compactly embedded into $L^p(\partial\Omega, \sigma)$, we now consider the case when $a + b = m - 1$.

According to (2.548), one has

$$C(\phi \odot \dot{g})_{(a,b)} + (\dot{C}'_{m-1}(\phi \odot \dot{g}))_{(a,b)}. \quad (2.550)$$

By the definitions in (2.511), we have

$$(\phi \odot \dot{g})_{(a,b)} = \sum_{\substack{\beta, \gamma \in \mathbb{N}_0^2 \\ \beta + \gamma = (a,b)}} \frac{a!b!}{\beta!\gamma!} (\partial_{\mathbb{C}}^\beta \psi)|_{\partial\Omega} \cdot g_\gamma. \quad (2.551)$$

Since $g_\gamma \in L_1^p(\partial\Omega, \sigma)$ if $|\gamma| < m - 1$, thus the classical Cauchy operator acting on $(\phi \odot \dot{g})_{(a,b)}$ has the compactness property over the summation for $|\gamma| < m - 1$. Therefore, to verify the compactness of the commutator in (2.541), the singular integral operator in (2.550) is reduced to

$$\begin{aligned} & C(\phi \cdot g_{(a,b)}) \quad (2.552) \\ & - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{1}{j+1} \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{j+1}}{(\zeta - z)^{j+1}} \right] (\phi \odot \dot{g})_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta). \end{aligned}$$

Applying the similar argument to the second term in (2.552), it suffices to consider

$$\begin{aligned} & C(\phi \cdot g_{(a,b)}) \quad (2.553) \\ & - \sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{1}{j+1} \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{j+1}}{(\zeta - z)^{j+1}} \right] \phi(\zeta) \cdot g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta). \end{aligned}$$

In summary, the remaining terms of the commutator $[M_\phi, \dot{C}'_{m-1}]\dot{g}$ as in (2.541) which are not verified as compact operators are for $a, b \in \mathbb{N}_0$ with $a + b = m - 1$

$$\begin{aligned} & \phi(z) \cdot Cg_{(a,b)}(z) - C(\phi \cdot g_{(a,b)})(z) - \left[\sum_{j=0}^{a-1} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{1}{j+1} \times \right. \\ & \times \left(\phi(z) \cdot \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{j+1}}{(\zeta - z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) \right. \\ & \left. \left. - \partial_{\tau(\zeta)} \left[\frac{(z - \zeta)^{j+1}}{(\zeta - z)^{j+1}} \right] \phi(\zeta) \cdot g_{(a-1-j, m-a+j)}(\zeta) \right) d\sigma(\zeta) \right], \quad (2.554) \end{aligned}$$

at σ -a.e. point $z \in \partial\Omega$. Using the commutator, the expression in (2.554) can be identifies as

$$[M_\phi, C]g_{(a,b)}(z) - \sum_{j=0}^{a-1} [M_\phi, C'_j]g_{(a-1-j, m-a+j)}(z), \quad (2.555)$$

where for $0 \leq j \leq a-1$,

$$\begin{aligned} C'_j g_{(a-1-j, m-a+j)}(z) &:= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta \in \partial\Omega \\ |\zeta - z| > \varepsilon}} \frac{1}{j+1} \times \\ &\times \partial_{\tau(\zeta)} \left[\frac{(z-\zeta)^{j+1}}{(\zeta-z)^{j+1}} \right] g_{(a-1-j, m-a+j)}(\zeta) d\sigma(\zeta). \end{aligned} \quad (2.556)$$

Observe that the operators C and C'_j for $0 \leq j \leq a-1$ are Calderón-Zygmund type operators. In other words, the kernel of the operator is smooth, odd, and positive homogeneous of degree -1 except at 0. Applying the property of the commutator that for a continuous function ϕ on $\partial\Omega$, the commutator of a Calderón-Zygmund type operator CZ with ϕ is compact operator on $L^p(\partial\Omega, \sigma)$ for all $p \in (1, \infty)$, we obtain that

$$[M_\phi, C] - \sum_{j=0}^{a-1} [M_\phi, C'_j] \text{ is compact operator on } \bigoplus_{k=1}^{a+1} L^p(\partial\Omega, \sigma) \quad (2.557)$$

if $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is a UR domain with compact boundary. In conclusion, $[M_\phi, \dot{C}_{m-1}]$ is compact operator on $CWA_{m-1}[L^p(\partial\Omega, \sigma)]$ provided $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is a UR domain with compact boundary. This completes the proof. \square

Now, we turn our attention to the Riemann-Hilbert problems for polyanalytic functions. Let $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ be a bounded UR domain. Fix a natural number $m \in \mathbb{N}$ and an integrability exponent $p \in (1, \infty)$. Consider a function $\phi : \partial\Omega \rightarrow \mathbb{R}$ which is a restriction of some function ψ which is of class \mathcal{C}^{m-1} in a neighborhood of $\partial\Omega$ and does not vanish on the boundary, that is,

$$\begin{aligned} \psi &\in \mathcal{C}^{m-1} \text{ in a neighborhood of } \partial\Omega, \\ \psi(z) &\neq 0 \quad \forall z \in \partial\Omega, \\ \phi &= \psi \Big|_{\partial\Omega}. \end{aligned} \quad (2.558)$$

Consider a complex Whitney array $\dot{g} = \{g_{(a,b)}\}_{\substack{a,b \in \mathbb{N}_0 \\ a+b \leq m-1}} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. The Riemann-Hilbert problem for polyanalytic function takes the form

$$\left\{ \begin{array}{l} \bar{\partial}^m u^\pm = 0 \text{ in } \Omega^\pm, \\ u^- \text{ radiates at } \infty, \\ \mathcal{N}_\kappa(\nabla^\ell u^\pm) \in L^p(\partial\Omega, \sigma), \text{ for } \ell \in \{0, 1, \dots, m-1\}, \\ \phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+) - \text{Tr}_{m-1}^{\mathbb{C}}(u^-) = 0, \end{array} \right. \quad (2.559)$$

where $\Omega^+ := \Omega$ and $\Omega^- := \mathbb{R}^2 \setminus \bar{\Omega}$. According to the Fatou type result and the integral representation formula in Theorem 2.7, one has

$$u^\pm = \mathcal{E}_{m-1}(\text{Tr}_{m-1}^{\mathbb{C}}(u^\pm)) \text{ in } \Omega. \quad (2.560)$$

According to the jump relation in (2.307), one has

$$\text{Tr}_{m-1}^{\mathbb{C}}(u^\pm) = \left(\frac{1}{2}I \pm \dot{C}_{m-1} \right) \text{Tr}_{m-1}^{\mathbb{C}}(u^\pm) \text{ at } \sigma\text{-a.e. points on } \partial\Omega. \quad (2.561)$$

This yields

$$\frac{1}{2}\text{Tr}_{m-1}^{\mathbb{C}}(u^+) = \dot{C}_{m-1}(\text{Tr}_{m-1}^{\mathbb{C}}(u^+)). \quad (2.562)$$

In addition, combining (2.318) with (2.561) provides

$$\left(\frac{1}{2}I + \dot{C}_{m-1} \right) \text{Tr}_{m-1}^{\mathbb{C}}(u^-) = 0, \quad (2.563)$$

at σ -a.e. points on $\partial\Omega$. Recall the boundary condition in the Riemann-Hilbert problem for polyanalytic function

$$\phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+) - \text{Tr}_{m-1}^{\mathbb{C}}(u^-) = 0. \quad (2.564)$$

This implies

$$\text{Tr}_{m-1}^{\mathbb{C}}(u^-) = \phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+). \quad (2.565)$$

According to (2.560) and (2.565), one can conclude that finding u^+ suffices to solve the problem (2.559). Taking the operator $(1/2)I + \dot{C}_{m-1}$ to the equation (2.565) forces

$$\left(\frac{1}{2}I + \dot{C}_{m-1} \right) \text{Tr}_{m-1}^{\mathbb{C}}(u^-) = \left(\frac{1}{2}I + \dot{C}_{m-1} \right) (\phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+)). \quad (2.566)$$

Substituting (2.563) into (2.566), we have

$$0 = \left(\frac{1}{2}I + \dot{C}_{m-1} \right) (\phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+)). \quad (2.567)$$

In other words,

$$0 = \frac{1}{2}\phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+) + \dot{C}_{m-1} (\phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+)). \quad (2.568)$$

Using the commutator as in (2.541), this can be rewritten as

$$0 = \frac{1}{2}\phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+) + \phi \odot \left(\dot{C}_{m-1} (\text{Tr}_{m-1}^{\mathbb{C}}(u^+)) \right) - [M_\phi, \dot{C}_{m-1}] \text{Tr}_{m-1}^{\mathbb{C}}(u^+). \quad (2.569)$$

Applying (2.562), one can conclude that

$$0 = \phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+) - [M_\phi, \dot{C}_{m-1}] \text{Tr}_{m-1}^{\mathbb{C}}(u^+). \quad (2.570)$$

Consequently,

$$\text{Tr}_{m-1}^{\mathbb{C}}(u^+) \in \ker \left[\odot_\phi - [M_\phi, \dot{C}_{m-1}] \right], \quad (2.571)$$

where \odot_ϕ is defined as in (2.529). Since we are assuming that $\psi(z) \neq 0$ for any point $z \in \partial\Omega$, according to Corollary 2.58, \odot_ϕ is invertible on the complex Whitney array space $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. In particular, \odot_ϕ is Fredholm operator of index 0 on $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. In addition, according to the proposition 2.59, one has $[M_\phi, \dot{C}_{m-1}]$ is compact operator on $\text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$. Consequently,

$$\odot_\phi - [M_\phi, \dot{C}_{m-1}] \text{ is Fredholm of index 0 on } \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]. \quad (2.572)$$

Applying Fredholm alternative, we obtain either that the Riemann-Hilbert problem for polyanalytic function with zero boundary data

$$\left\{ \begin{array}{l} \bar{\partial}^m u^\pm = 0 \text{ in } \Omega^\pm, \\ u^- \text{ radiates at } \infty, \\ \mathcal{N}_\kappa(\nabla^\ell u^\pm) \in L^p(\partial\Omega, \sigma), \text{ for } \ell \in \{0, 1, \dots, m-1\}, \\ \phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+) - \text{Tr}_{m-1}^{\mathbb{C}}(u^-) = 0, \end{array} \right. \quad (2.573)$$

has nontrivial solutions $u^\pm \in \text{PA}_m(\Omega^\pm)$ or for each $\dot{f} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)]$ the Riemann-Hilbert problem for polyanalytic function with given data \dot{f}

$$\left\{ \begin{array}{l} \bar{\partial}^m u^\pm = 0 \text{ in } \Omega^\pm, \\ u^- \text{ radiates at } \infty, \\ \mathcal{N}_\kappa(\nabla^\ell u^\pm) \in L^p(\partial\Omega, \sigma), \text{ for } \ell \in \{0, 1, \dots, m-1\}, \\ \phi \odot \text{Tr}_{m-1}^{\mathbb{C}}(u^+) - \text{Tr}_{m-1}^{\mathbb{C}}(u^-) = \dot{f} \in \text{CWA}_{m-1}[L^p(\partial\Omega, \sigma)], \end{array} \right. \quad (2.574)$$

has uniquely determined solutions $u^\pm \in \text{PA}_m(\Omega^\pm)$.

CHAPTER 3

The Neumann Problem for the bi-Laplacian in Infinite Sectors

In this chapter, we study the Neumann problem for the bi-Laplacian in infinite sectors in the plane. More precisely, we investigate the solvability of the L^p Neumann problem for the bi-Laplacian, for $p \in (1, \infty)$, using Mellin transform techniques. We explicitly describe the Mellin symbol of a singular integral operator naturally associated with the problem for arbitrary apertures $\theta \in (0, 2\pi)$ and Poisson ratios $\eta \in [-1, 1)$ and investigate when its determinant doesn't vanish. The analysis carried out here uses Mellin transform techniques and properties of hypergeometric functions of gamma, beta, Ferrers and Gauss type. As a result of this analysis, we derive information on the range of $p \in (1, \infty)$ for the aforementioned boundary value problem is well-posed for a number of more tractable apertures.

3.1 Introduction

The focus of this chapter is on the Neumann problem for the bi-Laplacian in infinite sectors in \mathbb{R}^2 via the layer potential method. To set the stage, fix $\theta \in (0, 2\pi)$ and let $\Omega \subseteq \mathbb{R}^2$ be the region above the graph of the function

$\phi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\phi(x) := |x| \cot(\theta/2), \quad x \in \mathbb{R}. \quad (3.1)$$

For each $X \in \partial\Omega \setminus \{0\}$ there exist outward unit normal and unit tangential vectors denoted by $\nu(X)$ and $\tau(X)$ respectively. Then the Neumann problem for the bi-Laplacian in Ω with L^p boundary datum, with integrability index $p \in (1, \infty)$ and aperture parameter $\kappa \in (0, \infty)$, can be formulated as:

$$(NBH_p) \left\{ \begin{array}{l} \Delta^2 u = 0 \quad \text{in } \Omega, \\ \mathcal{N}_\kappa(\nabla^2 u) \in L^p(\partial\Omega), \\ \eta \Delta u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} + (1 - \eta) \frac{\partial^2 u}{\partial \nu^2} = f \in L^p(\partial\Omega), \\ \frac{\partial \Delta u}{\partial \nu} + (1 - \eta) \frac{\partial}{\partial \tau} \left[\frac{\partial^2 u}{\partial \tau \partial \nu} \right] = \Lambda \in (\dot{L}_1^{p'}(\partial\Omega))^*, \end{array} \right. \quad (3.2)$$

where $1 < p' < \infty$ is such that $1/p + 1/p' = 1$. Here, with \mathcal{H}^1 standing for the 1-dimensional Hausdorff measure in \mathbb{R}^2 , set $\sigma := \mathcal{H}^1 \llcorner \partial\Omega$ for the surface measure on $\partial\Omega$. Then the space $L^p(\partial\Omega)$ is that of Lebesgue p -integrable functions on $\partial\Omega$ with respect to σ and $(\dot{L}_1^{p'}(\partial\Omega))^*$ is the dual space of all locally integrable functions on $\partial\Omega$ with tangential derivative in $L^{p'}(\partial\Omega)$. In addition, \mathcal{N}_κ denotes the non-tangential maximal function defined in (1.14) and by $\cdot \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}$ we have denoted the nontangential restriction to the boundary introduced in (1.15). Finally, in the formulation of the boundary conditions in (3.2) the convention for repeated normal and/or tangential derivatives as introduced in (3.33) has been used, while $\eta \in [-1, 1)$ is a given real constant called the Poisson ratio.

The formulation of the Neumann problem (3.2) follows from the Kirchhoff-Love theory of thin plates and the problem has been previously studied in the literature in a number of important cases. J. Giroire and J. C. Nédélec, see [9], considered the geometric setting of bounded smooth planar domains and a variational setting formulation, and their well-posedness approach relied on establishing coerciveness of the corresponding boundary bilinear form. C. Nazaret, see [33], extended their results (in the variational setting) to bounded

polygonal domains. The case when $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is a bounded Lipschitz domain and the integrability index p is near 2 has been treated by G. Verchota in [40], employing layer potential theory and Rellich-type identities. Shortly thereafter, Z. Shen proved in [36] L^p versions of Verchota's results when Ω is a bounded Lipschitz domain in \mathbb{R}^n , $n \geq 4$, and $\frac{2(n-1)}{n+1} - \varepsilon < p < 2$, for some $\varepsilon > 0$ depending on Ω . A different formulation of the Neumann problem for the bi-Laplacian, with boundary datum from duals of Whitney-Lebesgue spaces, in bounded Lipschitz domains with connected boundary can be found in [[23], Theorem 6.55, p.388]. There the authors establish well-posedness whenever $p \in (2 - \varepsilon, \infty)$ if $n \in \{2, 3\}$ and $p \in (2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon)$ if $n \geq 4$, where again $\varepsilon > 0$ depends on Ω .

To put the formulation (3.2) of the Neumann problem for Δ^2 into perspective, in connection with the study of the problem (3.2), for each $r \in \mathbb{R}$, consider the bilinear form \mathcal{B}_r associated to Δ^2 in \mathbb{R}^2 given by

$$\mathcal{B}_r(u, v) := c_r \sum_{i,j=1}^2 \int_{\Omega} [(\partial_i \partial_j + r \delta_{ij} \Delta)u](X) [(\partial_i \partial_j + r \delta_{ij} \Delta)v](X) dX, \quad (3.3)$$

$$c_r := \frac{1}{1 + 2r + 2r^2},$$

where $u, v \in W^{2,2}(\Omega)$ (the L^2 based Sobolev space with smoothness index 2) and δ_{ij} stands for the standard Kronecker symbol. It can be checked that, indeed,

$$\mathcal{B}_r(u, v) = \int_{\Omega} (\Delta^2 u)(X) v(X) dX, \quad \text{for all } u, v \in \mathcal{C}_0^\infty(\Omega) \text{ and all } r \in \mathbb{R}. \quad (3.4)$$

If u is a biharmonic function in Ω such that $\mathcal{N}_\kappa(\nabla^2 u), \mathcal{N}_\kappa(\nabla^3 u) \in L^2(\partial\Omega)$, and $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, integrating by parts one obtains

$$\int_{\partial\Omega} \mathcal{B}_r(u, v) d\sigma = \int_{\partial\Omega} \left(M_r(u) \frac{\partial v}{\partial \nu} - N_r(u) v \right) d\sigma, \quad (3.5)$$

where

$$N_r(u) := \frac{\partial(\Delta u)}{\partial \nu} + c_r \frac{\partial}{\partial \tau} \left[\frac{\partial^2 u}{\partial \tau \partial \nu} \right], \quad (3.6)$$

$$M_r(u) := c_r (2r + 2r^2) \Delta u \Big|_{\partial\Omega}^{\kappa\text{-n.t.}} + c_r \frac{\partial^2 u}{\partial \nu^2}.$$

Note that, setting $\eta := \frac{2(r+r^2)}{1+2r+2r^2} \in [-1, 1)$, the boundary conditions in (3.2) can be rephrased as $M_r(u) = f \in L^p(\partial\Omega)$, and $N_r(u) = \Lambda \in \left(\dot{L}_1^{p'}(\partial\Omega)\right)^*$, respectively.

In Section 3.2.1 we employ the layer potential method for the treatment of (3.2). Concretely, we show how seeking a solution of (3.2) as a single layer potential leads to the consideration of the invertibility properties of a boundary-to-boundary matrix-valued integral operator $T : (L^p(\partial\Omega))^2 \rightarrow (L^p(\partial\Omega))^2$ defined below. Specifically, for each $F, G \in L^p(\partial\Omega)$, set

$$T(F, G) := \begin{pmatrix} -\eta K_1 + (\eta - 1)K_2 & -\frac{1}{2}I - \eta K_3 + (\eta - 1)K_4 \\ -K_5 + (\eta - 1)K_6 - \frac{1}{2}I & K_7 + (\eta - 1)K_8 \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}, \quad (3.7)$$

where I stands for the identity operator, and, for $f \in L^p(\partial\Omega)$,

$$K_i f(P) := \int_{\partial\Omega} k_i(P, Q) f(Q) d\sigma(Q) \quad \text{for each } i \in \{1, \dots, 8\}, \quad (3.8)$$

with the kernels k_i , for $i \in \{1, \dots, 8\}$, defined on $\partial\Omega \times \partial\Omega \setminus \text{diag}(\partial\Omega)$ as follows

$$k_1(P, Q) = k_7(P, Q) := \frac{\partial}{\partial\tau(Q)} [E(P - Q)], \quad (3.9)$$

$$k_2(P, Q) := \frac{\partial^3}{\partial\nu^2(P)\partial\tau(Q)} [B(P - Q)], \quad (3.10)$$

$$k_3(P, Q) = k_5(P, Q) := \frac{\partial}{\partial\nu(Q)} [E(P - Q)], \quad (3.11)$$

$$k_4(P, Q) := \frac{\partial^3}{\partial\nu^2(P)\partial\nu(Q)} [B(P - Q)], \quad (3.12)$$

$$k_6(P, Q) := \frac{\partial^3}{\partial\tau(P)\partial\nu(P)\partial\tau(Q)} [B(P - Q)], \quad (3.13)$$

$$k_8(P, Q) := \frac{\partial^3}{\partial\tau(P)\partial\nu(P)\partial\nu(Q)} [B(P - Q)]. \quad (3.14)$$

Above, E is the classical radial fundamental solution for the Laplacian and B is the classical radial fundamental solution for the bi-Laplacian in the plane.

Specifically

$$E : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}, \quad E(X) := \frac{1}{2\pi} \log |X|, \quad \forall X \in \mathbb{R}^2 \setminus \{0\}, \quad (3.15)$$

and

$$B : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}, \quad B(X) := -\frac{1}{8\pi} |X|^2 (1 - \log |X|), \quad \forall X \in \mathbb{R}^2 \setminus \{0\}. \quad (3.16)$$

In particular, there holds that $\Delta E = \Delta^2 B = \delta$ in the sense of distributions in \mathbb{R}^2 , where δ is the Dirac delta distribution with mass at the origin, and that $\Delta B = E$.

A preliminary step in the analysis of the operator T from (3.7) is “transporting” this operator from one acting on functions defined on $\partial\Omega$ to an operator acting on functions defined on $\mathbb{R}_+ := [0, \infty)$. This is done by exploiting the fact that Ω is the region above the graph of the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ from (3.1), and as such $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, where

$$\partial\Omega_1 := \{s\vec{v}_1(\theta) : s \in \mathbb{R}_+\} \quad \text{and} \quad \partial\Omega_2 := \{s\vec{v}_2(\theta) : s \in \mathbb{R}_+\}, \quad (3.17)$$

with

$$\vec{v}_1(\theta) := \left(-\sin \frac{\theta}{2}, \cos \frac{\theta}{2}\right) \quad \text{and} \quad \vec{v}_2(\theta) := \left(\sin \frac{\theta}{2}, \cos \frac{\theta}{2}\right). \quad (3.18)$$

Note that the vectors

$$\vec{v}_3(\theta) := \left(-\cos \frac{\theta}{2}, -\sin \frac{\theta}{2}\right) \quad \text{and} \quad \vec{v}_4(\theta) := \left(\cos \frac{\theta}{2}, -\sin \frac{\theta}{2}\right) \quad \text{satisfy} \quad (3.19)$$

$$\vec{v}_1(\theta) \perp \vec{v}_3(\theta) \quad \text{and} \quad \vec{v}_2(\theta) \perp \vec{v}_4(\theta), \quad \text{and point outwardly to } \Omega.$$

The two pieces of the boundary $\partial\Omega_j$, $j = 1, 2$, are then identified with \mathbb{R}_+ via the mappings

$$\iota_j : \partial\Omega_j \longrightarrow \mathbb{R}_+, \quad \iota_j(P) := |P|, \quad \text{for each } P \in \partial\Omega_j. \quad (3.20)$$

As a result, for each $p \in (1, \infty)$ the space $L^p(\partial\Omega)$ is in turn identified with the product space $L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+)$ via $\mathcal{I} : L^p(\partial\Omega) \longrightarrow L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+)$ given by

$$\mathcal{I}(f) := \left(\begin{array}{c} f \Big|_{\partial\Omega_1} \circ \iota_1^{-1} \\ f \Big|_{\partial\Omega_2} \circ \iota_2^{-1} \end{array} \right), \quad \forall f \in L^p(\partial\Omega). \quad (3.21)$$

This and explicit trigonometrical calculations lead us to consider the operator

$$\mathcal{T} : \left(L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+) \right)^2 \longrightarrow \left(L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+) \right)^2,$$

$$\mathcal{T} \begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{pmatrix} (s) = \int_0^\infty R(s, t) \cdot \begin{pmatrix} f_1(t) \\ f_2(t) \\ g_1(t) \\ g_2(t) \end{pmatrix} dt, \quad (3.22)$$

where the kernel $R : \mathbb{R}_+ \times \mathbb{R}_+ \setminus \text{diag}(\mathbb{R}_+) \longrightarrow M_{4 \times 4}(\mathbb{R})$ is given by

$$R := \begin{pmatrix} -\eta k_1 + (\eta - 1)k_2 & -\frac{1}{2}I_{2 \times 2} - \eta k_3 + (\eta - 1)k_4 \\ -k_5 + (\eta - 1)k_6 - \frac{1}{2}I_{2 \times 2} & k_7 + (\eta - 1)k_8 \end{pmatrix}, \quad (3.23)$$

with the matrix-valued functions k_i , $i \in \{1, \dots, 8\}$, explicitly described in (3.93) and Lemma 3.13 in Section 3.3. Since the operator \mathcal{T} is merely a re-interpretation of the operator T , there holds

$$T : \left(L^p(\partial\Omega) \right)^2 \longrightarrow \left(L^p(\partial\Omega) \right)^2 \text{ is invertible if and only if} \quad (3.24)$$

$$\mathcal{T} : \left(L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+) \right)^2 \longrightarrow \left(L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+) \right)^2 \text{ is invertible.}$$

A further analysis of \mathcal{T} reveals that this operator belongs to the algebra of integral operators generated by Hardy kernel operators and the Hilbert transform, making the analysis of its invertibility properties amenable to Mellin transform techniques. This is discussed in subsection 3.2.2, where we record the invertibility criterion in Theorem 3.10. Keeping (3.24) in mind, Theorem 3.10 brings into focus the necessity of computing the determinant of the Mellin symbol of the operator R , i.e., $\det(\mathcal{M}R(\cdot, 1))(z)$ for $z \in \Gamma_{0,1}$ (an open vertical strip in \mathbb{C} defined as in (3.62)). This symbol is explicitly derived in Section 3.4 and the calculations leading to the formula for $\det(\mathcal{M}R(\cdot, 1))(z)$ obtained in Theorem 3.27 involve hypergeometric functions of gamma, beta, and Ferrers types, as well as Gauss' hypergeometric functions. In Theorem 3.27 we establish explicit formulas for $\det(\mathcal{M}R(\cdot, 1))(z)$ for *all* aperture values

$\theta \in (0, 2\pi)$. In addition, we show that $\det(\mathcal{M}R(\cdot, 1))(z)$ is the product of two simpler expressions $(\Phi_\theta(z, \eta) + \Psi_\theta(z, \eta))$ and $(\Phi_\theta(z, \eta) - \Psi_\theta(z, \eta))$, where $\Phi_\theta(z, \eta)$ and $\Psi_\theta(z, \eta)$ are as in (3.186) and (3.187). Generally speaking, finding the zeroes of $\det(\mathcal{M}R(\cdot, 1))(z)$ is a very difficult task, thus obtaining a factorization of $\det(\mathcal{M}R(\cdot, 1))(z)$ helps with this analysis. Indeed, utilizing the results aforementioned, we have located the zeros of $\det(\mathcal{M}R(\cdot, 1))(z) = 0$ in $\Gamma_{0,1}$ in a number of particular cases such as $\theta = \pi$, $\theta = \pi/2$, and $\theta = \pi/4$, ultimately obtaining solvability results for the problem (3.2) of the following types:

Theorem 3.1. *Let Ω be the upper half-plane. Then for $\eta \in \mathbb{R} \setminus \{-3, 1\}$ and for $p \in (1, \infty)$,*

$$(NBH_p) \text{ has a solution.} \quad (3.25)$$

Theorem 3.2. *Let Ω be the interior of an infinite upright sector of aperture $\frac{\pi}{2}$ in the plane. Then for $\eta \in [-1, 1)$ and for $p \in (1, \infty)$, the following implication holds*

$$p \in (1, \infty) \setminus \{\alpha, \beta\} \implies (NBH_p) \text{ has a solution,} \quad (3.26)$$

where

$$\alpha^{-1} \text{ is the unique real root of } z - 1 + \cos\left(\frac{3\pi z}{2}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.27)$$

$$\beta^{-1} \text{ is the unique real root of } z - 1 - \cos\left(\frac{3\pi z}{2}\right) = 0 \text{ in } \Gamma_{0,1}. \quad (3.28)$$

In particular $\alpha, \beta \in \mathbb{R}$ and $\alpha \in (9, 12)$ while $\beta \in (2, 3)$.

Validated numerics considerations show that the critical index α lies in (10.92, 10.94), while the critical index β lies in (2.194, 2.196).

Theorem 3.3. *Let Ω be the interior of an infinite upright sector of aperture $\frac{\pi}{4}$ in the plane. Then for $\eta = -1$ and for $p \in (1, \infty)$, the following implication holds*

$$p \in (1, \infty) \setminus \{\gamma, \delta\} \implies (NBH_p) \text{ has a solution,} \quad (3.29)$$

where

$$\gamma^{-1} \text{ is the unique real root of } z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.30)$$

$$\delta^{-1} \text{ is the unique real root of } z - 1 - \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \text{ in } \Gamma_{0,1}. \quad (3.31)$$

In particular, $\gamma, \delta \in \mathbb{R}$ and $\gamma \in (2.8, 3)$ while $\delta \in (2, 2.1)$.

Validated numerics allows us to show that the critical index $\gamma \in (2.92, 2.94)$ while the critical index $\delta \in (2.01, 2.03)$.

3.2 Preliminaries

In this section we introduce notation and record a number of useful results. Throughout, $\theta \in (0, 2\pi)$ is fixed and $\Omega \subseteq \mathbb{R}^2$ is the upper graph of the function ϕ from (3.1).

Regarding directional derivatives, if w is a differentiable function in Ω and if $\langle \cdot, \cdot \rangle$ is the canonical inner product in \mathbb{R}^n , we have set

$$\frac{\partial w}{\partial \nu} := \left\langle (\nabla w) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}, \nu \right\rangle, \quad (3.32)$$

with ν being the outward unit normal vector to Ω . In addition, in (3.2) and (3.6) the following convention for repeated normal and/or tangential derivatives has been used. If u is a sufficiently smooth function in Ω , given a collection of vectors $V := \{v_1, \dots, v_m\} \subset \mathbb{R}^n$ on $\partial\Omega$ such that $v_j = (v_j^1, v_j^2, \dots, v_j^n)$ for each $j \in \{1, \dots, m\}$, and $f(V)$ is an expression depending solely on v_1, \dots, v_m , then for all $j \in \{1, \dots, m\}$ we set

$$\frac{\partial}{\partial v_j} [f(V)u] := \sum_{k=1}^n v_j^k f(V) (\partial_k u) \Big|_{\partial\Omega}^{\kappa\text{-n.t.}}. \quad (3.33)$$

Thus, the iterated normal and tangential derivatives appearing in the formulation of the Neumann problem (3.2) take the form

$$\frac{\partial^2 u}{\partial \nu^2} = \sum_{i,j=1}^2 \nu_i \nu_j (\partial_i \partial_j u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}}, \quad (3.34)$$

$$\frac{\partial \Delta u}{\partial \nu} = \nu_1 (\partial_1^3 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} + \nu_1 (\partial_1 \partial_2^2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} + \nu_2 (\partial_1^2 \partial_2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} + \nu_2 (\partial_2^3 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}}, \quad (3.35)$$

$$\frac{\partial^2 u}{\partial \tau^2} = \nu_2^2 (\partial_1^2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} - 2\nu_1 \nu_2 (\partial_1 \partial_2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} + \nu_1^2 (\partial_2^2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}}, \quad (3.36)$$

$$\frac{\partial^2 u}{\partial \tau \partial \nu} = -\nu_1 \nu_2 (\partial_1^2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} + (\nu_1^2 - \nu_2^2) (\partial_1 \partial_2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} + \nu_1 \nu_2 (\partial_2^2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}}, \quad (3.37)$$

$$\begin{aligned} \frac{\partial^3 u}{\partial \tau^2 \partial \nu} &= \nu_1 \nu_2^2 (\partial_1^3 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} + (\nu_2^3 - \nu_1^2 \nu_2) (\partial_1^2 \partial_2 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} \\ &\quad + (\nu_1^3 - \nu_2^2 \nu_1) (\partial_2^2 \partial_1 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}} + \nu_1^2 \nu_2 (\partial_2^3 u) \Big|_{\partial \Omega}^{\kappa\text{-n.t.}}. \end{aligned} \quad (3.38)$$

3.2.1 The Single Layer Potential for the Bi-Laplacian

Here we shall introduce the (modified) single layer potential operator associated with the bi-Laplacian and record some of its basic properties such as jump formulas and non-tangential maximal function estimates. To get started, recall B , the fundamental solution of the bi-Laplacian in \mathbb{R}^2 from (3.16). Fix $X_o \in \Omega$ and for each $Y \in \partial \Omega$ consider the following affine (in the variable $X \in \mathbb{R}^2 \setminus \partial \Omega$) correction of B :

$$\mathcal{B}(X, Y) := B(X - Y) - [B(X_o - Y) + \langle (\nabla B)(X_o - Y), X_o - X \rangle]. \quad (3.39)$$

A double application of the Fundamental Theorem of Calculus yields

$$\mathcal{B}(X, Y) = \sum_{j,k=1}^2 (X - X_o)_j (X - X_o)_k \int_0^1 \int_0^1 t (\partial_j \partial_k B)(X_o + st(X - X_o) - Y) ds dt. \quad (3.40)$$

For each $p \in (1, \infty)$, consider the modified single layer potential operator \mathcal{S} acting on a pair $(\Lambda, F) \in \dot{L}_{-1}^p(\partial\Omega) \times L^p(\partial\Omega)$ given at each point $X \in \mathbb{R}^2 \setminus \partial\Omega$ by

$$\mathcal{S}(\Lambda, F)(X) := \langle \mathcal{B}(X, \cdot), \Lambda \rangle - \int_{\partial\Omega} \partial_{\nu(Q)}[\mathcal{B}(X, Q)]F(Q)d\sigma(Q), \quad (3.41)$$

where the two terms are understood in the following manner. Given a functional $\Lambda \in \dot{L}_{-1}^p(\partial\Omega) = (\dot{L}_1^{p'}(\partial\Omega))^*$ and $G \in L^p(\partial\Omega)$ such that $\partial_\tau G = \Lambda$, the pairing $\langle \mathcal{B}(X, \cdot), \Lambda \rangle$ is interpreted as

$$\begin{aligned} \langle \mathcal{B}(X, \cdot), \Lambda \rangle := & - \int_{\partial\Omega} \sum_{j,k=1}^2 (X - X_o)_j (X - X_o)_k G(Y) \\ & \times \left(\int_0^1 \int_0^1 t \partial_{\tau(Y)} [(\partial_j \partial_k B)(X_o + st(X - X_o) - Y)] ds dt \right) d\sigma(Y). \end{aligned} \quad (3.42)$$

Note that in (3.42) the kernel $\nabla^3 B \in L^{p'}(\partial\Omega)$, with $1/p + 1/p' = 1$, in light of (3.80)-(3.81). Also, using (3.40) and (3.80)-(3.81), the integrand in the second term in the right hand side of (3.41) is absolutely integrable. This makes the formula of \mathcal{S} in (3.41) meaningful. In addition, it is straightforward to check that there holds

$$\Delta^2(\mathcal{S}(\Lambda, F)) = 0 \quad \text{on } \mathbb{R}^2 \setminus \partial\Omega, \quad (3.43)$$

and, using [40, Lemma 8.4, p. 241], there holds $\mathcal{N}_\kappa(\nabla^2(\mathcal{S}(\Lambda, F))) \in L^p(\partial\Omega)$.

Next, for $r \in \mathbb{R}$ set $\eta := \frac{2(r + r^2)}{1 + 2r + 2r^2}$ and recall the operators M_r, N_r from (3.6). Then the following jump-formulas hold for σ -a.e. $P \in \partial\Omega$ (see [40, Section 9]):

$$\begin{aligned} M_r(\mathcal{S}(\Lambda, F))(P) = & -\frac{1}{2}F(P) - \eta \int_{\partial\Omega} \frac{\partial}{\partial\nu(Q)}[E(P - Q)]F(Q) d\sigma(Q) \\ & + (\eta - 1) \int_{\partial\Omega} \frac{\partial^3}{\partial\nu^2(P)\partial\nu(Q)}[B(P - Q)]F(Q) d\sigma(Q) \\ & + \eta \langle E(P - \cdot), \Lambda \rangle + (1 - \eta) \left\langle \frac{\partial^2}{\partial\nu^2(P)}[B(P - \cdot)], \Lambda \right\rangle, \end{aligned} \quad (3.44)$$

and

$$\begin{aligned}
& N_r(\mathcal{S}(\Lambda, F))(P) \tag{3.45} \\
&= -\frac{1}{2}\Lambda(P) - \int_{\partial\Omega} \frac{\partial^2}{\partial\nu(P)\partial\nu(Q)} [E(P-Q)]F(Q) d\sigma(Q) \\
&\quad - (1-\eta) \int_{\partial\Omega} \frac{\partial^4}{\partial\tau^2(P)\partial\nu(P)\partial\nu(Q)} [B(P-Q)]F(Q) d\sigma(Q) \\
&\quad + \langle \partial_{\nu(P)}[E(P-\cdot)], \Lambda \rangle + (1-\eta) \left\langle \frac{\partial^3}{\partial\tau^2(P)\partial\nu(P)} [B(P-\cdot)], \Lambda \right\rangle.
\end{aligned}$$

Next, for each $\Lambda \in \dot{L}_{-1}^p(\partial\Omega)$ let

$$G \in L^p(\partial\Omega) \text{ be such that } \Lambda = \partial_\tau G. \tag{3.46}$$

Using this, integrating by parts in (3.44) and in (3.45), and the properties of the duality pairing between $\dot{L}_{-1}^p(\partial\Omega)$ and $\dot{L}_1^{p'}(\partial\Omega)$, we obtain that for σ -a.e. $P \in \partial\Omega$ there holds

$$\begin{aligned}
M_r(\mathcal{S}(\Lambda, F))(P) &= -\frac{1}{2}F(P) - \eta \int_{\partial\Omega} \frac{\partial}{\partial\nu(Q)} [E(P-Q)]F(Q) d\sigma(Q) \tag{3.47} \\
&\quad + (\eta-1) \int_{\partial\Omega} \frac{\partial^3}{\partial\nu^2(P)\partial\nu(Q)} [B(P-Q)]F(Q) d\sigma(Q) \\
&\quad - \eta \int_{\partial\Omega} \frac{\partial}{\partial\tau(Q)} [E(P-Q)]G(Q) d\sigma(Q) \\
&\quad + (\eta-1) \int_{\partial\Omega} \frac{\partial^3}{\partial\tau(Q)\partial\nu^2(P)} [B(P-Q)]G(Q) d\sigma(Q),
\end{aligned}$$

and

$$\begin{aligned}
N_\tau(\mathcal{S}(\Lambda, F))(P) &= -\frac{1}{2}\partial_\tau G(P) - \int_{\partial\Omega} \frac{\partial^2}{\partial\nu(P)\partial\nu(Q)} [E(P-Q)]F(Q) d\sigma(Q) \\
&\quad + (\eta-1) \int_{\partial\Omega} \frac{\partial^4}{\partial\tau^2(P)\partial\nu(P)\partial\nu(Q)} [B(P-Q)]F(Q) d\sigma(Q) \\
&\quad - \int_{\partial\Omega} \frac{\partial^2}{\partial\tau(Q)\partial\nu(P)} [E(P-Q)]G(Q) d\sigma(Q) \tag{3.48} \\
&\quad + (\eta-1) \int_{\partial\Omega} \frac{\partial^4}{\partial\tau(Q)\partial\tau^2(P)\partial\nu(P)} [B(P-Q)]G(Q) d\sigma(Q).
\end{aligned}$$

At this point let us observe that a direct calculation based on (3.33) shows that for σ -a.e. points $P, Q \in \partial\Omega$ such that $P \neq Q$ there holds

$$\frac{\partial^2}{\partial\nu(P)\partial\tau(Q)} [E(P-Q)] = \frac{\partial^2}{\partial\tau(P)\partial\nu(Q)} [E(P-Q)], \tag{3.49}$$

$$\frac{\partial^2}{\partial\nu(P)\partial\nu(Q)} [E(P-Q)] = -\frac{\partial^2}{\partial\tau(P)\partial\tau(Q)} [E(P-Q)]. \tag{3.50}$$

Thus

$$\begin{aligned}
&\int_{\partial\Omega} \frac{\partial^2}{\partial\nu(P)\partial\tau(Q)} [E(P-Q)]G(Q) d\sigma(Q) \tag{3.51} \\
&= \partial_{\tau(P)} \int_{\partial\Omega} \partial_{\nu(Q)} [E(P-Q)]G(Q) d\sigma(Q),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\partial\Omega} \frac{\partial^2}{\partial\nu(P)\partial\nu(Q)} [E(P-Q)]F(Q) d\sigma(Q) \tag{3.52} \\
&= -\partial_{\tau(P)} \int_{\partial\Omega} \partial_{\tau(Q)} [E(P-Q)]F(Q) d\sigma(Q).
\end{aligned}$$

Recall next the kernels (3.9)-(3.14). In this notation, formula (3.47) be-

comes

$$\begin{aligned} M_r(\mathcal{S}(\Lambda, F))(P) = & -\frac{1}{2}F(P) + (\eta - 1) \int_{\partial\Omega} (k_2(P, \cdot)G + k_4(P, \cdot)F) d\sigma \quad (3.53) \\ & - \eta \int_{\partial\Omega} (k_1(P, \cdot)G + k_3(P, \cdot)F) d\sigma, \end{aligned}$$

while (3.48) can be written as

$$\begin{aligned} N_r(\mathcal{S}(\Lambda, F))(P) & \quad (3.54) \\ = & -\frac{1}{2}\partial_{\tau(P)}G(P) + (\eta - 1)\partial_{\tau(P)}\left(\int_{\partial\Omega} (k_6(P, \cdot)G + k_8(P, \cdot)F) d\sigma\right) \\ & - \partial_{\tau(P)}\left(\int_{\partial\Omega} (k_5(P, \cdot)G + k_7(P, \cdot)F) d\sigma\right). \end{aligned}$$

Consequently, with I standing for the identity operator, there holds

$$\begin{aligned} \begin{pmatrix} M_r(\mathcal{S}(\Lambda, F)) \\ N_r(\mathcal{S}(\Lambda, F)) \end{pmatrix} = & \quad (3.55) \\ \begin{pmatrix} -\eta K_1 + (\eta - 1)K_2 & -\frac{1}{2}I - \eta K_3 + (\eta - 1)K_4 \\ \partial_{\tau}(-K_5 + (\eta - 1)K_6 - \frac{1}{2}I) & \partial_{\tau}(K_7 + (\eta - 1)K_8) \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}, \end{aligned}$$

where for a generic function $f \in L^p(\partial\Omega)$, and k_i , $i \in \{1, \dots, 8\}$, as in (3.9)-(3.14), we have set

$$K_i f(P) := \int_{\partial\Omega} k_i(P, Q) f(Q) d\sigma(Q) \quad \text{for each } i \in \{1, \dots, 8\}. \quad (3.56)$$

Recall the operator $T : (L^p(\partial\Omega))^2 \longrightarrow (L^p(\partial\Omega))^2$ from (3.7). Given a functional $\Lambda \in (\dot{L}_1^p(\partial\Omega))^* \equiv \dot{L}_{-1}^p(\partial\Omega)$, consider

$$G \in L^p(\partial\Omega) \quad \text{such that} \quad \partial_{\tau}G = \Lambda. \quad (3.57)$$

Then, the jump relations (3.55) can be re-written as

$$\begin{pmatrix} M_r(\mathcal{S}(\Lambda, F)) \\ N_r(\mathcal{S}(\Lambda, F)) \end{pmatrix} = T(F, G). \quad (3.58)$$

In light of this discussion, given $f \in L^p(\partial\Omega)$ and $\Lambda \in (\dot{L}_1^{p'}(\partial\Omega))^*$, seeking a solution of the Neumann problem (NBH_p) in (3.2) as a single layer potential operator, $\mathcal{S}(\tilde{\Lambda}, H)$, with $H \in L^p(\partial\Omega)$ and $\tilde{\Lambda} \in (\dot{L}_1^{p'}(\partial\Omega))^*$, is then reduced to finding a solution of the boundary integral equation

$$T(H, \tilde{G}) = (f, G), \quad (3.59)$$

where G is as in (3.57), and \tilde{G} is associated with $\tilde{\Lambda}$ also as in (3.57). Indeed, if $(H, \tilde{G}) \in L^p(\partial\Omega) \times L^p(\partial\Omega)$ is a solution of (3.59), then $u := \mathcal{S}(\partial_\tau \tilde{G}, H)$ solves the Neumann problem (NBH_p) in (3.2).

3.2.2 Hardy Kernel Operators and the Mellin Transform

The goal here is to recall the class of Hardy kernels and Hardy kernel operators on $L^p(\mathbb{R}_+)$, for $p \in [1, \infty)$, and to introduce the Mellin transform as a tool to study spectral properties of Hardy kernel operators.

Definition 3.4. *Let $k : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a Lebesgue measurable function. Then k is a Hardy kernel for $L^p(\mathbb{R}_+)$ for some $1 \leq p < \infty$ provided that:*

1. *k is a homogeneous function of degree -1 , i.e., for any $\lambda > 0$ and any $x, y \in \mathbb{R}_+$, there holds $k(\lambda x, \lambda y) = \lambda^{-1}k(x, y)$;*
2. $\int_0^\infty |k(1, y)|y^{-1/p}dy < \infty$.

Furthermore, if $\ell, m \in \mathbb{N}$, a matrix-valued function $k : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}^{\ell \times m}$ whose entries are measurable is called a Hardy kernel for $(L^p(\mathbb{R}_+))^m$ provided that each entry k_{ij} , for $i \in \{1, \dots, \ell\}$ and for $j \in \{1, \dots, m\}$, is a Hardy kernel for $L^p(\mathbb{R}_+)$.

With Definition 3.4 in hand, let k be a Hardy kernel for $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$ and for any $f \in L^p(\mathbb{R}_+)$ define the action of the Hardy kernel operator \mathfrak{T} with kernel k on the function f by setting

$$\mathfrak{T}f(x) := \int_0^\infty k(x, y)f(y)dy, \quad x \in \mathbb{R}_+. \quad (3.60)$$

The setup of the vector-valued case follows a similar blueprint. Concretely, fix two integers $\ell, m \in \mathbb{N}$ and let $k = (k_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be a Hardy kernel for $(L^p(\mathbb{R}_+))^m$. For any vector-valued function $\vec{f} \in (L^p(\mathbb{R}_+))^m$, define the action of the operator \mathfrak{T} , called a Hardy kernel operator with kernel k , on \vec{f} by setting

$$\mathfrak{T}\vec{f}(s) := \int_0^\infty k(s, t) \cdot \vec{f}(t) dt, \quad s \in \mathbb{R}_+, \quad (3.61)$$

where above \cdot denotes matrix multiplication.

Next we recall the Mellin transform. Given $b, c \in \mathbb{R}$ satisfying $b < c$, we shall denote by $\Gamma_{b,c}$ the infinite strip

$$\Gamma_{b,c} := \{z \in \mathbb{C} : \Re z \in (b, c)\}, \quad (3.62)$$

where \Re denotes real part.

Definition 3.5. Let $a, b, c \in \mathbb{R}$ such that $b < c$ and set

$$\mathcal{A}_a := \{f : \mathbb{R}_+ \rightarrow \mathbb{C} \text{ measurable function} : \int_0^\infty |f(x)|x^{a-1}dx < \infty\}, \quad (3.63)$$

and

$$\mathcal{A}_{(b,c)} := \bigcap_{a \in (b,c)} \mathcal{A}_a. \quad (3.64)$$

Definition 3.6. Assume that $a \in \mathbb{R}$ and that $f \in \mathcal{A}_a$. Set

$$\mathcal{M}f(z) := \int_0^\infty x^{z-1}f(x)dx \quad \text{for each } z \in a + i\mathbb{R}. \quad (3.65)$$

$\mathcal{M}f(z)$ is called the Mellin transform of the function f at z . The fundamental strip of $\mathcal{M}f$ is the largest strip $\Gamma_{b,c}$ with $b, c \in \mathbb{R}$, $b < c$, such that $f \in \mathcal{A}_{(b,c)}$.

Consider next the Hilbert transform of the positive semi-axis. Concretely, for a fixed $y \in \mathbb{R}_+$, set

$$h : \mathbb{R}_+ \setminus \{y\} \rightarrow \mathbb{R}, \quad h(x) := \frac{1}{x - y}. \quad (3.66)$$

Note that $h \notin \mathcal{A}_c$ for any $c \in \mathbb{R}$ because for a fixed $c \in \mathbb{R}$ and for $0 < \varepsilon < y$ there holds

$$\lim_{t \rightarrow y^-} \int_\varepsilon^t \frac{x^{c-1}}{x - y} dx = -\infty. \quad (3.67)$$

However $\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}_+ \setminus B(y, \varepsilon)} h(x)x^{z-1} dx$ exists whenever $z \in \Gamma_{0,1}$. In particular, since the function h does not fall under the scope of Definition 3.6, it is meaningful to set,

$$\mathcal{M}h(z) := \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}_+ \setminus B(y, \varepsilon)} h(x)x^{z-1} dx \quad \text{for each } z \in \Gamma_{0,1}. \quad (3.68)$$

In relation to this, the following result can be found in [34].

Proposition 3.7. *Fix $y \in \mathbb{R}_+$ and let h be as in (3.66). Then,*

$$\mathcal{M}h(z) = -\pi y^{z-1} \cot(\pi z) \quad \text{for each } z \in \Gamma_{0,1}. \quad (3.69)$$

The following result, see [5] and [8], allows one to explicitly determine the spectrum of bounded linear operators which belong to the algebra generated by Hardy kernel operators and the Hilbert transform on \mathbb{R}_+ .

Theorem 3.8. *Let $k = (k_{ij})_{1 \leq i, j \leq m}$ be a Hardy kernel for $(L^p(\mathbb{R}_+))^m$, for some $p \in (1, \infty)$ and $m \in \mathbb{N}$, and let A, B be $m \times m$ matrices with real entries. Consider the operator \mathcal{R} defined as*

$$\mathcal{R}f(s) := Af + \int_0^\infty \mathfrak{R}(s, t)f(t)dt, \quad s \in \mathbb{R}_+, \quad (3.70)$$

with

$$\mathfrak{R}(s, t) := k(s, t) + \frac{1}{s-t} B, \quad \forall s, t \in \mathbb{R}_+ \text{ such that } s \neq t. \quad (3.71)$$

Then the operator

$$\mathcal{R} : (L^p(\mathbb{R}_+))^m \rightarrow (L^p(\mathbb{R}_+))^m \text{ is well defined, linear and bounded} \quad (3.72)$$

with spectrum

$$\sigma(\mathcal{R}; (L^p(\mathbb{R}_+))^m) = \overline{\mathcal{O}}, \quad (3.73)$$

with $\overline{\mathcal{O}}$ standing for the closure in \mathbb{C} of the set \mathcal{O} , which is defined as

$$\mathcal{O} := \left\{ \omega \in \mathbb{C} : \det(\omega I - (A + \mathcal{M}\mathfrak{R}(\cdot, 1))(\frac{1}{p} + i\xi)) = 0, \text{ for some } \xi \in \mathbb{R} \right\}. \quad (3.74)$$

A useful consequence of Theorem 3.8 is singled out below (see e.g. [32] for a proof).

Corollary 3.9. *Retain the setting of Theorem 3.8 and make the additional assumption that $\det(A - \pi iB) \neq 0$. Then the operator \mathcal{R} is invertible on $(L^p(\mathbb{R}_+))^m$ if and only if*

$$\det(A + \mathcal{M}\mathfrak{R}(\cdot, 1)(\frac{1}{p} + i\xi)) \neq 0 \text{ for each } \xi \in \mathbb{R}. \quad (3.75)$$

Corollary 3.9 ultimately allows us to establish the following invertibility result for the operator \mathcal{T} from (3.22).

Theorem 3.10. *Let Ω be the interior of an infinite sector of aperture θ in $(0, 2\pi)$ and fix $\eta \in [-1, 1)$ and $p \in (1, \infty)$. Consider the operator \mathcal{T} from (3.22) with matrix-valued kernel R as in (3.23). Then*

$$\begin{aligned} \mathcal{T} \text{ is invertible on } (L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+))^2 \text{ if and only if} \\ \text{for each } \xi \in \mathbb{R} \text{ there holds } \det(\mathcal{M}R(\cdot, 1)(\frac{1}{p} + i\xi)) \neq 0. \end{aligned} \quad (3.76)$$

Proof. Using Lemma 3.13 the operator \mathcal{T} is of the form discussed in Theorem 3.8 for

$$B := \frac{\eta+1}{4\pi} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } A := -\frac{1}{2} \begin{pmatrix} O_{2 \times 2} & I_{2 \times 2} \\ I_{2 \times 2} & O_{2 \times 2} \end{pmatrix}, \quad (3.77)$$

where $O_{2 \times 2}$ is the 2×2 zero matrix and $I_{2 \times 2}$ is the 2×2 identity matrix (and A is understood as a 4×4 matrix). With (3.77) in hand, a simple calculation shows $\det(A - \pi iB) = \left(-\frac{(\eta+1)^2}{16} + \frac{1}{4}\right)^2 \neq 0$ for all $\eta \in [-1, 1)$. Thus Corollary 3.9 applies and gives (3.76). \square

3.3 Explicit Formulas for the Operator \mathcal{T}

A preliminary step in the analysis of the operator T from (3.7) is determining explicit expressions for the kernel functions k_i for $i \in \{1, \dots, 8\}$ from

(3.9)-(3.14), which then via the identifications (3.20) lead to a more explicit formula for the operator \mathcal{T} when Ω is the infinite sector above the graph of the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ in (3.1) of arbitrary aperture $\theta \in (0, 2\pi)$.

Using formulas (3.15)-(3.16), it is straightforward to compute at each point $X = (X_1, X_2) \in \mathbb{R}^2 \setminus \{0\}$ and for each $j \in \{1, 2\}$ that

$$\begin{aligned}\partial_j E(X) &= \frac{X_j}{2\pi|X|^2}, \\ \partial_j B(X) &= -\frac{X_j}{8\pi} (1 - \log(|X|^2)),\end{aligned}\tag{3.78}$$

$$\begin{aligned}\partial_1 \partial_2 B(X) &= \frac{X_1 X_2}{4\pi|X|^2}, \\ \partial_j^2 B(X) &= -\frac{1}{8\pi} \left(1 - \log(|X|^2) - \frac{2X_j^2}{|X|^2}\right),\end{aligned}\tag{3.79}$$

and that

$$\begin{aligned}\partial_1^2 \partial_2 B(X) &= \frac{X_2^3 - X_1^2 X_2}{4\pi|X|^4}, \\ \partial_1 \partial_2^2 B(X) &= \frac{X_1^3 - X_1 X_2^2}{4\pi|X|^4},\end{aligned}\tag{3.80}$$

$$\begin{aligned}\partial_1^3 B(X) &= \frac{X_1^3 + 3X_1 X_2^2}{4\pi|X|^4}, \\ \partial_2^3 B(X) &= \frac{X_2^3 + 3X_1^2 X_2}{4\pi|X|^4}.\end{aligned}\tag{3.81}$$

Then (3.33), (3.9)-(3.14) and (3.78)-(3.81) allow us to obtain the following formulas.

Lemma 3.11. *Assume $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain with outward unit normal vector $\nu = (\nu_1, \nu_2)$. For each $P = (P_1, P_2) \in \partial\Omega$ and $Q = (Q_1, Q_2) \in \partial\Omega$,*

$P \neq Q$, the functions k_i , $i \in \{1, \dots, 8\}$, introduced in (3.9)-(3.14) satisfy

$$k_1(P, Q) = k_7(P, Q) = \frac{\nu_2(Q)(P_1 - Q_1) - \nu_1(Q)(P_2 - Q_2)}{2\pi|P - Q|^2}, \quad (3.82)$$

$$k_2(P, Q) = (P_1 - Q_1)^3 \cdot \frac{\nu_2(Q) - 2\nu_1(Q)\nu_1(P)\nu_2(P)}{4\pi|P - Q|^4} \quad (3.83)$$

$$\begin{aligned} &+ (P_2 - Q_2)^3 \cdot \frac{-\nu_1(Q) + 2\nu_2(Q)\nu_1(P)\nu_2(P)}{4\pi|P - Q|^4} \\ &+ (P_1 - Q_1)(P_2 - Q_2)^2 \cdot \frac{\nu_2(Q)(3\nu_1^2(P) - \nu_2^2(P))}{4\pi|P - Q|^4} \\ &+ (P_1 - Q_1)^2(P_2 - Q_2) \cdot \frac{\nu_1(Q)(-3\nu_2^2(P) + \nu_1^2(P))}{4\pi|P - Q|^4} \\ &- \frac{(P_1 - Q_1)(P_2 - Q_2)\nu_1(P)\nu_2(P)}{|P - Q|^2} \cdot k_1(P, Q), \end{aligned}$$

$$k_3(P, Q) = k_5(P, Q) = -\frac{\langle P - Q, \nu(Q) \rangle}{2\pi|P - Q|^2}, \quad (3.84)$$

$$k_4(P, Q) = -(P_1 - Q_1)^3 \cdot \frac{\nu_1(Q) + 2\nu_1(P)\nu_2(P)\nu_2(Q)}{4\pi|P - Q|^4} \quad (3.85)$$

$$\begin{aligned} &- (P_2 - Q_2)^3 \cdot \frac{\nu_2(Q) + 2\nu_1(P)\nu_2(P)\nu_1(Q)}{4\pi|P - Q|^4} \\ &- (P_1 - Q_1)(P_2 - Q_2)^2 \cdot \frac{\nu_1(Q)(3\nu_1^2(P) - \nu_2^2(P))}{4\pi|P - Q|^4} \\ &- (P_1 - Q_1)^2(P_2 - Q_2) \cdot \frac{\nu_2(Q)(3\nu_2^2(P) - \nu_1^2(P))}{4\pi|P - Q|^4} \\ &- \frac{(P_1 - Q_1)(P_2 - Q_2)\nu_1(P)\nu_2(P)}{|P - Q|^2} \cdot k_3(P, Q), \end{aligned}$$

$$k_6(P, Q) := -(P_1 - Q_1)^3 \frac{(\nu_1^2(P) - \nu_2^2(P))\nu_1(Q)}{4\pi|P - Q|^4} \quad (3.86)$$

$$\begin{aligned} &+ \frac{2(P_1 - Q_1)(P_2 - Q_2)\nu_1(P)\nu_2(P)}{|P - Q|^2} \cdot k_3(P, Q) \\ &- \frac{(P_1 - Q_1)(P_2 - Q_2)(\nu_1^2(P) - \nu_2^2(P))}{2|P - Q|^2} \cdot k_1(P, Q) \\ &+ (P_2 - Q_2)^3 \frac{(\nu_1^2(P) - \nu_2^2(P))\nu_2(Q)}{4\pi|P - Q|^4}, \end{aligned}$$

and

$$\begin{aligned}
k_8(P, Q) &= -(P_1 - Q_1)^3 \frac{(\nu_1^2(P) - \nu_2^2(P))\nu_2(Q)}{4\pi|P - Q|^4} \\
&\quad - (P_2 - Q_2)^3 \frac{(\nu_1^2(P) - \nu_2^2(P))\nu_1(Q)}{4\pi|P - Q|^4} \\
&\quad - \frac{2(P_1 - Q_1)(P_2 - Q_2)\nu_1(P)\nu_2(P)}{|P - Q|^2} \cdot k_1(P, Q) \\
&\quad - \frac{(P_1 - Q_1)(P_2 - Q_2)(\nu_1^2(P) - \nu_2^2(P))}{2|P - Q|^2} \cdot k_3(P, Q).
\end{aligned} \tag{3.87}$$

The following remark will be useful in the sequel.

Remark 3.12. Let Ω be the set above the graph of the function ϕ from (3.1). With $(\partial\Omega)_1$ and $(\partial\Omega)_2$ being the left and right rays of $\partial\Omega$, respectively, with corresponding direction vectors $v_1(\theta)$ and $v_2(\theta)$ as in (3.17)-(3.18), pick two points $P, Q \in \partial\Omega = (\partial\Omega)_1 \cup (\partial\Omega)_2$ such that $P \neq Q$ and let $s := |P|$ and $t := |Q|$. Recall the outward unit normal vectors $\vec{v}_3(\theta)$ and $\vec{v}_4(\theta)$ as in (3.19). Then clearly:

$$\text{if } P, Q \in (\partial\Omega)_1 : P = s\vec{v}_1(\theta), Q = t\vec{v}_1(\theta) \text{ and } \nu(P) = \nu(Q) = \vec{v}_3(\theta), \tag{3.88}$$

$$\text{if } P, Q \in (\partial\Omega)_2 : P = s\vec{v}_2(\theta), Q = t\vec{v}_2(\theta) \text{ and } \nu(P) = \nu(Q) = \vec{v}_4(\theta), \tag{3.89}$$

$$\text{if } P \in (\partial\Omega)_1, Q \in (\partial\Omega)_2 : \begin{cases} P = s\vec{v}_1(\theta), \\ Q = t\vec{v}_2(\theta), \end{cases} \text{ and } \begin{cases} \nu(P) = \vec{v}_3(\theta), \\ \nu(Q) = \vec{v}_4(\theta), \end{cases} \tag{3.90}$$

$$\text{if } P \in (\partial\Omega)_2, Q \in (\partial\Omega)_1 : \begin{cases} P = s\vec{v}_2(\theta), \\ Q = t\vec{v}_1(\theta), \end{cases} \text{ and } \begin{cases} \nu(P) = \vec{v}_4(\theta), \\ \nu(Q) = \vec{v}_3(\theta). \end{cases} \tag{3.91}$$

Moving on, for each $j \in \{1, \dots, 8\}$ and each $l, m \in \{1, 2\}$, we shall introduce

$$k_j^{lm} : \mathbb{R}_+ \times \mathbb{R}_+ \setminus \text{diag}(\mathbb{R}_+) \longrightarrow \mathbb{R} \text{ given by } k_j^{lm}(s, t) := k_j(P, Q) \quad (3.92)$$

for $P \in (\partial\Omega)_l$ and $Q \in (\partial\Omega)_m$ such that $|P| = s$ and $|Q| = t$,

where the functions k_j , $j \in \{1, \dots, 8\}$, are as in (3.9)-(3.14). With this notation in hand, let us consider the matrix-valued functions k_j acting on $\mathbb{R}_+ \times \mathbb{R}_+ \setminus \text{diag}(\mathbb{R}_+)$ with values in $M_{2 \times 2}(\mathbb{R})$ given by

$$k_j := \begin{pmatrix} k_j^{11} & k_j^{12} \\ k_j^{21} & k_j^{22} \end{pmatrix}, \quad j \in \{1, \dots, 8\}. \quad (3.93)$$

A direct consequence (whose elementary proof we omit) of Lemma 3.11 and (3.88)-(3.91) in Remark 3.12 is the following.

Lemma 3.13. *Fix $\theta \in (0, 2\pi)$ and for each $j \in \{1, \dots, 8\}$ and $l, m \in \{1, 2\}$ let k_j^{lm} be as in (3.92) and k_j as in (3.93). Then, corresponding to $j = 1$ and $j = 2$, for each $s, t \in \mathbb{R}_+$ with $s \neq t$ there holds*

$$\begin{cases} k_1^{11}(s, t) = -k_1^{22}(s, t) = \frac{1}{2\pi(s-t)}, \\ -k_1^{12}(s, t) = k_1^{21}(s, t) = \frac{s \cos \theta - t}{2\pi(s^2 - 2st \cos \theta + t^2)}, \end{cases} \quad (3.94)$$

$$\begin{cases} k_2^{11}(s, t) = -k_2^{22}(s, t) = \frac{1}{4\pi(s-t)}, \\ -k_2^{12}(s, t) = k_2^{21}(s, t) = \frac{f(s, t)}{4\pi(s^2 - 2st \cos \theta + t^2)^2}, \end{cases} \quad (3.95)$$

where

$$\begin{aligned} f(s, t) := & -(s+t)^3(1+2\cos^2(\theta/2))\sin^4(\theta/2) + (s-t)^3(1+2\sin^2(\theta/2))\cos^4(\theta/2) \\ & - 2s(s+t)(s-t)\sin^2(\theta/2)\cos^2(\theta/2)\cos\theta. \end{aligned} \quad (3.96)$$

Corresponding to $j = 3$ and $j = 4$, for each $s, t \in \mathbb{R}_+$ with $s \neq t$ there holds

$$\begin{cases} k_3^{11}(s, t) = k_3^{22}(s, t) = 0, \\ k_3^{12}(s, t) = k_3^{21}(s, t) = \frac{s \sin \theta}{2\pi(s^2 - 2st \cos \theta + t^2)}, \end{cases} \quad (3.97)$$

$$\begin{cases} k_4^{11}(s, t) = k_4^{22}(s, t) = 0, \\ k_4^{12}(s, t) = k_4^{21}(s, t) = \frac{g(s, t)}{4\pi(s^2 - 2st \cos \theta + t^2)^2}, \end{cases} \quad (3.98)$$

where

$$\begin{aligned} g(s, t) := & (s+t)^3 \sin^3(\theta/2) \cos(\theta/2) \cos \theta - (s-t)^3 \sin(\theta/2) \cos^3(\theta/2) \cos \theta \\ & + (s+t)^2(s-t) \sin^2(\theta/2) \cos(\theta/2) [\sin(\theta/2) \cos^2(\theta/2) + 3 \sin^3(\theta/2)] \\ & + (s+t)(s-t)^2 \sin(\theta/2) \cos^2(\theta/2) [\sin^2(\theta/2) \cos(\theta/2) + 3 \cos^3(\theta/2)]. \end{aligned} \quad (3.99)$$

Corresponding to $j = 6$, for each $s, t \in \mathbb{R}_+$ with $s \neq t$ there holds

$$\begin{cases} k_6^{11}(s, t) = k_6^{22}(s, t) = 0, \\ k_6^{12}(s, t) = k_6^{21}(s, t) = \frac{h(s, t)}{4\pi(s^2 - 2st \cos \theta + t^2)^2}, \end{cases} \quad (3.100)$$

where

$$\begin{aligned} h(s, t) := & (s+t)^3 \sin^3(\theta/2) \cos(\theta/2) \cos \theta - (s-t)^3 \sin(\theta/2) \cos^3(\theta/2) \cos \theta \\ & - (s+t)^2(s-t) \sin^2(\theta/2) \cos(\theta/2) [3 \sin(\theta/2) \cos^2(\theta/2) + \sin^3(\theta/2)] \\ & - (s+t)(s-t)^2 \sin(\theta/2) \cos^2(\theta/2) [3 \sin^2(\theta/2) \cos(\theta/2) + \cos^3(\theta/2)]. \end{aligned} \quad (3.101)$$

Corresponding to $j = 8$, for each $s, t \in \mathbb{R}_+$ with $s \neq t$ there holds

$$\begin{cases} k_8^{11}(s, t) = -k_8^{22}(s, t) = \frac{1}{4\pi(s-t)}, \\ -k_8^{12}(s, t) = k_8^{21}(s, t) = \frac{\ell(s, t)}{4\pi(s^2 - 2st \cos \theta + t^2)^2}, \end{cases} \quad (3.102)$$

where

$$\begin{aligned} \ell(s, t) := & (s+t)^3 \sin^4(\theta/2) \cos \theta + (s-t)^3 \cos^4(\theta/2) \cos \theta \\ & - \frac{1}{2}s(s+t)(s-t) \sin^2 \theta \cos \theta \\ & - (s+t)^2(s-t) \sin^2 \theta \sin^2(\theta/2) + (s+t)(s-t)^2 \sin^2 \theta \cos^2(\theta/2). \end{aligned} \quad (3.103)$$

Corresponding to $j = 5$ and $j = 7$, for each $s, t \in \mathbb{R}_+$ with $s \neq t$ there holds

$$k_5(s, t) = k_3(s, t) \quad \text{and} \quad k_7(s, t) = k_1(s, t). \quad (3.104)$$

Moreover, for each $j \in \{1, \dots, 8\}$ and each $l, m \in \{1, 2\}$, the function k_j^{lm} is either a Hardy kernel or a scalar multiple of the kernel of the Hilbert transform.

3.4 Mellin Symbol Computations

The main goal of this section is to compute the Mellin symbol of the matrix-valued functions k_i , $i \in \{1, \dots, 8\}$, introduced in (3.92)-(3.93), and explicitly calculated in Lemma 3.13.

Proposition 3.14. *Fix a parameter $\theta \in (0, 2\pi)$ and consider the functions $F_j : \mathbb{R}_+ \rightarrow \mathbb{R}$, $j \in \{1, \dots, 4\}$, given at $x \in \mathbb{R}_+$ by*

$$F_1(x) := \frac{1}{x^2 + 2x \cos(\pi - \theta) + 1}, \quad (3.105)$$

$$F_2(x) := xF_1(x), \quad F_3(x) := F_1'(x) \quad \text{and} \quad F_4(x) := F_2'(x).$$

Then,

$$F_1 \in \mathcal{A}_{(0,2)}, \quad F_2 \in \mathcal{A}_{(-1,1)}, \quad F_3 \in \mathcal{A}_{(1,3)} \quad \text{and} \quad F_4 \in \mathcal{A}_{(0,2)}, \quad (3.106)$$

and, if $\theta \in (0, 2\pi) \setminus \{\pi\}$,

$$\mathcal{M}F_1(z) = \pi \csc \theta \csc(\pi z) \sin(\theta + z(\pi - \theta)), \quad \text{for each } z \in \Gamma_{0,2}, \quad (3.107)$$

$$\mathcal{M}F_2(z) = \pi \csc \theta \csc(\pi z) \sin(z(\pi - \theta)), \quad \text{for each } z \in \Gamma_{-1,1}, \quad (3.108)$$

$$\mathcal{M}F_3(z) = -\frac{\pi(z-1) \csc \theta}{\sin(\pi z)} \sin(2\theta + z(\pi - \theta)), \quad \text{for each } z \in \Gamma_{1,3}, \quad (3.109)$$

$$\mathcal{M}F_4(z) = -\frac{\pi(z-1) \csc \theta}{\sin(\pi z)} \sin(z(\pi - \theta) + \theta), \quad \text{for each } z \in \Gamma_{0,2}. \quad (3.110)$$

When $\theta = \pi$, the following hold

$$\mathcal{M}F_1(z) = -\frac{\pi(z-1)}{\sin(\pi z)}, \text{ for each } z \in \Gamma_{0,2}, \quad (3.111)$$

$$\mathcal{M}F_2(z) = \frac{\pi z}{\sin(\pi z)}, \text{ for each } z \in \Gamma_{-1,1}, \quad (3.112)$$

$$\mathcal{M}F_3(z) = -\frac{\pi(z-2)}{\sin(\pi z)}, \text{ for each } z \in \Gamma_{1,3}, \quad (3.113)$$

$$\mathcal{M}F_4(z) = -\frac{\pi(z-1)}{\sin(\pi z)}, \text{ for each } z \in \Gamma_{0,2}. \quad (3.114)$$

Proof. When $\theta \neq \pi$, that $F_1 \in \mathcal{A}_{(0,2)}$ and that (3.107) holds are consequences of [34, formula (2.54), p. 23] applied for $\pi - \theta \in (-\pi, \pi) \setminus \{0\}$ in place of θ . The case (3.111) when $\theta = \pi$, follows from [34, formula (2.5), p. 13]. Regarding F_2 , since $F_2(x) = xF_1(x)$ and $F_1 \in \mathcal{A}_{(0,2)}$, basic properties of the Mellin transform yield that $F_2 \in \mathcal{A}_{(-1,1)}$ and that

$$\mathcal{M}F_2(z) = \mathcal{M}F_1(z+1) \text{ for each } z \in \Gamma_{-1,1}. \quad (3.115)$$

This together with (3.107), (3.111) and the fact that $\sin(\pi(z+1)) = -\sin(\pi z)$ gives (3.108) and (3.112).

Turning our attention to F_3 we notice first that

$$\lim_{x \rightarrow 0} F_1(x)x^{z-1} = 0 \text{ and } \lim_{x \rightarrow \infty} F_1(x)x^{z-1} = 0 \text{ for each } z \in \Gamma_{0,2}. \quad (3.116)$$

Thus, [34, formula (1.9), p. 11] guarantees that $F_3 \in \mathcal{A}_{(1,3)}$ and

$$\mathcal{M}F_3(z) = -(z-1)\mathcal{M}F_1(z-1) \text{ for each } z \in \Gamma_{1,3}. \quad (3.117)$$

Using this, (3.107), and (3.111), we obtain (3.109) and (3.113). The function F_2 satisfies similar properties to those of F_1 in (3.116) formulated this time in the strip $\Gamma_{-1,1}$. Appealing again to [34, formula (1.9), p. 11] it follows that $F_4 \in \mathcal{A}_{(0,2)}$ and

$$\mathcal{M}F_4(z) = -(z-1)\mathcal{M}F_2(z-1) \text{ for each } z \in \Gamma_{0,2}. \quad (3.118)$$

This together with (3.108) and (3.112) yield (3.110) and (3.114). \square

Going further, for $z \in \Gamma_{0,1}$ and $\theta \in (0, 2\pi)$ set

$$a(z) := \frac{\cos(\pi z)}{2 \sin(\pi z)}, \quad b_\theta(z) := \frac{\cos((\pi - \theta)z)}{2 \sin(\pi z)}, \quad d_\theta(z) := \frac{\sin((\pi - \theta)z)}{2 \sin(\pi z)}. \quad (3.119)$$

With this notation in hand, we provide next explicit formulas for the Mellin transform of the kernels k_1 , k_3 , k_5 , and k_7 .

Lemma 3.15. *Let $\theta \in (0, 2\pi)$ be fixed and recall the matrix-valued functions k_i for $i \in \{1, \dots, 8\}$ from (3.93). For each $z \in \Gamma_{0,1}$ there holds*

$$\mathcal{M}(k_1(\cdot, 1))(z) = \mathcal{M}(k_7(\cdot, 1))(z) = \begin{pmatrix} -a(z) & b_\theta(z) \\ -b_\theta(z) & a(z) \end{pmatrix}, \quad (3.120)$$

and

$$\mathcal{M}(k_3(\cdot, 1))(z) = \mathcal{M}(k_5(\cdot, 1))(z) = d_\theta(z) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.121)$$

Proof. With an eye towards proving (3.120), recall the functions F_1, F_2 from (3.105) and let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be given by $h(x) := 1/x$. Notice that using the first set of identities in (3.94), we may write

$$k_1^{11}(s, 1) = -k_1^{22}(s, 1) = \frac{1}{2\pi} h(s-1) \quad \text{for each } s \in \mathbb{R}_+. \quad (3.122)$$

Using the second set of identities in (3.94), there holds

$$-k_1^{12}(s, 1) = k_1^{21}(s, 1) = \frac{1}{2\pi} ((\cos \theta)F_2(s) - F_1(s)) \quad \text{for each } s \in \mathbb{R}_+. \quad (3.123)$$

Appealing now to (3.122)-(3.123), (3.107) and (3.111), and (3.108) and (3.112), together with Proposition 3.7 and the first equality in (3.104), we obtain that (3.120) is valid for each $z \in \Gamma_{0,1}$.

Turning our attention to the statement made in (3.121), using (3.97) we have

$$\left. \begin{aligned} k_3^{11}(s, 1) &= k_3^{22}(s, 1) = 0, \\ k_3^{12}(s, 1) &= k_3^{21}(s, 1) = \frac{\sin \theta}{2\pi} F_2(s), \end{aligned} \right\} \quad \text{for each } s \in \mathbb{R}_+. \quad (3.124)$$

Next, using (3.124) and formulas (3.108) and (3.112) from Proposition 3.14 along with the second equality in (3.104), we obtain that (3.121) holds for each $z \in \Gamma_{0,1}$. This completes the proof of the Lemma. \square

The next result will be useful when computing the Mellin transform of the remaining kernels.

Lemma 3.16. *For $\theta \in (-\pi, \pi)$, consider the function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, given by*

$$g(x) := \frac{1}{(x^2 + 2x \cos \theta + 1)^2}. \quad (3.125)$$

Then $g \in \mathcal{A}_{(0,4)}$. In addition, whenever $z \in \Gamma_{0,4} \setminus \{1, 2, 3\}$, and the angle $\theta \in (-\pi, \pi) \setminus \{0\}$, there holds

$$\mathcal{M}g(z) = \frac{\pi}{4(\sin^3 \theta) \sin(\pi z)} [(z-3) \sin((z-1)\theta) - (z-1) \sin((z-3)\theta)], \quad (3.126)$$

and, when $\theta = 0$,

$$\mathcal{M}g(z) = -\pi \frac{(z-1)(z-2)(z-3)}{6 \sin(\pi z)}. \quad (3.127)$$

Proof. Using [34, formula (2.58), p. 24], the function g satisfies $g \in \mathcal{A}_{(0,4)}$. Next, let us observe that if $\theta = 0$, then $g(x) = (x+1)^{-4}$ and on grounds of [34, formula (2.5), p. 13], there holds

$$\mathcal{M}g(z) = -\pi \frac{(z-1)(z-2)(z-3)}{6 \sin(\pi z)}. \quad (3.128)$$

When $\theta \in (-\pi, \pi) \setminus \{0\}$ we invoke again [34, formula (2.58), p. 24] to write that for each $z \in \Gamma_{0,4}$

$$\mathcal{M}g(z) = 2^{3/2}(\sin \theta)^{-3/2} \Gamma(\frac{5}{2}) \cdot B(z, 4-z) \cdot \mathbf{P}_{z-5/2}^{-3/2}(\cos \theta), \quad (3.129)$$

where Γ is the gamma function, B is the beta function, and \mathbf{P}_ν^μ is the Ferrers function (evaluated at $\cos \theta \in (-1, 1)$). Since $B(z, 4-z) = \frac{\Gamma(z)\Gamma(4-z)}{\Gamma(4)}$, using Euler's reflection formula (here we use that $z \notin \mathbb{Z}$) we obtain

$$B(z, 4-z) = \pi \frac{(3-z)(2-z)(1-z)}{6 \sin(\pi z)}. \quad (3.130)$$

Appealing to [34, p. 262] and elementary trigonometry, we may write

$$\begin{aligned} P_{z-5/2}^{-3/2}(\cos \theta) &= \frac{1}{\Gamma(\frac{5}{2})} \left[\frac{1 + \cos \theta}{1 - \cos \theta} \right]^{-3/4} {}_2F_1\left(-z + \frac{5}{2}, z - \frac{3}{2}; \frac{5}{2}; \frac{1 - \cos \theta}{2}\right) \quad (3.131) \\ &= \frac{1}{\Gamma(\frac{5}{2})} \left[\frac{\sin(\theta/2)}{\cos(\theta/2)} \right]^{3/2} {}_2F_1\left(-z + \frac{5}{2}, z - \frac{3}{2}; \frac{5}{2}; \sin^2\left(\frac{\theta}{2}\right)\right), \end{aligned}$$

where ${}_2F_1$ is Gauss' hypergeometric function. At this point we find it useful to recall the two following formulas from [1, formula (15.2.6), p. 557] (see also [35, (15.5.E6)]), and [38, formula (6c), p. 2] (see also [35, (15.4.E16)]),

$${}_2F_1\left(\frac{1}{2} + a, \frac{1}{2} - a; \frac{3}{2}; \sin^2 \alpha\right) = \frac{\sin(2a\alpha)}{2a \sin \alpha} \quad \text{for any } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad (3.132)$$

and, when c is not a non-positive integer and $|x| < 1$,

$$\frac{d}{dx}[(1-x)^{a+b-c} {}_2F_1(a, b; c; x)] = \frac{(c-a)(c-b)}{c} (1-x)^{a+b-c-1} {}_2F_1(a, b; c+1; x). \quad (3.133)$$

Using (3.132), with $a = 2 - z$ and $\alpha = \theta/2$, gives

$${}_2F_1\left(-z + \frac{5}{2}, z - \frac{3}{2}; \frac{3}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) = \frac{\sin((2-z)\theta)}{2(2-z) \sin \frac{\theta}{2}}. \quad (3.134)$$

Letting $F(x) := {}_2F_1\left(-z + \frac{5}{2}, z - \frac{3}{2}; \frac{3}{2}; x\right)$ and $x := \sin^2(\theta/2)$, using the chain rule we may write $\frac{dF}{dx}(x) = \frac{dF}{d\theta}(x) \frac{d\theta}{dx}(x)$. Differentiating with respect to θ in (3.134) and using that $\frac{d\theta}{dx}(x) = 2/\sin \theta$, this gives

$$\frac{dF}{dx}(x) = \frac{(2-z) \cos((2-z)\theta) \sin \frac{\theta}{2} - \frac{1}{2} \sin((2-z)\theta) \cos \frac{\theta}{2}}{(2-z)(\sin \theta) \sin^2 \frac{\theta}{2}}. \quad (3.135)$$

Applying (3.133) for the parameters $a = -z + \frac{5}{2}$, $b = z - \frac{3}{2}$ and $c = \frac{3}{2}$ gives that, on the one hand

$$\frac{d}{dx}[(1-x)^{-1/2} F(x)] = \frac{(z-1)(3-z)}{(3/2)} (1-x)^{-3/2} {}_2F_1\left(-z + \frac{5}{2}, z - \frac{3}{2}; \frac{5}{2}; x\right). \quad (3.136)$$

Using the chain rule, the definition of F , (3.135), and elementary trigonometry, we conclude on the other hand that

$$\begin{aligned} \frac{d}{dx}[(1-x)^{-1/2} F(x)] \Big|_{x=\sin^2(\frac{\theta}{2})} &= \frac{2}{(2-z) \sin^3 \theta} \quad (3.137) \\ &\times [(2-z) \cos((2-z)\theta) \sin \theta - \sin((2-z)\theta) \cos \theta]. \end{aligned}$$

Combining (3.136) (evaluated at $x = \sin^2(\frac{\theta}{2})$) with (3.137) implies that

$${}_2F_1\left(-z + \frac{5}{2}, z - \frac{3}{2}; \frac{5}{2}; \sin^2\left(\frac{\theta}{2}\right)\right) = \frac{3}{8(1-z)(2-z)(3-z)\sin^3(\theta/2)} \quad (3.138)$$

$$\times [\sin((2-z)\theta)\cos\theta - (2-z)\cos((2-z)\theta)\sin\theta].$$

Now (3.138) and (3.131) imply

$$P_{z-5/2}^{-3/2}(\cos\theta) = \frac{1}{\Gamma(5/2)} \frac{3}{(1-z)(2-z)(3-z)\sin^3\theta} \quad (3.139)$$

$$\times [\sin((2-z)\theta)\cos\theta - (2-z)\cos((2-z)\theta)\sin\theta].$$

Combining (3.129), (3.130) and (3.139) we can finally conclude that when the parameter $\theta \in (-\pi, \pi) \setminus \{0\}$, the formula for $\mathcal{M}g(z)$ from (3.126) holds, completing the proof of the Lemma 3.16. \square

Remark 3.17. Note that (3.126)-(3.127) imply that $\mathcal{M}g$ is continuous with respect to $\theta \in (-\pi, \pi)$.

Lemma 3.18. For each parameter $\theta \in (0, 2\pi)$ and each number $m \in \{0, \dots, 3\}$, consider the function $g_m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$g_m(x) := \frac{x^m}{(x^2 - 2x\cos\theta + 1)^2}. \quad (3.140)$$

Then for each $m \in \{0, \dots, 3\}$ there holds that $g_m \in \mathcal{A}_{(-m, 4-m)}$. In addition, if $\theta \in (0, 2\pi) \setminus \{\pi\}$, and $z \in \Gamma_{-m, 4-m} \setminus \mathbb{Z}$ for each of the formulas for $\mathcal{M}g_m(z)$ below, there holds:

$$\mathcal{M}g_0(z) = \frac{\pi [(z-3)\sin((z-1)(\pi-\theta)) - (z-1)\sin((z-3)(\pi-\theta))]}{4(\sin^3\theta)\sin(\pi z)}, \quad (3.141)$$

$$\mathcal{M}g_1(z) = -\frac{\pi [(z-2)\sin(z(\pi-\theta)) - z\sin((z-2)(\pi-\theta))]}{4(\sin^3\theta)\sin(\pi z)}, \quad (3.142)$$

$$\mathcal{M}g_2(z) = \frac{\pi [(z-1)\sin((z+1)(\pi-\theta)) - (z+1)\sin((z-1)(\pi-\theta))]}{4(\sin^3\theta)\sin(\pi z)}, \quad (3.143)$$

$$\mathcal{M}g_3(z) = -\frac{\pi [z\sin((z+2)(\pi-\theta)) - (z+2)\sin(z(\pi-\theta))]}{4(\sin^3\theta)\sin(\pi z)}. \quad (3.144)$$

Finally, when $\theta = \pi$ and $z \in \Gamma_{-m,4-m} \setminus \mathbb{Z}$ for each of the formulas for $\mathcal{M}g_m(z)$ below, there holds:

$$\mathcal{M}g_0(z) = -\frac{\pi(z-1)(z-2)(z-3)}{6 \sin(\pi z)}, \quad (3.145)$$

$$\mathcal{M}g_1(z) = \frac{\pi z(z-1)(z-2)}{6 \sin(\pi z)}, \quad (3.146)$$

$$\mathcal{M}g_2(z) = -\frac{\pi z(z-1)(z+1)}{6 \sin(\pi z)}, \quad (3.147)$$

$$\mathcal{M}g_3(z) = \frac{\pi z(z+1)(z+2)}{6 \sin(\pi z)}. \quad (3.148)$$

Proof. These statements immediately follow from Lemma 3.16 and elementary properties of the Mellin transform. \square

Lemma 3.19. Fix $\theta \in (0, 2\pi)$ and recall the kernel k_2 in (3.95). For each $z \in \Gamma_{0,1}$ there holds

$$\mathcal{M}(k_2(\cdot, 1))(z) = \begin{pmatrix} -\frac{1}{2}a(z) & A_\theta(z) \\ -A_\theta(z) & \frac{1}{2}a(z) \end{pmatrix}, \quad (3.149)$$

where, when $\theta \neq \pi$,

$$\begin{aligned} A_\theta(z) := & \frac{1}{16(\sin^3 \theta) \sin(\pi z)} \{ (\cos \theta) [z \sin((z+2)(\pi-\theta)) - (z+2) \sin(z(\pi-\theta))] \\ & + 3 [(z-1) \sin((z+1)(\pi-\theta)) - (z+1) \sin((z-1)(\pi-\theta))] \\ & + (3 + 2 \sin^2 \theta)(\cos \theta) [(z-2) \sin(z(\pi-\theta)) - z \sin((z-2)(\pi-\theta))] \\ & + [(z-3) \sin((z-1)(\pi-\theta)) - (z-1) \sin((z-3)(\pi-\theta))] \}, \end{aligned} \quad (3.150)$$

and when $\theta = \pi$,

$$A_\pi(z) := \frac{1}{4 \sin(\pi z)}. \quad (3.151)$$

Proof. Using Proposition 3.7 along with the definition of a from (3.119), we may write for each $z \in \Gamma_{0,1}$

$$\mathcal{M}(k_2^{11}(\cdot, 1))(z) = -\mathcal{M}(k_2^{22}(\cdot, 1))(z) = -\frac{\cos(\pi z)}{4 \sin(\pi z)} = -\frac{1}{2}a(z), \quad (3.152)$$

taking care of the diagonal entries in (3.149) for each $\theta \in (0, 2\pi)$. From (3.95), (3.96), straightforward algebra, and the definition of the functions g_m , $m \in \{0, \dots, 3\}$, from (3.140), note that for each $s \in \mathbb{R}_+$ we have

$$\begin{aligned} -k_2^{12}(s, 1) &= k_2^{21}(s, 1) \\ &= \frac{1}{4\pi} [(\cos \theta)g_3(s) - 3g_2(s) + (3 + 2\sin^2 \theta)(\cos \theta)g_1(s) - g_0(s)]. \end{aligned} \quad (3.153)$$

Consequently

$$\begin{aligned} & -\mathcal{M}(k_2^{12}(\cdot, 1))(z) \\ &= \mathcal{M}(k_2^{21}(\cdot, 1))(z) \\ &= \frac{1}{4\pi} [(\cos \theta)\mathcal{M}g_3(z) - 3\mathcal{M}g_2(z) + (3 + 2\sin^2 \theta)(\cos \theta)\mathcal{M}g_1(z) - \mathcal{M}g_0(z)]. \end{aligned} \quad (3.154)$$

In concert, (3.154), formulas (3.141)-(3.144), and straightforward algebra allow us to obtain that, for $\theta \in (0, 2\pi) \setminus \{\pi\}$, there holds

$$-\mathcal{M}(k_2^{12}(\cdot, 1))(z) = \mathcal{M}(k_2^{21}(\cdot, 1))(z) = -A_\theta(z), \quad \forall z \in \Gamma_{0,1}, \quad (3.155)$$

where $A_\theta(z)$ as in (3.150), completing the proof of (3.149) when θ lies in $(0, 2\pi) \setminus \{\pi\}$. Finally, the formulas for the off diagonal entries in (3.149) when $\theta = \pi$ follow from (3.154) and formulas (3.145)-(3.148). \square

Lemma 3.20. *Fix $\theta \in (0, 2\pi)$ and recall the kernel k_4 from (3.98). Then, for each $z \in \Gamma_{0,1}$ there holds*

$$\mathcal{M}(k_4(\cdot, 1))(z) = B_\theta(z) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.156)$$

where, when $\theta \neq \pi$,

$$B_\theta(z) := \frac{-1}{16(\sin^2 \theta) \sin(\pi z)} \{ [z \sin((z+2)(\pi-\theta)) - (z+2) \sin(z(\pi-\theta))] \quad (3.157)$$

$$\begin{aligned} & - (2 + \cos(2\theta)) [(z-2) \sin(z(\pi-\theta)) - z \sin((z-2)(\pi-\theta))] \\ & - 2(\cos \theta) [(z-3) \sin((z-1)(\pi-\theta)) - (z-1) \sin((z-3)(\pi-\theta))] \}, \end{aligned}$$

and when $\theta = \pi$,

$$B_\pi(z) := 0. \quad (3.158)$$

Proof. In light of (3.98), there holds that $k_4^{11}(s, 1) = k_4^{22}(s, 1) = 0$ for each $s \in \mathbb{R}_+$, taking care immediately of the diagonal entries in (3.156) for each $\theta \in (0, 2\pi)$. Going further, (3.98), and elementary algebra and trigonometry give that

$$k_4^{12}(s, 1) = k_4^{21}(s, 1) = \frac{\sin \theta}{4\pi} [g_3(s) - (2 + \cos(2\theta))g_1(s) + 2(\cos \theta)g_0(s)], \quad (3.159)$$

where the functions g_m , $m \in \{0, \dots, 3\}$, are as in (3.140). Consequently, for each $z \in \Gamma_{0,1}$,

$$\begin{aligned} \mathcal{M}(k_4^{12}(\cdot, 1))(z) &= \mathcal{M}(k_4^{21}(\cdot, 1))(z) \\ &= \frac{\sin \theta}{4\pi} [\mathcal{M}g_3(z) - (2 + \cos(2\theta))\mathcal{M}g_1(z) + 2(\cos \theta)\mathcal{M}g_0(z)], \end{aligned} \quad (3.160)$$

and applying formulas (3.141)-(3.144), after straightforward algebra we obtain

$$\mathcal{M}(k_4^{12}(\cdot, 1))(z) = \mathcal{M}(k_4^{21}(\cdot, 1))(z) = B_\theta(z), \quad \forall z \in \Gamma_{0,1}, \quad (3.161)$$

where $B_\theta(z)$ is as in (3.157). This proves (3.156) whenever $\theta \in (0, 2\pi) \setminus \{\pi\}$. To deal with the off diagonal entries in (3.156) when $\theta = \pi$, appeal again to (3.160) and notice that its right-hand side vanishes in this case. \square

Lemma 3.21. *Fix $\theta \in (0, 2\pi)$ and recall the kernel k_6 from (3.100). For each $z \in \Gamma_{0,1}$ there holds*

$$\mathcal{M}(k_6(\cdot, 1))(z) = C_\theta(z) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.162)$$

where, when $\theta \neq \pi$,

$$\begin{aligned} C_\theta(z) &:= \frac{1}{16(\sin^2 \theta) \sin(\pi z)} \{ [z \sin((z+2)(\pi-\theta)) - (z+2) \sin(z(\pi-\theta))] \\ &\quad + 2(\cos \theta) [(z-1) \sin((z+1)(\pi-\theta)) - (z+1) \sin((z-1)(\pi-\theta))] \\ &\quad + \cos(2\theta) [(z-2) \sin(z(\pi-\theta)) - z \sin((z-2)(\pi-\theta))] \}, \end{aligned} \quad (3.163)$$

and when $\theta = \pi$,

$$C_\pi(z) := 0. \quad (3.164)$$

Proof. Formula (3.100) gives that $k_6^{11}(s, 1) = k_6^{22}(s, 1) = 0$ for each $s \in \mathbb{R}_+$, justifying the diagonal entries in the right hand side of (3.162). In addition,

$$k_6^{12}(s, 1) = k_6^{21}(s, 1) = \frac{\sin \theta}{4\pi} [-g_3(s) + 2(\cos \theta)g_2(s) - \cos(2\theta)g_1(s)], \quad (3.165)$$

where the functions g_m , $m \in \{0, \dots, 3\}$, are as in (3.140). Consequently, for each $z \in \Gamma_{0,1}$,

$$\begin{aligned} \mathcal{M}(k_6^{12}(\cdot, 1))(z) &= \mathcal{M}(k_6^{21}(\cdot, 1))(z) \\ &= \frac{\sin \theta}{4\pi} \left[-\mathcal{M}g_3(z) + 2(\cos \theta)\mathcal{M}g_2(z) - \cos(2\theta)\mathcal{M}g_1(z) \right]. \end{aligned} \quad (3.166)$$

Now, (3.166) combined with (3.142)-(3.144) and straightforward algebra and trigonometry yield

$$\mathcal{M}(k_6^{12}(\cdot, 1))(z) = \mathcal{M}(k_6^{21}(\cdot, 1))(z) = C_\theta(z), \quad \forall z \in \Gamma_{0,1}, \quad (3.167)$$

where $C_\theta(z)$ is as in (4.99). If $\theta = \pi$, the right hand side of (3.166) clearly vanishes, finishing the proof. \square

Lemma 3.22. *Fix $\theta \in (0, 2\pi)$ and recall the kernel k_8 from (3.102). For each $z \in \Gamma_{0,1}$ there holds*

$$\mathcal{M}(k_8(\cdot, 1))(z) = \begin{pmatrix} -\frac{1}{2}a(z) & D_\theta(z) \\ -D_\theta(z) & \frac{1}{2}a(z) \end{pmatrix}, \quad (3.168)$$

where, when $\theta \neq \pi$,

$$\begin{aligned} &D_\theta(z) \\ &:= \frac{1}{16(\sin^3 \theta) \sin(\pi z)} \left\{ (\cos \theta) [z \sin((z+2)(\pi-\theta)) - (z+2) \sin(z(\pi-\theta))] \right. \\ &\quad + (2 + \cos(2\theta)) [(z-1) \sin((z+1)(\pi-\theta)) - (z+1) \sin((z-1)(\pi-\theta))] \\ &\quad + (\cos \theta)(2 + \cos(2\theta)) [(z-2) \sin(z(\pi-\theta)) - z \sin((z-2)(\pi-\theta))] \\ &\quad \left. + \cos(2\theta) [(z-3) \sin((z-1)(\pi-\theta)) - (z-1) \sin((z-3)(\pi-\theta))] \right\}, \end{aligned} \quad (3.169)$$

and when $\theta = \pi$,

$$D_\pi(z) := \frac{1}{4 \sin(\pi z)}. \quad (3.170)$$

Proof. Combining Proposition 3.7, the definition of a from (3.119), and (3.102), we obtain that for $z \in \Gamma_{0,1}$

$$\mathcal{M}(k_8^{11}(\cdot, 1))(z) = -\mathcal{M}(k_8^{22}(\cdot, 1))(z) = -\frac{\cos(\pi z)}{4 \sin(\pi z)} = -\frac{1}{2}a(z), \quad (3.171)$$

taking care of the diagonal entries in (3.168). For the other entries in $k_8(\cdot, 1)$, appeal again to (3.102) and elementary trigonometry to write for each $s \in \mathbb{R}_+$,

$$\begin{aligned} & -k_8^{12}(s, 1) \quad (3.172) \\ & = k_8^{21}(s, 1) \\ & = \frac{1}{4\pi} \left[(\cos \theta)g_3(s) - (2 + \cos(2\theta))g_2(s) + (\cos \theta)(2 + \cos(2\theta))g_1(s) \right. \\ & \quad \left. - \cos(2\theta)g_0(s) \right], \end{aligned}$$

where the functions g_0, g_1, g_2 and g_3 are as in (3.140). Consequently

$$\begin{aligned} & -\mathcal{M}(k_8^{12}(\cdot, 1))(z) \quad (3.173) \\ & = \mathcal{M}(k_8^{21}(\cdot, 1))(z) \\ & = \frac{1}{4\pi} \left[(\cos \theta)\mathcal{M}g_3(z) - (2 + \cos(2\theta))\mathcal{M}g_2(z) \right. \\ & \quad \left. + (\cos \theta)(2 + \cos(2\theta))\mathcal{M}g_1(z) - \cos(2\theta)\mathcal{M}g_0(z) \right]. \end{aligned}$$

Combining (3.173) with (3.141)-(3.144), and using straightforward algebra and trigonometry, we ultimately arrive at

$$-\mathcal{M}(k_8^{12}(\cdot, 1))(z) = \mathcal{M}(k_8^{21}(\cdot, 1))(z) = -D_\theta(z), \quad \forall z \in \Gamma_{0,1},$$

where $D_\theta(z)$ as in (3.169). This completes the proof of (3.168) when the angle $\theta \in (0, 2\pi) \setminus \{\pi\}$. Finally, when $\theta = \pi$, invoke again (3.173) and formulas (3.145)-(3.148). \square

Remark 3.23. *Straightforward calculations show that, with A_θ as in (3.150)-(3.151), B_θ as in (3.157)-(3.158), C_θ as in (4.99)-(3.164), and D_θ as in (3.169)-(3.170), there holds*

$$\lim_{\theta \rightarrow \pi} A_\theta = A_\pi, \quad \lim_{\theta \rightarrow \pi} B_\theta = B_\pi, \quad \lim_{\theta \rightarrow \pi} C_\theta = C_\pi, \quad \text{and} \quad \lim_{\theta \rightarrow \pi} D_\theta = D_\pi. \quad (3.174)$$

In addition, in the case when the aperture $\theta = \pi/2$, there holds

$$\begin{aligned} b_{\pi/2}(z) &= \frac{\cos\left(\frac{\pi z}{2}\right)}{2 \sin(\pi z)}, \quad d_{\pi/2}(z) = \frac{\sin\left(\frac{\pi z}{2}\right)}{2 \sin(\pi z)}, \\ A_{\pi/2}(z) &= \frac{(z+1) \cos\left(\frac{\pi z}{2}\right)}{4 \sin(\pi z)}, \quad B_{\pi/2}(z) = \frac{z \sin\left(\frac{\pi z}{2}\right)}{4 \sin(\pi z)}, \\ C_{\pi/2}(z) &= -\frac{z \sin\left(\frac{\pi z}{2}\right)}{4 \sin(\pi z)}, \quad D_{\pi/2}(z) = \frac{(z-1) \cos\left(\frac{\pi z}{2}\right)}{4 \sin(\pi z)}. \end{aligned} \quad (3.175)$$

Having computed explicitly $\mathcal{M}(k_j(\cdot, 1))(z)$ for each $j \in \{1, \dots, 8\}$ and each $z \in \Gamma_{0,1}$, we are ready to compute the Mellin symbol of the kernel matrix R from (3.23).

Lemma 3.24. *Fix $\theta \in (0, 2\pi)$ and $\eta \in \mathbb{R}$, and consider the matrix R as in (3.23). Then*

$$\mathcal{M}(R(\cdot, 1))(z) = \begin{pmatrix} M_{11}(z) & M_{12}(z) \\ M_{21}(z) & M_{22}(z) \end{pmatrix}, \quad \forall z \in \Gamma_{0,1}, \quad (3.176)$$

where, with a, b_θ, d_θ as in (3.119), A_θ as in (3.150)-(3.151), B_θ as in (3.157)-(3.158), C_θ as in (4.99)-(3.164), and D_θ as in (3.169)-(3.170),

$$M_{11}(z) := \begin{pmatrix} \frac{1+\eta}{2} a(z) & -\eta \cdot b_\theta(z) + (\eta - 1) \cdot A_\theta(z) \\ \eta \cdot b_\theta(z) - (\eta - 1) \cdot A_\theta(z) & -\frac{1+\eta}{2} a(z) \end{pmatrix}, \quad (3.177)$$

$$M_{12}(z) := \begin{pmatrix} -\frac{1}{2} & -\eta \cdot d_\theta(z) + (\eta - 1) \cdot B_\theta(z) \\ -\eta \cdot d_\theta(z) + (\eta - 1) \cdot B_\theta(z) & -\frac{1}{2} \end{pmatrix}, \quad (3.178)$$

$$M_{21}(z) := \begin{pmatrix} -\frac{1}{2} & -d_\theta(z) + (\eta - 1) \cdot C_\theta(z) \\ -d_\theta(z) + (\eta - 1) \cdot C_\theta(z) & -\frac{1}{2} \end{pmatrix}, \quad (3.179)$$

and

$$M_{22}(z) := \begin{pmatrix} -\frac{1+\eta}{2} a(z) & b_\theta(z) + (\eta - 1) \cdot D_\theta(z) \\ -b_\theta(z) - (\eta - 1) \cdot D_\theta(z) & \frac{1+\eta}{2} a(z) \end{pmatrix}. \quad (3.180)$$

Proof. This follows directly from (3.23), Lemma 3.19, Lemma 3.20, Lemma 3.21, and Lemma 3.22. \square

The following straightforward linear algebra result will be useful for our further analysis.

Lemma 3.25. Consider 2×2 matrices $A, D \in \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ and $B, C \in \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A \cdot D + B \cdot \tilde{C}), \quad (3.181)$$

where

$$\tilde{C} := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot C \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.182)$$

Our next results are focused on properties of the determinant of the Mellin symbol of the matrix-valued kernel R from (3.23).

Lemma 3.26. Let $\theta \in (0, 2\pi)$ and let R be as in (3.23). Then, with $M_{ij}(z)$, $i, j \in \{1, 2\}$, as in (3.177)-(3.180), the following holds

$$\det(\mathcal{M}(R(\cdot, 1)))(z) = \det(M_{11}(z) \cdot M_{22}(z) + M_{12}(z) \cdot \tilde{M}_{21}(z)), \quad \forall z \in \Gamma_{0,1}, \quad (3.183)$$

where

$$\widetilde{M}_{21}(z) := \begin{pmatrix} \frac{1}{2} & -d_\theta(z) + (\eta - 1) \cdot C_\theta(z) \\ -d_\theta(z) + (\eta - 1) \cdot C_\theta(z) & \frac{1}{2} \end{pmatrix}. \quad (3.184)$$

Proof. This is a direct application of Lemma 3.24 and Lemma 3.25. \square

Theorem 3.27. Fix $\theta \in (0, 2\pi)$ and $\eta \in \mathbb{R}$ and let R be as in (3.23). Then for each $z \in \Gamma_{0,1}$ there holds

$$\det(\mathcal{M}(R(\cdot, 1)))(z) = (\Phi_\theta(z, \eta) + \Psi_\theta(z, \eta))(\Phi_\theta(z, \eta) - \Psi_\theta(z, \eta)), \quad (3.185)$$

where, with $a(z), b_\theta(z), d_\theta(z)$ as in (3.119), and A_θ as in (3.150)-(3.151), B_θ as in (3.157)-(3.158), C_θ as in (4.99)-(3.164), and D_θ as in (3.169)-(3.170),

$$\begin{aligned} \Phi_\theta(z, \eta) := & -\frac{1}{4} [(1 + \eta)^2 a^2(z) + 1] \\ & + (\eta \cdot b_\theta(z) - (\eta - 1) \cdot A_\theta(z)) \cdot (b_\theta(z) + (\eta - 1) \cdot D_\theta(z)) \\ & + (\eta \cdot d_\theta(z) - (\eta - 1) \cdot B_\theta(z)) \cdot (d_\theta(z) - (\eta - 1) \cdot C_\theta(z)), \end{aligned} \quad (3.186)$$

and

$$\Psi_\theta(z, \eta) := \frac{1 - \eta^2}{2} a(z) [b_\theta(z) - A_\theta(z) - D_\theta(z)] + \frac{1 - \eta}{2} [d_\theta(z) - B_\theta(z) + C_\theta(z)]. \quad (3.187)$$

Proof. With $z \in \Gamma_{0,1}$ and the matrices $M_{11}(z)$ and $M_{22}(z)$ as in (3.177) and (3.180), first note that performing the matrix multiplication $M_{11}(z) \cdot M_{22}(z)$ and elementary algebra give

$$M_{11}(z) \cdot M_{22}(z) = \begin{pmatrix} U(z) & V(z) \\ V(z) & U(z) \end{pmatrix}, \quad (3.188)$$

where

$$U(z) := -\frac{(1 + \eta)^2}{4} a^2(z) + (\eta \cdot b_\theta(z) - (\eta - 1) \cdot A_\theta(z)) \cdot (b_\theta(z) + (\eta - 1) \cdot D_\theta(z)), \quad (3.189)$$

and

$$V(z) := \frac{1 - \eta^2}{2} a(z) \cdot [b_\theta(z) - (A_\theta(z) + D_\theta(z))]. \quad (3.190)$$

Second, multiplying the matrices $M_{12}(z)$ and $\widetilde{M}_{21}(z)$ from (3.178) and (3.184) yields

$$M_{12}(z) \cdot \widetilde{M}_{21}(z) = \begin{pmatrix} \widetilde{U}(z) & \widetilde{V}(z) \\ \widetilde{V}(z) & \widetilde{U}(z) \end{pmatrix}, \quad (3.191)$$

where

$$\widetilde{U}(z) := -\frac{1}{4} + (\eta \cdot d_\theta(z) - (\eta - 1) \cdot B_\theta(z)) \cdot (d_\theta(z) - (\eta - 1) \cdot C_\theta(z)), \quad (3.192)$$

and

$$\widetilde{V}(z) := \frac{1}{2} [(1 - \eta)d_\theta(z) + (\eta - 1) \cdot (B_\theta(z) - C_\theta(z))]. \quad (3.193)$$

Using now formula (3.183) in Lemma 3.24 along with identities (3.188) and (3.191), for each $z \in \Gamma_{0,1}$ we may write

$$\det(\mathcal{M}(R(\cdot, 1))(z)) = \det \begin{pmatrix} U(z) + \widetilde{U}(z) & V(z) + \widetilde{V}(z) \\ V(z) + \widetilde{V}(z) & U(z) + \widetilde{U}(z) \end{pmatrix}. \quad (3.194)$$

Finally (3.185) follows as soon as we notice that

$$\Phi_\theta(z, \eta) = U(z) + \widetilde{U}(z) \quad \text{and} \quad \Psi_\theta(z, \eta) = V(z) + \widetilde{V}(z), \quad (3.195)$$

finishing the proof of the Theorem 3.27. \square

In particular, the two factors in the right hand side of (3.185) are

$$\begin{aligned} \Phi_\theta(z, \eta) + \Psi_\theta(z, \eta) &= -\frac{1}{4} [(1 + \eta)^2 a^2(z) + 1] \\ &\quad + (\eta \cdot b_\theta(z) - (\eta - 1) \cdot A_\theta(z)) \cdot (b_\theta(z) + (\eta - 1) \cdot D_\theta(z)) \\ &\quad + (\eta \cdot d_\theta(z) - (\eta - 1) \cdot B_\theta(z)) \cdot (d_\theta(z) - (\eta - 1) \cdot C_\theta(z)) \\ &\quad + \frac{1 - \eta^2}{2} a(z) \cdot [b_\theta(z) - A_\theta(z) - D_\theta(z)] \\ &\quad + \frac{1 - \eta}{2} \cdot [d_\theta(z) - B_\theta(z) + C_\theta(z)], \end{aligned} \quad (3.196)$$

and

$$\begin{aligned}
\Phi_\theta(z, \eta) - \Psi_\theta(z, \eta) = & -\frac{1}{4} [(1 + \eta)^2 a^2(z) + 1] \\
& + (\eta \cdot b_\theta(z) - (\eta - 1) \cdot A_\theta(z)) \cdot (b_\theta(z) + (\eta - 1) \cdot D_\theta(z)) \\
& + (\eta \cdot d_\theta(z) - (\eta - 1) \cdot B_\theta(z)) \cdot (d_\theta(z) - (\eta - 1) \cdot C_\theta(z)) \\
& - \frac{1 - \eta^2}{2} a(z) \cdot [b_\theta(z) - A_\theta(z) - D_\theta(z)] \\
& - \frac{1 - \eta}{2} \cdot [d_\theta(z) - B_\theta(z) + C_\theta(z)].
\end{aligned} \tag{3.197}$$

3.5 Solvability results for the biharmonic Neumann problem in infinite sectors

In this section, utilizing the Mellin analysis results, we investigate the solvability for the Neumann problem for the bi-Laplacian in two-dimensional infinite sectors and establish the solvability range of the integrability exponent $p \in (1, \infty)$ for more tractable aperture cases. The following theorem states the relationship between the Mellin symbol and the existence of the solution for the biharmonic Neumann problem.

Theorem 3.28. *Let Ω be the interior of an infinite upright sector of aperture $\theta \in (0, 2\pi)$ in the plane, fix $\eta \in [-1, 1)$, and recall the operator R as in (3.23). Then if $p \in (1, \infty)$, the following implication holds*

$$\det(\mathcal{M}R(\cdot, 1))(z) \neq 0 \quad \forall z \in \Gamma_{0,1}, \quad \Re z = \frac{1}{p} \implies (NBH_p) \text{ has a solution.} \tag{3.198}$$

Proof. Fix $p \in (1, \infty)$ and assume that

$$\det(\mathcal{M}R(\cdot, 1))(z) \neq 0 \quad \forall z \in \Gamma_{0,1}, \quad \Re z = \frac{1}{p}, \tag{3.199}$$

and consider the boundary value problem (NBH_p) from (3.2) with given datum $(f_0, f_1) \in L^p(\partial\Omega) \times (\dot{L}_1^{p'}(\partial\Omega))^*$ where $1/p + 1/p' = 1$. Since $f_1 \in \dot{L}_{-1}^p(\partial\Omega)$ there exists $g \in L^p(\partial\Omega)$ such that $f_1 = \partial_\tau g$ and the conditions on p from

the hypothesis ensure that Theorem 3.10 applies. Combining this with (3.24) allows us to define $(h_0, h_1) \in L^p(\partial\Omega) \times L^p(\partial\Omega)$ via

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} := T^{-1} \begin{pmatrix} f_0 \\ g \end{pmatrix}. \quad (3.200)$$

With this in hand, set $u := \mathcal{S}(\partial_\tau h_0, h_1)$ in Ω , where \mathcal{S} is the bi-harmonic single layer potential operator introduced in (3.41). Then, as discussed in Section 3.2.1,

$$\Delta^2 u = 0 \text{ in } \Omega \text{ and } \mathcal{N}_\kappa(\nabla^2 u) \in L^p(\partial\Omega). \quad (3.201)$$

Pick next $r \in \mathbb{R}$ such that $\eta = \frac{2(r+r^2)}{1+2r+2r^2} \in [-1, 1)$ (note that for $\eta \in [-1, 1)$ this is always possible), and recall the boundary operators M_r and N_r from (3.6). Using the jump formulas (3.55) we have

$$\begin{pmatrix} M_r(u) \\ N_r(u) \end{pmatrix} = \begin{pmatrix} -\eta K_1 - 2K_2 & -\frac{1}{2}I - \eta K_3 - 2K_4 \\ \partial_\tau(-K_5 - 2K_6 - \frac{1}{2}I) & \partial_\tau(K_7 - 2K_8) \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}. \quad (3.202)$$

Let

$$\mathfrak{t} : L^p(\partial\Omega) \times L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \times \dot{L}_{-1}^p(\partial\Omega), \quad (3.203)$$

be given by $\mathfrak{t}(F, G) := (F, \partial_\tau G)$, $\forall (F, G) \in L^p(\partial\Omega) \times L^p(\partial\Omega)$.

Then using the definition of T from (3.7) together with (3.202), we obtain that

$$\begin{pmatrix} M_r(u) \\ N_r(u) \end{pmatrix} = \mathfrak{t} \circ T \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}. \quad (3.204)$$

This and (3.200) ultimately yield

$$M_r(u) = f_0 \text{ and } N_r(u) = f_1 \text{ } \sigma\text{-a.e. on } \partial\Omega. \quad (3.205)$$

Finally, (3.201), (3.205), and the definition of M_r and N_r show that u is a solution for (NBH_p) with datum $(f_0, f_1) \in L^p(\partial\Omega) \times (\dot{L}_1^{p'}(\partial\Omega))^*$, completing the proof of the Theorem 3.28. \square

3.5.1 The case of the upper half-plane

In this section, we consider the upper half-plane domain. In other words, Ω is an infinite sector with an angle $\theta = \pi$. According to (3.119),

$$a(z) = \frac{\cos(\pi z)}{2 \sin(\pi z)}, \quad b_\pi(z) = \frac{1}{2 \sin(\pi z)}, \quad d_\pi(z) = 0. \quad (3.206)$$

For simplicity, we introduce the following notations. For $\theta \in (0, 2\pi)$,

$$\begin{aligned} G_0 &= G_0(z, \theta) := z \sin((z+2)(\pi-\theta)) - (z+2) \sin(z(\pi-\theta)), \\ G_1 &= G_1(z, \theta) := (z-1) \sin((z+1)(\pi-\theta)) - (z+1) \sin((z-1)(\pi-\theta)), \\ G_2 &= G_2(z, \theta) := (z-2) \sin(z(\pi-\theta)) - z \sin((z-2)(\pi-\theta)), \\ G_3 &= G_3(z, \theta) := (z-3) \sin((z-1)(\pi-\theta)) - (z-1) \sin((z-3)(\pi-\theta)). \end{aligned} \quad (3.207)$$

Going further,

$$\begin{aligned} G_0 &= z \sin(z\gamma) \cos(2\theta) - z \cos(z\gamma) \sin(2\theta) - (z+2) \sin(z\gamma), \\ G_1 &= (z-1) \sin((z-1)\gamma) \cos(2\theta) - (z-1) \cos((z-1)\gamma) \sin(2\theta) \\ &\quad - (z+1) \sin((z-1)\gamma), \\ G_2 &= (z-2) \sin(z\gamma) - z \sin(z\gamma) \cos(2\theta) - z \cos(z\gamma) \sin(2\theta), \\ G_3 &= (z-3) \sin((z-1)\gamma) - (z-1) \sin((z-1)\gamma) \cos(2\theta) \\ &\quad - (z-1) \cos((z-1)\gamma) \sin(2\theta), \end{aligned} \quad (3.208)$$

where $\gamma = \pi - \theta$. Note that for $i = 0, 1, 2, 3$,

$$\lim_{\theta \rightarrow \pi} \frac{G_i}{\sin^3(\theta)} = -\frac{2}{3}(z-i)(z+1-i)(z+2-i), \quad (3.209)$$

for $z \in \Gamma_{0,1}$. Combining this with (3.150), (3.157), (4.99), and (3.169), we obtain for $z \in \Gamma_{0,1}$

$$\begin{aligned} A_\pi(z) &= D_\pi(z) = \frac{1}{4 \sin(\pi z)}, \\ B_\pi(z) &= C_\pi(z) = 0. \end{aligned} \quad (3.210)$$

In particular,

$$\Phi_\pi(z, \eta) = \frac{(\eta + 3)(\eta - 1)}{16}, \quad (3.211)$$

and

$$\Psi_\pi(z, \eta) = 0. \quad (3.212)$$

This forces the following solvability result for the biharmonic Neumann problem for any $p \in (1, \infty)$ whenever $\eta \neq -3, 1$.

Theorem 3.29. *Let Ω be the upper half-plane. Then for $\eta \in \mathbb{R} \setminus \{-3, 1\}$ and for $p \in (1, \infty)$,*

$$(NBH_p) \text{ has a solution.} \quad (3.213)$$

Proof. Applying (3.206)-(3.212) to Theorem 3.27 gives for each $z \in \Gamma_{0,1}$ there holds

$$\det(\mathcal{M}(R(\cdot, 1)))(z) = \left[\frac{(\eta + 3)(\eta - 1)}{16} \right]^2. \quad (3.214)$$

This forces for $\eta \in \mathbb{R} \setminus \{-3, 1\}$ and for $p \in (1, \infty)$, there holds

$$\det(\mathcal{M}(R(\cdot, 1))) \left(\frac{1}{p} + i\xi \right) \neq 0, \quad (3.215)$$

for each $\xi \in \mathbb{R}$. The Theorem 3.28 with (3.215) completes the proof of the theorem. \square

3.5.2 The case of a quadrant

Recall that Ω is the region above the graph of the function ϕ from (3.1) for $\theta := \frac{\pi}{2}$, i.e.

$$\Omega \text{ is the upper graph of } \phi : \mathbb{R} \longrightarrow \mathbb{R} \text{ given by } \phi(x) := |x|. \quad (3.216)$$

The main goal of this section is to investigate the solvability of the boundary value problem (3.2) in the case when Ω is a first quadrant. The first order of business is to identify the values of $p \in (1, \infty)$ for which the operator \mathcal{T} in (3.22) is invertible. Our main result in this direction is as follows.

Theorem 3.30. *Let Ω be the interior of the infinite sector of aperture $\frac{\pi}{2}$ from (3.216). Then for $\eta \in [-1, 1)$ and for $p \in (1, \infty)$, there holds*

$$\mathcal{T} \text{ is invertible on } \left(L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+) \right)^2 \iff p \in (1, \infty) \setminus \{\alpha, \beta\}, \quad (3.217)$$

where $\alpha \in (9, 12)$ and $\beta \in (2, 3)$ with

$$\alpha^{-1} \text{ is the unique root of } z - 1 + \cos\left(\frac{3\pi z}{2}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.218)$$

$$\beta^{-1} \text{ is the unique root of } z - 1 - \cos\left(\frac{3\pi z}{2}\right) = 0 \text{ in } \Gamma_{0,1}. \quad (3.219)$$

where $\Gamma_{0,1}$ denotes the complex strip as in (3.62).

The proof of Theorem 3.30 requires several preliminary steps. We start by stating a direct consequence of Theorem 3.27 in particular the case when $\theta = \frac{\pi}{2}$.

Proposition 3.31. *Let Ω be an infinite sector as in (3.216) (with aperture $\theta = \frac{\pi}{2}$), let R as in (3.23) and recall the functions a , $b_{\pi/2}$, $d_{\pi/2}$, $A_{\pi/2}$, $B_{\pi/2}$, $C_{\pi/2}$, $D_{\pi/2}$ from (3.175). Then, for each $z \in \Gamma_{0,1}$ the following holds*

$$\begin{aligned} \det(\mathcal{M}(R(\cdot, 1))(z)) &= \frac{(1-\eta)^2}{16^2 \sin^4(\pi z)} \left(z - 1 - \cos\left(\frac{3\pi z}{2}\right) \right) \left(z - 1 + \cos\left(\frac{3\pi z}{2}\right) \right) \\ &\quad \times \left[(1-\eta) \left(\cos\left(\frac{\pi z}{2}\right) - z + 1 \right) - 4 \cos\left(\frac{\pi z}{2}\right) \right] \\ &\quad \times \left[(1-\eta) \left(-\cos\left(\frac{\pi z}{2}\right) - z + 1 \right) + 4 \cos\left(\frac{\pi z}{2}\right) \right]. \end{aligned} \quad (3.220)$$

Proof. Using (3.196) and (3.197), and trigonometric manipulations that we omit we obtain

$$\begin{aligned} \Phi_{\pi/2}(z, \eta) + \Psi_{\pi/2}(z, \eta) &= \frac{(1-\eta)}{16 \sin^2(\pi z)} \left(z - 1 - \cos\left(\frac{3\pi z}{2}\right) \right) \\ &\quad \times \left[(1-\eta) \left(\cos\left(\frac{\pi z}{2}\right) - z + 1 \right) - 4 \cos\left(\frac{\pi z}{2}\right) \right], \end{aligned} \quad (3.221)$$

and

$$\begin{aligned} \Phi_{\pi/2}(z, \eta) - \Psi_{\pi/2}(z, \eta) &= \frac{(1-\eta)}{16 \sin^2(\pi z)} \left(z - 1 + \cos\left(\frac{3\pi z}{2}\right) \right) \\ &\quad \times \left[(1-\eta) \left(\cos\left(\frac{\pi z}{2}\right) - z + 1 \right) + 4 \cos\left(\frac{\pi z}{2}\right) \right]. \end{aligned} \quad (3.222)$$

Next, using (3.185), (3.221) and (3.222) we obtain the desired result. \square

Lemma 3.32. *For each $\eta \in [-1, 1)$ the equation*

$$(1 - \eta)\left(\cos\left(\frac{\pi z}{2}\right) - z + 1\right) - 4 \cos\left(\frac{\pi z}{2}\right) = 0 \quad \text{has no root in } \Gamma_{0,1}. \quad (3.223)$$

Proof. Introduce the function

$$F : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (3.224)$$

given by

$$F(x, y, \xi) := \Re \left((1 - \xi) \left(\cos\left(\frac{\pi(x + iy)}{2}\right) - (x + iy) + 1 \right) - 4 \cos\left(\frac{\pi(x + iy)}{2}\right) \right). \quad (3.225)$$

Straightforward algebra gives

$$F(x, y, \xi) = -(1 - \xi)x + (1 - \xi) - (3 + \xi) \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right) \quad (3.226)$$

$$\forall (x, y, \xi) \in [0, 1] \times \mathbb{R} \times \mathbb{R}.$$

Notice next that $\frac{\partial F}{\partial \xi}(x, y, \xi) = x - 1 - \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right)$. Since $x \in (0, 1)$, and the trigonometric functions $\cos\left(\frac{\pi x}{2}\right) > 0$ and $\cosh\left(\frac{\pi y}{2}\right) > 0$ for $y \in \mathbb{R}$, it follows that

$$\frac{\partial F}{\partial \xi}(x, y, \xi) < 0, \quad \forall (x, y, \xi) \in (0, 1) \times \mathbb{R} \times \mathbb{R}. \quad (3.227)$$

Going further and evaluating F at $\xi = \pm 1$ we obtain

$$F(x, y, 1) = -4 \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right) < 0, \quad \forall (x, y) \in (0, 1) \times \mathbb{R}, \quad (3.228)$$

and

$$F(x, y, -1) = 2 \left(-x + 1 - \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right) \right), \quad \forall (x, y) \in (0, 1) \times \mathbb{R}. \quad (3.229)$$

We claim next that

$$F(x, y, -1) < 0, \quad \forall (x, y) \in (0, 1) \times \mathbb{R}. \quad (3.230)$$

Assume (3.230) for a moment. Notice that the inequalities (3.228) and (3.230) in concert with (3.227) guarantee that $F(x, y, \xi) \neq 0$ for any triplet $(x, y, \xi) \in$

$(0, 1) \times \mathbb{R} \times [-1, 1)$. Thus (given that $z \in \Gamma_{0,1}$ forces $x = \Re(z) \in (0, 1)$, the definition of F from (3.225), and the fact that by hypothesis $\eta \in [-1, 1)$) the statement made in (3.223) holds.

Turn now to showing (3.230). To this end, taking partial derivatives with respect to the variable x twice in (3.226) (and, subsequently letting $\xi = -1$), we obtain

$$\frac{\partial^2 F}{\partial x^2}(x, y, -1) = \frac{\pi^2}{2} \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right). \quad (3.231)$$

Notice that (3.231) ensures

$$\frac{\partial^2 F}{\partial x^2}(x, y, -1) > 0, \quad \forall (x, y) \in [0, 1) \times \mathbb{R}. \quad (3.232)$$

it follows that, for each $y \in \mathbb{R}$ fixed, the function $F(x, y, -1)$ is concave up in $x \in [0, 1]$. Since $F(0, y, -1) = 2 - 2 \cosh\left(\frac{\pi y}{2}\right) \leq 0$ and $F(1, y, -1) = 0$ for each $y \in \mathbb{R}$, the claim (3.230) is then an immediate consequence of the concavity of $F(\cdot, y, -1)$ on $[0, 1]$. The proof of Proposition 3.32 is now complete. \square

Lemma 3.33. *For each $\eta \in [-1, 1)$ the equation*

$$(1 - \eta) \left(-\cos\left(\frac{\pi z}{2}\right) - z + 1 \right) + 4 \cos\left(\frac{\pi z}{2}\right) = 0, \quad (3.233)$$

has no root in the strip $\Gamma_{0,1}$.

Proof. For each $z = x + iy$, $x, y \in \mathbb{R}$, and $\eta \in \mathbb{R}$, denote by $G(x, y, \eta)$ the real part of the left-hand side of (3.233). Direct calculations give

$$\begin{aligned} G(x, y, \eta) &= (1 - \eta) \left(-\cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right) + 1 - x \right) \\ &\quad + 4 \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right). \end{aligned} \quad (3.234)$$

Thus

$$\frac{\partial G}{\partial \eta}(x, y, \eta) = x - 1 + \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right), \quad (3.235)$$

and

$$\frac{\partial^3 G}{\partial \eta \partial x^2}(x, y, \eta) = -\left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right). \quad (3.236)$$

Since the expression in the right-hand side of (3.236) is strictly negative for $x \in [0, 1)$ and $y \in \mathbb{R}$ it follows that for each fixed $y, \eta \in \mathbb{R}$ the function $\frac{\partial G}{\partial \eta}(x, y, \eta)$ is concave down in the variable x . Going further, notice that by (3.235) we have

$$\frac{\partial G}{\partial \eta}(0, y, \eta) = -1 + \cosh\left(\frac{\pi y}{2}\right) \geq 0 \quad \text{and} \quad \frac{\partial G}{\partial \eta}(1, y, \eta) = 0, \quad \forall y, \eta \in \mathbb{R}. \quad (3.237)$$

In turn, (3.237) and the concavity property for $\frac{\partial G}{\partial \eta}(x, y, \eta)$ give

$$\frac{\partial G}{\partial \eta}(x, y, \eta) > 0 \quad \forall x \in (0, 1) \quad \text{and} \quad \forall y, \eta \in \mathbb{R}. \quad (3.238)$$

Going further, (3.238) ensures that

$$G(x, y, \eta) \geq G(x, y, -1) = 2 \left(-x + 1 + \cos\left(\frac{\pi x}{2}\right) \cosh\left(\frac{\pi y}{2}\right) \right) > 0 \quad (3.239)$$

$$\forall x \in (0, 1), y \in \mathbb{R}, \eta \in [-1, \infty).$$

Consequently, the real part of the equation in the left-hand side of (3.233) is strictly positive for η as in the hypothesis and $z \in \Gamma_{0,1}$, finishing the proof of Lemma 3.33. \square

Lemma 3.34. *The equation*

$$z - 1 - \cos\left(\frac{3\pi z}{2}\right) = 0, \quad \text{for } z \in \mathbb{C}. \quad (3.240)$$

has precisely one root in $\Gamma_{0,1}$ and, denoting it by x , this satisfies

$$x \in \left(\frac{1}{3}, \frac{1}{2}\right). \quad (3.241)$$

Proof. We start the proof by analyzing the function

$$\Lambda : \overline{\Gamma_{0,1}} \longrightarrow \mathbb{C} \quad \text{given by} \quad \Lambda(z) := z - 1 - \cos\left(\frac{3\pi z}{2}\right), \quad (3.242)$$

where as usual bar denotes closure. Keeping in mind the natural identification of \mathbb{C} with the plane \mathbb{R}^2 , we start by introducing $F, G : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, the real and imaginary parts of the function Λ in (3.242). It is straightforward to see that

$$F(x, y) = x - 1 - \cos\left(\frac{3\pi x}{2}\right) \cosh\left(\frac{3\pi y}{2}\right), \quad \forall (x, y) \in [0, 1] \times \mathbb{R}, \quad (3.243)$$

and

$$G(x, y) = y + \sin\left(\frac{3\pi x}{2}\right) \sinh\left(\frac{3\pi y}{2}\right), \quad \forall (x, y) \in [0, 1] \times \mathbb{R}. \quad (3.244)$$

Thus,

$$(\partial_1 F)(x, y) = 1 + \frac{3\pi}{2} \sin\left(\frac{3\pi x}{2}\right) \cosh\left(\frac{3\pi y}{2}\right), \quad \forall (x, y) \in [0, 1] \times \mathbb{R}, \quad (3.245)$$

and

$$(\partial_1 G)(x, y) = \frac{3\pi}{2} \cos\left(\frac{3\pi x}{2}\right) \sinh\left(\frac{3\pi y}{2}\right) \quad \forall (x, y) \in [0, 1] \times \mathbb{R}. \quad (3.246)$$

Fix $y \in \mathbb{R}$, and consider first the behavior of the function $F(\cdot, y)$ on $[0, 1]$.

We claim that

$$(\partial_1 F)(\cdot, y) = 0 \quad \text{has a unique solution in } [0, 1], \quad (3.247)$$

and denoting this by x_o , this satisfies $x_o \in (2/3, 1)$.

To see this notice first that, since the cosine hyperbolic function never vanishes,

$$(\partial_1 F)(x, y) = 0 \iff -\frac{2}{3\pi \cosh\left(\frac{3\pi y}{2}\right)} = \sin\left(\frac{3\pi x}{2}\right), \quad (3.248)$$

and clearly

$$-\frac{2}{3\pi \cosh\left(\frac{3\pi y}{2}\right)} \in (-1, 0) \quad \text{and} \quad \frac{3\pi x}{2} \in \left[0, \frac{3\pi}{2}\right] \quad \text{for } x \in [0, 1]. \quad (3.249)$$

Based on (3.248), (3.249) and keeping in mind that $\sin x$ is nonnegative on $[0, \pi]$, $\sin \pi = 0$, $\sin\left(\frac{3\pi}{2}\right) = -1$ and the fact that the sinus function is a bijection of $[\pi, \frac{3\pi}{2}]$ onto $[-1, 0]$, the first line in the claim (3.247) immediately follows. In addition denoting by $x_o \in [0, 1]$ the root of the equation $(\partial_1 F)(\cdot, y) = 0$, this satisfies $\frac{3\pi x_o}{2} \in (\pi, \frac{3\pi}{2})$, finishing the proof of (3.247).

An immediate consequence of (3.247) and (3.245) is that

$$F(\cdot, y) \text{ is increasing on the interval } [0, x_o], \quad (3.250)$$

and $F(\cdot, y)$ is decreasing on $[x_o, 1]$.

Notice also that (3.243) immediately yields

$$F(0, y) < 0, \quad F\left(\frac{1}{3}, y\right) = -\frac{2}{3} < 0, \quad F\left(\frac{1}{2}, y\right) > 0 \quad \text{and} \quad F(1, y) = 0. \quad (3.251)$$

Combining now (3.250) with (3.251) we conclude that

$$\begin{aligned} \text{the equation } F(\cdot, y) = 0 \text{ has only one solution in } (0, 1) \\ \text{and denoting it by } x_1, \text{ this satisfies } x_1 \in \left(\frac{1}{3}, \frac{1}{2}\right). \end{aligned} \quad (3.252)$$

Next, fix $y \in (0, \infty)$ and consider the function $G(\cdot, y)$ on the interval $[0, 1]$. Since the sinus hyperbolic function is strictly positive on $(0, \infty)$ from (3.246) we see that the sign of the expression $(\partial_1 G)(x, y)$ is the same as that of $\cos\left(\frac{3\pi x}{2}\right)$ and, as such,

$$(\partial_1 G)(x, y) > 0 \text{ on } \left(0, \frac{1}{3}\right] \text{ and } (\partial_1 G)(x, y) < 0 \text{ on } \left(\frac{1}{3}, 1\right). \quad (3.253)$$

Consequently, on the one hand,

$$\text{the function } G(\cdot, y) \text{ is increasing on } \left(0, \frac{1}{3}\right] \text{ and decreasing on } \left(\frac{1}{3}, 1\right). \quad (3.254)$$

On the other hand, (3.244) immediately gives

$$G(0, y) = y > 0, \quad G\left(\frac{1}{3}, y\right) = y + \sinh\left(\frac{3\pi y}{2}\right) > 0 \text{ and } G\left(\frac{2}{3}, y\right) = y > 0. \quad (3.255)$$

At this point, (3.254) and (3.255) show that

$$\begin{aligned} \text{whenever } y > 0, \text{ the equation } G(\cdot, y) = 0 \\ \text{has no solution in the interval } \left[0, \frac{2}{3}\right]. \end{aligned} \quad (3.256)$$

Since $G(x, y) = -G(x, -y)$ on $[0, 1] \times \mathbb{R}$, based on (3.256) we can conclude that

$$\begin{aligned} \text{for each } y \in \mathbb{R} \setminus \{0\}, \text{ the equation } G(\cdot, y) = 0 \\ \text{has no solution in the interval } \left[0, \frac{2}{3}\right]. \end{aligned} \quad (3.257)$$

Thus, using (3.252) and (3.257), the real and imaginary parts of the function Λ are never simultaneously zero in $\Gamma_{0,1} \setminus \{z \in \mathbb{C} : \text{Im } z = 0\}$ proving that

$$\Lambda(z) \neq 0, \quad \forall z \in \Gamma_{0,1} \setminus \{z \in \mathbb{C} : \text{Im } z = 0\}. \quad (3.258)$$

Using (3.258), in order to finish the proof of the lemma, it suffices to show that the equation $\Lambda(x) = 0$ has a unique root on the interval $(0, 1)$ and this satisfies $x \in \left(\frac{1}{3}, \frac{1}{2}\right)$. This is immediate from the fact that $\Lambda(x) = F(x, 0)$ for each $x \in (0, 1)$ and (3.252). \square

Lemma 3.35. *The equation*

$$z - 1 + \cos\left(\frac{3\pi z}{2}\right) = 0 \quad \text{for } z \in \mathbb{C}. \quad (3.259)$$

has precisely one root in the set $\Gamma_{0,1}$ and, denoting it by x , this satisfies

$$x \in \left(\frac{1}{12}, \frac{1}{9}\right). \quad (3.260)$$

Proof. Introduce the function

$$\Gamma : \overline{\Gamma_{0,1}} \longrightarrow \mathbb{C} \quad \text{given by} \quad \Gamma(z) := z - 1 + \cos\left(\frac{3\pi z}{2}\right), \quad (3.261)$$

and denote by $F_2, G_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, its real and imaginary parts. It is immediate that, for each pair $(x, y) \in [0, 1] \times \mathbb{R}$ there holds

$$F_2(x, y) = x - 1 + \cos\left(\frac{3\pi x}{2}\right) \cosh\left(\frac{3\pi y}{2}\right) \quad (3.262)$$

$$\text{and } G_2(x, y) = y - \sin\left(\frac{3\pi x}{2}\right) \sinh\left(\frac{3\pi y}{2}\right).$$

Thus, by differentiating

$$(\partial_1 F_2)(x, y) = 1 - \frac{3\pi}{2} \sin\left(\frac{3\pi x}{2}\right) \cosh\left(\frac{3\pi y}{2}\right), \quad \forall (x, y) \in [0, 1] \times \mathbb{R}, \quad (3.263)$$

and

$$(\partial_1^2 F_2)(x, y) = -\frac{9\pi^2}{4} \cos\left(\frac{3\pi x}{2}\right) \cosh\left(\frac{3\pi y}{2}\right), \quad \forall (x, y) \in [0, 1] \times \mathbb{R}. \quad (3.264)$$

Fix $y \in \mathbb{R}$, and consider the equation $F_2(x, y) = 0$ for $x \in [0, 1]$. Using (3.264) it is easy to see that the sign of $(\partial_1^2 F_2)(x, y)$ is the same as that of $-\cos\left(\frac{3\pi x}{2}\right)$, and consequently,

$$(\partial_1^2 F_2)(x, y) < 0 \quad \text{for } x \in \left(0, \frac{1}{3}\right), \quad \text{and } (\partial_1^2 F_2)(x, y) > 0 \quad \text{for } x \in \left(\frac{1}{3}, 1\right). \quad (3.265)$$

In addition, direct evaluations in (3.263) yield

$$\begin{aligned} (\partial_1 F_2)(0, y) &= 1 > 0, \quad (\partial_1 F_2)\left(\frac{1}{3}, y\right) < 0, \\ (\partial_1 F_2)\left(\frac{2}{3}, y\right) &= 1 > 0 \quad \text{and} \quad (\partial_1 F_2)(1, y) > 0, \end{aligned} \quad (3.266)$$

where for the second and last inequalities we have used that $\cosh(\frac{3\pi y}{2}) \geq 1$. In concert, (3.265) and (3.266) allow us to deduce that $(\partial_1 F_2)(\cdot, y) = 0$ has precisely two roots in the interval $[0, 1]$ which are distinct and, denoting these by α_1, α_2 with $\alpha_1 < \alpha_2$, they satisfy $\alpha_1 \in (0, \frac{1}{3})$ and $\alpha_2 \in (\frac{1}{3}, \frac{2}{3})$. Moreover,

$$\begin{aligned} (\partial_1 F_2)(x, y) &> 0 \text{ for } x \in (0, \alpha_1) \cup (\alpha_2, 1), \\ \text{and } (\partial_1 F_2)(x, y) &< 0 \text{ for } x \in (\alpha_1, \alpha_2). \end{aligned} \quad (3.267)$$

Next, direct evaluations in (3.262) combined with (3.267) and the fact that the cosine hyperbolic function takes values in $[1, \infty)$ allow us to write

$$\begin{aligned} F_2(0, y) > 0, \quad F_2(\alpha_1, y) > 0, \quad F_2(\frac{1}{3}, y) = -\frac{2}{3} < 0, \\ F_2(\alpha_2, y) < 0 \text{ and } F_2(1, y) = 0. \end{aligned} \quad (3.268)$$

Using (3.267) and (3.268) we can therefore conclude

$$\begin{aligned} \text{the equation } F_2(\cdot, y) = 0 \text{ has precisely one solution in the} \\ \text{interval } (0, 1) \text{ and, denoting this by } x_o, \text{ this satisfies } x_o \in (\alpha_1, \frac{1}{3}). \end{aligned} \quad (3.269)$$

In particular, this shows that

$$\Gamma(z) \neq 0 \quad \forall z \in \Gamma_{0,1} \cap \left\{ z \in \mathbb{C} : \operatorname{Re} z \in (0, \alpha_1] \cup \left[\frac{1}{3}, 1\right] \right\}. \quad (3.270)$$

Consider now the case when $z \in \Gamma_{0,1}$, $\operatorname{Re} z \in (\alpha_1, \frac{1}{3})$ and $\operatorname{Im} z \in (0, \infty)$. Differentiating G_2 with respect to the variable y in (3.262) gives

$$(\partial_2 G_2)(x, y) = 1 - \frac{3\pi}{2} \sin\left(\frac{3\pi x}{2}\right) \cosh\left(\frac{3\pi y}{2}\right), \quad \forall (x, y) \in [0, 1] \times \mathbb{R}, \quad (3.271)$$

and

$$(\partial_2^2 G_2)(x, y) = -\frac{9\pi^2}{4} \sin\left(\frac{3\pi x}{2}\right) \sinh\left(\frac{3\pi y}{2}\right) \quad \forall (x, y) \in [0, 1] \times \mathbb{R}. \quad (3.272)$$

With (3.269), it is enough to study $G_2(x, y)$ for $x \in (\alpha_1, \frac{1}{3})$, since otherwise $F_2(x, y) \neq 0$. Without loss of generality we let $y \in \mathbb{R}_+ \setminus \{0\}$, and fix $x \in (\alpha_1, \frac{1}{3})$. Hence, using (3.272) we deduce that $\partial_2^2 G_2(x, y) < 0$ for all $y \in \mathbb{R}_+ \setminus \{0\}$. This

in turn implies that $\partial_2 G_2(x, y)$ is decreasing. Moving on, for $x \in (\alpha_1, \frac{1}{3})$, using (3.267) and the fact that

$$\partial_2 G_2(x, y) = \partial_1 F_2(x, y), \quad (3.273)$$

we have that $\partial_2 G_2(x, y) < 0$. Thus, $\partial_2 G_2(x, y) < 0$ for all $y \in \mathbb{R}_+ \setminus \{0\}$. But $G_2(x, 0) = 0$, hence we deduce that $G_2(x, y) \neq 0$ on $(\alpha_1, \frac{1}{3}) \times \mathbb{R}_+ \setminus \{0\}$. Thus, the real and imaginary parts of $z - 1 + \cos(\frac{3\pi z}{2})$ are never simultaneously 0. Furthermore, for $y = 0$, we have

$$x - 1 + \cos\left(\frac{3\pi x}{2}\right) = F_2(x, 0), \quad (3.274)$$

In conclusion, we deduce that

$$\begin{aligned} x - 1 + \cos\left(\frac{3\pi x}{2}\right) = 0 \text{ has exactly one solution in } \Gamma_{0,1}, \\ \text{and this satisfies } z \in \left(0, \frac{1}{3}\right). \end{aligned} \quad (3.275)$$

In addition, direct calculation yields

$$F_2(1/12, 0) = \frac{-11 + 6\sqrt{2 + \sqrt{2}}}{12} > 0, \quad F_2(1/9, 0) = -\frac{8}{9} + \frac{\sqrt{3}}{2} < 0, \quad (3.276)$$

finishing the proof of the lemma. \square

Now we are ready to prove Theorem 3.30

Proof of Theorem 3.30. According to the proposition 3.31, for each $p \in (1, \infty)$, $\xi \in \mathbb{R}$ there holds

$$\begin{aligned} \det(\mathcal{M}(R(\cdot, 1))(z)) &= \frac{(1 - \eta)^2}{16^2 \sin^4(\pi z)} \left(z - 1 - \cos\left(\frac{3\pi z}{2}\right) \right) \left(z - 1 + \cos\left(\frac{3\pi z}{2}\right) \right) \\ &\quad \times \left[(1 - \eta) \left(\cos\left(\frac{\pi z}{2}\right) - z + 1 \right) - 4 \cos\left(\frac{\pi z}{2}\right) \right] \\ &\quad \times \left[(1 - \eta) \left(-\cos\left(\frac{\pi z}{2}\right) - z + 1 \right) + 4 \cos\left(\frac{\pi z}{2}\right) \right], \end{aligned} \quad (3.277)$$

where $z = \frac{1}{p} + i\xi \in \Gamma_{0,1}$. Applying the lemma 3.32-3.35, we obtain that

$$\det(\mathcal{M}(R(\cdot, 1)))(z) \neq 0 \iff z \in \Gamma_{0,1} \setminus \{x_0, x_1\}, \quad (3.278)$$

where $x_0 \in (1/12, 1/9)$, $x_1 \in (1/3, 1/2)$ with

$$x_0 \text{ is the unique root of the equation } z - 1 + \cos\left(\frac{3\pi z}{2}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.279)$$

$$x_1 \text{ is the unique root of the equation } z - 1 - \cos\left(\frac{3\pi z}{2}\right) = 0 \text{ in } \Gamma_{0,1}. \quad (3.280)$$

Combining this with Theorem 3.10 yields

$$\mathcal{T} \text{ is invertible on } (L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+))^2 \iff p \in (1, \infty) \setminus \{x_0^{-1}, x_1^{-1}\}, \quad (3.281)$$

where $x_0^{-1} \in (9, 12)$ and $x_1^{-1} \in (2, 3)$. This completes the proof of Theorem 3.30.

At this point we are ready to present the proof of Theorem 3.2

Proof of Theorem 3.2. According to Theorem 3.30, for $p \in (1, \infty)$, $\eta \in [-1, 1)$, there holds

$$\mathcal{T} \text{ is invertible on } (L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+))^2 \iff p \in (1, \infty) \setminus \{\alpha, \beta\}, \quad (3.282)$$

where $\alpha \in (9, 12)$ and $\beta \in (2, 3)$ with

$$\alpha^{-1} \text{ is the unique root of the equation } z - 1 + \cos\left(\frac{3\pi z}{2}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.283)$$

$$\beta^{-1} \text{ is the unique root of the equation } z - 1 - \cos\left(\frac{3\pi z}{2}\right) = 0 \text{ in } \Gamma_{0,1}. \quad (3.284)$$

Consequently, this combining together with Theorem 3.28 finishes the proof of the theorem.

□

3.5.3 The case of an infinite sector with angle $\pi/4$

In this section, we investigate the values of $p \in (1, \infty)$ for which the operator \mathcal{T} in (3.22) is invertible. Our main result for the invertibility range of \mathcal{T}

in this section is as follows.

Theorem 3.36. *Let Ω be the interior of the infinite sector of aperture $\frac{\pi}{4}$ from (3.216). Then for $\eta = -1$ and for $p \in (1, \infty)$, there holds*

$$\mathcal{T} \text{ is invertible on } (L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+))^2 \iff p \in (1, \infty) \setminus \{\gamma, \delta\}, \quad (3.285)$$

where $\gamma \in (2.8, 3)$ and $\delta \in (2, 2.1)$ with

$$\gamma^{-1} \text{ is the unique root of } z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.286)$$

$$\delta^{-1} \text{ is the unique root of } z - 1 - \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.287)$$

where $\Gamma_{0,1}$ denotes the complex strip as in (3.62).

In order to prove Theorem 3.36, we need several preliminary steps. We start by applying the theorem 3.27 when the domain Ω is an infinite sector of an angle $\pi/4$. According to (3.119),

$$a(z) := \frac{\cos(\pi z)}{2 \sin(\pi z)}, \quad b_{\pi/4}(z) := \frac{\cos\left(\frac{3\pi z}{4}\right)}{2 \sin(\pi z)}, \quad d_{\pi/4}(z) := \frac{\sin\left(\frac{3\pi z}{4}\right)}{2 \sin(\pi z)}. \quad (3.288)$$

Going further, from (3.208), we have

$$\begin{aligned} G_0 &= -z \cos\left(\frac{3\pi z}{4}\right) - (z+2) \sin\left(\frac{3\pi z}{4}\right), \\ G_1 &= \sqrt{2} \cdot \left[z \cos\left(\frac{3\pi z}{4}\right) + \sin\left(\frac{3\pi z}{4}\right) \right], \\ G_2 &= -z \cos\left(\frac{3\pi z}{4}\right) + (z-2) \sin\left(\frac{3\pi z}{4}\right), \\ G_3 &= \sqrt{2} \cdot \left[\cos\left(\frac{3\pi z}{4}\right) - (z-2) \sin\left(\frac{3\pi z}{4}\right) \right]. \end{aligned} \quad (3.289)$$

Combining this with (3.150), (3.157), (4.99), and (3.169) provides

$$\begin{aligned}
A_{\pi/4}(z) &= \frac{1}{4\sqrt{2}\sin(\pi z)} \left(\frac{\sqrt{2}}{2} \cdot G_0 + 3 \cdot G_1 + 2\sqrt{2} \cdot G_2 + G_3 \right) \\
&= \frac{1}{8\sin(\pi z)} \left((z+2) \cos\left(\frac{3\pi z}{4}\right) + z \sin\left(\frac{3\pi z}{4}\right) \right), \\
B_{\pi/4}(z) &= -\frac{1}{8\sin(\pi z)} \left(G_0 - 2 \cdot G_2 - \sqrt{2} \cdot G_3 \right) \\
&= -\frac{1}{8\sin(\pi z)} \left((z-2) \cos\left(\frac{3\pi z}{4}\right) - (z+2) \sin\left(\frac{3\pi z}{4}\right) \right), \\
C_{\pi/4}(z) &= \frac{1}{8\sin(\pi z)} \left(G_0 + \sqrt{2} \cdot G_1 \right) \\
&= \frac{1}{8\sin(\pi z)} \left(z \cos\left(\frac{3\pi z}{4}\right) - z \sin\left(\frac{3\pi z}{4}\right) \right), \quad \text{and} \\
D_{\pi/4}(z) &= \frac{1}{4\sqrt{2}\sin(\pi z)} \left(\frac{\sqrt{2}}{2} \cdot G_0 + 2 \cdot G_1 + \sqrt{2} \cdot G_2 \right) \\
&= \frac{1}{8\sin(\pi z)} \left(z \cos\left(\frac{3\pi z}{4}\right) + (z-2) \sin\left(\frac{3\pi z}{4}\right) \right). \tag{3.290}
\end{aligned}$$

Proposition 3.37. *Let Ω be an infinite sector as in (3.216) with aperture $\theta = \frac{\pi}{4}$, let R as in (3.23), recall the functions $a, b_{\pi/4}, d_{\pi/4}$ from (3.288), and $A_{\pi/4}, B_{\pi/4}, C_{\pi/4}, D_{\pi/4}$ from (3.290). Then, for each $z \in \Gamma_{0,1}$, $\eta \in [-1, 1]$ there holds*

$$\begin{aligned}
&\det(\mathcal{M}(R(\cdot, 1)))(z) \\
&= \left(\frac{1}{32\sin^2(\pi z)} \right)^2 \times \\
&\quad \times \left((\eta-1)z - (F_+ + E)(z, \eta) \right) \left((\eta-1)z - (F_- + E)(z, \eta) \right) \times \\
&\quad \times \left((\eta-1)z - (F_+ - E)(z, \eta) \right) \left((\eta-1)z - (F_- - E)(z, \eta) \right). \tag{3.291}
\end{aligned}$$

Here,

$$F_{\pm}(z, \eta) := (\eta-1) \pm \sqrt{\Delta/4}, \quad E(z, \eta) := \frac{(\eta+1)\alpha}{2} - 2\beta, \tag{3.292}$$

$$\Delta/4 = \frac{1}{16} \left[(\eta - 3)^2 (\alpha_1 + \beta_1)^2 + (\eta + 5)^2 (\alpha_2 - \beta_2)^2 - 2(7\eta^2 + 14\eta - 9) \cdot (\alpha_1 + \beta_1)(\alpha_2 - \beta_2) \right], \quad (3.293)$$

where

$$\begin{aligned} \alpha_1 &= \cos\left(\frac{7\pi z}{4}\right), & \alpha_2 &= \cos\left(\frac{\pi z}{4}\right), \\ \beta_1 &= \sin\left(\frac{7\pi z}{4}\right), & \beta_2 &= \sin\left(\frac{\pi z}{4}\right). \end{aligned} \quad (3.294)$$

Proof. According to (3.186) and (3.187), we have

$$\begin{aligned} &\Phi_{\pi/4}(z, \eta) \quad (3.295) \\ &= \frac{(1 - \eta)}{16 \sin^2(\pi z)} \cdot \left\{ (\eta - 1) \cos^2(\pi z) - 4 \sin^2(\pi z) \right. \\ &\quad \left. + \frac{(\eta - 1)}{2} \cdot \left[z^2 - 2z + \frac{4}{(\eta - 1)} \right] + (\eta + 3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \right\}, \end{aligned}$$

and

$$\begin{aligned} \Psi_{\pi/4}(z, \eta) &= \frac{(1 - \eta)}{16 \sin^2(\pi z)} \cdot \left\{ \frac{(1 + \eta)(1 - z)}{2} \cos(\pi z) \cos\left(\frac{3\pi z}{4}\right) \right. \\ &\quad \left. + \frac{(1 + \eta)(1 - z)}{2} \cos(\pi z) \sin\left(\frac{3\pi z}{4}\right) \right. \\ &\quad \left. + 2(1 - z) \sin(\pi z) \sin\left(\frac{3\pi z}{4}\right) - 2(1 - z) \sin(\pi z) \cos\left(\frac{3\pi z}{4}\right) \right\}. \end{aligned} \quad (3.296)$$

This forces

$$\begin{aligned}
& \Phi_{\pi/4}(z, \eta) + \Psi_{\pi/4}(z, \eta) \tag{3.297} \\
&= \frac{(1-\eta)}{16 \sin^2(\pi z)} \cdot \left\{ (\eta-1) \cos^2(\pi z) - 4 \sin^2(\pi z) + \frac{(\eta-1)z^2}{2} - z(\eta-1) \right. \\
&\quad + 2 + (\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \\
&\quad + \frac{(1+\eta)(1-z)}{2} \cos(\pi z) \cos\left(\frac{3\pi z}{4}\right) + \frac{(1+\eta)(1-z)}{2} \cos(\pi z) \sin\left(\frac{3\pi z}{4}\right) \\
&\quad \left. + 2(1-z) \sin(\pi z) \sin\left(\frac{3\pi z}{4}\right) - 2(1-z) \sin(\pi z) \cos\left(\frac{3\pi z}{4}\right) \right\}.
\end{aligned}$$

In particular,

$$\begin{aligned}
& 2 \times \left\{ (\eta-1) \cos^2(\pi z) - 4 \sin^2(\pi z) + \frac{(\eta-1)z^2}{2} \right. \tag{3.298} \\
&\quad - z(\eta-1) + 2 + (\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \\
&\quad + \frac{(1+\eta)(1-z)}{2} \cos(\pi z) \cos\left(\frac{3\pi z}{4}\right) + \frac{(1+\eta)(1-z)}{2} \cos(\pi z) \sin\left(\frac{3\pi z}{4}\right) \\
&\quad \left. + 2(1-z) \sin(\pi z) \sin\left(\frac{3\pi z}{4}\right) - 2(1-z) \sin(\pi z) \cos\left(\frac{3\pi z}{4}\right) \right\} \\
&= (\eta-1)z^2 + 2 \left\{ -\eta + 1 - \frac{1+\eta}{2} \cos(\pi z) \cdot \left[\cos\left(\frac{3\pi z}{4}\right) + \sin\left(\frac{3\pi z}{4}\right) \right] \right. \\
&\quad \left. + 2 \sin(\pi z) \cdot \left[\cos\left(\frac{3\pi z}{4}\right) - \sin\left(\frac{3\pi z}{4}\right) \right] \right\} z \\
&\quad + 2 \cdot \left\{ (\eta-1) \cos^2(\pi z) - 4 \sin^2(\pi z) + 2 + (\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \right. \\
&\quad + \frac{1+\eta}{2} \cos(\pi z) \cdot \left[\cos\left(\frac{3\pi z}{4}\right) + \sin\left(\frac{3\pi z}{4}\right) \right] \\
&\quad \left. - 2 \sin(\pi z) \cdot \left[\cos\left(\frac{3\pi z}{4}\right) - \sin\left(\frac{3\pi z}{4}\right) \right] \right\}.
\end{aligned}$$

This forces, the discriminant Δ for the equation

$$\frac{32 \sin^2(\pi z)}{(1-\eta)} \cdot \left[\Phi_{\pi/4}(z, \eta) + \Psi_{\pi/4}(z, \eta) \right] = 0 \tag{3.299}$$

satisfies

$$\begin{aligned}
& \Delta/4 \tag{3.300} \\
&= \left\{ -\eta + 1 - \frac{1+\eta}{2} \cos(\pi z) \cdot \left[\cos\left(\frac{3\pi z}{4}\right) + \sin\left(\frac{3\pi z}{4}\right) \right] \right. \\
&\quad \left. + 2 \sin(\pi z) \cdot \left[\cos\left(\frac{3\pi z}{4}\right) - \sin\left(\frac{3\pi z}{4}\right) \right] \right\}^2 - 2(\eta-1) \cdot \left\{ (\eta-1) \cos^2(\pi z) \right. \\
&\quad \left. - 4 \sin^2(\pi z) + 2 + (\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \right. \\
&\quad \left. + \frac{1+\eta}{2} \cos(\pi z) \cdot \left[\cos\left(\frac{3\pi z}{4}\right) + \sin\left(\frac{3\pi z}{4}\right) \right] \right. \\
&\quad \left. - 2 \sin(\pi z) \cdot \left[\cos\left(\frac{3\pi z}{4}\right) - \sin\left(\frac{3\pi z}{4}\right) \right] \right\} \\
&= \left(\frac{(1+\eta)}{2} \alpha' - 2\beta' \right)^2 + (\eta-1) \cdot \left[(\eta-5) - 2(\eta-1) \cos^2(\pi z) + 8 \sin^2(\pi z) \right. \\
&\quad \left. - 2(\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \right],
\end{aligned}$$

where

$$\begin{aligned}
\alpha' &= \cos(\pi z) \left(\cos\left(\frac{3\pi z}{4}\right) + \sin\left(\frac{3\pi z}{4}\right) \right), \\
\beta' &= \sin(\pi z) \left(\cos\left(\frac{3\pi z}{4}\right) - \sin\left(\frac{3\pi z}{4}\right) \right). \tag{3.301}
\end{aligned}$$

In addition,

$$\begin{aligned}
& (\eta-5) - 2(\eta-1) \cos^2(\pi z) + 8 \sin^2(\pi z) - 2(\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \\
&= -(\eta+3) \cdot \left[\cos(2\pi z) + \sin\left(\frac{3\pi z}{2}\right) \right]. \tag{3.302}
\end{aligned}$$

This further implies that

$$\Delta/4 = \left(\frac{(1+\eta)}{2} \alpha' - 2\beta' \right)^2 - (\eta-1)(\eta+3) \cdot \left[\cos(2\pi z) + \sin\left(\frac{3\pi z}{2}\right) \right]. \tag{3.303}$$

Going further, we use the following trigonometric identities.

$$\begin{aligned}\alpha' &= \frac{1}{2}(\alpha_1 + \alpha_2 + \beta_1 - \beta_2), \\ \beta' &= \frac{1}{2}(\alpha_1 - \alpha_2 + \beta_1 + \beta_2), \\ \cos(2\pi z) &= \alpha_1\alpha_2 - \beta_1\beta_2, \\ \sin\left(\frac{3\pi z}{2}\right) &= \alpha_2\beta_1 - \alpha_1\beta_2,\end{aligned}\tag{3.304}$$

where α_1, β_1 are as in (3.294). This yields,

$$\begin{aligned}\Delta/4 &= \frac{(1+\eta)^2}{16}(\alpha_1 + \alpha_2 + \beta_1 - \beta_2)^2 + (\alpha_1 - \alpha_2 + \beta_1 + \beta_2)^2 \\ &\quad - \frac{(1+\eta)}{2}(\alpha_1 + \alpha_2 + \beta_1 - \beta_2)(\alpha_1 - \alpha_2 + \beta_1 + \beta_2) \\ &\quad - (\eta - 1)(\eta + 3) \cdot [\alpha_1\alpha_2 - \beta_1\beta_2 + \alpha_2\beta_1 - \alpha_1\beta_2] \\ &= \frac{1}{16} \left[(\eta - 3)^2(\alpha_1 + \beta_1)^2 + (\eta + 5)^2(\alpha_2 - \beta_2)^2 \right. \\ &\quad \left. - 2(7\eta^2 + 14\eta - 9) \cdot (\alpha_1 + \beta_1)(\alpha_2 - \beta_2) \right].\end{aligned}\tag{3.305}$$

This implies that

$$\begin{aligned}\frac{32 \sin^2(\pi z)}{1 - \eta} \cdot [\Phi_{\pi/4}(z, \eta) + \Psi_{\pi/4}(z, \eta)] &= 0 \\ \iff z = \frac{1}{(\eta - 1)} \cdot [F_{\pm}(z, \eta) + E(z, \eta)],\end{aligned}\tag{3.306}$$

where $F_{\pm}(z, \eta), E(z, \eta)$ are as in (3.292). This forces

$$\begin{aligned}\Phi_{\pi/4}(z, \eta) + \Psi_{\pi/4}(z, \eta) \\ = -\frac{1}{32 \sin^2(\pi z)} \cdot \left((\eta - 1)z - (F_+ + E)(z, \eta) \right) \left((\eta - 1)z - (F_- + E)(z, \eta) \right).\end{aligned}\tag{3.307}$$

Similarly,

$$\begin{aligned}
& \Phi_{\pi/4}(z, \eta) - \Psi_{\pi/4}(z, \eta) \tag{3.308} \\
&= \frac{(1-\eta)}{16 \sin^2(\pi z)} \cdot \left\{ (\eta-1) \cos^2(\pi z) - 4 \sin^2(\pi z) + \frac{(\eta-1)z^2}{2} - z(\eta-1) + 2 \right. \\
&\quad + (\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \\
&\quad - \frac{(1+\eta)(1-z)}{2} \cos(\pi z) \cos\left(\frac{3\pi z}{4}\right) - \frac{(1+\eta)(1-z)}{2} \cos(\pi z) \sin\left(\frac{3\pi z}{4}\right) \\
&\quad \left. - 2(1-z) \sin(\pi z) \sin\left(\frac{3\pi z}{4}\right) + 2(1-z) \sin(\pi z) \cos\left(\frac{3\pi z}{4}\right) \right\},
\end{aligned}$$

and

$$\begin{aligned}
& 2 \times \left\{ (\eta-1) \cos^2(\pi z) - 4 \sin^2(\pi z) + \frac{(\eta-1)z^2}{2} \right. \tag{3.309} \\
&\quad - z(\eta-1) + 2 + (\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) \\
&\quad - \frac{(1+\eta)(1-z)}{2} \cos(\pi z) \cos\left(\frac{3\pi z}{4}\right) - \frac{(1+\eta)(1-z)}{2} \cos(\pi z) \sin\left(\frac{3\pi z}{4}\right) \\
&\quad \left. - 2(1-z) \sin(\pi z) \sin\left(\frac{3\pi z}{4}\right) + 2(1-z) \sin(\pi z) \cos\left(\frac{3\pi z}{4}\right) \right\} \\
&= (\eta-1)z^2 + 2 \left\{ -\eta + 1 + \frac{(1+\eta)}{2} \alpha - 2\beta \right\} z \\
&\quad + 2 \cdot \left\{ (\eta-1) \cos^2(\pi z) - 4 \sin^2(\pi z) + 2 \right. \\
&\quad \left. + (\eta+3) \cdot \cos\left(\frac{3\pi z}{4}\right) \sin\left(\frac{3\pi z}{4}\right) - \frac{(1+\eta)}{2} \alpha + 2\beta \right\}.
\end{aligned}$$

Going further, based on the calculation the discriminant for the equation

$$\frac{32 \sin^2(\pi z)}{(1-\eta)} \cdot \left[\Phi_{\pi/4}(z, \eta) - \Psi_{\pi/4}(z, \eta) \right] = 0 \tag{3.310}$$

is identically equal to the discriminant Δ for the equation in (3.299) which has

$$\begin{aligned} \Delta/4 = \frac{1}{16} & \left[(\eta - 3)^2(\alpha_1 + \beta_1)^2 + (\eta + 5)^2(\alpha_2 - \beta_2)^2 \right. \\ & \left. - 2(7\eta^2 + 14\eta - 9) \cdot (\alpha_1 + \beta_1)(\alpha_2 - \beta_2) \right]. \end{aligned} \quad (3.311)$$

This yields the solution for the equation in (3.310) satisfies

$$z = \frac{1}{(\eta - 1)} \cdot \left[F_{\pm}(z, \eta) - E(z, \eta) \right], \quad (3.312)$$

where $F_{\pm}(z, \eta)$, $E(z, \eta)$ are as in (3.292). In particular,

$$\begin{aligned} & \Phi_{\pi/4}(z, \eta) - \Psi_{\pi/4}(z, \eta) \\ &= - \frac{1}{32 \sin^2(\pi z)} \cdot \left((\eta - 1)z - (F_+ - E)(z, \eta) \right) \left((\eta - 1)z - (F_- - E)(z, \eta) \right). \end{aligned} \quad (3.313)$$

According to Theorem 3.27, for each $z \in \Gamma_{0,1}$ and R as in (3.23) we have

$$\begin{aligned} & \det(\mathcal{M}(R(\cdot, 1)))(z) \\ &= \left(\frac{1}{32 \sin^2(\pi z)} \right)^2 \times \\ & \quad \times \left((\eta - 1)z - (F_+ + E)(z, \eta) \right) \left((\eta - 1)z - (F_- + E)(z, \eta) \right) \times \\ & \quad \times \left((\eta - 1)z - (F_+ - E)(z, \eta) \right) \left((\eta - 1)z - (F_- - E)(z, \eta) \right), \end{aligned} \quad (3.314)$$

where $F_{\pm}(z, \eta)$, $E(z, \eta)$ are as in (3.292). This completes the proof of the proposition. \square

Next, we investigate the solvability range of the integrability exponent p for the Neumann problem in (3.2) when the Poisson ratio $\eta \in [-1, 1)$ is the endpoint -1 .

Proposition 3.38. *Let the Poisson ratio η is -1 . With the same background*

hypotheses as in Proposition 3.37 for each $z \in \Gamma_{0,1}$ there holds

$$\begin{aligned} & \det(\mathcal{M}(R(\cdot, 1)))(z) \tag{3.315} \\ &= \left(\frac{1}{8 \sin^2(\pi z)} \right)^2 \times \\ & \times \left(z - 1 + \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right) \right) \left(z - 1 - \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right) \right) \times \\ & \times \left(z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) \right) \left(z - 1 - \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) \right). \end{aligned}$$

Proof. Let us assume that $\eta = -1$. From (3.292), we have

$$F_{\pm}(z, -1) := F_{\pm}(z) = -2 \pm \sqrt{\Delta/4}, \quad E(z, -1) := E(z) = -2\beta', \tag{3.316}$$

where

$$\Delta/4 = (\alpha_1 + \beta_1 + \alpha_2 - \beta_2)^2 = (2\alpha')^2, \tag{3.317}$$

which forces

$$F_{\pm}(z) = -2 \pm 2\alpha', \quad E(z) = -2\beta', \tag{3.318}$$

where α', β' are as in (3.301). Substituting these into (3.291), we obtain that

$$\begin{aligned} & \det(\mathcal{M}(R(\cdot, 1)))(z) \tag{3.319} \\ &= \left(\frac{1}{8 \sin^2(\pi z)} \right)^2 \left(z - (1 - \alpha' + \beta') \right) \left(z - (1 + \alpha' + \beta') \right) \times \\ & \times \left(z - (1 - \alpha' - \beta') \right) \left(z - (1 + \alpha' - \beta') \right). \end{aligned}$$

According to (3.301),

$$\begin{aligned} \alpha' &= \cos(\pi z) \left(\cos\left(\frac{3\pi z}{4}\right) + \sin\left(\frac{3\pi z}{4}\right) \right), \\ \beta' &= \sin(\pi z) \left(\cos\left(\frac{3\pi z}{4}\right) - \sin\left(\frac{3\pi z}{4}\right) \right). \end{aligned} \tag{3.320}$$

We rewrite these as,

$$\begin{aligned} \alpha' &= \sqrt{2} \cos(\pi z) \cos\left(\frac{\pi}{4}(1 - 3z)\right), \\ \beta' &= \sqrt{2} \sin(\pi z) \sin\left(\frac{\pi}{4}(1 - 3z)\right). \end{aligned} \tag{3.321}$$

Note that

$$\begin{aligned}\alpha' + \beta' &= \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right), \\ \alpha' - \beta' &= \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right).\end{aligned}\tag{3.322}$$

Combining these with (3.319), we have

$$\begin{aligned}& \det(\mathcal{M}(R(\cdot, 1)))(z) \\ &= \left(\frac{1}{8 \sin^2(\pi z)}\right)^2 \times \\ & \quad \times \left(z - 1 + \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right)\right) \left(z - 1 - \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right)\right) \times \\ & \quad \times \left(z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right)\right) \left(z - 1 - \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right)\right),\end{aligned}\tag{3.323}$$

which finishes the proof of the proposition. \square

Turning our attention to find the roots of $\det(\mathcal{M}(R(\cdot, 1)))(z)$ in (3.315) to treat the biharmonic Neumann problem with specific domain and Poisson ratio.

Lemma 3.39. *The equations*

$$z - 1 + \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right) = 0,\tag{3.324}$$

$$z - 1 - \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right) = 0,\tag{3.325}$$

have no root in the strip $\Gamma_{0,1}$.

Proof. For $x \in (0, 1)$, $y \in \mathbb{R}$, we consider the function $\zeta_1(x, y)$ given by

$$\zeta_1(x, y) := x - 1 + \sqrt{2} \cos\left(\frac{(x+1)\pi}{4}\right) \cosh\left(\frac{y\pi}{4}\right),\tag{3.326}$$

which indicates the real part of the left-hand side of (3.324). Then

$$(\partial_1^2 \zeta_1)(x, y) = -\sqrt{2} \left(\frac{\pi}{4}\right)^2 \cos\left(\frac{(x+1)\pi}{4}\right) \cosh\left(\frac{y\pi}{4}\right).\tag{3.327}$$

Since $x \in (0, 1)$, $\cos\left(\frac{(x+1)\pi}{4}\right) > 0$, this implies that $(\partial_1^2 \zeta_1)(x, y) < 0$ for any $x \in (0, 1)$, $y \in \mathbb{R}$. This forces $\zeta_1(x, y)$ is concave down in the variable x in $(0, 1)$. Going further, in (3.326), notice that

$$\zeta_1(0, y) = -1 + \cosh\left(\frac{y\pi}{4}\right) \geq 0, \quad \zeta_1(1, y) = 0, \quad \forall y \in \mathbb{R}. \quad (3.328)$$

Combining this with the concavity gives

$$\zeta_1(x, y) > 0, \quad \forall x \in (0, 1), y \in \mathbb{R}, \quad (3.329)$$

which implies that the real part of the left-hand side of (3.324) has no root in $\Gamma_{0,1}$. This proves the equation (3.324) has no root in the strip $\Gamma_{0,1}$. Similarly, for $x \in (0, 1)$, $y \in \mathbb{R}$, consider $\zeta_2(x, y)$ given by

$$\zeta_2(x, y) := x - 1 - \sqrt{2} \cos\left(\frac{(x+1)\pi}{4}\right) \cosh\left(\frac{y\pi}{4}\right), \quad (3.330)$$

which is the real part of the left-hand side of (3.325). With the similar argument, we conclude that $\zeta_{\textcircled{a}}(x, y)$ is concave up in the variable x in $(0, 1)$ and

$$\zeta_2(x, y) > 0, \quad \forall x \in (0, 1), y \in \mathbb{R}, \quad (3.331)$$

which forces the real part of the left-hand side of (3.325) has no root in $\Gamma_{0,1}$. This completes the proof of the lemma. \square

Lemma 3.40. *The equation*

$$z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \quad (3.332)$$

has precisely one root in the strip $\Gamma_{0,1}$ and, denoting it by x , this satisfies

$$x \in \left(\frac{1}{3}, \frac{5}{14}\right). \quad (3.333)$$

Proof. For $z \in \overline{\Gamma_{0,1}}$ define the function Λ_3 by

$$\Lambda_3(z) := z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right), \quad (3.334)$$

where bar denotes closure. For $x \in [0, 1]$, $y \in \mathbb{R}$, we consider the functions $\zeta_3(x, y)$, $\xi_3(x, y)$ given by

$$\zeta_3(x, y) := x - 1 + \sqrt{2} \cos\left(\frac{(7x-1)\pi}{4}\right) \cosh\left(\frac{7y\pi}{4}\right), \quad (3.335)$$

and

$$\xi_3(x, y) := y - \sqrt{2} \sin\left(\frac{(7x-1)\pi}{4}\right) \sinh\left(\frac{7y\pi}{4}\right), \quad (3.336)$$

which are real and imaginary parts of Λ_3 , respectively. Then,

$$(\partial_1 \zeta_3)(x, y) = 1 - \frac{7\sqrt{2}\pi}{4} \sin\left(\frac{(7x-1)\pi}{4}\right) \cosh\left(\frac{7y\pi}{4}\right), \quad (3.337)$$

and

$$(\partial_1 \xi_3)(x, y) = -\frac{7\sqrt{2}\pi}{4} \cos\left(\frac{(7x-1)\pi}{4}\right) \sinh\left(\frac{7y\pi}{4}\right), \quad (3.338)$$

for $(x, y) \in [0, 1] \times \mathbb{R}$. Fix $y \in \mathbb{R}$. Since $\cosh\left(\frac{7y\pi}{4}\right) > 0$, we get

$$(\partial_1 \zeta_3)(x, y) = 0 \quad (3.339)$$

$$\iff \frac{4}{7\sqrt{2}\pi \cosh\left(\frac{7y\pi}{4}\right)} = \sin\left(\frac{(7x-1)\pi}{4}\right), \quad (3.340)$$

and note that

$$\frac{4}{7\sqrt{2}\pi \cosh\left(\frac{7y\pi}{4}\right)} \in \left(0, \frac{\sqrt{2}}{2}\right) \quad \text{and} \quad \frac{(7x-1)\pi}{4} \in \left[-\frac{\pi}{4}, \frac{3\pi}{2}\right]. \quad (3.341)$$

Since the left-hand side of (3.340) is in $(0, \sqrt{2}/2)$, the range of $\frac{(7x-1)\pi}{4}$ has to be $(0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$ which implies that $x \in (\frac{1}{7}, \frac{2}{7}) \cup (\frac{4}{7}, \frac{5}{7})$. Suppose that $(\partial_1 \zeta_3)(\cdot, y) = 0$ has roots in $(4/7, 5/7)$. Since sinus function is bijective in $[\pi/2, 3\pi/2]$, this yields

$$(\partial_1 \zeta_3)(\cdot, y) = 0 \quad \text{has a unique solution in } [3/7, 1], \quad (3.342)$$

and denoting this by τ_0 , this satisfies $\tau_0 \in (4/7, 5/7)$.

Combining this with (3.337), we conclude that

$$\zeta_3(\cdot, y) \quad \text{is decreasing on the interval } [3/7, \tau_0], \quad (3.343)$$

and $\zeta_3(\cdot, y)$ is increasing on $(\tau_0, 1]$.

According to (3.335), we notice that

$$\zeta_3(3/7, y) = -4/7 < 0, \quad \zeta_3(1, y) = 0. \quad (3.344)$$

Combining this with (3.343) we conclude that

$$\zeta_3(\cdot, y) \text{ has no root in } [3/7, 1). \quad (3.345)$$

Next, suppose that $(\partial_1 \zeta_3)(\cdot, y) = 0$ has roots in $(1/7, 2/7)$. Since sinus function is bijective in $[0, \pi/2]$ which yields,

$$(\partial_1 \zeta_3)(\cdot, y) = 0 \text{ has a unique solution in } [0, 3/7], \quad (3.346)$$

and denoting this by τ_0' , this satisfies $\tau_0' \in (1/7, 2/7)$.

Combining this with (3.337), one can conclude that

$$\zeta_3(\cdot, y) \text{ is increasing on the interval } [0, \tau_0'), \quad (3.347)$$

and $\zeta_3(\cdot, y)$ is decreasing on $(\tau_0', 3/7]$.

According to (3.335), one can notice that

$$\zeta_3(0, y) \geq 0, \quad \zeta_3(1/7, y) > 0, \quad \zeta_3(2/7, y) > 0, \quad \zeta_3(3/7, y) = -4/7 < 0. \quad (3.348)$$

Combining this with (3.343), one can conclude that

$$\text{the equation } \zeta_3(\cdot, y) = 0 \text{ has only one solution in } (0, 3/7], \quad (3.349)$$

and denoting it by τ_1 , this satisfies $\tau_1 \in \left(\frac{2}{7}, \frac{3}{7}\right)$.

Consequently, on the one hand, combining (3.349) with (3.345) we get

$$\text{the equation } \zeta_3(\cdot, y) = 0 \text{ has only one solution } \tau_1 \text{ in } (0, 1), \quad (3.350)$$

and this satisfies $\tau_1 \in \left(\frac{2}{7}, \frac{3}{7}\right)$.

Fix $y \in (-\infty, 0)$ and consider the function $\xi_3(\cdot, y)$ on the interval $[0, 1]$. Since the sinus hyperbolic function is strictly negative on $(-\infty, 0)$, combining

this with (3.338) provides that $(\partial_1 \xi_3)(x, y)$ has same sign as that of $\cos\left(\frac{(7x-1)\pi}{4}\right)$ and, as such

$$(\partial_1 \xi_3)(x, y) \geq 0 \quad \text{on} \quad \left[0, \frac{3}{7}\right] \quad \text{and} \quad (\partial_1 \xi_3)(x, y) \leq 0 \quad \text{on} \quad \left[\frac{3}{7}, 1\right]. \quad (3.351)$$

This yields,

$$\text{the function } \xi_3(\cdot, y) \text{ is increasing on } \left[0, \frac{3}{7}\right] \text{ and decreasing on } \left[\frac{3}{7}, 1\right]. \quad (3.352)$$

On the other hand, since $x > \sinh(x)$ for $x < 0$, (3.336) immediately gives for $y < 0$

$$\xi_3(2/7, y) = y - \sinh\left(\frac{7y\pi}{4}\right) > 0, \quad \xi_3(3/7, y) = y - \sqrt{2} \sinh\left(\frac{7y\pi}{4}\right) > 0, \quad (3.353)$$

which forces

$$\text{whenever } y < 0, \text{ the equation } \xi_3(\cdot, y) = 0 \quad (3.354)$$

$$\text{has no root in the interval } \left(\frac{2}{7}, \frac{3}{7}\right).$$

Since $\xi_3(x, y) = -\xi_3(x, -y)$ on $[0, 1] \times \mathbb{R}$, based on (3.354) one can conclude that

$$\text{for each } y \in \mathbb{R} \setminus \{0\}, \text{ the equation } \xi_3(\cdot, y) = 0 \quad (3.355)$$

$$\text{has no solution in the interval } \left(\frac{2}{7}, \frac{3}{7}\right).$$

Thus, using (3.355) with (3.350), we prove that

$$\Lambda_3(z) \neq 0, \quad \forall z \in \Gamma_{0,1} \setminus \{z \in \mathbb{C} : \text{Im } z = 0\}. \quad (3.356)$$

In addition, based on the direct calculation, one can conclude that

$$\zeta_3(1/3, 0) = 1/3 > 0, \quad \zeta_3(5/14, 0) = -1/2 + \sqrt{2 - \sqrt{2}} < 0. \quad (3.357)$$

Going further, since $\Lambda_3(x) = \zeta_3(x, 0)$ we obtain that

$$\Lambda_3(x) = 0 \quad \text{has a unique root } x \in \left(\frac{1}{3}, \frac{5}{14}\right) \text{ on the interval } (0, 1). \quad (3.358)$$

This finishes the proof of the lemma. \square

Lemma 3.41. *The equation*

$$z - 1 - \sqrt{2} \cos \left(\frac{(7z - 1)\pi}{4} \right) = 0, \quad (3.359)$$

has precisely one root in the strip $\Gamma_{0,1}$ and, denoting it by x , this satisfies

$$x \in \left(\frac{10}{21}, \frac{1}{2} \right). \quad (3.360)$$

Proof. Fix $z \in \overline{\Gamma_{0,1}}$ define the function Λ_4 by

$$\Lambda_4(z) := z - 1 - \sqrt{2} \cos \left(\frac{(7z - 1)\pi}{4} \right), \quad (3.361)$$

where bar denotes closure. Fix $x \in [0, 1]$, $y \in \mathbb{R}$, we consider the function $\zeta_4(x, y)$, $\xi_4(x, y)$ given by

$$\zeta_4(x, y) := x - 1 - \sqrt{2} \cos \left(\frac{(7x - 1)\pi}{4} \right) \cosh \left(\frac{7y\pi}{4} \right), \quad (3.362)$$

and

$$\xi_4(x, y) := y + \sqrt{2} \sin \left(\frac{(7x - 1)\pi}{4} \right) \sinh \left(\frac{7y\pi}{4} \right), \quad (3.363)$$

which are real and imaginary parts of Λ_4 , respectively. Then,

$$(\partial_1 \zeta_4)(x, y) = 1 + \frac{7\sqrt{2}\pi}{4} \sin \left(\frac{(7x - 1)\pi}{4} \right) \cosh \left(\frac{7y\pi}{4} \right), \quad (3.364)$$

and

$$(\partial_1 \xi_4)(x, y) = \frac{7\sqrt{2}\pi}{4} \cos \left(\frac{(7x - 1)\pi}{4} \right) \sinh \left(\frac{7y\pi}{4} \right), \quad (3.365)$$

for $(x, y) \in [0, 1] \times \mathbb{R}$. Fix $y \in \mathbb{R}$. Since $\cosh \left(\frac{7y\pi}{4} \right) > 0$, we get

$$(\partial_1 \zeta_4)(x, y) = 0 \quad (3.366)$$

$$\iff -\frac{4}{7\sqrt{2}\pi \cosh \left(\frac{7y\pi}{4} \right)} = \sin \left(\frac{(7x - 1)\pi}{4} \right), \quad (3.367)$$

and note that

$$-\frac{4}{7\sqrt{2}\pi \cosh \left(\frac{7y\pi}{4} \right)} \in \left(-\frac{\sqrt{2}}{2}, 0 \right) \quad \text{and} \quad \frac{(7x - 1)\pi}{4} \in \left[-\frac{\pi}{4}, \frac{3\pi}{2} \right]. \quad (3.368)$$

Since the left-hand side of (3.367) is in $(-\sqrt{2}/2, 0)$, the range of $\frac{(7x-1)\pi}{4}$ has to be $(-\frac{\pi}{4}, 0) \cup (\pi, \frac{5\pi}{4})$ which implies that $x \in (0, \frac{1}{7}) \cup (\frac{5}{7}, \frac{6}{7})$. Suppose that $(\partial_1 \zeta_4)(\cdot, y) = 0$ has roots in $(5/7, 6/7)$. Since sinus function is bijective in $[\pi/2, 3\pi/2]$, this yields

$$(\partial_1 \zeta_4)(\cdot, y) = 0 \text{ has a unique solution in } [3/7, 1], \quad (3.369)$$

and denoting this by τ_2 , this satisfies $\tau_2 \in (5/7, 6/7)$.

Combining this with (3.364), one can conclude that

$$\zeta_4(\cdot, y) \text{ is increasing on the interval } [3/7, \tau_2), \quad (3.370)$$

and $\zeta_4(\cdot, y)$ is decreasing on $(\tau_2, 1]$.

According to (3.362), one can notice that

$$\begin{aligned} \zeta_4(3/7, y) &= -4/7 < 0, & \zeta_4(4/7, y) &> 0, \\ \zeta_4(5/7, y) &> 0, & \zeta_4(6/7, y) &> 0, & \zeta_4(1, y) &= 0. \end{aligned} \quad (3.371)$$

This implies that

$$\zeta_4(\cdot, y) \text{ has only one solution root in } [3/7, 1), \quad (3.372)$$

and denoting this by τ_3 , this satisfies $\tau_3 \in \left(\frac{3}{7}, \frac{4}{7}\right)$.

Next, suppose that $(\partial_1 \zeta_4)(\cdot, y) = 0$ has roots in $(0, 1/7)$. Since sinus function is bijective in $[-\pi/4, \pi/2]$, this yields

$$(\partial_1 \zeta_4)(\cdot, y) = 0 \text{ has a unique solution in } [0, 3/7], \quad (3.373)$$

and denoting this by τ'_2 , this satisfies $\tau'_2 \in (0, 1/7)$.

Combining this with (3.364), one can conclude that

$$\zeta_4(\cdot, y) \text{ is decreasing on the interval } [0, \tau'_2), \quad (3.374)$$

and $\zeta_4(\cdot, y)$ is increasing on $(\tau'_2, 3/7]$.

According to (3.362), one can notice that

$$\zeta_4(0, y) < 0, \quad \zeta_4(3/7, y) = -4/7 < 0. \quad (3.375)$$

Combining this with (3.374), one can conclude that

$$\zeta_4(\cdot, y) \text{ has no root in } (0, 3/7]. \quad (3.376)$$

Consequently, on the one hand, combining (3.376) with (3.372) we get

$$\text{the equation } \zeta_4(\cdot, y) = 0 \text{ has only one solution } \tau_3 \text{ in } (0, 1), \quad (3.377)$$

$$\text{and this satisfies } \tau_3 \in \left(\frac{3}{7}, \frac{4}{7}\right).$$

Fix $y \in (0, \infty)$ and consider the function $\xi_4(\cdot, y)$ on the interval $[0, 1]$. Since the sinus hyperbolic function is strictly positive on $(0, \infty)$, combining this with (3.365) provides that $(\partial_1 \xi_4)(x, y)$ has same sign as that of $\cos\left(\frac{(7x-1)\pi}{4}\right)$ and, as such

$$(\partial_1 \xi_4)(x, y) \geq 0 \text{ on } \left[0, \frac{3}{7}\right] \text{ and } (\partial_1 \xi_4)(x, y) \leq 0 \text{ on } \left[\frac{3}{7}, 1\right]. \quad (3.378)$$

This yields,

$$\text{the function } \xi_4(\cdot, y) \text{ is increasing on } \left[0, \frac{3}{7}\right] \text{ and decreasing on } \left[\frac{3}{7}, 1\right]. \quad (3.379)$$

On the other hand, (3.363) immediately gives, for $y < 0$

$$\xi_4(3/7, y) = y + \sqrt{2} \sinh\left(\frac{7y\pi}{4}\right) > 0, \quad \xi_4(4/7, y) = y + \sinh\left(\frac{7y\pi}{4}\right) > 0, \quad (3.380)$$

which forces

$$\begin{aligned} \text{whenever } y > 0, \text{ the equation } \xi_4(\cdot, y) = 0 \quad (3.381) \\ \text{has no solution in the interval } \left(\frac{3}{7}, \frac{4}{7}\right). \end{aligned}$$

Since $\xi_4(x, y) = -\xi_4(x, -y)$ on $[0, 1] \times \mathbb{R}$, based on (3.381) one can conclude that

$$\begin{aligned} \text{for each } y \in \mathbb{R} \setminus \{0\}, \text{ the equation } \xi_4(\cdot, y) = 0 \quad (3.382) \\ \text{has no solution in the interval } \left(\frac{3}{7}, \frac{4}{7}\right). \end{aligned}$$

Thus, using (3.382) with (3.377), we prove that

$$\Lambda_4(z) \neq 0, \quad \forall z \in \Gamma_{0,1} \setminus \{z \in \mathbb{C} : \text{Im } z = 0\}. \quad (3.383)$$

In addition, based on the direct calculation, one can conclude that

$$\zeta_4(10/21, 0) = \frac{-43 + 21\sqrt{3}}{42} < 0, \quad \zeta_4(1/2, 0) = \frac{-1 + \sqrt{4 - 2\sqrt{2}}}{2} > 0. \quad (3.384)$$

Going further, since $\Lambda_4(x) = \zeta_4(x, 0)$ we obtain that

$$\Lambda_4(x) = 0 \quad \text{has a unique root } x \text{ on the interval } (0, 1) \text{ and} \quad (3.385)$$

$$\text{this satisfies } x \in \left(\frac{10}{21}, \frac{1}{2}\right).$$

This completes the proof of the lemma. \square

Proof of Theorem 3.36. According to the proposition 3.38, for each $p \in (1, \infty)$, $\xi \in \mathbb{R}$ there holds

$$\begin{aligned} & \det(\mathcal{M}(R(\cdot, 1)))(z) \quad (3.386) \\ &= \left(\frac{1}{8 \sin^2(\pi z)}\right)^2 \times \\ & \times \left(z - 1 + \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right)\right) \left(z - 1 - \sqrt{2} \cos\left(\frac{(z+1)\pi}{4}\right)\right) \times \\ & \times \left(z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right)\right) \left(z - 1 - \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right)\right), \end{aligned}$$

where $z = \frac{1}{p} + i\xi \in \Gamma_{0,1}$. Applying the lemma 3.39, 3.40, and 3.41 provides

$$\det(\mathcal{M}(R(\cdot, 1)))(z) \neq 0 \iff z \in \Gamma_{0,1} \setminus \{\tau_1, \tau_3\}, \quad (3.387)$$

where $\tau_1 \in (1/3, 5/14)$ and $\tau_3 \in (10/21, 1/2)$ with

$$\tau_1 \text{ is the unique root of } z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.388)$$

$$\tau_3 \text{ is the unique root of } z - 1 - \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \text{ in } \Gamma_{0,1}. \quad (3.389)$$

Combining this with Theorem 3.10 yields

$$\mathcal{T} \text{ is invertible on } (L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+))^2 \iff p \in (1, \infty) \setminus \{\tau_1^{-1}, \tau_3^{-1}\}, \quad (3.390)$$

where $\tau_1^{-1} \in (2.8, 3)$ and $\tau_2 \in (2, 2.1)$. This completes the proof of Theorem 3.36. \square

At this point we are ready to present the proof of Theorem 3.3.

Proof of Theorem 3.3. According to Theorem 3.36, for $\eta = -1$ and for the integrability exponent $p \in (1, \infty)$, there holds

$$\mathcal{T} \text{ is invertible on } (L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+))^2 \iff p \in (1, \infty) \setminus \{\gamma, \delta\}, \quad (3.391)$$

where $\gamma \in (2.8, 3)$ and $\delta \in (2, 2.1)$ with

$$\gamma^{-1} \text{ is the unique root of } z - 1 + \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \text{ in } \Gamma_{0,1}, \quad (3.392)$$

$$\delta^{-1} \text{ is the unique root of } z - 1 - \sqrt{2} \cos\left(\frac{(7z-1)\pi}{4}\right) = 0 \text{ in } \Gamma_{0,1}. \quad (3.393)$$

In conclusion, combining this together with Theorem 3.28 the desired result follows. \square

CHAPTER 4

The Spectral Properties of Layer Potentials for Second-Order Elliptic Systems

In this chapter, we investigate the coefficient tensors associated with second order elliptic operators in two dimensional infinite sectors and properties of the corresponding singular integral operators, employing Mellin transform. Concretely, we explore the relationship between distinguished coefficient tensors and L^p spectral and Hardy kernel properties of the associated singular integral operators.

4.1 Layer potential theory in the infinite sector

In \mathbb{R}^2 , the conormal derivative for L of u with respect to the coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}}$ is given by

$$(\partial_\nu^A u)_\alpha(x) := \nu_r a_{rs}^{\alpha\beta} \partial_s u_\beta \Big|_{\partial\Omega}^{n.t.}(x), \quad (4.1)$$

where ν is the outward unit normal vector to Ω . In addition, the double layer potential operator K_A is given by

$$(K_A g)(x) := \int_{\partial\Omega} \left(\partial_\nu^{A^T} (E_{L^T}(x - \cdot)) \right)^T (y) g(y) d\sigma(y), \quad x \in \partial\Omega \quad (4.2)$$

where $g = (g_1, g_2, \dots, g_M) : \partial\Omega \rightarrow \mathbb{R}^M$, E_L is the fundamental solution with respect to the differential operator L . Here, superscript T stands the transposition of the matrix. Moreover, the kernel of the double layer potential operator K associated with A and Ω has the form

$$\left(\partial_\nu^{A^T} (E_{L^T}(x - \cdot)) \right)_{1 \leq \gamma, \alpha \leq M} (y) = \left(-\nu_s(y) a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x - y) \right)_{1 \leq \gamma, \alpha \leq M}. \quad (4.3)$$

Let $p \in (1, \infty)$, Ω be the infinite sector of an angle $\theta \in (0, 2\pi)$ in \mathbb{R}^2 . The following theorem provides a relationship between the Hardy kernel and the distinguished coefficient tensor.

Theorem 4.1. *Let $p \in (1, \infty)$, $\theta \in (0, 2\pi)$ and L be a homogeneous, second-order, constant complex-coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 , consider $\Omega \subseteq \mathbb{R}^2$ the infinite sector of an angle θ and a coefficient tensor $A \in \mathfrak{A}_L$. If $A \in \mathfrak{A}_L^{\text{dis}}$, then the kernel of the double layer potential operator K_A associated with A and Ω is a Hardy kernel on $L^p(\mathbb{R}_+)$.*

Proof. Since $A \in \mathfrak{A}_L^{\text{dis}}$, from (1.64), there exists a matrix-valued function

$$k = \{k_{\gamma\alpha}\}_{1 \leq \gamma, \alpha \leq M} : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{C}^{M \times M} \quad (4.4)$$

such that

$$\begin{aligned} \left(-\nu_s(y) a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x - y) \right)_{1 \leq \gamma, \alpha \leq M} &= - \left(\nu_s(y) (x_s - y_s) k_{\gamma\alpha}(x - y) \right)_{1 \leq \gamma, \alpha \leq M} \\ &= - \langle \nu(y), x - y \rangle k(x - y) \\ &:= k^A(x - y), \end{aligned} \quad (4.5)$$

where $k^A = \{k_{\gamma\alpha}^A\}_{1 \leq \gamma, \alpha \leq M}$. Since k belongs to $\mathcal{C}^\infty(\mathbb{R}^2 \setminus \{0\})$, even and positive homogeneous of degree -2 , the kernel k^A also belongs to $\mathcal{C}^\infty(\mathbb{R}^2 \setminus \{0\})$, odd and positive homogeneous of degree -1 .

According to the equation (4.5), we have for any $1 \leq \gamma, \alpha \leq M$, $x, y \in \mathbb{R}^2$ with $x \neq y$,

$$\begin{aligned} k_{\gamma\alpha}^A(x-y) &= -\langle \nu(y), x-y \rangle k_{\gamma\alpha}(x-y) \\ &= -\langle \nu(y), x-y \rangle \cdot |x-y|^{-2} k_{\gamma\alpha} \left(\frac{x-y}{|x-y|} \right). \end{aligned} \quad (4.6)$$

This yields, $\forall x, y \in \mathbb{R}^2$ with $x \neq y$,

$$|k_{\gamma\alpha}^A(x-y)| \leq \left(\sup_{S^1} |k_{\gamma\alpha}| \right) \frac{|\langle \nu(y), x-y \rangle|}{|x-y|^2} \quad (4.7)$$

Since Ω is the infinite sector of an angle θ , Ω is the upper graph of the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\phi(x) := |x| \cot(\theta/2), \quad x \in \mathbb{R}. \quad (4.8)$$

Hereafter, we denote by $(\partial\Omega)_1$ and $(\partial\Omega)_2$ the left and the right side of the angle $\partial\Omega$, respectively. Concretely,

$$\begin{aligned} (\partial\Omega)_1 &:= \left\{ \left(-s \sin \frac{\theta}{2}, s \cos \frac{\theta}{2} \right) : s \in \mathbb{R}_+ \right\} \text{ and} \\ (\partial\Omega)_2 &:= \left\{ \left(s \sin \frac{\theta}{2}, s \cos \frac{\theta}{2} \right) : s \in \mathbb{R}_+ \right\}. \end{aligned} \quad (4.9)$$

In this notation one can naturally identify the two pieces of the boundary $(\partial\Omega)_j$, $j = 1, 2$, with \mathbb{R}_+ via the mapping

$$\iota_1 : (\partial\Omega)_1 \rightarrow \mathbb{R}_+, \quad \iota_1(P) := |P|, \text{ for each } P \in (\partial\Omega)_1, \quad (4.10)$$

$$\iota_2 : (\partial\Omega)_2 \rightarrow \mathbb{R}_+, \quad \iota_2(P) := |P|, \text{ for each } P \in (\partial\Omega)_2. \quad (4.11)$$

Consequently for each $p \in (0, \infty)$ given, identify $L^p(\partial\Omega)$ with the space $L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+)$ via the mapping

$$\mathcal{F} : L^p(\partial\Omega) \rightarrow L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+), \quad \mathcal{F}(f) := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \forall f \in L^p(\partial\Omega),$$

$$\text{where } f_j = f \Big|_{(\partial\Omega)_j} \circ \iota_j^{-1}, \text{ for each } j \in \{1, 2\}.$$

We may also find it to introduce the following notation for the restriction operator

$$\mathcal{R} : L^p(\partial\Omega) \rightarrow L^p((\partial\Omega)_1) \times L^p((\partial\Omega)_2), \quad \mathcal{R}(f) := (f_1, f_2),$$

where $f_j := f|_{(\partial\Omega)_j}$, for each $j \in \{1, 2\}$,

and

$$\mathcal{I} : L^p((\partial\Omega)_1) \times L^p((\partial\Omega)_2) \rightarrow L^p(\partial\Omega), \quad \mathcal{I} := \mathcal{R}^{-1}, \quad (4.12)$$

for its inverse.

Now, we compute the kernel of the double layer potential on the boundary. If $x, y \in (\partial\Omega)_1$ or $x, y \in (\partial\Omega)_2$, then $\langle \nu(y), x - y \rangle = 0$. If $x \in (\partial\Omega)_1, y \in (\partial\Omega)_2$, that is,

$$\begin{aligned} x &= \left(-s \sin \frac{\theta}{2}, s \cos \frac{\theta}{2} \right), \\ y &= \left(t \sin \frac{\theta}{2}, t \cos \frac{\theta}{2} \right), \end{aligned} \quad (4.13)$$

for some $s, t \in \mathbb{R}$, then

$$\nu(y) = \left(\cos \frac{\theta}{2}, -\sin \frac{\theta}{2} \right). \quad (4.14)$$

Then, we get

$$-\langle \nu(y), x - y \rangle = s \sin \theta. \quad (4.15)$$

Moreover, from (4.13) we obtain

$$\begin{aligned} x - y &= \left(-(s + t) \sin \frac{\theta}{2}, (s - t) \cos \frac{\theta}{2} \right), \\ |x - y|^2 &= s^2 - 2st \cos \theta + t^2. \end{aligned} \quad (4.16)$$

In particular, from (4.7) we get,

$$|k_{\gamma\alpha}^A(s, t)| \leq C(k) \frac{s}{s^2 - 2st \cos \theta + t^2}, \quad (4.17)$$

where $k_{\gamma\alpha}^A(s, t) = k_{\gamma\alpha}^A\left(-\frac{s+t}{2}\sin\frac{\theta}{2}, \frac{s-t}{2}\cos\frac{\theta}{2}\right)$.

Now, we apply the estimate (4.17) to the following integral

$$\int_0^\infty |k_{\gamma\alpha}^A(1, t)|t^{-1/p}dt \leq C(k) \int_0^\infty \frac{1}{1 - 2t \cos \theta + t^2} t^{-1/p} dt. \quad (4.18)$$

Going further note that

$$t^2 - 2t \cos \theta + 1 = (t - \cos \theta)^2 + \sin^2 \theta > 0 \text{ for } \theta \in (0, 2\pi). \quad (4.19)$$

On the behavior of the integrand on the right hand side near 0 and ∞ , we have

$$\frac{1}{1 - 2t \cos \theta + t^2} t^{-1/p} \approx t^{-1/p} \text{ near } 0, \quad (4.20)$$

$$\frac{1}{1 - 2t \cos \theta + t^2} t^{-1/p} \approx t^{-(2+1/p)} \text{ near } \infty, \quad (4.21)$$

which yield

$$\int_0^\infty |k_{\gamma\alpha}^A(1, t)|t^{-1/p}dt < +\infty. \quad (4.22)$$

Combining this with the homogeneity, we show that $k_{\gamma\alpha}^A$ is a Hardy kernel on $L^p(\mathbb{R}_+)$ for $p \in (1, \infty)$.

Similarly, if $x \in (\partial\Omega)_2$, $y \in (\partial\Omega)_1$, $k_{\gamma\alpha}^A$ is also a Hardy kernel on $L^p(\mathbb{R}_+)$ for $p \in (1, \infty)$. This completes the proof of Theorem 4.1. \square

Our next goal is to prove a somewhat converse statement of Theorem 4.1. More precisely, we will show that if a coefficient tensor $A \in \mathfrak{A}_L$ is not distinguished, then there exists an infinite sector Ω such that the kernel of the double layer potential operator K associated with A and Ω is not a Hardy kernel on L^p for any $p \in (1, \infty)$.

We start by introducing some notation and proving an auxiliary lemma. Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Given an open ball $B \subset \mathbb{R}^2$ that does not contain the origin, we define the double cone Γ associated with B by

$$\Gamma := \{y \in \mathbb{R}^2 : y = \lambda z, \text{ for some } z \in B \text{ and } \lambda \in \mathbb{R}^*\}. \quad (4.23)$$

Note that $B \subset \Gamma$ and that Γ is an open subset of \mathbb{R}^2 .

Lemma 4.2. *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 and consider $A \in \mathfrak{A}_L$. If $A \notin \mathfrak{A}_L^{\text{dis}}$, then there exist a constant $C > 0$, indices $\alpha, \gamma \in \{1, \dots, M\}$ and an open double cone Γ such that*

$$|(y_2 a_{r_1}^{\beta\alpha} - y_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(y)| \geq C \quad \text{for all } y \in \Gamma. \quad (4.24)$$

Proof. Note that if $n = 2$, then item (a) from Proposition 1.16 implies that $A \in \mathfrak{A}_L^{\text{dis}}$ only if for each $\alpha, \gamma \in \{1, \dots, M\}$ there holds

$$(x_2 a_{r_1}^{\beta\alpha} - x_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(x) = 0 \quad \text{for all } x = (x_j)_{1 \leq j \leq 2} \in \mathbb{R}^2 \setminus \{0\}. \quad (4.25)$$

Consequently, if $A \in \mathfrak{A}_L$ is not a distinguished coefficient tensor for L in \mathbb{R}^2 , then there exist a constant $C > 0$, indices $\alpha, \gamma \in \{1, \dots, M\}$ and a point $x' \in \mathbb{R}^2 \setminus \{0\}$ such that

$$|(x'_2 a_{r_1}^{\beta\alpha} - x'_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(x')| \geq 2C. \quad (4.26)$$

Since the function on the left-hand side of the above inequality is continuous on $\mathbb{R}^2 \setminus \{0\}$, we get that there exists an open ball $B_{x'}$ centered at x' and such that

$$|(z_2 a_{r_1}^{\beta\alpha} - z_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(z)| \geq C \quad \text{for all } z \in B_{x'}. \quad (4.27)$$

Consider the double cone

$$\Gamma := \{y \in \mathbb{R}^2 : y = \lambda z, \text{ for some } z \in B_{x'} \text{ and } \lambda \in \mathbb{R}^*\}. \quad (4.28)$$

We claim that

$$|(y_2 a_{r_1}^{\beta\alpha} - y_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(y)| \geq C \quad \text{for all } y \in \Gamma. \quad (4.29)$$

Indeed, suppose $y \in \Gamma$. Then there exist $\lambda \in \mathbb{R}^*$ and $z \in B_{x'}$ such that $y = \lambda z$. Since $(\partial_r E_{\gamma\beta})$ is homogeneous of degree -1 , it follows that if $\lambda > 0$ we have

$$(\partial_r E_{\gamma\beta})(y) = (\partial_r E_{\gamma\beta})(\lambda z) = \frac{1}{\lambda} (\partial_r E_{\gamma\beta})(z), \quad (4.30)$$

thus

$$(y_2 a_{r_1}^{\beta\alpha} - y_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(y) = (z_2 a_{r_1}^{\beta\alpha} - z_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(z), \quad (4.31)$$

so by (4.27) we have

$$|(y_2 a_{r_1}^{\beta\alpha} - y_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(y)| \geq C. \quad (4.32)$$

If $\lambda < 0$, then the homogeneity of $(\partial_r E_{\gamma\beta})$ together with the fact that $E_{\gamma\beta}$ is an even function, so $(\partial_r E_{\gamma\beta})$ is odd, allow us to write $y = \lambda z = (-\lambda)(-z)$ to conclude that

$$(\partial_r E_{\gamma\beta})(y) = (\partial_r E_{\gamma\beta})((-\lambda)(-z)) = \frac{1}{(-\lambda)}(\partial_r E_{\gamma\beta})(-z) = \frac{1}{\lambda}(\partial_r E_{\gamma\beta})(z), \quad (4.33)$$

which implies

$$(y_2 a_{r_1}^{\beta\alpha} - y_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(y) = (z_2 a_{r_1}^{\beta\alpha} - z_1 a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(z). \quad (4.34)$$

Hence the final conclusion follows once again from (4.27) and this finishes the proof. \square

We are now ready to prove the aforementioned result of non-distinguished coefficient tensors and the kernel of the double layer potential associated with it.

Theorem 4.3. *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 and consider $A \in \mathfrak{A}_L$. If $A \notin \mathfrak{A}_L^{dis}$, then there exists $\theta \in (0, 2\pi)$ such that the kernel of the double layer potential associated with A and Ω ,*

$$\Omega := \{y \in \mathbb{R}^2 : y_2 > |x| \cot(\theta/2), x \in \mathbb{R}\}, \quad (4.35)$$

is not a Hardy kernel on $L^p(\mathbb{R}_+)$ for $p \in (1, \infty)$.

Proof. According to the equation (4.3), the kernel of the double layer associated with A and Ω has the form

$$(-\nu_s(y) a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x - y))_{1 \leq \gamma, \alpha \leq M}. \quad (4.36)$$

Let $k_{\gamma\alpha}^A(x-y) := -\nu_s(y)a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)$ for $x, y \in \partial\Omega$. More explicitly, for each $\alpha, \gamma \in \{1, \dots, M\}$ we have

$$k_{\gamma\alpha}^A(x-y) = (-\nu_1(y)a_{r1}^{\beta\alpha} - \nu_2(y)a_{r2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(x-y). \quad (4.37)$$

Let $x, y \in (\partial\Omega)_1$, that is

$$\begin{aligned} x &= \left(-s \sin \frac{\theta}{2}, s \cos \frac{\theta}{2} \right), \\ y &= \left(-t \sin \frac{\theta}{2}, t \cos \frac{\theta}{2} \right), \\ \nu(y) &= \left(-\cos \frac{\theta}{2}, -\sin \frac{\theta}{2} \right), \end{aligned} \quad (4.38)$$

for some $s, t \in \mathbb{R}_+$. Note that

$$x_1 - y_1 = (t - s) \sin \frac{\theta}{2} \quad \text{and} \quad x_2 - y_2 = (s - t) \cos \frac{\theta}{2}, \quad (4.39)$$

and that $|x - y| = |s - t|$. Inserting equations (4.38) into (4.37) we get

$$k_{\gamma\alpha}^A(s, t) = \left(\cos \frac{\theta}{2} a_{r1}^{\beta\alpha} + \sin \frac{\theta}{2} a_{r2}^{\beta\alpha} \right) (\partial_r E_{\gamma\beta}) \left((t - s) \sin \frac{\theta}{2}, (s - t) \cos \frac{\theta}{2} \right). \quad (4.40)$$

Assume that $x \neq y$, that is, $s \neq t$. Then multiplying and dividing the above equality by $(s - t)$ gives us

$$\begin{aligned} k_{\gamma\alpha}^A(s, t) &= \frac{1}{(s - t)} \left[(s - t) \cos \frac{\theta}{2} a_{r1}^{\beta\alpha} - (t - s) \sin \frac{\theta}{2} a_{r2}^{\beta\alpha} \right] \times \\ &\quad \times (\partial_r E_{\gamma\beta}) \left((t - s) \sin \frac{\theta}{2}, (s - t) \cos \frac{\theta}{2} \right). \end{aligned} \quad (4.41)$$

Thus taking absolute values in the above equality gives us

$$|k_{\gamma\alpha}^A(x-y)| = \frac{1}{|x-y|} \left| \left((x_2 - y_2)a_{r1}^{\beta\alpha} - (x_1 - y_1)a_{r2}^{\beta\alpha} \right) (\partial_r E_{\gamma\beta})(x-y) \right|, \quad (4.42)$$

for all $x, y \in (\partial\Omega)_1$, $x \neq y$.

Now, if $A \notin \mathfrak{A}_L^{\text{dis}}$, then from Lemma 4.2 there exist a constant $C > 0$, indices $\alpha', \gamma' \in \{1, \dots, M\}$ and an open double cone Γ such that

$$|(z_2 a_{r1}^{\beta\alpha'} - z_1 a_{r2}^{\beta\alpha'}) (\partial_r E_{\gamma'\beta})(z)| \geq C \quad \text{for all } z \in \Gamma. \quad (4.43)$$

Since Γ is a double cone there exists $\theta \in (0, 2\pi)$ such that $(\partial\Omega)_1 \subset \Gamma$. Thus for $x, y \in (\partial\Omega)_1$, $x \neq y$, we have $(x - y) \in \Gamma$ and we conclude that

$$|k_{\gamma'\alpha'}^A(x - y)| \geq \frac{C}{|x - y|} \quad \text{for all } x, y \in (\partial\Omega)_1, x \neq y. \quad (4.44)$$

In particular, for $s, t \in \mathbb{R}_+$ with $s \neq t$ we get

$$|k_{\gamma'\alpha'}^A(s, t)| \geq \frac{C}{|s - t|}, \quad (4.45)$$

thus for $p \in (1, \infty)$

$$\int_0^\infty |k_{\gamma'\alpha'}^A(1, t)| t^{-1/p} dt \geq C \int_0^\infty \frac{1}{|1 - t|} t^{-1/p} dt, \quad (4.46)$$

and this last integral diverges, so $k_{\gamma'\alpha'}^A$ is not a Hardy kernel for every $p \in (1, \infty)$ and this finishes the proof. \square

Theorems 4.1 and 4.3 give us the following corollary.

Corollary 4.4. *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 . A coefficient tensor $A \in \mathfrak{A}_L$ is distinguished if and only if whenever Ω is an infinite sector on \mathbb{R}^2 we have that the double layer potential associated with A and Ω is a Hardy kernel on $L^p(\mathbb{R}_+)$ for all $p \in (1, \infty)$.*

4.2 The case of half-planes

Let $A \in \mathfrak{A}_L$. Set $D_{\alpha\gamma}^A : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{C}$ to be

$$D_{\alpha\gamma}^A(x) := (x_2 a_{r1}^{\beta\alpha} - x_1 a_{r2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(x). \quad (4.47)$$

Then $D_{\alpha\gamma}^A$ is even and homogeneous of degree 0, which means that $D_{\alpha\gamma}^A$ is constant over any straight line that crosses the origin. This observation allow us to prove the following proposition.

Proposition 4.5. *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 and $A \in \mathfrak{A}_L$. Whenever Ω is a half plane, the kernel of the double layer potential associated with A and Ω is a matrix multiple of the Hilbert transform.*

Proof. Without loss of generality, we may assume that Ω is a half plane with $0 \in \partial\Omega$ and $A \in \mathfrak{A}_L$. We parametrize $\partial\Omega$ using $\omega \in [0, \pi)$ and $s, t \in \mathbb{R}$. That is, for $x, y \in \partial\Omega$ we have

$$\begin{aligned} x &= (s \cos \omega, s \sin \omega), \\ y &= (t \cos \omega, t \sin \omega), \\ \nu(y) &= (\sin \omega, -\cos \omega). \end{aligned} \tag{4.48}$$

Thus we may write the kernel of the double layer potential associated with A and Ω in terms of $s, t \in \mathbb{R}$, $s \neq t$ to obtain

$$k_{\gamma\alpha}^A(s, t) = (-\sin \omega a_{r_1}^{\beta\alpha} + \cos \omega a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})((s-t) \cos \omega, (s-t) \sin \omega). \tag{4.49}$$

Multiplying and dividing by $s - t$ gives us

$$k_{\gamma\alpha}^A(s, t) = \frac{C_{\gamma\alpha}(\omega)}{s - t}, \tag{4.50}$$

where for each $1 \leq \gamma, \alpha \leq M$,

$$C_{\gamma\alpha}(\omega) = (-\sin \omega a_{r_1}^{\beta\alpha} + \cos \omega a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(\cos \omega, \sin \omega). \tag{4.51}$$

Since this holds for all $\alpha, \gamma \in \{1, \dots, M\}$ we conclude the result. \square

Let Ω be a half plane domain. Then if $A \in \mathfrak{A}_L$ is distinguished, we have that the double layer operator (boundary-to-boundary) is the zero operator. If $A \in \mathfrak{A}_L$ is not distinguished, then

$$k_{\gamma\alpha}^A(s, t) = \frac{C_{\gamma\alpha}(\omega)}{s - t}. \tag{4.52}$$

Theorem 4.6. *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 . A coefficient tensor $A \in \mathfrak{A}_L$ is distinguished if and only if whenever Ω is a half plane on \mathbb{R}^2 the kernel of the double layer potential operator K associated with A and Ω is identically zero in Ω .*

Proof. Applying the same argument of the characterization theorem of the distinguished coefficient tensor in section 2, it can be shown that a coefficient tensor $A \in \mathfrak{A}_L$ is distinguished if and only if whenever Ω is a half plane of an angle $\omega \in [0, \pi)$ on \mathbb{R}^2 we have that the double layer potential associated with A and Ω is a Hardy kernel on $L^p(\mathbb{R}_+)$ for all $p \in (1, \infty)$. Combining this with (4.52), we obtain that A is distinguished if and only if the kernel of the double layer potential associated with A and half plane Ω of and angle $\omega \in [0, \pi)$ is identically zero in Ω . This finishes the proof of the Theorem 4.6. \square

In this section, we assume that $p \in (1, \infty)$ and L be a homogeneous, second-order, constant complex-coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 , and $\Omega \subseteq \mathbb{R}^2$ is a half plane at an angle $\omega \in [0, \pi)$ from x -axis and a coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}} \in \mathfrak{A}_L$.

To get our hands at the spectrum of k_A we may use the Mellin transform technique.

Theorem 4.7. *Fix $p \in (1, \infty)$. Let K be an element in the algebra of Mellin convolution operators generated by Hardy kernels and the Hilbert transform for $(L^p(\mathbb{R}_+))^2$. Then K is a bounded operator on $(L^p(\mathbb{R}_+))^2$ and its spectrum is the closure of the range of the Mellin transform $\mathcal{M}k(1/p + i\xi)$, i.e., it is the closure in the plane of the set of all points $\lambda \in \mathbb{C}$ such that*

$$\det(\lambda I - \mathcal{M}k)(1/p + iy) = 0 \quad \text{for some } y \in \mathbb{R}. \quad (4.53)$$

Above, k is the kernel of the operator K , I is the identity matrix operator, and $\mathcal{M}k := (\mathcal{M}k_{\alpha\gamma})_{\alpha, \gamma=1, \dots, M}$.

We parametrize $\partial\Omega$ using $\omega \in [0, \pi)$ and $s, t \in \mathbb{R}$ as before. Our main goal is to find an explicit description of the spectrum of the operator K_A . To apply Theorem 4.7 we distinguish four cases. Without loss of generality, let us assume that Ω is a half plane containing $0 \in \partial\Omega$.

Case 1: We consider $x, y \in (\partial\Omega)_2$, that is

$$\begin{aligned} x &= (s \cos \omega, s \sin \omega), \\ y &= (t \cos \omega, t \sin \omega), \end{aligned} \tag{4.54}$$

for some $s, t \in \mathbb{R}_+$. In that case we have

$$k_{\gamma\alpha}^A(s, t) = \frac{C_{\gamma\alpha}(\omega)}{s - t}, \tag{4.55}$$

hence the Mellin transform of the symbol $k_{\gamma\alpha}$ is

$$\mathcal{M}k_{\gamma\alpha}(z) = \mathcal{M}(k_{\gamma\alpha}(\cdot, 1))(z) = -\pi C_{\gamma\alpha}(\omega) \frac{\cos(\pi z)}{\sin(\pi z)}. \tag{4.56}$$

Case 2: We consider $x \in (\partial\Omega)_2$ and $y \in (\partial\Omega)_1$, that is

$$\begin{aligned} x &= (s \cos \omega, s \sin \omega), \\ y &= (-t \cos \omega, -t \sin \omega), \end{aligned} \tag{4.57}$$

for some $s, t \in \mathbb{R}_+$. In that case we have

$$k_{\gamma\alpha}^A(s, t) = \frac{C_{\gamma\alpha}(\omega)}{s + t}, \tag{4.58}$$

hence the Mellin transform of the symbol $k_{\gamma\alpha}^A$ is

$$\mathcal{M}k_{\gamma\alpha}^A(z) = \mathcal{M}(k_{\gamma\alpha}^A(\cdot, 1))(z) = \pi C_{\gamma\alpha}(\omega) \frac{1}{\sin(\pi z)}. \tag{4.59}$$

Case 3: We consider $x \in (\partial\Omega)_1$ and $y \in (\partial\Omega)_2$, that is

$$\begin{aligned} x &= (-s \cos \omega, -s \sin \omega), \\ y &= (t \cos \omega, t \sin \omega), \end{aligned} \tag{4.60}$$

for some $s, t \in \mathbb{R}_+$. In that case we have

$$k_{\gamma\alpha}^A(s, t) = -\frac{C_{\gamma\alpha}(\omega)}{s + t}, \tag{4.61}$$

hence the Mellin transform of the symbol $k_{\gamma\alpha}^A$ is

$$\mathcal{M}k_{\gamma\alpha}^A(z) = \mathcal{M}(k_{\gamma\alpha}^A(\cdot, 1))(z) = -\pi C_{\gamma\alpha}(\omega) \frac{1}{\sin(\pi z)}. \tag{4.62}$$

Case 4: We consider $x, y \in (\partial\Omega)_1$, that is

$$\begin{aligned} x &= (-s \cos \omega, -s \sin \omega), \\ y &= (-t \cos \omega, -t \sin \omega), \end{aligned} \quad (4.63)$$

for some $s, t \in \mathbb{R}_+$. In that case we have

$$k_{\gamma\alpha}^A(s, t) = -\frac{C_{\gamma\alpha}(\omega)}{s - t}, \quad (4.64)$$

hence the Mellin transform of the symbol $k_{\gamma\alpha}^A$ is

$$\mathcal{M}k_{\gamma\alpha}^A(z) = \mathcal{M}(k_{\gamma\alpha}^A(\cdot, 1))(z) = \pi C_{\gamma\alpha}(\omega) \frac{\cos(\pi z)}{\sin(\pi z)}. \quad (4.65)$$

Based on (4.56)–(4.65), define the $M \times M$ matrices

$$(B_1(z))_{\gamma\alpha} = -C_{\gamma\alpha}(\omega) \cos(\pi z) \quad (4.66)$$

and

$$(B_2)_{\gamma\alpha} = C_{\gamma\alpha}(\omega) \quad (4.67)$$

Then we may write the Mellin transform of the kernel of the operator K as the $2M \times 2M$ matrix

$$\mathcal{M}k(z) = \frac{\pi}{\sin(\pi z)} \begin{bmatrix} B_1(z) & B_2 \\ -B_2 & -B_1(z) \end{bmatrix}. \quad (4.68)$$

Note that for $\operatorname{Re}(z) \neq 0$ we have $\sin(\pi z) \neq 0$, hence the denominator in the above expression is not vanishing.

From (4.68), we obtain that for $z = \frac{1}{p} + iy$, $y \in \mathbb{R}$,

$$wI - (\mathcal{M}k)(z) = wI - \frac{\pi}{\sin(\pi z)} \begin{bmatrix} B_1(z) & B_2 \\ -B_2 & -B_1(z) \end{bmatrix} \quad (4.69)$$

$$= \frac{\pi}{\sin(\pi z)} \begin{bmatrix} \frac{\sin(\pi z)}{\pi} wI_{M \times M} - B_1(z) & -B_2 \\ B_2 & \frac{\sin(\pi z)}{\pi} wI_{M \times M} + B_1(z) \end{bmatrix}, \quad (4.70)$$

which implies that

$$\begin{aligned}
& \det(wI - (\mathcal{M}k)(z)) \tag{4.71} \\
&= \frac{\pi^{2M}}{\sin^{2M}(\pi z)} \det\left(\frac{\sin^2(\pi z)}{\pi^2} w^2 I_{M \times M} - B_1^2(z) + B_2^2\right) \\
&= \frac{\pi^{2M}}{\sin^{2M}(\pi z)} \det\left(\frac{\sin^2(\pi z)}{\pi^2} w^2 I_{M \times M} - \cos^2(\pi z) B_2^2(z) + B_2^2\right) \\
&= \frac{\pi^{2M}}{\sin^{2M}(\pi z)} \det\left(\frac{\sin^2(\pi z)}{\pi^2} w^2 I_{M \times M} + \sin^2(\pi z) B_2^2\right) \\
&= \pi^{2M} \det\left(\frac{w^2}{\pi^2} I_{M \times M} + B_2^2\right) \\
&= \det(w^2 I_{M \times M} + (\pi B_2)^2).
\end{aligned}$$

Note that

$$\sigma(K_A; L^p(\mathbb{R}_+)) = \{\pm i\pi\lambda; \lambda \in \mathbb{C}, \lambda \text{ is eigenvalue for } (C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}\}. \tag{4.72}$$

Combining this with (4.53), we conclude that for $p \in (1, \infty)$,

$$\begin{aligned}
& \sigma(K_A; L^p(\mathbb{R}_+)) = \{0\} \\
& \iff (C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M} \text{ does not have nonzero eigenvalue} \\
& \iff (C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M} \text{ is nilpotent matrix.} \tag{4.73}
\end{aligned}$$

In the following theorems, we investigate the relation between the spectrum and the coefficient tensor.

Theorem 4.8. *If $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}} \in \mathfrak{A}_L^{dis}$, then whenever $\Omega \subseteq \mathbb{R}^2$ is a half plane the spectrum of the double layer K_A associated with A and Ω on $L^p(\mathbb{R}_+)$*

$$\sigma(K_A; L^p(\mathbb{R}_+)) = \{0\}. \tag{4.74}$$

Proof. According to the Theorem 4.6, if A is a distinguished coefficient tensor we have that for all $1 \leq \gamma, \alpha \leq M$, $C_{\gamma\alpha}(\omega) = 0$, $\forall \omega \in [0, \pi)$ which implies that the Mellin symbol in (4.68) is identically zero whenever $\Omega \subseteq \mathbb{R}^2$ is a half plane.

Applying the Theorem 4.7, we obtain that whenever $\Omega \subseteq \mathbb{R}^2$ is a half plane $\sigma(K_A; L^p(\mathbb{R}_+)) = \{0\}$ which proves the theorem 4.8. \square

Next, we state and prove the sufficient condition on the integral kernel for which the trivial spectrum implies the coefficient tensor is distinguished.

Theorem 4.9. *Let $\Omega \subseteq \mathbb{R}^2$ be a half plane of an angle $\omega \in [0, \pi)$. If for any $\omega \in [0, \pi)$, $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ defined in (4.51) is diagonalizable over \mathbb{C} , then*

$$\sigma(K_A; L^p(\mathbb{R}_+)) = \{0\}, \quad \forall \omega \in [0, \pi) \implies A \in \mathfrak{A}_L^{\text{dis}}. \quad (4.75)$$

Proof. According to (4.73), we obtain that $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ is nilpotent matrix for any $\omega \in [0, \pi)$. Since every nilpotent diagonalizable matrix is zero matrix, we have for all $1 \leq \gamma, \alpha \leq M$, $C_{\gamma\alpha}(\omega) = 0 \quad \forall \omega \in [0, \pi)$ which implies that $A \in \mathfrak{A}_L^{\text{dis}}$ according to the Theorem 4.6. \square

Theorem 4.8 and 4.9 give us the following equivalence condition for the trivial spectrum under a certain type of the integral kernel.

Theorem 4.10. *Let $p \in (1, \infty)$ and L be a homogeneous, second-order, constant complex-coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 , let us consider $\Omega \subseteq \mathbb{R}^2$ a half plane of an angle $\omega \in [0, \pi)$ and a coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}} \in \mathfrak{A}_L$. If for any $\omega \in [0, \pi)$, $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ defined in (4.51) is diagonalizable over \mathbb{C} , then*

$$\sigma(K_A; L^p(\mathbb{R}_+)) = \{0\}, \quad \forall \omega \in [0, \pi) \iff A \in \mathfrak{A}_L^{\text{dis}}. \quad (4.76)$$

In the next example, we apply the Theorem 4.10 to the Lamé operator in \mathbb{R}_+^2 .

Example 4.11. Let us assume that the system (1.26) becomes the Lamé system, that is

$$a_{rs}^{\alpha\beta} = \mu \delta_{rs} \delta_{\alpha\beta} + (\mu + \lambda - \ell) \delta_{r\alpha} \delta_{s\beta} + \ell \delta_{r\beta} \delta_{s\alpha}, \quad (4.77)$$

for $\ell \in \mathbb{C}$ in half plane $\Omega \subseteq \mathbb{R}^2$ at angle $\omega \in [0, \pi)$. Comparing the coefficients on the (4.77), we have

$$\begin{bmatrix} (a_{11}^{\alpha\beta})_{1 \leq \alpha, \beta \leq 2} & (a_{12}^{\alpha\beta})_{1 \leq \alpha, \beta \leq 2} \\ (a_{21}^{\alpha\beta})_{1 \leq \alpha, \beta \leq 2} & (a_{22}^{\alpha\beta})_{1 \leq \alpha, \beta \leq 2} \end{bmatrix} = \begin{bmatrix} 2\mu + \lambda & 0 & 0 & \mu + \lambda - \ell \\ 0 & \mu & \ell & 0 \\ 0 & \ell & \mu & 0 \\ \mu + \lambda - \ell & 0 & 0 & 2\mu + \lambda \end{bmatrix}.$$

This implies that $a_{rs}^{\alpha\beta}$ is symmetric and $a_{rs}^{\alpha\beta} = a_{\alpha\beta}^{rs}$ for any $1 \leq r, s, \alpha, \beta \leq 2$. Moreover, we obtain that

$$\mu = a_{11}^{22} \quad (4.78)$$

$$\lambda = a_{11}^{11} - 2a_{11}^{22} \quad (4.79)$$

$$\ell = a_{11}^{11} - a_{11}^{22} - a_{12}^{12}. \quad (4.80)$$

Since the Lamé system is weakly elliptic if and only if $\mu \neq 0$, $2\mu + \lambda \neq 0$, we assume that μ , $2\mu + \lambda$ are nonzero. Let us further assume that $3\mu + \lambda \neq 0$ for the existence of the distinguished coefficient tensor. Indeed, applying the result found in [30], we have $\mathfrak{A}_L^{\text{dis}} \neq \emptyset$. According to the equation (4.51), we have

$$\begin{aligned} & [C_{\gamma\alpha}(\omega)]_{1 \leq \gamma, \alpha \leq 2} \quad (4.81) \\ &= \begin{bmatrix} \alpha_r^\beta (\partial_r E_{1\beta})(\cos \omega, \sin \omega) & \beta_r^\beta (\partial_r E_{1\beta})(\cos \omega, \sin \omega) \\ \alpha_r^\beta (\partial_r E_{2\beta})(\cos \omega, \sin \omega) & \beta_r^\beta (\partial_r E_{2\beta})(\cos \omega, \sin \omega) \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \alpha_r^\beta &= -\sin \omega a_{r1}^{\beta 1} + \cos \omega a_{r2}^{\beta 1}, \\ \beta_r^\beta &= -\sin \omega a_{r1}^{\beta 2} + \cos \omega a_{r2}^{\beta 2}. \end{aligned} \quad (4.82)$$

Since $E_{\gamma\beta}$ is the fundamental solution associated with the Lamé operator, we have for each $1 \leq r, \gamma, \beta \leq 2$, $n \geq 2$, $X \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} (\partial_r E_{\gamma\beta})(X) &= \frac{1}{2\mu(2\mu + \lambda)\omega_{n-1}} \left[(3\mu + \lambda) \frac{\delta_{\gamma\beta} X_r}{|X|^n} + n(\mu + \lambda) \frac{X_\gamma X_\beta X_r}{|X|^{n+2}} \right. \\ &\quad \left. - (\mu + \lambda) \frac{X_\beta \delta_{\gamma r} + X_\gamma \delta_{\beta r}}{|X|^n} \right]. \end{aligned} \quad (4.83)$$

Thus, we have

$$[C_{\gamma\alpha}(\omega)]_{1 \leq \gamma, \alpha \leq 2} = -\frac{\gamma(\ell)}{2\pi} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (4.84)$$

where $\gamma(\ell) := \frac{\mu(\mu+\lambda) - \ell(3\mu+\lambda)}{2\mu(2\mu+\lambda)}$. Note that the kernel of the double layer potential associated with the Lamé operator and the coefficient tensor in (4.77) is

$$k_\ell(s, t) = -\frac{\gamma(\ell)}{2\pi} \cdot \frac{1}{s-t} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (4.85)$$

This implies that for any $\omega \in [0, \pi)$, $[C_{\gamma\alpha}(\omega)]_{1 \leq \gamma, \alpha \leq 2}$ is antisymmetric which yields $[C_{\gamma\alpha}(\omega)]_{1 \leq \gamma, \alpha \leq 2}$ is diagonalizable over \mathbb{C} . Applying the result in Theorem 4.10, we get

$$\sigma(K_A; L^p(\mathbb{R}_+)) = \{0\}, \quad \forall \omega \in [0, \pi) \iff A \in \mathfrak{A}_L^{\text{dis}}. \quad (4.86)$$

Moreover, from the previous argument

$$A \in \mathfrak{A}_L^{\text{dis}} \iff C_{\gamma\alpha}(\omega) = 0, \quad \forall 1 \leq \gamma, \alpha \leq 2 \quad (4.87)$$

$$\iff \ell = \frac{\mu(\mu + \lambda)}{3\mu + \lambda}. \quad (4.88)$$

In conclusion,

$$\sigma(K_A; L^p(\mathbb{R}_+)) = \{0\}, \quad \forall \omega \in [0, \pi) \iff \ell = \frac{\mu(\mu + \lambda)}{3\mu + \lambda}, \quad (4.89)$$

where $\ell = \frac{\mu(\mu + \lambda)}{3\mu + \lambda}$ gives rise to the so called pseudostress conormal derivative.

Next, we reduce the conditions for $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ to the sufficient conditions of the coefficient tensor.

Lemma 4.12. *If $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}}$ is either symmetric or antisymmetric, then for any $\omega \in [0, \pi)$, $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ in (4.51) is antisymmetric or symmetric, respectively.*

Proof. Since Ω is a half plane, for any $x, y \in \partial\Omega$, we have $\nu(x) = \nu(y)$. If $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}}$ is symmetric, that is $A^T = A$ which further implies that $L^T = L$. Since the integral kernel of the transpose double layer potential operator has the form

$$\left(k_{\gamma\alpha}^{A^T}(x, y) \right)_{1 \leq \alpha, \gamma \leq M} = \left(\nu_s(x) a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x - y) \right)_{1 \leq \alpha, \gamma \leq M}, \quad (4.90)$$

by the previous argument in (4.50) and (4.48), for $s, t \in \mathbb{R}$, $s \neq t$,

$$\begin{aligned} & \left(k_{\gamma\alpha}^{A^T}(s, t) \right)_{1 \leq \alpha, \gamma \leq M} \quad (4.91) \\ &= \left((\sin \omega a_{r1}^{\beta\alpha} - \cos \omega a_{r2}^{\beta\alpha}) (\partial_r E_{\gamma\beta})((s - t) \cos \omega, (s - t) \sin \omega) \right)_{1 \leq \alpha, \gamma \leq M} \\ &= \left(-\frac{C_{\gamma\alpha}(\omega)}{s - t} \right)_{1 \leq \alpha, \gamma \leq M}. \end{aligned}$$

Since $A^T = A$, $L^T = L$, we get for any $s, t \in \mathbb{R}$, $s \neq t$, $1 \leq \alpha, \gamma \leq M$,

$$C_{\gamma\alpha}(\omega) = -C_{\alpha\gamma}(\omega), \quad (4.92)$$

which implies that $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ is antisymmetric.

Indeed, if $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}}$ is antisymmetric, one can conclude $A^T = -A$, $L^T = -L$, and applying the previous argument provides $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ is symmetric. This completes the proof of this lemma. \square

Corollary 4.13. *Let $p \in (1, \infty)$ and L be a homogeneous, second-order, constant complex-coefficient, weakly elliptic $M \times M$ system in \mathbb{R}^2 , let us consider $\Omega \subseteq \mathbb{R}^2$ a half plane of an angle $\omega \in [0, \pi)$ and a coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}} \in \mathfrak{A}_L$. If $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}}$ is either symmetric or antisymmetric, then*

$$\sigma(K_A; L^p(\mathbb{R}_+)) = \{0\}, \quad \forall \omega \in [0, \pi) \iff A \in \mathfrak{A}_L^{dis}. \quad (4.93)$$

Proof. According to the Lemma 4.12, if $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq M \\ 1 \leq r, s \leq 2}}$ is either symmetric or antisymmetric, then for any $\omega \in [0, \pi)$, $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ in (4.51) is antisymmetric or symmetric, respectively. This implies that $(C_{\gamma\alpha}(\omega))_{1 \leq \gamma, \alpha \leq M}$ is diagonalizable over \mathbb{C} . Combining this with Theorem 4.10, equivalence (4.93) holds. This proves the Corollary 4.13. \square

The following example in [41] shows that (4.93) does not hold if the system is not symmetric nor antisymmetric.

Example 4.14. We consider the following system:

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda}{k^2} \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} 0 & \frac{\lambda - k^2}{k} \\ \frac{\lambda - 1}{k} & 0 \end{pmatrix} \frac{\partial^2}{\partial x \partial y} + \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial y^2} \right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (4.94)$$

where $\lambda \neq 0, 1, k^2$ and $0 < k \leq 1$. The system (4.94) is nonsymmetric where

the coefficient tensor $A = (a_{rs}^{\alpha\beta})_{\substack{1 \leq \alpha, \beta \leq 2 \\ 1 \leq r, s \leq 2}}$ is

$$\begin{aligned} (a_{11}^{\alpha\beta})_{1 \leq \alpha, \beta \leq 2} &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda}{k^2} \end{pmatrix}, \quad (a_{12}^{\alpha\beta})_{1 \leq \alpha, \beta \leq 2} = \begin{pmatrix} 0 & \frac{\lambda - k^2}{2k} - r_1 \\ \frac{\lambda - 1}{2k} + r_2 & 0 \end{pmatrix} \\ (a_{21}^{\alpha\beta})_{1 \leq \alpha, \beta \leq 2} &= \begin{pmatrix} 0 & \frac{\lambda - k^2}{2k} + r_1 \\ \frac{\lambda - 1}{2k} - r_2 & 0 \end{pmatrix}, \quad (a_{22}^{\alpha\beta})_{1 \leq \alpha, \beta \leq 2} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (4.95)$$

where $r_1, r_2 \in \mathbb{R}$. In addition, for $\xi = (\xi_r)_{1 \leq r \leq 2} \in \mathbb{R}^2 \setminus \{0\}$,

$$(a_{rs}^{\alpha\beta} \xi_r \xi_s)_{1 \leq \alpha, \beta \leq 2} = \begin{pmatrix} \xi_1^2 + \xi_2^2 \lambda & \xi_1 \xi_2 \left(\frac{\lambda - k^2}{k} \right) \\ \xi_1 \xi_2 \left(\frac{\lambda - 1}{k} \right) & \xi_1^2 \frac{\lambda}{k^2} + \xi_2^2 \end{pmatrix}, \quad (4.96)$$

which yields

$$\det[(a_{rs}^{\alpha\beta} \xi_r \xi_s)_{1 \leq \alpha, \beta \leq 2}] = \lambda \left(\frac{1}{k^2} \xi_1^4 + \frac{k^2 + 1}{k^2} \xi_1^2 \xi_2^2 + \xi_2^4 \right). \quad (4.97)$$

Since $\xi \in \mathbb{R}^2 \setminus \{0\}$, $\lambda, k \neq 0$, we have $\det[(a_{rs}^{\alpha\beta} \xi_r \xi_s)_{1 \leq \alpha, \beta \leq 2}] \neq 0$ which implies that the system (4.94) is weakly elliptic for any $\lambda \neq 0, 1, k^2$ and $0 < k \leq 1$.

We further assume that $\lambda \neq -k$.

Let us assume that $k \neq 1$ and Ω is a half plane with an angle ω in $[0, \pi)$. According to (4.51), for each $1 \leq \gamma, \alpha \leq 2$,

$$C_{\gamma\alpha}(\omega) = (-\sin \omega a_{r_1}^{\beta\alpha} + \cos \omega a_{r_2}^{\beta\alpha})(\partial_r E_{\gamma\beta})(\cos \omega, \sin \omega). \quad (4.98)$$

Substituting $a_{rs}^{\beta\alpha}$ in (4.95) yields

$$\begin{aligned} C_{11}(\omega) &= -\sin \omega \left(\partial_1 E_{11} + \left(\frac{\lambda - 1}{2k} - r_2 \right) \partial_2 E_{12} \right) \\ &\quad + \cos \omega \left(\left(\frac{\lambda - 1}{2k} + r_2 \right) \partial_1 E_{12} + \lambda \partial_2 E_{11} \right) \\ C_{12}(\omega) &= -\sin \omega \left(\left(\frac{\lambda - k^2}{2k} + r_1 \right) \partial_2 E_{11} + \frac{\lambda}{k^2} \partial_1 E_{12} \right) \\ &\quad + \cos \omega \left(\left(\frac{\lambda - k^2}{2k} - r_1 \right) \partial_1 E_{11} + \partial_2 E_{12} \right), \end{aligned} \quad (4.99)$$

and

$$\begin{aligned}
C_{21}(\omega) &= -\sin \omega \left(\partial_1 E_{21} + \left(\frac{\lambda - 1}{2k} - r_2 \right) \partial_2 E_{22} \right) \\
&\quad + \cos \omega \left(\left(\frac{\lambda - 1}{2k} + r_2 \right) \partial_1 E_{22} + \lambda \partial_2 E_{21} \right) \\
C_{22}(\omega) &= -\sin \omega \left(\frac{\lambda}{k^2} \partial_1 E_{22} + \left(\frac{\lambda - k^2}{2k} + r_1 \right) \partial_2 E_{21} \right) \\
&\quad + \cos \omega \left(\left(\frac{\lambda - k^2}{2k} - r_1 \right) \partial_1 E_{21} + \partial_2 E_{22} \right),
\end{aligned} \tag{4.100}$$

where for each $1 \leq r, \gamma, \beta \leq 2$, $\partial_r E_{\gamma\beta}$ is $(\partial_r E_{\gamma\beta})(\cos \omega, \sin \omega)$.

Moreover, for $X = (x, y)$, we obtain

$$\begin{aligned}
\partial_1 E_{11}(x, y) &= \frac{1}{2\pi\lambda(a+b)} \left[a \frac{x}{|X|^2} + bk^3 \frac{x}{k^2x^2 + y^2} \right], \\
\partial_2 E_{11}(x, y) &= \frac{1}{2\pi\lambda(a+b)} \left[a \frac{y}{|X|^2} + bk \frac{y}{k^2x^2 + y^2} \right], \\
\partial_1 E_{12}(x, y) &= \frac{ak(k-1)}{2\pi\lambda(a+b)} \frac{y(y^2 - kx^2)}{|X|^2(k^2x^2 + y^2)}, \\
\partial_2 E_{12}(x, y) &= -\frac{ak(k-1)}{2\pi\lambda(a+b)} \frac{x(y^2 - kx^2)}{|X|^2(k^2x^2 + y^2)}, \\
\partial_1 E_{21}(x, y) &= -\frac{bk(k-1)}{2\pi\lambda(a+b)} \frac{y(y^2 - kx^2)}{|X|^2(k^2x^2 + y^2)}, \\
\partial_2 E_{21}(x, y) &= \frac{bk(k-1)}{2\pi\lambda(a+b)} \frac{x(y^2 - kx^2)}{|X|^2(k^2x^2 + y^2)}, \\
\partial_1 E_{22}(x, y) &= \frac{1}{2\pi\lambda(a+b)} \left[bk^2 \frac{x}{|X|^2} + ak^3 \frac{x}{k^2x^2 + y^2} \right], \\
\partial_2 E_{22}(x, y) &= \frac{1}{2\pi\lambda(a+b)} \left[bk^2 \frac{y}{|X|^2} + ak \frac{y}{k^2x^2 + y^2} \right],
\end{aligned} \tag{4.101}$$

where $a = \lambda - k^2$, $b = 1 - \lambda$. Substituting $X = (\cos \omega, \sin \omega)$, we have

$$\begin{aligned}
\partial_1 E_{11}(\cos \omega, \sin \omega) &= \frac{1}{2\pi\lambda(a+b)} \left[a \cos \omega + bk^3 \frac{\cos \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right], \quad (4.102) \\
\partial_2 E_{11}(\cos \omega, \sin \omega) &= \frac{1}{2\pi\lambda(a+b)} \left[a \sin \omega + bk \frac{\sin \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right], \\
\partial_1 E_{12}(\cos \omega, \sin \omega) &= \frac{ak(k-1)}{2\pi\lambda(a+b)} \frac{\sin \omega(\sin^2 \omega - k \cos^2 \omega)}{(k^2 \cos^2 \omega + \sin^2 \omega)}, \\
\partial_2 E_{12}(\cos \omega, \sin \omega) &= -\frac{ak(k-1)}{2\pi\lambda(a+b)} \frac{\cos \omega(\sin^2 \omega - k \cos^2 \omega)}{(k^2 \cos^2 \omega + \sin^2 \omega)} \\
\partial_1 E_{21}(\cos \omega, \sin \omega) &= -\frac{bk(k-1)}{2\pi\lambda(a+b)} \frac{\sin \omega(\sin^2 \omega - k \cos^2 \omega)}{(k^2 \cos^2 \omega + \sin^2 \omega)}, \\
\partial_2 E_{21}(\cos \omega, \sin \omega) &= \frac{bk(k-1)}{2\pi\lambda(a+b)} \frac{\cos \omega(\sin^2 \omega - k \cos^2 \omega)}{(k^2 \cos^2 \omega + \sin^2 \omega)}, \\
\partial_1 E_{22}(\cos \omega, \sin \omega) &= \frac{1}{2\pi\lambda(a+b)} \left[bk^2 \cos \omega + ak^3 \frac{\cos \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right], \\
\partial_2 E_{22}(\cos \omega, \sin \omega) &= \frac{1}{2\pi\lambda(a+b)} \left[bk^2 \sin \omega + ak \frac{\sin \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right].
\end{aligned}$$

Combining this with (4.99) and (4.100), we further obtain that

$$\begin{aligned}
&C_{11}(\omega) \tag{4.103} \\
&= -\sin \omega \left(\frac{1}{2\pi\lambda(a+b)} \left[a \cos \omega + bk^3 \frac{\cos \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right] \right. \\
&\quad \left. - \left(\frac{\lambda-1}{2k} - r_2 \right) \frac{ak(k-1)}{2\pi\lambda(a+b)} \frac{\cos \omega(\sin^2 \omega - k \cos^2 \omega)}{(k^2 \cos^2 \omega + \sin^2 \omega)} \right) \\
&+ \cos \omega \left(\left(\frac{\lambda-1}{2k} + r_2 \right) \frac{ak(k-1)}{2\pi\lambda(a+b)} \frac{\sin \omega(\sin^2 \omega - k \cos^2 \omega)}{(k^2 \cos^2 \omega + \sin^2 \omega)} \right. \\
&\quad \left. + \lambda \frac{1}{2\pi\lambda(a+b)} \left[a \sin \omega + bk \frac{\sin \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right] \right) \\
&= \frac{ab}{2\pi\lambda(a+b)} \sin \omega \cos \omega \left(\frac{k}{k^2 \cos^2 \omega + \sin^2 \omega} - 1 \right) \\
&\quad - \frac{ab(k-1)}{2\pi\lambda(a+b)} \sin \omega \cos \omega \frac{\sin^2 \omega - k \cos^2 \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \\
&= 0.
\end{aligned}$$

Similarly,

$$C_{22}(\omega) \tag{4.104}$$

$$\begin{aligned} &= -\sin \omega \left(\frac{1}{2\pi k^2(a+b)} \left[bk^2 \cos \omega + ak^3 \frac{\cos \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right] \right. \\ &\quad \left. + \left(\frac{\lambda - k^2}{2k} + r_1 \right) \frac{bk(k-1) \cos \omega (\sin^2 \omega - k \cos^2 \omega)}{2\pi \lambda(a+b) (k^2 \cos^2 \omega + \sin^2 \omega)} \right) \\ &+ \cos \omega \left(- \left(\frac{\lambda - k^2}{2k} - r_1 \right) \frac{bk(k-1) \sin \omega (\sin^2 \omega - k \cos^2 \omega)}{2\pi \lambda(a+b) (k^2 \cos^2 \omega + \sin^2 \omega)} \right. \\ &\quad \left. + \frac{1}{2\pi \lambda(a+b)} \left[bk^2 \sin \omega + ak \frac{\sin \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right] \right) \tag{4.105} \\ &= \frac{ab}{2\pi \lambda(a+b)} \sin \omega \cos \omega \left(\frac{k}{k^2 \cos^2 \omega + \sin^2 \omega} - 1 \right) \\ &\quad - \frac{ab(k-1)}{2\pi \lambda(a+b)} \sin \omega \cos \omega \frac{\sin^2 \omega - k \cos^2 \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \\ &= 0. \end{aligned}$$

Moreover,

$$C_{12}(\omega) \tag{4.106}$$

$$\begin{aligned} &= -\sin \omega \left(\left(\frac{\lambda - k^2}{2k} + r_1 \right) \frac{1}{2\pi \lambda(a+b)} \left[a \sin \omega + bk \frac{\sin \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right] \right. \\ &\quad \left. + \frac{\lambda}{k^2} \frac{ak(k-1) \sin \omega (\sin^2 \omega - k \cos^2 \omega)}{2\pi \lambda(a+b) (k^2 \cos^2 \omega + \sin^2 \omega)} \right) \\ &+ \cos \omega \left(\left(\frac{\lambda - k^2}{2k} - r_1 \right) \frac{1}{2\pi \lambda(a+b)} \left[a \cos \omega + bk^3 \frac{\cos \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right] \right. \\ &\quad \left. - \frac{ak(k-1) \cos \omega (\sin^2 \omega - k \cos^2 \omega)}{2\pi \lambda(a+b) (k^2 \cos^2 \omega + \sin^2 \omega)} \right) \\ &= \frac{1}{2\pi \lambda(a+b)(k^2 \cos^2 \omega + \sin^2 \omega)} \left[(\lambda - k^2)k^2 \left(\frac{\lambda + k^2 - 2k}{2k} - r_1 \right) \cos^4 \omega \right. \\ &\quad + (\lambda - k^2)((k-1)^2(\lambda + k^2) - (k^2 + 1)r_1) \sin^2 \omega \cos^2 \omega \\ &\quad + (1 - \lambda)k^3 \left(\frac{\lambda - k^2}{2k} - r_1 \right) \cos^2 \omega - (1 - \lambda)k \left(\frac{\lambda - k^2}{2k} + r_1 \right) \sin^2 \omega \\ &\quad \left. + (\lambda - k^2) \left(\frac{\lambda + k^2 - 2k\lambda}{2k} - r_1 \right) \sin^4 \omega \right] \\ &:= \frac{1}{2\pi \lambda(a+b)(k^2 \cos^2 \omega + \sin^2 \omega)} [(k-1)(\lambda + k)(k^2 \cos^2 \omega + \sin^2 \omega)r_1 + I], \end{aligned}$$

where

$$\begin{aligned}
I &= \frac{a^2k}{2} \cos^4 \omega - \frac{a^2k}{2} \cos^2 \omega \sin^2 \omega + \frac{a^2}{2k} \cos^2 \omega \sin^2 \omega - \frac{a^2}{2k} \sin^4 \omega \quad (4.107) \\
&\quad + \frac{abk^2}{2} \cos^2 \omega - \frac{ab}{2} \sin^2 \omega - \frac{a(k-1)\lambda}{k} \sin^4 \omega + ak^2(k-1) \cos^4 \omega \\
&\quad + a(k-1)\lambda \sin^2 \omega \cos^2 \omega - ak(k-1) \sin^2 \omega \cos^2 \omega \\
&:= I_1 + I_2 + I_3,
\end{aligned}$$

with

$$I_1 = \frac{a^2k}{2} \cos^4 \omega - \frac{a^2k}{2} \cos^2 \omega \sin^2 \omega + \frac{a^2}{2k} \cos^2 \omega \sin^2 \omega - \frac{a^2}{2k} \sin^4 \omega \quad (4.108)$$

$$I_2 = \frac{abk^2}{2} \cos^2 \omega - \frac{ab}{2} \sin^2 \omega \quad (4.109)$$

$$\begin{aligned}
I_3 &= -\frac{a(k-1)\lambda}{k} \sin^4 \omega + ak^2(k-1) \cos^4 \omega \\
&\quad + a(k-1)\lambda \sin^2 \omega \cos^2 \omega - ak(k-1) \sin^2 \omega \cos^2 \omega. \quad (4.110)
\end{aligned}$$

First, we have

$$\begin{aligned}
I_1 &= \frac{a^2k}{2} \cos^4 \omega - \frac{a^2k}{2} \cos^2 \omega \sin^2 \omega + \frac{a^2}{2k} \cos^2 \omega \sin^2 \omega - \frac{a^2}{2k} \sin^4 \omega \\
&= \frac{a^2}{2k} (\cos^2 \omega - \sin^2 \omega) (k^2 \cos^2 \omega + \sin^2 \omega). \quad (4.111)
\end{aligned}$$

Next,

$$\begin{aligned}
I_2 &= \frac{abk^2}{2} \cos^2 \omega - \frac{ab}{2} \sin^2 \omega \quad (4.112) \\
&= \frac{abk^2}{2} \cos^2 \omega + \frac{ab}{2} \sin^2 \omega - ab \sin^2 \omega \\
&= \frac{ab}{2} (k^2 \cos^2 \omega + \sin^2 \omega) - ab \sin^2 \omega.
\end{aligned}$$

In addition,

$$\begin{aligned}
I_3 &= -\frac{a(k-1)\lambda}{k} \sin^4 \omega \quad (4.113) \\
&\quad + ak^2(k-1) \cos^4 \omega + a(k-1)\lambda \sin^2 \omega \cos^2 \omega - ak(k-1) \sin^2 \omega \cos^2 \omega \\
&= a(k-1)(k^2 \cos^2 \omega + \sin^2 \omega) \left(\cos^2 \omega - \frac{1}{k} \sin^2 \omega \right) \\
&\quad + ab(k-1) \left(\frac{1}{k} \sin^4 \omega - \sin^2 \omega \cos^2 \omega \right).
\end{aligned}$$

In conclusion, we obtain that

$$\begin{aligned}
I &= \frac{a^2}{2k}(\cos^2 \omega - \sin^2 \omega)(k^2 \cos^2 \omega + \sin^2 \omega) + \frac{ab}{2}(k^2 \cos^2 \omega + \sin^2 \omega) \\
&\quad + a(k-1)(k^2 \cos^2 \omega + \sin^2 \omega) \left(\cos^2 \omega - \frac{1}{k} \sin^2 \omega \right) \\
&\quad + ab(k-1) \left(\frac{1}{k} \sin^4 \omega - \sin^2 \omega \cos^2 \omega \right) - ab \sin^2 \omega. \tag{4.114}
\end{aligned}$$

In particular,

$$\begin{aligned}
&ab(k-1) \left(\frac{1}{k} \sin^4 \omega - \sin^2 \omega \cos^2 \omega \right) - ab \sin^2 \omega \\
&= - \frac{ab \sin^2 \omega}{k} (k^2 \cos^2 \omega + \sin^2 \omega). \tag{4.115}
\end{aligned}$$

Combine this with (4.114), we get

$$\begin{aligned}
I &= a(k^2 \cos^2 \omega + \sin^2 \omega) \left[\frac{a}{2k}(\cos^2 \omega - \sin^2 \omega) + \frac{b}{2} \right. \\
&\quad \left. + (k-1) \left(\cos^2 \omega - \frac{1}{k} \sin^2 \omega \right) - \frac{b \sin^2 \omega}{k} \right]. \tag{4.116}
\end{aligned}$$

Substituting $a = \lambda - k^2$, $b = 1 - \lambda$ into (4.116), one has

$$I = \frac{(\lambda - k^2)(\lambda - k)(1 - k)}{2k} (k^2 \cos^2 \omega + \sin^2 \omega). \tag{4.117}$$

This forces

$$C_{12}(\omega) = - \frac{(\lambda + k)}{2\pi\lambda(1 + k)} \left[r_1 - \frac{(\lambda - k^2)(\lambda - k)}{2k(\lambda + k)} \right]. \tag{4.118}$$

Similarly,

$$\begin{aligned}
& C_{21}(\omega) \tag{4.119} \\
&= -\sin \omega \left(-\frac{bk(k-1)}{2\pi\lambda(a+b)} \frac{\sin \omega(\sin^2 \omega - k \cos^2 \omega)}{(k^2 \cos^2 \omega + \sin^2 \omega)} \right. \\
&\quad \left. + \left(\frac{\lambda-1}{2k} - r_2 \right) \frac{1}{2\pi\lambda(a+b)} \left[bk^2 \sin \omega + ak \frac{\sin \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right] \right) \\
&\quad + \cos \omega \left(\left(\frac{\lambda-1}{2k} + r_2 \right) \frac{1}{2\pi\lambda(a+b)} \left[bk^2 \cos \omega + ak^3 \frac{\cos \omega}{k^2 \cos^2 \omega + \sin^2 \omega} \right] \right. \\
&\quad \left. + \lambda \frac{bk(k-1)}{2\pi\lambda(a+b)} \frac{\cos \omega(\sin^2 \omega - k \cos^2 \omega)}{(k^2 \cos^2 \omega + \sin^2 \omega)} \right) \\
&= \frac{1}{2\pi\lambda(a+b)(k^2 \cos^2 \omega + \sin^2 \omega)} \left[bk^2(k^2 \cos^2 \omega + \sin^2 \omega) \left(\cos^2 \omega \left(\frac{\lambda-1}{2k} + r_2 \right) \right. \right. \\
&\quad \left. \left. - \sin^2 \omega \left(\frac{\lambda-1}{2k} - r_2 \right) \right) - \frac{abk^2}{2} \cos^2 \omega + r_2 ak^3 \cos^2 \omega + \frac{ab}{2} \sin^2 \omega \right. \\
&\quad \left. + r_2 ak \sin^2 \omega + b\lambda k(k-1) \sin^2 \omega \cos^2 \omega + bk(k-1) \sin^4 \omega \right. \\
&\quad \left. - b\lambda k^2(k-1) \cos^4 \omega - bk^2(k-1) \sin^2 \omega \cos^2 \omega \right] \\
&= \frac{1}{2\pi\lambda(a+b)(k^2 \cos^2 \omega + \sin^2 \omega)} \left[k(\lambda+k)(1-k)(k^2 \cos^2 \omega + \sin^2 \omega)r_2 \right. \\
&\quad \left. - \frac{b}{2}(\lambda-k)(k-1)(k^2 \cos^2 \omega + \sin^2 \omega) \right] \\
&= \frac{k(\lambda+k)}{2\pi\lambda(1+k)} \left[r_2 - \frac{(\lambda-1)(\lambda-k)}{2k(\lambda+k)} \right].
\end{aligned}$$

In conclusion,

$$[C_{\gamma\alpha}(\omega)]_{1 \leq \gamma, \alpha \leq 2} = \frac{(\lambda+k)}{2\pi\lambda(1+k)} \begin{bmatrix} 0 & -\left(r_1 - \frac{(\lambda-k^2)(\lambda-k)}{2k(\lambda+k)} \right) \\ k \left(r_2 - \frac{(\lambda-1)(\lambda-k)}{2k(\lambda+k)} \right) & 0 \end{bmatrix}. \tag{4.120}$$

Since $\lambda \neq 0, 1, k^2, -k$ and $0 < k < 1$, $\frac{(\lambda+k)}{2\pi\lambda(1+k)}$ is nonzero. According to the Theorem 4.6, the coefficient tensor in (4.95) associated with the system (4.94)

is distinguished if and only if

$$r_1 = \frac{(\lambda - k^2)(\lambda - k)}{2k(\lambda + k)}, \quad r_2 = \frac{(\lambda - 1)(\lambda - k)}{2k(\lambda + k)}. \quad (4.121)$$

In particular, if we choose the parameters r_1, r_2 as

$$r_1 = \frac{(\lambda - k^2)(\lambda - k)}{2k(\lambda + k)}, \quad r_2 = 1 + \frac{(\lambda - 1)(\lambda - k)}{2k(\lambda + k)}, \quad (4.122)$$

then $[C_{\gamma\alpha}(\omega)]_{1 \leq \gamma, \alpha \leq 2}$ in (4.120) turns out to be

$$\frac{(\lambda + k)}{2\pi\lambda(1 + k)} \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix} \quad (4.123)$$

which is nilpotent nonzero matrix for any $\omega \in [0, \pi)$. According to the Theorem 4.6 and (4.73),

$$\sigma(K_A; L^p(\mathbb{R}_+)) = \{0\}, \quad \text{but } A \text{ is not distinguished coefficient tensor.} \quad (4.124)$$

This provides a counterexample to the Corollary 4.13 in the absence of symmetry and antisymmetry.

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