GEOMETRY REPRESENTATIONS IN A TEXTBOOK

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Waldemar Stepnowski
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Dissertation Approvals:
Timothy Fukawa-Connelly, Dissertation Advisor, Department of Teaching and Learning
Julie Booth, Committee Member, Department of Teaching and Learning
Kristie Jones Newton, Committee Member, Department of Teaching and Learning
Timothy Patterson, Committee Member Department of Teaching and Learning

ABSTRACT

The purpose of the study is to enumerate the representations in high school geometry, to narrow the focus to the most common ones, and to describe how the common representations are coordinated. To achieve this purpose, I analyzed several sections of a popular geometry textbook using thematic analysis, semiotics, and a pragmatic approach to capture the variety of representations into categories and to use descriptive statistics to narrow the focus to the most common representations and coordinations. The major findings are the bringing into prominence of representations like textbook gestures, ordered pairs, written language, tables, and their uses, and at the same time understating the importance of less common representations of physical objects. Other important findings are: (1) exposing which representations are most often coordinated like written language (WL), diagrams (D), numbers (N), and ordered pairs (OP), short geometry symbols (Sy), e.g., WL to D, WL+Sy to D, and N+OP to D; (2) some of the mechanisms in that coordination like using, numbers, point names, and textbook gestures, which include color, arrows, font changes, etc. Clarifying the representations in high school geometry and narrowing the scope to the most common ones allows researchers to study various combinations of representations and their impact students.

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CHAPTER 1
INTRODUCTION

> *... different representations support different ways of thinking about and manipulating a mathematical object. An object can be better understood when viewed through multiple lenses.* (NCTM, 2000)

The above quote gets at the heart of the matter of the importance of representations, or the way we represent or discuss mathematics, and for the purposes of this study geometry. Some representations are ideal in certain situations, while some are not. Teachers, students, and seasoned mathematicians must make choices in how they represent a mathematical object so that it helps them attain their goal. For a teacher, that goal may be to explain a property of a triangle, for a student it may be to solve a problem, and for a professional mathematician it may be to write the clearest proof. To be able to make choices in the use of geometry representations, teachers, students, and mathematicians need to know which representations are available, how to use them, and what are some benefits of using one over another. That is precisely the goal of my research: to list and describe useful representations in high school geometry and to identify how they are presented and coordinated in a geometry textbook, Larson and Boswell’s (2015) *Geometry: A Common Core Curriculum*. I do that by applying thematic analysis to find distinctions among the representations, looking methodically for how those representations are introduced, coding for those representations in coordinations and recording them in a spreadsheet, and, finally, analyzing the mechanisms that aid in the most common coordinations.

Besides the above, there are a few more reasons why studying representations is valuable. First, we all want students to solve problems with a high degree of accuracy and efficiency, and representations help us manage mathematical information. Students who coordinate different mathematical representations well solve mathematical problem better than those who do not (Gagatsis & Shiakalli, 2004). Second, there is a difference between changing within one representation and changing or coordinating between different representations. While students have difficulty changing representations (conversion), they have less difficulty in manipulating (treatment) a given representation (Duval, 2006). Therefore, studying which representations appear to be coordinated most often and what are some ways that they are coordinated may help geometry teachers, publishers, and researchers devise strategies that aid students with conversion. Third, to solve problems students need to use aids or tools like representations to reduce the complexity of a problem. Humans can "extend the operation of memory beyond the biological dimensions of the human nervous system and permit it to incorporate artificial, or self-generated, stimuli, which we call *signs*" (Vygotsky, 1978, p. 39). Signs, as I explain later, are an important lens through which I analyze how math objects are introduced and manipulated. We use representations as a kind of external memory, where we can offload some of the information from our working memory (Scaife & Rogers, 1996). Chandler and Sweller's (1991) seminal work on cognitive load theory show via experiments that integrated diagrams with text, in contrast to a diagram with text below it, reduced the cognitive load, and reducing cognitive load improves learning. Fourth, in the digital age, there are also practical reasons to study geometry representations. We are beginning to develop digital textbooks, online environments, etc., where the choice of representations is important (Presmeg, 2016). Because digital books and websites are more flexible, we might be able to invent new representation and improved ways to coordinate among the present ones.

One of the most important reasons why it is crucial to study geometry representations is so teachers can more effectively teach a concept, using the most appropriate representation for their students. Sometimes that means using physical objects, diagrams, animations, and other representations, but sometimes it means using more abstract symbols. Students in a given zone of proximal development can learn using signs appropriate for that zone (Vygotsky, 1978). Knowing how teachers interpret students' use of sign-vehicles (representations), we can create better instructional sequences (Sáenz-Ludlow & Zellweger, 2016). Signs, as I describe later in more detail, are a 'triangulation' of external representation (1) that combined with our interpretation (2) stand for a mathematical object (3).

Those that seem to know representations in terms of content knowledge and pedagogical content knowledge are teams of authors and experts that publishers trust to write textbooks. Authors are called that for a reason; they are the authority in the content area they write about. To see how authors of textbooks, who are advised by teachers, mathematicians, and researchers, use and explain representations, I analyzed a popular textbook. When using the same textbooks, students see the same representations when examining examples and solving problems. Also, publishers devoted years, even decades, improving their books applying countless suggestion from many educators, academics, and other professionals.

Above, I describe some of the many reasons why we should study math representations. I will next discuss the formalist perspective in mathematics and how mathematics and its representations are invented. After that, I provide definitions of object, representation, visualization, and certain terms in semiotics. Then, I will discuss the different types of representations, like language, diagrams, and algebra, and how they might be more beneficial in certain classroom situations and detrimental in others. Then, I will analyze how those representations are used together. Finally, I will discuss where the gaps are and how I analyzed a popular textbook to see how American students learn geometry and to answer my research questions.

Ontology and Epistemology

Before we can discuss representations, we need to define them and their ontology in mathematics. That definition will depend on our philosophy of mathematics. According to Davis and Hersch (1998), most mathematics is done in the Platonist paradigm, in which mathematicians believe that there are perfect ideas or mathematical objects and we can only glimpse at their shadows. In this context, representations are these shadows. For example, line AB, a pencil mark made along straightedge, and y=3x-2 each try to give us insight into this idea of a line that exists in the land of perfection. If we did not have these various representations, we would have limited access into the idea of a line that is infinitely long, perfectly straight, and has no thickness. We can only communicate our thoughts using words, sketches and other symbols that do not represent all the attributes of a line.

Platonists believe pure mathematical objects exist and, using the Plato's allegory of the cave, we can only learn about them by observing shadows. Formalists are very similar to Platonists. Formalists say that there may be these mathematical objects, but that we should not be concerned with that fact and simply build a mathematical system based on assumptions. Formalists simply define and assume certain ideas. If those ideas lead to interesting mathematics and help solve problems in the real or ideal world, it is purely a coincidence. On the other hand, constructivists (in mathematics, not mathematics education) are quite different; constructivists work completely in the finite, real world (Davis & Hersch, 1998). Constructivists require construction proofs, not proofs by contradiction (Ernest, 1998). This paper takes the formalist view that not all mathematical objects or concepts can be constructed, and that formal symbols only provide us with a certain perspective of the objects.

The ontology of representations and mathematical objects helps explain what mathematics *is*, but we need an epistemology that will explain how we *know* or learn representations. Later, I will discuss Peirce's semiotics, but here his and John Dewey's pragmatic approach of acquiring knowledge are relevant. Peirce's idea of abduction is, in short, the process of creating hypotheses, in contrast to deduction, which produces necessary consequences from hypotheses (Otte, 2006). Like the original thinkers who invented geometry, students learn geometry, need to understand this abduction process, where, definitions, axioms and postulates (hypotheses) are invented. For example, students might struggle with what undefined terms like line and point mean, and how to represent them in an exploration activity. Induction would mean students draw many lines and they induce that they are straight, infinite, and are a point thick. No lines that they could possibly draw are like that. They must therefore abduct, or form a hypothesis, a probable conclusion, from what seems to be a line. This seems to be how all definitions and postulates come into being. Postulates can also be taken away like the parallel postulate to form new mathematical system like non-Euclidean geometry. Peirce's pragmatist view of how we know, i.e., coming up with hypotheses and living with the consequence, is significantly different than that of geometric (deductive) method, i.e., all geometry is deduced from axioms or 'self-evident truths;' they are not self-evident. Although we know facts are true (epistemology) through the three ways (induction, deduction, and abduction), Peirce's abduction is often overlooked. We induce facts by observing some phenomena repeatedly, and we deduce by using logic to combine facts in some logical way. Abduction is a mental process that happens before induction, where we spontaneously generate hypotheses, especially finding similarities between different concepts (Melrose, 1995). Abduction is a kind of brainstorming based on initial evidence, choosing the best hypothesis from many, but it is not basing it on a great amount of data as in induction, nor deducing it from agreed-to facts. Researchers think of hypotheses constantly, and so do students as they solve a problem, thinking of which representation or algorithm to use.

To a pragmatist, knowledge is a relationship between an action and its consequence; it is not an external truth (Biesta, 2010). For example, choosing different axioms (action) leads to different mathematics (consequence). This way of knowing through action is contrasted with the rationalism and the mind/body dualism of Descartes—who was instrumental in developing our current representations of graph and algebra—and at the same time doubting and moving away from language as a representation (O'Halloran, 2008). Pragmatists have more freedom in choosing methods, and truth to them is what works (Creswell, 2009). The focus of the pragmatist is using a method that works in a given situation. For example, as we will see later, I chose to use Pimm's (2002) list of 12 initial representations rather than Duval's (2006) 4 representations. That was choice that has consequences, and it is based on my hypothesis that having more diverse list is able analyze them at a more granular level. Therefore, not only do we need to choose which hypotheses to explore from abduction while doing research, but we also need to choose the best representation in teaching and in problem-solving.

Cognitive load theory provides a relevant perspective in my research. According to cognitive load theory, students make use of all three components of working memory, namely central executive, visual-spatial sketchpad and an auditory loop (Sweller, Ayres, & Kalyga, 2011). When first reading a problem, a student determines a goal (usually from instructions), and at that point he or she has an auditory loop running in the working memory and devising a plan in the central executive. At the same time, the student is looking at the images like sketches, more text, numerals, or other visual representations. Any one component of the three does not necessarily take away from capacity of our working memory of the other two components. For example, planning to execute a solution does not necessarily take away from our capacity to visualize a figure. Yet, our working memory is limited to a sentence, maybe two, an equation or two, a small diagram, a few ordered pairs, etc. As we will see in the methods, a sentence, or more specifically an independent clause, was my unit of analysis.

Definitions and Semiotic Perspective

I have been using the word object, but it is difficult to define. I think starting with objectification makes more sense. According to Sfard (2007) objectification is stripping everyday human activity from mathematical ideas. For example, Sfard mentions how numbers, which in everyday life are adjectives describing the quantity of apples or feet, are turned into objects that mathematicians use as nouns. Radford (2002) takes a slightly different twist and describes objectification as “a process aimed at bringing something in front of someone's attention or view” by any means (p.14). Thus, we can think of a mathematical *object* as an abstracted concept that can only be apprehended through representations.

At its most basic meaning in mathematics education, ‘to represent,’ ‘to symbolize,’ or ‘to stand for’ means to make present in the mind something that is not physically present (Sáenz-Ludlow, 2002). Therefore, a *representation* is some method or symbol that helps us understand something that we do not have in front of us to examine.

Semiotics also provides us some tools to analyze representations of mathematical objects. In terms of Saussure’s semiology, a representation (word or sound) is the signifier, whereas the object is the signified (Presmeg et al., 2016). Goldin and Kaput (1996) have a similar understanding, where the signified is the internal representation (IR) and the signifier is the external representation (ER). In Peirce's semiotics, the object is what is signified, the representation is the signifier or representamen, and a student's understanding of the object is the interpretant (Sáenz-Ludlow, 2002; Schreiber, 2013). Sáenz-Ludlow (2002) points out that representations are a process (semiosis) among the object, the representamen, and the interpretant. In the case of mathematics, there is little or no direct contact with the object, and therefore a student coordinates representamen ('symbols') and interpretants (students' understanding of the object) to understand the properties of the object or even the object itself.

As mentioned, there is nothing in the physical world that is exactly a mathematical object like a line or a circle in the same sense as physical objects like atoms and light. Because geometric objects do not exist in the physical world, it is difficult for students to describe them and to study them. Mathematical constructs are not accessible to our senses (Sfard, 1991). Physicists, chemists, biologists can refer to the physical world. Mathematicians cannot; mathematicians (formalists) affirm that some immaterial axioms, and based on those axioms, they build an entire mathematical system. For example, all Euclidean geometry is based on statements as in "Let's say that one can draw a straight line through two points." Although that statement seems simple enough, that idea of a straight line has many underlying concepts like no thickness, perfectly straight, and infinite in both directions, and these concepts are considered self-evident. No such line exists in the universe; even a ray of light begins and ends, bends, and is a photon thick. For all these reasons, it is difficult to define a *mathematical object* (i.e., any definition leads to a representation which is not the object itself because it only imperfectly describes it from one perspective).

Because students need to imagine most geometrical objects, visualization has been considered to be important and has been studied. Visualization is defined as a process of creating and changing mental visual images (Gal & Linchevski, 2010). Gal and Linchevski (2010) describe three phases of visualizing: organization, recognition, and representation. During organization, a student extracts shapes from the visual scene; during recognition, the student recognizes the shapes; and during representation, the student forms a working image of the situation (like Peirce's interpretant). Hershkowitz, Parzysz, and Van Dormolen, (1996) categorize visualization into: (1) interacting with real shapes, (2) transferring objects and their concepts into representations, and vice versa, and (3) relating shapes with other mathematical areas, science, engineering, etc. The difference between the two descriptions of phases is that the former separates the first phase into two, namely organization and recognition, while the latter describes the phase as interacting with shapes; also, the latter list of phases adds a phase, where a student relates the visualization to another field or different mathematical concept or area. Although the other categories of visualizations are important and are relevant to a high school geometry class in general, the second category, i.e., transferring objects and their concepts into representations, is relevant to my study.

Some researchers stretch the boundaries of a mathematical object to include "any entity which is involved in some way in mathematical practice or activity, and which can be separated or individualized" (Font, Godino, & Gallardo, 2013, p. 108). This way of looking at objects will provide flexibility when dealing with certain representations, especially language. We consider mathematical objects to be nouns like point, equation, or perpendicular bisector, but verbs and adjectives can be troublesome. For example, a student may be asked to calculate. Although 'calculate' can be changed to 'calculation,' it is still an action, a process. On the other hand, a point and a perpendicular bisector (seemingly immobile objects) can be considered an action: a point can be thought of as a process of locating, and perpendicular bisector can be thought of a process of creating a line that both intersects at a right angle and through the midpoint of a segment. The point is that some mathematical objects are named after processes, actions, or relationships. The perpendicular bisector is not named after some property based on location of points like the property that all points on it are equidistant from a segment's endpoints. It is named after the relationship of the segment and the bisector, i.e., the angle between them is 90°. It is also named after a dynamic action, i.e., it divides the segment into two equal parts. It could have been named something like "endpoint equidistancer" to link its meaning with a static set of points based on location, stating that the set of point are located the same distance away from both points. The point is not that perpendicular bisector is a misleading name, but that it is based on a relationship and a *dynamic* action, rather than *static* relationship among a set of points.

Font, Godino, and Gallardo (2013) list the following primary mathematical objects: (1) linguistic elements, or terms, expressions, notations, graphs, etc. in their various registers, (2) situations/problems, or extra-mathematical applications, tasks, exercises, examples, etc., (3) concepts/definitions, introduced by means of definitions or descriptions, explicit or otherwise, (4) propositions, statements about concepts, etc., (5) procedures, or algorithms, operations, techniques of calculation, etc.,, (6) arguments, statements used to validate or explain the propositions and procedures, whether deductive or of another kind. The above list underlines my point that mathematical objects do not need to be thought of in the same way we think of physical objects. The working definition of *object* is the abstract mathematical idea that we are trying to represent in some way or another, to visualize it. Some authors use other words like figure, construct, or idea, but I will use the word *object* to stand for Kaput's represented world or Peirce's signified.

There are many words used for representations. Sfard (2007) uses mediators to underline the fact representations mediate the process of communicating objects in a mathematical discourse. Sáenz-Ludlow and Zellweger (2016) expand Peirce's triad and relate it to math education; they develop some important vocabulary for the semiotics needed to discuss representations, namely sign-vehicle (representation), real/dynamic/immediate sign-object (RO, do, io), and sign-interpretant, and how they are utilized in different ways by students, teachers, and mathematicians. The real object (RO) is what the professional mathematics community understands the object to be, the dynamic object (do) is how an individual interprets the sign-vehicle, and the immediate object (io) is an aspect or two that an individual focuses on at a given time (Perry et al., 2016). Some aspects of Kellman and Massey's (2013) perceptual learning (PL) are similar to immediate object (io), where at the time of perception experts have a deeper view of the representation and what actions can be taken than novices. Perry et al. (2016) also introduced the concept of didactic dynamic object (odd), which is not a teacher's knowledge about the real object, but what they anticipate the students in their class know. For example, a mathematician and often a teacher may quickly devise a sign-vehicle (representation) to solve a problem or represent a situation, having great knowledge of the object defined by the math community, and not having to perform iterations of immediate and dynamic sign-objects, while a student may guess multiple times at what aspects—and therefore time-consuming sign-vehicles—are useful to move forward. Textbooks often show the most efficient and direct way to model, solve, and communicate a problem, meaning they were devised by a professional mathematics educator. They attempt to teach students this efficient way often via modelling the use. The alternative would be to display less direct routes to a problem, maybe showing student work, where students see that others explores methods that led to dead ends.

Cobb, Yackel, and Wood (1992) argue that representations that are natural for experts are introduced too early and confuse students rather than help explain a concept. They explain that Dienes blocks (physical objects) as a representation are natural and self-evident for experts to represent place value (numerals), but many students do better if they used a representation that are culturally and socially situated. It is therefore difficult for students to make a connection (conversion) between the Dienes blocks and numbers. For that reason, we must be careful in choosing representations that seem obvious to experts, especially thinking that physical representations are best. Mathematics is socially and culturally situated, and children perform much better when they are taught through culturally and socially situated activity (Cobb, 1994). For example, Brazilian children were doing calculations in their native environments very quickly and accurately, but they struggled using the algorithms learned in school (Carraher, Carraher, & Schliemann, 1985).

Teachers and professional mathematicians can access representations easily, but these concepts, these models, these schemas in our minds are quite different than those in other fields. To access the concept of line, individuals use *only* representations to develop understanding in mathematics (Duval, 2006). Physicists, on the other hand, can smash atoms to analyze how they behave, and historians can read old newspapers to discover what the weather was on July 4th, 1776, but mathematicians can only use certain symbols and diagrams that describe certain aspects of the mathematical object in question. This leads to another difference between science and mathematics, and that proof in science is a collection of facts, while a proof in math is a deduction based on axioms. Chazan (1993) found that students had difficulty with realizing that evidence, as in diagrams of a midsegment being parallel to the third side of a triangle, was not a proof, and that a geometric proof was not evidence; they thought the representations (sign-vehicles) were the object, and, therefore, the proof. Therefore, students need to be taught that representations are not the objects, and many times do not express all its aspects and generalities. Hoffman (2006) sums it up well:

> Mathematization means representing problems or facts by means of symbols, indices, and relational representations as provided by the history of mathematics; calculation means transforming those representations according to the rules of a certain system of representation; proving means representing a theorem as implied by other theorems within a consistent system of representation; and generalization is restructuring such systems of representation to include new, symbolically designated mathematical objects and relations (p. 279).

I have provided some working definitions for object, representation, visualization, and some semiotic terms like signifier/signified, representamen and interpretant. These terms will help me analyze the different geometry representations in the section below.

CHAPTER 2

REVIEW OF LITERATURE

Some representations are better than others for a given task (Ainsworth, 1999; Lesh, 1983). Representations are important for representing mathematical objects, but some are more critical when the student is learning and less important in later study of formal mathematics (Kuzniak & Rauscher, 2011). For example, students learn new signs (e.g., exponents like 3^2) based on previous representations (e.g., 3×3), and as they get comfortable with the notation, they can finally make other abstractions (e.g., $x^{-1.2}$) (Goldin, 1998).

Various researchers have grouped mathematical representations differently. The most common way is that of the typical algebraic representations, i.e., graphs, equations (algebraic expressions), and tables (Acevedo et al., 2012; Chang et al., 2014). Yet, those three representations would be limiting in geometry because of the many diagrams, geometric symbols, and solids that students study. Duval (2006) divides representations into four categories: multifunctional discursive (e.g., language), mono-functional discursive (computation), multifunctional non-discursive (e.g., drawings), and mono-functional non-discursive (e.g., graphs); therefore, at the minimum, Duval’s list expands from the three algebraic representations above by adding language and drawing/diagrams Lesh, Lindau, and Hamilton (1983) categorize representations as: (a) real life experiences, (b) manipulative models, (c) pictures and diagrams, (d) spoken words, and (e) written symbols; their list adds the real-world or physical component and other symbols that can include what I termed short geometry symbols. Representational systems can also be described as “well-established and carefully connected collections of SIGNS that extend across an extremely wide range of vocabularies, notations, algorithms, tables, graphs, diagrams, metaphors, analogies, models, arguments, proofs, etc." (Sáenz-Ludlow and Zellweger, 2016, p. 51). "Semiotic means of objectification may include material mathematical signs (e.g., alphanumeric formulas and sentences, graphs, etc.) objects, gestures, perceptual activity, written language, speech, the corporeal position of the students and the teacher, rhythm, and so on" (Presmeg, 2016, p. 17). Researchers group representations into system that work together, but ultimately, separating "systems of representation as distinct from each other, rather than as part of a larger system is a matter of convenience and convention" (p. 403, Goldin & Kaput, 1996). The above three lists add representations that seem to be beyond what I can study, namely analogies, arguments, proofs, corporeal positions, rhythm, etc.

Geometry Representations

I initially follow Pimm (2002) and draw as many distinct representations as possible, but I did not ignore the other lists, because they might contribute to develop a representation, e.g., Presmeg's (2016) 'perceptual activity' is part of what I termed meta-language later. Based on Pimm (2002), I chose to start with thirteen representations of geometry objects: (1) spoken language, (2) written language, (3) gestures, (4) sketches/diagrams, (5) symbols in statements, (6) numerals, (7) algebraic expressions, (8) graphs on coordinate grid, (9) construction with compass and straightedge, (10) physical objects, including manipulatives, (11) tables, (12) construction in dynamic geometry software, and (13) animations. Each representation has its attributes, and some are better than others at representing a certain aspect of a geometrical object easily and at being used pedagogically to explain properties and applications of mathematical objects to students.

Language

A natural and easily accessible representation is language, but it has limitations. From a mathematical perspective, we can understand very quickly when someone says or writes the word "circle with a radius of three," and many of us can even calculate in our head the circumference (6π) or area (9π). That holding of information in one's mind is what Peirce calls the interpretant. We can think of the definition: a circle is a set of points equidistant from a given point called the center. Parzysz (1988) claims that text is what defines figures, or mathematical objects, and that in any drawing there is a loss of information. This aspect is crucial, especially in the formalist perspective, where postulates, definitions, and theorems are often stated in language and then visualized in diagrams. Language has more redundancy than other representations, meaning that it allows us to say the same idea in many ways and perspectives (Goldin & Kaput, 1996). Yet, language is most useful pedagogically when explaining to students which properties they need to use or how to continue in a given problem. In the same respect, language allows students to "conjure and control personal mathematical images, as well as convey them to others" (p. 40, Pimm, 2002).

O'Keeffe (2013) described six word signifiers within language at the level of words: (1) general vocabulary, word signs used regularly in daily life, (2) mathematical terms, terms with specific mathematical meaning, (3) technical vocabulary, word signs peculiar to math like Heptagon, multiple, (4) special vocabulary, word signs used in daily life which have different mathematical meaning, e.g., set, group, or figure, (5) abbreviations, shortened or abbreviated technical words such as cm, km, GCF, etc., and (6) letters, alphabetical letters which represent numbers, lines. I grouped the last word signifier of language into short geometry symbols or algebra, which are described below.

Language can be divided into spoken and written. According to Chafe (1987), spoken language contains more personal references, repetition, shorter thoughts, and shorter words, making spoken language more exciting, but, on the other hand, written language is more diverse, denser with ideas, and more abstract. Spoken language has limitations in geometry. Spoken language is temporal, i.e., as soon as it is uttered/heard, the sound dissipates; whereas written text allows for information to be processed in a non-linear manner, allowing some reflection (Goldin & Kaput, 1996). When listening to language, “inflection, tone, and other information-carrying dimensions come directly into play” (Goldin & Kaput, 1996, p. 411). On the other hand, written language can convey information through its typography, its font; for example, **bold**, *italicized*, non-serif, etc. text could represent meaning (Van Leeuwen, 2006).

Schreiber’s (2012) study illustrates the above point about reflection by using MathChat and taking away the oral communication between students, to analyze how using only text, and not speech, which is harder to reflect on than written language, students learn while not speaking, and noticed that students were more reflective in their discussion. Although Schreiber (2012) focused mostly on Peirce’s abduction, his study is interesting in that it tries to eliminate one of the representations (spoken language) and to analyze how that affects learning. The most relevant aspect of Schreiber’s (2012) research is his method of analyzing communication between two students was reduced to writing and his interesting concept of “legend” for the student diagrams. The “legend” seems to explain what the diagrams mean, and most importantly, it did not require explicit explanations from one student to the other, that is students understood the new diagrams and symbols through exposure.

Sketches and Diagrams

Sketches and diagrams are visual representations of a locus of points and, sometimes, a of a relationship between or among loci of points. They are powerful representations because they facilitate us solving problems visually. In other words, language may have the similar information as a sketch, but it does not provide us with the same computational efficiency (Larkin & Simon, 1987). What Larkin and Simon mean by computation is not only mean manipulating numbers, but also the indexing of information on diagrams so that it can be used efficiently in analyzing and determining measurements, relationships, etc. For example, in the diagram below if segments AD and CD are tangent to the circle and angle B measures 140°, we can use the diagram to find the measure of angle D. Knowing that tangents are perpendicular to radii, we can place the perpendicular symbol (a little square) where segments AD and AB meet, and the same for segments BC and CD.

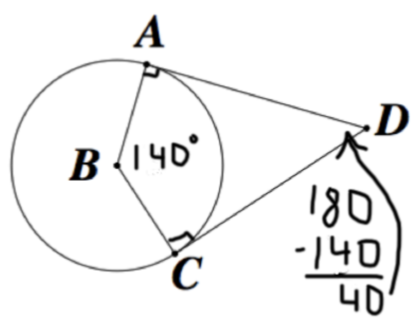


Figure 1. Diagram of circle and tangent segments.

Placing a square is the indexing of information on diagrams that Larkin and Simon mention. Knowing that the sum of the angles of a quadrilateral is 360° and that the sum of angles A and C is 180°, we know that angles B and D are 360°-180° or 180°. Therefore, angle D is 180° - 140° or 40°. This kind of indexing would be more difficult to understand using spoken language alone without labeling the angles and their measures on the diagram.

Dimmel and Herbst (2015) studied 2,300 diagrams in 22 geometry textbooks from 1913 to 2001 to determine how diagrams convey geometry concepts. They organized their “analysis of geometry diagrams by systems of semiotic choice and use[d] these systems to conduct an inventory of the choices that are realized in diagrams in different textbooks” (p. 157, Dimmel & Herbst, 2015). Their study produced categories of diagrams like stroke (lines and curves), region, dot, and symbol (letters, numbers), emphasis (weight or thickness, gauge or size, and transparency), position (distance and orientation), difference (color, pattern, fill, and style), attribute (relational, operational, and existential). They found that, over the 9 decades, diagrams in textbooks had more labeled points, more other labels, more properties labeled, and used color more often. This study is similar to what I did in my study except that it only focused on triangles in the geometry curriculum, on diagrams as a representation, and not on the conversion to and from different representations.

“The fundamental difference between our diagrammatic and sentential representations is that the diagrammatic representation preserves explicitly the information about the topological and geometric relations among the components of the problem, while the sentential representation does not” (Larkin & Simon, 1987, p. 66). The corollary to the statement above is that it is difficult to draw a diagram that displays a generic case, i.e., a diagram that shows all the possible cases of a concept. Textbook writers and teachers strive to present the generic case (Sinclair, 2003). On the other hand, diagrams are easier to search and often provide more information (shape, color, position, etc.) with a quick glance than text (Larkin & Simon, 1987). For example, a student can easily tell that, in figure 1, angle B is a central angle with B and C being the endpoints of each side and located on the circle, assuming that the student has that background knowledge; that is a lot of text just to explain one part of the diagram, making it difficult to draw information from.

Diagrams and sketches are conventionally different in that sketches are drawn freehand like the perpendicular symbol on the diagram above; whereas diagrams are usually drawn more carefully, but neither is necessarily drawn to scale. At times, I will use the term sketches to underline that they are not drawn to scale and the lines do not need to be straight, but they are interchangeable in this study. Although sketches are not always true to the objects they represent, they do provide us with a quick depiction of the relationship between sides, angles, points, arcs, etc. We benefit from a sketch for most proofs in geometry, so that we can label objects and refer to those labels in statements. In contrast, constructions (discussed below) with compass and straightedge, although more precise, take more time.

Mathematicians use diagrams in many ways. One study combined graphical representations like Cartesian graphs and diagrams together, and it “revealed a total of four ways in which these mathematicians used diagrams: noticing properties and generating conjectures, estimating the truth of an assertion, suggesting a proof approach, and instantiating an idea or assertion” (Samkoff, Lai, & Weber, 2012, p. 64). Many mathematicians consider diagrams essential in solving problems, because they display implicit and explicit properties (Sáenz-Ludlow & Kadunz, 2016).

Booth and Koedinger (2012) study explored how middle-school students solved problems with various combinations of representations, namely algebraic equations, story alone, and story with diagrams. They found that improved performance depended on problem difficulty, developmental stage, and representations used to present the problem. One of the relevant findings is that diagrams with story problems did not offer an advantage, except with more difficult problems. Also, age seemed to make a difference, where sixth graders performed better with story problems, whereas seventh and eighth graders did better with diagrams in the problem. Such studies introduce the complexity that comes with education research; developmental stages, content area (quadrilateral properties, proofs, solids, trigonometry, coordinate geometry, etc.), all representation combinations, SES, etc. It is, therefore, difficult to choose an insightful combination without some guidance that narrows the focus and has a high impact on learning (Koedinger, Booth, & Klahr, 2013).

Finally, three-dimensional objects pose a difficulty in the two-dimensional space of a textbook, especially in drawing diagrams. Students often study 2D depictions of 3D solids, where only one perspective is shown, so certain properties may not be easily visualized, e.g., the altitude of a 2D drawing of a pyramid being perceived as not shown perpendicular to the base (Parzysz, 1988). In a later study, Parzysz (1991) supports parallel perspective because it is how we see geometric objects, because it preserves many properties and because it is less error prone.

Numbers

In geometry, points, sets of points, planes, etc. (i.e., the basic building blocks of geometry) and their relationships dominate, but, as can be seen in figure 1, numbers are also used often. Numbers are object that are used to count or to measure.

Although the focus of this review is high school level geometry, numerical representations are still valuable in many instances in any high school course. Arithmetic, a pre-algebraic skill, allows students solve a large percentage of problems in high school geometry class. Oftentimes, as in the problem in figure 1, students do not write any formulas; instead, they perform a calculation on the side. As far as learning relationships among numbers, students seem to learn better with number, e.g., they learn multiplication with negative numbers better with numerical examples than with algebraic rules (Sfard, 2007). Our decimal number system is based on place value, and Dienes blocks try to represent it, but they fail to do it completely because the blocks would work even without place value (Pimm, 2002).

Numerals have a long history that goes back thousands of years and rival written language itself. Our modern decimal based system has its roots in India (early Middle Ages) and came through the Middle East to Europe by the late Middle Ages. Similarly, as language, numerals are difficult to separate from other representations. They can be reference for angles, e.g., $\angle 2$, or they can be values with or without units. They are found in representations such as tables, sentences, diagrams, and equations.

Short Geometry Symbols

To discuss sketches, we could use a language like English, or we could use geometric symbols in statements, which are much shorter and easier to read. They tend to be a more immediate, more visual representation than written language, i.e., they often, but not always, e.g., the congruent symbol (≅), visually represent relationships between or among loci of points, and they can be written within the flow of written language. For example, we could write ‘line AB is perpendicular to ray AC’ or $\overleftrightarrow{AB} \perp \overrightarrow{AC}$, which is much shorter and provides a more visual connection. These shorter statements are common in writing proofs that teach logic, and geometry has historically held the position of training students in logical thinking (Wu, 1996). A student could see from the statement of short geometry symbols that AB is a line and extends in both directions, AC is ray, which starts at A and passes through B, and they are 90° to each other. Students understand more if they use the symbol rather than just the arbitrary-sounding five-syllable word ‘perpendicular.’ Yet, the symbol ‘⊥’ is not exactly like a diagram, especially if it represents two segments intersecting at their endpoints.

The symbol for perpendicular (⊥) did not come into usage until 1634, almost 2000 years after Euclid (Cajori, 1928), but it was not used conventionally until the end of the 19th century in America (Herbst, 2002). In fact, there was, quite a vigorous discussion between academics like John Wallis and Thomas Hobbes, where Hobbes thought that symbols create unnecessary double load on the mind: they translate from symbols to words, then to the ideas they signify (Cajori, 1928).

Students write most proofs in high school geometry with these short symbolic statements in the standard two-column proof. There have been many studies conducted about proofs. Jahnke and Wambach (2013) studied students' understanding of the nature of proof. Sears and Chávez (2014) explored how different US geometry textbooks provide different opportunities in solving proofs. They found that one textbook presented proofs in paragraph form more than another, that the textbook was the primary source of proofs shown to students, and that proof were presented in variety of forms. In two different textbooks, proofs were presented as a flowchart (4% and 18%), in paragraphs (22% and 41%), a two-column table (73% and 41%), and proofs included diagrams, pictures, tables or figures 64% and 87% of the time. Recently, studies on proofs focused on how students learn proofs in DGE, mostly through exploration (Leung, 2009; Olivero, & Robutti, 2007; Sinclair, & Robutti, 2013).

The use of short geometry symbols increased over the last 100 years or so. Herbst (2002) provides a history of the two-column proof, and how it developed from students memorizing paragraph proofs and diagrams in the early 1800s, through early adaptations of the two-column proof in the late 1800s due to a change in pedagogy where students were expected to perform proofs, to contemporary two-column proofs. In late 1800s and early 1900s, the two-column proof was being refined, especially in its conciseness:

> The notion that a proof was composed of 'concise steps' (Beman and Smith, 1899, p. 20) could operate normatively to prevent students from interjecting unwarranted arguments. Yet, to do so it had to be supported by available language that permitted such conciseness. Hence, it became customary to write proofs using notation more flexibly (e.g., various ways of denoting angles). It also became customary to shorten the statements of special propositions (e.g., 'a. s. a' for the theorem that asserts triangle congruence based on the congruence of two angles and their common side). It seems that such abbreviations would support students' random access to previous theorems to quote them as reasons when producing a proof. All of these changes would help students use productively the notion that a proof was composed of statements and reasons. A diversity of notation and a plurality of technical words would protect from the risk of having to shift the activity of proof production into one of searching for the canonical way of writing a statement or quoting a reason. Instead, random and rapid availability of propositions that could be used as reasons could be used heuristically to think of statements that might be possible to make. (Herbst, 2002, p. 303)

Through this period of creating concise representations of longer sentences we see the development of a new type of representation, which I will call symbolic geometry statements, as opposed to algebraic statements or expressions. Some symbols like $\odot$ for a circle are as old as the temples of ancient Greece (over 2000 years ago), while others like $\cong$ for congruence began to be used (first by Leibniz, about 300 years ago) much later (Cajori, 1928). The short history also explains the reasons, or affordances, that these symbolic geometry statements were introduced, namely standardization and quick access to theorems. These geometry statements are not only present in proofs, but also as given information needed to solve a problem and as solutions. They are less linear than the sentential representations, which are more difficult to draw information from (Larkin & Simon, 1987).

Unfortunately, there is little research on short geometry representations e.g., $\overleftrightarrow{AB} \perp \overrightarrow{AC}$. Herbst (2002) devoted a paper to the history of two-column proofs and abbreviations that came out of them; the paper alludes that the abbreviations are just shortened version of written language, but from the pilot study to my research, I feel these representations need to be separated from written language, algebra, and diagrams, because the former are different from the latter. As stated above, geometry symbols were developed to prevent students from making unwarranted arguments and to make information more concise, e.g., instead of 'angle A' writing ∠A, saving 5 characters if the space is included. Geometry symbols are different than algebra because they focus on the representation of figures and their relationships in space rather than focusing on relationship between numbers.

Construction

Construction, or drawing with only a compass and a straightedge, was vital for centuries. Constructing was and is still considered fundamental in geometry. It is different from sketches in that a construction is drawn to scale, but it may be considered a sub-category of sketches because the notations for both are similar, e.g., naming of points, labeling measurements. Some books differentiate constructing from drawing in that in drawing we can use measuring tools like a protractor or a ruler (Serra, 2003). There is little difference between the two, because in both we are trying to draw as precise and accurate a figure as possible. We might draw a more precise depiction with a compass and straightedge than a ruler and a stencil, but sometimes it is the other way around. One key difference is that drawings usually display measurements of the object whereas constructions do not. Therefore, with constructions and drawings students can analyze properties differently, i.e., with measurement, which would not be possible with a sketch. Measurement with tools like rulers and protractors is an affordance that makes constructions distinct; the only other representations that comes close to this capability are physical objects made with high precision like Lego blocks.

Dynamic Geometry Environment (DGE)

With the invention of computers, we can be as precise in construction as the computing power of the computer allows, but the new technology does not allow for use of hand-held tools like a ruler. The new technology of dynamic geometry software or system (DGS), or dynamic geometry environment (DGE), like GeoGebra and Geometer's SketchPad has made construction much easier and dynamic. DGE is an overarching representation that allows other representations like graphs, diagrams, tables, algebra, ordered pairs, and numbers to be dynamic. With the click and drag of a mouse we can draw a 'perfect' circle; the limiting factor is the size of the pixels and the rounding of the variables in the software code. Such precision would only be possible if we drew very large diagrams with a very fine pencil on extremely flat surface. Besides the precision, DGS provides a platform where we can move parts while some relationships (invariants) that were defined stay the same (Mariotti, 2013). Moreover, all the transformations can be done by dragging follow the Euclidean rules of construction on DGE, and they can be used to prove theorems as dragging shows the theoretical correctness of the constructions (Mariotti, 2000). For example, in figure 1 above we could change the measure of angle B by moving point A, and segment AD could still stay tangent, if we had defined it that way; or we could deduce that the central and the exterior angles are supplementary. DGS affords students to easily keep track of construction history so they can check which part is a 'parent' (Pimm, 2002). Although Straesser (2002) described research claiming most DGS is the same as construction on paper, he admits that Cabri-Geometre (DGS) deeply changes geometry as a discipline. Such DGS software creates affordances that previous representations did not have. Hollebrands (2007) describes reactive and proactive strategies that DGS affords through its dragging and measuring capabilities, where, for example, students can drag randomly and reactively to explore, or they can drag proactively to test a conjecture. DGS can also provide a common language in the classroom for certain actions like ‘mark mirror’ in Geometer’s SketchPad (Hollebrands, 2007). Sinclair (2003) discusses some of the benefits of using DGS: drawing attention through affordances such as color and motion (animation is discussed below), supporting experimentation, and providing alternative paths. Yet, she describes how important it is to provide support through text (language representation) on lab sheets and the presence of another student (spoken language representation) to guide the student exploration.

Although this literature review is mostly about external representations, it is impossible to discuss representations without considering the mind and imagination. When students perform constructions of lines or rays, Pimm (2002) claims that those actions may imprint those images into their imaginations. Some students have difficulty imagining such situations, and DGE provides the space in which students can see such situations (Sinclair & Robutti, 2013). For example, two consecutive and congruent sides can be reflected to show a rhombus has symmetry. Another example of using DGS is to teach inscribed angles and how they are related to central angles, where not only a student sees the construction but also the angle measure relationships (usually in the representation of a short geometric statement, e.g., $m\angle ABC=140^{o}$) as the angles change (Andreasen & Haciomeroglu, 2014).

Both static and dynamic constructions are probably more valuable in pedagogy than in rigorous mathematics. Students can learn through hands-on activities how space is or can be organized. They can separate two-dimensional space by lines and arcs, and they can learn inductively through investigations how certain angles are congruent to each other, or that an inscribed angle is half the central angle.

Although Google SketchUp and other CAD-type programs are not usually considered as typical DGS, it has shown to improve visualization skills; students who learned to visualize three dimensional solids with SketchUp performed statistically better on a posttest than those students taught traditionally with pencil and paper (Kurtulus & Uygan, 2010). Wasserman (2014) lists many other implementations of SketchUp in the classroom, including volume of similar solids or solids with congruent bases and different heights. Yet, Laborde and Laborde (2014) describe the difficulties in how to represent three-dimensional objects on a computer screen in their Cabri 3D software, which shows we are living in a time where the DGS representations are being developed.

Animations

Animations can be defined as one image turning into another. Some researchers have noted the value of animation. Many concepts are difficult to visualize without motion and animations, and animations appeal to students (ChanLin, 2000). Dynamic displaying of a process or a procedure can compensate for a student's difficulty to imagine motions; "[t]hus, the animation provides an external model for a mental representation" (Höffler & Leutner, 2007, p. 723). Animation "produces effects far closer to the seamless dynamics of mental images in the mind" (Pimm, 2002, p. 45). From a practical teacher perspective, animations are less expensive and take up less space (only computer memory) than physical objects that show similar changes (discussed later).

Dynamic Geometry Software can present some of this motion, but not all, at least not easily and not as artistically. For example, I observed a teacher use the theme song of Transformers to students to introduce transformations, and it had the robots transforming in three-dimensional space and grid lines in background; it was engaging, and it introduced the coordinate grid in a dynamic way. Although Transformers changing their shape not an isometry, but they also rotate and translate on the grid, which is what the teacher emphasized. Sinclair (2003) discussed students' inability to prove congruency in triangles that overlap, are rotated, reflected or translated, and to help students she created action button animations in Geometer's SketchPad so that those triangles are easier to compare. The most relevant findings are: (1) dynamic sketches must draw attention through color, motion, markings so that students focus on parts relevant to the lesson, (2) students were not familiar with describing visual information, (3) students often stopped exploring sketches through dragging (Sinclair, 2003). These findings imply that not only do dynamic diagrams need to be designed in a way to attract attention to relevant parts, but students need to learn language to describe their exploration and learn how to analyze dynamic diagrams through motion.

Equations, Tables and Coordinate Geometry

Coordinate or analytic geometry is more precise than some of the representations above, and it is strongly connected to algebra. These representations involve equations, tables and the coordinate grid. Equations like $x^2 + y^2 = r^2$ for a circle account for the infinite number of points that constitute a circle, and the coordinate grid provides us with precise locations of points, and if solved with algebra they are infinitely precise locations. In sketches, we do not even attempt at any precision, and in constructions we may come to within a thousandth, possibly millionth with excellent instruments, part of the entire construction. In our example algebraic equation of a circle, if the circle is centered at the origin of coordinate grid, its radius is 5, and segment AD is defined by $y = 5$, then point A is exactly at (0,5). It is not at (0, 5.0000000000000000001); it is exactly at 5 (i.e., using the concept of significant figures it is five, decimal point and followed by infinitely many zeroes). Therefore, coordinate geometry allows for perfect precision.

There are three representations related to algebra, namely tables, graphs, and equations (Bell, 1981). Algebraic formulas allow for manipulation and substitution unparalleled by any other representation, even language. For example, if y=x, then students can replace y with x without any loss of information; in language if we replace screen with a synonym like monitor, there could be an important difference, and the same is true for two segments even if they are congruent. Students can substitute numbers or expressions for variables, they can rearrange terms, change signs using valid algebraic steps, of course. In fact, algebra evolved so that we can manipulate expressions or equations to suit our needs, in contrast to graphs and tables, which mostly display information (Goldin & Kaput, 1996).

Graphs of equations are different from sketches and diagrams, because they are on a grid with two perpendicular axes. Even carefully drawn graphs are certainly no more accurate on paper than constructions and sometimes even sketches, and many times in algebra students are simply asked to sketch a line or a parabola, labeling relevant points like the intercepts. They are different in that graphs have a specific location and orientation. For example, $x^2/4 + y^2 = 1$ is an ellipse that is different from $(x-1)^2 + (y+0.5)^2/4 = 1$, because the graph of first ellipse is oriented horizontally and centered at the origin, and the graph of the second ellipse is oriented vertically and centered at (1, -0.5).

Orientation and location are crucial in transformations by rotation, reflection, etc. From the three representations, students see the graphs as the representation that gives an equation ‘meaning,’ because it is the most absorbing (Pimm, 2002).

Tables of values like number of sides in a polygon and the sum of all internal angles are useful when students look for patterns inductively. For example, in the sum of the degree measures of angles in polygons, once the table is set up, students can easily find the pattern that the sum increases by 180º. Some students when looking for a pattern in n x n square of matches use tables to find how many more matches were added to subsequent squares (Hershkowitz, Arcavi, & Bruckheimer, 2001). In Acevedo's (2012) study, students chose tables to solve slope problems 62% of the time; he found that one explanation is those students did it that way in class.

Physical Objects

Physical objects are objects that seem to take up three-dimensional space and to appear tangible. They are the most practical—in the sense of being connected to the real world—representations of mathematical objects. Using physical object as representations can help students see why it important to study geometry, i.e., how practical it is and how it is connected to their reality (Hershkowitz, Parzysz, & Van Dormolen, 1996). For example, an engineer could use simple geometry to calculate the length of a belt in a car. One drawback is that physical representations are further removed from ideal mathematical object than many of the other representations. The belt may twist or sag, it is rather thick, the wheel may be irregular, the other wheel is not really a point, it is troublesome to hold the belt in one place, etc.

Not only can physical objects motivate students and connect geometry to the real world, we seem to share the innate ability to perceive physical objects and space, another affordance of physical space. Spelke, Lee, and Izard (2010) describe it well:

> Research on young human children, nonhuman animals, and human adults in diverse cultures provides evidence for at least two core systems of geometry that are present and

> functional early in human development, that predate the evolution of humans as a species, and that remain universally present in human adults. Research on older children provides evidence for the emergence of capacities to relate these systems. At 4 years, children appear to relate the shapes of objects and of the surface layout only with respect to their common distance relations. With development, however, children also relate these representations on the basis of angle, and by adulthood, direction (p. 879).

One system is associated with our processing large-scale environmental spatial layouts, and the other helps us represent small-scale objects and forms (Spelke, Lee, & Izard, 2010). Therefore, it seems that evolution has pre-wired us to relate shapes, surface layouts, and length. Our perception of angle, to a degree, seems to be learned, and direction seems to be the most difficult to learn.

This innate ability may be used to teach more abstract aspects in geometry. We usually think of mathematical objects like rectangular prism are representations of physical objects like a cardboard box, but in math class students can use physical objects to represent mathematical objects. Continuing with the circle and tangent segments example, we can explore properties of tangents by stretching a rubber band on a wheel and pulling it away with a pencil to form tangent segment. Nelson (1993) has an interesting resource where students can prove theorems like the Pythagorean Theorem by cutting shapes and rearranging to prove one area is equal to another. Although these are two-dimensional preprinted constructions, once students cut them out, they become physical representations that exist in three-dimensional space. This ability to move physical objects is similar to the affordance of animations that helps students who have difficulty imagining certain transformations. Moving physical objects can stimulate students to form their own visualization of an object.

Again, as the other representations, physical objects have benefits and drawbacks. One benefit is the obvious connection to the real world that students can touch and manipulate (Pimm, 2002). Yet, it was not until the late 1800s that 'concrete geometry' was introduced into the elementary curriculum in the United States to "dispel questions regarding the existence of geometric objects" (Herbst, 2002, p. 302). Learning properties of 3D objects is better on small objects, or micro-space, but students learn argumentation better on larger objects, or meso-space (Hershkowitz, Parzysz, & Van Dormolen, 1996). When students deal with larger objects, they are pushed to go beyond their sense (not so when handling small objects), and, thus, they need to resort to reflection and argumentation. Yoon and Miskell (2016) studied whether students would grasp the cubic relationship of volume as the lengths are doubled, and they found that using multifix cubes (similar to Lego blocks) students grasped the 3D aspect of volume better than just with diagrams. Arici and Aslan-Tutak (2015) have found that creating origami figures (physical representations) during lessons about geometric concepts like the triangular inequality theorem improved students' spatial visualization, geometric achievement, and geometric reasoning. "The fold diagrams might give an opportunity to make transitions from visual to formal statements…" (Arici & Aslan-Tutak, 2015, p. 196).

Gestures

Gestures are movements of the body, especially hands, that emphasize an object, attempt to show an object, or some action of an object. They are not a representation that many mathematicians think about, and they do not appear in textbooks, but in math education researchers have taken notice, and so they are included here for completeness. Teachers use gestures all the time in class, especially when a static drawing or formula does not suffice. Students and teachers can use their bodies in various ways: "… our eyes coordinating with muscles, giving rise to notions of straight, vertical, solid… elbows, and the rotating of other joints, images of turning and angles" (Pimm, 2002, p. 17). Students often use gestures to describe 'imaginary objects' (Chen & Herbst, 2013). Radford (2008) claims that some students who are in the process of learning new representation benefit from the use of gestures. "Bobby's grasping of the mathematical meaning of the graph – its *objectification* – was accomplished through spoken words, gestures, bodily actions, artifacts (the pen, etc.) and mathematical signs" (Radford, 2008, p.119). According to Radford (2008), gesturing is crucial in this objectification stage and in the next stage of explaining and thinking, but less so in the semiotic node.

Using our body and gestures is also very insightful for students because they can become part of the representation instead of just seeing one on paper or screen. In the Embodied Mathematical Imagination and Cognition (EMIC) working group at PME-NA (Nathan et al., 2017) multiple researchers presented their newest findings in how students represent objects with their whole bodies, hands, or even among many students. Smith et al. (2014) demonstrated the difference between forming a parallelogram from blocks and from the bodies of 8 participants; it was clearly different in that the participants felt part of the structure and encoded the properties of the shape from their perspective as a side or a vertex. In a study on angles, she found that students estimated and drew angles better after they formed angles with their own bodies, and it allowed for using personal experiences during discussions (Smith, King, & Hoyte, 2014).

Manipulating Multiple Representations

Having a list of representations, we can now discuss what problems come about using them and how we manipulate them, i.e., how we change within one representation (treatment) and how we change from one representation to another (conversion). It may be beneficial to distinguish treatment, conversion, and translation. Treatment is when a change occurs within a representation as in “a polygon with three sides” → “triangle.” (p. 112, Duval, 2006). A conversion is a change from one representation to another as in “the set of points whose ordinate is greater than their abscissa” → “$y > x$.” Translation is similar to conversion, but rather than just changing representations, the meaning is changed as in “the set of points whose abscissa and ordinate have the same sign” → “$x \cdot y > 0$,” where there was no mention of multiplication nor of zero. O’Halloran (2008) uses intrasemiosis for treatment and intersemiosis for conversion. Although some researchers use translation to mean conversion (Chang & Cromley, 2015), I will try to avoid it unless such cases as above arise for two reasons. One reason is that it is confusing with the geometric concept of translating the position of a figure. The other is that some researchers use it to mean Duval’s (2006) conversion.

Sometimes a solution to a problem is impossible or, at least very difficult without conversion to a different representation. For example, in Figure 1 suppose there is another point F on line AD that is involved in a second problem; as a result, we can say “line AF is perpendicular to segment AB” instead of “line AD is perpendicular to segment AB,” which is a manipulation, or a treatment as Duval (2006) would call it, of the language representation. We may be able to imagine the above statement, but very often we need to sketch or construct a figure that will demonstrate it visually. For that reason, we need to use at least two representations in geometry, a discursive one and a visual one (Duval, 2006; Lesh, 1983).

Treatments, or manipulation of one representation, can also be complex and difficult. Algebraic simplification and projection of 3D onto 2D are two examples. Algebra students know that simplifying some equations can take five, ten, or even more steps. Students also find it difficult to stay within the diagram representation and to draw a projection of a 3D object in two-dimensions, and for that reason 3D models may be useful (Parzysz, 1991). Drawing auxiliary lines, measuring them and their angles, and constructing arcs are all treatments in the drawing/construction representation, but that usually leads to a hybrid, where measurements (numerals) are written down.

As stated above, conversions, or the changing between two representations or even among multiple ones, is very difficult for students. Yet, it seems students are more accurate when they do convert. In a recent study, high school students coordinated three algebraic representations, namely graph, table, and equation, with more accuracy when spending more time and using more strategies (Zahner et al., 2017). Also, these researchers found that students who applied evaluation strategies finished tasks faster than those who checked coordinate by coordinate.

Graph, equation, and even table conversions are becoming easier with DGS, other programs, and graphing calculators, because they can automatically convert them for us. For example, drawing a line in GeoGebra automatically produces a linear equation in a side panel, and as we change the slope or y-intercept of the line, the equation changes accordingly, and vice versa. Language parsing software is getting better as well. Wong et al. (2010) were able to convert natural language of geometry problems into accurate constructions 90% of the time. The recent software allows us to convert more easily from one representation to another.

According to Ainsworth (2006), there are many reasons why we need multiple representations—she does write about multimedia systems, but her description is general enough to include mathematics education. Information may be redundant, overlapping, or mutually exclusive on representations, and that forces us to make choices among which representation or representations will be best for a given situation. Goldin and Kaput (1996) mention affordances, constraints, and efficiency of representations. For example, a construction would provide all the information of a sketch, and tools could be used to measure other segments or angles on the construction but not on the sketch; on the other hand, constructions are more labor intensive and require more time, which discourages students from solving a problem. Laborde and Laborde (2014) write about Cabri (a DGS), which almost by default displays multiple representations, and they describe how students can perform computations on the constructions and how there is feedback through other representations. For example, a student can change an angle in a parallelogram by dragging a point, and he or she can see the measure of that angles change but not the sum of two consecutive angles.

Language and diagrams are often paired. Yet, sometimes language skills and diagram conventions in geometry conflict with one another, e.g., some students name polygons as one would read text (left to right, next line) instead of the convention of listing consecutive vertices (Gal & Linchevski, 2010). For example, students might incorrectly name the kite in figure 1 kite ABDC, and not ABCD, because, on the diagram, D is to the right of B and C is below as in written text. Yet, although the standard rules for parsing language and diagrams sometimes conflict, they are inseparable when solving a problem or solving a proof. Oftentimes, mathematicians and students need to show that all cases of a given situations were considered, and language is used "to 'point' with words, so as to communicate with others and to describe and analyze complexity" (Pimm, 2002, p. 39).

Converting from one representation to another is difficult for students. Students studying geometry transformations, for example, preferred to use algebraic transformation rules that they memorized rather than think visually (Bansilal & Naidoo, 2012). When student do convert, they are better at it in one direction than the other. Expert teachers in China recognize the importance of converting representation by having students change language to geometrical symbol language, text or oral statements to diagrams, diagram and/or geometrical symbol language to verbal description (Ding, Jones, & Zhang, 2013). The difference between expert and novice may be best explained with Cognitive Load Theory (CLT). Sweller, Ayres, Kalyuga (2011) explain that:

> … when processing biologically secondary information, human cognition includes a working memory that is limited in capacity and duration if dealing with novel information but unlimited in capacity and duration if dealing with familiar information previously stored in a very large long-term memory. Instruction needs to consider the limitations of working memory so that information can be stored effectively in long-term memory.

Therefore, the difference between a novice and an expert is that an expert has biologically secondary information (geometry representations, algorithms, etc.) stored in his or her long-term memory.

Because transforming representations is vital for solving problems in mathematics, it may need to be explicitly taught to students. Students are shown the representations, but I have not come across research that studied whether students are taught how to convert between them. Students may benefit in knowing how representations are different, which methods are more sophisticated, effective, and elegant, and which representations are acceptable in mathematics.

A distinction between procedural knowledge and conceptual knowledge may clarify more. Hiebert (2013) defines conceptual knowledge as knowledge between concepts either of the same or different abstraction levels and procedural knowledge as knowledge of representations and the rules, algorithms, or procedures that are associated with those representations. As stated above students may learn procedure to change representations in one direction, possibly not knowing the conceptual connection between representations, especially at different abstraction levels.

Teachers may see student improvement if they set the norm of distinguishing representations, discussing them explicitly, or applying them implicitly to objects introduced early like lines, points, and segments. For example, when talking, using symbols, and drawing sketches, the teacher and students may work out an effective way of switching representations and how one representation maps onto another. Yet, as Goldin (1998) warns, there may be some ambiguity between certain representations, especially language and algebra or diagrams. Although students need to use all representations to learn major geometric objects like parallel and perpendicular lines, circles, different types of triangles, quadrilaterals, etc., certain representations are more sophisticated, effective, and elegant for a given task. For example, given equations for two lines and asked to find whether they are perpendicular, comparing their slopes is a sophisticated, effective, and elegant method, but graphing them carefully and measuring the angle may be useful in teaching the concept. Students would likely benefit from being aware and constantly asked which representation they are using and why until it becomes second nature.

The representation's utility for an object is determined by its ability to represent a given situation (Duval, 2006, Sáenz-Ludlow, A., & Kadunz, G., 2015). When that utility is exhausted, we can change to a different representation. Chinnapan (1998) links effective representation use with broad prior knowledge in terms of schemas. Surprisingly, lower-performing students recalled more schemas, and therefore representations, than better-performing students in open-ended questions, but the high-achievers recalled more relevant schemas in problems (Chinnapan, 1998). In the example above, comparing the slopes would have been the easiest and fastest way to solve the problem. The transformation of the slopes, an algebraic representation that is mono-functional, would suffice to solve the problem. Mono-functional representations are algorithmic, and the student would have just followed a procedure to find and then to compare the slopes (Duval, 2006). O'Halloran (2008) would add that mathematical symbolism (what I call algebra) is the dynamic representation that can do the calculations that language and visual representations cannot do, and that symbolism has been designed to do that starting around the time of Descartes and later Newton and Leibniz. The dynamic ability of transformation given to symbols was due to a long process of thought and experimentation of different symbols. To continue with the example above, some students might not remember that lines with slopes that are negative reciprocals are perpendicular. They have exhausted the representation of equation, and must therefore convert to another, a graph in this case. If they are savvy enough, they could convert the equation to a graph directly, and if not, they would need a pair of points for each line, or possibly a table of many points. In either case, they would need to be meticulous in plotting and drawing the lines to get the proper measure of the angle between the lines.

Multiple representations are also necessary to reduce the amount of ambiguity, which makes solving, or even understanding, problems difficult. Natural language and imagistic systems are often ambiguous, we use more formal language or formal mathematical representations like algebra to reduce or even eliminate any ambiguity (Goldin, 2002). Physical objects, for example, reduce a great amount of ambiguity as opposed to a description by simply having more detail like color, texture, mass, hardness, opaqueness, etc. that might be ignored in the description. Yet, a student having to see all that information (color, texture, mass, etc.) may be distracting; for that reason, at times and when the connection with the real world is not deemed necessary, an abstracted image of the physical object may be a better choice.

Finally, let me describe O'Halloran's (2008) use of Systemic Functional Linguistics to parse the representations in mathematics. Although she grouped the representations into only three, namely language, mathematical symbolism, and visual images, she provided a detailed 'grammar' of how we operate within and among those resources. To list all the examples would require a book, but I will list some that are of special importance in my study. First and foremost, she, as Duval, distinguish between treatment (intrasemiosis), or changes within a representation, and conversions (intersemiosis), or changes between representations. Another example is her description of how metaphors - historically, during learning, and during problem-solving - expand meaning in each of the representations by adding new 'participants.' O'Halloran's (2008) example of this is a word problem that asked to find the width of a river from a cliff, where the language description of the lines of sight were converted to triangles. That metaphor, which applies a description to something that is not literally connected to, converted an *imaginary* triangle that 'exists' between the eye of the observer and the bank of the river and an *imaginary* horizontal line; this contrasts with the angle that is *literally* formed from a photon that bounces off the river and enters the observers eye. Yet, because of that metaphor, a student can use the new expanded 'grammar' of triangles and later trigonometry to solve the problem. Third, O'Halloran (2008) describes mechanisms of intersemiosis (conversions), namely semiotic cohesion, semiotic mixing (e.g., △ABC), semiotic adoption (use x on graph and algebra), juxtaposition (placement in space allows for easy reading and coordination of representations), and semiotic transition (e.g.,△CBR from a description of a diagram). Again, O'Halloran's (2008) entire book is full of these semiotic descriptions (too many to describe here).

Gaps and Research Questions

Shen, Bao, and Zhang (2019) performed a meta-analysis of all the studies on mathematical representations, they found that only 9% of the 211 articles were based on textbooks, and only 4 of those 21 studied geometry representations. Moreover Shen, Bao, and Zhang (2019) state that:

> … current researches on the representation of mathematics textbooks are still in its infancy. The current research problems regarding to the representation of textbooks are relatively simplistic. (p. 309)

From the above review of articles related to geometry representations we can see there is a great amount of theory (Ainsworth, 2006; Duval, 2006; Hoffman, 2006; Lesh, 1987; Presmeg et al., 2016; Radford 2002; Sáenz-Ludlow et al., 2016). Many of these theories are very broad (Ainsworth, 2006; Duval, 2006), and they miss much of the nuance in geometry representations and possibly newer representations like dynamic geometry. Some of the articles mention experimental studies, but few articles directly report on empirical research that studied geometry representations in high school geometry (Chen & Herbst, 2013; Gal & Linchevski, 2010; Naidoo, & Bansilal, 2010; Dimmel & Herbst, 2015). If there are studies about geometry—and there are an abundance of them—they are not from the theoretical perspective of semiotics (Abdelfatah, 2011; Andreasen, & Haciomeroglu, 2014; Arici, & Aslan-Tutak, 2015; Chazan, 1993; Ding, Jones, & Zhang, 2013; Fukawa-Connelly & Silverman, 2015; Gagatsis, & Demetriadou, 2001; Jahnke, & Wambach, 2013, etc.). Dimmel and Herbst (2015) analyzed diagrams in geometry, and they did not explore other objects or how diagrams are coordinated with other representations Those studies that focus on representations and related to geometry textbooks are very narrow in focus. O'Halloran (2008) described math representations and their grammars with great detail, but she limited her representations to three, namely language, symbols (mostly algebra), and visual (mostly graphs), ignoring for the most part pictures, physical objects, geometry symbols, and tables; she also claimed that more detailed research is necessary.

Those studies that focus on representations in textbooks are very narrow in focus. Chang, Cromley, and Tran (2016) studied representations of only functions in four chapters of a calculus textbook, and they only studied the "four canonical representations of functions," namely text (Te), graph (G), algebra (A), and table (Ta). Moreover, the distinction "between situational or real-world text versus plain text descriptions in verbal forms of functions" is ignored for the purpose of their study (Chang, Cromley, & Tran, 2016).

Therefore, to add to the current research, I address the following research questions:

1. What are some representations introduced in textbooks currently?
2. Which geometry representations are coordinated often in a geometry textbook?
3. What are some possible mechanisms among representations during coordination?

CHAPTER 3

METHODS

Overview

To answer the research questions, this study of a geometry textbook provided much evidence as to which representations, at what frequency of use, and which combinations students see as they learn geometry. I studied a high school geometry textbook initially using thematic analysis to find distinctions among the representations and the way they were introduced. Then, using those distinctions, I coded for the coordinations. Finally, I focused on the most common coordinations to expose any common mechanisms that aid in coordination.

From my experience as expressed in the positional statement and according to research, teachers use textbooks often, and, therefore, they use the representations in them. Teachers, especially the ones beginning their careers including mine, rely heavily on textbooks, and most education programs suggest only using textbooks as a resource (Ball & Feiman-Nemser, 1988). Textbooks seem to predict what is happening in the classroom. Because teachers use textbooks extensively and students mostly solve problems from textbooks, thematic analysis of a textbook would show which representations are most likely used in the classroom and what are the ways that we coordinate among those representations. I chose Larson and Boswell's (2015) *Geometry: A Common Core Curriculum*, which is a popular textbook that was presented at multiple math education conferences, that is listed on many reviews of geometry textbooks, and that is on the Texas adopted curriculum list. Ron Larson is also a well-known author of math textbooks, including calculus textbooks that I used in college, and I found the textbooks to be clear and comprehensive.

Positional Statement

Because much of my research is qualitative, it may aid the reader to know who I am and how I began my teaching and academic career. Even in high school some 30 years ago, geometry was my favorite course, partly because of the teacher and partly because I preferred solving probably in space. I remember being confused when my teacher could not see that obtuse angles with each side is in a different plane could be drawn in two perpendicular planes. After high school, I initially majored in computer science at Penn State, and that is where I learned more advance mathematics. Actually, Ron Larson, one of the authors of the textbook taught at Penn State-Behrend, where I attended for one year and used his textbook for calculus. Yet, after three semesters, I decided to study philosophy. Studying philosophy trained me to be introspective and not to take everything as it seems to be, i.e., to question what seems obvious to others. At the same time, I studied epistemology, especially in Modern Philosophy (Bacon, Descartes, Berkeley, etc.) and ontology from those of Aristotle through Spinoza to Quine. After college and mostly in Poland, I taught English as a Second Language (ESL), which, among other things, helped me hone my grammar.

After returning to the US, I decide to become a math teacher. Initially, I taught Algebra 1 which I did not like for a few reasons: I was teaching immature freshmen, it is not as interesting as geometry, and I was not prepared to teach at a school to which I was assigned. After transferring to a different school, I was able to teach geometry and technology, both of which were more engaging. It was also more rewarding, because, having taken more education courses and having some experience, I was better prepared to teach, the students were older and more engaged, and I was more able to try different teaching strategies. At this time, one of the first things that I noticed was that students had difficulty changing from symbols to diagrams; for that reason, my student often did exercises where they changed from representations like AB || CD and CD=5 to a diagram, or vice versa. In the meantime, I took coursework in chemistry education at University of Pennsylvania, graduating with a master's degree in chemistry education. During the coursework, I again gravitated to topics like spectroscopy and molecular symmetry, i.e., those that involved some geometry.

Finally, after teaching math, physics, technology, and later chemistry in a high school setting for about 7 years, I started my academic career at Temple University. Many of the courses exposed me to literature about math and science education. I also spent a year as research assistant using eye-tracking and think-alouds to study how students coordinate representations when they see two representations of a function, namely table, graph, and equation. At this time, I also attended and presented at a few conferences, e.g., at PME-NA 39 I present a poster on my study of teachers and how they used representations, including gestures, in geometry classes.

Data Corpus

The textbook in this study is Larson and Boswell's (2015) *Geometry: A Common Core Curriculum*. I chose to study the chapter one (Basics of Geometry), chapter 2 (Reasoning and Proofs), chapter 3 (parallel and perpendicular lines), chapter 4 (Transformations), and most of chapter 11 (Circumference, Area, and Volume), because all the representations were either introduced in those chapters or used extensively enough to study. To determine that selection I had to analyze representations thoroughly in the first chapter, where the majority of representations were introduced both explicitly and implicitly.

From multiple initial passes through the textbook, I found many possible distinctions of how objects and concepts are represented in geometry. I also noticed that many chapters used the same representations with little variation (mostly written language and diagram). Therefore, the following chapters were omitted: (chapter 5) - congruent triangles, (6) relationships with triangles, (7) quadrilaterals with other polygons, (8) similarity, (9) right triangles and trigonometry, and (10) circles. I excluded these chapters and focused on deeper analysis of coordination and the mechanisms that aid that coordination in chapters 1, 2, 3, 4, and 11.

I focused on how geometry knowledge is introduced, not reviewed, and coordinated following that introduction. For that reason, I did not analyze the exercises at the end of the sections and chapters, the preface, index, appendix, glossary, etc. Instead, I analyzed the expository text, postulates/theorems, worked examples, explorations, constructions, etc., that are meant to introduce students to new mathematical content.

Some representations, namely Venn diagrams (used twice in the entire textbook) and logic symbols (only in section 2.1-conditional statements), are used so sparingly that each data point can be analyzed individually or as a small group. One representation, 2D diagrams of 3D figures, appears seldom until chapter 11. Chapter 11 has a combination of topics, namely circumference, area and volume. The distinction between 2D diagrams of 2D figures versus 3D figures (solids, cross-sections) was analyzed as a sub-category of the diagram (D) representation. Sections 11.4 to 11.8 included many such diagrams.

Methods for Analysis

The analysis was based on mixed-methods. I used thematic analysis (Braun, & Clarke, 2006) to create a comprehensive list of geometry representations, and, at the same time, I made some notes where they were introduced. From the extant literature, Pimm (2002) and others have provided me with many examples to start. During the initial phases of thematic analysis more codes emerged, some were removed, and some refined. After this first stage of defining the representations, I looked for how they were introduced. Then, with a refine list of representations, I coded almost 4,000 coordinations into a spreadsheet, giving me the ability to look for common types of coordinations like written language to diagram (WL→D).

Table 1. *Stages and Phases of the Research Methods*

Stage	Phase	Description
1		Thematic Analysis
	1	Familiarize with textbook – scan textbook
	2	Initial representation – mostly from Pimm (2002) and initial scanning of textbook
	3	Search for themes • printed many pages from book • cut snippets and grouped them
	4 & 5	Review and Define • number, ordered pair, sub-categories of language developed • definitions analyzed
	6	Report – the coding manual was prepared (in the appendix)
2	1	Look for and analyze introduction
	2	Look for and analyze coordination • create a spreadsheet • use spreadsheet to group and count common coordinations
	3	Look for mechanism using a shorter list of common coordinations

Finally, having those coordinations, I looked for common mechanisms that could aid in the coordination. This was not a linear process, meaning I had to return to previous stages to refine data, definitions, coordinations, etc.

The unit of analysis was established during the fifth phase of thematic analysis. I adopted a similar micro-level scope as Chang, Cromley, and Tran (2016), i.e., “any instance where students are required to coordinate or translate between two different canonical representations of a function” (p. 1485). The major difference is that I broadened it to any mathematical object as defined in the extant literature, that I did not limit myself just to the four canonical representations of graph, table, algebra, and text, and that I did not limit myself to coordination between two representations. O’Halloran (2008) states that the clause is the basic unit and that mostly language leads the learning process in mathematical texts. For that reason, my unit of analysis is an independent clause.

The first step I took was to print each page of the textbook sometimes multiple times and cut out each independent clause and any associated representations like graphs, diagrams, algebraic expressions, etc. I labeled each snippet with the representations present, and I analyzed in which order they could be read by a student. The initial coding manual is outlined in table one, and the final version is in the appendix. I experimented with various new representations (e.g., Ordered Pair, textbook Gestures), divided more frequent ones (e.g., Written Language, Diagrams) and refined others (e.g., Number). Through analysis I have shown that possible candidates for separate representations, namely vectors, trigonometric functions, and flowcharts, can be represented in terms of the other representations.

Thematic Analysis (Stage 1)

The first step was to be clear on what the representations were. As I said above, thematic analysis was used for that, but first I will discuss how I used a journal to keep a record for any insights I had along the way.

To externalize my thoughts and return to them at a later time, I kept a journal during the entire analysis. I wrote ideas that came to me as I analyzed the textbook, dating each item and often writing the page number and item that inspired the thought. Many of the new representations, mechanisms of coordination, organization, etc. started as an entry in the journal. For example, on April 1st, I described sub-codes for written language, and I created a four by two matrix attempting to describe all the language I had encountered up to that point. Another example is when on April 16, I decided to code for what I called gestures, and listed six different ones like colors, italics/bold text, check mark, arrows, distance indicator, and construction marks.

I also noticed that I need to code for exercises and their possible answers during the instructional part of each section of the textbook. I described qualitatively the representations that students are expected to create. I also created a prefix ‘m.’ before other codes to show that the representation would be made by a student, e.g., ‘m.a’ would be an algebraic expression made by a student. Chang and Cromley (2016) used the word ‘construct’, but I used ‘made by student’ to avoid the confusion with the geometry representation of construction. From my pilot review of the textbook, this creating of representations was most common in ‘Explorations’ and ‘Monitoring Progress’ sections of the textbook. In ‘Explorations,’ students are expected to explore properties of figures mostly in DGS, constructions, and graphs. I performed the instructions to see which representations would be created or used, e.g., cm graph paper, compass, mouse/trackpad clicking and dragging, etc., and I took notes. In ‘Monitoring Progress’ exercises, I assume that students perform similar steps as in the worked examples that are usually directly above them, but I did not include the explanations that are along the solution in the worked example, if they were not specifically requested. These assumptions are based on my teaching experience. I assumed how an above-average student would process the textbook, using the concept of didactic dynamic object (odd), which is not a teacher’s knowledge about the real object, but what they anticipate the students know (Perry et al., 2016).

Before I created such a coding system, I analyzed the textbook scanning for different types of representations. Some were already differentiated in the extant literature, but I looked for more distinctive representation using thematic analysis. Thematic analysis is composed of roughly six phases: (1) familiarize yourself with the data, (2) generate initial codes, (3) search for themes, (4) review themes, (5) define and name themes, and finally (6) produce the report (Braun & Clarke, 2006). As Braun and Clarke (2006) write, thematic analysis is not linear, but a more recursive process. Although I had not familiarized myself with the data (phase 1), I did look through the first chapter of the textbook, and some theory (Pimm, 2002; Goldin, 1998; Duval, 2006) has guided me to prepare some codes. That stated, as discussed below, the data drove more codes to be developed.

In phase 1 and 2 of the thematic analysis (stage 1), I familiarized myself more with the data and generated initial codes, some of which were already described above. Figure 2 shows my initial idea of coding. As Braun and Clarke (2006) describe, thematic analysis is both inductive and theoretical, meaning the initial codes are what I found in the theory of the extant literature. In figure 2, I coded the written language, diagram, and short symbolic statement, and I also write that ‘two arrowheads’ is meta language that describes the representation and not part of a strict mathematical definition. This meta language is describing the representation, explaining to the student explicitly that the diagram representation conveys the concept of infinity (“extends without end”) with arrowheads.

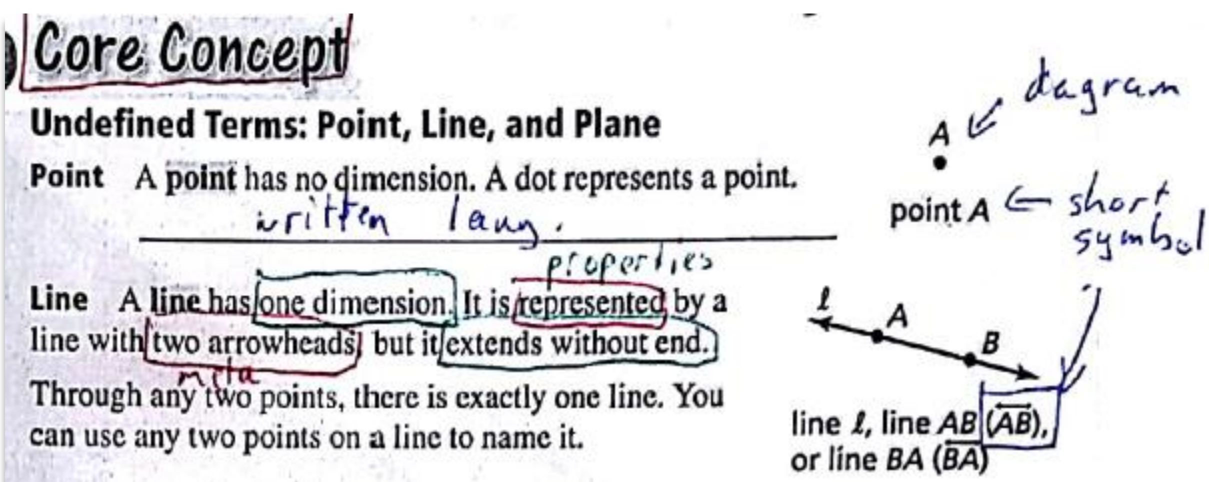


Figure 2. Sample of initial analysis of textbook.

During phase 3 of thematic analysis (stage 1), I looked for sub-representations, especially in language, which is ubiquitous in textbooks. The basic representations mentioned in the literature were initially coded as follows: written language (WL), sketches/diagrams (D), symbols in statements (Sy), numerals (N), algebraic expressions (A), graphs on coordinate grid (G), construction with compass, straightedge and/or measuring tools (C), physical objects, including manipulatives (P), tables (T). Spoken language (SL) and animations are not directly possible in a textbook, and from further analysis, those codes were not warranted. For some tasks, students are asked to "draw several points… some lines, line segments, and rays" in dynamic geometry software (p. 3, Larson & Boswell, 2015). I performed those tasks fully, but I did not do anything not in the direct instruction. I took notes while following instruction, and I coded the representation based on those notes. Table 1 was my initial coding manual (the final version is in the appendix).

Table 2. *Explanation of initial codes*

Code	Description

Written Language (WL)	1. Any text in typical paragraphs will be coded as WL.
	    a. Ignore text leading to website, e.g., "Help in … Spanish at BigIdeasMath.com"
	2. Headings, text in margins, and vocabulary printed in typical left to right will be coded WL.
	3. Letters on diagrams that represent points, lines, planes, etc. will not be coded WL
	    a. they are part of the diagram representation.
	    b. when written in a paragraph (e.g., point A is on line BC), they will be coded as WL.
	4. Subcategories
	    a. Pure math language
	        i. Any language that represents abstract math objects
	        ii. e.g., The five-pointed star has a regular pentagon at its center
	    b. Applied math language
	        i. Language that describes the real world, or
	        ii. language that is a mixture of real-world description and abstract math concepts
	    c. Metalanguage

	i. Language describing more generalized ideas of what it means to do math or how one becomes proficient at doing math 1. e.g., To be proficient in math you need to understand definitions … ii. Language used as commands, directions, instructions in general terms 1. e.g., Work with a partner 2. e.g., verbs in imperative sentences like: use, name, explain, give,
Diagrams (D)	1. Any 2D or 3D figure drawn in 2D that contains points, lines, rays, segments, and or planes will be coded as D, unless it is drawn to scale and so labeled. 2. Subcategory of Diagrams: a. Any diagram where the author explicitly states that it is drawn to scale or with a compass, straightedge, ruler, protractor, or any other measuring tool will be coded C.
Geometric Symbols in Statements (Sy)	Any geometry symbolic statement with capital letters that reference points, especially with the following symbols: angle, ray, segment, line, parallel, perpendicular. e.g., $\overline{AB} \cong \overline{CD}$ or $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$. Combined with written language will be coded as mixed, but not if there is no symbol for a specific object as in circle P. Some books do use the symbol $\odot$ for a circle and other objects, which will be coded as Sy.
Numerals (N)	Any numbers processed in a calculation, but not part of an algebraic expression will be coded N. e.g., 5(3)-6=9 Numerals will not be coded, if they are used for reference. e.g., $\angle 3$, example problem 2.
Algebraic Expression (A)	Any expression with variables and operation symbols like $+, -, \cdot, /, (), =$, etc. will be coded as A. Coefficients and constants will not be coded as numerals (N). e.g., $y=4x-7$ or $a^2 + b^2 = c^2$ If numbers are substituted for all variables, it will be coded as N.
Coordinate Grid (G)	Any diagram with an x-axis and y-axis will be coded as graph (G). If the graph is drawn to scale on a coordinate grid and labeled so, it will be a mixture of grid (G) and construction (C).
Physical Objects (P)	Any pictures of 3D objects will be coded P. e.g., a picture of a pyramid Any descriptions where a student is supposed to look at a 3D object or imagine it will be coded P. e.g., Give examples of each type of line intersection formed by the walls floor and ceiling.

	Any paper or other flat object that the student is instructed to fold to create a 3D object will be coded P.
Table (T)	Any list of numbers and/or variables separated into columns and/or rows will be coded T. Truth table of conditional or other statements. This may be different than table of values of a function that may have a more defined pattern. Table with text, e.g., number of sides and name of polygon. Similarities among various tables: • Easy to compare two or more dimensions using rows and columns Becomes overly complicated when more cells are added, especially columns.
Dynamic Geometry Software (DGS)	Any instructions to use dynamic geometry software like Geogebra or Geometer's SketchPad, especially if there is a picture of it, will be coded as DGS. The coder will follow the instructions in Geogebra to see what the experience entails.
Spoken Language (SL)	in speech bubbles will be coded as SL.
Logical Symbols (L)	Any time a conditional, biconditional, etc. statement is condensed into symbols. e.g., If the metal is red, then it's hot. $p \rightarrow q$

As the guidelines of thematic analysis (phase 4) instruct, I added more codes or modified them when text, an image, a diagram, etc. did not fit a specific code (Braun & Clarke, 2006). By analyzing the first chapter of Larson and Boswell (2015), for example, I realized that the written language (WL) was often not representing mathematical objects or even relationships between those objects; it sometimes described a more general thinking strategy that can be applied not just to mathematics. I coded that language as meta-declarative written language (WDM). Another example is a description of a chemical that is "the most potent greenhouse gas in the world," (p. 7, Larson & Boswell, 2015) i.e., information that has little mathematical relevance, but that might motivate a student geometry or prepare him or her for symmetry in chemistry. This language I coded as real-world declarative language (WDR) or, if written as a question or instructions/directions, I coded it as real-world interrogative/imperative written language (WIR).

At this point (phase 5 of thematic analysis), the codes, or definitions, of the representations were forming. I wrote them in a coding manual. Some aspects had to be operationalized. For example, I had to decide that for a code to be real-world written language (WDR), the meaning of the sentence and the majority of the words had to reflect the physical real-world. The finished coding manual was the report (phase 6) of the thematic analysis.

Analysis of Representations and Their Relationships (Stage 2)

During the second stage (phase 1) I continued looking for the introductions of distinct representations. To remind the reader, this process was not linear, and therefore I had already recorded many of the introductions. I looked more conscientiously at how they were introduced, whether implicitly through examples, explicitly through meta-language, or some other way.

During the second stage (phase 2), to answer the other two research questions, I recorded which representations are present on a page, their frequency, and in which order they are coordinated. I recorded each coordination in a spreadsheet typing in the codes for initial representations and then codes for subsequent representations. Later, I would use this spreadsheet to look for common coordinations, and where I can find them so that I can analyze them as a group looking for mechanisms (phase 3). The spreadsheet was a helpful quantitative component of the analysis: counting representations, percentages, common coordinations, etc., ordering representations and coordinations, and collapsing certain representations, e.g., the sub-categories for written language in one code WL to make the data less complicated for analysis.

This part of the study had a qualitative description along with the quantitative data. Conversions among representations that a student must perform is where students struggle most (Duval, 2006). For example, students find it more difficult to change “twice one number is equal to 8” to “2x=8” than actually solving it, “x=4.” I therefore recorded how many representations are used in a coordination to quantify that aspect.

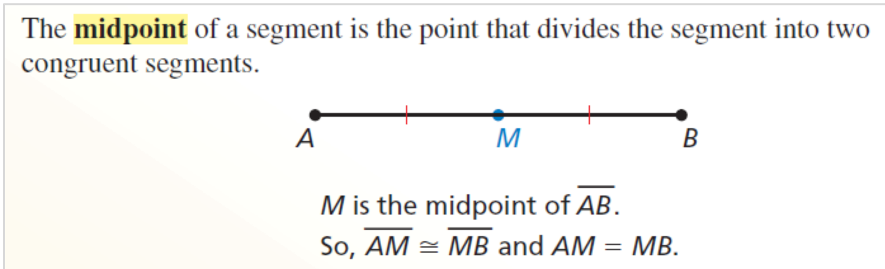


Figure 3. Use of Written Language (WL), geometric Symbols, and diagram (2D)

For example, figure 3 above displays three different representations (WL, Sy, and 2D), and a student needs to coordinate those representations in multiple ways. There are at least three conversions, WL to 2D, WL/Sy to 2D twice. When conversions are needed, I counted the shortest possible string. In this example, a student would need to convert the bar above AB to the word ‘segment,’ a short geometry symbol to language conversion.

Peirce’s semiotic theory of the sign, which consists of the object, the representamen, and the interpretant, also aided in the analysis. For example, in figure 2 the line is described as one dimensional, yet obviously for us to see a line on a page needs to be two dimensional. Peirce’s theory helps us to understand the coordination of the signs, the representamen. In figure 2, a student needs to coordinate the idea of a line being one dimensional in written language (WL) and two dimensional as a diagram (2D). With each representation having different affordances, they underscore different aspects of the object. I described that in the qualitative description of coordination among the representations.

Cognitive load theory (CLT) was useful in analyzing how a person’s working memory may process the representations. According to Sweller, Ayres, and Kalyga (2011), during the learning process students learn best with limited load on their working memory, which oftentimes means as few representations as possible. Yet, mathematics requires at least two representations—often more than two—to display sufficient attributes of mathematical objects (Duval, 2006). According to cognitive load theory, students do not need to learn biologically primary knowledge (e.g., recognizing faces and speech) explicitly, but they need to learn biologically secondary knowledge (e.g., mathematics and second language) with considerable effort and conscientious attention. During this stage and the first stage I looked at how mathematics is introduced explicitly, how they are modeled in worked examples, and then how practice problems (sub-sections called "Progress Monitoring") reinforce that learning. Good textbook design will aid students to form lasting knowledge in long-term memory so that they can use that mathematical knowledge (schemas) during problem-solving instead of overworking their working memory. Problem-solving strategies like means-end are biologically primary knowledge, which does not need to be taught, but if a student does not know a domain specific situation like x+4=7 and how to solve for x, that student will try various means like guess-and-check to attain the goal, x=11 (Sweller, Ayres, & Kalyga, 2011). Means-end strategies consume time and mental energy.

I also compared quantitatively how many representations were used at each unit of analysis. The number of representations may quantify the load on a student's working memory, i.e., the more representations a student coordinates, the higher the load. When recording which representation is read first, I choose the one that is highest and most to the left on the page. The only exception were the comments that are written in the side bar, which I assume a student would read after the main body of the text is read. There were also instances where due to page restrictions images and diagrams were further away, but they were clearly referenced or described in the text.

Finally, in phase 3 of stage 2, I focused on the mechanism. O'Halloran's (2008) linguistic approach, along with Peirce's semiotics, aided to understand some of the mechanism that occur during coordination of representations. By mechanism, as I explained earlier, I mean the method through which we change within one representation or convert one representation to another. O'Halloran (2008) provides many mechanisms as described above that helped with the analysis of some of the instances of coordination. For example, the caption (WL) in figure 2 "line *l*, line AB…" describes the diagram of a line with two labeled points, 'A' and 'B.' If we call the line '*l*', we do not consider the included points, whereas calling it line 'AB', we would already introduce two points that can be used to describe an intersection; at the same time, thinking of line AB instead of line *l* puts more the focus on the points. The points are usually equally spaced out between the arrowheads, whereas the single lower-case letter *l* is away from the line near one of the arrowheads. Also, notice that "line *l*, line AB…" is written with a different font (sans serif), except the *l*, which is a serif font. The italic font and the location of names helps with the focus on the line, distinguishing it from the capital roman (non-italized) font that emphasize the points. These differences provide students, teachers, and authors of textbooks to emphasize, or even de-emphasize, a certain type of coordination.

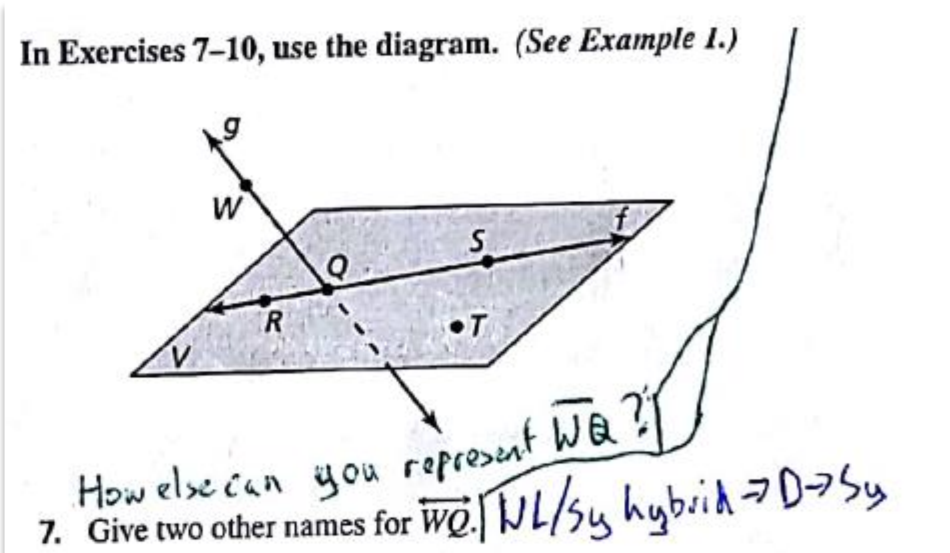


Figure 4. Sample of coding problems at end of a section.

Figure 4 shows the order of the representations used and the dilemma of how to code when representations are presented simultaneously. Coordination is needed when either the utility of one representation is exhausted or the when the other representation aids in finishing a task. In this example, a student would start with instructions to use a diagram for specific questions in written language (WL). Then, exercise 7 again uses written language (WL) and geometric symbols (Sy) to direct a student to look at the diagram, find line WQ, meaning the student would focus on the diagram (D). Finally, the student would write the answer in symbols and written language. Therefore, the overall order of representation use is WL → WL + Sy → D → WL + Sy.

CHAPTER 4
RESULTS

Thematic analysis comprises of six phases: (1) familiarizing with data, (2) generating initial codes, (3) searching for themes, (4) reviewing potential themes, (5) defining and naming themes, and (6) producing the report (Braun & Clarke, 2012). After familiarizing myself with the data (1), I generated the initial codes (2). As I searched for themes, I developed more themes and refined some codes (3), which are listed in the table below. The major differences between the initial codes and the finalized ones are (1) expanding written language (WL), (2) refining the code for number (N), (3) separating 2-dimensional (2D) and 3-dimensional diagrams, (4) separating physical objects (P), picture of physical object (PP), and physical tool (PT), (5) adding codes for ordered pair (OP), student-made representations (M.), and textbook gestures (GST). Written language was the most common representation, and I divided into subcategories. I changed the name of the numeral code to a code called number to underline that the code stands for a value rather than any instance of Arabic numerals (0-9). For example, 1 can be used to represent an angle (∠1), where '1' is an arbitrary symbol that has nothing to do with a value of 1 unit of something. While table 2 provides a brief description of the codes for representations, the representations are described in more detail with specific examples in the descriptions of the figures in the section below.

Table 3 *Codes before grouping into broader themes*

	Representation	Description
1	WDR	Written Declarative language of Real-world physical objects
2	WDP	Written Declarative language of Pure mathematical objects or concepts
3	WDM	Written Declarative language of Metacognition
4	WDD	Written Declarative language incorporated in a Diagram

5	WIR	Written Imperative/Interrogative language of Real-world physical objects
6	WIP	Written Imperative/Interrogative language of Pure mathematical objects or concepts
7	2D	Diagrams or sketches of 2 dimensional figures like lines, segments, polygons, etc.
8	3D	Diagrams or sketches of 3 dimensional figures like solids, 2 or more planes, or non-coplanar lines, rays, segments, etc.
9	G	Similar to diagram except Graphed on a coordinate grid
10	C	Similar to diagram except Constructed to scale using tools (PT) allowing for measurement
11	N	Number values that stand for a measurement or a value of a variable
12	OP	Order Pair, consisting of an x-value and a y-value, e.g., (3, -2)
13	A	Algebraic expression involving at least one operation and one variable, e.g.,2x
14	Sy	Geometry short Symbols like $\overrightarrow{AB}, \cong, \angle$D
15	PP	Picture of a Physical object, including graphic illustrations of physical objects like maps or room, where the artist removes much of the detail a physical object would have.
16	PT	Physical object that is a Tool like a compass, ruler, or protractor
17	P	Physical object that students are instructed to use like the paperclip in figure 6
18	T	Tables that organize data or the logical argument of a proof
19	E	Dynamic Geometry Environment (I use E, since D is used in diagrams and declarative). This will be used together with D, OP, G, and A, e.g., ED or EOP.
20	Gst	Textbook gestures such as arrows connecting written language and diagrams, or checks implying correct solution

The following sections in this chapter answer the three research questions after completing the thematic analysis of a geometry textbook and then coding 3,967 coordinations among the representations outlined in table 3. The first section builds on the codes developed in thematic analysis, and it explains how the authors of the textbook introduced each of those coded representations. By introduce, I mean describe, whether implicitly or explicitly, the representation, its affordances, and constraints. I use the number of instances in coordination to show the frequency, but for some representations that appear less often like PP, G, or C, I also mention the actual count in the data corpus and/or in the entire textbook. The second section answers both the question about common coordinations among representations and the one about mechanisms used in coordinations.

Introduction of Representations in Textbooks

The first research question is how representations are introduced in a textbook. This study has answered this research question and found most representations are introduced implicitly through examples; some affordances of these representations are discussed, but almost no constraints are mentioned. In the following paragraphs, I provide examples of how representations are introduced and explained in the textbook. To introduce an idea like a representation, one must know what it is, and explain it explicitly by explicitly describing it in detail, or by implicitly displaying examples. In each of the representations, I will present the frequency of use, I will describe attributes (an affordance, a constraint, sometimes neither) individually, and then how or whether the authors introduced it. It is difficult to separate representations because they are so closely linked with coordination; figures often contain many representations because they would often not make sense by themselves. Written language often introduces the other representations and itself, but occasionally another representation like gestures help introduce written language.

The first sentence of chapter 1 is "How can you use dynamic geometry software to visualize geometric concepts?" (p. 3, Larson & Boswell, 2015). Although it is written in language which we will discuss later, the sentence introduces dynamic geometry environment (DGE).

Table 4 *Frequency and Percentage of Representations in Coordinations*

Initial Representations			Subsequent Representations		Total	
code	frequency	percent	frequency	percent	frequency	percent
2D	208	2.8%	649	8.7%	857	5.7%
3D	71	0.9%	259	3.5%	330	2.2%
A	540	7.2%	474	6.4%	1014	6.8%
C	30	0.4%	187	2.5%	217	1.4%
EG	42	0.6%	47	0.6%	89	0.6%
EN	24	0.3%	25	0.3%	49	0.3%
EOP	25	0.3%	25	0.3%	50	0.3%
G	111	1.5%	264	3.5%	375	2.5%
GST	823	10.9%	832	11.2%	1655	11.0%
M.A	62	0.8%	61	0.8%	123	0.8%
M.C	11	0.1%	31	0.4%	42	0.3%
M.EG	21	0.3%	89	1.2%	110	0.7%
M.EN	20	0.3%	52	0.7%	72	0.5%
M.EOP	20	0.3%	44	0.6%	64	0.4%
M.G	40	0.5%	162	2.2%	202	1.3%
M.N	107	1.4%	169	2.3%	276	1.8%
M.OP	65	0.9%	74	1.0%	139	0.9%
M.SY	43	0.6%	95	1.3%	138	0.9%
M.WDP	25	0.3%	342	4.6%	367	2.4%
N	1012	13.4%	1007	13.5%	2019	13.5%
OP	356	4.7%	229	3.1%	585	3.9%
P			29	0.4%	29	0.2%
PP	19	0.3%	133	1.8%	152	1.0%
PT	45	0.6%	147	2.0%	192	1.3%
SY	904	12.0%	484	6.5%	1388	9.3%
T	180	2.4%	166	2.2%	346	2.3%
WDD	26	0.3%	158	2.1%	184	1.2%
WDM	76	1.0%	30	0.4%	106	0.7%
WDP	1356	18.0%	844	11.3%	2200	14.7%
WDR	76	1.0%	54	0.7%	130	0.9%
WIP	1099	14.6%	236	3.2%	1335	8.9%

The table displays the data of the frequency and percentage of representations in each coordination. For example, a coordination of WDP + Sy → 2D would add another instance to WDP and Sy in the initial representation frequency column, and 2D would add another instance to the 2D in the subsequent frequency column. Representation less than 0.2% were removed.

Dynamic Geometry Environment

In this section, I will describe the Dynamic Geometry Environment (DGE) representation. In the following paragraphs, I describe various aspects of DGE, their frequency and how they are introduced.

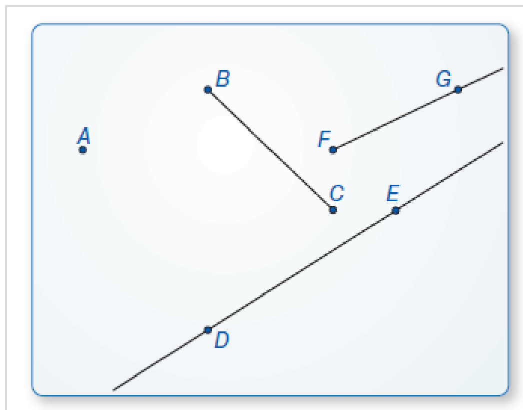
Description. After multiple comparisons, I decided that the major difference between a dynamic diagram (and its attributes like length and equations) in DGE and a static diagram on paper is that it is *dynamic*, i.e., it can be changed with typing a different value, a click, or a drag. Obviously, students cannot construct DGE figures in a printed textbook, but neither can the textbook contain the walls of the classroom when they are described. I assumed that a student would perform the actions described in the directions just as he or she would perform constructions or other directions. This dynamic diagram allows for dragging and was, therefore, coded E2D ('E' for DGE and '2D' for a 2D diagram).

The geometric figures and other aspects in DGE are different colors. All polygons are salmon in color and the area inside is a lighter shade of salmon, and 66% of all DGE images contained polygons. Lines and segments are either black or red, but if there are two lines constructed, then red and blue are used. Circles are black, but arcs of interest are red. Axes are always black (79% had axes), but not one had grid lines. The border is blue with curved corners and shadowing so that it appears coming out of the textbook. The background has a slight hint of blue. The names of points are blue, and so are the points themselves, represented by a tiny filled-in circle, or represented as an intersection without a circle (see fig. 5 below). Angles named with numbers and angle measures are green. Symbols for angle congruence, segment congruence, perpendicular, and parallel are different colors (red, black, green) with little consistency. Wide yellow arrows (EGest) were used in transformations like reflection, translation, and rotation. Besides the graphs or diagrams in DGE, other representations were displayed to the right or below the images. The font of the ordered pairs (EOP) was blue, lengths (ENS, i.e., DGE Number Symbol) were red, angle measures (ENSy) were green, and algebraic equations (EA) were black. The authors did not explain the colors of the DGE images.

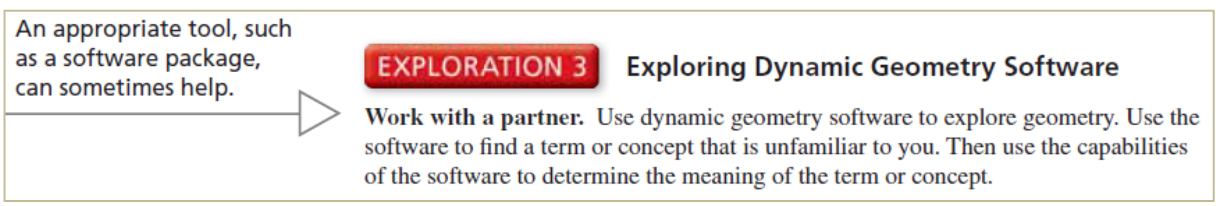
Graphing calculators may not come to mind when coding for DGE, but in the one instance on page 124 where they were employed in graphing a line and not dragging, a basic graphing calculator would be equivalent to a powerful DGE. Dynamic geometry software is no longer just available on computers; tablets and smartphone can run DGE programs. The authors did not state any similarities between the graphing feature on calculators and DGE.

Frequency. In the five chapters studied, the frequency of DGE is 80 images, and dynamic geometry environment is included in 2.1% of initial representation and 3.7% of subsequent representations of in the coordinations. For comparison there are 84 images of graphs and 112 images of constructions (both discussed later), but, with constructions, there were usually three or four images in a row to show the steps of the construction. Of the 80 images of DGE, which are all used in the 174 instances of coordination, 75 are used in explorations, where students are asked to construct figures, and then they are to answer questions or make conjectures based on dragging, measurements, equations, etc. The other 5 are in worked examples.

Introduction. The image below is the first image presented in section 1.1, meaning it is introduced implicitly at this point. The directions in the exploration required to create such a construction in a program like GeoGebra or Geometer's SketchPad. Figure 5 consists of two images from the same page. The image of the dynamic diagram (ED) is about 4 inches above the lower image of the written language, but they are related.



… [4 inches]


An appropriate tool, such as a software package, can sometimes help.

EXPLORATION 3 Exploring Dynamic Geometry Software

Work with a partner. Use dynamic geometry software to explore geometry. Use the software to find a term or concept that is unfamiliar to you. Then use the capabilities of the software to determine the meaning of the term or concept.

Figure 5. DGE and explanation of why to use certain tools (p. 3, Larson & Boswell, 2015)

In figure 5, the authors explain that dynamic geometry environment (DGE) can be used to explore geometry and through the process of exploring find the meaning of terms and concepts. In the side note, the authors also suggest that in this case (students explored the undefined terms of point, line, and plane in DGE) it is an appropriate tool. Here, the authors explicitly state an affordance of DGE, i.e., "to explore geometry." This exploration is a page before the introduction of undefined terms, where the authors write explicitly what a line is. For example, before the authors of the textbook explicitly explain to students that lines "extend without end" (fig. 6), they can drag a line during the whole lesson in one direction, and it will never end.

In fact, I zoomed out to the limit in GeoGebra —and there is a limit—and the maximum numbers are in the magnitude of 10^9. Zooming in, I reached a limit of 10^{-13}. Visually the major difference with infinite objects like lines or rays is that on paper infinity is represented with an arrow, whereas in DGE it extends to the edge of the program's window (see figure 5). The authors do not contrast DGE's ability to zoom in and out with the diagram's (2D) inability to do so. They do not explain the DGE's affordance of zooming out to so that students can explore the concept of infinity, nor do they point out the constraint of some limit in that zooming out.

DGE constructions appear in various content areas. In the textbook, the next few constructions in DGE are to explore transformations. Transformations are shown with large yellow arrows (Gst). Students are asked to use, draw, and copy, translate (slide), reflect, rotate, hide, and dilate. These gestures do not appear in the actual software and are not explicitly explained.

DGE consists of multiple representations that can change dynamically, which is quite different than those on paper. Changing one representation like an equation changes a graph, lengths, angle measure, table values, and coordinates of points. Depending which view a user turns on and what is to be displayed, the dynamic geometry environment can be a diagram, graph, a construction, a number, an ordered pair, and/or an algebraic expression, or all of them at the same time. If DGE is used to show a dynamic affordance of algebraic equation of a line, for example, then it was coded EA for dynamic (E) Algebra; EOP for dynamic (E) ordered pairs; and EN for dynamic numbers like length or angle measure. To remind the reader, if the textbook provided directions to do an activity like a construction, solving a problem, or using DGE, I performed those actions and coded them. The dynamic geometry environment provides a plethora of dynamic information by default like coordinates of points, lengths of segments, and algebraic representation of lines. Any time a student creates a point, its coordinates appear in a sidebar, and then when a point moves due to dragging, the coordinates change accordingly. The same is true for measurements of angles and lengths and algebraic expressions. Again, the authors did not state these affordances explicitly.

Rotating figures on paper by angles that are not multiples of 90 is difficult and, therefore, not very practical in the classroom, but in DGE it is as easy a clicking a button and entering the angle of rotation. Another affordance is the ease with which to correct mistakes with undo or deleting unwanted points, lines, and other objects. Moreover, the ability to hide steps or objects that are not needed at a given moment reduces the complexity of the amount of representations on the screen. Construction on paper is more difficult because it takes more time, it is not as forgiving with mistakes (e.g., we cannot just press the undo button), it does not allow for reducing complexity by hiding, etc. Again, these affordances were introduced by example, and some like hiding, deleting, undo, nor redo were not introduced at all.

One of the most important affordances is the ease and speed many simple and complicated constructions can be made. The ease of drawing in DGE reduces the amount of working memory needed to do a task. Speed is relevant practically in the classroom because of time limitations of a lesson. For example, a student can construct a triangle on a DGE coordinate grid in a fraction of the time it would take on paper simply by clicking and dragging; just to create a grid takes effort on paper, but it is a default feature in most DGEs. I performed a few tasks like construction, translation, measuring, and most took about half the time, e.g., constructing a triangle on a grid took me 61 seconds on paper and 25 seconds in GeoGebra. Yet, measuring took about the same time and required the knowledge (meaning load on working memory) that to measure non-reflex angles (less than 180°), a user of GeoGebra must click on the point or segments in counterclockwise order. Although time spent is not directly connected to working memory, I noticed in my practical experience as a teacher that student are less willing to persevere through an activity, exercise, or problem the more time-consuming the problem is.

Error in construction in DGE does not compound itself as it does in drawings on paper because DGE internal algorithms create a very precise construction. Students, even the most conscientious ones with very precise tools, necessarily draw each step of a drawing or construction a fraction of a millimeter off, and after a few steps that error can become great enough to make the construction inaccurate. That error only occurs in DGE when estimates or incorrect procedures are used.

There is no more detailed discussion about the utility of DGE until page 172. Up to page 172, students learn about the affordances of DGE by following instructions. For example, there is no explicit mention of dragging, arguably the most valuable affordance of DGE, until chapter 6 (relationships within triangles), and there is no mention of comparing values as they change as some point or segment is dragged. Moreover, the book does not introduce any other capabilities like three-dimensional construction or tables. On page 172, the authors of the textbook do list some of the capabilities of the DGE, like drawing, measuring, copying, sliding, and reflecting objects, but, as stated above, many other affordances are disregarded.

Although there are many affordances in DGE, one major constraint is that, compared to paper, it is not easy to write in it. First, DGE is mostly limited to the keyboard characters, more for some savvy computer users. Therefore, it is more difficult to take notes, to use arrows to point, to highlight, to underline, to circle, etc. Also, it is not useful for notes that students write so they can study later. If a student had taken notes and had made comments in DGE files, those notes would be spread out over many files and not easily perused like a notebook or a binder. Therefore, DGE might be less advantageous than paper and pencil for notetaking and certain kind of problem-solving, because paper leaves a written record that is more suitable for studying or referring to later.

Lastly, when learning we know that sometimes limiting the amount of complexity reduces cognitive load, and that reduction often helps students to learn. DGE can do that in at least two ways. First, we can hide unnecessary objects that might have been used to construct a figure or were at one point needed, e.g., hide a circle that was used to construct an isosceles triangle. Second, we can hide the other representations that often appear by default, e.g., ordered pairs, algebra, length, etc. On the other hand, these other representations do often appear by default, and would require the student (or teacher) to remind him/herself to hide those components. That is often difficult because the student often thinks more information is better, while the teacher is trained to ignore irrelevant information and might forget to remind students to hide unnecessary representations.

Most of the above comments on DGE were my observations during execution of the instructions in the textbook. Some, like the list of capabilities on page 172, were explicitly stated in the textbook, but most affordances and constraints were implicit in the instruction or vague like the instruction in figure 5, “use the capabilities of the software to determine the meaning of the term or concept.” Part of the reason may be that different teachers, schools, or school districts may use various DGEs. If the authors provided more specific directions, they may not be aligned with a different DGE program. Another reason could be that all uses of DGE occurred in the exploration sub-sections at the beginning of a new topic. The authors might want to encourage exploration. Yet, the relevant geometry knowledge that was to be discovered in the exploration is often discussed explicitly on the following page.

Written Language and Textbook Gestures

In this section, I will describe written language (WL), its subcategories (WIP, WIR, WDP, WDR, WDM), and textbook gestures (Gst) are the focus. In the following paragraphs, I describe various aspects of DGE, their frequency and how they are introduced. Although written language is not introduced in similar ways as DGE, the textbook does mention some affordances and does introduce vocabulary in certain ways.

Description. Language being the most complex and most ubiquitous representation, it is not possible to describe it all in one dissertation. For that reason, I describe only the most common and the most prominent introduction and use of language. I will use examples to show how it is introduced. Following the introduction of the subcategories of written language, three common objects (points, lines, and, later in the diagrams section, segments) are described along with the language needed to describe them.

Gestures are difficult to isolate, especially when the occur with written language. They may be a change in font, font color, font background, use of italicized or bold fonts; they may be arrows, check marks, perpendicular or parallel symbols on diagrams, etc. Many times, they are a layer on another representation. They are mostly used to emphasize an idea or help with coordination.

Language is very complex, flexible, and often ambiguous. As we saw above in figure 5—or any figure for that matter—it would be next to impossible to convey much of the meaning of geometry with images, diagrams, let alone algebra, but the ease of language to connect ideas together allows it to convey that meaning. The flexibility to put words together is unparalleled in relation to the other representations, allowing language to change direction at any moment. Yet, ambiguity can make it difficult to understand; for example, “a ruler’ could mean ‘a king,’ or ‘to draw’ could mean ‘to pull out’ as in from a bag. This ability of language to be complex, flexible, and ambiguous may put a great amount of cognitive load on a learner trying to decide which meaning to consider in a given situation.

In figure 5, we can notice various types of written language (WL). In the side note, the written language is one level removed from the actual procedures of the exploration. It discusses the utility of a tool like DGE, in more general terms, expecting students to step out of the process of performing the exploration and look at what they are doing and how helpful the tool is. I call this written declarative meta-language (WDM), because it describes meta-cognition that the student can utilize to solve problems. In this case, a student is introduced to a tool (software) that may be useful. The only hint that this language is different is that it is in a side note and that it is a different font (sans-serif), and a different font is a textbook gesture.

Frequency. Written language was 35.1% of initial representation and 21.4% of subsequent representations of in the coordinations. Of the 3967 instances of coordination only 938 did not involve written language, and of those 924 were the student produced answers (M.G, M.N, M.Sy, etc.). For those I assumed students would not explain each step of their answer as the authors had in worked example or constructions. That means almost all coordination printed in the textbook involved written language, which is reasonable in analyzing a **TEXT**book. Textbook gestures occur in 11% of the instances of coordination.

Introduction. In the following paragraph, I will explain how the different sub-categories of language and textbook gestures were introduced.

Meta-language. Side comments allow the student to step out of a question, exercise, exploration, etc. and think about what is happening in that question, exercise, or exploration, and the arrow pointing to what the meta-language was describing. That type of side comment appears above in figure 5. It states, “To be proficient in math, you need to understand definitions and previously established results” (p. 3, Larson & Boswell, 2015). Here, the authors ask a student to notice that they are using DGE to explore the meaning of points, lines, segments, and rays, and at the same time, they are establishing results in those meanings that will be used later the student’s learning math. The book does not go deeper than that, though, and language itself is not mentioned as the medium of those meanings. Reading the side comment (WDM), the student might recognize more broadly that he or she is using a tool to understand geometric concepts.

Although the side comments and the meta-language use is not explained in the chapters I studied, the authors do mention them in the preface (DGE and other representations are not discussed in the preface):

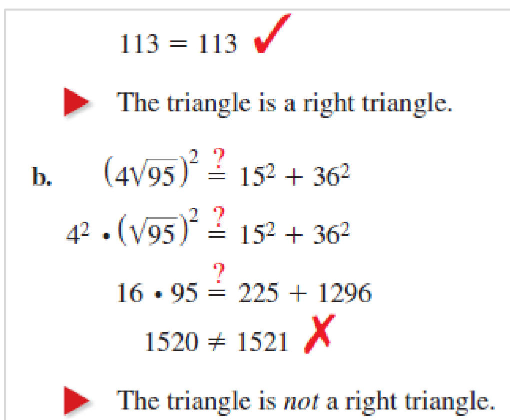
> Look for STUDY TIPS, COMMON ERRORS, and suggestions looking at problems ANOTHER WAY throughout the lessons. We will provide you with guidance for accurate mathematical READING and concept details you should REMEMBER (p. xxi, Larson & Boswell, 2015).

The font in the preface for the side comment headings is gray and all capital letters (again textbook gestures) in the text above just as it is in the rest of the book. Here the words ‘suggestions’ and ‘guidance’ hint at the meta-language aspect of these side-comments.

Pure imperative written language. The second sentence in the paragraph (fig. 5) states “Use dynamic geometry software…” A student is instructed to *do* something, perform an action, an imperative sentence. I add interrogative because question like “What is the distance?” and

"Find the distance," are semantically similar, and they require the same action of the student. I decided to make a distinction between the two types (declarative and interrogative/imperative), because (1) language is the most ubiquitous representation needing some sub-categories and (2) it causes a student to take action or follow the authors' actions besides adding information into his/her working memory as he or she would with a declarative sentence. This sentence was coded written language-imperative pure (WIP), pure as in pure mathematics and opposed to real-world or applied mathematics. There is no introduction of this other WIP language category.

Textbook gestures. The first gesture in the textbook is the arrow pointing from meta-language to the instructions in an exploration in figure 5, and the gesture is not discussed. Figure 6 provides an example of what I call other 'textbook gestures,' ones not related to the font. Many textbook gestures are a non-black color and within paragraphs like the triangles above or italized text like '*not*', but sometimes they are in diagrams, graphs, algebra, etc.



$113 = 113$ ✓

▶ The triangle is a right triangle.

b. $\left(4\sqrt{95}\right)^2 \stackrel{?}{=} 15^2 + 36^2$

$4^2 \cdot \left(\sqrt{95}\right)^2 \stackrel{?}{=} 15^2 + 36^2$

$16 \cdot 95 \stackrel{?}{=} 225 + 1296$

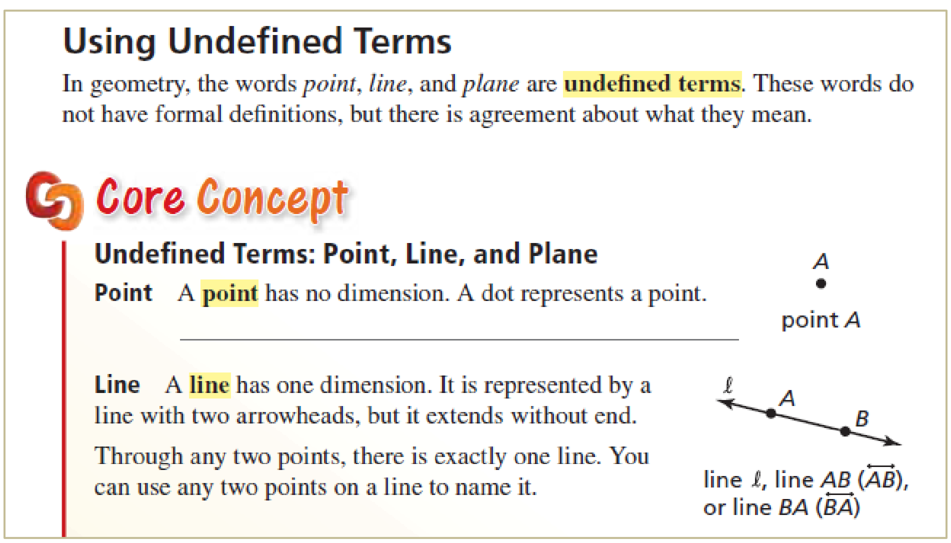
$1520 \neq 1521$ ✗

▶ The triangle is *not* a right triangle.

Figure 6. Textbook Gestures (p. 466, Larson & Boswell, 2015).

In this case, the red check mark ✔ means that sum of the legs squares equals the hypotenuse squared. In other cases, it means that something was done correctly, but in all cases, it means something positive. On the other hand, the red X means that the two sides of the equal sign are not equal, and therefore the triangle is not a right triangle. I call these textbook gestures (Gest) because they are analogous to a teacher giving a thumbs up or down. Another example is the red question mark. The textbook seems to gesture inquisitiveness. Another common textbook gesture is an arrow pointing. Figures 1, 15, and 20 contain examples of arrows pointing. Most of these gestures are implicitly introduced, but some like the symbols on diagrams are explicitly introduced. Moreover, I would not expect the authors to use the term textbook gesture, because I developed it through the thematic analysis of my research, and they are not mentioned in research outside of body gestures.

Written declarative pure math language. In the next few paragraphs, I will not only explore written language, but I will also focus on the three most common objects based on a locus of points, namely point, line, and segment. They are ubiquitous, appearing in almost all graphs, diagrams, constructions, written language, short geometry symbols, and DGE. In figure 7, the authors mention the language representation, by using words like "undefined terms … words … formal definitions … [and] what they mean."


Using Undefined Terms

In geometry, the words *point*, *line*, and *plane* are **undefined terms**. These words do not have formal definitions, but there is agreement about what they mean.

Core Concept

Undefined Terms: Point, Line, and Plane

Point A **point** has no dimension. A dot represents a point.

Line A **line** has one dimension. It is represented by a line with two arrowheads, but it extends without end.

Through any two points, there is exactly one line. You can use any two points on a line to name it.

Figure 7. First coordination of WL, 2D, and Sy (p. 4, Larson & Boswell, 2015)

The basis of any language are words, and here the authors discuss the language representation by pointing out to the reader that geometry is not only presented in diagrams, numbers, and equations, but also in words, in language. Yet, the authors do not discuss much beyond that observation about language. There is no mention of how metaphors are used, of how concepts are changed (treatment) into short two- or three-word theorems (an example of objectification), of how powerful language is in explaining ideas, of how certain words mean a specific idea in pure mathematics and something else in common language, etc. Students may know that already, and they may be able to learn it through examples.

Definitions, descriptions, postulates, and theorems are often in written declarative pure math language (WDP) and displayed on diagrams (2D). I mention diagrams here because almost all non-language representations are coordinated with language, and because it helps to see the contrast between representations and how they are introduced.

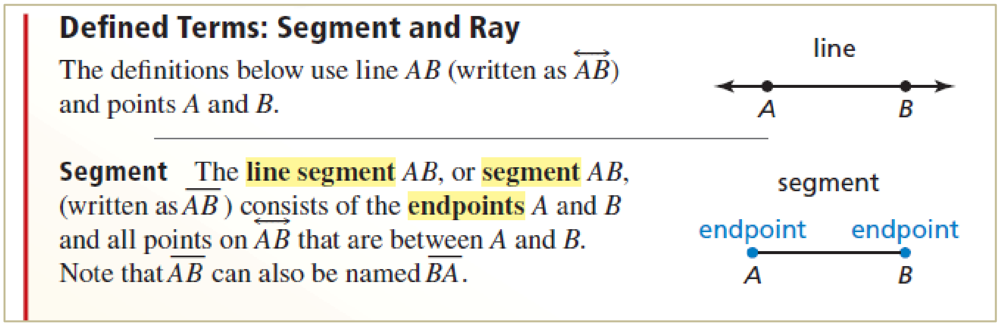
The first two sentences coordinate examples of undefined terms, namely point, line, and plane, and a short description. They are followed by each term being described in more detail and how it can be represented. The first term, point, is described as dimensionless, but that sentence is immediately followed by “A dot represents a point.” A dot, a small circle, is obviously two dimensional, but it is one way we can represent a point. In terms of Peirce’s semiotics, a student must struggle with that paradox of a point being dimensionless (object) and being represented by a small circle (representamen) in their mind (interpretant). The representamen of a point is a dot, the object (the mathematical idea agreed to by the mathematical community) is a dimensionless location in space, and the interpretant is what the student interprets those representations to be. Moreover, the student is also exposed to the naming convention of points, which is not explicit in written language, but can be inferred to be capital letters. The convention is consistent throughout the book so students would see hundreds of times.

The line, just as the point, is also explained in written language (WDP) first, and then displayed as a diagram and geometric symbols. Again, a student must wrestle with the object of line as it is described having one dimension (WL) and it being displayed in two dimensions in the diagram (2D). The author explains that arrowheads in the diagram representation mean that the line "extends without end." Although there is no mention of the straightness of a line, a student can infer it from the diagram (2D) that a line has no curves; again, lines are always drawn without any curvature so if a student did not notice it at this first instance, it is reinforced continuously in the book. Finally, the short geometry symbol (Sy), $\overleftrightarrow{AB}$, is also introduced for the first time to name the line. There is no explanation of the symbol above AB, just an example. Yet, there is an explanation (WL) that we can represent lines with short geometry symbols (Sy) with any two points, and through the two names of lines, the author shows that the points can be in any order, namely $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$.

Although the authors of the textbook do not write much about some of the properties or capabilities of written language as they did with DGE, some are implied. Language has the power and flexibility to explain ideas. As with the example above, if the authors just printed a point and a line without any explanation of what they are, readers of the textbook would be confused and would not know the full meaning of those symbols. Another affordance is that we do not have symbols for all kinds of objects. For example, it is common to see △ABC in the textbook, but there is no symbol for pentagon, transversal, cube, etc. and those words need to be used. Again, that ability to explain properties, affordances, constraints, etc., and to name things that do not have symbols is not explicitly stated.

One introduction is an implicit one about written language (WL) and how it is used to invent constructs. The authors are not explicit about the power of language, but the example of the postulate (fig. 13) demonstrates that power of simply stating something: “points on a line **can** [emphasis added] be matched one to one with the real numbers.” (p. 11, Larson & Boswell, 2013). The word ‘can’ is crucial here, because it creates the basis of the number line and eventually graphs. That creation is an important aspect in mathematics since all mathematics come from such statements. In traditional Euclidean geometry, such statements would be a phrase more like this: “**Let** [emphasis added] the points be matched …” That simple assumption leads to some powerful geometry; it is the foundation of coordinate geometry. Although this is a very subtle moment, it is absolutely one of the most crucial moments in learning geometry and understanding how indispensable language is.

Moreover, the sentence above, especially the word ‘matched,’ hides some very fundamental ideas and in a complicated way uses a metaphor. Again, these fundamental ideas of the mathematical community can only be expressed through language, and this affordance of language is not introduced. First, who matches the numbers with the points? Is it the students, the teacher, the author, the mathematical community? Or maybe the points match themselves? Second, the word ‘matched’ implies some agency. In this case, it seems to be the author or the learner, but in other cases points, lines, segments seem to do some action. In other parts of the book, for example, midpoints divide, arcs indicate, transversals cut, diagrams suggest, lines pass through, etc. Lines, for example, are a set of point; they do not move like light that can *pass through* things. These are metaphors, which connect the abstract world of geometry with everyday life. ‘Pass through’ is usually used in situations like: a bus passing through a city, cable passing through a wall, or thread passing through the eye of a needle. Saying “line *m* passes though the origin” is certainly different than “line *m* contains the origin” or “the origin is part of the set of points that constitute line *m*.” Thread is not a part of the eye of a needle, but a point is a part of a line. Moreover, passing through implies motion. Again, the reader of the textbook is exposed to none of these concepts explicitly.


Defined Terms: Segment and Ray

The definitions below use line *AB* (written as $\overleftrightarrow{AB}$) and points *A* and *B*.

Segment The **line segment** *AB*, or **segment** *AB*, (written as $\overline{AB}$) consists of the **endpoints** *A* and *B* and all points on $\overleftrightarrow{AB}$ that are between *A* and *B*. Note that $\overline{AB}$ can also be named $\overline{BA}$.

Figure 8. Definition of a term. (p. 5, Larson & Boswell, 2015)

Figure 8 shows one of the first objects fully defined, not just described as with the undefined terms. The author continues to introduce more properties and examples of the written language (WL), diagram (D) and short geometry symbol (Sy) representations. In this case, the author uses previous terms to define a new one. The concept of line is invoked to state that a segment "consists of the endpoints A and B and all the points on $\overleftrightarrow{AB}$ that are between A and B." The author could have defined a segment only in written language, e.g., "a part of a line consisting of all points between two endpoints, inclusive." As we will see, the authors consistently define many terms using the combination of written language (WL), diagram (2D), and geometric symbols (Sy). Geometric symbols (Sy) mediate between the diagram (2D) and written language (WL), i.e., they resemble the diagram with less detail, but they are small enough to fit in with the flow of words (WL). With a few exceptions, written language (WL) is coordinated with diagrams (2D)—the other times it is coordinated with graphs (G)—in definitions, postulates, theorems, and core concepts.

Language allows us to transform itself from more descriptive, even action-based, ideas into more concrete-sounding ideas. For example, the book defines a segment bisector as "a point, ray, … that intersects a segment at its midpoint" (p. 20, Larson & Boswell, 2015). Here an action of dividing a segment into two equal parts became a segment bisector, a noun, something that sounds more like an object in common language. The book abounds with theorems that describe properties or actions, but they are recalled with a short name. For example, "the sum of the … angles of a triangle is 180°" becomes the Triangle Sum Theorem, which further in the book is used as a reason in proofs or solutions (p. 233, Larson & Boswell, 2015). This objectification of ideas, an affordance of language, is not introduced nor explained in the book.

Finally, metaphors are an important tool of language that helps students understand concepts. For example, rise and run are a common way, including in this textbook, to explain the slope and its related changes in the y- and x-values. 'Rise' describes something moving up, and 'run' describes something moving horizontally. Not only does rise not include the meaning of change in a downward direction, but it also implies movement. Slope does not imply movement; it is steepness, a static property. Yet, metaphors help students understand mathematics, and it is helpful for student to think that the difference between the y-values is the rise, even if it is a drop or fall when it is negative. The authors do not address the use of metaphors.

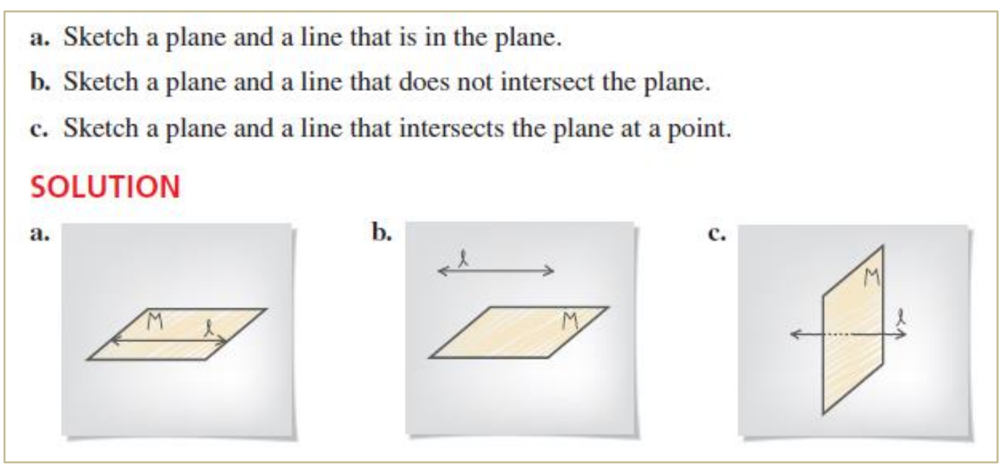
Diagrams

In the following paragraphs, I describe various aspects of diagrams, their frequency and how they are introduced.

Description. Diagrams depict the spatial components of a geometric figure without being precise, or without needing to be precise. Most diagrams were black, including points, lines, line segments, names of points, circles, names of lines, and names of planes. Yet, sometimes they were different colors, especially the points and names of points. Numbers and algebraic expressions standing for lengths and angle measures were usually blue, and arcs accentuating angles, numeral names of angles, parallel symbols, perpendicular symbol, and congruent symbols were usually red. Interior of polygons and surfaces of solids were usually shades of beige.

Many of the other conventions of diagrams were described where the explanatory ability of written language was discussed. Some of these descriptions were explicit, some not. For example, whereas the fact that lines were infinite in both directions as denoted by the arrowheads on both ends was explicit, the fact that lines are straight (no curves) was not explicit. Another example that might be confusing is the explanation that an intersection of two lines represents a point. Although the book does explain that the intersection of two lines is a point, it has an example where the intersection has a small blue dot, but on the previous page an intersection did not have a small dot and it was labeled with a capital letter and referred to in the text.

Frequency. Diagrams are common, and they account for 4.8% of initial representations in coordination and 12.5% of the subsequent ones. That implies that diagrams are referred to, usually by written language, more often than they refer to something else.


a. Sketch a plane and a line that is in the plane.
b. Sketch a plane and a line that does not intersect the plane.
c. Sketch a plane and a line that intersects the plane at a point.

SOLUTION

a. **b.** **c.**

Figure 9. Example of a sketch both 2D and 3D. (p. 6, Larson & Boswell, 2015)

Inroduction. To be precise the first instance of a diagram is the DGE 2D diagram in figure 5, which is introduced by asking the student to do a task. The second diagram is of two planes, where a student is again asked to do a task, i.e., to describe how two planes can intersect. In neither example are the diagrams introduced explicitly. They are discussed more explicitly in other places like figure 10.

In the extant literature, I combined sketch and diagram into one representation, because neither needs to be drawn to scale nor with precision. Figure 9 shows the first use of the word ‘sketch’ and some examples of sketches. ‘Sketch’ is not defined, which implies the author either is confident that students know what it means or that examples are enough for readers to define it for themselves. The sketches are not much different than diagrams. The major difference is that the letters *l* and *M* are cursive or hand-written rather than a computer font. Also, the plane is shaded in rather than having a uniform color as in the other diagram. Finally, the sketch is on paper that seems to come out of the page. Yet, these conventions are not continued throughout the rest of the book, and, when the word sketch is used, typical diagrams are displayed. Therefore, it gives me reason to combine the representations sketch and diagram into one, and I use the words interchangeably. The authors were not explicit about drawings, sketches, and diagrams, but, as we will see later, they were explicit about constructions with compass and straightedge. Moreover, they were not consistent with using the terms draw and sketch, e.g., they used the word draw with constructions, drawings, and diagrams.

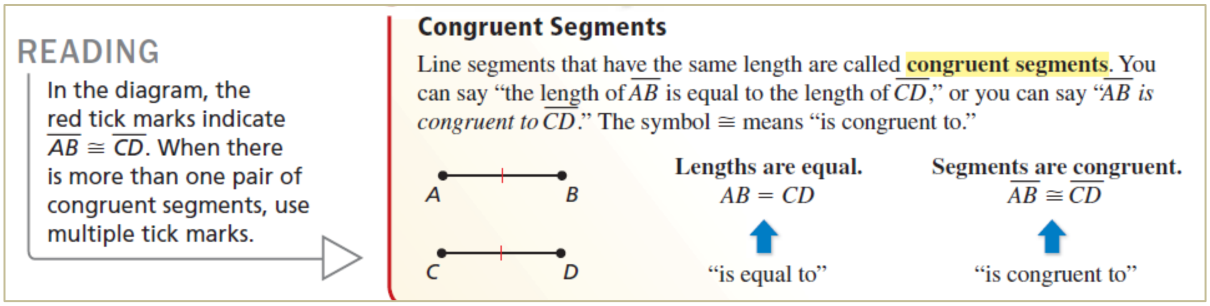


Figure 10. Definition, symbols, and diagram attributes. (p. 13, Larson & Boswell, 2015)

In Figure 10, the authors of the textbook display the interconnections among three key representations, namely written language (WL), diagram (D), and geometric symbols (Sy), to introduce how the object of congruent segments will be represented in each. Although figure 10 contains many a coordination, that coordination is needed to introduce the diagram (D) and short geometry symbols (Sy). We need multiple representation to understand an object, and we need other representations to understand a representation; in this case we need language to understand the two other representation (D and Sy). The concept of congruence, which is a relationship between typical objects, is introduced in the figure. Relationships are difficult to represent and communicate, and partly for that reason we have multiple ways to represent congruence, i.e., in words (WL), on a diagram (D) with tick marks (some say hash marks or congruence marks) (Gst), two ways in short geometry symbols (Sy), and later with equal values. The authors were explicit with the tick marks, a gesture on the diagram representation, and they are explicit about the perpendicular, congruent angle, and parallel symbols. The authors of the textbook define congruent segments as ones having the same length. The same concept is presented in two ways in short geometric symbols $\mathrm{AB} = \mathrm{CD}$ and $\overline{AB} \cong \overline{CD}$. The former compares values, length in this case, and the latter compares figures, segments in this case.

Diagrams appear mostly in worked examples, postulates, theorems, and definitions.

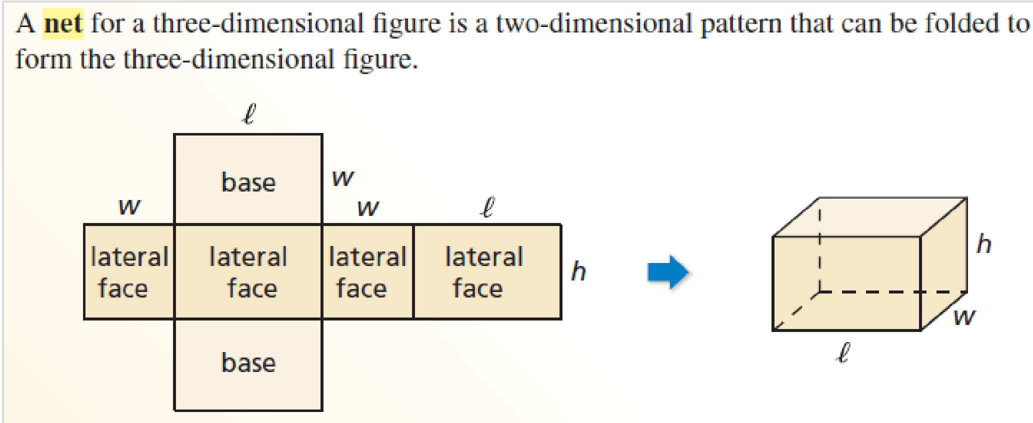


Figure 11. Net and 3D diagram (p. 23, Larson & Boswell, 2015)

Nets and three-dimensional (3D) diagram are difficult to visualize because they attempt to display 3D aspects of a solid in 2D space. The net does not display which edges are attached, and the 3D diagram is not displayed to scale. In the rectangular prism on the right, for example, it is impossible to draw every part to scale or to draw so all the faces are rectangles, and some must be parallelograms; angles, due to perspective, will not be the correct measures. 2D diagrams can and usually are drawn to scale, especially the angles. In fact, I measured 20 angles arbitrarily chosen diagrams in the textbook, and their measures were identical to the measures written. The lengths were usually not the actual lengths, which is practical; a 25-inch segment would not fit in a typical textbook. Moreover, lengths in constructions and drawing are usually not drawn to the exact measurements; they are usually drawn to a reduced size, but not always similar. For example, the centimeter graph paper in the textbook, which is a hybrid of a graph and a construction, the units are 6 millimeters apart in the textbook. Although the authors of the book warn students not to think angles are a certain measure or that lengths are drawn to scale, they usually are. Therefore, students can use that affordance of two-dimensional diagrams (2D) to gain insight into some aspect of a given diagram. That is not the case with three-dimensional diagrams (3D).

Physical Objects

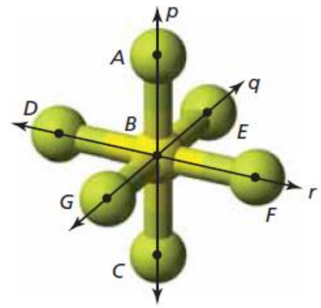
In the following paragraphs, I describe various aspects of physical objects or pictures of them, their frequency and how they are introduced. The authors of the textbook introduce physical objects that students would use to learn geometry minimally.

Description. Physical objects tend to bring a more concrete, more tangible, and less abstract way of looking at geometry. Depictions and even representations of physical objects vary the most. First, there are different sub-categories like pictures of physical objects (PP), physical object (P) that a student would use or see in class, physical tools (PT) like ruler and compass. Second, they are represented in different ways, sometimes a picture, sometimes just a description, and many times both.

Frequency. They account for 0.8% of initial representations in coordination and 4.1% of the subsequent ones. There are 44 images of physical tools (PT) that I assume a student would use in class. There are 11 pictures of and/or descriptions of physical objects (P) that are meant to be used in the classroom, e.g., walls to describe planes. There are 139 pictures of physical objects (PP). Diagrams superimposed on pictures or illustrations of physical objects (PP) appeared 72 times in the entire book.

The diagram shows a molecule of sulfur hexafluoride, the most potent greenhouse gas in the world. Name two different planes that contain line *r*.

Electric utilities use sulfur hexafluoride as an insulator. Leaks in electrical equipment contribute to the release of sulfur hexafluoride into the atmosphere.



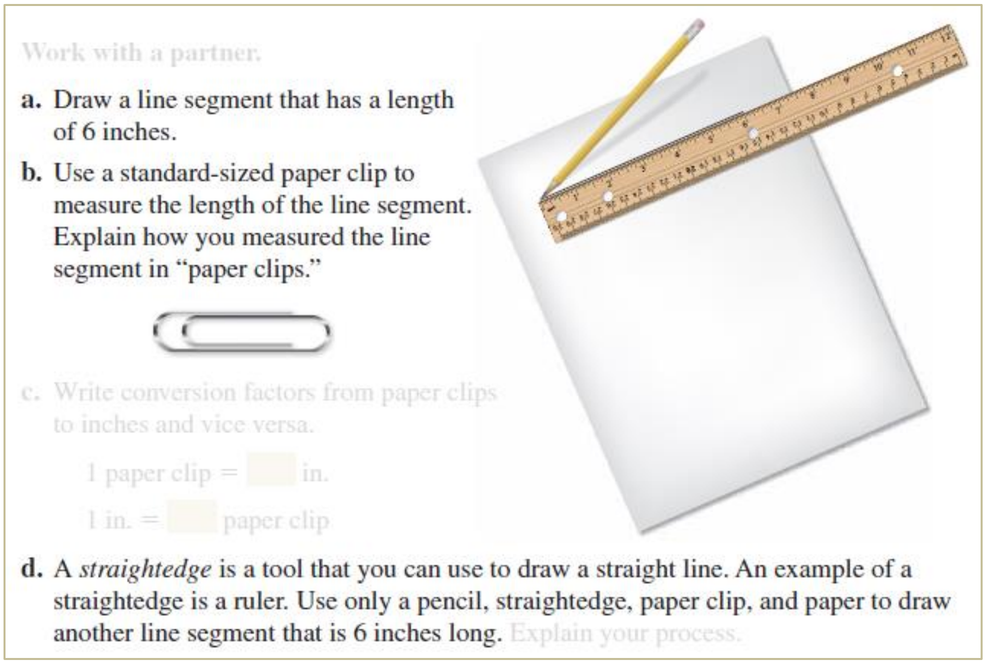
SOLUTION

1. **Understand the Problem** In the diagram, you are given three lines, *p*, *q*, and *r*, that intersect at point *B*. You need to name two different planes that contain line *r*.
2. **Make a Plan** The planes should contain two points on line *r* and one point not on line *r*.

Figure 12. Example of a physical object. (p. 7, Larson & Boswell, 2015)

Introduction. Figure 12 contains the first picture of a physical object and the first three-dimensional model, which is a physical object—although the model is computer generated, there are many molecular model sets that are similar to this model, which is why I treat it as a physical object—Although these visuals are not actual physical objects, they do have most of the features of a physical model, namely detail, shadow, perspective, a sense of texture, a sense of weight, and most importantly a connection to physical reality. Obviously, we are not standing in front of electrical equipment, listening to some buzzing sound coming from the transformers or touching the cold, or possibly hot, surface of the gray metal cases, but we can imagine we are there. In coding I used PP for this type of representation standing for Picture of a Physical object. The most important fact about the picture on the left in figure 12 is that it is not directly relevant to the mathematics in the worked example. In fact, in all the instructional (not exercises) part of the entire book, there are 21 instances of pictures that are not directly relevant to the math described in the text. In figure 12, the author conveys the fact that in the real world this substance, namely sulfur hexafluoride, is used, and the fact that it is “the most potent greenhouse gas.” The picture seems to be there for motivation, i.e., to show that scientist study objects using geometry and that is why it is important to study geometry. The use of pictures of physical objects is often not directly related to the mathematical concept discussed nor used to solve an exercise in the textbook. The authors do not explain that picture is not relevant to the problem, and that it should be ignored when solving the problem.

The other physical object (PP) is the model of a molecule, which is relevant to the exercise. It is interesting because the model overlaps with a diagram. This overlapping is common in the book, and it is used 118 in the entire book. Again, the model is not technically a physical object, e.g., we cannot see it from underneath, we cannot feel its weight, we cannot pull the atoms off, etc., but with the shadows and the shortened bonds along line q, we get a sense of a physical model. My point is that it is not simply a diagram, totally abstract mathematical construct. If students know some chemistry, they could even notice the yellow sulfur atom in the center and the six green fluorine atoms attached to the sulfur. Another interesting aspect of this use of physical objects is that they are used to teach a mathematical concept, in a sense the reverse of how mathematics is usually used in science and in other disciplines. In the other disciplines like science, engineering, finance, medicine, etc., math tends to model a situation, a structure, a physical phenomenon, etc., and not the opposite, i.e., the structure being used to teach mathematics. Use of physical objects is very similar to the use of metaphors. Just as in the rise and run being used to explain slope, a model of an atom—atoms are not static and not really points—is used to help students learn to visualize and name planes. Neither the aspect that physical models are used to show a mathematical idea rather than the reverse, nor the aspect that physical models are a type of metaphor is not discussed explicitly.



Work with a partner.

a. Draw a line segment that has a length of 6 inches.

b. Use a standard-sized paper clip to measure the length of the line segment. Explain how you measured the line segment in "paper clips."

c. Write conversion factors from paper clips to inches and vice versa.

1 paper clip = ☐ in.

1 in. = ☐ paper clip

d. A *straightedge* is a tool that you can use to draw a straight line. An example of a straightedge is a ruler. Use only a pencil, straightedge, paper clip, and paper to draw another line segment that is 6 inches long. Explain your process.

Figure 13. Example of a physical object, construction, and number. (p. 11, Larson & Boswell, 2015)

In Figure 13, I have faded the color of the irrelevant parts to accentuate the tools of construction (PT), construction (C) itself, number (N), and an actual physical object (P). The ruler used as a ruler in the first step and as a straightedge in the last step is a physical object used as a tool (PT). Besides paper and a writing utensil—which are almost never referred to—the most common tools (PT) are straightedge, ruler, compass, and protractor. The second step (b) in Figure 13 asks a student to "[u]se a standard-sized paper clip to measure the length..." This is the first time a student is asked to use a physical object (P), a paper clip, besides the ruler (PT) in the textbook. A student would measure the length in paper clips. This activity shows us that length can be measured in any units, including 'paper clips.' The ability to place and pick up a paper clip gives a student a way to measure the segment in "paper clip" units. There are 44 instances in the entire book where physical objects are used as tools (PT). There are 11 instances where a student is asked to use a physical object (P) like a paper clip. Other examples of using physical object (P) in class are using walls and floor to represent planes, reflecting with a reflective surface, using string, folding paper, or cutting an index card.

The authors do not discuss the tools explicitly explaining their utility. It may be obvious to many, but the use of tools and drawings could be contrasted with sketches and the lack of tools. Then, drawing could be shown to have the affordance of verifying that certain parts actually measure what postulates assume and theorems have logically proven. This way of verifying is rarely done in the book. After the ruler and protractor are introduced, they are seldom used to measure. Similarly, as we shall see in the next section, students usually follow directions in constructions (C), occasionally use those algorithms to construct something in an exercise. Again, rarely are they used to prove a theorem or verify in the physical world that a theorem is applicable.

Constructions and Drawings

In the following paragraphs, I describe various aspects of construction and drawings, their frequency and how they are introduced. The majority of these are drawings with a compass and a straightedge, meaning they are what we conventionally call constructions.

Frequency. Constructions and drawings, which is closely related with physical tools (PT) discussed in the previous section, They account for 0.5% of initial representations in coordination and 2.9% of the subsequent ones. 88% of the instances of coordination with constructions (C) are coordinated with written imperative language (WIP). On the other hand, there are only 14 images of drawings, 6 of which are segments to be measured on one page, 3 are pictures of a protractor with angles superimposed, 4 are pictures of centimeter grid paper used for a graph (only in one is a student asked to measure a length), and one is of a hexagon that a student is asked to draw with tools. Therefore, drawings are not common.

Introduction. Construction is introduced with the following: "A construction is a geometric drawing that uses a limited set of tools, usually a *compass* and *straightedge*" (p. 13, Larson & Boswell, 2015). Drawing is not defined, but implied earlier when the ruler and the protractor were introduced. Therefore, drawings allow for other tools.

Images of construction with implied movements often come in a series that could be thought about like an animation, but I count these as three images. Just like a cartoon, animations typically are images that change in time in one frame.

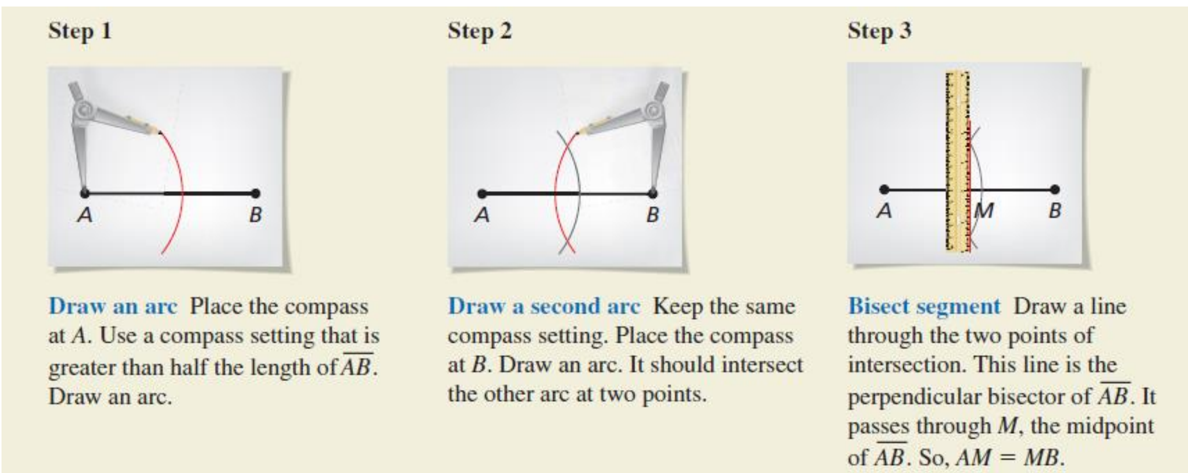


Figure 14. Example of C, PT, WL, and Sy. (p. 149, Larson & Boswell, 2015)

In a printed textbook that is not possible, but the authors used images with implied movement of successive steps that show how the figures would be constructed in time. In figure 14, the compass appears on the left, and it draws an arc, which is red and seems to stand out. In the second step, the compass is on the right and is constructing a new red arc, while the previous arc is now gray. Finally, in the third step, a straightedge appears and draws a red line. Although these steps are very similar to an animation, they are not the same, and I therefore did not code for animations.

The steps imply this progression, but why the arcs are red is not explained. These steps are very procedural in nature, and there is no explanation why these steps actually construct a perpendicular bisector. Measuring of length and of angle is, in my opinion, the single most important affordance of constructions and drawings. The affordances and constraints are not explained explicitly, neither are the techniques of using the tool. There is no mention of rotating the compass to construct the arc, which seems to be implicitly shown on the picture. Drawing an arc is an affordance of a compass because an arc is a locus of points equidistant from a given point. Many solutions or proofs require congruent segments or points being equidistant. The reader needs to follow the directions, written in imperative sentences (WIP), and learn to create constructions (C) by doing.

Graphs & Ordered Pair

In the following paragraphs, I describe various aspects of graphs and ordered pairs, their frequency and how they are introduced.

Description. Graphs (G), or figures that contained an x-axis and a y-axis, and ordered pairs (OP), or a two numbers or variables in parentheses separated by a comma, are linked closely, which is why they are grouped together.

Frequency. Graphs that were not represented in a dynamic geometry environment account for 2.0% (0.3% in DGE) of initial representations in coordination and 5.7% (1.2% in DGE) of the subsequent ones. They numbered 84 images in the entire book. Ordered pairs not in account for 5.6% (0.6% in DGE) of initial representations in coordination and 4.0% (0.9% in DGE) of the subsequent ones. Only 6 lacked grid lines, all of which either contained algebraic rules in ordered pairs (AOP) or algebraic equations (A). Algebra appeared in ordered pairs (AOP) 9 times and as equations 9 times with no overlap. There were only 2 graphs (G) superimposed on a picture (PP). Ordered pairs with numbers (NOP) appeared 37 times. An ordered pairs often appears (1) near a point on a graph, (2) in proximity to a graph both in the flow of text and (3) by itself, or any combination of the three. All graphs used at least one other color besides black, of which 32 (34%) contained one other color (usually blue), 35 (44%) contained two (usually blue and red), 15 (17%) contained three colors (usually red, blue, green), and 2 (5%) contained four colors. Points were usually represented by a filled in small black circle (47 times or 56%), but there were graphs where points were only represented as intersections of segments and/or lines (37 times or 44%). The color was implicitly introduced; it aids in coordination with other representations.

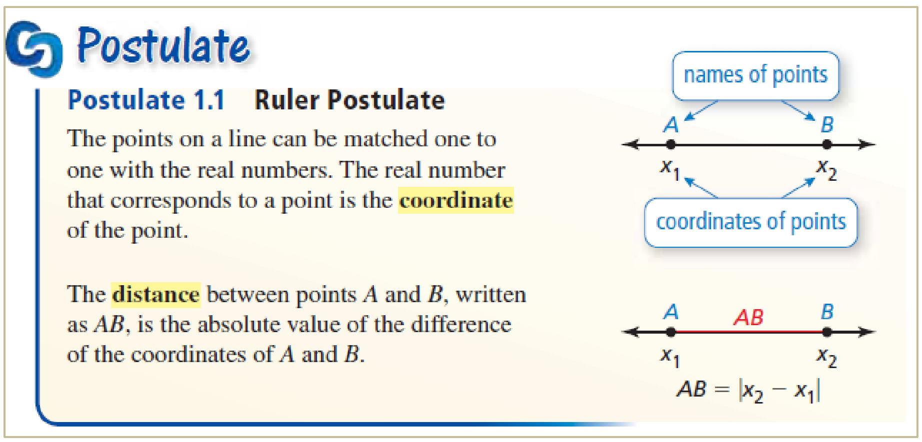


Figure 15. Postulate, coordinates and algebra. (p. 11, Larson & Boswell, 2015)

Introduction. The establishment of coordinates is the basis of graphs (G), algebra (A), and other mathematics, and it deserves attention. It implicitly establishes the unit, which is then used to determine the distance from the x-axis or y-axis, i.e., the x-value and the y-value, respectively. In figure 15, the coordinates set up a variable *x*, which consist of all the real numbers as described in the first paragraph. In the diagram (fig. 15), "coordinates of points" textbox *points* to x_1 and x_2. The variables x_1 and x_2 are used in the algebraic equation AB = $|x_1 - x_2|$, which is also the first time that algebra (A) appears in the textbook.

Once we add another number line, or the y-axis, we have a coordinate grid. Not only does coordinate grid (G) allow for utilizing the distance along perpendicular grid lines defined above, it also fixes a figure in a specific orientation, which is not the case in diagrams and constructions/drawings. An individual figure in diagrams and construction in a way floats without a fixed position; it can be rotated reflected and none of its properties change. On the other hand, if a graphed figure is translated, reflected, or rotated, its coordinates change. To borrow from physics, they are located relative to a frame of reference.

There was one instance where a diagram (2D) of a vector was on a grid without axes, and it was coded as a graph since the gird was used to count the squares.

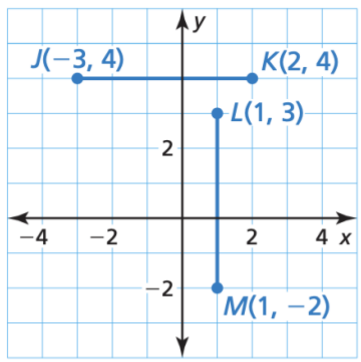


Figure 16. Use of cm graph paper. (p. 13, Larson & Boswell, 2015)

Figure 16 shows the first instance of a graph (G), where students are asked to graph four points and then to find the length of the segment. Student are not presented with how to plot points nor with how to coordinate between ordered pairs and a graph on a coordinate grid. Teachers as well textbook publishers may assume many prerequisite skills like graphing because high school students usually take geometry after at least a full year course of algebra, and a skill taught in elementary school.

In the following few pages of the textbook after the above figure both the midpoint formula and the distance formula are introduced. Slope is not introduced until chapter 3. Those three concepts are a great affordance on a coordinate grid and with ordered pairs, i.e., in coordinate geometry. Those formulas are common statements establishing segments being congruent, parallel, or perpendicular. The authors do introduce coordinate proofs (in chapter 5), and they are explicit about choosing variables and how these proofs can be general enough to prove properties of any figure of a specific type. Coordinate proofs/geometry are arguably the most complex activity in geometry, even without variables, because they use so many representations (WL, A, G, OP, N) and there is so much accounting for the ordered pairs and algebra.

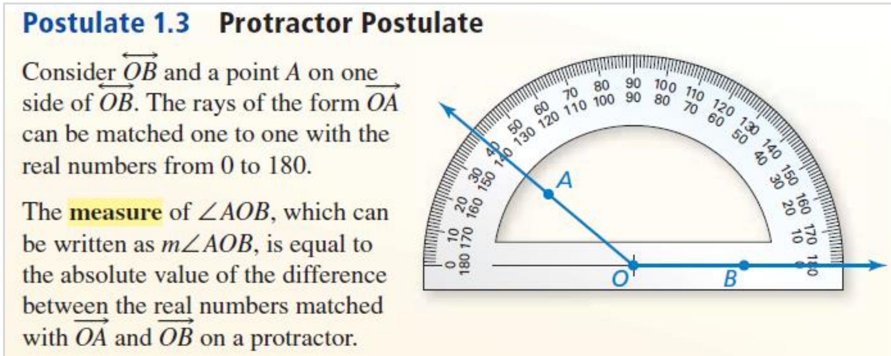
Numbers

In the following paragraphs, I describe various aspects of numbers, their frequency and how they are introduced.

Description. Through the thematic analysis, I chose to focus my analysis on how the text using numbers represented three objects or attributes of objects, namely distance, rotation/angle measure, and a count of objects.

Frequency. Numbers are the second most common representation used in the textbook, and they account for 14.8% (0.6% in DGE) of initial representations in coordination and 15.6% (1.0% in DGE) of the subsequent ones. Distance is defined with the ruler postulate above and the implied number system. It is implied that every unit is the same distance away, e.g., 2 is the same distance away from 1 as it is from 3.

Introduction. Numbers are first introduced as review of arithmetic with absolute value. Then, they appear as another review of units, e.g., 1 foot = 12 inches. Yet, they appear in the main body of text as in an exploration, asking the student to draw a 6 inch segment. Although they appear earlier it is not until the ruler and protractor postulates that numbers come to have the meaning that they represent in most of the book.



Postulate 1.3 Protractor Postulate

Consider $\overleftrightarrow{OB}$ and a point A on one side of $\overleftrightarrow{OB}$. The rays of the form $\overrightarrow{OA}$ can be matched one to one with the real numbers from 0 to 180.

The **measure** of $\angle AOB$, which can be written as $m\angle AOB$, is equal to the absolute value of the difference between the real numbers matched with $\overrightarrow{OA}$ and $\overrightarrow{OB}$ on a protractor.

Figure 17. Postulate, angle measure. (p. 39, Larson & Boswell, 2015)

Rotation or angle measure is defined analogously to the ruler postulate, but instead of numbers being evenly distributed along a line segment, they are evenly distributed along a semicircle, a protractor (fig. 17). In this case, the endpoints of a segments are not used to define the measure, but rather the sides of the angle intersecting specific locations along the marked semicircle. To be explicit, the blue rays of the angle lie on top of the numbered hash marks along the protractor. Area and volume are concepts based on length; one unit of area is a square with one unit of length along its sides; volume is the same concept except with a cube. Finally, numbers are used to count objects like sides of polygons. These values (length, angle measure, and count) are introduced as demonstrated in figures 15 and 17, but the authors of the textbook do not mention that numbers primarily represent those three aspects.

It is important to point out that I did not code numbers in algebraic expressions, but there is overlap. A number could stand for a length, an angle measure (and concepts derived from them like area, volume, etc.), or a count (N), but once it was used in an expression or equation, I considered it algebra (A). I chose to do that because in an expression it becomes abstract, losing it meaning. It is difficult to make such a decision, because many of us might consider that number to have meaning even later in an algebraic treatment. For example, a problem might ask to find the value of x, if two complementary angles are x and 2x. In this case, we would write the following equation x+2x=90. At this point, it may still be clear that the 90 represents the total measure of two complementary angle, but as we begin algebraic treatment, i.e., 3x=90 and x=30, that meaning is often forgotten, irrelevant or both. For that reason, I decided to make a clear dividing line that when a number becomes or is a part of an algebraic expression, it is coded as algebra (A), not a number (N) that stands for a length, an angle measure, or a count. The authors do show this at time when they leave units like the degree symbol in the equation to show that those values have meaning in some other way besides a relationship in an algebraic equation.

On the other hand, if numbers were substituted into a formula as in the slope or distance formulas, I coded the arithmetic as a number rather than algebra. Most of these calculations contained short geometry symbols (Sy) like AB for length instead of algebraic variables like x, e.g., $m\angle \mathrm{A} + 9 = 40$, because they, like in this case $m\angle \mathrm{A}$, retain the meaning of angle measure and not a forgotten or irrelevant variable. We want student to think of the measure as a number while in algebra, we often want students to use x as a variable, being able to take on multiple values.

Algebra

In the following paragraphs, I describe various aspects of algebra, its frequency and how it is introduced. Algebra represents many different objects in the textbook.

Description. Algebra use mostly lower-case letters to represent unknowns or sometimes a variable representing a set of numbers. The first way is to calculate or apply a formula for some attribute of that locus of points, as in slope, perimeter, circumference, arclength, area, volume, etc. The second way, which is not used often in this textbook, is to display a locus of points like a line. The third way is the distance formula. The fourth way is to define relationship between or among segments and angles like triangle sum theorem, proportions in similarity, trigonometric ratios, special right triangles, the Pythagorean Theorem, etc. The fifth way is to set up and/or solve problems and exercises with an unknown, where, for example, segments or an angle measure are given as an algebraic expression like AB=2x-4. Finally, the sixth way to use algebra is to describe physical properties like density or financial modeling of interest, both of which only appear once. Algebra's most powerful affordance is to solve for an unknown, which is quite difficult in most other representations.

Frequency. Algebra accounts for 8.0% (0.03% in DGE) of initial representations in coordination and 7.1% (0.2% in DGE) of the subsequent ones.

Introduction. The first algebraic expression is introduced with the ruler postulate, i.e., $AB = |x_2 - x_1|$ in an implicit way. Yet, I think the most important aspect of the analysis is that algebra was never introduced explicitly for any of the uses I mentioned above. Moreover, it used very little for a locus of points, namely line and circle, and that is how it mostly studied in other research of representations.

First, algebra is used to calculate or apply a formula for some attribute of a locus of points, as in slope, perimeter, circumference, arclength, area, volume, etc. For example, slope tells us how a line is oriented to the axes and area tells us how many square units there are within some figure. Therefore, if we consider a locus of points like a rectangle to be the object, then area is simply some quantification of that figure. The authors of the book use color to shade in part of the plane inside figures to objectify area as we can see in the figure below.

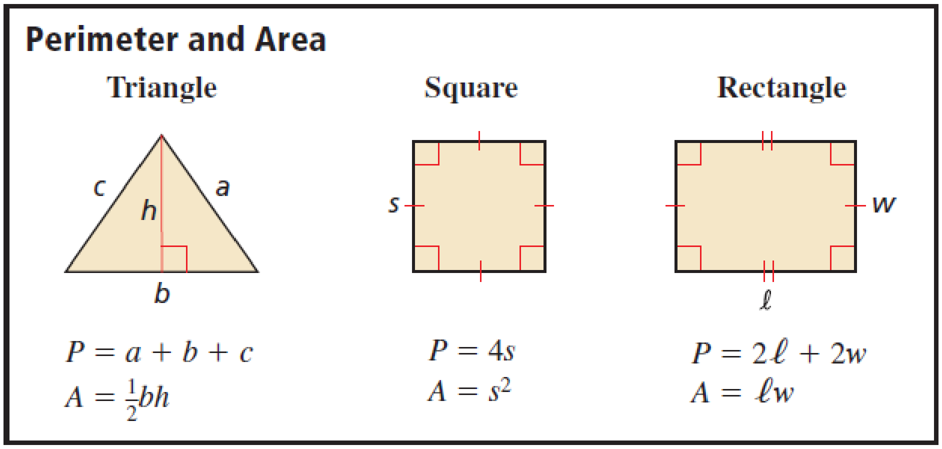


Figure 18. Algebra, diagram, and written language and concept of area. (p. 31, Larson & Boswell, 2015)

The perimeter formulas refer to the one-dimensional segments that constitute the sides, while the area formulas refer to the shaded two-dimensional parts of the plane within the figures.

Second, algebra often describes a locus of points like a line or a circle. In the book, there are only three types of objects described in this way, namely line, circle, and parabola. Parabola is outside the scope of my analysis because it located in “Additional Topics” on pages 721-728, just before the answers to exercises. As a result, I focus my analysis on the line and circle. A line is represented mostly in slope intercept form (y=mx+b) or in standard form (Ax+By=C). From those two forms, slope and the y-intercept are easily extracted, and the line can be graphed. A student can graph the line by using values for x and y to create ordered pairs that can be plotted; these ordered pairs can be presented in a table. In the entire book, not just the five chapters a table was not used once for that purpose. Equations (A) represented lines 43 times and ordered pairs (OP) 23 times. The other situation that algebra is used to describe a locus of points is with the circle. The book defines the “standard equation of a circle” as $(x-h)^2+(y-k)^2=r^2$, where (h, k) is the center, and r is the radius (p. 576, Larson & Boswell, 2015). Just as with the line, all the points on the circle are accounted for by the equation. There is only one section on this topic (5 pages), with a total of 8 representations of the equation or its application. Students are asked to write equations, knowing the center and the radius, or knowing the center and a point on the circle; they are also asked to graph a circle given an equation.

The third way is the distance formula. Although the distance formula could be considered in the category above along with slope, there are two differences. Like slope the distance formula uses ordered pairs, but, unlike slope, the distance formula requires ordered pairs.

The fourth way is to define relationship between or among segments and angles like triangle sum theorem $(x + y + z = 180°)$, proportions in similarity, trigonometric ratios, special right triangles $(x, \sqrt{3}x, 2x)$, etc. like the Pythagorean Theorem.

The fifth way is to set up and/or solve problems and exercises with an unknown, which is different from the above use of *defining* relationship. Many exercises require student to solve for an unknown. For example, a problem may have two congruent segments; one labeled with a length of 2x-3 and the other labeled x+1. Students are to create an equation and solve for x. In the instance above, once the equation, 2x-3=x+1, is created the idea of length and congruent segments is not relevant, or, at least, not the focus of the student. A new mathematical object is created with different rules, constraints, and affordances. It is easier to manipulate, but it is also abstract.

Finally, the sixth way to use algebra is to describe physical properties like density or financial modeling of interest. The authors use density when volume is introduced, probably to connect these abstract equations to the real world. Also, the interest formula, $p(r + 1) = n$, was used in a worked example where algebraic properties were introduced. It seemed to be used to connect the properties to the real world.

Thus, algebra and its related representations, namely graph, and ordered pair, are used to represent different types of objects. Objects may be loci of points like lines and circles, which we typically considered as objects. They may be aspects of those like general formulas for area, volume, perimeter, etc. Algebra may simply be used as a mechanism to prove one equation from another, showing how coherent mathematics is, or it may be used to define relationships between segment or angles. Once again, the authors do not explicitly discuss these affordances.

Tables

Description. Tables are most often used to organized data or information into rows and columns. Usually there are at least two lines separating at a minimum two rows and two columns, both of which, especially the columns, are usually labeled. The lines are usually thin and black like in figure 19. When they are used in two-column proofs, they just have two lines intersecting, a horizontal line directly below the headings of 'statements' and 'reasons,' and a vertical one dividing the two columns.

Frequency. Tables account for 2.4% of initial representations in coordination and 2.2% of the subsequent ones.

Number of sides	Type of polygon
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon

Figure 19. Table (p. 66, Larson & Boswell, 2015).

Introduction. Figure 19 is the first table in the book. The authors of the book do explain the representation; they refer to it with these words "… as shown in the table." As discussed in the extant literature, tables organize information well and in some order. In this case, polygons are ordered by the number of sides. The most common use of table is in the two-column proofs.

Arguably the most important affordance of tables is to be able to view information organized in a specific way so that one can scan both horizontally and vertically. For example, in figure 19, we can scan the number of sides and names of polygons vertically. That often helps in finding patterns like the fact that polygons starting with pentagon end with ‘gon.’ With proofs the statement column is usually scanned more often vertically to connect statements logically, while we scan horizontally to the right, i.e., to the reason column, for justification of the statement. In tables with numbers such as the one listing the number of sides and the sum of the angle measures. These affordances are not explicitly explained in the textbook.

Tables are not used to represent a list of ordered pairs that could later be graphed or converted into an equation; they are mostly used to organize two-column proofs. There are four instances that tables are used to directly coordinate with algebra, three of which are to show that trigonometric ratios depend on the angle not on the lengths of the sides. The other instance is when students convert the number of sides of a polygon and the corresponding sum of the interior angles into a function, $f(n)=180(n-2)$. Thus, I do not focus on tables of values as a representation as it is typically discussed in mathematics education research, namely as a list of coordinates that are coordinated among a graph and/or an algebraic equation.

Coordination of Representations and Mechanisms

I have shown how representations are introduced in the section above. In this section I focus on the next two research questions: which combinations of representations are coordinated and what the mechanisms are. This study has answered the second research question and found that there some individual representations that are very common like written language (WL), diagrams (D), numbers (N), and some in combination like written language and short geometry symbols (WLSy), written language and numbers (WLN), and numbers and ordered pairs (NOP). This study has answered the third research question and found that there are a few common mechanisms to aid coordination like names of points, numbers, algebra, and textbook gestures like color, font change, arrows, etc. I have coded 3,967 individual instances of coordination between or among various representations. The descriptive statistics of the most common representations or combinations of representations along with some examples are shown on the table below. To be clear, the table does not show the frequency of representations in the textbook; some representations like diagrams were counted multiple times because other representations referred to them multiple times. Initial representation means that I considered that representation leading to another, the subsequent representation. Table 3 lists combinations of representations that appeared more than 100 times in the coding. The statistics of the combinations of codes helped me choose which coordination of representation combinations to analyze in more detail; it is more pragmatic to study those combinations of representations that students are most likely to come across. Each combination is referred to by lower case letters in italics, e.g., *a, b, c*, and they will be placed before subsections that are related or the same as the common coordinations in table 3. Because some combinations like WL to WL are more frequent, I discuss them in more detail and with more examples.

Table 3 shows coding where all sub-codes for written language (WIP, WDP, WDM, WDR, WDD, WIR, and student made responses) were group in a broader theme of written language (WL) and where diagrams (2D and 3D), constructions (C), graphs (G), their equivalents in dynamic geometry environment (EG, ED2), and those made by students were grouped into one theme of diagrams (D). Although I grouped them into broader themes for the statistics, for the qualitative descriptions of coordination, I use the more detailed codes. Finally, I focus on textbook gestures when they are part of figures, but I do not include them when they are not because they are typically a coordination mechanism.

Table 5 *Descriptive statistics of the frequency of representations during coordination*

Representation Combination	Initial	Subsequent	Total	Example (frequency)
WL	1521	797	2318	*a*. WL → WL (471), *b*. WL → D (257)
D	258	844	1102	*c*. NOP → D (131), *d*. D → WL (86)
WLSy	471	154	625	*e*. WLSy → D (231), *f*. WL → WLSy (59)
N	141	243	384	*g*. NA → N (55), *h*. N → N (51)
WLN	181	182	363	*i*. WL → WLN (46), N → WLN (36)
NOP	204	123	327	*j*. NOP → NOP (42), OPA → NOP (28)
DN	78	121	199	*k*. DN → NSy (21), DN → N (16)
NSy	98	101	199	*l*. D → NSy (15), DNP → NSy (12)
A	118	78	196	*m*. A → WLA (27), A → A (21)
DP		179	179	*p*. WL → DP (89), WLSy → DP (49)
NA	115	61	176	*q*. NA → WLN (15), NA → A (10)
WLA	76	98	174	*r*. WL → WLA (44), A → WLA (27)
Sy	90	55	145	*s*. Sy → D (30), Sy → WL (27)
WLD		118	118	*t*. WL → WLD (95)
ASy	67	40	107	*u*. ASy → WL (25), ASy → ASy (13)

I have chosen to discuss the following coordinations because they appear often in the textbook. The following examples are not exactly what I have as the most common coordinations, but I tried to choose a representative sample that was not too large and cover most of the common coordinations. They are numbered below and in the following paragraphs.

1. WDP → WDP, WDP → WIP, which are variations of *a.* WL→ WL
2. WDP+Sy → P+2D, which is similar to *p* WLSy → DP
3. WDP+Sy → D, which is a variation of *b.* WL → D*, d.* D → WL*, e*. WLSy → D
4. Sy → WDP+ Sy, which is *f.* Sy → WLSy
5. WIP+Sy → 2D, which is *e.* WLSy → D*,* and similar to *s.* Sy → D
6. Sy+N→2D+N, which is similar to *l.* D → NSy
7. WIP+Sy→2D, which is another *e,* Similar to *t.* WL→WLD
8. WIP→2D, which is another variation of *b*. WL→D
9. 2D→Sy+N, which is similar to *l*. D→NSy
10. 2D+N→Sy+N+A, which is similar to *k,* DN→NSy, and *u*, ASy→WL

11. N→N, which is *h*
12. WDP+N→G+N+OP, similar to *c.* NOP→D
13. WL+N→WL+N, similar to *i* and *q*
14. A+OP→N+OP and N+OP→N+OP, *j*
15. WIP→C+PT+Gst

1. WL → WL

Written language was the most common representation in both the initial representations and the subsequent representations. Written language to written language (1. WL → WL, more specifically WDP → WDP) was the most common coordination. The two sentences below demonstrate two concepts in language that are in proximity to each other. A reader would read the two clauses in a linear fashion, one after the other.

> A *straightedge* is a tool that you can use to draw a straight line. An example of a straightedge is a ruler. (p. 11, Larson & Boswell, 2015)

Both clauses are declarative written language (WDP), and '*straightedge*' is emphasized with italics, a textbook gesture (GST). The gesture seems to underscore the word 'straightedge,' in a similar way a teacher would point to it with her finger as gesture emphasizing the word's importance. Other important vocabulary words have a yellow background, and the font is in bold. With the italics, a textbook gesture (Gst), we coordinate these two sentences focusing not only on *straightedge* because it is the subject of the first sentence and part of the noun phrase of the second one, but also because it is emphasized through a gesture. Gestures, repeated use of words, and sentence structure (like definitions) aid in connecting representations through emphasis and an easy way to refer to an object when scanning for it. This is a mechanism where a student can retain the word in their working memory for longer period of time while reading and thus help coordinate between the ideas of the two sentences.

The example above also displays some mechanisms within written language. The first one is the gesture of italics discussed above, which helps emphasize the focus of both sentences, i.e., the straightedge. The second is using the same word, a synonym, an example, or an instance to connect one sentence to another. In the example above, straightedge is repeated, and 'a ruler' is a specific instance of 'a tool.' A third mechanism in the example above is the form used for definitions that are numerous in geometry: "[specific word, e.g., straightedge] is a [general word, e.g., tool] that [description, e.g., draws lines]." This fill-in-the-blank construction for definitions (… is a … that …) helps students coordinate definitions in various situations. Gestures, repeated use of words, sentence structure (like definitions) aid in connecting representations through emphasis and an easy way to refer to an object when scanning for it.

Another common 1. WL to WL coordination is when students need to answer a question or follow some instruction. The following is a question from the textbook that we can use as an example: "3. How can you find the perimeter and area of a polygon in a coordinate plane?" (p. 29, Larson & Boswell, 2015). To answer the first part, we can expect a student would say something like, "First use the distance formula to find the lengths of the sides, which you then add together to find the perimeter." The student reads the instructions, which are imperative written language of pure math (WIP), and then responded in declarative written language (M.WDP – to remind the reader 'M.' means the student *made* the representation).

Another example of 1. WL to WL coordination presented is when there are instructions (WIP) given consecutively:

> Find the values of the indicated variables. Do not use a protractor to measure the angles (p. 47, Larson & Boswell, 2015).

In this instance, a student would probably read the first sentence and then the second one (WIP → WIP). Reading the first sentence a student may consider using a protractor to find the measures, but that strategy is restrained by the second sentence. Had that second sentence not been there, the student would most likely move on to the diagram that is associated with the "indicated variables." Moreover, there is a treatment of the words "Find the values of the indicated variable" to "measure the angles;" here a generalized 'Find … variables' is transformed (treated) into a specific 'measure the angles.' For an expert that connection is immediate, but for a student learning this topic it may be difficult to make that connection.

Conditional statements are also a mechanism by which readers can coordinate written language. Conditional statements are introduced as simple statements, e.g., if A, then, B. They are used in postulates, in theorems, later, in proofs and in explaining how to solve problems. They are the logical link between many mathematical ideas, where one idea necessarily yields another idea. They are most commonly joined logically through a syllogism (if A, then B; A; ergo, B).

Although they are usually composed of two clauses, contracted and more complex versions are often used in the textbook. For example, the second and third sentences on figure 20 could be written in a wordier way so that they would appear more like a syllogism ($p \rightarrow q$; p; $\therefore$ q): "If angles are a linear pair, then they are supplementary (Linear Pair Postulate); ∠5 and ∠6 are a linear pair; therefore, ∠5 and ∠6 are supplementary." In figure 20, the syllogisms are overlapping, which is more concise, but it may also be more difficult to separate those ideas for the learner. Because language is so flexible and complex, the mechanisms by which these ideas and objects are coordinated are difficult to uncover.

Figure 20 incorporates more than just coordination of written language (WL) and diagrams (D), but I will use it explain the very common WL → D coordination, and those coordinations labeled with letters *b, d, e, f* and *p* on table 3—I think the readers of this dissertation can think abstractly enough to ignore the picture of the airport when I discuss only the diagram—Therefore, I will discuss many different combinations of written language (WL), short geometry symbols (Sy), physical objects (P), and diagrams (D).

2. WDP+Sy → P+2D

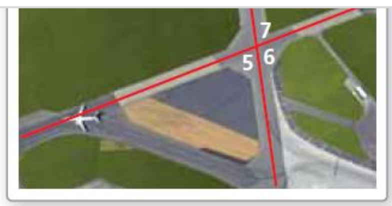
The picture of airport runways with the red lines and angles (2D) is an example of how physical objects (P) can be used to represent mathematical objects. The diagram overlaps the runways, which are clearly wider than a line, and for that reason a thinner line must be drawn somewhere on the broad runway; the authors chose to draw it in the middle of the runway.

STUDY TIP

In paragraph proofs, *transitional words* such as *so*, *then*, and *therefore* help make the logic clear.

Given $\angle 5$ and $\angle 7$ are vertical angles.

Prove $\angle 5 \cong \angle 7$



Paragraph Proof

$\angle 5$ and $\angle 7$ are vertical angles formed by intersecting lines. As shown in the diagram, $\angle 5$ and $\angle 6$ are a linear pair, and $\angle 6$ and $\angle 7$ are a linear pair. Then, by the Linear Pair Postulate, $\angle 5$ and $\angle 6$ are supplementary and $\angle 6$ and $\angle 7$ are supplementary. So, by the Congruent Supplements Theorem, $\angle 5 \cong \angle 7$.

Figure 20. WMP, syllogism, Sy, Picture of Physical object (PP). (p. 110, Larson & Boswell, 2015).

This runway example shows the problem with using physical objects (P) to represent mathematical ones; it is more difficult to discuss the abstract features of geometry when coordinating with physical objects than diagrams alone.

As far as coordination of language with physical objects, there are different variations. Some are like the one in figure 20 where no language describes the picture of the runway; only the diagram (2D) is described. There are also situations where only the physical aspects are described and not any geometric ones. Finally, there are situations where no picture is present, but physical objects are described.

3. WDP+Sy → 2D

Figure 20 also exemplifies coordination of written language (WL), short geometry symbols (Sy), and diagrams (2D). Because WLSy includes WL, this example explains both coordinations. In figure 20, written pure declarative language (WDP) describes the diagram (2D) when the authors of the textbook use vertical angles to describe angles 5 and 7. By this point in the textbook, students are familiar with the term 'vertical angles,' meaning they should have a schema in their long-term memory recognizing the situation. Therefore, language (WDP) begins the recall of the term and associated definition (and maybe diagram), and the diagram (2D) should confirm that recalled term with the specific names ∠5 and ∠7 in short geometry symbols (Sy). Coordinations of language and diagrams is often aided with symbols (Sy). In this case numerals (5 and 7) help the reader of the textbook coordinate the idea of vertical angles with the angles on the diagram. Other diagrams (2D), written language (WL), and symbols (Sy) use points and color to aid in the coordination. Thus, the most common mechanisms in coordination between written language and diagrams are the transferring of numerals, points, color, and symbols themselves.

4. Sy → WDP+Sy

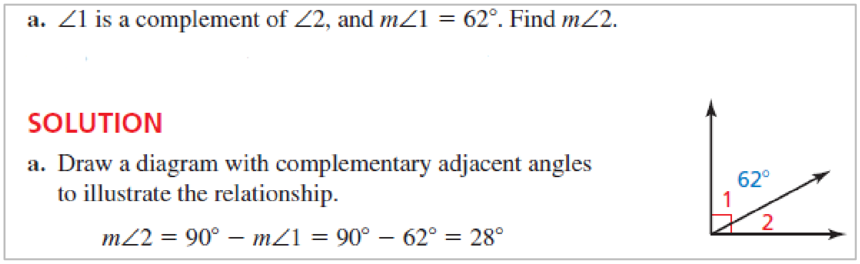
Figure 21 is an example of the complexity of how definitions written in declarative sentences (WDP) are used, especially in proofs, and how they are coordinated with short geometry symbols (Sy) in a table (T) in a proof—I use this figure because it is as simple a use of a definition as I found in the textbook, it shows how tables aid, and, although present, the diagram is not needed.

2. $\overline{ST} \cong \overline{TU}$	2. Definition of midpoint
3. $ST = TU$	3. Definition of congruent segments

Figure 21. Use of definitions and table (p. 100, Larson & Boswell, 2015).

Although the definition of terms is a single sentence (x is … that …), not two independent clauses (my unit of analysis) needed for coordination, the use of definitions in proofs usually requires a coordination. In proofs, an instance of the defined concept, e.g., congruent segments, would be in the 'statement' column (on the left), and a reference to the definition in the 'reason' column (on the right). For example, the third line of the proof in figure 21 uses the definition of congruent segments, which is "segments that have the same length" (p. 13, Larson & Boswell, 2015). Although it seems simple for an expert, many notions are at play in the situation. First, the reader of this proof must recall the definition of congruent segments, i.e., that the definition must enter the working memory. Second, the definition must be separated into the 'congruent segments' idea and the 'same length' idea. Third, based on that separation the reader can apply that knowledge to $\overline{ST} \cong \overline{TU}$ (congruent segment idea) to determine that $ST = TU$ (same length idea). Therefore, although definitions are in the form "A is a B that [has some property]," an independent clause with a relative clause, language easily allows A, B, and the property to be separated into distinct ideas as in the proof above.

The table and the short geometry symbols are part of the mechanism that helps coordinate $\overline{ST} \cong \overline{TU}$ and $ST = TU$. The table allows the student glance up to see $\overline{ST} \cong \overline{TU}$ (Sy) and notice the change to $ST = TU$ (Sy) while seeing that it is based on the definition of congruent segments (WL) in the column on the right. Notice that the two changes between the two statements are that the congruent symbol changes to an equal sign and that the line segment symbols above ST and TU disappear. I interpret this change provides a mechanism where we notice an abstraction of focusing on the length of a segment rather than the segment itself; in a way we are stripping the segment of all its point and focusing on the distance between the endpoints, e.g., S and T.


a. ∠1 is a complement of ∠2, and $m\angle 1 = 62°$. Find $m\angle 2$.

SOLUTION

a. Draw a diagram with complementary adjacent angles to illustrate the relationship.

$$m\angle 2 = 90° - m\angle 1 = 90° - 62° = 28°$$

Figure 22. Example of A, D, Sy, and N combination (p. 49, Larson & Boswell, 2015).

With figure 22, I would like to demonstrate a series of coordinations, labeled 5 to 13 in the subheadings, so that the reader can see a fuller picture of coordination and mechanism in a worked example. In this case, there are 5 representations (A, D, WL, N, and Sy) coordinated. In this worked example, I coded for six different instances of coordination between or among representations: the four independent clauses are coordinated with the diagram, two instances of diagram to algebra, and one treatment of number arithmetic.

5. WIP+Sy → 2D

In the first clause in the directions a student is told that one angle is a complement of another. A student may coordinate that with the diagram below or continue reading. I assume that he or she would glance at the diagram. Therefore, ∠1 and ∠2 (Sy) would be coordinated with the diagram (2D). As I was reading through the worked example, I thought about the auditory loop saying, “angles, complementary… angles, complementary.” Soon after that, I was thinking of “complementary, 90 degrees …” Then, “the two angles, the perpendicular symbol, that they were adjacent.” The perpendicular symbol (Gst) on the diagram is the part of the mechanism the authors use to show that ∠1 and ∠2 add up to 90°. They could have kept them separate and written on the side that they were complementary (a different mechanism).

6. Sy+N→2D+N

The second clause is a short geometric symbol, $m\angle 1$ (Sy), and its angle measure as a number, 62° (N). The cursive *m* in front of the angle symbol is not an iconic sign but rather a symbolic sign, meaning it is an arbitrary symbol connected to another arbitrarily named concept of *measurement*. The value of 62° (N) of the measure is on the diagram (2D) slightly further away than the name of the angle '1.' The measure of the angle, being placed in the interior of the angles, could be confusing, but the author uses colors, blue for the angle measure and red for the name of the angle, to differentiate the two, thus aiding in coordination. As we have seen color is used often as a mechanism for coordination.

7. WIP+Sy→2D

At the end of the directions, the goal of the worked example is provided, "Find $m\angle 2$." This time written language, i.e., "Find" (WIP), is combined with the short geometry symbol of $m\angle 2$ (Sy). It can be coordinated with the angle name without a measure on the diagram (2D) unlike the coordination above. To emphasize that subtle point, a student has one mechanism to coordinate $\angle 2$ with the diagram, whereas $\angle 1$ has two, namely the name ($\angle 1$) and the measure (62°).

8. WIP→2D

In the first statement of the solution to the worked example, students are asked in written language (WIP) to "[d]raw a diagram with complementary adjacent angles to illustrate the relationship." First, this diagram already exists since it is a *worked* example. Yet, the ideal student would probably follow the process as the author of the book models it. Second, the author is explicit about converting the information in the directions, i.e., 'complementary,' into the diagram (2D) representation. Third, the author chooses to represent 'complementary' by using adjacent angles, which slightly adds to the initial description by making the angles adjacent rather than non-adjacent. The use of adjacent might be helpful, though, because it uses the diagram's symbol for a right angle. If they were not adjacent, the diagram would require other information like text or an equation ($90° = m\angle 1 + m\angle 2$) which would increase the number of representations, and, therefore, the cognitive load. Also, notice that in the diagram representation, we must choose whether we sketch adjacent or non-adjacent angles, while in written language we do not need to specify it.

9. 2D→Sy+N

A student would coordinate the equation with a diagram. The first three terms of the equation and the diagram are one unit of analysis. For such a short expression, it is packed with information from the diagram that is not explicitly stated. First, reading the diagram (2D) a student would see that angle 1 and angle 2 form a right angle with the aid of the little red square. Second, combining that fact and the angle addition postulate, the authors state that the $m\angle 2$ (Sy) is equal to the difference of 90° (N) and the measure of $m\angle 1$ (Sy). The expression above could have been written with each equal sign on a separate line and a step added to explain the first part of the equation like so:

1. $m\angle 1 + m\angle 2 = 90°$ Angle Addition Postulate. (I added this line for completeness)
2. $m\angle 2 = 90° - m\angle 1$ Subtract $m\angle 1$ from both sides.
3. $= 90° - 62°$ Substitute for $m\angle 1$.
4. $= 28°$ Add.

The more contracted way of $m\angle 2 = 90° - m\angle 1 = 90° - 62° = 28°$ assumes that students can read that $m\angle 1$ subtracted from 90° is equal to $m\angle 2$ from the diagram. It requires the reader of the book associate the situation with the Angle Addition Postulate, i.e., the postulate and its related concepts are already stored in the student's long-term memory for easy access and manipulation internally. The mechanisms of algebra are often very procedural and are based on schema of how to deal with various algebraic situations and solving for the unknown.

10. 2D+N→Sy+N+A

The next part of the equation $90°\text{-}m\angle 1=90°\text{-}62°$ is another unit of analysis. It is a substitution of $62°$ for $m\angle 1$. A student can retrieve this information from two places, the diagram or the original directions. I assume that a student would retrieve the information from the diagram in this case because the authors provide a diagram that is closer than the original directions and information. Again, as in many instances above, a student uses the measure of the angle (N) or the numeral with a short geometry symbol (Sy) to coordinate the representations and notice that 62° was substituted for $m\angle 1$.

11. N→N

The final coordination of representations is simplifying a subtraction of 90° - 62° and presenting it as 28°. I decided not to code this for algebra (A) in this coordination for three reasons. The first reason is that numbers (N) are substituted for a geometry symbol (Sy). Secondly, the degree symbol is not removed, allowing a reader to continuously coordinate with the angle measures (62°), and not treating those numerals abstractly in an algebraic expression. Thirdly, there is no algebraic manipulation, only arithmetic (90° - 62°).

As far as mathematical objects, there are three typical ones, namely $\angle 1$, $\angle 2$, and the right angle. The right angle was not described in the directions of the worked example; it was added later in the diagram. Then, there was a conversion of the objects from the diagram to an equation with unknowns (A). The algebraic expression, $m\angle 2=90°\text{-}m\angle 1$, is a new overarching object that encapsulates the relationship among the angles in the diagram, namely the fact that they are complements. The most important difference between the diagram and the equation is that only the objects' measures are used in the equation, an abstraction. The representations of the measures on the diagram can be confusing, and the equation is ready for a calculation, a treatment (90°-62°) with a minor conversion (62° substituted). First, the right angle does not have a name so only its measure (90°) in number form (N) appears in the equation. Second, ∠1 had two possibilities, namely $m\angle 1$ (Sy) or 62° (N), and third, ∠2 was not known so it could only be written as $m\angle 2$. At this point the objects are either numbers (N) or geometric symbols (Sy), and the conversion is to a certain extent completed. The objects are not completely converted, because the reader of the exercise would most likely look back and forth at the diagram to check where the symbols, numbers, and the relationship came from. When a student finally substitutes 62° (N) for $m\angle 1$ (Sy), he or she is finished analyzing the objects in the diagram.

In Figure 23, a graph (G) is used to display some of the information in a worked example. To remind the reader, the number of representations for table 3 was reduced; graph (G), construction (C), 2D, and 3D are grouped under one representation diagrams (D).

Determine which of the lines are parallel and which of the lines are perpendicular.

SOLUTION

Find the slope of each line.

Line a: $m = \frac{3-2}{0-(-3)} = \frac{1}{3}$

Line b: $m = \frac{0-(-1)}{2-0} = \frac{1}{2}$

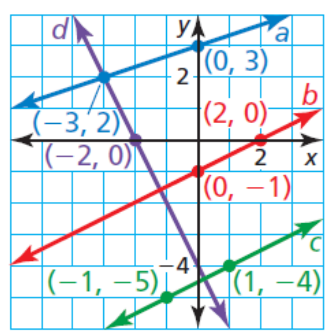


Figure 23. Example of WIP, G, OP, and N combination (p. 157, Larson & Boswell, 2015).

We can notice many features in figure 23: use of color for different objects, ordered pairs, points, grid, short segment (gesture) that point to a point (see point at (-3, 2)). Other graphs have other aspects: equations, variables in ordered pairs, angle names and measures. Of the 84 graphs, 40 had, as in figure 23, ordered pairs with numbers (NOP) representing the position of the points relative to the axes, and 9 of those had equations (A). Numbers (N) and ordered pairs (OP) often appear together. I coded the values of ordered pairs as numbers (N) because they represent the distance of a point from an axis. Graph representations with variables occurred 7 times, and they were used to define transformations and the circle, and one was used in a worked example. Figure 23 and the next two paragraphs explain how ordered pair with numbers (NOP) were coordinated with written language and arithmetic of numbers. Figure 24 and the next two paragraphs explain algebra and ordered pair (AOP) coordination.

12. WDP+N→G+N+OP

As in figure 23, numbers (N) and ordered pairs (OP) are often coordinated. In this case, 'Line *a*' (WL) and the substituted x- and y-values (N) would be coordinated with the graph (G) and the ordered pairs (NOP). Line *a* is not only labeled with the letter '*a*' on the graph but it is also colored blue for easier coordination. Again, such mechanisms are used often in the book, especially when a concept is being explained for the first or second time. A student would need to have the schema of slope (change of y divided by change in x) well established to follow the worked example, or he or she would need to look back at the formula. In this figure, the student would look back and forth possibly a few times to see that the y-values are 3 and 2 (numerator), and that the x-values (denominator) are 0 and -3. Because we are comparing slopes of various lines to see if they are parallel the lines are differentiated by color; when slope was reviewed earlier in the textbook (p. 123) the change in the x- and y-values was color-coded in a worked example.

13. WDP+N→WDP+N

After the slopes of lines a and b are calculated, a student can compare the values 1/3 and ½ to determine whether the lines are parallel. At this point, the student would look at "line a

(WL) … 1/3 (N)" and "line b (WL) … ½ (N)." Noticing that they are different values, the student could decide that they are not parallel. The mechanism that allows for easy comparison is that the slopes are aligned vertically the way a table is organized. A student can scan across to compare the line name with the slope, and vertically to compare the values of the slope.

The following example shows a typical transformation on a coordinate grid with a rule.

Use the coordinate rule for a 270° rotation to find the coordinates of the vertices of the image. Then graph quadrilateral *RSTU* and its image.

$(a, b) \rightarrow (b, -a)$

$R(3, 1) \rightarrow R'(1, -3)$

$S(5, 1) \rightarrow S'(1, -5)$

$T(5, -3) \rightarrow T'(-3, -5)$

$U(2, -1) \rightarrow U'(-1, -2)$

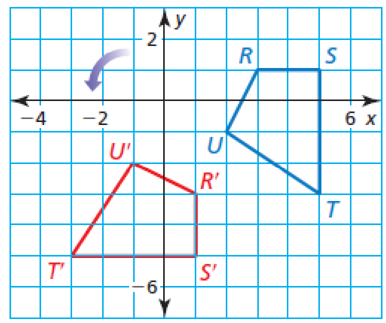


Figure 24. Example of WIP, G, OP, and N combination (p. 157, Larson & Boswell, 2015).

14. A+OP→N+OP

Initially, a reader would read the instructions (WIP) and look at the graph (G), but he or she would eventually look at the coordinate rule for rotation, which is used often in the chapter on transformations. After familiarizing oneself with the rule, which is in the form of an ordered pair (OP) and algebra (A), the reader would look at individual points. Again, color and capital letters for points are a mechanism that aids in coordination between the graph and the ordered pairs. Yet, in the current coordination, the location in the ordered pair and the variables (*b*, -*a*) and the minus on *a* (A) provide an algorithm of how the following 4 points are coordinated with their points on the image.

Number (N) and ordered pair (OP) are common representations in the textbook. In the figure above, right after seeing the rule for rotation, the student would probably coordinate the original ordered pairs with their images, e.g., R(3, 1) with R′(1, -3). The algebra/ordered pair rule, (a, b) → (a, -b), might be referred to or it might be stored in the student's working memory, and be reinforced with the four examples of ordered pairs being transformed. After, or possibly during, the application of the rule to the four points of RSTU, the student would refer to the graph (G) with its textbook gestures (Gst). The two textbook gestures are the purple arrow showing that the figure is rotated and the color-coding of the points.

In the next few paragraphs, I would like to describe the coordination of written language , construction (C), and physical tools (PT). To remind the reader, constructions are ultimately diagrams which are drawn to scale and where measurement tools can be used to compare length and angle measures, among other things. As typical, the language (WL) seems to lead the coordination among itself, geometry symbols (Sy), the construction (C), physical tools (PT), and textbook gestures.

15. WIP→C+PT+Gst

In figure 25, a construction of a perpendicular bisector, we can see as written language guides the reader to what is happening in the image, i.e., a compass (PT) is used to construct (C) an arc.

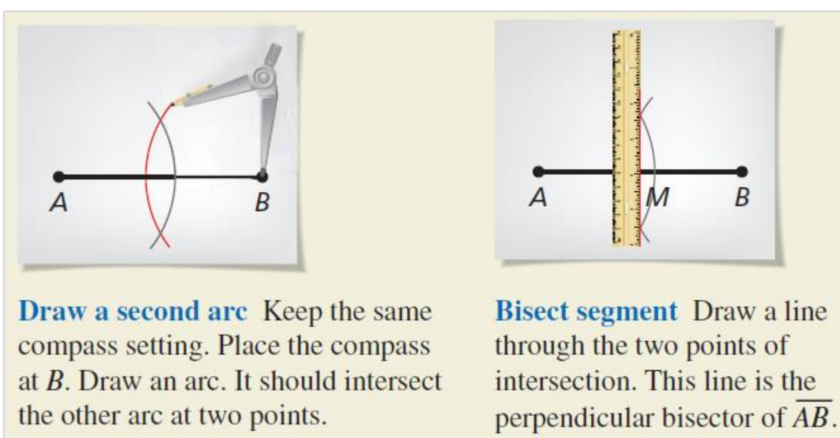


Figure 25. Example of C, WL, PT, and Sy combination (p. 149, Larson & Boswell, 2015).

First, “Keep the same compass setting” is coordinated with the radii of the two arcs. The distance between the pencil and the metal tip is the primary concept use, and it shows that points are equidistant or that segments are congruent. In this case, the important four congruent segments, which are not constructed, are between the two labeled points (A and B) and the unlabeled points (intersections of red and gray arcs). The intersection of the arcs, without small dots, is a common representation of points in constructions. If the four segments were drawn, there would be a rhombus, which would allow us to prove that the construction is of a perpendicular bisector. Although the segments are not drawn, the representation gives students the affordance of measuring them, and any segments from points A and B to verify the property that all points on the perpendicular bisector are equidistant from the endpoints. “Place the compass at B” is coordinated with point B on the construction and the compass, a physical tool (PT).

The new arc is red, a textbook gesture that is different from the gray arc (constructed in a previous step). The gesture of the red color (Gst) focuses the attention on what is constructed in that step, aiding in coordination between “Draw an arc” and the new red arc on the construction (C). In these directions short geometry symbols (Sy) like $\overline{AB}$ are often used; the most common ones are segments, followed by angles.

Common Mathematical Objects

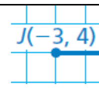
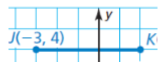
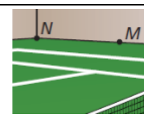
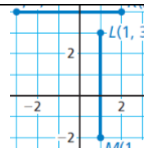
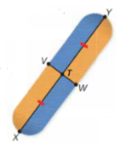
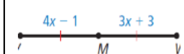
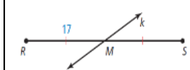
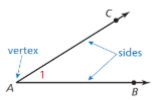
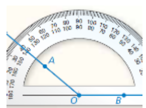
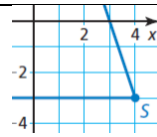
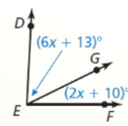
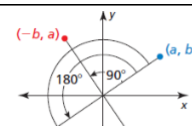
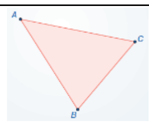
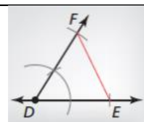
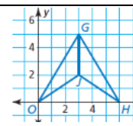
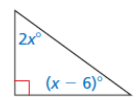
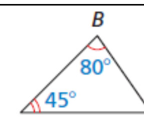
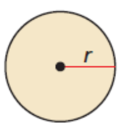
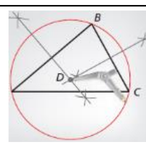
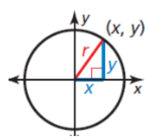
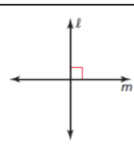
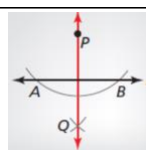
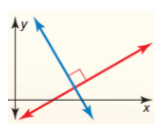
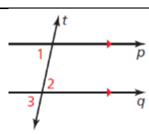
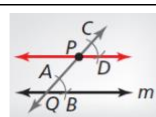
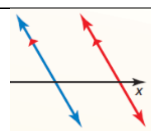
After reviewing the various common coordinations, it is worthwhile to point out the common mathematical objects. That is the point of coordination to learn or make clearer some aspect of an object the original representation could not do fully. There are many mathematical objects presented in the textbook, but there are a few that appear more frequently than the others. The most common typical objects that are loci of points are lines, angles, polygons, circles, and solids; objects related to polygons like angle bisectors or midsegment and related to circles like radii and tangents are included. I would like to devote this section describing these common objects, some of which are presented on table 4 below. Other objects that are not loci of points are properties or aspects of the above-mentioned typical objects, and the most common ones are length, angle measure, concepts derived from them (e.g., area or volume), or their generalizations (e.g., angles in a triangle, a+b+c=180). Oftentimes, the typical objects are defined by length or angle measure, which are typically numbers, but are not always expressed as numbers (e.g., angle measures are irrelevant in an angle bisector as long as the two angles formed are congruent). It helps to make this distinction between types of objects, especially when discussing coordination among the representations, because some types of objects appear more often as one representation rather than another.

One example of a typical object is a line, which is undefined, but it is often describe by three qualities: infinite in both directions, straight (no curves), and no thickness. A part of a line that has two endpoints is a segment, and it has two of the line's qualities, namely straightness and no thickness. Instead of being infinite, a segment has gained the quality of measurable length. Representations of a segment stress different aspects (see table below). For example, in language (WL), geometry symbols (Sy), and diagram (2D), the segment's quality of a set of points is stressed. Actually, it is more complicated than that; language can express many aspects, in contrast to number, which for the majority of the instances represents the non-locus objects like length, angle measure, and count. Algebra (A) and ordered pairs mostly represented non-locus-based objects. In the textbook, algebra was most often used in finding length, angle measure, slope, etc., and not in defining a set of points. Ordered pairs are used to display distances from the two axes, and rarely as a list of points that define some figure, and if they do try to define a figure, they do not define the whole figure (e.g., the coordinates of the three vertices of a triangle do not define all the points of the sides of a triangle).

A few very common objects are listed in the table below—the overarching representation of a table is useful in this situation because we can scan horizontally to compare the object and vertically to compare the representation. Although the instances of representations across the table do not always correspond, they do show how that object is represented in a specific representation.

On diagrams (D), measurement is displayed in a variety of ways. On sketches (2D and 3D) numbers and algebra are often used to quantify length or degree measure. Also, gestures for parallel, perpendicular, and congruent either directly or through reasoning provide information about measurement. The perpendicular symbol gesture in the shape of a little square on diagrams—and the short geometry symbol (⊥)—directly provides information that the angle formed is 90°. Parallel symbol gestures on a diagram indirectly provide information that certain angles have the same measure or are supplementary, e.g., corresponding angles are congruent. On sketches (2D and 3D), measurements appear often in the form of number (N) or algebra (A). On sketches that overlap physical objects (PP), numbers (N) appeared often, perpendicular and congruent symbols (Gst) less, and parallel symbols rarely. On graphs, physical objects also appear in the form of coordinates (often), length (less); algebra only appears to describe the locus of points of lines, not as measurement. On the other hand, measurement rarely appears in construction (C). The only time a number appeared on a construction was when a protractor was used. The symbolic gesture for perpendicular appeared once, and it was rather irrelevant to the lesson the construction was trying to convey. Although constructions are drawn to scale, which allows for precise measuring, the authors ask only once in the constructions to measure a length.

Table 6. *Different representation of the same concept*

	WL	Sy	2D	C	G	P	A	OP	N
1	point A	--	A •	A •			(a,b) → (a+h,b+k)	(-3, 4)	3 mi west 4 mi north
2	segment AB	$\overline{AB}$					35 = GH+21	J(-3,4) K(2,4)	AB=8
3	congruent segments	$\overline{AB} \cong \overline{CD}$						J(-3,4) K(2,4); L(1,3), M(1,-2)	
4	angle	∠1 ∠BAC							Angles $m\angle A = 33.69°$ $m\angle B = 90°$ $m\angle C = 56.31°$
5	triangle (polygon)	△ABC						O(0,0) G(3,5) J(3,2)	
6	circle	⊙V					$x^2 + y^2 = r^2$	point (-5,6) is on a circle with center (-1,3)	radius: 7
7	perpen-dicular	$\ell \perp m$					$m_1 \cdot m_2 = -1$	l: (0,0),(3,6) m: (0,0),(-4,2)	90°, $^{\pi}/_{2}$
8	parallel	$p \parallel q$					$m_1 = m_2$	l: (0,0),(2,1) m: (0,1),(2,2)	$m\angle 1 = m\angle 2 = 40°$

Other Mechanisms in Coordination

The most common mechanism in coordinating objects is using the point. Segments, lines, rays, circles, intersections, polygons, etc., are displayed on diagrams (D) containing named points (see table 4). They are then referred to in written language (WL) or short geometry statements (Sy) using those points, that is a student needs to look at specific points on a diagram

(D) to see where an object is. For example, a student would look for the three points A, B, and C of △ABC (Sy) on the diagram.

Other mechanism between representations are variables, color, measurement (N), numerals/letters for angles, gestures, and other symbols in statements. Variables or even algebraic expressions (A) often appear as measures of angles or length and then in a paragraph (WL) or statement (Sy, and maybe Sy) near the diagram. Students can then use those variables to coordinate between the two representations. Numbers (N) are also used in a similar manner, often appearing as length or angle measure both on the diagram and in written language and/or statements. Numbers are also crucial in coordination between graphs, points as ordered pairs (NOP), and written language. Students can not only refer to numbers on the axes, but they can also count the squares on the graph to determine how many units a point, line, center, etc., is away from the x- or y-axis. Numerals and cursive lower-case letters also help with coordination between representations for angles and lines, respectively. Finally, other symbols (Sy) like △, ∠, ∥, $\overline{AB}$ (the bar above AB), ≅, etc., aid in coordination. Many are iconic, visually close to the objects, but others like the congruence symbol (≅) are arbitrary symbols.

Many words convey meaning in geometry through being connected to physical reality, a mechanism that aids in coordination. That connections may aid in students remembering the word. For example, a word like ray is usually thought in connection to light. In common language, a base is usually defined as the bottom part of some structure similar to bases in triangles; they appear at the bottom of a corresponding altitude. There are other words from the textbook that mean something in the physical world that is different in varying degrees to what they mean in geometry: line, compass, construction, equal, side, vertical, adjacent, reflect, intersection, slope, etc.

Many of these words provide a mechanism to remember and understand the concept they are describing. They often provide mechanisms by helping remind a student of the geometric meaning of a word, based on the common meaning. For example, ‘side’ often means “closer to a wall” or “the face of a building,” and in geometry it means a segment in a polygon (or a ray in an angle). A student may use the common meaning to remind him/herself that just as a building has walls or sides, a polygon has sides. Therefore, when the textbook introduces “the opposite sides of a parallelogram,” even though there are very few buildings in the shape of parallelograms, a students may refer in his or her experience of seeing opposite walls or sides of a building.

Movement as a Mechanism

Movement is not often discussed in geometry class; geometry is usually static, but many of its concepts involve motion. As I analyzed the language, diagrams, constructions, and other representations, movement seems to pervade them. In the rotation above (fig. 26) it seems clear, but the language and ordered pairs disguise it. An arrow is drawn to show the motion of the quadrilateral about the origin. Yet, it is slightly hidden with the rule, i.e., (a, b) → (b, -a), which stands for where the location of the points is to be calculated. Location implies immobility, i.e., that those points appear there without them moving through the rest of quadrant I, completely though quadrant II, and into quadrant III. Most of the chapter on transformations or rigid motion deals with movement, and rigid motion leads into proofs for congruent triangles and triangle properties and later into properties of quadrilaterals and other polygons. Those properties lead to area of all polygons and volume of all polyhedra.

There are many concepts that movement appears in. Dynamic geometry and construction obviously involve motion to construct, and with DGE to drag even after construction. Slope, a less direct example connection to movement, is often described as “rise over run,” which implies moving up and then over; even the purer mathematical language of “change in y” implies upward or downward movement. On graphs arrows often show that to move from some point A to some point B, you need to go up a certain number of units and over a certain number of units. Roads or airplane paths, which are superimposed on diagrams, often contain cars or planes (see fig. 29) that are implied to be moving from one point to another. So, when a student thinks of a point or an intersection, he or she might think of a car or a plane passing through it. In fact, in purer mathematical language problems often describe a situation as in a line passes through a point. There are also multiple problems in the book where a beam of light, a person looking along a path, or the path of a billiard ball is represented in a diagram. Again, that implies an initial or terminal point (a source of light, an eyeball, original location of ball, etc.) and, through some movement, ending up at some end (reflecting in a mirror, bouncing of an embankment, etc.). As another example, term ‘bisect’ is defined as “cut into … parts,” which implies a kind of slicing of a segment or an angle.

Addressing these depictions of movement in geometry concepts, we must consider that formal definitions, concepts, properties, etc. are static sets of points. For example, a line contains a point, not passes through it; a reflection is the set of points the same distance away on the other side of a line, not a flipping of a figure; a line bisects a segment when one of line’s points is the midpoint of the segment, not a cutting or a slicing of a segment like a knife. These metaphors may aid students to remember and to connect concepts, or they may confuse them. The point is that representations may complement, contradict, mirror each other, and that is part of the mechanism that helps us coordinate between representations. For example, if a student understands bisecting a segment as cutting into two equal parts, then he or she may think that the segment like a piece of rope is not whole but rather two separate smaller segments because when he or she cuts a segment of rope in half, the piece of rope is no longer whole. On the other hand, from similar experiences, the student's idea of equal parts of a piece of rope may help him or her to remember that bisect means exactly in half.

To make this idea clearer, let us focus on congruent figure as mentioned above. "Two geometric figures are congruent figures if and only if there is … a composition of rigid motions that maps one of the figures onto the other" (p. 200, Larson & Boswell, 2015). Figure 26 is an example of two rigid motions that map the triangle on the left to the triangles on the right.

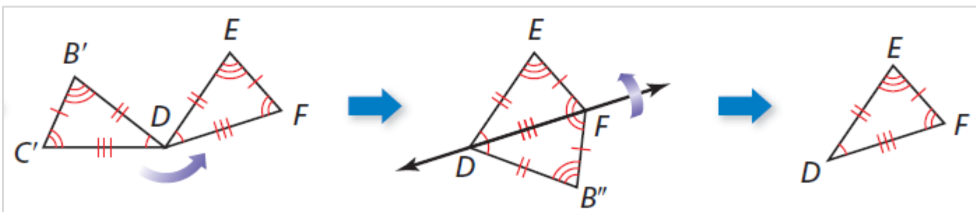


Figure 26. Example of 2D and gestures combination (p. 240, Larson & Boswell, 2015).

First the left triangle is rotated so that the sides with the three congruence marks ($\overline{C'D}$ & $\overline{DF}$) overlap. Then, point B" along with the angle and the two sides is reflected so that it overlaps point E. Note that the purple arrow (Gst) implies a flip rather than a reflection. Throughout the process we imagine cutout of the triangle as it is rotated and then flipped onto the other triangle to completely cover it. This kind of imagining and aligning is common in our lives, e.g., cards aligned to form a deck or dimes stacked to form a roll of coins.

Therefore, there are many ways movement is represented in geometry. These representations may be helpful because they may aid with remembering processes like rotating or flipping because we practice such activities in our daily lives. Yet, they may also confuse us by making it more difficult to understand the formal definitions, postulates, and theorem because the formal concepts like reflecting could be quite different from our daily practice like flipping to align objects.

The analysis in this chapter displayed the representations found in a geometry textbook, how the authors introduced them in the textbook, which coordinations were most common, and what were some mechanisms that aided in coordination. Most introduction of representations was implicit with little discussion of the affordances and constraints (almost never). Movement, font, color, letters of points or lines, iconic symbols (Sy), and physical objects are the most common mechanism applied in the textbook to aid coordination among representations. These mechanisms among representations aid in forming student's interpretants of mathematical objects.

CHAPTER 5

DISCUSSION

Analysis of geometry representations is very complex in many ways. First, there are a variety of representations. Second, there are a great number of mathematical objects throughout a typical geometry textbook and course; the objects can be typical like points and segments, they can be aspects of those objects like length, they can be relationships like the ratio between two lengths, they can be more general like explanations of the logic of a mathematical system, etc. Third, there are a great number of combinations of the various representations. My research tried to clarify this specific field of study on the one hand by making specific distinctions among representations; that was accomplished by studying how and which representations were introduced in the textbook (first research question). Then, I tried to simplify the complexity by focusing on the most common coordinations (second research question). Finally, while analyzing those coordinations, I looked for common mechanisms (third research question) that aided in the coordination between or among representations.

The analysis of the introduction of representations, especially the introduction of their affordances, shows that certain representations like written language and graphs are not introduced with detail and with explicitness, while others like dynamic geometry environments (DGE) have more sections devoted to explanations of use and affordances. Yet, the textbook very rarely provides the constraints of specific representations. Having the descriptive statistics and knowing what coordinations of representations that students are exposed to in a geometry course most often, I was able to focus on some mechanisms in high school geometry. For example, coordination of language, diagrams, etc. in many different combinations is very useful, but coordination of a table of values and a graph is not because it does not appear often in the textbook.

Interpretations and Implications

This study explored the introduction of representations in a textbook, but to do that I needed to make distinctions among those representations and go beyond those found in the extant literature. Therefore, while the representations were introduced, I was searching for specific differences between the ones that appeared earlier in the textbook with those that followed, and also with those in the extant literature. If they were similar the ones that came before, there was no introduction, if they were different, I analyzed the introduction. In this section, I would like to compare the representations introduced in the textbook with those in the extant literature, describe how they were introduced, and describe possible implications of both.

My representations and the literature

In analyzing how representations were introduced, I came across certain representations that are not written much in literature, or that they are not made distinct enough in the literature. For example, what I termed textbook gestures like arrows or perpendicular symbols are often grouped as part of diagrams. Dimmel and Herbst (2015) have written exhaustively about diagrams, but my analysis diverges from theirs in specific ways. First, they do not include graphs and DGE, which as shown in the results that they are different, although there are many overlapping aspects. Second, they contend that many of the markings on diagrams are part of the diagram, but I contend that they are distinct enough to be a different representation, and, they are used outside of the diagrammatic register.

> "[*M*]*athematical* symbols (e.g., small squares to mark right angles, arrows to indicate a rotation, construction traces, or algebraic operations that are nested within labels) are the

> resources through which geometric properties and mathematical operations are communicated by the diagram" (p. 160, Dimmel & Herbst, 2015).

Dimmel and Herbst (2015) treated these 'resources' as part of a diagram register, a much broader concept than my diagram representation.

The concept of perpendicular may elucidate the difference between my use of diagram as a representation and theirs. First, there would be no difference in our meaning of the diagram representation (2D, 3D, or C) in the two strokes, names of points (e.g., A, B), and/or names of the entire line (e.g., *l*, *m*), needed to sketch or construct a figure like an angle. The difference would lie in how we display that they are perpendicular. Dimmel and Herbst (2015) would treat the square box, 90°, or the arcs made by a compass in a construction as part of the diagram register. In contrast, I make a distinction between the textbook gesture (Gst) of the square box in a sketch, the number 90° (N) in a drawing with a protractor, and the arcs made with a compass during construction (C). Moreover, I grouped graph (G) and DGE (E) to a theme of diagram (D) when thematic analysis showed they are alike. That grouping allowed for other ways to distinguish perpendicular lines, namely slope (N) on a graph and dynamic capabilities on DGE.

This difference between the interpretation of Dimmel and Herbst (2015) and mine has some implications. When teaching, discussing, or studying the representation of perpendicular lines, a teacher or a researcher can focus on representing them in 2D or 3D sketches, in construction (C), in a graph (G), or in a dynamic diagram (E2D). My distinction provides a teacher in a classroom the explicit difference among the representations to decide what might be appropriate for students. A researcher is provided the same difference to study which representation might influence some outcome. Some gestures or numbers cannot be used with other representation (e.g., arcs of a compass or measurement on a sketch unless used to describe the procedure), but others that are not typically used can (e.g., 90° on a construction, a construction with arcs on a cm grid paper with specific coordinates on a graph). It opens possibilities of mixing representations like algebra and construction, which are not typically coordinated.

In this textbook study, language is the representation that is assumed to lead the reader through its content, to explain and to introduce other representations, and even itself. Other representations are described in mostly declarative sentences that are at the same level of abstraction, but there are some instances where the textbook explains how objects can be represented in more meta-cognitive language that describes their affordances, but not their constraints.

Another important outcome of this study is that some representations need more attention and would be more useful if separated from a larger category. Duval (2006) provides one category for language, i.e., multipurpose discursive. Pimm (2002) categorized language into spoken and written, but because of the ubiquity of written language (spoken is obviously not present in textbooks) more categories are useful so that language can be discussed in more detail and with more nuance. I divide written language into pure/real/meta and declarative/imperative.

Finally, some representations need to be analyzed differently than in other courses like algebra or calculus, e.g., tables are more often used in proofs and lists of vocabulary terms rather than as lists of x-values and y-values that define a function. Chang, Cromley, and Tran (2016) analyzed coordination between tables, graphs, algebra, and text, and tables represented values of a function. This is the most common aspect of tables that researchers study. As the results show, tables in high school geometry are mostly used in two-column proofs, and not to represent a function, and algebra is mostly used to represent aspects other than a locus of points on a graph, primarily to solve for an unknown. The implication of how these representations are used in more varied ways in geometry than just describing algebraic functions is that the affordances and constraints of tables need to be more broadly analyzed and the coordination between or among them incorporate those other uses. To be exact, we need to broaden the type of mathematical objects that are represented by certain representations, like algebra and tables.

Introduction of Representations

After discussing which representations describe which object and how that compares with the literature, I will describe the introductions themselves. Some representations like *DGE* are introduced with some detail while others like written language with less detail. The textbook devotes a good portion of a page to describe the affordances of DGE further in the book on page 172, but the focus is on what it can construct and not on DGE's ability to change a variety of representations like graph, ordered pairs, algebraic equation, and tables of values as points or lines are dragged. A student would benefit from being aware that he or she can offload some of that information by using such a representation, and that that information changes as values are changed or parts are dragged. Constructions (on paper) have their place as a proof of a certain theorem, and, as Mariotti (2013) alludes to, DGE constructions can validate theorems. The book usually describes that DGE can help, but it does not underscore that constructions can validate theorems.

Constructions (C) with compass and straightedge are also described with less detail than DGE, and but they are displayed over a few steps with new additions in red while previous step in black or gray. Construction with tools is a way to introduce geometry to beginning students because it encourages measurement. Students and teachers may need to explicitly discuss the aspect so that construction and theoretical proofs with diagrams are not confusing. Teachers are often not aware that they switch between the measurement paradigm of Geometry I and more theoretical paradigm of Geometry II (Kuzniak & Rauscher, 2011). For students to be able to transition to a more theoretical stage, they may benefit from being made aware of the stage they are in and what it is comprised of. Most of the constructions in the textbook were procedural in nature, simply explaining what steps need to be taken to construct a given figure using WIP (imperative and pure mathematical language).

The affordances of *language* are to some degree explicitly introduced in the book. There is mention of how proper definitions need to be written, but there is little mention of the use of metaphors and objectification. O'Halloran (2008) describes the use of metaphors, and there are many in the textbook that are like her examples. The book mentions that we can model certain situations, but most of the situations are provided as a matter of fact. For example, in the molecule worked example (fig. 9) the authors of the textbook do not mention that the sticks connecting nuclei are metaphors for lines. In fact, "diagram of a molecule of sulfur …" is the only reference to modeling of a sticks representing bonds. The authors do not explicitly state that that the *bonds of the model are lines*, which is a metaphor. The concept of objectification (Sfard, 2007; Radford, 2002) is not mentioned in the book. Theorems and postulates, which are later used as math objects, would be a suitable place to describe objectification (representing an idea usually written as an if-then statement in a two- or three-word name for a theorem).

The authors introduced *diagrams* to the reader in varying degrees of detail. For example, many of what Dimmel and Herbst (2015) calls type, attribute, position, and prominence systems are not explicitly explained. Movement arrows (e.g., rotation, reflection), color of lines, regions, etc., styles of lines (solid, dashed, dotted), fonts, weight of lines, etc. were minimally introduced. Many of the relational attributes like hash marks for congruent segments or arrows for parallel lines, nominal labels like angle names, and construction traces are explicitly explained. The textbook also does not underscore that diagrams are easier to search and often provide more information (shape, color, position, etc.) with a quick glance than text (Larkin & Simon, 1987). Although having less content when authors choose to print fewer explicit explanations may reduce the cognitive load for a student while learning, it does not teach the student to use metacognitive strategies to regulate his or her learning. For example, in figure 5, the authors printed a side comment (increasing content on a page and therefore possibly cognitive load) that makes readers aware of appropriate tools like DGE aiding to grasp math terms. There are few such comments that introduce the affordances, constraints, and other properties of representations despite the literature (e.g., Dimmel & Herbst, 2015; Larkin & Simon, 1987).

The introduction of short geometry symbols for lines, rays, congruence, circle, etc. (e.g., $\overline{AB} \cong \overline{CD}$) is introduced well, but not its affordances and constraints. The book describes most of these symbols, explaining what they stand for and what the equivalent representations are in a diagram and language. For example, in figure 16 congruent segments are defined and an arrow points from "is congruent to" and the congruent symbol ($\cong$), connecting the representations of written language and short geometry symbols. Yet, the authors of the textbook do not explain how and why these short symbols were developed. Whether an explicit description of symbol development aids in students understanding needs to be studied. Herbst (2002) explained how by refining the use of the two-column proof, we developed shorter symbols. Cajori (1928) explains well the development of these symbols and the reasons for the changes. Although a course in geometry must not include this history of symbols, that history and development does explain their meaning and utility of these condensed forms. That knowledge may benefit students in building a coherent picture of what mathematics is.

The other representations (graphs, physical objects, and textbook gestures) are introduced by example and practice only. For example, the first time a graph appears, students are asked to graph $\overline{AB}$ and then find its midpoint. Pictures sometimes were not even referred to, e.g., a picture of a skateboarder had no caption, and another image was used for the worked example. Physical objects and pictures of them were treated not as a representation of mathematical objects, but they were to be analyzed by mathematical objects as it is done in a science textbook. This way of presenting shows the utility of mathematics, but it does not explicitly teach what mathematical objects are. It may be worthwhile to study whether describing these representations explicitly and whether utilizing physical object to represent math objects (not vice-versa) is beneficial in some way to geometry students.

The analysis of the frequency of how the various representations are introduced exposes either the limited knowledge textbook authors have of the affordances, constraints, and other qualities of representations or their intuition that certain representations do not need to be introduced to students. The former is relatively easy to address by adding an expert on the textbook team that would help write clear explanations of the affordances and constraints into the textbook. The latter requires more studies to research whether the authors' intuition that students learn the power of certain representation through various examples or descriptions rather than through explicit instruction about representations.

Coordination and Mechanisms

It is helpful to discuss coordination (second research question) with Duval's distinction of treatment and conversion, and with Peirce's triad of object, representamen, and interpretant. In this textbook examination, most coordination involved conversion, where different representations were compared. Some coordinations (about 10%) were treatments, or coordinations within a representation, but most of it was within written language and a small amount of algebra. It was possible to read an implied reference to Peirce’s triad where the authors discuss “interpreting a diagram,” and list assumptions that a student can and cannot conclude. For example, we cannot conclude that two segments are congruent because they appear to be. This is the closest that the authors come to making clear that the reader has a role to play in the interpretation of representations. There is a representation (representamen) of two segments that appear to be the same length. There is an interpretant, where a student might think that they are or are not congruent. There is the object that whoever designed the diagram for an exercise or a worked example either planned them to be congruent or not, and that the math community has a specific idea of a congruent segments. Neither idea was discussed explicitly even with more teenage-friendly vocabulary in the textbook. The closest explanation of conversion is the use of ‘words’ and ‘symbols’ next to those different representations of an object. Again, the implication for future research is to study whether such descriptions of Peirce’s triad or Duval’s treatment/conversion distinction, or similar ideas more suitable for adolescents in high school geometry, improves student learning. The distinction might provide students with meta-language to process their learning and understanding, which in turn might improve their self-regulating techniques.

From my perspective as a teacher, I found some mechanisms that aided coordination (the third research question). One was using colors to compare relative parts like x- and y-values in ordered pairs and in equations or on a graph, new arcs on construction, highlighting specific words, etc. Another was sentence structure like conditional (if/then) statements or definitions to aid students connect ideas in a familiar way. A third was using textbook gesture like arrows to point to specific locations or parts or to show a transformation or other movement. Fourth, algebra has its refined mechanisms. Fifth, many of the short geometry symbols like parallel (||) are iconic so students can see a smaller version in the flow of text or as separate statements. Sixth, the names of points are the most traditional way of coordinating among representations.

Textbook gestures seem to overlap with mechanisms and studying them more may change how we use them in the classroom, in future research, and in textbooks. First, studying whether they increase cognitive load may warrant using them less or more. For example, having multiple lines on the same diagram like figure 23, color as a textbook gesture may reduce the cognitive load, but there is probably a point when the number of lines overload a student working memory. What is the best number when introducing a new topic like slope of parallel lines? Two? Three? More? For which students, more advanced less? Where is the most practical balance? Another important question is whether the color itself is distracting. It may be that displaying those three or four lines would reduce the load, or displaying them in black may reduce load. With digital textbooks being more and more common a third way is possible, being able to turn the color off or on, or displaying it before or after displaying the figures in black. A study using pre- and post-tests could help answer some of these questions.

As with the coordinations, this is not an exhaustive list, and it certainly needs to be studied through the eyes of actual high school students seeing some of these representations for the first time or in new light. Many of the mechanisms are what I termed gestures, e.g., color, arrows, type of font, and they appear in both representations or join them in some way.

Limitations

Due to choosing to analyze as broad a swath of geometry content, I was not able to compare many textbooks. Analyzing one textbook is an obvious limitation, but having taught from multiple textbooks, much of the representations, content, and worked examples are similar.

Another limitation is that I was the person determining what the coordinations are. Again, I analyzed a broad area of geometry content, and time and financial constraints would not have allowed me to have multiple participants analyze almost 4000 coordinations. Despite that limitation, the purpose was to improve our understanding of which representation are most common and how are they coordinated, and to provide possible mechanism in those common coordinations. The purpose was not to draw statistical inferences from the data.

Recommendations

Besides the recommendations for further research, teacher training and textbook authors could benefit from this research. Teacher colleges could train teachers to fill the gaps of textbooks that do not address explicit introduction and coordination of representations. In math content and methods courses, future teachers could explore the affordances and constraints of representations and ways of introducing representations to students. Readers of textbooks may benefit by authors adding explicit section or possibly side comments written with meta-language that explain what a representation is conveying or how to coordinate between or among representations. We all want students to learn geometry and mathematics in general, and, as Campbell et al. (2014) claim, content knowledge (CK) and pedagogical content knowledge (PCK) lead to improved achievement. When teachers learn to use convenient representations and when they why they are convenient, teacher improve both their CK and PCK.

At the same time, researchers can learn more about whether explicit introduction to the various representations and their affordances and constraints, explanations of conversions among representations, and teacher knowledge about these topics, leads to better student outcomes. We may discover that students learn these concepts through example, or that teachers fill in the gaps in the textbook well enough that explicit explanations may not be necessary in a textbook.

The diagram representation (2D and 3D, not graph and construction) could benefit being displayed more as a sketch than an accurately and precisely drawn depiction of a mathematical object. Diagrams, especially those explaining properties, definitions, postulates, and theorems, are often meant to be general, standing for any angle measure, any lengths, any size, etc. They could be sketched with rougher, slightly curved lines, imprecise circles, etc., that you would expect someone sketch freehand on paper. Not only could that convey to students that diagrams are not meant to show precise and accurate depictions that students could use tools to measure angles and segments, but it reduce pressure on students to produce very precise, accurate, and time-consuming drawings when taking notes and solving problems.

A recommendation to future studies of geometry symbols (Sy), as well as other representations, would be to introduce how some were developed historically, that they are not rigid and that other countries use different symbols. Cajori (1928) explains well the development of many of the representations, but that knowledge is rarely written about in textbooks including the one studied in this dissertation. For example, perpendicular (⊥) comes from the Latin word ‘perpendiculum,’ which means ‘plumb line.’ It may be beneficial for students to see a plumb line (a physical representation of a line perpendicular to the floor), learn about how the plumb line was abstracted to the symbol (⊥), and that the strange sounding word comes from an object used for thousands of years. In my opinion, student will form a more robust connection among terms, symbols, physical objects, and other representations.

Finally, with digital textbooks and websites becoming more abundant, the use of animation, sound, and possibly other representations are possible. Not only is it worthwhile to study digital textbooks and websites to find more dynamic mechanisms than the ones in a printed textbook, but I hope that researchers and textbook and website designers attempt to add more and improved mechanisms that make learning coordination clearer and more engaging. I have a vision of an animation where a short symbolic statement morphs into a diagram just as Dr. Banner morphs into the Hulk.

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APPENDIX

Table 7. *Explanation of codes*

Code	Description

Written Language (WL)	1. Any text in typical paragraphs will be coded as WL.
	    a. Ignore text leading to website, e.g., "Help in … Spanish at BigIdeasMath.com"
	2. Headings, text in margins, and vocabulary printed in typical left to right will be coded WL.
	3. Letters on diagrams that represent points, lines, planes, etc. will not be coded WL
	    a. they are part of the diagram representation.
	    b. when written in a paragraph (e.g., point A is on line BC), they will be coded as WL.
	4. Pure, Real, or Meta. Written language contained sub-codes.
	    a. Pure math language (P)
	        i. Any language that represents abstract math objects
	            1. e.g., The five-pointed star has a regular pentagon at its center (WDP)
	        ii. If there is any language about the real physical world, it is coded as real-world language
	            1. These instances were coded as real when a physical object was referred to
	            2. They were not coded if physical metaphors were use like "the line touches the circle"
	            3. If only one or two words referred to a physical object, especially if there was no picture of the object, the instance was coded as pure (P). For example, a prism like a box has six faces.
	    b. Real-world language (R)
	        i. Language that describes the real world, or
	            1. e.g., "A park, a store … are located in order on a city street." (WDR)
	        ii. language that is a mixture of a real-world description and abstract math concepts
	            1. e.g., "The distance between the shoe store and … is the same as …" (WDR)
	    c. Metalanguage (M)
	        i. Language describing more generalized ideas on what it means to do math or how to be proficient

	at doing math, language that is one abstract level removed from solving a problem or describing pure math 1. e.g., To be proficient in math you need to understand definitions … (WDM) 5. Declarative or Imperative/interrogative a. Declarative (the D in WDP, WDR, WDM) i. Declarative sentence that state facts or convey information ii. e.g., b. Language used as commands, directions, instructions i. e.g., Work with a partner ii. e.g., verbs in imperative sentences like: use, name, explain, give,
Diagrams (2D or 3D)	Any 2D or 3D figure drawn in 2 dimensions that contains points, lines, rays, segments, and or planes will be coded as 2D or 3D, unless it is drawn to scale and so labeled.
Construction (C)	Any diagram where the author explicitly states that it is drawn to scale or with a compass, straightedge, ruler, protractor, or any other measuring tool will be coded as a construction (C).
Geometric Symbols in Statements (Sy)	Any geometry symbolic statement with capital letters that reference points, especially with the following symbols: angle, ray, segment, line, parallel, perpendicular. e.g., $\overline{AB} \cong \overline{CD}$ or $\overrightarrow{EF} \,\|\|\, \overrightarrow{GH}$. Combined with written language will be coded as mixed, but not if there is no symbol for a specific object as in circle P. Some books do use the symbol $\odot$ for a circle and other objects, which will be coded as Sy.
Numbers (N)	Any number representing a length, angle measure, or count of objects. Any numbers processed in a calculation, but not part of an algebraic expression will be coded N. e.g., 5(3)-6=9 will be coded only as number (N). y=4x-5 will be coded as algebra (A). Numerals will not be coded, if they are used for reference. e.g., $\angle 3$, example problem 2.
Algebraic Expression (A)	Any expression with variables and operation symbols like $+, -, \cdot, /, (), =$, tan, sin, cos, etc. will be coded as A. Coefficients and constants will not be coded as numerals (N). e.g., y=4x-7 or $a^2 + b^2 = c^2$ If numbers are substituted for all variables, it will be coded as N.
Coordinate Grid/Graph (G)	Any diagram with an x-axis and y-axis will be coded as a grid. If the graph is drawn to scale on a coordinate grid and labeled so, it will be a mixture of grid (G) and construction (C).
Physical Objects (P)	Any pictures of 3D objects will be coded P. e.g., a picture of a pyramid Any descriptions where a student is supposed to look at a 3D object or imagine it will be coded P.

	e.g., Give examples of each type of line intersection formed by the walls floor and ceiling. Any paper or other flat object that the student is instructed to fold to create a 3D object will be coded P.
Table (T)	Any list of numbers and/or variables separated into columns and/or rows will be coded T. Truth table of conditional or other statements. This may be different than table of values of a function that may have a more defined pattern. Table with text, e.g., number of sides and name of polygon. Similarities among various tables: • Easy to compare two or more dimensions using rows and columns Becomes overly complicated when more cells are added, especially columns.
Dynamic Geometry Software (E)	Any instructions to use dynamic geometry software like Geogebra or Geometer's SketchPad, especially if there is a picture of it, will be coded as DGS. The coder will follow the instructions in Geogebra to see what the experience entails.