GT-SHADOWS RELATED TO FINITE QUOTIENTS OF THE FULL MODULAR GROUP

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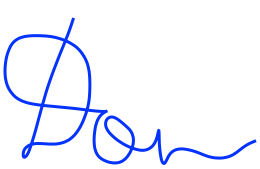

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ABSTRACT

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GT-shadows are tantalizing objects that can be thought of as "approximations" to elements of the mysterious Grothendieck-Teichmueller group $\widehat{\mathsf{GT}}$ introduced by V. Drinfeld in 1990 [5]. GT-shadows [4] form a groupoid whose objects are certain finite index normal subgroups of Artin's braid group B_4 on 4 strands. In this thesis we introduce GT-shadows for the gentle version $\widehat{\mathsf{GT}}_{gen}$ of the Grothendieck-Teichmueller group. These entities are morphisms of a groupoid GTSh whose objects are certain finite index normal subgroups of Artin's braid group B_3 on 3 strands. We explore the connected components of GTSh for subgroups of B_3 coming from the standard homomorphism from B_3 to $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$, where q is a power of an odd prime integer > 3.

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To my mother,

Li Lejing

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CHAPTER 1

INTRODUCTION

1.1 Why we study the Grothendieck-Teichmueller group $\widehat{\mathsf{GT}}$

Grothendieck-Teichmueller group and its versions [1], [2], [5], [6], [7], [8], [9], [18], [19], [23], [24], [26], [30] are among the most mysterious objects in mathematics. The profinite version $\widehat{\mathsf{GT}}$ of the Grothendieck-Teichmueller group receives an injective homomorphism [19] from the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the field of rationals and it acts on Grothendieck's child's drawings[1] [6], [11], [12], [13], [17], [21], [27], [28], [31]. The pro-unipotent version of $\widehat{\mathsf{GT}}$ is related to graph complexes [30] and finite type invariants of tangles [22].

Let us denote by B_n (resp. PB_n) Artin's braid group (resp. the pure braid group) on n strands. We denote by $\mathsf{NFI}_{\mathrm{PB}_n}(\mathrm{B}_n)$ the poset of finite index normal subgroups $N \trianglelefteq \mathrm{B}_n$ satisfying the condition $N \le \mathrm{PB}_n$.

In paper [4], the authors introduced[2] GT-shadows. These are morphisms of a groupoid whose objects are elements of $\mathsf{NFI}_{\mathrm{PB}_4}(\mathrm{B}_4)$. To every element $\hat{T} \in \widehat{\mathsf{GT}}$ and $N \in \mathsf{NFI}_{\mathrm{PB}_4}(\mathrm{B}_4)$, one can naturally assign a GT-shadow T_N in GTSh with the target N, and T_N is an approximation of $\hat{T}$. GT-shadows

[1] The list of references is far from complete.

[2] Very similar objects were also studied in [15], [16] and [17].

produced in this way from elements of $\widehat{\mathsf{GT}}$ are called **genuine** and they are the most interesting ones. In paper [4], the authors described various properties of genuine GT-shadows and proved that $\widehat{\mathsf{GT}}$ is the limit of a functor constructed using GT-shadows.

In this thesis, we develop similar constructions for the gentle version $\widehat{\mathsf{GT}}_{gen}$ of the Grothendieck-Teichmueller group. As far as we know, the group $\widehat{\mathsf{GT}}_{gen}$ was introduced in [18, Introduction] and, in [18], it is denoted by $\widehat{\mathsf{GT}}_0$. $\widehat{\mathsf{GT}}_{gen}$ naturally contains $\widehat{\mathsf{GT}}$ as a subgroup.

More precisely, we define analogs of GT-pairs and GT-shadows for $\widehat{\mathsf{GT}}_{gen}$. We construct a groupoid GTSh whose objects are element of the poset $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and whose morphisms are GT-shadows for $\widehat{\mathsf{GT}}_{gen}$. We define the concept of an isolated element in $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and prove versions of statements from [4] for this set-up.

Finally, we consider the standard homomorphism $\mathrm{PB}_3 \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ (for $q \in \mathbb{Z}_{\geq 1}$) and denote by $N^{(q)}$ the kernel of this homomorphism. Since the homomorphism $\mathrm{PB}_3 \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ comes from the standard homomorphism $\mathrm{B}_3 \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ and the group $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ is finite,

$$N^{(q)} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3), \qquad \forall \;\; q \in \mathbb{Z}_{\geq 1}.$$

We prove that the assignment $q \mapsto N^{(q)}$ gives us a map of posets from the divisibility poset to $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. We give an explicit description of the set of charming GT-pairs with the target $N^{(q)}$ in the case when q is a power of an odd prime integer > 3. (See Theorem 4.2). We also formulate a conjecture about the connected component of $N^{(q)}$ in the groupoid GTSh. (See Conjecture 4.5).

Remark 1.1 *In the light of [4], we should have used a different notational convention for the set of* GT*-shadows with the target* N *and for the groupoid of* GT*-shadows. For simplicity of presentation, in this thesis, we tacitly assume that* GT*-shadows are* GT*-shadows for the gentle version* $\widehat{\mathsf{GT}}_{gen}$ *of the Grothendieck-Teichmueller group. So, for* $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$*,* $\mathsf{GT}(N)$ *denotes the set of* GT*-shadows for* $\widehat{\mathsf{GT}}_{gen}$ *with the target* N*. Similarly* GTSh *denotes* *the groupoid of* GT*-shadows for* $\widehat{\mathsf{GT}}_{gen}$*. In particular, in this thesis, the set of objects of* GTSh *is* $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$*, unlike in [4], where the set of objects of* GTSh *is* $\mathsf{NFI}_{\mathrm{PB}_4}(\mathrm{B}_4)$*. So* $\mathsf{GT}(N)$ *and* GTSh*, in this thesis, should not be confused with the same notational conventions from [4]*

1.2 Plan of the thesis

In Chapter 2, we introduce GT-pairs, charming GT-pairs and the groupoid GTSh. In the beginning of this chapter, we describe two ways of constructing elements of the poset $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. Then we introduce GT-pairs and discuss consequences of the hexagon relations. We define GT-shadows (for $\widehat{\mathsf{GT}}_{gen}$) and prove that they form a groupoid with respect to a natural composition law (see Theorem 2.12). The set of objects of the groupoid GTSh is $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. We say that an element $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ is isolated if its connected component in GTSh has exactly one object. In Section 2.4, we describe a way to construct isolated elements of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ (see Proposition 2.15).

In Chapter 3, we define the gentle version $\widehat{\mathsf{GT}}_{gen}$ of the Grothendieck-Teichmueller group and establish a link between $\widehat{\mathsf{GT}}_{gen}$ and the groupoid GTSh. We start this chapter with introducing a structure of a monoid on the set $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$. Then we define a submonoid $\widehat{\mathsf{GT}}_{gen,mon}$ of $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ and introduce $\widehat{\mathsf{GT}}_{gen}$ as the group of invertible elements of $\widehat{\mathsf{GT}}_{gen,mon}$.

In Section 3.2, we define a right action of $\widehat{\mathsf{GT}}_{gen}$ on $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and construct a functor from the corresponding transformation groupoid $\widehat{\mathsf{GT}}_{gen,\mathsf{NFI}}$ to the groupoid GTSh.

In Chapter 4, we use the standard homomorphism $\mathrm{B}_3 \to \mathrm{SL}_2(\mathbb{Z})$ and the divisibility poset to define a natural subposet of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. More precisely, to every $q \in \mathbb{Z}_{\geq 1}$, we assign an element $N^{(q)} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. In this chapter, we also give an explicit description of the set of charming GT-pairs with the target $N^{(q)}$ in the case when q is a power of an odd prime integer > 3. (See Theorem 4.2). Theorem 4.2 is the main result of this thesis. We conclude Chapter 4 with a conjecture about the connected component of $N^{(q)}$ in GTSh.

1.3 Notational conventions

For a set X with an equivalence relation and $a \in X$, we denote by $[a]$ the equivalence class which contains the element a. For a group G, the notation $\mathcal{Z}(G)$ (resp. $[G, G]$) is reserved for the center of G (resp. the commutator subgroup of G). For a subgroup $H \leq G$, $|G : H|$ denotes the index of H in G. For a normal subgroup $H \trianglelefteq G$ of finite index, we denote by $\mathsf{NFI}_H(G)$ the poset of finite index normal subgroups N in G such that $N \leq H$. Moreover, $\mathsf{NFI}(G) := \mathsf{NFI}_G(G)$, i.e. $\mathsf{NFI}(G)$ is the poset of normal finite index subgroups of a group G. For a subgroup $H \leq G$, the notation $\mathrm{Core}_G(H)$ is reserved for the normal core of H in G.

For a group G, the notation $\widehat{G}$ is reserved for the profinite completion of G. If G is residually finite, then we tacitly identify G with the corresponding subgroup of $\widehat{G}$. Every finite group is tacitly considered with the discrete topology.

For a group G and $N \trianglelefteq G$, the notation $\mathcal{P}_N$ is reserved for the standard group homomorphism $\mathcal{P}_N : G \to G/N$. Moreover, if $N \leq H$ and $N \trianglelefteq G$, $H \trianglelefteq G$, then the notation $\mathcal{P}_{N,H}$ is reserved for the standard group homomorphism $\mathcal{P}_{N,H} : G/N \to G/H$. $\mathcal{P}_{N,H}$ is defined by the formula

$$\mathcal{P}_{N,H}(gN) := gH.$$

The notation $\widehat{\mathcal{P}}_N$ is reserved for the standard continuous group homomorphism $\widehat{\mathcal{P}}_N : \widehat{G} \to G/N$.

For groups G, H, $\mathrm{Hom}(G, H)$ denotes the set of all group homomorphisms from G to H.

B_n (resp. PB_n) denotes the Artin braid group on n strands (resp. the pure braid group on n strands) [20]. S_n denotes the symmetric group on n letters. Symbols $\sigma_1, \ldots, \sigma_{n-1}$ (resp. x_{ij} for $1 \leq i < j \leq n$) denote the standard generators of B_n (resp. the standard generators of PB_n). The notation ρ is reserved for the standard homomorphism from B_n to S_n, i.e.

$$\rho(\sigma_i) := (i, i+1), \qquad 1 \leq i \leq n-1.$$

For example,

$$\mathrm{B}_3 := \langle\, \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \,\rangle$$

and

$$x_{12} := \sigma_1^2, \qquad x_{23} := \sigma_2^2, \qquad x_{13} := \sigma_2\sigma_1^2\sigma_2^{-1}\,. \tag{1.1}$$

Recall that $\mathcal{Z}(\mathrm{B}_n) = \mathcal{Z}(\mathrm{PB}_n)$ and $\mathcal{Z}(\mathrm{PB}_n)$ is isomorphic to $\mathbb{Z}$. For example, $\mathcal{Z}(\mathrm{PB}_3) = \langle\, c \,\rangle$, where

$$c := x_{23}x_{12}x_{13} = x_{12}x_{13}x_{23} = (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_1)^3\,. \tag{1.2}$$

It is known that $\langle\, x_{12}, x_{23} \,\rangle$ is isomorphic to the free group F_2 on two generators and we tacitly identify F_2 with the subgroup $\langle\, x_{12}, x_{23} \,\rangle$ of PB_3. It is known that $\mathrm{PB}_3 \cong \mathrm{F}_2 \times \langle\, c \,\rangle$ with $\langle\, c \,\rangle \cong \mathbb{Z}$.

We often denote x_{12} (resp. x_{23}) by x (resp. y) and set $z := y^{-1}x^{-1}$. We denote by θ (resp. τ) the following endomorphisms of F_2:

$$\theta(x) := y, \qquad \theta(y) := x, \tag{1.3}$$

$$\tau(x) := y, \qquad \tau(y) := z. \tag{1.4}$$

Note that $\tau(z) = \tau(y^{-1}x^{-1}) = z^{-1}y^{-1} = xyy^{-1} = x$. Hence $\tau^3 = \mathrm{id}$. It is also easy to see that $\theta^2 = \mathrm{id}$. Thus θ and τ are actually automorphisms of F_2.

We denote by Δ the following element of B_3

$$\Delta := \sigma_1\sigma_2\sigma_1. \tag{1.5}$$

It is easy to see that Δ satisfies the following identities

$$\Delta^2 = c, \qquad \sigma_1\Delta = \Delta\sigma_2, \qquad \sigma_2\Delta = \Delta\sigma_1. \tag{1.6}$$

Hence, for every $f \in \mathrm{F}_2 \leq \mathrm{PB}_3$, we have

$$\Delta f \Delta^{-1} = \theta(f). \tag{1.7}$$

Recall that B_2 is an infinite cyclic group generated by σ_1 and PB_2 is an infinite cyclic group generated by $x_{12} := \sigma_1^2$. Depending on the context, σ_1

(resp. x_{12}) denotes an element of B_3 (resp. PB_3) or an element of B_2 (resp. PB_2).

For a category $\mathcal{C}$, $\mathrm{Ob}(\mathcal{C})$ denotes the set of objects of $\mathcal{C}$. For $a, b \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(a, b)$ often denotes the set of morphisms with the source (resp. the target) a (resp. b). For a groupoid $\mathcal{G}$, the expression $\gamma \in \mathcal{G}$ means that γ is a **morphism** of $\mathcal{G}$.

CHAPTER 2

THE GROUPOID GTSh OF GT-SHADOWS

2.1 The poset $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$

To explore the groupoid of GT-shadows for $\widehat{\mathsf{GT}}_{gen}$, we need a large supply of elements of the poset $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

The first obvious source of elements of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ comes from group homomorphisms from B_3 to finite groups. Indeed, given a group homomorphism $\psi : \mathrm{B}_3 \to G$, where G is finite,

$$\ker(\psi) \cap \mathrm{PB}_3 \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3).$$

The starting point of another approach is a group homomorphism ψ from PB_3 to a finite group G. Since $\mathrm{PB}_3 \cong \mathrm{F}_2 \times \langle\, c \,\rangle$ with $\langle\, c \,\rangle \cong \mathbb{Z}$, the set $\mathrm{Hom}(\mathrm{PB}_3, G)$ of group homomorphisms from PB_3 to G is in bijection with the set of triples

$$\{(g_{12}, g_{23}, g_c) \in G^3 \mid g_c g_{12} = g_{12} g_c, \quad g_c g_{23} = g_{23} g_c\}. \tag{2.1}$$

The homomorphism $\psi : \mathrm{PB}_3 \to G$ corresponding to a triple (g_{12}, g_{23}, g_c) in (2.1) is defined by the equations

$$\psi(x_{12}) := g_{12}, \qquad \psi(x_{23}) := g_{23}, \qquad \psi(c) := g_c\,. \tag{2.2}$$

In general, $\ker(\psi)$ is not normal in B_3. However we have an explicit way to compute the normal core of $\ker(\psi)$ in B_3.

Proposition 2.1 *Let G be a finite group and $\psi : \mathrm{PB}_3 \to G$ be a homomorphism corresponding to a triple (g_{12}, g_{23}, g_c) in (2.1). The formulas*

$$\begin{aligned} \psi^\sharp(x_{12}) &:= (g_{12},\, g_{12},\, g_{23}^{-1}g_{12}^{-1}g_c,\, g_{23}^{-1}g_{12}^{-1}g_c,\, g_{23},\, g_{23}),\\ \psi^\sharp(x_{23}) &:= (g_{23},\, g_{12}^{-1}g_{23}^{-1}g_c,\, g_{23},\, g_{12},\, g_{12}^{-1}g_{23}^{-1}g_c,\, g_{12}) \\ \psi^\sharp(c) &:= (g_c, g_c, g_c, g_c, g_c, g_c). \end{aligned} \tag{2.3}$$

define a group homomorphism $\psi^\sharp : \mathrm{PB}_3 \to G^6$ and

$$\ker(\psi^\sharp) = \mathrm{Core}_{\mathrm{B}_3}(\ker(\psi)). \tag{2.4}$$

In particular, $\ker(\psi^\sharp) \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ *and* $\ker(\psi^\sharp) \le \ker(\psi)$.

Proof. Since

$$\rho(1) = \mathrm{id}_{S_3} \quad \rho(\sigma_1) = (1,2), \quad \rho(\sigma_2) = (2,3),$$

$$\rho(\sigma_1\sigma_2) = (1,2,3), \quad \rho(\sigma_2\sigma_1) = (1,3,2), \quad \rho(\sigma_1\sigma_2\sigma_1) = (1,3),$$

the elements $1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \Delta := \sigma_1\sigma_2\sigma_1$ are representatives of the 6 distinct cosets in $\mathrm{B}_3/\mathrm{PB}_3 \cong S_3$.

Combining this observation with the fact that the subgroup $K := \ker(\psi)$ is normal is PB_3, we conclude that the normal core $\mathrm{Core}_{\mathrm{B}_3}(K)$ of K in B_3 coincides with the finite intersection

$$K \cap \sigma_1 K \sigma_1^{-1} \cap \sigma_2 K \sigma_2^{-1} \cap \sigma_1\sigma_2 K (\sigma_1\sigma_2)^{-1} \cap \sigma_2\sigma_1 K (\sigma_2\sigma_1)^{-1} \cap \Delta K \Delta^{-1}. \tag{2.5}$$

For $w \in \mathrm{B}_3$, we denote by ψ_w the homomorphism $\psi_w : \mathrm{B}_3 \to G$ defined by the formula:

$$\psi_w(v) := \psi(w^{-1} v w).$$

Since for every $w \in \mathrm{B}_3$, $\ker(\psi_w) = wKw^{-1}$, the subgroup $\mathrm{Core}_{\mathrm{B}_3}(K)$ coincides with the intersection

$$\ker(\psi) \cap \ker(\psi_{\sigma_1}) \cap \ker(\psi_{\sigma_2}) \cap \ker(\psi_{\sigma_1\sigma_2}) \cap \ker(\psi_{\sigma_2\sigma_1}) \cap \ker(\psi_\Delta). \tag{2.6}$$

Combining this observation with

$$\sigma_1^{-1} x_{12} \sigma_1 = x_{12}, \qquad \sigma_2^{-1} x_{12} \sigma_2 = x_{23}^{-1} x_{12}^{-1} c, \quad \sigma_2^{-1} \sigma_1^{-1} x_{12} \sigma_1 \sigma_2 = x_{23}^{-1} x_{12}^{-1} c,$$

$$\sigma_1^{-1} x_{23} \sigma_1 = x_{12}^{-1} x_{23}^{-1} c \qquad \sigma_2^{-1} x_{23} \sigma_2 = x_{23}, \qquad \sigma_2^{-1} \sigma_1^{-1} x_{23} \sigma_1 \sigma_2 = x_{12},$$

$$\sigma_1^{-1} \sigma_2^{-1} x_{12} \sigma_2 \sigma_1 = x_{23}, \qquad \Delta x_{12} \Delta^{-1} = x_{23}$$

$$\sigma_1^{-1} \sigma_2^{-1} x_{23} \sigma_2 \sigma_1 = x_{12}^{-1} x_{23}^{-1} c \quad \Delta x_{23} \Delta^{-1} = x_{12}\,,$$

we conclude that $\mathrm{Core}_{\mathrm{B}_3}(K)$ coincides with the kernel of the homomorphism $\psi^\sharp : \mathrm{PB}_3 \to G^6$ defined by the formulas:

$$\begin{aligned} \psi^\sharp(x_{12}) &:= (g_{12},\, g_{12},\, g_{23}^{-1} g_{12}^{-1} g_c,\, g_{23}^{-1} g_{12}^{-1} g_c,\, g_{23},\, g_{23}), \\ \psi^\sharp(x_{23}) &:= (g_{23},\, g_{12}^{-1} g_{23}^{-1} g_c,\, g_{23},\, g_{12},\, g_{12}^{-1} g_{23}^{-1} g_c,\, g_{12}) \\ \psi^\sharp(c) &:= (g_c, g_c, g_c, g_c, g_c, g_c). \end{aligned} \tag{2.7}$$

To prove that $\ker(\psi^\sharp) \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, we just observe that, for every finite index subgroup K of a group H, the subgroup $\mathrm{Core}_H(K)$ also has a finite index in H. Finally, the inclusion $\ker(\psi^\sharp) \le \ker(\psi)$ is obvious. □

2.2 GT-pairs and GT-shadows

For $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, we denote by N_{ord} the least common multiple of the orders of the elements $x_{12}N, x_{23}N, cN \in \mathrm{PB}_3/N$. We also set

$$N_{\mathrm{F}_2} := N \cap \mathrm{F}_2. \tag{2.8}$$

Definition 2.2 *A* GT-***pair with the target*** N *is*

$$(m + N_{\mathrm{ord}}\mathbb{Z}, f N_{\mathrm{F}_2}) \;\in\; \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathrm{F}_2/N_{\mathrm{F}_2}$$

satisfying the ***hexagon relations****:*

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f\, N \;=\; f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m\, N, \tag{2.9}$$

$$f^{-1} \sigma_2^{2m+1} f\, \sigma_1^{2m+1}\, N \;=\; \sigma_2 \sigma_1 x_{23}^{-m} c^m\, f\, N. \tag{2.10}$$

We denote by $\mathsf{GT}_{pr}(N)$ *the set of all* GT*-pairs with the target* N*.*

Since $N_{\rm ord}$ is the least common multiple of the orders of the elements $x_{12}N$, $x_{23}N$, cN and $N_{\mathrm{F}_2} \le N$, we see that, if a pair $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$ satisfies (2.9) and (2.10), then so does the pair $(m + tN_{\rm ord}, fh)$ for any $t \in \mathbb{Z}$ and any $h \in N_{\mathrm{F}_2}$.

It is convenient to denote by $[m, f]$ the GT-pair represented by $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$.

Due to the hexagon relations, every GT-pair $[m, f] \in \mathsf{GT}_{pr}(N)$ gives us a group homomorphism $\mathrm{B}_3 \to \mathrm{B}_3/N$. More precisely,

Proposition 2.3 *For every $[m, f] \in \mathsf{GT}_{pr}(N)$, the formulas*

$$T_{m,f}(\sigma_1) := \sigma_1^{2m+1} N, \qquad T_{m,f}(\sigma_2) := f^{-1}\sigma_2^{2m+1} f N \tag{2.11}$$

define a group homomorphism $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/N$.

Proof. Since $\mathrm{B}_3 = \langle\, \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \,\rangle$, it suffices to verify that

$$T_{m,f}(\sigma_1)T_{m,f}(\sigma_2)T_{m,f}(\sigma_1) \stackrel{?}{=} T_{m,f}(\sigma_2)T_{m,f}(\sigma_1)T_{m,f}(\sigma_2). \tag{2.12}$$

Using (2.9), we rewrite the left hand side of (2.12) as

$$(\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f)\sigma_1^{2m+1}\, N = f^{-1}\sigma_1\sigma_2 x_{12}^{-m} c^m\, \sigma_1^{2m+1}\, N = f^{-1}\Delta c^m\, N, \tag{2.13}$$

where $\Delta = \sigma_1\sigma_2\sigma_1$.

Using (2.9) once again, we rewrite the right hand side of (2.12) as

$$f^{-1}\sigma_2^{2m+1} f(\sigma_1^{2m+1} f^{-1}\sigma_2^{2m+1} f)\, N = f^{-1}\sigma_2^{2m+1} f(f^{-1}\sigma_1\sigma_2 x_{12}^{-m} c^m) N =$$

$$f^{-1}\sigma_2^{2m}\sigma_2\sigma_1\sigma_2 x_{12}^{-m} c^m\, N = f^{-1}\sigma_2^{2m}\Delta x_{12}^{-m} c^m\, N = f^{-1}\Delta c^m\, N.$$

In the last step, we used the identity $\sigma_2\Delta = \Delta\sigma_1$.

Relation (2.12) is proved. □

Since $N \le \mathrm{PB}_3$, the standard homomorphism $\rho : \mathrm{B}_3 \to S_3$ induces the homomorphism

$$\rho_N : \mathrm{B}_3/N \to S_3. \tag{2.14}$$

It is easy to see that, for every $[m, f] \in \mathsf{GT}_{pr}(N)$, the composition $\rho_N \circ T_{m,f}$ coincides with ρ. Hence the restriction of $T_{m,f}$ to PB_3 gives us a homomorphism

$$T^{\mathrm{PB}_3}_{m,f} := T_{m,f}\big|_{\mathrm{PB}_3} : \mathrm{PB}_3 \to \mathrm{PB}_3/N \tag{2.15}$$

and

$$\ker(T^{\mathrm{PB}_3}_{m,f}) = \ker(T_{m,f}). \tag{2.16}$$

In particular,

$$\ker(T^{\mathrm{PB}_3}_{m,f}) \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3), \qquad \forall \;\; [m, f] \in \mathsf{GT}_{pr}(N). \tag{2.17}$$

A direct computation shows that, for every $[m, f] \in \mathsf{GT}_{pr}(N)$,

$$T^{\mathrm{PB}_3}_{m,f}(x_{12}) = x_{12}^{2m+1} N, \qquad T^{\mathrm{PB}_3}_{m,f}(x_{23}) = f^{-1} x_{23}^{2m+1} f N, \tag{2.18}$$

$$T^{\mathrm{PB}_3}_{m,f}(c) = c^{2m+1} N.$$

Moreover, the restriction of $T^{\mathrm{PB}_3}_{m,f}$ to $\mathrm{F}_2 \le \mathrm{PB}_3$ gives us a homomorphism

$$T^{\mathrm{F}_2}_{m,f} := T^{\mathrm{PB}_3}_{m,f}\big|_{\mathrm{F}_2} : \mathrm{F}_2 \to \mathrm{F}_2/N_{\mathrm{F}_2}, \tag{2.19}$$

where $N_{\mathrm{F}_2} := N \cap \mathrm{F}_2$.

The equations in (2.18) imply that

$$T^{\mathrm{F}_2}_{m,f}(x) = x^{2m+1} N_{\mathrm{F}_2}\,, \qquad T^{\mathrm{F}_2}_{m,f}(y) = f^{-1} y^{2m+1} f N_{\mathrm{F}_2}. \tag{2.20}$$

Recall that θ is the automorphism of F_2 defined by the formulas $\theta(x) := y$ and $\theta(y) := x$.

We will need this proposition:

Proposition 2.4 *Let* $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. *If a pair* $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$ *satisfies hexagon relations* (2.9) *and* (2.10) *(modulo* N*) then*

$$f\theta(f) \in N_{\mathrm{F}_2}. \tag{2.21}$$

Equation (2.21) is often written in the form $f(x,y)f(y,x) \in N_{\mathrm{F}_2}$.

Proof. Applying (2.9) to the left hand side of (2.12), we get

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1} N = f^{-1} \Delta c^m N. \tag{2.22}$$

Applying (2.10) and the identity $\sigma_1 \Delta = \Delta \sigma_2$ to the left hand side of (2.12), we get

$$\sigma_1^{2m+1} (f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1}) N = \sigma_1^{2m+1} \sigma_2 \sigma_1 c^m x_{23}^{-m} f N =$$

$$\sigma_1^{2m} \Delta c^m x_{23}^{-m} f N = \Delta f c^m N.$$

Therefore we conclude that $\Delta f N = f^{-1} \Delta N$. Hence using (1.7) we get that

$$f \theta(f) \in N.$$

Since $f\theta(f) \in \mathrm{F}_2$, the desired property follows. □

Definition 2.5 *Let $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. A GT-pair $[m, f] \in \mathsf{GT}_{pr}(N)$ is called* ***charming*** *if*

1. $2m+1$ *represents a unit in the ring $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$ and*
2. *the coset fN_{F_2} can be represented by an element in the commutator subgroup $[\mathrm{F}_2, \mathrm{F}_2]$ or, equivalently, $fN_{\mathrm{F}_2} \in [\mathrm{F}_2/N_{\mathrm{F}_2}, \mathrm{F}_2/N_{\mathrm{F}_2}]$.*

We denote by $\mathsf{GT}_{pr}^{\heartsuit}(N)$ the set of charming GT-pairs with the target N.

The second condition in the above definition plays a role in the proof of the following proposition:

Proposition 2.6 *Let $(m, f) \in \mathbb{Z} \times [\mathrm{F}_2, \mathrm{F}_2]$, $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. Then (m, f) satisfies hexagon relations (2.9), (2.10) (modulo N) if and only if*

$$f\theta(f) \in N_{\mathrm{F}_2} \tag{2.23}$$

and

$$\tau^2(y^m f)\tau(y^m f) y^m f \in N_{\mathrm{F}_2}\,, \tag{2.24}$$

where θ and τ are the automorphisms of F_2 defined in (1.3) and (1.4), respectively.

We call relations (2.23), (2.24) **the simplified hexagon relations**.

Proof. Recall that

$$x := x_{12}, \qquad y := x_{23}, \qquad z := y^{-1}x^{-1}\,.$$

It is also convenient to set $w := x^{-1}y^{-1}$.

We would like to rewrite (2.9) in the form $\sigma_1\sigma_2 hN = \sigma_1\sigma_2\tilde{h}N$, where $hN, \tilde{h}N \in \mathrm{PB}_3/N$.

Using the relations

$$\sigma_2^{-1}x_{12}\sigma_2 = x_{23}^{-1}x_{12}^{-1}c, \qquad \sigma_2^{-1}x_{23}\sigma_2 = x_{23},$$

and

$$\sigma_2^{-1}\sigma_1^{-1}x_{12}\sigma_1\sigma_2 = x_{23}^{-1}x_{12}^{-1}c, \qquad \sigma_2^{-1}\sigma_1^{-1}x_{23}\sigma_1\sigma_2 = x_{12},$$

we see that (2.9) is equivalent to

$$\sigma_1\sigma_2(x_{23}^{-1}x_{12}^{-1})^m f^{-1}(x_{23}^{-1}x_{12}^{-1}c, x_{23})x_{23}^m f c^m N \;=\;$$

$$\sigma_1\sigma_2 f^{-1}(x_{23}^{-1}x_{12}^{-1}c, x_{12})x_{12}^{-m}c^m N. \tag{2.25}$$

Hence (2.9) is equivalent to

$$x^m f(zc, x) z^m f^{-1}(zc, y) y^m f \;\in\; N. \tag{2.26}$$

Since $f \in [\mathrm{F}_2, \mathrm{F}_2]$, f is a product of words of form $w_1w_2w_1^{-1}w_2^{-1}$ where $w_i \in \mathrm{F}_2$. Hence, since $c \in \mathcal{Z}(\mathrm{PB}_3)$, we conclude that $f(zc, x) = f(z, x)$ and $f(zc, y) = f(z, y)$.

Thus (2.9) is equivalent to

$$x^m f(z, x) z^m f^{-1}(z, y) y^m f \;\in\; N. \tag{2.27}$$

Now we would like to rewrite (2.10) in the form $\sigma_2\sigma_1 hN = \sigma_2\sigma_1\tilde{h}N$, where $hN, \tilde{h}N \in \mathrm{PB}_3/N$.

Using the relations

$$\sigma_1^{-1}\sigma_2^{-1}x_{12}\sigma_2\sigma_1 = x_{23}, \qquad \sigma_1^{-1}\sigma_2^{-1}x_{23}\sigma_2\sigma_1 = x_{12}^{-1}x_{23}^{-1}c$$

$$\sigma_1^{-1} x_{12} \sigma_1 = x_{12}, \qquad \sigma_1^{-1} x_{23} \sigma_1 = x_{12}^{-1} x_{23}^{-1} c,$$

one can show that (2.10) is equivalent to

$$\sigma_2\sigma_1\, f^{-1}(x_{23}, x_{12}^{-1}x_{23}^{-1}c)(x_{12}^{-1}x_{23}^{-1})^m f(x_{12}, x_{12}^{-1}x_{23}^{-1}c) x_{12}^m c^m\, N =$$

$$\sigma_2\sigma_1\, x_{23}^{-m} f c^m\, N. \tag{2.28}$$

Using the property $f \in [\mathrm{F}_2, \mathrm{F}_2]$, we see that relation (2.28) is clearly equivalent to

$$x^m f^{-1} y^m f^{-1}(y, w) w^m f(x, w) \ \in \ N. \tag{2.29}$$

Conjugating (2.29) with $\Delta := \sigma_1\sigma_2\sigma_1$ and then conjugating the resulting relation with $f(y, x)y^{-m}$, we see that (2.29) is equivalent to

$$x^m f^{-1}(x, z) z^m f(y, z) y^m f^{-1}(y, x) \ \in \ N. \tag{2.30}$$

Thus (2.10) is equivalent to (2.30).

Let us assume that equations (2.9) and (2.10) are satisfied. Due to Proposition 2.4, relation (2.23) is satisfied.

Conjugating (2.23) with $\sigma_1\sigma_2$ and with $(\sigma_1\sigma_2)^2$, and using the property $f \in [\mathrm{F}_2, \mathrm{F}_2]$, we deduce that

$$f(z, y)N = f^{-1}(y, z)N \tag{2.31}$$

and

$$f(x, z)N = f^{-1}(z, x)N. \tag{2.32}$$

Combining (2.27) with (2.31), we conclude that (2.24) is satisfied.

Let us now assume that (2.23) and (2.24) are satisfied. We proved that (2.23) implies (2.31) and (2.32). Combining (2.24) with (2.31) and (2.32), we conclude that (2.27) and (2.30) are satisfied.

Since (2.27) and (2.30) are equivalent to (2.9) and (2.10), respectively, we deduced (2.9) and (2.10) from (2.23) and (2.24). □

Let us prove the following statement:

Proposition 2.7 *Let* $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ *and* $[m,f]$ *be a charming* GT*-pair in* $\mathsf{GT}_{pr}(N)$*. Then the following statements are equivalent:*

1. *The homomorphism* $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/N$ *is onto.*

2. *The homomorphism* $T^{\mathrm{PB}_3}_{m,f} : \mathrm{PB}_3 \to \mathrm{PB}_3/N$ *is onto.*

3. *The homomorphism* $T^{\mathrm{F}_2}_{m,f} : \mathrm{F}_2 \to \mathrm{F}_2/N_{\mathrm{F}_2}$ *is onto.*

Proof. We will prove that 1) $\iff$ 2) and 2) $\iff$ 3).

1) $\Rightarrow$ 2). Recall that, since $N \le \mathrm{PB}_3$, the standard homomorphism $\rho : \mathrm{B}_3 \to S_3$ induces the homomorphism $\rho_N : \mathrm{B}_3/N \to S_3$. Moreover, for every $[m,f] \in \mathsf{GT}_{pr}(N)$,

$$\rho_N \circ T_{m,f} = \rho. \tag{2.33}$$

Let $w \in \mathrm{PB}_3$. Since $T_{m,f}$ is onto, there exists $v \in \mathrm{B}_3$ such that $T_{m,f}(v) = wN$. Applying ρ_N to $T_{m,f}(v) = wN$ and using (2.33), we conclude that $v \in \mathrm{PB}_3$. Thus we proved the implication 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 1). Let $w \in \mathrm{B}_3$. Equation (2.33) implies that

$$T_{m,f}(w^{-1})wN \in \mathrm{PB}_3/N.$$

Since the homomorphism $T^{\mathrm{PB}_3}_{m,f} := T_{m,f}\big|_{\mathrm{PB}_3}$ is onto, there exists $h \in \mathrm{PB}_3$ such that

$$T_{m,f}(w^{-1})wN = T_{m,f}(h).$$

Hence $wN = T_{m,f}(wh)$. Thus we proved the implication 2) $\Rightarrow$ 1).

2) $\Rightarrow$ 3). It suffices to show $x_{12}N_{\mathrm{F}_2}, x_{23}N_{\mathrm{F}_2} \in \mathrm{F}_2/N_{\mathrm{F}_2}$. Since $\mathrm{ord}(x_{12}N_{\mathrm{F}_2})$, $\mathrm{ord}(x_{23}N_{\mathrm{F}_2}) = \mathrm{ord}(f^{-1}x_{23}fN_{\mathrm{F}_2})$ are coprime with $2m+1$, there are integers q_1, q_2 such that $x_{12}N_{\mathrm{F}_2} = T^{\mathrm{F}_2}_{m,f}(x_{12}^{q_1}), f^{-1}x_{23}fN_{\mathrm{F}_2} = T^{\mathrm{F}_2}_{m,f}(x_{23}^{q_2})$. The surjectivity of $T^{\mathrm{PB}_3}_{m,f}$ implies that there exists $w \in \mathrm{PB}_3$ such that $fN = T^{\mathrm{PB}_3}_{m,f}(w)$. Write $w = \tilde{w} \cdot c^k, \tilde{w} \in \mathrm{F}_2$ and set $v := \tilde{w}^{-1}x_{23}^{q_2}\tilde{w} \in \mathrm{F}_2$. Then, using $c \in \mathcal{Z}(\mathrm{PB}_3)$ and $fN = T^{\mathrm{PB}_3}_{m,f}(w)$, we get

$$T^{\mathrm{F}_2}_{m,f}(v) = T^{\mathrm{PB}_3}_{m,f}(\tilde{w}^{-1}x_{23}^{q_2}\tilde{w}) = T^{\mathrm{PB}_3}_{m,f}(w^{-1}x_{23}^{q_2}w) = ff^{-1}x_{23}ffN = x_{23}N.$$

Since we identify F_2/N_{F_2} with the subgroup of PB_3/N, the above calculation shows that $x_{23}N_{F_2}$ belongs to the image of $T^{F_2}_{m,f}$. The implication 2) $\Rightarrow$ 3) is proved.

3) $\Rightarrow$ 2). Since PB_3/N is generated by $x_{12}N$, $x_{23}N$ and cN, it suffices to show that $x_{12}N, x_{23}N$ and cN belong to the image of $T^{PB_3}_{m,f}$.

Since $T^{F_2}_{m,f}$ is onto, $x_{12}N, x_{23}N$ belong the image of $T^{PB_3}_{m,f}$.

We have $T^{PB_3}_{m,f}(c) = c^{2m+1}N$. Since $\mathrm{ord}(cN)$ is coprime with $2m+1$ and $c^{2m+1}N \in T^{PB_3}_{m,f}(PB_3)$, we conclude that cN also belongs to the image of $T^{PB_3}_{m,f}$. The implication 3) $\Rightarrow$ 2) is proved. □

Definition 2.8 *Let $N \in \mathsf{NFI}_{PB_3}(B_3)$. A* GT*-**shadow with the target** N is a charming* GT*-pair $[m, f] \in \mathsf{GT}_{pr}(N)$ for which one of the three equivalent conditions of Proposition 2.7 is satisfied.*

We denote by $\mathsf{GT}(N)$ the set of GT-shadows with the target N.

Remark 2.9 *Note that, for every $N \in \mathsf{NFI}_{PB_3}(B_3)$, the pairs $(0, 1_{F_2})$ and $(-1, 1_{F_2})$ represent elements in*[1] $\mathsf{GT}(N)$*. In particular,* $\mathsf{GT}(N)$ *non-empty for every $N \in \mathsf{NFI}_{PB_3}(B_3)$. We also have*

$$T_{-1,1_{F_2}}(\sigma_1) = \sigma_1^{-1}N, \qquad T_{-1,1_{F_2}}(\sigma_2) = \sigma_2^{-1}N,$$

and

$$T^{PB_3}_{-1,1_{F_2}}(x_{12}) = x_{12}^{-1}N, \quad T^{PB_3}_{-1,1_{F_2}}(x_{23}) = x_{23}^{-1}N, \quad T^{PB_3}_{-1,1_{F_2}}(c) = c^{-1}N.$$

Let $N \in \mathsf{NFI}_{PB_3}(B_3)$, $[m, f] \in \mathsf{GT}(N)$ and

$$K := \ker(B_3 \xrightarrow{T_{m,f}} B_3/N).$$

Due to (2.17), $K \in \mathsf{NFI}_{PB_3}(B_3)$. Moreover

$$K_{F_2} = \ker(F_2 \xrightarrow{T^{F_2}_{m,f}} F_2/N_{F_2}).$$

[1] In fact, the verification of the hexagon relations for the pair $(-1, 1_{F_2})$ requires a little bit of work.

Since $T_{m,f}$ is onto, it induces an isomorphism

$$T^{\mathrm{isom}}_{m,f} : \mathrm{B}_3/K \xrightarrow{\simeq} \mathrm{B}_3/N \tag{2.34}$$

for which the diagram

$$\begin{array}{ccc} \mathrm{B}_3 & & \\ {\scriptstyle \mathcal{P}_K}\downarrow & \searrow^{T_{m,f}} & \\ \mathrm{B}_3/K & \xrightarrow{T^{\mathrm{isom}}_{m,f}} & \mathrm{B}_3/N \end{array} \tag{2.35}$$

commutes.

Similarly, the homomorphisms $T^{\mathrm{PB}_3}_{m,f}$ and $T^{\mathrm{F}_2}_{m,f}$ induce isomorphisms

$$T^{\mathrm{PB}_3,\mathrm{isom}}_{m,f} : \mathrm{PB}_3/K \xrightarrow{\simeq} \mathrm{PB}_3/N \tag{2.36}$$

and

$$T^{\mathrm{F}_2,\mathrm{isom}}_{m,f} : \mathrm{F}_2/K_{\mathrm{F}_2} \xrightarrow{\simeq} \mathrm{F}_2/N_{\mathrm{F}_2} \tag{2.37}$$

respectively.

This way we proved the first three statements of the following proposition:

Proposition 2.10 *Let* $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, $[m, f] \in \mathsf{GT}(N)$ *and*

$$K := \ker(\mathrm{B}_3 \xrightarrow{T_{m,f}} \mathrm{B}_3/N).$$

Then

a) the finite groups B_3/K *and* B_3/N *are isomorphic;*

b) the finite groups PB_3/K *and* PB_3/N *are isomorphic;*

c) the finite groups $\mathrm{F}_2/K_{\mathrm{F}_2}$ *and* $\mathrm{F}_2/N_{\mathrm{F}_2}$ *are isomorphic;*

d) $K_{\mathrm{ord}} = N_{\mathrm{ord}}$.

Proof. Due to the above discussions, it remains to prove statement *d)*.

Since the isomorphism $T^{\mathrm{PB}_3,\mathrm{isom}}_{m,f} : \mathrm{PB}_3/K \xrightarrow{\simeq} \mathrm{PB}_3/N$ is induced by the homomorphism $T^{\mathrm{PB}_3}_{m,f}$, equations (2.18) imply that

$$\begin{aligned} T^{\mathrm{PB}_3,\mathrm{isom}}_{m,f}(x_{12}K) &= x_{12}^{2m+1}N, \\ T^{\mathrm{PB}_3,\mathrm{isom}}_{m,f}(x_{23}K) &= f^{-1}x_{23}^{2m+1}fN, \\ T^{\mathrm{PB}_3,\mathrm{isom}}_{m,f}(cK) &= cN. \end{aligned} \tag{2.38}$$

Therefore

$$\begin{aligned} \mathrm{ord}(x_{12}K) &= \mathrm{ord}(x_{12}^{2m+1}N), \\ \mathrm{ord}(x_{23}K) &= \mathrm{ord}(f^{-1}x_{23}^{2m+1}fN) = \mathrm{ord}(x_{23}^{2m+1}N), \\ \mathrm{ord}(cK) &= \mathrm{ord}(c^{2m+1}N). \end{aligned} \tag{2.39}$$

Combining these observations with the fact that $2m+1$ is coprime with $\mathrm{ord}(x_{12}N)$, $\mathrm{ord}(x_{23}N)$ and $\mathrm{ord}(cN)$, we conclude that

$$\mathrm{ord}(x_{12}K) = \mathrm{ord}(x_{12}N), \quad \mathrm{ord}(x_{23}K) = \mathrm{ord}(x_{23}N), \quad \mathrm{ord}(cK) = \mathrm{ord}(cN).$$

Thus $K_{\mathrm{ord}} = N_{\mathrm{ord}}$, i.e. statement *d)* is proved. □

2.3 The composition of GT-shadows and the groupoid GTSh

For every pair $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$, the formulas

$$E_{m,f}(x) := x^{2m+1}, \qquad E_{m,f}(x) := f^{-1}x^{2m+1}f$$

define an endomorphism of F_2

$$E_{m,f} : \mathrm{F}_2 \to \mathrm{F}_2\,. \tag{2.40}$$

A direct computation shows that, for all $(m_1, f_1), (m_2, f_2) \in \mathbb{Z} \times \mathrm{F}_2$,

$$E_{m_1,f_1} \circ E_{m_2,f_2} = E_{m,f}\,, \tag{2.41}$$

where

$$m := 2m_1m_2 + m_1 + m_2, \qquad f := f_1E_{m_1,f_1}(f_2). \tag{2.42}$$

Let us prove that

Proposition 2.11 *The set $\mathbb{Z} \times \mathrm{F}_2$ is a monoid with respect to the binary operation defined by the formula*

$$(m_1, f_1) \bullet (m_2, f_2) := (2m_1m_2 + m_1 + m_2, f_1E_{m_1,f_1}(f_2)). \tag{2.43}$$

Moreover, the assignment $(m, f) \mapsto E_{m,f}$ defines a homomorphism of monoids $(\mathbb{Z} \times \mathrm{F}_2, \bullet) \to \mathrm{End}(\mathrm{F}_2)$.

Proof. Equations (2.41) and (2.42) imply that

$$E_{m_1,f_1} \circ E_{m_2,f_2} = E_{(m_1,f_1)\bullet(m_2,f_2)}\,. \tag{2.44}$$

In other words, the assignment $(m, f) \mapsto E_{m,f}$ is a homomorphism of magmas $(\mathbb{Z} \times \mathrm{F}_2, \bullet) \to \mathrm{End}(\mathrm{F}_2)$.

Let $(m_1, f_1), (m_2, f_2), (m_3, f_3) \in \mathbb{Z} \times \mathrm{F}_2$. We have

$$\big((m_1, f_1) \bullet (m_2, f_2)\big) \bullet (m_3, f_3) = \big(2m_1m_2 + m_1 + m_2, f_1E_{m_1,f_1}(f_2)\big) \bullet (m_3, f_3) =$$

$$\big(2(2m_1m_2 + m_1 + m_2)m_3 + 2m_1m_2 + m_1 + m_2 + m_3, fE_{m,f}(f_3)\big) =$$

$$\big(4m_1m_2m_3 + 2m_1m_2 + 2m_2m_3 + 2m_1m_3 + m_1 + m_2 + m_3, fE_{m,f}(f_3),\big)$$

where $(m, f) := (m_1, f_1) \bullet (m_2, f_2)$.

Using (2.44), we get

$$\big((m_1, f_1) \bullet (m_2, f_2)\big) \bullet (m_3, f_3) =$$

$$\big(4m_1m_2m_3 + 2m_1m_2 + 2m_2m_3 + 2m_1m_3 + m_1 + m_2 + m_3, fE_{m_1,f_1}E_{m_2,f_2}(f_3)\big) =$$

$$\big(4m_1m_2m_3 + 2m_1m_2 + 2m_2m_3 + 2m_1m_3 + m_1 + m_2 + m_3, f_1E_{m_1,f_1}(f_2)E_{m_1,f_1}E_{m_2,f_2}(f_3)\big).$$

Since E_{m_1,f_1} is a group homomorphism, we can simplify the expression

$$\big((m_1, f_1) \bullet (m_2, f_2)\big) \bullet (m_3, f_3)$$

further as

$$\big((m_1, f_1) \bullet (m_2, f_2)\big) \bullet (m_3, f_3) = \tag{2.45}$$

$$\big(4m_1m_2m_3+2m_1m_2+2m_2m_3+2m_1m_3+m_1+m_2+m_3,\, f_1E_{m_1,f_1}(f_2E_{m_2,f_2}(f_3))\big).$$

Unfolding the expression $(m_1, f_1) \bullet \big((m_2, f_2) \bullet (m_3, f_3)\big)$, we see that

$$(m_1, f_1) \bullet \big((m_2, f_2) \bullet (m_3, f_3)\big) =$$

$$\big(4m_1m_2m_3+2m_1m_2+2m_2m_3+2m_1m_3+m_1+m_2+m_3,\, f_1E_{m_1,f_1}(f_2E_{m_2,f_2}(f_3))\big).$$

Thus the binary operation $\bullet$ is associative. It is easy to see that $(0, 1_{\mathrm{F}_2})$ is the identity element of $(\mathbb{Z} \times \mathrm{F}_2, \bullet)$ and $E_{0,1_{\mathrm{F}_2}} = \mathrm{id}_{\mathrm{F}_2}$. □

Let (m, f) be a pair that represents a GT-shadow with the target $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. Equations (2.20) imply that

$$T^{\mathrm{F}_2}_{m,f}(w) = E_{m,f}(w)N_{\mathrm{F}_2}\,. \tag{2.46}$$

This important link between the endomorphism $E_{m,f}$ and the homomorphism $T^{\mathrm{F}_2}_{m,f} : \mathrm{F}_2 \to \mathrm{F}_2/N_{\mathrm{F}_2}$ is used below in the proof of Theorem 2.12.

We are now ready to construct the groupoid GTSh. The objects of GTSh are elements of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. Furthermore, for $K, N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, we set

$$\mathsf{GTSh}(K, N) := \{[m, f] \in \mathsf{GT}(N) \mid \ker(T_{m,f}) = K\}. \tag{2.47}$$

For $N^{(1)}, N^{(2)}, N^{(3)} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, $[m_1, f_1] \in \mathsf{GTSh}(N^{(2)}, N^{(1)})$ and $[m_2, f_2] \in \mathsf{GTSh}(N^{(3)}, N^{(2)})$ we set

$$m := 2m_1m_2 + m_1 + m_2, \qquad f := f_1E_{m_1,f_1}(f_2). \tag{2.48}$$

We claim that

Theorem 2.12 *The assignment*

$$[m_1, f_1] \bullet [m_2, f_2] := \big[2m_1m_2 + m_1 + m_2\ ,\ f_1E_{m_1,f_1}(f_2)\,\big] \tag{2.49}$$

defines a composition of morphisms in GTSh*. With respect to this composition law,* GTSh *is a groupoid. Moreover, if* $m = 2m_1m_2 + m_1 + m_2$ *and* $f = f_1 E_{m_1,f_1}(f_2)$*, then*

$$T_{m,f} = T^{\text{isom}}_{m_1,f_1} \circ T_{m_2,f_2}\,, \qquad T^{\text{isom}}_{m,f} = T^{\text{isom}}_{m_1,f_1} \circ T^{\text{isom}}_{m_2,f_2}\,, \tag{2.50}$$

$$T^{\mathrm{PB}_3}_{m,f} = T^{\mathrm{PB}_3,\text{isom}}_{m_1,f_1} \circ T^{\mathrm{PB}_3}_{m_2,f_2}\,, \qquad T^{\mathrm{PB}_3,\text{isom}}_{m,f} = T^{\mathrm{PB}_3,\text{isom}}_{m_1,f_1} \circ T^{\mathrm{PB}_3,\text{isom}}_{m_2,f_2}\,, \tag{2.51}$$

and

$$T^{\mathrm{F}_2}_{m,f} = T^{\mathrm{F}_2,\text{isom}}_{m_1,f_1} \circ T^{\mathrm{F}_2}_{m_2,f_2}\,, \qquad T^{\mathrm{F}_2,\text{isom}}_{m,f} = T^{\mathrm{F}_2,\text{isom}}_{m_1,f_1} \circ T^{\mathrm{F}_2,\text{isom}}_{m_2,f_2}\,. \tag{2.52}$$

Proof. Since $\mathsf{GTSh}(N^{(3)}, N^{(2)})$ and $\mathsf{GTSh}(N^{(2)}, N^{(1)})$ are non-empty, statement *d)* of Proposition 2.10 implies that $N^{(1)}_{\text{ord}} = N^{(2)}_{\text{ord}} = N^{(3)}_{\text{ord}}$. So we simply set $N_{\text{ord}} := N^{(1)}_{\text{ord}}$.

We set

$$(m, f) := \big(2m_1m_2 + m_1 + m_2\,,\ f_1 E_{m_1,f_1}(f_2)\big). \tag{2.53}$$

First we want to show that the composition is well defined, i.e.

- the residue class $m + N_{\text{ord}}\mathbb{Z}$ depends only on the residue classes of m_1 and m_2 modulo N_{ord};
- the coset $fN^{(1)}$ depends only on the cosets $f_1N^{(1)}$ and $f_2N^{(2)}$;
- the pair (m, f) represents a charming GT-pair (with the target $N^{(1)}$), i.e. (m, f) satisfies hexagon relations (2.9), (2.10) modulo $N^{(1)}$, $2m+1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$ and $fN^{(1)}$ belongs to $[\mathrm{F}_2/N^{(1)}, \mathrm{F}_2/N^{(1)}]$;
- the group homomorphism $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/N^{(1)}$ is onto and its kernel is $N^{(3)}$.

The statement about the residue class of m is obvious.

Since $N^{(2)}_{\mathrm{F}_2} = \ker(\mathrm{F}_2 \stackrel{T^{\mathrm{F}_2}_{m_1,f_1}}{\longrightarrow} \mathrm{F}_2/N^{(1)}_{\mathrm{F}_2})$, $E_{m_1,f_1}(h) \in N^{(1)}$ for every $h \in N^{(2)}$. Hence the statement about the independence of the coset $f_1E_{m_1,f_1}(f_2)\,N^{(1)}$ on representatives f_1 and f_2 follows.

Since

$$2m+1 = 2(2m_1m_2 + m_1 + m_2) + 1 = (2m_1+1)(2m_2+1), \tag{2.54}$$

and $(2m_1+1), (2m_2+1)$ represent units in $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$, $2m+1$ also represents a unit in $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$.

Let us prove that the pair (m, f) satisfies the hexagon relations modulo $N^{(1)}$.

To prove that (m, f) satisfies (2.9), we apply $T^{\mathrm{isom}}_{m_1,f_1} : \mathrm{B}_3/N^{(2)} \to \mathrm{B}_3/N^{(1)}$ to the first hexagon relation for (m_2, f_2):

$$\sigma_1^{2m_2+1} f_2^{-1} \sigma_2^{2m_2+1} f_2 N^{(2)} = f_2^{-1} \sigma_1 \sigma_2 x_{12}^{-m_2} c^{m_2} N^{(2)} \,.$$

Using (2.46), we get

$$\begin{aligned} \sigma_1^{(2m_1+1)(2m_2+1)} E_{m_1,f_1}(f_2)^{-1} f_1^{-1} \sigma_2^{(2m_1+1)(2m_2+1)} f_1 E_{m_1,f_1}(f_2)\, N^{(1)} \ &= \\ E_{m_1,f_1}(f_2)^{-1} \sigma_1^{2m_1+1} f_1^{-1} \sigma_2^{2m_1+1} f_1 x_{12}^{-m_2(2m_1+1)} c^{m_2(2m_1+1)}\, N^{(1)}. \end{aligned} \tag{2.55}$$

Due to (2.54), the left hand side of (2.55) is precisely the left hand side of (2.9) for (m, f).

Using the first hexagon relation for (m_1, f_1), we can rewrite the right hand side of (2.55) as follows

$$E_{m_1,f_1}(f_2)^{-1} \sigma_1^{2m_1+1} f_1^{-1} \sigma_2^{2m_1+1} f_1 x_{12}^{-m_2(2m_1+1)} c^{m_2(2m_1+1)}\, N^{(1)} =$$

$$E_{m_1,f_1}(f_2)^{-1} f_1^{-1} \sigma_1 \sigma_2 x_{12}^{-m_1} c^{m_1} x_{12}^{-m_2(2m_1+1)} c^{m_2(2m_1+1)}\, N^{(1)} \ = \ f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^{m}\, N^{(1)} \,.$$

Thus the right hand side of (2.55) coincides with the right hand side of the first hexagon relation for (m, f).

We proved that the pair (m, f) satisfies (2.9) modulo $N^{(1)}$.

Applying $T^{\mathrm{isom}}_{m_1,f_1} : \mathrm{B}_3/N^{(2)} \to \mathrm{B}_3/N^{(1)}$ to the second hexagon relation for (m_2, f_2), using (2.46) and the second hexagon relation for (m_1, f_1) we see that (m, f) also satisfies (2.10) modulo $N^{(1)}$.

Let us prove that $fN^{(1)}$ belongs to the commutator subgroup $[\mathrm{F}_2/N^{(1)}, \mathrm{F}_2/N^{(1)}]$.

We may assume, without loss of generality, that $f_1, f_2 \in [\mathrm{F}_2, \mathrm{F}_2]$. Therefore $f_1 E_{m_1,f_1}(f_2) \in [\mathrm{F}_2, \mathrm{F}_2]$ and hence $f N^{(1)} \in [\mathrm{F}_2/N^{(1)}, \mathrm{F}_2/N^{(1)}]$.

We proved that the pair (m, f) represents a charming GT-pair with the target $N^{(1)}$.

Applying $T_{m,f}$ and $T^{\mathrm{isom}}_{m_1,f_1} \circ T_{m_2,f_2}$ to the generators σ_1, σ_2, we see that the first equation in (2.50) holds. Using this equation, it is easy to see that the homomorphism $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/N^{(1)}$ is onto and its kernel is $N^{(3)}$.

We proved that the formula (2.49) defines a composition of GT-shadows.

Since $(0, 1_{\mathrm{F}_2})$ is the identity element of the monoid $(\mathbb{Z} \times \mathrm{F}_2, \bullet)$, the GT-shadow $[0, 1_{\mathrm{F}_2}] \in \mathsf{GT}(N)$ is the identity morphism in $\mathsf{GTSh}(N, N)$.

The associativity of the composition of GT-shadows follows from the associativity of the operation $\bullet$ on $\mathbb{Z} \times \mathrm{F}_2$.

We already proved that the first equation in (2.50) holds. It is easy to see that the first equation in (2.50) implies the second equation in (2.50).

Similarly, applying $T^{\mathrm{PB}_3}_{m,f}$ (resp. $T^{\mathrm{F}_2}_{m,f}$) and $T^{\mathrm{F}_2,\mathrm{isom}}_{m_1,f_1} \circ T^{\mathrm{F}_2}_{m_2,f_2}$ to the generators x_{12}, x_{23}, c (resp. $x := x_{12}$, $y := x_{23}$), we see that the first equation in (2.51) (resp. in (2.52)) holds. It is not hard to see that the second equation in (2.51) (resp. in (2.52)) follows from the first one.

It remains to prove that, for every $[m, f] \in \mathsf{GTSh}(K, N)$, there exists $[m', f'] \in \mathsf{GTSh}(N, K)]$ such that $[m', f'] \bullet [m, f] = [0, 1_{\mathrm{F}_2}]$ and $[m, f] \bullet [m', f'] = [0, 1_{\mathrm{F}_2}]$.

As above, since $\mathsf{GTSh}(K, N)$ is non-empty, Proposition 2.10 implies that $K_{\mathrm{ord}} = N_{\mathrm{ord}}$.

Since $2m + 1$ represents a unit in $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$, there exists $m' \in \mathbb{Z}$ such that

$$(2m + 1)m' \equiv -m \bmod N_{\mathrm{ord}} \,. \tag{2.56}$$

Therefore $4mm' + 2m' + 2m \equiv 0 \bmod N_{\mathrm{ord}}$ and hence

$$4mm' + 2m' + 2m + 1 = (2m + 1)(2m' + 1) \equiv 1 \bmod N_{\mathrm{ord}} \,. \tag{2.57}$$

In particular, $2m'+1$ represents a unit in $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$.

Since $T^{\mathrm{F}_2}_{m,f} : \mathrm{F}_2 \to \mathrm{F}_2/N_{\mathrm{F}_2}$ is onto and $fN_{\mathrm{F}_2} \in [\mathrm{F}_2/N_{\mathrm{F}_2}, \mathrm{F}_2/N_{\mathrm{F}_2}]$, there exists $f' \in [\mathrm{F}_2, \mathrm{F}_2]$ such that

$$T^{\mathrm{F}_2}_{m,f}(f') = f^{-1} N_{\mathrm{F}_2} \,. \tag{2.58}$$

Hence f' satisfies the equation

$$f N_{\mathrm{F}_2}\, T^{\mathrm{F}_2}_{m,f}(f') = 1_{\mathrm{F}_2/N_{\mathrm{F}_2}} \,. \tag{2.59}$$

Moreover, since $\ker(T^{\mathrm{F}_2}_{m,f}) = K_{\mathrm{F}_2}$, a solution f' of (2.59) is unique up to multiplication by elements of K_{F_2}.

To prove that (m', f') satisfies

$$\sigma_1^{2m'+1}\,(f')^{-1}\sigma_2^{2m'+1} f'\, K \stackrel{?}{=} (f')^{-1}\sigma_1\sigma_2 x_{12}^{-m'} c^{m'}\, K, \tag{2.60}$$

we apply $T^{\mathrm{F}_2,\mathrm{isom}}_{m,f}$ to both sides of (2.60).

Using (2.46), we get

$$\begin{gathered} \sigma_1^{(2m'+1)(2m+1)}\, E_{m,f}(f')^{-1} f^{-1} \sigma_2^{(2m'+1)(2m+1)} f E_{m,f}(f')\, N \stackrel{?}{=} \\ E_{m,f}(f')^{-1} \sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f x_{12}^{-m'(2m+1)} c^{m'(2m+1)}\, N. \end{gathered} \tag{2.61}$$

Using (2.9) for $[m, f]$, (2.56) and $fE_{m,f}(f') \in N_{\mathrm{F}_2}$, we rewrite the right hand side of (2.61) as follows

$$\begin{gathered} E_{m,f}(f')^{-1} \sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f x_{12}^{-m'(2m+1)} c^{m'(2m+1)}\, N = \\ E_{m,f}(f')^{-1} f^{-1} \sigma_1\sigma_2 x_{12}^{-m} c^m x_{12}^{-m'(2m+1)} c^{m'(2m+1)}\, N = \\ \sigma_1\sigma_2 x_{12}^{-2mm'-m-m'} c^{2mm'+m+m'}\, N = \sigma_1\sigma_2 N. \end{gathered} \tag{2.62}$$

Using (2.57) and $fE_{m,f}(f') \in N_{\mathrm{F}_2}$ we see that

$$\sigma_1^{(2m'+1)(2m+1)}\, E_{m,f}(f')^{-1} f^{-1} \sigma_2^{(2m'+1)(2m+1)} f E_{m,f}(f')\, N \;=\; \sigma_1\sigma_2 N.$$

Combining this observation with (2.62), we conclude that (2.61) holds. Since (2.61) is equivalent to (2.60), equation (2.60) also holds.

Similarly, to prove that (m', f') satisfies

$$(f')^{-1}\sigma_2^{2m'+1} f'\, \sigma_1^{2m'+1}\, K \overset{?}{=} \sigma_2\sigma_1 x_{23}^{-m'} c^{m'} f'\, K, \tag{2.63}$$

we apply $T^{\mathrm{F}_2,\mathrm{isom}}_{m,f}$ to both sides of (2.63).

Using (2.46), we get

$$\begin{aligned} E_{m,f}(f')^{-1} f^{-1}\sigma_2^{(2m'+1)(2m+1)} f E_{m,f}(f')\,\sigma_1^{(2m'+1)(2m+1)}\, N \overset{?}{=} \\ f^{-1}\sigma_2^{2m+1} f\, \sigma_1^{2m+1} f^{-1} x_{23}^{-m'(2m+1)} f c^{m'(2m+1)} E_{m,f}(f')\, N. \end{aligned} \tag{2.64}$$

Using (2.10) for $[m, f]$, (2.56), $f E_{m,f}(f') \in N_{\mathrm{F}_2}$, and (2.57), we deduce that both sides of (2.64) equal $\sigma_2\sigma_1\, N$. Therefore, since (2.64) is equivalent to (2.63), we conclude that (2.63) holds.

We proved that the pair (m', f') represents a charming GT-pair with the target K.

Equation (2.56) and $\mathrm{ord}(\sigma_1^2 N)|N_{\mathrm{ord}}$, $\mathrm{ord}(\sigma_2^2 N)|N_{\mathrm{ord}}$ imply that

$$\sigma_1^{(2m+1)(2m'+1)} N = \sigma_1 N, \qquad \sigma_2^{(2m+1)(2m'+1)} N = \sigma_2 N. \tag{2.65}$$

Applying $T^{\mathrm{isom}}_{m,f} \circ T_{m',f'}$ to the generators σ_1, σ_2, and using $f E_{m,f}(f') \in N_{\mathrm{F}_2}$ and (2.65), we see that

$$T^{\mathrm{isom}}_{m,f} \circ T_{m',f'} = \mathcal{P}_N \tag{2.66}$$

and hence $T_{m',f'} : \mathrm{B}_3 \to \mathrm{B}_3/N$ is onto and $\ker(T_{m',f'}) = N$. Thus the pair (m', f') represents an element in $\mathsf{GTSh}(N, K)$.

Identity (2.66) also implies that

$$T^{\mathrm{F}_2}_{m,f} \circ E_{m',f'} = \mathcal{P}_{N_{\mathrm{F}_2}} \ :\ \mathrm{F}_2 \to \mathrm{F}_2/N_{\mathrm{F}_2}\,. \tag{2.67}$$

It remains to prove that

$$[m', f'] \bullet [m, f] = [0, 1_{\mathrm{F}_2}]. \tag{2.68}$$

We set

$$k := 2mm' + m + m', \qquad h := f' E_{m',f'}(f).$$

Equation (2.56) implies that[2] $k \equiv 0 \bmod K_{\mathrm{ord}}$. Thus it remains to prove that $h \in K_{\mathrm{F}_2}$.

Applying $T^{\mathrm{F}_2}_{m,f}$ to h, and using (2.58) and (2.67), we get

$$T^{\mathrm{F}_2}_{m,f}(h) = T^{\mathrm{F}_2}_{m,f}(f')T^{\mathrm{F}_2}_{m,f}(E_{m',f'}(f)) = T^{\mathrm{F}_2}_{m,f}(f')fN = 1_{\mathrm{F}_2/N_{\mathrm{F}_2}}.$$

Therefore, since $\ker(T^{\mathrm{F}_2}_{m,f}) = K_{\mathrm{F}_2}$, we conclude that $h \in K_{\mathrm{F}_2}$ and equation (2.68) is proved.

Theorem 2.12 is proved. □

Let us prove the following statement:

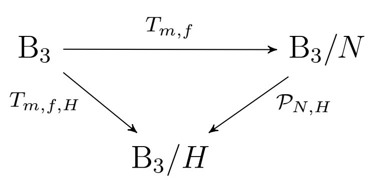
Proposition 2.13 *Let $N, H \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, $N \le H$ and $(m,f) \in \mathbb{Z} \times \mathrm{F}_2$ represent a GT-pair with the target N. Then $H_{\mathrm{ord}} | N_{\mathrm{ord}}$, $N_{\mathrm{F}_2} \le H_{\mathrm{F}_2}$ and*

a) the same pair (m,f) also represents an element in $\mathsf{GT}_{pr}(H)$;

b) if the GT-pair $[m,f] \in \mathsf{GT}_{pr}(N)$ is charming then so is the corresponding GT-pair in $\mathsf{GT}_{pr}(H)$;

c) if the pair (m,f) represents a GT-shadow with the target N, then (m,f) also represents a GT-shadow with the target K.

Let us denote by $T_{m,f,H}$ the group homomorphism $\mathrm{B}_3 \to \mathrm{B}_3/H$ corresponding to $[m,f] \in \mathsf{GT}_{pr}(H)$. In the set-up of statement a), the following diagram



(2.69)

commutes.

Proof. Since

$$\mathcal{P}_{N,H}(x_{12}N) = x_{12}H, \qquad \mathcal{P}_{N,H}(x_{23}N) = x_{23}H, \qquad \mathcal{P}_{N,H}(cN) = cH,$$

[2]Recall that $K_{\mathrm{ord}} = N_{\mathrm{ord}}$.

$\mathrm{ord}(x_{12}H)|\mathrm{ord}(x_{12}N)$, $\mathrm{ord}(x_{23}H)|\mathrm{ord}(x_{23}N)$ and $\mathrm{ord}(cH)|\mathrm{ord}(cN)$. Hence H_{ord} divides N_{ord}. It is obvious that $N_{\mathrm{F}_2} \le H_{\mathrm{F}_2}$.

a). Since the pair (m, f) represents an element in $\mathsf{GT}_{pr}(N)$, (m, f) satisfies hexagon relation (2.9), (2.10). Applying the homomorphism $\mathcal{P}_{N,H} : \mathrm{B}_3/N \to \mathrm{B}_3/H$ to (2.9) and (2.10), we see that the pair (m, f) also satisfies the hexagon relations modulo H. Thus the same pair (m, f) also represents an element in $\mathsf{GT}_{pr}(H)$.

As above, we denote by $T_{m,f,H}$ the group homomorphism $\mathrm{B}_3 \to \mathrm{B}_3/H$ corresponding to $[m, f] \in \mathsf{GT}_{pr}(H)$. Applying $T_{m,f,H}$ and $\mathcal{P}_{N,H} \circ T_{m,f}$ to the generators σ_1, σ_2, we see that the diagram in (2.69) indeed commutes.

b). Since $2m + 1$ represents a unit in $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$ then $2m + 1$ also represents a unit in $\mathbb{Z}/H_{\mathrm{ord}}\mathbb{Z}$. Since fN_{F_2} belongs to the commutator subgroup $[\mathrm{F}_2/N_{\mathrm{F}_2}, \mathrm{F}_2/N_{\mathrm{F}_2}]$, we have

$$fH_{\mathrm{F}_2} \in [\mathrm{F}_2/H_{\mathrm{F}_2}, \mathrm{F}_2/H_{\mathrm{F}_2}].$$

Thus (m, f) represents a charming GT-pair with the target H.

c). This statement follows easily from the commutativity of the diagram in (2.69) and the surjectivity of the homomorphism $\mathcal{P}_{N,H}$. $\square$

2.4 Isolated elements of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$

The groupoid GTSh is highly disconnected. Indeed, if N and K have different indices in PB_3, then the set of morphisms $\mathsf{GTSh}(K, N)$ is empty. (See Proposition 2.10.) We denote by $\mathsf{GTSh}^{\mathrm{conn}}_N$ the connected component of N in GTSh. It is easy to see that, for every $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, the groupoid $\mathsf{GTSh}^{\mathrm{conn}}_N$ is finite. Elements of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ for which $\mathsf{GTSh}^{\mathrm{conn}}_N$ has exactly one object play a special role.

Definition 2.14 *Let $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. A GT-shadow $[m,f] \in \mathsf{GT}(N)$ is called* ***settled*** *if*

$$\ker(T_{m,f}) = N,$$

i.e. $[m,f]$ is an automorphism of N in GTSh. An element $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ is called ***isolated*** *if every GT-shadow $[m,f] \in \mathsf{GT}(N)$ is settled.*

It is clear that $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ is isolated if and only if $\mathsf{GTSh}^{\mathrm{conn}}_N$ has exactly one object. Of course, in this case, the set $\mathsf{GT}(N)$ is a group.

Due to the following proposition, the subposet $\mathsf{NFI}^{isolated}_{\mathrm{PB}_3}(\mathrm{B}_3)$ of isolated elements in $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ is cofinal:

Proposition 2.15 *For every $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$*

$$N^{\diamond} := \bigcap_{K \in \mathrm{Ob}(\mathsf{GTSh}^{\mathrm{conn}}_N)} K$$

is an isolated element of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

Proof. Note that, for every $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, $\mathsf{GT}(N)$ is a finite set and, for every $K \in \mathrm{Ob}(\mathsf{GTSh}^{\mathrm{conn}}_N)$, there is $[m,f] \in \mathsf{GT}(N)$ such that

$$K = \ker(T_{m,f}).$$

Thus $\mathrm{Ob}(\mathsf{GTSh}^{\mathrm{conn}}_N$ is also finite.

Since $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ is closed under finite intersections, $N^{\diamond}$ belongs to $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

To prove that $N^{\diamond}$ is isolated, we consider $[m,f] \in \mathsf{GT}(N^{\diamond})$ and $K \in \mathrm{Ob}(\mathsf{GTSh}^{\mathrm{conn}}_N)$.

Since $N^{\diamond} \le K$, Proposition 2.13 implies that the pair (m,f) also represents a GT-shadow with the target K. Just as in Proposition 2.13, we denote by $T_{m,f,K}$ the group homomorphism $\mathrm{B}_3 \to \mathrm{B}_3/K$ corresponding to $[m,f] \in \mathsf{GT}(K)$. Let us also recall that

$$T_{m,f,K} = \mathcal{P}_{N^{\diamond},K} \circ T_{m,f}\,. \tag{2.70}$$

Let $w \in N^{\diamond}$. Since $w \in H$ for every $H \in \mathrm{Ob}(\mathsf{GTSh}^{\mathrm{conn}}_N)$, we have

$$w \in \ker(T_{m,f,K})$$

Let $w^\diamond \in \mathrm{B}_3$ be a representative of the coset $T_{m,f}(w) \in \mathrm{B}_3/N^\diamond$. Using (2.70) we conclude that $w^\diamond \in K$ for every $K \in \mathrm{Ob}(\mathsf{GTSh}_N^{\mathrm{conn}})$. Therefore $w^\diamond \in N^\diamond$ and hence $w \in \ker(\mathrm{B}_3 \xrightarrow{T_{m,f}} \mathrm{B}_3/N^\diamond)$.

We proved that $N^\diamond \le \tilde{K}$, where $\tilde{K} := \ker(\mathrm{B}_3 \xrightarrow{T_{m,f}} \mathrm{B}_3/N^\diamond)$. Since $|\mathrm{B}_3 : \tilde{K}| = |\mathrm{B}_3 : N^\diamond|$ (see Proposition 2.10) and $N^\diamond$ has finite index in B_3, we conclude that $\ker(\mathrm{B}_3 \xrightarrow{T_{m,f}} \mathrm{B}_3/N^\diamond) = N^\diamond$. $\square$

CHAPTER 3

THE GENTLE VERSION $\widehat{\mathsf{GT}}_{gen}$ OF THE GROTHENDIECK-TEICHMUELLER GROUP AND ITS LINK TO GTSh

3.1 The group $\widehat{\mathsf{GT}}_{gen}$

To introduce $\widehat{\mathsf{GT}}_{gen}$, we observe that, for every pair $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$, the group homomorphism

$$E_{\hat{m},\hat{f}}(x) := x^{2\hat{m}+1}, \qquad E_{\hat{m},\hat{f}}(y) := \hat{f}^{-1} y^{2\hat{m}+1} \hat{f} \tag{3.1}$$

$E_{\hat{m},\hat{f}} : \mathrm{F}_2 \to \widehat{\mathrm{F}}_2$ extends uniquely to a continuous endomorphism of $\widehat{\mathrm{F}}_2$:

$$E_{\hat{m},\hat{f}} : \widehat{\mathrm{F}}_2 \to \widehat{\mathrm{F}}_2 \,. \tag{3.2}$$

Moreover, if we compose two homomorphisms of this form, we also get a homomorphism of the form $E_{\hat{m},\hat{f}}$. Indeed, a direct computation shows that, for all $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2) \in \widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$,

$$E_{\hat{m}_1,\hat{f}_1} \circ E_{\hat{m}_2,\hat{f}_2} = E_{\hat{m},\hat{f}}, \tag{3.3}$$

where

$$\hat{m} := 2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2, \qquad \hat{f} := \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2).$$

This motivates us to define the following binary operation $\bullet$ on $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$

$$(\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2) := \big(2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2,\ \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2) \big). \tag{3.4}$$

Let us prove that

Proposition 3.1 *The set $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ is a monoid with respect to the binary operation $\bullet$ (3.4). The element $(0, 1_{\widehat{\mathrm{F}}_2})$ is the identity element of this monoid. Moreover, the assignment*

$$(\hat{m}, \hat{f}) \mapsto E_{\hat{m},\hat{f}}$$

defines a homomorphism of monoids $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2 \to \mathrm{End}(\widehat{\mathrm{F}}_2)$, where $\mathrm{End}(\widehat{\mathrm{F}}_2)$ is the monoid of continuous endomorphisms of $\widehat{\mathrm{F}}_2$.

Proof. It is easy to see that $(0, 1_{\widehat{\mathrm{F}}_2})$ is the identity element of the magma $\big(\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2, \bullet\big)$. So let us prove the associativity of $\bullet$.

For $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2), (\hat{m}_3, \hat{f}_3) \in \widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$, we have

$$\big((\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)\big) \bullet (\hat{m}_3, \hat{f}_3) = \big(2\hat{q}\hat{m}_3 + \hat{q} + \hat{m}_3, \hat{g} E_{\hat{q},\hat{g}}(\hat{f}_3)\big) \tag{3.5}$$

and

$$(\hat{m}_1, \hat{f}_1) \bullet \big((\hat{m}_2, \hat{f}_2) \bullet (\hat{m}_3, \hat{f}_3)\big) = (2\hat{m}_1\hat{k} + \hat{m}_1 + \hat{k}, \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}(\hat{h})), \tag{3.6}$$

where $(\hat{q}, \hat{g}) := (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$ and $(\hat{k}, \hat{h}) := (\hat{m}_2, \hat{f}_2) \bullet (\hat{m}_3, \hat{f}_3)$.

Using $\hat{q} := 2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2$ and $\hat{k} := 2\hat{m}_2\hat{m}_3 + \hat{m}_2 + \hat{m}_3$, it is easy to see that

$$2\hat{q}\hat{m}_3 + \hat{q} + \hat{m}_3 = 2\hat{m}_1\hat{k} + \hat{m}_1 + \hat{k}.$$

Using (3.3) and the fact that $E_{\hat{m}_1,\hat{f}_1}$ is an endomorphism of $\widehat{F}_2$, we can rewrite $\hat{g} E_{\hat{q},\hat{g}}(\hat{f}_3)$ as follows

$$\hat{g} E_{\hat{q},\hat{g}}(\hat{f}_3) = \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2)\, E_{\hat{m}_1,\hat{f}_1} \circ E_{\hat{m}_2,\hat{f}_2}(\hat{f}_3) = \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}\big(\hat{f}_2 E_{\hat{m}_2,\hat{f}_2}(\hat{f}_3)\big).$$

Thus $\hat{g} E_{\hat{q},\hat{g}}(\hat{f}_3) = \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}(\hat{h})$ and the associativity of $\bullet$ is proved.

Since $E_{0,1_{\widehat{\mathrm{F}}_2}} = \mathrm{id}_{\widehat{\mathrm{F}}_2}$, the last statement of the proposition follows from (3.3).

□

Remark 3.2 *Note that, if* $\hat{m} = 2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2$, *then*

$$2\hat{m} + 1 = (2\hat{m}_1 + 1)(2\hat{m}_2 + 1). \tag{3.7}$$

Let us denote by $\widehat{\mathsf{GT}}_{gen,mon}$ the subset of $\widehat{\mathbb{Z}} \times \widehat{\mathrm{F}}_2$ that consists of pairs

$$(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times [\widehat{\mathrm{F}}_2, \widehat{\mathrm{F}}_2]^{top.cl.} \tag{3.8}$$

satisfying the hexagon relations

$$\sigma_1^{2\hat{m}+1}\hat{f}^{-1}\sigma_2^{2\hat{m}+1}\hat{f} = \hat{f}^{-1}\sigma_1\sigma_2\, x_{12}^{-\hat{m}}c^{\hat{m}}\,, \tag{3.9}$$

$$\hat{f}^{-1}\sigma_2^{2\hat{m}+1}\hat{f}\,\sigma_1^{2\hat{m}+1} = \sigma_2\sigma_1 x_{23}^{-\hat{m}}c^{\hat{m}}\,\hat{f}\,. \tag{3.10}$$

Let us prove the following statement:

Proposition 3.3 *For every* $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen,mon}$, *the formulas*

$$T_{\hat{m},\hat{f}}(\sigma_1) := \sigma_1^{2\hat{m}+1}\,, \qquad T_{\hat{m},\hat{f}}(\sigma_2) := \hat{f}^{-1}\sigma_2^{2\hat{m}+1}\hat{f} \tag{3.11}$$

define a group homomorphism $T_{\hat{m},\hat{f}} : \mathrm{B}_3 \to \widehat{\mathrm{B}}_3$. *The homomorphism* $T_{\hat{m},\hat{f}}$ *extends uniquely to a continuous endomorphism of* $\widehat{\mathrm{B}}_3$ *and*

$$T_{\hat{m},\hat{f}}\big|_{\widehat{\mathrm{F}}_2} = E_{\hat{m},\hat{f}}. \tag{3.12}$$

Proof. It suffices to prove that

$$T_{\hat{m},\hat{f}}(\sigma_1)T_{\hat{m},\hat{f}}(\sigma_2)T_{\hat{m},\hat{f}}(\sigma_1) \stackrel{?}{=} T_{\hat{m},\hat{f}}(\sigma_2)T_{\hat{m},\hat{f}}(\sigma_1)T_{\hat{m},\hat{f}}(\sigma_2),$$

or equivalently

$$\sigma_1^{2\hat{m}+1}\hat{f}^{-1}\sigma_2^{2\hat{m}+1}\hat{f}\sigma_1^{2\hat{m}+1} \stackrel{?}{=} \hat{f}^{-1}\sigma_2^{2\hat{m}+1}\hat{f}\,\sigma_1^{2\hat{m}+1}\hat{f}^{-1}\,\sigma_2^{2\hat{m}+1}\hat{f}. \tag{3.13}$$

Note that, since the relation

$$\Delta x_{12}^t = x_{23}^t\Delta$$

holds for every $t \in \mathbb{Z}$, we also have

$$\Delta x_{12}^t\, N = x_{23}^t \Delta\, N$$

for every $N \in \mathsf{NFI}(\mathrm{B}_3)$. Hence we have the following identity in $\widehat{\mathrm{B}}_3$:

$$\Delta x_{12}^{\hat{t}} = x_{23}^{\hat{t}}\Delta, \qquad \forall \;\; \hat{t} \in \widehat{\mathbb{Z}}. \tag{3.14}$$

Using (3.9), we rewrite the left hand side of (3.13) as follows:

$$\sigma_1^{2\hat{m}+1}\hat{f}^{-1}\sigma_2^{2\hat{m}+1}\hat{f}\sigma_1^{2\hat{m}+1} = \hat{f}^{-1}\Delta c^{\hat{m}}\,.$$

Using (3.9) and (3.14), we rewrite the right hand side of (3.13) as follows:

$$\hat{f}^{-1}\sigma_2^{2\hat{m}+1}\hat{f}\,\sigma_1^{2\hat{m}+1}\,\hat{f}^{-1}\sigma_2^{2\hat{m}+1}\hat{f} = \hat{f}^{-1}x_{23}^{\hat{m}}\Delta x_{12}^{-\hat{m}}c^{\hat{m}} = \hat{f}^{-1}\Delta c^{\hat{m}}\,.$$

Thus relation (3.13) holds.

The proofs of the remaining two statements of the proposition are straightforward. □

We will need the following statement:

Proposition 3.4 *The subset* $\widehat{\mathsf{GT}}_{gen,mon}$ *of* $\widehat{\mathbb{Z}}\times\widehat{\mathrm{F}}_2$ *is a submonoid of* $\big(\widehat{\mathbb{Z}}\times\widehat{\mathrm{F}}_2, \bullet\big)$. *The assignment*

$$(\hat{m}, \hat{f}) \mapsto T_{\hat{m},\hat{f}}$$

is a homomorphism of monoids from $\widehat{\mathsf{GT}}_{gen,mon}$ *to the monoid of continuous endomorphisms of* $\widehat{\mathrm{B}}_3$. *Similarly, the assignment*

$$(\hat{m}, \hat{f}) \mapsto E_{\hat{m},\hat{f}}$$

defines a homomorphism of monoids from $\widehat{\mathsf{GT}}_{gen,mon}$ *to the monoid of continuous endomorphisms of* $\widehat{\mathrm{F}}_2$.

Proof. Let $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2) \in \widehat{\mathsf{GT}}_{gen,mon}$ and $(\hat{m}, \hat{f}) := (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$, i.e.

$$\hat{m} := 2\hat{m}_1\hat{m}_2 + \hat{m}_1 + \hat{m}_2, \qquad \hat{f} := \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2).$$

Since $E_{\hat{m}_1,\hat{f}_1}$ is a continuous group homomorphism and $\hat{f}_2$ belongs to the topological closure of $[\widehat{\mathrm{F}}_2, \widehat{\mathrm{F}}_2]$, $E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2)$ also belongs to the topological closure of $[\widehat{\mathrm{F}}_2, \widehat{\mathrm{F}}_2]$. Hence

$$\hat{f} \in [\widehat{\mathrm{F}}_2, \widehat{\mathrm{F}}_2]^{top.cl.}\,.$$

Let us prove that the pair $(\hat{m}, \hat{f})$ satisfies hexagon relations (3.9) and (3.10).

Applying $T_{\hat{m}_1,\hat{f}_1}$ to the first hexagon relation for $(\hat{m}_2, \hat{f}_2)$ and using (3.12), we get

$$\begin{gathered}\sigma_1^{(2\hat{m}_2+1)(2\hat{m}_1+1)} E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2)^{-1} \hat{f}_1^{-1} \sigma_2^{(2\hat{m}_2+1)(2\hat{m}_1+1)} \hat{f}_1 E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2) = \\ E_{\hat{m}_1,\hat{f}_1}(\hat{f}_2)^{-1} \sigma_1^{2\hat{m}_1+1} \hat{f}_1^{-1} \sigma_2^{2\hat{m}_1+1} \hat{f}_1\, x_{12}^{-\hat{m}_2(2\hat{m}_1+1)} c^{\hat{m}_2(2\hat{m}_1+1)}\,.\end{gathered} \tag{3.15}$$

Using (3.7), the first hexagon relation for $(\hat{m}_1, \hat{f}_1)$ and (3.15), we get

$$\sigma_1^{2\hat{m}+1} \hat{f}^{-1} \sigma_2^{2\hat{m}+1} \hat{f} \;=\; \hat{f}^{-1} \sigma_1 \sigma_2 x_{12}^{-\hat{m}} c^{\hat{m}}\,.$$

Thus the pair $(\hat{m}, \hat{f})$ satisfies (3.9).

Similarly, applying $T_{\hat{m}_1,\hat{f}_1}$ to the second hexagon relation for $(\hat{m}_2, \hat{f}_2)$, and using (3.12), the second hexagon relation for $(\hat{m}_1, \hat{f}_1)$ and (3.7), we see that the pair $(\hat{m}, \hat{f})$ also satisfies (3.10).

We proved that the subset $\widehat{\mathsf{GT}}_{gen,mon}$ is closed with respect to the binary operation $\bullet$.

It is easy to see that the pair $(0, 1_{\widehat{\mathrm{F}}_2})$ satisfies hexagon relations (3.9) and (3.10). Thus the first statement of the proposition is proved.

To prove the second statement of the proposition, we need to show that,

$$T_{0,1_{\widehat{\mathrm{F}}_2}} = \mathrm{id}_{\widehat{\mathrm{B}}_3} \tag{3.16}$$

and, for all $(\hat{m}_1, \hat{f}_1)$, $(\hat{m}_2, \hat{f}_2) \in \widehat{\mathsf{GT}}_{gen,mon}$, we have

$$T_{\hat{m}_1,\hat{f}_1} \circ T_{\hat{m}_2,\hat{f}_2} = T_{\hat{m},\hat{f}}, \tag{3.17}$$

where $(\hat{m}, \hat{f}) = (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$.

Applying $T_{\hat{m}_1,\hat{f}_1} \circ T_{\hat{m}_2,\hat{f}_2}$ and $T_{\hat{m},\hat{f}}$ to the generators σ_1, σ_2 of B_3, we see that

$$T_{\hat{m}_1,\hat{f}_1} \circ T_{\hat{m}_2,\hat{f}_2}\big|_{\mathrm{B}_3} = T_{\hat{m},\hat{f}}\big|_{\mathrm{B}_3}.$$

Since the maps $T_{\hat{m}_1,\hat{f}_1} \circ T_{\hat{m}_2,\hat{f}_2}$ and $T_{\hat{m},\hat{f}}$ are continuous, they agree on a dense subset B_3 of $\widehat{\mathrm{B}}_3$ and $\widehat{\mathrm{B}}_3$ is Hausdorff, equation (3.17) holds.

The similar argument works for (3.16).

The statement about the assignment $(\hat{m}, \hat{f}) \to E_{\hat{m},\hat{f}}$ follows from Proposition 3.1. Indeed, the restriction of a homomorphism of monoids to a submonoid is always a homomorphism of monoids. □

Definition 3.5 *$\widehat{\mathsf{GT}}_{gen}$ is the group of invertible elements of the monoid $\widehat{\mathsf{GT}}_{gen,mon}$.*

The following proposition shows the importance of the invertibility condition in the definition of $\widehat{\mathsf{GT}}_{gen}$:

Proposition 3.6 *For every $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen}$, $2\hat{m} + 1$ is a unit of the ring $\widehat{\mathbb{Z}}$. The assignments $(\hat{m}, \hat{f}) \mapsto T_{\hat{m},\hat{f}}$ and $(\hat{m}, \hat{f}) \mapsto E_{\hat{m},\hat{f}}$ are group homomorphisms from $\widehat{\mathsf{GT}}$ to the group of continuous automorphisms of $\widehat{\mathrm{B}}_3$ and to the group of continuous automorphisms of $\widehat{\mathrm{F}}_2$, respectively.*

Proof. If $(\hat{m}', \hat{f}')$ is the inverse of $(\hat{m}, \hat{f})$ then

$$2\hat{m}\hat{m}' + \hat{m} + \hat{m}' = 0$$

and hence

$$2(2\hat{m}\hat{m}' + \hat{m} + \hat{m}') + 1 = (2\hat{m} + 1)(2\hat{m}' + 1) = 1.$$

The remaining two statements are consequences of the last two statements of Proposition 3.4. Indeed, if $\varphi : A \to \tilde{A}$ is a homomorphism of monoids, then the restriction of φ to the group $A^{\times}$ of invertible elements of A is a group homomorphism from $A^{\times}$ to $\tilde{A}^{\times}$. □

3.2 The action of $\widehat{\mathsf{GT}}_{gen}$ on $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and the link between $\widehat{\mathsf{GT}}_{gen}$ and the groupoid GTSh

Let $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. Recall that $\widehat{\mathcal{P}}_N$ denotes the standard (continuous) group homomorphism from $\widehat{\mathrm{B}}_3$ to B_3/N.

Using the fact that B_3 is dense in $\widehat{\mathrm{B}}_3$, one can easily prove that, for every $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen}$, the group homomorphism

$$\widehat{\mathcal{P}}_N \circ T_{\hat{m},\hat{f}}\big|_{\mathrm{B}_3} : \mathrm{B}_3 \to \mathrm{B}_3/N \tag{3.18}$$

is onto.

In the following proposition, we define a right action of $\widehat{\mathsf{GT}}_{gen}$ on $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$:

Proposition 3.7 *Let $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. For every $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen}$, the pair*

$$\big(\widehat{\mathcal{P}}_{N_{\mathrm{ord}}}(\hat{m}), \widehat{\mathcal{P}}_{N_{\mathrm{F}_2}}(\hat{f})\big)$$

is a GT-shadow with the target N. Furthermore, the assignment

$$N^{(\hat{m},\hat{f})} := \ker(\widehat{\mathcal{P}}_N \circ T_{\hat{m},\hat{f}}) \tag{3.19}$$

defines a right action of $\widehat{\mathsf{GT}}_{gen}$ on $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

Proof. Let m (resp. f) be any representative of the residue class $\widehat{\mathcal{P}}_{N_{\mathrm{ord}}}(\hat{m})$ (resp. the coset $\widehat{\mathcal{P}}_{N_{\mathrm{F}_2}}(\hat{f})$).

Since the pair $(\hat{m}, \hat{f})$ satisfies (3.9) and (3.10), the pair (m, f) satisfies hexagon relations (2.9) and (2.10) modulo N.

Since $2\hat{m}+1$ is a unit in $\widehat{\mathbb{Z}}$, the integer $2m+1$ represents a unit in $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z}$.

The property $\hat{f} \in [\widehat{\mathrm{F}}_2, \widehat{\mathrm{F}}_2]^{top.cl.}$ implies that

$$f N_{\mathrm{F}_2} \;\in\; [\mathrm{F}_2/N_{\mathrm{F}_2}, \mathrm{F}_2/N_{\mathrm{F}_2}].$$

Finally, it is easy to see that the homomorphism $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/N$ coincides with $\widehat{\mathcal{P}}_N \circ T_{\hat{m},\hat{f}}$:

$$T_{m,f} = \widehat{\mathcal{P}}_N \circ T_{\hat{m},\hat{f}}\big|_{\mathrm{B}_3}. \tag{3.20}$$

In particular, $T_{m,f}$ is onto.

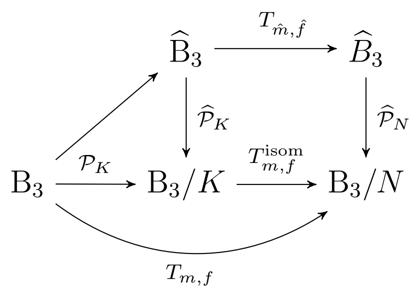
We proved that the pair $(m + N_{\mathrm{ord}}\mathbb{Z}, f N_{\mathrm{F}_2})$ is a GT-shadow with the target N and

$$N^{(\hat{m},\hat{f})} = \ker(T_{m,f}).$$

Hence $N^{(\hat{m},\hat{f})} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

We say that the GT-shadow $[m, f] \in \mathsf{GT}(N)$ **comes from** the element $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen}$.

Let $K := \ker(T_{m,f})$. Using (2.35) and (3.20), we see that the diagram



(3.21)

commutes. In (3.21), the slanted straight arrow is the standard inclusion map $\mathrm{B}_3 \to \widehat{\mathrm{B}}_3$.

Indeed, the outer "curved" rectangle commutes due to (3.20). The lower "curved" triangle commutes due to (2.35). The left triangle commutes by definition of $\widehat{\mathrm{B}}_3$. Finally, the continuous maps $\widehat{\mathcal{P}}_N \circ T_{\hat{m},\hat{f}}$ and $T^{\mathrm{isom}}_{m,f} \circ \widehat{\mathcal{P}}_K$ agree on the dense subset $\mathrm{B}_3 \subset \widehat{\mathrm{B}}_3$ and B_3/N is Hausdorff. Hence the inner square in (3.21) also commutes.

It is clear that

$$\widehat{\mathcal{P}}_N \circ T_{0,1_{\widehat{\mathrm{F}}_2}}\big|_{B_3} = \mathcal{P}_N.$$

Hence $N^{(0,1_{\widehat{\mathrm{F}}_2})} = N$.

It remains to prove, for all $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2) \in \widehat{\mathsf{GT}}_{gen}$,

$$\left(N^{(\hat{m}_1,\hat{f}_1)}\right)^{(\hat{m}_2,\hat{f}_2)} = N^{(\hat{m},\hat{f})}\,, \tag{3.22}$$

where $(\hat{m}, \hat{f}) := (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$.

For this purpose, we will use the inner square of the diagram in (3.21). We set $K := N^{(\hat{m}_1,\hat{f}_1)}$ and $H := K^{(\hat{m}_2,\hat{f}_2)}$. Then, putting together the "squares" corresponding to $(\hat{m}_1, \hat{f}_1)$ and $(\hat{m}_2, \hat{f}_2)$, adding the obvious "triangle with the vertex" B_3 and the "curved triangle" with the arrow $\widehat{\mathcal{P}}_N \circ T_{\hat{m},\hat{f}}\big|_{\mathrm{B}_3}$, and using

(3.17), we get the following commutative diagram:

$$\begin{array}{ccccccc} & & \widehat{\mathrm{B}}_3 & \xrightarrow{T_{\hat{m}_2,\hat{f}_2}} & \widehat{B}_3 & \xrightarrow{T_{\hat{m}_1,\hat{f}_1}} & \widehat{B_3} \\ & \nearrow & \downarrow \widehat{\mathcal{P}}_H & & \downarrow \widehat{\mathcal{P}}_K & & \downarrow \widehat{\mathcal{P}}_N \\ \mathrm{B}_3 & \xrightarrow{\mathcal{P}_H} & \mathrm{B}_3/H & \xrightarrow{T^{\mathrm{isom}}_{m_2,f_2}} & \mathrm{B}_3/K & \xrightarrow{T^{\mathrm{isom}}_{m_1,f_1}} & \mathrm{B}_3/N \end{array} \qquad \mathrm{B}_3 \xrightarrow{\widehat{\mathcal{P}}_N \circ T_{\hat{m},\hat{f}}|_{\mathrm{B}_3}} \mathrm{B}_3/N \tag{3.23}$$

where $[m_1, f_1] \in \mathsf{GT}(N)$ and $[m_2, f_2] \in \mathsf{GT}(K)$ are the GT-shadows coming from $(\hat{m}_1, \hat{f}_1)$ and $(\hat{m}_2, \hat{f}_2)$, respectively.

The lower "curved" triangle shows us that $H = N^{(\hat{m},\hat{f})}$. Thus identity (3.22) holds. □

We denote by $\widehat{\mathsf{GT}}_{gen,\mathsf{NFI}}$ the transformation groupoid of the action of $\widehat{\mathsf{GT}}_{gen}$ on $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, i.e. $\mathrm{Ob}(\widehat{\mathsf{GT}}_{gen,\mathsf{NFI}}) = \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and

$$\widehat{\mathsf{GT}}_{gen,\mathsf{NFI}}(K, N) := \{(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen} \mid N^{(\hat{m},\hat{f})} = K\}.$$

Definition 3.8 *Let $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and $[m, f] \in \mathsf{GT}(N)$. We say that the GT-shadow $[m, f]$ is* ***genuine*** *if there exists $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen}$ such that $[m, f]$ comes from $(\hat{m}, \hat{f})$, i.e.*

$$m + N_{\mathrm{ord}}\mathbb{Z} = \widehat{\mathcal{P}}_{N_{\mathrm{ord}}}(\hat{m}), \qquad f N_{\mathrm{F}_2} = \widehat{\mathcal{P}}_{N_{\mathrm{F}_2}}(\hat{f}).$$

Otherwise, the GT-shadow is called ***fake****.*

The following theorem gives us a link between $\widehat{\mathsf{GT}}_{gen}$ and the groupoid GTSh:

Theorem 3.9 *Let $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. The assignments*

$$\mathcal{PR}(N) := N, \qquad \mathcal{PR}(\hat{m}, \hat{f}) = \big(\widehat{\mathcal{P}}_{N_{\mathrm{ord}}}(\hat{m}), \widehat{\mathcal{P}}_{N_{\mathrm{F}_2}}(\hat{f})\big) \tag{3.24}$$

define a functor from the transformation groupoid $\widehat{\mathsf{GT}}_{gen,\mathsf{NFI}}$ to GTSh.

Proof. Let $(m, f) \in \mathbb{Z} \times [\mathrm{F}_2, \mathrm{F}_2]$ be a pair that represents $\big(\widehat{\mathcal{P}}_{N_{\mathrm{ord}}}(\hat{m}), \widehat{\mathcal{P}}_{N_{\mathrm{F}_2}}(\hat{f})\big)$.

Due to the first statement of Proposition 3.7, $[m, f]$ is a GT-shadow with the target N. Moreover, since

$$\ker(T_{m,f}) = N^{(\hat{m},\hat{f})},$$

$[m, f]$ is indeed a morphism from $N^{(\hat{m},\hat{f})}$ to N in GTSh.

It is clear that $\mathcal{PR}(0, 1_{\widehat{\mathrm{F}}_2}) = [0, 1_{\mathrm{F}_2}]$, i.e. the functor $\mathcal{PR}$ sends the identity morphisms of $\widehat{\mathsf{GT}}_{gen,\mathsf{NFI}}$ to the identity morphisms of GTSh.

It remains to prove that, for all $(\hat{m}_1, \hat{f}_1), (\hat{m}_2, \hat{f}_2) \in \widehat{\mathsf{GT}}_{gen}$ and $N \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$,

$$\mathcal{PR}(\hat{m}_1, \hat{f}_1) \bullet \mathcal{PR}(\hat{m}_2, \hat{f}_2) = \mathcal{PR}(\hat{m}, \hat{f}), \tag{3.25}$$

where $(\hat{m}, \hat{f}) = (\hat{m}_1, \hat{f}_1) \bullet (\hat{m}_2, \hat{f}_2)$, $(\hat{m}_1, \hat{f}_1)$ is viewed as a morphism from $K := N^{(\hat{m}_1,\hat{f}_1)}$ to N and $(\hat{m}_2, \hat{f}_2)$ is viewed as a morphism from $K^{(\hat{m}_2,\hat{f}_2)}$ to K.

Let (m_1, f_1) and (m_2, f_2) be pairs that represent the GT-shadows $\mathcal{PR}(\hat{m}_1, \hat{f}_1)$ and $\mathcal{PR}(\hat{m}_2, \hat{f}_2)$. Since the source of $[m_1, f_1]$, K, coincides with the target of $[m_2, f_2]$, the GT-shadows $[m_1, f_1]$ and $[m_2, f_2]$ can be composed in this order $[m_1, f_1] \bullet [m_2, f_2]$ and $[m_1, f_1] \bullet [m_2, f_2]$ is an element of $\mathsf{GTSh}(K^{(\hat{m}_2,\hat{f}_2)}, N)$

We set

$$m := 2m_1 m_2 + m_1 + m_2, \qquad f := f_1 E_{m_1,f_1}(f_2)$$

and notice that

$$m + N_{\mathrm{ord}}\mathbb{Z} = \widehat{\mathcal{P}}_{N_{\mathrm{ord}}}(\hat{m})$$

and

$$f N_{\mathrm{F}_2} = \widehat{\mathcal{P}}_{N_{\mathrm{F}_2}}(\hat{f}).$$

Hence $\mathcal{PR}(\hat{m}, \hat{f}) = [m, f]$, i.e. equation (3.25) indeed holds. $\square$

CHAPTER 4

EXPLORATION OF THE CONNECTED COMPONENTS OF GTSh COMING FROM FINITE QUOTIENTS OF THE FULL MODULAR GROUP

4.1 The subposet $\mathfrak{M}$ of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ coming from the full modular group

Recall [20] that the formulas

$$\psi(\sigma_1) := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \psi(\sigma_2) := \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \tag{4.1}$$

define a surjective homomorphism ψ from B_3 to the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

For a positive integer q, we denote by $\mathcal{P}_q$ the standard homomorphism

$$\mathcal{P}_q : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) \tag{4.2}$$

and denote by ψ_q the composition

$$\psi_q := \mathcal{P}_q \circ \psi : \mathrm{B}_3 \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}). \tag{4.3}$$

We set

$$N^{(q)} := \mathrm{PB}_3 \cap \ker(\psi_q) \tag{4.4}$$

and observe that $N^{(q)}$ is an element of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. In this chapter, the notation $N^{(q)}$ (for $q \in \mathbb{Z}_{\geq 1}$) is reserved for this specific element of the poset $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

Let us prove that

Proposition 4.1 *The assignment $q \mapsto N^{(q)}$ defines a map of posets from the divisibility poset $\mathbb{Z}^{div}_{\geq 1}$ to $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.*

Proof. Let $k, q \in \mathbb{Z}_{\geq 1}$ and $k|q$. We denote by $\mathcal{P}_{q,k}$ the standard canonical homomorphism $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/k\mathbb{Z})$. And by definition of ψ_k and ψ_q,

$$\psi_k = \mathcal{P}_{q,k} \circ \psi_q \,.$$

Hence $\ker(\psi_q) \subset \ker(\psi_k)$, i.e. $N^{(q)} \subset N^{(k)}$. We proved that the assignment $q \mapsto N^{(q)}$ defines a map of posets. □

Let us denote by $\mathfrak{M}$ the image of the above map of posets $\mathbb{Z}^{div}_{\geq 1} \to \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. In the next section, we study the connected components of GTSh for specific elements of the poset $\mathfrak{M}$.

4.2 The description of the set $\mathsf{GT}^{\heartsuit}_{pr}(N^{(q)})$ of charming GT-pairs for the case when q is a prime power

Let p be a prime integer > 3 and $q = p^t$ for $t \in \mathbb{Z}_{\geq 1}$. In this section, $\bar{n}$ denotes the residue class of the integer n in $\mathbb{Z}/q\mathbb{Z}$.

Since the orders of the elements $x_{12}N^{(q)}$, $x_{23}N^{(q)}$ and $cN^{(q)}$ coincide with the orders of the elements

$$\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{1} \end{bmatrix}, \qquad \begin{bmatrix} \bar{1} & \bar{0} \\ -\bar{2} & \bar{1} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -\bar{1} & \bar{0} \\ \bar{0} & -\bar{1} \end{bmatrix},$$

respectively, and 2 is coprime with q,

$$N_{\mathrm{ord}}^{(q)} = 2q. \tag{4.5}$$

Recall[1] [3, Theorem 3.2] that the standard homomorphism $\mathcal{P}_q : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ is onto and $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Combining this observation with $\gcd(2, q) = 1$, we see that the group $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ is generated by the elements

$$\psi_q(x_{12}) := \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{1} \end{bmatrix}, \qquad \psi_q(x_{23}) := \begin{bmatrix} \bar{1} & \bar{0} \\ -\bar{2} & \bar{1} \end{bmatrix}.$$

Therefore, we may simply identify the quotient group $\mathrm{F}_2/N_{\mathrm{F}_2}^{(q)}$ with $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ and it is convenient to do so.

The main result of this thesis is the following description of the set of charming GT-pairs with the target $N^{(q)}$.

Theorem 4.2 *Let p be a prime integer > 3 and $q = p^t$ for $t \in \mathbb{Z}_{\geq 1}$. Furthermore, let*

$$\mathcal{X}_q := \{m \in \{0, 1, \dots, 2q-1\} \mid \gcd(2m+1, q) = 1\}.$$

Then $\mathsf{GT}^{\heartsuit}_{pr}(N^{(q)})$ consists of pairs $(m + 2q\mathbb{Z}, g_{m,b}) \in \mathbb{Z}/2q\mathbb{Z} \times \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$, where

$$g_{m,b} := \begin{bmatrix} (-1)^m(2\bar{m}+1)^{-1} & b \\ b & (-1)^m(2\bar{m}+1)(1+b^2) \end{bmatrix},$$

and

$$m \in \mathcal{X}_q \quad \textit{and} \quad b \in \mathbb{Z}/q\mathbb{Z}.$$

[1]Note that paper [3] is expository. So the statements from [3] we are using here can be found in other sources, including textbooks.

We should remark that, in the statement of the above theorem, we identified the quotient $\mathrm{F}_2/N^{(q)}_{\mathrm{F}_2}$ with $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ using the homomorphism $\psi_q\big|_{\mathrm{F}_2} : \mathrm{F}_2 \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$.

To prove Theorem 4.2, we need the following proposition:

Proposition 4.3 *For every* GT*-pair* $[m, f] \in \mathsf{GT}_{pr}(N^{(q)})$*,* $\psi_q(f)$ *is a symmetric matrix in* $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$*.*

Proof. Recall that θ is the automorphism of F_2 defined by the formulas $\theta(x) := y$ and $\theta(y) := x$.

Proposition 2.4 implies that

$$f\theta(f) \in N^{(q)}_{\mathrm{F}_2} \tag{4.6}$$

for every pair $(m, f) \in \mathbb{Z} \times \mathrm{F}_2$ that satisfies hexagon relations (2.9) and (2.10) (modulo $N^{(q)}$).

Note that the formula

$$\tilde{\theta}(A) := (A^T)^{-1} \tag{4.7}$$

defines a endomorphism of $\mathrm{SL}_2(R)$ for every commutative ring R. Here A^T denotes the transpose of the matrix A.

We also have $\tilde{\theta} \circ \tilde{\theta} = \mathrm{id}_{\mathrm{SL}_2(R)}$. Hence $\tilde{\theta}$ is an automorphism of $\mathrm{SL}_2(R)$.

We claim that

$$\psi^{\mathrm{F}_2}_q \circ \theta = \tilde{\theta} \circ \psi^{\mathrm{F}_2}_q\,, \tag{4.8}$$

where $\psi^{\mathrm{F}_2}_q := \psi_q\big|_{\mathrm{F}_2} : \mathrm{F}_2 \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$.

Indeed, both maps $\psi^{\mathrm{F}_2}_q \circ \theta$ and $\tilde{\theta} \circ \psi^{\mathrm{F}_2}_q$ are group homomorphisms from F_2 to $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$. Since F_2 is free and $\psi^{\mathrm{F}_2}_q \circ \theta(x) = \tilde{\theta} \circ \psi^{\mathrm{F}_2}_q(x)$ and $\psi^{\mathrm{F}_2}_q \circ \theta(y) = \tilde{\theta} \circ \psi^{\mathrm{F}_2}_q(y)$, we conclude that (4.8) indeed holds.

Let $g := \psi^{\mathrm{F}_2}_q(f)$. Applying $\psi^{\mathrm{F}_2}_q$ to $f\theta(f)$ and using (4.6), (4.8) we get

$$g\,\tilde{\theta}(g) = I_q, \tag{4.9}$$

where I_q is the identity element of $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$.

Identity (4.9) obviously implies that $g = g^T$. □

We also need the following statement:

Proposition 4.4 *Let $q = p^t$ be a power of some prime $p, p > 3$, then*

$$[\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})] = \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}). \tag{4.10}$$

Proof. Let $H := [\mathrm{SL}_2(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z})]$. It is well known that $H \trianglelefteq \mathrm{SL}_2(\mathbb{Z})$.

Due to [3, Theorem 3.8], $|\mathrm{SL}_2(\mathbb{Z}) : H| = 12$. Therefore

$$\psi(x)^{12} \in H, \qquad \psi(y)^{12} \in H \tag{4.11}$$

and hence

$$\psi_q(x)^{12} = \begin{bmatrix} \overline{1} & \overline{24} \\ \overline{0} & \overline{1} \end{bmatrix} \in [\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})],$$

$$\psi_q(y)^{12} = \begin{bmatrix} \overline{1} & \overline{0} \\ -\overline{24} & \overline{1} \end{bmatrix} \in [\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})].$$

Since q is coprime with $24 = 2^3 \cdot 3$, the subgroup $\langle\, \psi_q(x)^{12}, \psi_q(y)^{12} \,\rangle \leq \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ contains the elements

$$\begin{bmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \overline{1} & \overline{0} \\ -\overline{1} & \overline{1} \end{bmatrix}.$$

Since these elements generate $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$, we conclude that $\langle\, \psi_q(x)^{12}, \psi_q(y)^{12} \,\rangle = \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$. Thus $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ coincides with its commutator subgroup. □

Proof of Theorem 4.2. Let $[m, f] \in \mathsf{GT}^{\heartsuit}_{pr}(N^{(q)})$ and

$$g := \psi_q(f) \in \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}).$$

Due to Proposition 4.3, the matrix g is of the form

$$g = \begin{bmatrix} a & b \\ b & d \end{bmatrix},$$

where $a, b, d \in \mathbb{Z}/q\mathbb{Z}$ and

$$ad - b^2 = 1.$$

Since $2m + 1$ represents a unit in the ring $\mathbb{Z}/2q\mathbb{Z}$, $2m + 1$ also represents a unit in the ring $\mathbb{Z}/q\mathbb{Z}$. We set

$$u := \overline{2m + 1} \in \mathbb{Z}/q\mathbb{Z}.$$

Since the pair (m, f) satisfies (2.9), (2.10) modulo $N^{(q)}$, and $N^{(q)} = \ker(\psi_q\big|_{\mathrm{PB}_3})$, we have

$$\psi_q(\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f) = \psi_q(f^{-1}\sigma_1\sigma_2 x_{12}^{-m} c^m) \tag{4.12}$$

and

$$\psi_q(f^{-1}\sigma_2^{2m+1} f \sigma_1^{2m+1}) = \psi_q(\sigma_2\sigma_1 x_{23}^{-m} c^m f). \tag{4.13}$$

Rewriting (4.12) and (4.13), we get

$$\begin{bmatrix} 1 - u^2a^2 + uab & uad - u^2ab \\ -ua^2 & 1 - uab \end{bmatrix} = \begin{bmatrix} (-1)^m b & (-1)^m(d - ub) \\ -(-1)^m a & (-1)^m(ua - b) \end{bmatrix} \tag{4.14}$$

and

$$\begin{bmatrix} 1 + uab & ua(ub + d) \\ -ua^2 & 1 - u^2a^2 - uab \end{bmatrix} = \begin{bmatrix} (-1)^m(ua + b) & (-1)^m(ub + d) \\ -(-1)^m a & -(-1)^m b \end{bmatrix} \tag{4.15}$$

Note that, since the integer m is considered modulo an even integer $2q$, the expression $(-1)^m$ does not depend on the choice of a representative in $m+2q\mathbb{Z}$.

Comparing $(2, 1)$-entry of the matrices in (4.14) (or in (4.15)) we get the equation in $\mathbb{Z}/q\mathbb{Z}$:

$$a(ua - (-1)^m) = \bar{0}. \tag{4.16}$$

Let n_a be an integer that represents $a \in \mathbb{Z}/q\mathbb{Z}$. If $p|n_a$, then p cannot divide the integer

$$(2m + 1)n_a - (-1)^m.$$

Hence, we have the two excluding possibilities

- either $a = 0$ or
- $ua - (-1)^m = 0$.

Setting $a = 0$ and combining (4.14) with $ad - b^2 = 1$, we get an obvious contradiction. Hence we have $ua - (-1)^m = 0$ and, since u is a unit in $\mathbb{Z}/q\mathbb{Z}$, we get

$$a = (-1)^m u^{-1}. \tag{4.17}$$

If $a = (-1)^m u^{-1}$ then equations (4.14) and (4.15) are satisfied.

Using $ad - b^2 = 1$ and the invertibility of a we conclude that

$$d = (-1)^m u(1 + b^2). \tag{4.18}$$

Thus every charming GT-pair is of the form $(m + 2q\mathbb{Z}, g_{m,b})$, where $2m + 1 + 2q\mathbb{Z}$ is a unit of $\mathbb{Z}/2q\mathbb{Z}$ and

$$g_{m,b} := \begin{bmatrix} (-1)^m(2\bar{m}+1)^{-1} & b \\ b & (-1)^m(2\bar{m}+1)(1+b^2) \end{bmatrix}. \tag{4.19}$$

Let us now consider a pair $(m + 2q\mathbb{Z}, g_{m,b}) \in \mathbb{Z}/2q\mathbb{Z} \times \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ such that $2m + 1 + 2q\mathbb{Z}$ is a unit of $\mathbb{Z}/2q\mathbb{Z}$ and $g_{m,b}$ is given by equation (4.19).

A direct computation shows that the pair $(m + 2q\mathbb{Z}, g_{m,b})$ satisfies hexagon relations (2.9) and (2.10) modulo $N^{(q)}$. Proposition (4.4) implies that $g_{m,b} \in [\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})]$. Combining these observations with $2m + 1 + 2q\mathbb{Z} \in (\mathbb{Z}/2q\mathbb{Z})^{\times}$, we conclude that $(m + 2q\mathbb{Z}, g_{m,b})$ is a charming GT-pair with the target $N^{(q)}$.

Theorem 4.2 is proved. □

4.3 Future plans

Computer experiments support the following conjecture:

Conjecture 4.5 *Let p be a prime integer > 3 and $q = p^t$ for $t \in \mathbb{Z}_{\geq 1}$. Then*

- $\mathsf{GT}(N^{(q)}) = \mathsf{GT}^{\heartsuit}_{pr}(N^{(q)})$, *i.e. Theorem 4.2 gives us a complete description of the set of GT-shadows with the target $N^{(q)}$ and*
- *$N^{(q)}$ is isolated.*

In the future, we will continue working on this conjecture.

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