GEOMETRY AND SPECTRA OF CLOSED EXTENSIONS OF ELLIPTIC CONE OPERATORS

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Abstract. We study the geometry of the set of closed extensions of index 0 of an elliptic differential cone operator and its model operator in connection with the spectra of the extensions, and give a necessary and sufficient condition for the existence of rays of minimal growth for such operators.

1. Introduction

The purpose of this paper is to study the spectra and resolvents of the closed extensions of an elliptic differential cone operator A on a compact manifold M with boundary, and of its model operator $A_\wedge$. It is well known that the closed extensions of A are in one to one correspondence with the subspaces of a finite dimensional space, $\mathcal{D}_{\max}/\mathcal{D}_{\min}$, the spaces $\mathcal{D}_{\max}$ and $\mathcal{D}_{\min}$ being certain subpaces determined by A of an L^2 space on M, cf. Lesch [5]. It is thus natural to view the extensions as corresponding to points in the various Grassmannians associated with $\mathcal{D}_{\max}/\mathcal{D}_{\min}$. Extending this, we develop a viewpoint in which issues pertaining spectra and resolvents, both for the closed extensions of A and of $A_\wedge$, are expressed and examined in (finite-dimensional) geometric terms.

Cone differential operators are generalizations of the operators that arise when standard differential operators are written using polar coordinates. Their study is therefore of interest in the context of manifolds with conical singularities, both in themselves and as guiding examples in a general theory of analysis of differential operators on manifolds with other kinds of singularities, cf. Schulze [12].

Our motivation for undertaking this study comes from the desirability of executing Seeley's program [13] in the case of elliptic cone operators. This requires a detailed understanding of the resolvent in terms of the symbol of A and the domain of the extension. From the pseudodifferential point of view, the symbol of A is a pair consisting of $A_\wedge$ and its b-symbol, or more invariantly, its cone-symbol ${}^c\boldsymbol{\sigma}(A)$ as defined in Section 3. As in the standard theory of elliptic operators on a manifold without boundary, the statement that a given sector $\Lambda \subset \mathbb{C}$ is a sector of minimal growth for ${}^c\boldsymbol{\sigma}(A)$ is domain-insensitive. The operator $A_\wedge$, however, is a differential operator, so the analogous statement for $A_\wedge$ requires that a domain be specified. In Section 4 we shall construct a natural bijection from the set of domains of closed extensions of A to that of $A_\wedge$, and in [3] we use pseudodifferential techniques to show that if $\mathcal{D}_\wedge$ is the domain of $A_\wedge$ associated with the domain $\mathcal{D}$ of A, and if Λ is a sector of minimal growth for ${}^c\boldsymbol{\sigma}(A)$ and for $A_\wedge$ with domain $\mathcal{D}_\wedge$, then it is also a sector of minimal growth for A with domain $\mathcal{D}$. This of course brings up

2000 *Mathematics Subject Classification.* Primary: 58J50; Secondary: 35J70, 14M15.

Key words and phrases. Resolvents, manifolds with conical singularities, spectral theory, boundary value problems, Grassmannians.

the question of how to determine whether a given sector, or even a ray, is a sector of minimal growth for $A_\wedge$ with domain $\mathcal{D}_\wedge$. In connection with this we give, in Theorem 8.7, a necessary and sufficient condition for Λ to be a sector of minimal growth for $A_\wedge$ with domain $\mathcal{D}_\wedge$. The condition (8.8) of Theorem 8.7 is in principle verifiable.

Resolvents for cone-elliptic operators written as pseudodifferential operators have been constructed by other authors in special cases, e.g. Brüning-Seeley [1], Mooers [10], Gil [2], and Schrohe-Seiler [11], the last mentioned article being the one closest to our own aims in [3]. Also of interest is Loya [6] in the context of b-operators. Our goal, here and in [3], is to study the problem with minimal assumptions.

A description of the paper follows.

We shall be working with a fixed elliptic cone operator A acting on sections of a Hermitian vector bundle E over a manifold M; the latter is assumed to be compact of dimension n with nonempty boundary. The definition of cone operators is recalled in Section 2, where we also recall the definitions of the spaces on which cone operators act. In this section we introduce certain strongly continuous one-parameter groups of isometries κ_ϱ, one associated with M and one with the interior pointing part of the normal bundle of ∂M in M (where $A_\wedge$ lives). These actions generally play an important role in the analysis of degenerate elliptic operators, see Schulze [12], and they do so here as well.

The c-cotangent bundle, ${}^cT^*M$, is defined in Section 3. Its definition is analogous to that of the b-tangent bundle of Melrose [7, 8]. It is a vector bundle over M which is canonically isomorphic to T^*M over the interior of M. Cone operators have invariantly defined symbols, ${}^c\boldsymbol{\sigma}(A)$, defined on ${}^cT^*M$. We also recall in this section the definition of $A_\wedge$, and discuss some properties inherited by $A_\wedge$ from A. We also briefly recall the definition of the conormal symbol.

In Section 4 we first recall known facts about the closed extensions of cone-elliptic operators on compact manifolds, such as M, and sketch proofs of analogous results for the operator $A_\wedge$. Proofs are needed since $A_\wedge$, though elliptic in the proper sense, is not a Fredholm operator on the spaces naturally associated with it. For A, as is well known, there is a minimal closed extension with domain $\mathcal{D}_{\min}$, and there is a maximal extension with domain $\mathcal{D}_{\max}$. Likewise, for $A_\wedge$ there is the domain of the minimal extension, $\mathcal{D}_{\wedge,\min}$, and the maximal domain $\mathcal{D}_{\wedge,\max}$. In both cases, the minimal domain has finite codimension in the maximal domain (in fact the same codimension). The set of domains of closed extensions can be viewed as a Grassmannian variety, and there is a natural map Θ, cf. (4.22), one can use to pass from one variety to the other. This is most relevant in [3]; indeed, the meaning of the condition that ${}^c\boldsymbol{\sigma}(A)$ admits a ray of minimal growth is clear, but to express the analogous condition for $A_\wedge$ requires the specification of a domain for $A_\wedge$. This domain is the one associated by Θ with the given domain for A.

The analysis of the spectrum of a given closed extension of A is taken up in Section 5. It is natural to classify the set of extensions of A by the index. The ones with index 0 being the only relevant in the problem of studying the spectrum, we let $\mathfrak{G}$ be the set of domains $\mathcal{D}$ such that $\operatorname{ind} A_{\mathcal{D}} = 0$; here and elsewhere $A_{\mathcal{D}}$ means A with domain $\mathcal{D}$. The simple condition that both numbers $d'' = -\operatorname{ind} A_{\mathcal{D}_{\min}}$ and $d' = \operatorname{ind} A_{\mathcal{D}_{\max}}$ be nonnegative is necessary and sufficient for $\mathfrak{G}$ to be nonempty, see Lemma 5.1, and if this is the case, then $\mathfrak{G}$ can be viewed as a (complex) Grassmannian variety (based on $\mathcal{D}_{\max}/\mathcal{D}_{\min}$). An at first surprising fact is that if $\dim \mathfrak{G} > 0$, then for every $\lambda \in \mathbb{C}$ there is $\mathcal{D} \in \mathfrak{G}$ such that $\lambda \in \operatorname{spec} A_{\mathcal{D}}$, see Proposition 5.7.

Letting

$$\text{bg-spec}\, A = \bigcap_{\mathcal{D}\in\mathfrak{G}} \operatorname{spec} A_{\mathcal{D}}, \quad \text{bg-res}\, A = \mathbb{C}\setminus \text{bg-spec}\, A,$$

we get

$$\operatorname{spec} A_{\mathcal{D}} = \text{bg-spec}\, A \cup (\operatorname{spec} A_{\mathcal{D}} \cap \text{bg-res}\, A),$$

a disjoint union. It is the part of $\operatorname{spec} A_{\mathcal{D}}$ in bg-res A that is most amenable to study. For $\lambda \in$ bg-res A, the dimension of $\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda)$ is constant, equal to d', and

$$\lambda \in \operatorname{res} A_{\mathcal{D}} \iff \lambda \in \text{bg-res}\, A \text{ and } \mathcal{K}_\lambda \cap \mathcal{D} = 0,$$

cf. Lemma 5.10; by the same lemma, if $\mathcal{K}_\lambda \cap \mathcal{D} = 0$ then $\mathcal{D}_{\max} = \mathcal{K}_\lambda \oplus \mathcal{D}$. Let then $\pi_{\mathcal{K}_\lambda,\mathcal{D}}$ be the projection on $\mathcal{K}_\lambda$ according to this decomposition.

If $\lambda \in$ bg-res A, then $A_{\mathcal{D}_{\min}} - \lambda$ is injective and $A_{\mathcal{D}_{\max}} - \lambda$ is surjective (this property characterizes bg-res A). For such λ let $B_{\max}(\lambda)$ be the right inverse of $A_{\mathcal{D}_{\max}} - \lambda$ whose range is orthogonal to $\mathcal{K}_\lambda$ with respect to the inner product

$$(u, v)_A = (Au, Av)_{x^{-m/2}L^2_b} + (u, v)_{x^{-m/2}L^2_b},$$

and let $B_{\min}(\lambda)$ be the left inverse of $A_{\mathcal{D}_{\min}} - \lambda$ with kernel the orthogonal of $\operatorname{rg}(A_{\mathcal{D}_{\min}} - \lambda)$ in $x^{-m/2}L^2_b(M;E)$. Then, if $\lambda \in \operatorname{res} A_{\mathcal{D}}$, one has the formula

$$B_{\mathcal{D}}(\lambda) = B_{\max}(\lambda) - \big(I - B_{\min}(\lambda)(A-\lambda)\big)\pi_{\mathcal{K}_\lambda,\mathcal{D}} B_{\max}(\lambda)$$

for the resolvent $B_{\mathcal{D}}(\lambda) = (A_{\mathcal{D}} - \lambda)^{-1}$ of $A_{\mathcal{D}}$, cf. (5.19). This formula is evident if one notes that the factor in front of $\pi_{\mathcal{K}_\lambda,\mathcal{D}}$ is the identity on $\mathcal{K}_\lambda$. In principle both $B_{\min}(\lambda)$ and $B_{\max}(\lambda)$ can be written as pseudodifferential operators, a purely analytic problem, so inverting $A_{\mathcal{D}} - \lambda$ is reduced to an algebraic problem, indeed, a problem in a finite dimensional space, as follows.

Let $\mathcal{E}_{\max}$ be the orthogonal of $\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$ with respect to the inner product defined above; this is a finite dimensional space. Let $\pi_{\max} : \mathcal{D}_{\max} \to \mathcal{D}_{\max}$ be the orthogonal projection on $\mathcal{E}_{\max}$. Both $I - B_{\min}(\lambda)(A-\lambda)$ and $\pi_{\mathcal{K}_\lambda,\mathcal{D}}$ vanish on $\mathcal{D}_{\min}$, so

$$B_{\mathcal{D}}(\lambda) = B_{\max}(\lambda) - \big(I - B_{\min}(\lambda)(A-\lambda)\big)\pi_{\max}\, \pi_{\mathcal{K}_\lambda,\mathcal{D}}\, \pi_{\max} B_{\max}(\lambda).$$

On the other hand,

$$\lambda \in \operatorname{res} A_{\mathcal{D}} \iff \lambda \in \text{bg-res}\, A \text{ and } \pi_{\max}\mathcal{K}_\lambda \cap \pi_{\max}\mathcal{D} = 0,$$

and for such λ, $\mathcal{E}_{\max} = \pi_{\max}\mathcal{K}_\lambda \oplus \pi_{\max}\mathcal{D}$, cf. Lemma 5.10. The map $\pi_{\max}\, \pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}}$ is just the projection on $\pi_{\max}\mathcal{K}_\lambda$ according to this decomposition of $\mathcal{E}_{\max}$, cf. Lemma 5.21.

Organizing the information in terms of Grassmannians turns out to be quite useful. The set $\mathfrak{G}$ can be viewed as the Grassmannian $\operatorname{Gr}_{d''}(\mathcal{E}_{\max})$ of d''-dimensional subspaces of $\mathcal{E}_{\max}$, and the spaces $\mathcal{K}_\lambda$ (which are the fibers of a holomorphic vector bundle over bg-res A) give a holomorphic map $\lambda \mapsto \pi_{\max}\mathcal{K}_\lambda \in \operatorname{Gr}_{d'}(\mathcal{E}_{\max})$. The condition that $\lambda \in \text{bg-res}\, A \cap \operatorname{spec} A_{\mathcal{D}}$ is that $\pi_{\max}\mathcal{K}_\lambda$ belongs to the set

$$\mathfrak{V}_{\mathcal{D}} = \{\mathcal{V} \in \operatorname{Gr}_{d'}(\mathcal{E}_{\max}) : \mathcal{V} \cap \pi_{\max}(\mathcal{D}) \neq 0\}.$$

This is a complex analytic variety in $\operatorname{Gr}_{d'}(\mathcal{E}_{\max})$ of codimension 1. The condition that for some nonzero $\lambda_0 \in$ bg-res A, the ray $\{r\lambda_0 : r > R\}$ contains no point of $\operatorname{spec} A_{\mathcal{D}}$ is that the curve in $\operatorname{Gr}_{d'}(\mathcal{E}_{\max})$ given by $r \mapsto \pi_{\max}\mathcal{K}_{r\lambda_0}$ has no point in $\mathfrak{V}_{\mathcal{D}}$ when $r > R$. And if $\mathcal{V} \in \operatorname{Gr}_{d'}(\mathcal{E}_{\max})\setminus\mathfrak{V}_{\mathcal{D}}$, then the norm of the projection on

$\mathcal{V}$ using $\mathcal{E}_{\max} = \mathcal{V} \oplus \pi_{\max}\mathcal{D}$ can be estimated in simple terms. This can be useful for estimating the norm of the resolvent of $A_{\mathcal{D}}$ near a point in $\operatorname{spec} A_{\mathcal{D}} \cap \text{bg-res}\, A$.

In Section 6 we discuss some aspects of symmetric cone operators from the geometric perspective developed in Section 5. Among other things we show that for such operators, the set of domains of selfadjoint extensions is a real-analytic submanifold of $\mathfrak{G}$, and that if $\dim \mathfrak{G} > 0$, then for every real λ there is a selfadjoint extension of A with λ in its spectrum. This is so even if the operator with minimal domain is bounded below (or above). A more detailed study of geometric aspects of the spectrum of selfadjoint extensions will be taken up elsewhere.

In Section 7 we analyze $A_\wedge$, also from the perspective of Section 5. While $A_\wedge$ is not a Fredholm operator, the fact that it is homogeneous under the action of the one-parameter group κ_ϱ permits a rather complete analysis of the operator, its background spectrum and the resolvents of the various extensions with index 0. Theorem 8.7 gives a necessary and sufficient condition for a given extension of $A_\wedge$ to admit a sector of minimal growth.

We finish the paper proving Theorem 9.4, an analogue of Theorem 8.7 giving a necessary and sufficient condition for an extension of A to admit a sector of minimal growth. While the proofs of these theorems are quite similar, some assumptions in Theorem 9.4 are automatically satisfied in the case of Theorem 8.7.

Most of the nonstandard notation used in this paper, and not mentioned in this introduction, is presented in Sections 4 and 5. In general, objects associated with $A_\wedge$ have the symbol $\wedge$ as part of the notation. For example, $\mathcal{E}_{\wedge,\max}$ is the orthogonal of $\mathcal{D}_{\wedge,\min}$ in $\mathcal{D}_{\wedge,\max}$, and $\pi_{\wedge,\max}$ is the corresponding orthogonal projection. All the other projections will usually indicate the space on which they map: If $H = E \oplus F$, then $\pi_{E,F} : H \to H$ will denote the projection on E according to this decomposition, and π_E is the orthogonal projection on E.

2. Definitions and conventions

Throughout the paper M is a compact n-manifold with boundary, $\mathfrak{m}$ is a smooth b-measure, $E \to M$ is a Hermitian vector bundle, and ∇ a Hermitian connection on E. The boundary of M will be denoted by Y. By x we shall mean a smooth defining function of Y, positive in the interior $\mathring{M}$ of M. This function will be fixed shortly so as to have certain properties that simplify the calculations.

The b-tangent bundle of Melrose, ${}^bTM$, is the vector bundle over M whose space of sections is

$$C^\infty_{\tan}(M; TM) = \{X \in C^\infty(M, TM) : X \text{ is tangent to } \partial M\}, \tag{2.1}$$

see [7, 8]. The space $C^\infty_{\tan}(M; \mathbb{C}TM)$ is a Lie algebra over $\mathbb{C}$ under the usual Lie bracket, and the collection of elements of order $\leq m$ in its enveloping algebra is the space $\operatorname{Diff}^m_b(M)$ of totally characteristic differential operators of order $\leq m$. If $E \to M$ is a complex vector bundle and $\operatorname{Diff}^m(M; E)$ is the space of differential operators on $C^\infty(M; E)$ of order m, then $\operatorname{Diff}^m_b(M; E)$ denotes the subspace consisting of totally characteristic differential operators on $C^\infty(M; E)$ of order m, cf. Melrose [8].

The elements of $x^{-m} \operatorname{Diff}^m_b(M; E)$, that is, differential operators of the form $A = x^{-m} P$ with $P \in \operatorname{Diff}^m_b(M; E)$, are the cone operators of order m.

The Hilbert space $L^2_b(M;E)$ is the L^2 space of sections of E with respect to the Hermitian form on E and the density $\mathfrak{m}$. Thus the inner product is

$$(u,v)_{L^2_b} = \int (u,v)_E\, \mathfrak{m} \quad \text{if } u,\ v \in L^2_b(M;E).$$

For non-negative integers s the Sobolev spaces $H^s_b(M;E)$ are defined as

$$H^s_b(M;E) = \{u \in L^2_b(M;E) : Pu \in L^2_b(M;E)\ \forall P \in \operatorname{Diff}^s_b(M;E)\}.$$

The Hilbert space structure is defined using the vector fields in $C^\infty_{\tan}(M;TM)$ with the aid of the connection on E and a partition of unity. The spaces $H^s_b(M;E)$ for general $s \in \mathbb{R}$ are defined by interpolation and duality, and we set

$$H^\infty_b(M;E) = \bigcap_s H^s_b(M;E), \quad H^{-\infty}_b(M;E) = \bigcup_s H^s_b(M;E).$$

The weighted spaces

$$x^\mu H^s_b(M;E) = \{x^\mu u : u \in H^s_b(M;E)\}$$

are Hilbert spaces with the inner product for which $H^s_b(M;E) \ni u \mapsto x^\mu u \in x^\mu H^s_b(M;E)$ is an isometry. In the case of $s=0$ one has

$$x^\mu H^0_b(M;E) = x^\mu L^2_b(M;E) = L^2(M, x^{-2\mu}\mathfrak{m}; E),$$

and the Sobolev spaces based on $L^2(M, x^{-2\mu}\mathfrak{m}; E)$ and $\operatorname{Diff}^s_b(M;E)$ are isomorphic to $x^\mu H^s_b(M;E)$. The topological structure of these spaces is independent of the particular b-density on M, Hermitian structure and connection of E, and defining function x.

To simplify a number of computations and constructions it is convenient to introduce additional structure. Let $\pi : NY \to Y$ be the normal bundle of Y in M, $NY = T_Y M/TY$. Let $x : M \to \mathbb{R}$ be any defining function for Y, positive in $\mathring{M}$. Since dx vanishes on TY, dx defines a function $x_\wedge = dx : NY \to \mathbb{R}$. Define

$$Y^\wedge = \{v \in NY : x_\wedge v \geq 0\},$$

and let $\pi_+ : Y^\wedge \to Y$ be the restriction of π.

Let $x\partial_x$ denote the canonical section of ${}^bTM$ along Y.

Lemma 2.2. *Let $\mathfrak{m}_Y = x\partial_x \rfloor \mathfrak{m}$ be the contraction of $\mathfrak{m}$ by $x\partial_x$ along Y; $\mathfrak{m}_Y$ is a smooth positive density on Y. There is a tubular neighborhood map*

$$\Phi : V \subset Y^\wedge \to U \subset M \tag{2.3}$$

and a defining function x for Y in M such that

$$\Phi^*\mathfrak{m} = \frac{dx}{x} \otimes \pi_+^*\mathfrak{m}_Y \ \text{ in } V. \tag{2.4}$$

Proof. Pick some smooth but otherwise arbitrary tubular neighborhood map $\tilde{\Phi}$ and a defining function $\tilde{x}$. Trivialize N_+Y as $[0,\infty) \times Y$ by choosing some smooth vector field $\partial_{\tilde{x}}$ in M along Y such that $\partial_{\tilde{x}}\tilde{x} = 1$. Trivialized in this manner, $\tilde{x}_\wedge : [0,\infty) \times Y \to [0,\infty)$ is the canonical projection. The b-density $\frac{d\tilde{x}_\wedge}{\tilde{x}_\wedge} \otimes \pi_+^*\mathfrak{m}_Y$ is smooth, positive, and globally defined on $Y^\wedge$. Therefore, near $\tilde{x}_\wedge = 0$,

$$\tilde{\Phi}^*\mathfrak{m} = f\frac{d\tilde{x}_\wedge}{\tilde{x}_\wedge} \otimes \pi_+^*\mathfrak{m}_Y$$

with some smooth function f. From the fact that $\tilde\Phi$ is a tubular neighborhood map it follows that $f = 1$ when $\tilde x_\wedge = 0$. There is g smooth, defined near $\tilde x_\wedge = 0$, and equal to 1 at $\tilde x_\wedge = 0$, such that if

$$F(\tilde x_\wedge, y) = (\tilde x_\wedge g(\tilde x_\wedge, y), y),$$

then

$$F^*\big(f\frac{d\tilde x_\wedge}{\tilde x_\wedge}\otimes \pi_+^*\mathfrak{m}_Y\big) = \frac{d\tilde x_\wedge}{\tilde x_\wedge}\otimes \pi_+^*\mathfrak{m}_Y.$$

Indeed, this holds if g solves

$$\partial_{\tilde x_\wedge} g = \frac{g}{f(\tilde x_\wedge g, y)}\frac{1-f(\tilde x_\wedge g, y)}{\tilde x_\wedge}.$$

Since $f(0,y) = 1$, There is a smooth solution with initial condition $g(0,y) = 1$. Define $\Phi = \tilde\Phi\circ F$. Then Φ is a tubular neighborhood map satisfying (2.4). Let x be a smooth function on M, positive in $\mathring{M}$, that agrees with $\tilde x_\wedge\circ\tilde\Phi^{-1}$ near Y. Then Φ and x are as required. □

We fix a tubular neighborhood map (2.3) and defining function x for Y such that (2.4) holds, and take

$$\mathfrak{m}_\wedge = \frac{dx_\wedge}{x_\wedge}\otimes \pi_+^*\mathfrak{m}_Y \tag{2.5}$$

as density on $Y^\wedge$. We also fix $x_\wedge$ as defining function for Y in N_+Y. Both U and V contain Y.

Let $X_\wedge = \partial_{x_\wedge}$ be the canonical vertical vector field. Fix a smooth real vector field X on M which coincides with $d\Phi(X_\wedge)$ near Y. Shrinking V and U we assume that this holds in U.

Definition 2.6. An operator $P\in \operatorname{Diff}^m_b(M;E)$ is said to have coefficients independent of x near Y if $[P,\nabla_{xX}] = 0$ near Y.

The operators on M we are concerned with need not have coefficients independent of x. They appear, however, in the form of Taylor coefficients. Namely, if $P\in \operatorname{Diff}^m_b(M;E)$, then for any N there are operators P_k, $\tilde P_N\in \operatorname{Diff}^m_b(M;E)$ such that

$$P = \sum_{k=0}^{N-1} P_k x^k + \tilde P_N x^N \tag{2.7}$$

where each P_k has coefficients independent of x near Y. The operators P_k are uniquely determined near Y by P and our choices of connection on E, defining function x, and vector field X. These Taylor expansions will be used in the course of the construction of the map θ in Theorem 4.12.

If P has coefficients independent of x near Y then so does its formal adjoint $P^\star$ in $L^2_b(M;E)$. This follows immediately from

$$(\nabla_{xX}u, v)_{L^2_b(M;E)} = -(u,\nabla_{xX}v)_{L^2_b(M;E)},\quad u,\ v\in C_0^\infty(\mathring{U};E). \tag{2.8}$$

To see that this formula holds we note that

$$xX(u,v)_E = (\nabla_{xX}u,v)_E + (u,\nabla_{xX}v)_E$$

because the connection is Hermitian. Near Y, the Lie derivative $\mathcal{L}_{xX}\mathfrak{m}$ vanishes because of (2.4) and the choice of X. So if u and v are supported in U and $h = (u,v)_E$, then $xXh\mathfrak{m} = \mathcal{L}_{xX}h\mathfrak{m} = d(h\,xX\rfloor\mathfrak{m})$. Therefore, by Stokes' theorem, $\int xX\,(u,v)_E\,\mathfrak{m} = 0$ if $u,\ v\in C_0^\infty(\mathring{U};E)$. This gives (2.8).

Let $E^\wedge \to Y^\wedge$ be the vector bundle $\pi_+^*(E|_Y)$ and give it the canonical Hermitian metric and connection. An operator $P \in \mathrm{Diff}_b^m(Y^\wedge; E^\wedge)$ is said to have coefficients independent of $x_\wedge$ if it commutes with $\nabla_{x_\wedge X_\wedge}$. The spaces

$$x_\wedge^\mu H_b^s(Y^\wedge; E^\wedge)$$

are defined in a manner completely analogous to those associated with M, using operators with coefficients independent of $x_\wedge$; for nonnegative integers s they may be defined using smooth vector fields in ${}^bTY^\wedge$ that commute with $x^\wedge X^\wedge$. Since $Y^\wedge$ is non-compact, $x_\wedge^\mu L_b^2(Y^\wedge; E^\wedge)$ literally means the L^2-space corresponding to the measure $x_\wedge^{-2\mu}\mathfrak{m}_\wedge$.

Using the tubular neighborhood map Φ, define

$$\Phi_* : E^\wedge|_V \to E|_U$$

as follows: For $\eta = (p, \eta') \in E_p^\wedge$ with $p \in V$ and $\eta' \in E_{\pi_+(p)}$, let $\Phi_*\eta \in E_{\Phi(p)}$ be the element obtained by parallel transport of η' along the curve $t \mapsto \Phi(tp)$, $t \in [0,1]$. The map Φ_* is a smooth vector bundle isomorphism covering Φ, an isometry because ∇ is Hermitian. For this reason, and because of (2.4), the induced map

$$\Phi_* : x_\wedge^{-m/2} L_b^2(V; E^\wedge|_V) \to x^{-m/2} L_b^2(U; E|_U) \tag{2.9}$$

is an isometry.

Let χ_t be the one parameter group of diffeomorphisms of M generated by xX. If u is a section of E, let $(\kappa_\varrho^\parallel u)(p) \in E_p$ be the result of parallel transport of $u(\chi_{\log \varrho} p) \in E_{\chi_{\log\varrho} p}$ along the curve

$$[0,1] \ni s \mapsto \chi_{(1-s)\log\varrho}(p) \in M.$$

There is a unique smooth positive function $f_\varrho : M \to \mathbb{R}$ with the property that

$$f_\varrho^2 x^m \mathfrak{m} = \chi_{\log\varrho}^*(x^m\mathfrak{m}).$$

Definition 2.10. Let κ_ϱ act on $C_0^\infty(\mathring{M}; E)$ as $\kappa_\varrho u = f_\varrho \kappa_\varrho^\parallel$. Denote also by κ_ϱ the analogously defined family of maps on $C_0^\infty(\mathring{Y}^\wedge; E^\wedge)$ obtained using $\mathfrak{m}_\wedge$ and $x_\wedge \partial_{x_\wedge}$.

The context will indicate whether an instance of κ_ϱ means the operator on sections of E over M or sections of $E^\wedge$ over $Y^\wedge$. In the case of $Y^\wedge$, the function f_ϱ is $\varrho^{m/2}$. Because of the following lemma, the function f_ϱ, in the case of M, is equal to $\varrho^{m/2}$ near Y.

Lemma 2.11. *Let $u \in C_0^\infty(V; E^\wedge|_V)$. Then $\kappa_\varrho \Phi_* u = \Phi_* \kappa_\varrho u$ for all $\varrho \geq 1 - \varepsilon$ for some $\varepsilon > 0$ depending on u.*

This follows from the definitions of Φ_* and κ_ϱ, using that near Y, $\Phi^*\mathfrak{m} = \mathfrak{m}_\wedge$, $x^\wedge = x \circ \Phi$, and $\Phi_* \partial_{x_\wedge} = X$. The number ε serves only to ensure that the support of $\kappa_\varrho u$ is contained in V.

Lemma 2.12. *The family $\varrho \mapsto \kappa_\varrho$, initially defined on $C_0^\infty(\mathring{M}; E)$, extends to $x^{-m/2}L_b^2(M; E)$ as a strongly continuous one-parameter group of isometries.*

Proof. Let h denote the Hermitian metric on E. If u, $v \in C_0^\infty(\mathring{M}; E)$, then

$$\begin{aligned} h(\kappa_\varrho u, \kappa_\varrho v) x^m \mathfrak{m} &= h(f_\varrho \kappa_\varrho^\parallel u, f_\varrho \kappa_\varrho^\parallel v) x^m \mathfrak{m} \\ &= h(\kappa_\varrho^\parallel u, \kappa_\varrho^\parallel v) f_\varrho^2 x^m \mathfrak{m} = \chi_{\log\varrho}^*(h(u,v)x^m\mathfrak{m}) \end{aligned}$$

so κ_ϱ extends to $x^{-m/2}L^2_b(M;E)$ as an isometry. Next we note that

$$f_{\varrho'\varrho} = f_{\varrho'}\chi^*_{\log\varrho'} f_\varrho.$$

Indeed,

$$f^2_{\varrho'\varrho}x^m\mathfrak{m} = \chi^*_{\log\varrho\varrho'}x^m\mathfrak{m} = \chi^*_{\log\varrho'}\chi^*_{\log\varrho'}x^m\mathfrak{m} \\ = \chi^*_{\log\varrho'}(f^2_\varrho x^m\mathfrak{m}) = (\chi^*_{\log\varrho'}f^2_\varrho)f^2_{\varrho'}x^m\mathfrak{m}.$$

Thus

$$\kappa_{\varrho'\varrho} = f_{\varrho'\varrho}\kappa^{\|}_{\varrho'\varrho} = f_{\varrho'}(\chi^*_{\log\varrho'}f_\varrho)\kappa^{\|}_{\varrho'}\kappa^{\|}_\varrho = f_{\varrho'}\kappa^{\|}_{\varrho'}f_\varrho\kappa^{\|}_\varrho = \kappa_{\varrho'}\kappa_\varrho.$$

That $\varrho\mapsto\kappa_\varrho u$ is continuous follows from the fact that this holds when u belongs to the dense subspace $C^\infty_0(\mathring{M};E)$ of $x^{-m/2}L^2_b(M;E)$ and the continuity of each κ_ϱ. □

We end the section with a brief comment on what we mean by the Mellin transform of an element of $x^{-m/2}L^2_b(M;E)$. Fix $\omega\in C^\infty_0(-1,1)$ real valued, nonnegative and such that $\omega=1$ in a neighborhood of 0. Also fix a Hermitian connection ∇ on E. The Mellin transform of an element $u\in C^\infty_0(\mathring{M};E)$ is defined to be the entire function $\hat{u}:\mathbb{C}\to C^\infty(Y;E|_Y)$ such that for any $v\in C^\infty(Y;E|_Y)$

$$(x^{-i\sigma}\omega u, \pi^*_Y v)_{L^2_b(M;E)} = (\hat{u}(\sigma), v)_{L^2(Y;E|_Y)}.$$

By $\pi^*_Y v$ we mean the section of E over U obtained by parallel transport of v along the fibers of π_Y. As is well known, the Mellin transform extends to the spaces $x^\mu L^2_b(M;E)$ in such a way that if $u\in x^\mu L^2_b(M;E)$ then $\hat{u}(\sigma)$ is holomorphic in $\{\Im\sigma > -\mu\}$ and in $L^2(\{\Im\sigma=-\mu\}\times Y)$ with respect to $d\sigma\otimes\mathfrak{m}_Y$.

The density $\mathfrak{m}$, the map Φ, the function x and the Hermitian connection are fixed throughout the paper. For the sake of some notational simplification we will henceforth write x, $\mathfrak{m}$, and E instead of $x_\wedge$, $\mathfrak{m}_\wedge$, and $E^\wedge$. Fixing a defining function x for Y in M, as we have done, is equivalent to fixing a trivialization of $Y^\wedge$, a diffeomorphism $Y^\wedge\to[0,\infty)\times Y$.

3. The symbols of a cone operator

Let E, $F\to M$ be complex vector bundles over M. An operator

$$A\in x^{-m}\operatorname{Diff}^m_b(M;E,F)$$

is called c-elliptic if $P=x^mA$ is b-elliptic, which means that its b-symbol,

$${}^b\!\boldsymbol{\sigma}(x^mA)\in C^\infty({}^bT^*M\backslash 0;\operatorname{Hom}({}^b\pi^*E, {}^b\pi^*F))$$

(cf. Melrose, op. cit.), is invertible. Here ${}^b\pi : {}^bT^*M\backslash 0\to M$ is the projection map. This definition depends in a mild way on the choice of defining function: if $\tilde{x}$ is another defining function for ∂M, then

$${}^b\!\boldsymbol{\sigma}(\tilde{x}^mA) = (\tilde{x}/x)^m\,{}^b\!\boldsymbol{\sigma}(x^mA). \tag{3.1}$$

Alternatively, consider the following construction of the c-cotangent bundle of M, ${}^cT^*M$, motivated by Melrose's definition of ${}^bTM$, and definition of an invariant replacement of the b-symbol. Let $\iota:\partial M\to M$ be the inclusion map and define

$$C^\infty_{\mathrm{cn}}(M;T^*M) = \{\eta\in C^\infty(M,T^*M) : \iota^*\eta = 0\},$$

the space of smooth 1-forms on M which are, over ∂M, sections of the conormal bundle of ∂M in M. Just as with the b-tangent bundle, there is the c-cotangent bundle, ${}^cT^*M$, whose space of smooth sections is $C^\infty_{\mathrm{cn}}(M;T^*M)$, and a homomorphism

$$ {}^c\mathrm{ev} : {}^cT^*M \to T^*M $$

which is an isomorphism over the interior. The fiber over p is

$$ {}^cT^*_pM = C^\infty_{\mathrm{cn}}(M;T^*M)/\big(\mathcal{I}_p(M)\cdot C^\infty_{\mathrm{cn}}(M;T^*M)\big) $$

where $\mathcal{I}_p(M)$ is the ideal in $C^\infty(M)$ of functions vanishing at p, and the homomorphism ${}^c\mathrm{ev}$ is the one induced by

$$ C^\infty_{\mathrm{cn}}(M;T^*M) \ni \eta \mapsto \eta(p) \in T^*_pM. $$

Since the latter map has $\mathcal{I}_p(M)\cdot C^\infty_{\mathrm{cn}}(M;T^*M)$ in its kernel, it induces a map ${}^c\mathrm{ev}_p : {}^cT^*_pM \to T^*_pM$. Let ${}^cTM$ be the dual bundle and let ${}^c\pi : {}^cT^*M \to M$ be the projection map.

At this point it is convenient to recall that the b-tangent bundle of M is defined in a completely analogous manner using $C^\infty_{\tan}(M;TM)$, cf. (2.1), so that

$$ {}^bT_pM = C^\infty_{\tan}(M;TM)/\big(\mathcal{I}_p(M)\cdot C^\infty_{\tan}(M;TM)\big). $$

Thus we have a map ${}^b\mathrm{ev} : {}^bTM \to TM$.

Now let $A \in x^{-m}\operatorname{Diff}^m_b(M;E,F)$. Since A is a differential operator in the interior of M, it has a principal symbol there, given by the standard formula

$$ \boldsymbol{\sigma}(A)(\xi)(\phi(p)) = \lim_{\tau\to\infty} \tau^{-m} e^{-i\tau f(p)} A(e^{i\tau f}\phi)(p) $$

with f a real-valued smooth function such that $df(p) = \xi$ and with ϕ a smooth section of E. Suppose now that f is defined in a neighborhood of a point $p_0 \in \partial M$ and vanishes on ∂M, so that df is conormal to ∂M and therefore represents a local section of ${}^cT^*M$. If, with local coordinates $x, y_1, \dots, y_{n-1}$ and with respect to some frame $\phi_1 \dots, \phi_r$ of E and frame $\psi_1, \dots, \psi_s$ of F, near p_0, we have

$$ A(\sum_\mu h^\mu \phi_\mu) = x^{-m} \sum_{\mu,\nu} \sum_{k+|\alpha|\le m} a^\nu_{k\alpha\mu}(x,y) D^\alpha_y (xD_x)^k h^\mu\, \psi_\nu, $$

then, away from the boundary,

$$ \boldsymbol{\sigma}(A)(df)(\sum_\mu h^\mu\phi_\mu) = x^{-m} \sum_{\mu,\nu} \sum_{k+|\alpha|= m} a^\nu_{k\alpha\mu}(x,y) (\partial_y f)^\alpha (x\partial_x f)^k h^\mu\, \psi_\nu $$

where by $\partial_y f$ we mean the gradient of f in the y variables. Since $f = xg$ with smooth g, this is equal to

$$ \sum_{\mu,\nu} \sum_{k+|\alpha|= m} a^\nu_{k\alpha\mu}(x,y) (\partial_y g)^\alpha (g + x\partial_x g)^k h^\mu\, \psi_\nu, $$

which is smooth up to the boundary. Suppose that $\tilde f$ is another smooth function defined near p_0 and vanishing on the boundary, so that $\tilde f = x\tilde g$ for some $\tilde g$. Then the statement that

$$ df - d\tilde f \in \mathcal{I}_{p_0}(M)\cdot C^\infty_{\mathrm{cn}}(M;T^*M) $$

is equivalent to the statement that

$$ g(p_0) = \tilde g(p_0) \text{ and } \partial_{y_j} g(p_0) = \partial_{y_j}\tilde g(p_0) \text{ for } j = 1, \dots, n-1; $$

recall that $p_0 \in \partial M$. Thus if df and $d\tilde f$ represent the same element of ${}^cT_{p_0}M$, then

$$ \lim_{p\to p_0} \boldsymbol{\sigma}(A)(df(p))(\phi(p)) = \lim_{p\to p_0} \boldsymbol{\sigma}(A)(d\tilde f(p))(\phi(p)) $$

for any smooth section ϕ of E defined near p_0. It follows that the function

$$ {}^c T^*_{\mathring{M}} M \ni \eta \mapsto {}^c\boldsymbol{\sigma}(A)(\eta) = \boldsymbol{\sigma}(A)({}^c\mathrm{ev}(\eta)) $$

extends by continuity to a function

$$ {}^c T^* M \ni \eta \mapsto {}^c\boldsymbol{\sigma}(A)(\eta), $$

a section of $\mathrm{Hom}({}^c\pi^* E, {}^c\pi^* F)$ over ${}^cT^*M\backslash 0$. It is easy to see that ${}^c\boldsymbol{\sigma}(A)$ is smooth.

Definition 3.2. The section ${}^c\boldsymbol{\sigma}(A)$ is the c-symbol of A.

For example, with the notation above, taking $g = \xi + \sum \gamma_j y_j$ with ξ and γ_j real constants, and $f = xg$ we get

$$ \boldsymbol{\sigma}(A)(df)(\sum_\mu h^\mu \phi_\mu) = \sum_{\mu,\nu} \sum_{k+|\alpha|=m} a^\nu_{k\alpha\mu}(x,y)\gamma^\alpha \xi^k h^\mu\, \psi_\nu, $$

so if η is the element of ${}^cT^*M$ represented by df, then the right hand side of this formula is ${}^c\boldsymbol{\sigma}(A)(\eta)(\sum_\mu h^\mu \phi_\mu)$.

By the definition,

$$ {}^c\boldsymbol{\sigma}(A)(\eta) = \boldsymbol{\sigma}(A)({}^c\mathrm{ev}(\eta)), \quad \eta \in {}^cT^*_{\mathring{M}}M. $$

From the fact that ${}^c\mathrm{ev}$ is an isomorphism over the interior of M, invertibility of ${}^c\boldsymbol{\sigma}(A)$ over $\mathring{M}$ is equivalent to ellipticity of A in that set.

To relate the c-symbol of A and the b-symbol of $x^m A$ recall first that if $P \in \mathrm{Diff}^m_b(M; E, F)$, then

$$ {}^b\boldsymbol{\sigma}(P)({}^b\mathrm{ev}^*\eta) = \boldsymbol{\sigma}(P)(\eta), \quad \eta \in T^*M\backslash 0; $$

here ${}^b\mathrm{ev}^* : T^*M \to {}^bT^*M$ is the dual of ${}^b\mathrm{ev} : {}^bTM \to TM$. Thus, if $\eta \in {}^bTM$ projects on an interior point of M, then

$$ {}^b\boldsymbol{\sigma}(x^m A)(\tilde\eta) = \boldsymbol{\sigma}(P)(({}^b\mathrm{ev}^*)^{-1}(\tilde\eta)), \quad \tilde\eta \in {}^bT^*M\backslash 0. $$

The fact that $x^m A$ is totally characteristic implies that $\tilde\eta \mapsto \boldsymbol{\sigma}(P)(({}^b\mathrm{ev}^*)^{-1}(\tilde\eta))$ extends by continuity to the boundary. Let $\eta \in {}^cT^*M$ project over an interior point. Then

$$ \begin{aligned} {}^c\boldsymbol{\sigma}(A)(\eta) = \boldsymbol{\sigma}(A)({}^c\mathrm{ev}(\eta)) &= x^{-m}\, \boldsymbol{\sigma}(x^m A)({}^c\mathrm{ev}(\eta)) \\ &= \boldsymbol{\sigma}(x^m A)(x^{-1}\, {}^c\mathrm{ev}(\eta)) = {}^b\boldsymbol{\sigma}(x^m A)({}^b\mathrm{ev}^*(x^{-1}\, {}^c\mathrm{ev}(\eta))). \end{aligned} $$

Writing the map $\eta \mapsto {}^b\mathrm{ev}^*(x^{-1}\, {}^c\mathrm{ev}(\eta))$ in coordinates one sees that it extends as a smooth isomorphism $\mathbf{x}^{-1} : {}^cT^*M \to {}^bT^*M$, so

$$ {}^c\boldsymbol{\sigma}(A)(\eta) = {}^b\boldsymbol{\sigma}(x^m A)(\mathbf{x}^{-1}(\eta)). $$

In particular, invertibility of the c-symbol of A is equivalent to invertibility of the b-symbol of $x^m A$.

The isomorphism $\mathbf{x}^{-1} : {}^cT^*M \to {}^bT^*M$ is determined by the defining function x, so is not natural. Write $\mathbf{x}$ for its inverse. If $\tilde x$ is another defining function for ∂M then $\mathbf{x}^{-1}\tilde{\mathbf{x}}$ is multiplication by $\tilde x/x$; this is the reason for (3.1).

Definition 3.3. The operator $A \in x^{-m}\,\mathrm{Diff}^m_b(M; E, F)$ is called c-elliptic if

$$ {}^c\boldsymbol{\sigma}(A) \in C^\infty({}^cT^*M\backslash 0; \mathrm{Hom}({}^c\pi^*E, {}^c\pi^*F)) $$

is an isomorphism. If $F = E$, the family $\lambda \mapsto A - \lambda$ is called c-elliptic with parameter in a set $\Lambda \subset \mathbb{C}$ if

$$ {}^c\boldsymbol{\sigma}(A) - \lambda \in C^\infty(({}^cT^*M \times \Lambda)\backslash 0; \mathrm{End}(({}^c\pi \times id)^*E)) $$

is an isomorphism. Here ${}^c\pi \times id : ({}^cT^*M \times \Lambda)\backslash 0 \to M \times \Lambda$ is the canonical map.

Let χ_t be the one-parameter group of diffeomorphisms generated by the vector field xX, cf. Section 2. Fix t and let $\eta \in C^\infty_{\mathrm{cn}}(M;T^*M)$. Then $\chi_t^*\eta \in C^\infty_{\mathrm{cn}}(M;T^*M)$, since $\chi_t \circ \iota = \iota$. Since also $\chi_t^*\mathcal{I}_{\chi_t(p)}(M) = \mathcal{I}_p(M)$, we get a map

$$\chi_t^* : {}^cTM \to {}^cTM, \tag{3.4}$$

a vector bundle morphism covering χ_{-t}. It is not hard to see that this map is smooth. If $A \in x^{-m}\operatorname{Diff}^m_b(M;E,F)$, let $A_\varrho = \varrho^{-m}\kappa_\varrho^{-1}A\kappa_\varrho$. Then $A_\varrho \in x^{-m}\operatorname{Diff}^m_b(M;E)$ and

$${}^c\boldsymbol{\sigma}(A_\varrho) = \varrho^{-m}\kappa_\varrho^{-1}({}^c\boldsymbol{\sigma}(A) \circ \chi^*_{\log\varrho})\kappa_\varrho. \tag{3.5}$$

Thus A_ϱ is c-elliptic if A is.

We now recall the definitions of conormal and wedge symbols, and of boundary spectrum.

If $P \in \operatorname{Diff}^m_b(M;E)$ and if u is a smooth section of E that vanishes on $Y = \partial M$, then Pu also vanishes on Y. Therefore, if v is a section of E over Y and u is an extension of v, then $(Pu)|_Y$ does not depend on the extension. Thus, associated with P there is a differential operator

$$\hat{P}(0) : C^\infty(Y;E|_Y) \to C^\infty(Y;E|_Y)$$

of order m. Fix $\sigma \in \mathbb{C}$. Since $u \mapsto x^{-i\sigma}P(x^{i\sigma}u)$ is an operator in $\operatorname{Diff}^m_b(M;E)$, there is, for each $\sigma \in \mathbb{C}$, an operator $\hat{P}(\sigma) \in \operatorname{Diff}^m(Y;E|_Y)$. The conormal symbol of P is defined to be the operator-valued polynomial

$$\mathbb{C} \ni \sigma \mapsto \hat{P}(\sigma) \in \operatorname{Diff}^m(Y;E|_Y). \tag{3.6}$$

It is easy to verify that $\hat{P}(\sigma)$ is elliptic for every σ if P is b-elliptic. The boundary spectrum of P, cf. Melrose [8], Melrose-Mendoza [9], is

$$\operatorname{spec}_b(P) = \{\sigma \in \mathbb{C} : \hat{P}(\sigma) \text{ is not invertible}\}.$$

The definition of $\hat{P}(\sigma)$ depends on the choice of defining function x but different choices of defining functions give operators related by conjugation with multiplication by $e^{i\sigma g}$ for some smooth real-valued function g, so the particular choice of defining function to define the conormal symbol is not critical. The conormal symbol of $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$ is defined to be that of the totally characteristic operator x^mA, and the boundary spectrum of A is defined to be that of x^mA.

If $A \in x^{-m}\operatorname{Diff}^m_b(M;E,F)$, then $\Phi_*^{-1}A\Phi_*$ is a cone operator defined in V, cf. (2.9), and if $u \in C_0^\infty(\mathring{Y}^\wedge;E^\wedge)$, then the limit in the following definition exists in $C_0^\infty(\mathring{Y}^\wedge;F^\wedge)$.

Definition 3.7. The wedge symbol of A is the operator $A_\wedge \in x^{-m}\operatorname{Diff}^m_b(Y^\wedge;E,F)$ defined by

$$A_\wedge u = \lim_{\varrho\to 0}\varrho^m\kappa_\varrho(\Phi_*^{-1}A\Phi_*)\kappa_\varrho^{-1}u \tag{3.8}$$

Writing A as $x^{-m}P$ and expanding P as in (2.7) with $N = 1$ we get directly from the definition of $A_\wedge$ that

$$A_\wedge = x^{-m}P_{\wedge,0}$$

where $P_{\wedge,0}$ is the operator on $Y^\wedge$ that coincides with $\Phi_*^{-1}P_0\Phi_*$ near Y and satisfies

$$\kappa_\varrho(\Phi_*^{-1}P_{\wedge,0}\Phi_*)\kappa_\varrho^{-1} = P_{\wedge,0}$$

for all $\varrho > 0$. Near Y (in M), with the notation and conventions of Section 2, $P_0 = \sum_{|\alpha|+k\le m} a_{k\alpha}(y) D_y^\alpha (xD_x)^k$ near P_0 where $a_{k\alpha}(y)$ is independent of x. The operator $P_{\wedge,0}$ is the "same" operator (the pull-back by Φ), but on $Y^\wedge$, and so

$$A_\wedge = x^{-m} \sum_{|\alpha|+k\le m} a_{k\alpha}(y) D_y^\alpha (xD_x)^k$$

for all x. The conormal symbols of A and $A_\wedge$ are equal to each other, in coordinates the family

$$x^{-m} \sum_{|\alpha|+k\le m} a_{k\alpha}(y)\sigma^k D_y^\alpha, \quad \sigma\in\mathbb{C}.$$

The wedge symbol $A_\wedge$ of A inherits properties of A. Using the tubular neighborhood map Φ we also get a bundle isomorphism

$$^c T_V^* Y^\wedge \to {}^c T_U^* M$$

covering Φ. It is not hard to verify that, over $\partial Y^\wedge = Y = \partial M$ we have

$$^c\boldsymbol{\sigma}(A_\wedge)|_{^cT_Y^*Y^\wedge} = {}^c\boldsymbol{\sigma}(A)|_{^cT_Y^*M}.$$

Thus c-ellipticity is preserved.

Let $A^\star$ be the formal adjoint of A acting on $x^{-m/2}L_b^2(M;E)$. Since Φ_* and κ_ϱ are isometries (the former near Y),

$$(\varrho^m\kappa_\varrho(\Phi_*^{-1}A\Phi_*)\kappa_\varrho^{-1}u,v)_{x^{-m/2}L_b^2} = (u,\varrho^m\kappa_\varrho(\Phi_*^{-1}A^\star\Phi_*)\kappa_\varrho^{-1}v)_{x^{-m/2}L_b^2}$$

if $u,v\in C_0^\infty(\mathring{Y}^\wedge;E^\wedge)$ and ϱ is small. Thus, taking the limit as $\varrho\to 0$ we get

$$(A_\wedge)^\star = (A^\star)_\wedge. \tag{3.9}$$

Lemma 3.10. *Suppose that A is symmetric on $C_0^\infty(\mathring{M};E)$. Then $A_\wedge$ is symmetric on $C_0^\infty(\mathring{Y}^\wedge;E)$. If in addition A is semibounded from below, then $A_\wedge$ is semibounded from below by* 0.

The first assertion follows immediately from (3.9) and the hypothesis that $A^\star = A$. For the second, let $C\in\mathbb{R}$ be such that

$$(Au,u)_{x^{-m/2}L_b^2} \ge C\|u\|^2_{x^{-m/2}L_b^2}, \quad u\in C_0^\infty(\mathring{M};E).$$

Suppose that $u\in C_0^\infty(\mathring{Y};E^\wedge)$. Then

$$\begin{aligned}(\varrho^m\kappa_\varrho^{-1}(\Phi_*^{-1}A\Phi_*)\kappa_\varrho u,u)_{x^{-m/2}L_b^2} &= \varrho^m(A\Phi_*\kappa_\varrho u,\Phi_*\kappa_\varrho u)_{x^{-m/2}L_b^2}\\ &\ge C\varrho^m\|\Phi_*\kappa_\varrho u\|^2_{x^{-m/2}L_b^2} = C\varrho^m\|u\|^2_{x^{-m/2}L_b^2}\end{aligned}$$

Passing to the limit as $\varrho\to 0$ we thus get the second assertion of the lemma.

It also follows from (3.8) that the family $\lambda\mapsto A_\wedge-\lambda$ satisfies the homogeneity relation

$$A_\wedge - \varrho^m\lambda = \varrho^m\kappa_\varrho(A_\wedge-\lambda)\kappa_\varrho^{-1} \text{ for every } \varrho>0. \tag{3.11}$$

Definition 3.12. A family of operators $A(\lambda)$ acting on a κ-invariant space of distributions on $Y^\wedge$ will be called κ-homogeneous of degree ν if

$$A(\varrho^m\lambda) = \varrho^\nu\kappa_\varrho A(\lambda)\kappa_\varrho^{-1}$$

for every $\varrho>0$.

4. Closed extensions

In this section we recall some known facts about the closed extensions of a c-elliptic cone operator A on a compact manifold and, where needed, sketch proofs of analogous results for the closed extensions of its model operator $A_\wedge$. Theorem 4.12 gives a natural isomorphism between $\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$ and $\mathcal{D}_{\max}(A_\wedge)/\mathcal{D}_{\min}(A_\wedge)$ that will play an important role in [3] but not in the remainder of the present paper. The rest of the material in this section will be used at various points in all later sections.

Suppose that $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$, fix $\mu \in \mathbb{R}$ and consider A first as an unbounded operator

$$A : C_0^\infty(\mathring{M};E) \subset x^\mu L^2_b(M;E) \to x^\mu L^2_b(M;E) \tag{4.1}$$

Write $\mathcal{D}_{\min}(A)$ for the domain of the closure of this operator; with this domain, A is referred to as the minimal extension of A. The structure of $\mathcal{D}_{\min}(A)$ when A is c-elliptic was characterized in Gil-Mendoza [4, Proposition 3.6]. Define also

$$\mathcal{D}_{\max}(A) = \{u \in x^\mu L^2_b(M;E) : Au \in x^\mu L^2_b(M;E)\}$$

The maximal extension of A is

$$A : \mathcal{D}_{\max}(A) \subset x^\mu L^2_b(M;E) \to x^\mu L^2_b(M;E),$$

also a closed operator. The space $\mathcal{D}_{\min}(A)$ is a closed subspace of $\mathcal{D}_{\max}(A)$ in the graph norm defined by A, and all closed extensions of (4.1) have as domain a subspace of $\mathcal{D}_{\max}(A)$ containing $\mathcal{D}_{\min}(A)$.

It is well known, see Lesch [5], that if A is c-elliptic, then A with domain $\mathcal{D}_{\max}(A)$ is Fredholm, $\mathcal{D}_{\min}(A)$ has finite codimension in $\mathcal{D}_{\max}(A)$, and if $\mathcal{D}$ is a subspace of $\mathcal{D}_{\max}(A)$ containing $\mathcal{D}_{\min}(A)$, then

$$\operatorname{ind} A_{\mathcal{D}} = \operatorname{ind} A_{\mathcal{D}_{\min}} + \dim \mathcal{D}/\mathcal{D}_{\min}. \tag{4.2}$$

Here $A_{\mathcal{D}}$ means the operator

$$A : \mathcal{D} \subset x^\mu L^2_b(M;E) \to x^\mu L^2_b(M;E).$$

The problem we wish to consider is the nature of the spectrum and structure of the resolvent of the closed extensions of (4.1) of index zero (if any).

Since multiplication by x^ν is an isomorphism (in fact an isometry)

$$x^\nu : x^\mu L^2_b(M;E) \to x^{\mu+\nu} L^2_b(M;E)$$

we may conjugate A with such operators with no essential change of the problem. For convenience we will work with the operator $x^{-\mu-m/2}Ax^{\mu+m/2}$ so as to base all the analysis on $x^{-m/2}L^2_b(M;E)$. Clearly, $x^{-\mu-m/2}Ax^{\mu+m/2} \in x^{-m}\operatorname{Diff}^m_b(M;E)$. Since

$${}^c\boldsymbol{\sigma}(x^{-\mu-m/2}Ax^{\mu+m/2}) = {}^c\boldsymbol{\sigma}(A),$$

c-ellipticity is preserved by such conjugations. We thus assume that $\mu = -m/2$. The standing assumption, unless otherwise indicated, will be that A is c-elliptic.

We will usually abbreviate $\mathcal{D}_{\min}(A)$ to $\mathcal{D}_{\min}$ and $\mathcal{D}_{\max}(A)$ to $\mathcal{D}_{\max}$ when the operator is clear from the context. As already indicated, the operator A with domain $\mathcal{D}$ will be denoted by $A_{\mathcal{D}}$.

The inner product

$$(u,v)_A = (u,v)_{x^{-m/2}L^2_b} + (Au,Av)_{x^{-m/2}L^2_b} \tag{4.3}$$

on $\mathcal{D}_{\max}$ makes this space into a Hilbert space.

Definition 4.4. The orthogonal of $\mathcal{D}_{\min}(A)$ in $\mathcal{D}_{\max}(A)$ with respect to this inner product will be denoted $\mathcal{E}_{\max}(A)$, or $\mathcal{E}_{\max}$ if A is clear from the context. We denote by $\pi_{\max} : \mathcal{D}_{\max}(A) \to \mathcal{D}_{\max}(A)$ the orthogonal projection on $\mathcal{E}_{\max}(A)$.

Since $\mathcal{D}_{\min}$ is closed in $\mathcal{D}_{\max}$,

$$\mathcal{D}_{\max} = \mathcal{D}_{\min} \oplus \mathcal{E}_{\max}$$

and since $\mathcal{D}_{\min}$ has finite codimension in $\mathcal{D}_{\max}$, $\mathcal{E}_{\max}$ is a finite-dimensional space.

Lemma 4.5. *The space $\mathcal{E}_{\max}(A)$ is equal to $\mathcal{D}_{\max}(A) \cap \ker(A^\star A + I)$, where the kernel is computed in the space of extendable distributions.*

Here $A^\star$ is the formal adjoint of A in the space $x^{-m/2}L^2_b(M;E)$, that is,

$$(Au, v) = (u, A^\star v) \quad \forall u,\ v \in C_0^\infty(\mathring{M};E).$$

It is immediate from the definitions of minimal and maximal domains that the Hilbert space adjoint of

$$A : \mathcal{D}_{\min}(A) \subset x^{-m/2}L^2_b(M;E) \to x^{-m/2}L^2_b(M;E) \tag{4.6}$$

is

$$A^\star : \mathcal{D}_{\max}(A^\star) \subset x^{-m/2}L^2_b(M;E) \to x^{-m/2}L^2_b(M;E).$$

Proof. We first show that $\mathcal{E}_{\max} \subset \mathcal{D}_{\max}(A) \cap \ker(A^\star A + I)$. If $u \in \mathcal{E}_{\max}$, then $u \in \mathcal{D}_{\max}(A)$ and

$$(Au, Av)_{x^{-m/2}L^2_b} = -(u, v)_{x^{-m/2}L^2_b} \quad \forall v \in \mathcal{D}_{\min}.$$

Therefore the map

$$\mathcal{D}_{\min}(A) \ni v \mapsto (Au, Av)_{x^{-m/2}L^2_b} \in \mathbb{C}$$

is continuous in the norm of $x^{-m/2}L^2_b(M;E)$, and consequently, Au belongs to the domain of the Hilbert space adjoint of (4.6). Thus $u \in \mathcal{D}_{\max}(A^\star A)$ and the identity $(u, v)_A = 0$, $v \in \mathcal{D}_{\min}(A)$ gives

$$(A^\star Au, v) + (u, v) = 0 \quad \forall v \in C_0^\infty(\mathring{M};E)$$

which gives that $u \in \ker(A^\star A + I)$. Thus $\mathcal{E}_{\max} \subset \mathcal{D}_{\max}(A) \cap \ker(A^\star A + I)$.

To prove the opposite inclusion, suppose that $u \in \mathcal{D}_{\max}(A) \cap \ker(A^\star A + I)$. Then $Au \in x^{-m/2}L^2_b(M;E)$ (since $u \in \mathcal{D}_{\max}(A)$) and $A^\star(Au) = -u \in x^{-m/2}L^2_b(M;E)$, so $Au \in \mathcal{D}_{\max}(A^\star)$. Thus

$$(A^\star Au, v) = (Au, Av) \quad \forall v \in C_0^\infty(\mathring{M};E)$$

and it follows that $(u, v)_A = 0$ for all $v \in C_0^\infty(\mathring{M};E)$. Since the latter space is dense in $\mathcal{D}_{\min}$, we get that $u \in \mathcal{E}_{\max}$. □

In the course of the proof we also showed:

Lemma 4.7. *$\mathcal{E}_{\max}(A) \subset \mathcal{D}_{\max}(A^\star A)$.*

Since A is c-elliptic, so are $A^\star$ and $A^\star A + I$. It follows that the Mellin transform of any $u \in \mathcal{E}_{\max}$ is a meromorphic function defined on all of $\mathbb{C}$.

We now discuss analogous aspects for the operator $A_\wedge$. The space $\mathcal{D}_{\min}(A_\wedge)$ is the domain of the closure of

$$A_\wedge : C_0^\infty(\mathring{Y}^\wedge;E) \subset x^{-m/2}L^2_b(Y^\wedge;E) \to x^{-m/2}L^2_b(Y^\wedge;E)$$

and

$$\mathcal{D}_{\max}(A_\wedge) = \{u \in x^{-m/2}L^2_b(Y^\wedge;E) : A_\wedge u \in x^{-m/2}L^2_b(Y^\wedge;E)\}.$$

Since $A_\wedge$ need not be Fredholm with either of these domains, we discuss these in some detail. We will usually write $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$ for the minimal and maximal domains of $A_\wedge$.

Lemma 4.8. *If $u \in \mathcal{D}_{\max}(A_\wedge)$, then $(1-\omega)u \in \mathcal{D}_{\min}(A_\wedge)$ for every cut-off function ω with $\omega = 1$ near $x = 0$.*

In other words, as far as closed extensions are concerned, there is no essential structure at infinity.

Proof. Let $j : Y^\wedge \to Y^\wedge$ be the involution $(x, y) \mapsto (\xi, y) = (1/x, y)$. Under this map, $P_0 = x^m A_\wedge$ goes over to a certain other totally characteristic b-elliptic operator $\check{P}_0$, $A_\wedge$ goes to $\check{A}_\wedge = \xi^m \check{P}_0$, and

$$j^* : x^\nu H^\mu_b(Y^\wedge, E) \to \xi^{-\nu} H^\mu_b(Y^\wedge;E)$$

is an isomorphism. We'll write $\check{u}$ for j^*u. Since $\check{P}_0$ is b-elliptic, there are properly supported operators $\check{Q}$ and $\check{R}$ defined on extendable distributions such that for every ν the operators

$$\check{Q} : \xi^\nu L^2_b(Y^\wedge;E) \to \xi^\nu H^m_b(Y^\wedge;E), \quad \check{R} : \xi^\nu L^2_b(Y^\wedge;E) \to \xi^\nu H^\infty_b(Y^\wedge;E)$$

are continuous and

$$\check{Q}\check{P}_0 = I - \check{R}.$$

If $u \in \mathcal{D}_{\max}(A_\wedge)$ then $\check{u} \in \xi^{m/2}L^2_b(Y^\wedge;E)$ and $\xi^m \check{P}_0 \check{u} = f \in \xi^{m/2}L^2_b(Y^\wedge;E)$. From

$$\check{Q}\xi^{-m}\check{f} = \check{Q}\xi^{-m}\xi^m\check{P}_0\check{u} = \check{u} - \check{R}\check{u},$$

we get

$$\check{u} = \check{Q}\xi^{-m}\check{f} + \check{R}\check{u}$$

with $\check{Q}\xi^{-m}\check{f} \in \xi^{-m/2}H^m_b(Y^\wedge;E)$ and $\check{R}\check{u} \in \xi^{m/2}H^\infty_b(Y^\wedge;E)$. If ω is as in the statement of the lemma, then $(1-\check{\omega})$ is supported near $\xi = 0$, so $(1-\check{\omega})\check{R}\check{u} \in \xi^{m/2}H^m_b(Y^\wedge;E)$. Thus $(1-\check{\omega})\check{u} \in \xi^{m/2}L^2_b(Y^\wedge;E) \cap \xi^{-m/2}H^m_b(Y^\wedge;E)$. Let $\chi \in C^\infty_0(\mathbb{R})$ be such that $\chi(\xi) = 1$ near 0 and let $\chi_\ell(\xi) = \chi(\ell\xi)$. Define

$$\check{v}_\ell = (1-\chi_\ell)(1-\check{\omega})\check{u}.$$

Then $\check{v}_\ell \in \xi^{m/2}L^2_b(Y^\wedge;E) \cap \xi^{-m/2}H^m_b(Y^\wedge;E)$ and $\check{v}_\ell \to (1-\check{\omega})\check{u}$ as $\ell \to \infty$, in $\xi^{m/2}L^2_b(Y^\wedge;E)$ as well as in $\xi^{-m/2}H^m_b(Y^\wedge;E)$. From the latter we get that $\check{P}_0\check{v}_\ell$ converges in $\xi^{-m/2}L^2_b(Y^\wedge;E)$ to $\check{P}_0(1-\check{\omega})\check{u}$ as $\ell \to \infty$, and consequently, that $\check{A}_\wedge\check{v}_\ell$ converges in $\xi^{m/2}L^2_b(Y^\wedge;E)$ to $\check{A}_\wedge(1-\check{\omega})\check{u}$. This proves that $(1-\omega)u \in \mathcal{D}_{\min}(A_\wedge)$, since the $\check{v}_\ell$ are compactly supported. □

The structure of $\mathcal{D}_{\wedge,\min}$ near Y is described in the first two items of the following proposition, which can be proved using the same arguments as in the proof of [4, Proposition 3.6]. The third follows from an analysis of Mellin transforms that takes advantage of the fact that the conormal symbols of $A_\wedge$ and A are the same. An explicit, simple but fundamental isomorphism between the spaces $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$ is given in Theorem 4.12.

Proposition 4.9. *Let $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$ be c-elliptic. Then*

(*i*) $\mathcal{D}_{\wedge,\min} = \{u \in \mathcal{D}_{\wedge,\max} : \omega u \in x^{m/2-\varepsilon}H^m_b(Y^\wedge;E)\ \forall\varepsilon > 0\}$.

(ii) $\mathcal{D}_{\wedge,\min} = x^{m/2}H_b^m(Y^\wedge;E)\cap x^{-m/2}L_b^2(Y^\wedge;E)$ *if and only if* $\operatorname{spec}_b(A)\cap\{\Im\sigma = -m/2\}=\varnothing$.

(iii) $\dim \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} = \dim\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$.

On $\mathcal{D}_{\wedge,\max}$ we take, naturally,

$$(u,v)_{A_\wedge} = (A_\wedge u, A_\wedge v)_{x^{-m/2}L_b^2} + (u,v)_{x^{-m/2}L_b^2}, \tag{4.10}$$

as inner product.

Definition 4.11. The orthogonal of $\mathcal{D}_{\wedge,\min}$ in $\mathcal{D}_{\wedge,\max}$ with respect to this inner product will be denoted $\mathcal{E}_{\max}(A_\wedge)$, or $\mathcal{E}_{\wedge,\max}$ if $A_\wedge$ is clear from the context. We denote by $\pi_{\wedge,\max}:\mathcal{D}_{\max}(A_\wedge)\to\mathcal{D}_{\max}(A_\wedge)$ the orthogonal projection on $\mathcal{E}_{\max}(A_\wedge)$.

The proof of Lemma 4.5 gives that $\mathcal{E}_{\wedge,\max}=\mathcal{D}_{\wedge,\max}\cap\ker(A_\wedge^\star A_\wedge + I)$.

The following result, although elementary in nature, is of fundamental importance in expressing the relation between the domains of A and the domains of $A_\wedge$. Let

$$S=\{\sigma\in\mathbb{C} : -m/2<\Im\sigma<m/2\},$$

and for each $\sigma\in S$ let $N(\sigma)$ be the largest integer N such that $\Im\sigma - N > -m/2$. Let σ_j, $j=1,\dots,\nu$ be an enumeration of the elements of $\Sigma = \operatorname{spec}_b(A)\cap S$.

Theorem 4.12. *Let A be an arbitrary c-elliptic cone differential operator. There are canonical decompositions*

$$\mathcal{E}_{\max}(A) = \bigoplus_{j=1}^{\nu}\mathcal{E}_{\sigma_j}(A),\quad \mathcal{E}_{\max}(A_\wedge) = \bigoplus_{j=1}^{\nu}\mathcal{E}_{\sigma_j}(A_\wedge) \tag{4.13}$$

such that

(i) if $u\in\mathcal{E}_{\sigma_j}(A)$, then $\hat u|_{\{\Im\sigma>-m/2\}}$ has poles at most at $\sigma_j - i\vartheta$ for $\vartheta = 0,\dots,N(\sigma_j)$;

(ii) if $u\in\mathcal{E}_{\sigma_j}(A_\wedge)$, then $\hat u|_{\{\Im\sigma>-m/2\}}$ has a pole at most at σ_j;

(iii) if $u\in\mathcal{E}_{\sigma_j}(A)$ or $u\in\mathcal{E}_{\sigma_j}(A_\wedge)$ and $\hat u$ is holomorphic at σ_j, then $u=0$.

There is a natural isomorphism

$$\theta:\mathcal{E}_{\max}(A)\to\mathcal{E}_{\max}(A_\wedge) \tag{4.14}$$

such that for each j,

$$\theta\big|_{\mathcal{E}_{\sigma_j}(A)}:\mathcal{E}_{\sigma_j}(A)\to\mathcal{E}_{\sigma_j}(A_\wedge), \tag{4.15}$$

and for each j and for all $u\in\mathcal{E}_{\sigma_j}(A)$,

$$(\Phi_*^{-1}\omega u - \theta u)\widehat{\ }\ \text{ is holomorphic near } \sigma_j \tag{4.16}$$

where $\omega\in C_0^\infty(U;E)$ is such that $\omega=1$ near Y.

Proof. For any open set $U\subset\mathbb{C}$ let $\mathfrak{M}(U)$ be the space of meromorphic functions on U with values in $C^\infty(\partial M;E|_{\partial M})$. For $\sigma_0\in U$ let $\mathfrak{M}_{\sigma_0}(U)$ be the subspace of $\mathfrak{M}(U)$ consisting of elements with pole only at $\sigma_0\in U$. Finally let $\mathfrak{H}(U)$ be the subspace of holomorphic elements. We let

$$\mathrm{s}_{\sigma_0}:\mathfrak{M}_{\sigma_0}(U)\to\mathfrak{M}_{\sigma_0}(\mathbb{C})$$

be the map that sends an element in $\mathfrak{M}_{\sigma_0}(U)$ to its singular part at σ_0.

If $A = x^{-m}P$ with $P \in \operatorname{Diff}_b^m(M;E)$, then near $Y = \partial M = \partial Y^\wedge$ we have

$$x^m A = P = \sum_{k=0}^{m-1} P_k x^k + \tilde{P}_m x^m \tag{4.17}$$

where each P_k, $k < m$, has coefficients independent of x, cf. Definition 2.6. Then

$$x^m A_\wedge = \Phi_*^{-1} P_0 \Phi_*$$

near Y in $Y^\wedge$. Let $\hat{P}_k$ be the conormal symbol of P_k. The operator $\hat{P}_0$ is the conormal symbol of both A and $A_\wedge$.

Let $\sigma_0 \in \Sigma$, and let $U \subset S$ be a neighborhood of σ_0 such that $U \cap \operatorname{spec}_b(A) = \{\sigma_0\}$. Then $\hat{P}_0$ gives an operator

$$\mathcal{P} : \mathfrak{M}(U)/\mathfrak{H}(U) \to \mathfrak{M}(U)/\mathfrak{H}(U)$$

whose kernel is finite-dimensional. Since σ_0 is the only point of $\operatorname{spec}_b(A)$ in U, the elements in the kernel of $\mathcal{P}$ are represented by meromorphic functions on U with pole only at σ_0. By taking the singular part of such functions we get a space $\hat{\mathcal{E}}_{\sigma_0}(A_\wedge) \subset \mathfrak{M}_{\sigma_0}(\mathbb{C})$ with the property that $h \in \mathfrak{M}_{\sigma_0}(\mathbb{C})$ and $\hat{P}_0 h \in \mathfrak{H}(U)$ imply that there is a unique element $\psi \in \hat{\mathcal{E}}_{\sigma_0}(A_\wedge)$ such that $h - \psi \in \mathfrak{H}(U)$:

$$\hat{\mathcal{E}}_{\sigma_0}(A_\wedge) = \{\mathrm{s}_{\sigma_0}\big(\hat{P}_0(\sigma)^{-1} f(\sigma)\big) : f \in \mathfrak{H}(U)\}. \tag{4.18}$$

If $\psi \in \hat{\mathcal{E}}_{\sigma_0}(A_\wedge)$, then there is $u \in x^{-m/2} H_b^\infty(M;E)$ supported in U such that $\hat{u} - \psi$ is holomorphic in $\Im\sigma > -m/2$. Such u belongs to $x^{-m/2} L_b^2(M;E)$, and since $\hat{P}_0 \hat{u} = \hat{P}_0(\hat{u} - \psi)$ is holomorphic in $\Im\sigma > -m/2$, we get that $\Phi_*^{-1} u \in \mathcal{D}_{\max}(A_\wedge)$. If $v \in \mathcal{D}_{\max}(A_\wedge)$ also has the property that $\hat{v} - \psi$ is holomorphic in $\Im\sigma > -m/2$, then $\Phi_*^{-1} u - v \in \mathcal{D}_{\min}(A_\wedge)$, and consequently $\pi_{\wedge,\max}\Phi_*^{-1} u = \pi_{\wedge,\max} v$. Thus there is a well defined operator $F_{\wedge,\sigma_0} : \hat{\mathcal{E}}_{\sigma_0}(A_\wedge) \to \mathcal{E}_{\max}(A_\wedge)$, characterized by the property that

$$\psi - [F_{\wedge,\sigma_0}\psi]\widehat{\ } \quad \text{is holomorphic in } \Im\sigma > -m/2.$$

From this property one obtains that

$$\mathrm{s}_{\sigma_0}([F_{\wedge,\sigma_0}\psi]\widehat{\ }) = \psi$$

so the operator $F_{\wedge,\sigma_0}$ is an isomorphism onto its image. Define

$$\mathcal{E}_{\sigma_0}(A_\wedge) = F_{\wedge,\sigma_0} \hat{\mathcal{E}}_{\sigma_0}(A_\wedge). \tag{4.19}$$

Clearly, if σ_i, $\sigma_j \in \Sigma$ and $\sigma_i \neq \sigma_j$, then $\mathcal{E}_{\sigma_i}(A_\wedge) \cap \mathcal{E}_{\sigma_j}(A_\wedge) = 0$.

If $u \in \mathcal{E}_{\max}(A_\wedge)$, then $\hat{u}$ is meromorphic in $\Im\sigma > -m/2$ with poles in Σ, since $\hat{P}_0 \hat{u}$ is holomorphic in $\Im\sigma > -m/2$. Therefore $\mathrm{s}_{\sigma_j}(\hat{u}) \in \hat{\mathcal{E}}_{\sigma_j}(A_\wedge)$ and $\hat{u} - \sum_{\sigma_j \in \Sigma} \mathrm{s}_{\sigma_j}(\hat{u})$ is holomorphic in $\Im\sigma > -m/2$. Thus the Mellin transform of

$$v = u - \sum_{j=1}^{\nu} F_{\wedge,\sigma_j} \mathrm{s}_{\sigma_j}(\hat{u})$$

is holomorphic in $\Im\sigma > -m/2$, and therefore $v \in \mathcal{D}_{\min}(A_\wedge)$. But since v also belongs to $\mathcal{E}_{\max}(A_\wedge)$, $v = 0$. Thus we have (4.13) for the operator $A_\wedge$.

We now construct the spaces $\mathcal{E}_{\sigma_j}(A)$ for A. Pick $\sigma_0 \in \Sigma$ and let $\psi \in \hat{\mathcal{E}}_{\sigma_0}(A_\wedge)$. Thus $\hat{P}_0 \psi$ is entire. Define $\hat{\mathrm{e}}_{\sigma_0,0}$ as the identity map on $\hat{\mathcal{E}}_{\sigma_0}(A_\wedge)$ and inductively define

$$\hat{\mathrm{e}}_{\sigma_0,\vartheta} : \hat{\mathcal{E}}_{\sigma_0}(A_\wedge) \to \mathfrak{M}_{\sigma_0 - i\vartheta}(\mathbb{C}), \quad \vartheta = 1, \ldots, N(\sigma_0)$$

by
$$\hat{\mathrm{e}}_{\sigma_0,\vartheta}(\psi) = -\mathrm{s}_{\sigma_0-i\vartheta}\Big(\hat{P}_0(\sigma)^{-1}\sum_{\ell=0}^{\vartheta-1}\hat{P}_{\vartheta-\ell}(\sigma)\big(\hat{\mathrm{e}}_{\sigma_0,\ell}(\psi)(\sigma+i(\vartheta-\ell))\big)\Big).$$
Then
$$\sum_{\ell=0}^{\vartheta}\hat{P}_{\vartheta-\ell}(\sigma)\big(\hat{\mathrm{e}}_{\sigma_0,\ell}(\psi)(\sigma+i(\vartheta-\ell))\big) \tag{4.20}$$
is entire, and
$$\sum_{k=0}^{m-1}\hat{P}_k(\sigma)\sum_{\vartheta=0}^{N(\sigma_0)}\hat{\mathrm{e}}_{\sigma_0,\vartheta}(\psi)(\sigma+ik)$$
is holomorphic in $\Im\sigma > -m/2$. Define
$$\hat{\mathcal{E}}_{\sigma_0}(A) = \{\sum_{\vartheta=0}^{N(\sigma_0)}\hat{\mathrm{e}}_{\sigma_0,\vartheta}(\psi) : \psi\in\hat{\mathcal{E}}_{\sigma_0}(A_\wedge)\}. \tag{4.21}$$
Given $\psi\in\hat{\mathcal{E}}_{\sigma_0}(A_\wedge)$, choose for each ϑ and element $u_\vartheta\in x^{-m/2}H_b^\infty(M;E)$ such that $\hat{u}_\vartheta - \hat{\mathrm{e}}_{\sigma_0,\vartheta}(\psi)$ is entire. Then $u=\sum u_\vartheta\in\mathcal{D}_{\max}(A)$. If the $v_\vartheta\in x^{-m/2}H_b^\infty(M;E)$ also satisfy the condition that $\hat{v}_\vartheta - \hat{\mathrm{e}}_{\sigma_0,\vartheta}(\psi)$ is entire, then $\pi_{\max}(u)=\pi_{\max}(v)$, so again we have a well defined operator $F_{\sigma_0}:\hat{\mathcal{E}}_{\sigma_0}(A_\wedge)\to\mathcal{E}_{\max}(A)$. This operator is injective; we let $\mathcal{E}_{\sigma_0}(A)$ be its image. It is more tedious than hard to verify that (4.13) holds.

Define θ so that (4.15) holds, and on each $\mathcal{E}_{\sigma_j}(A)$, $\theta = F_{\wedge,\sigma_j}\circ F_{\sigma_j}^{-1}$. Then (4.16) also holds. ☐

Let
$$\mathfrak{D}(A) = \{\mathcal{D}\subset\mathcal{D}_{\max}(A) : \mathcal{D} \text{ is a vector space and } \mathcal{D}_{\min}(A)\subset\mathcal{D}\}.$$
The elements of $\mathfrak{D}(A)$ are in one-to-one correspondence with the subspaces of $\mathcal{E}_{\max}(A)$ via the map
$$\mathfrak{D}\ni\mathcal{D}\mapsto\mathcal{D}\cap\mathcal{D}_{\max}(A)=\pi_{\max}(\mathcal{D})\subset\mathcal{E}_{\max}(A),$$
so $\mathfrak{D}(A)$ can be viewed as the union of the Grassmannian varieties of various dimensions associated with $\mathcal{E}_{\max}(A)$. Likewise let
$$\mathfrak{D}_\wedge = \{\mathcal{D}\subset\mathcal{D}_{\wedge,\max} : \mathcal{D} \text{ is a vector space and } \mathcal{D}_{\wedge,\min}\subset\mathcal{D}\}.$$
With the map θ of Theorem 4.12 we then get a map
$$\Theta : \mathfrak{D}(A)\to\mathfrak{D}_\wedge. \tag{4.22}$$

5. Domains and spectra

We discuss here the spectra and resolvents of the closed extensions of a cone operator $A\in x^{-m}\operatorname{Diff}_b^m(M;E)$ in geometric terms. We continue to assume that A is c-elliptic and that M is compact. The results in this section will be relevant mostly in Sections 6 and 9. The conceptual point of view developed here will be taken up in Section 7 in the context of the model operator.

We begin with the elementary observation that only those extensions of A that have index zero may have nonempty resolvent set.

Lemma 5.1. *There is $\mathcal{D}\in\mathfrak{D}$ such that $\operatorname{ind} A_{\mathcal{D}}=0$ if and only if $\operatorname{ind} A_{\mathcal{D}_{\min}}\le 0$ and $\operatorname{ind} A_{\mathcal{D}_{\max}}\ge 0$.*

Proof. If there is a domain $\mathcal{D} \in \mathfrak{D}$ such that $\operatorname{ind} A_{\mathcal{D}} = 0$, then the relative index formula (4.2) gives

$$\begin{aligned} \operatorname{ind} A_{\mathcal{D}_{\min}} \leq \operatorname{ind} A_{\mathcal{D}_{\min}} + \dim \mathcal{D}/\mathcal{D}_{\max} = 0 \\ \leq \operatorname{ind} A_{\mathcal{D}_{\min}} + \dim \mathcal{D}_{\max}/\mathcal{D}_{\min} = \operatorname{ind} A_{\mathcal{D}_{\max}}. \end{aligned}$$

Conversely, suppose that $0 \leq -\operatorname{ind} A_{\mathcal{D}_{\min}}$ and $\operatorname{ind} A_{\mathcal{D}_{\max}} \geq 0$. Using (4.2) again we get

$$d = \operatorname{ind} A_{\mathcal{D}_{\max}} - \operatorname{ind} A_{\mathcal{D}_{\min}},$$

so $-\operatorname{ind} A_{\mathcal{D}_{\min}} \leq d$, and there is a subspace of $\mathcal{D}_{\max}/\mathcal{D}_{\min}$ of dimension $-\operatorname{ind} A_{\mathcal{D}_{\min}}$. This subspace corresponds to an element $\mathcal{D} \in \mathfrak{D}$ for which (4.2) gives

$$\operatorname{ind} A_{\mathcal{D}} = \operatorname{ind} A_{\mathcal{D}_{\min}} - \operatorname{ind} A_{\mathcal{D}_{\min}} = 0,$$

which proves the lemma. □

The domains $\mathcal{D} \in \mathfrak{D}$ on which A has index 0 are those in

$$\mathfrak{G} = \{\mathcal{D} \in \mathfrak{D} : \dim \mathcal{D}/\mathcal{D}_{\min} = -\operatorname{ind} A_{\mathcal{D}_{\min}}\}.$$

By the lemma, $\mathfrak{G}$ is empty unless

$$\operatorname{ind} A_{\mathcal{D}_{\min}} \leq 0 \text{ and } \operatorname{ind} A_{\mathcal{D}_{\max}} \geq 0. \tag{5.2}$$

Assuming this, let $d'' = -\operatorname{ind} A_{\mathcal{D}_{\min}}$. Then

$$\mathfrak{G} \ni \mathcal{D} \mapsto \mathcal{D} \cap \mathcal{E}_{\max} = \pi_{\max}\mathcal{D} \in \operatorname{Gr}_{d''}(\mathcal{E}_{\max}) \tag{5.3}$$

is a bijection between $\mathfrak{G}$ and the Grassmannian of d''-dimensional subspaces of $\mathcal{E}_{\max}$ which we use to give $\mathfrak{G}$ the structure of a complex manifold. Let $d' = \operatorname{ind} A_{\mathcal{D}_{\max}}$. Then $d = d' + d'' = \dim \mathcal{E}_{\max}$.

An initial classification of points in the spectrum of a closed extension of A begins with the notion of background spectrum.

Definition 5.4. The background spectrum of A is the set

$$\text{bg-spec}\, A = \{\lambda \in \mathbb{C} : \lambda \in \operatorname{spec} A_{\mathcal{D}}\ \forall \mathcal{D} \in \mathfrak{D}\}.$$

The complement of this set, bg-res A, is the background resolvent set.

Thus, if $\mathcal{D} \in \mathfrak{G}$, then $A_{\mathcal{D}}$ has as spectrum the (disjoint) union of bg-spec A and a subset of bg-res A. Note that the resolvent set $\operatorname{res} A_{\mathcal{D}}$ of $A_{\mathcal{D}}$, $\mathcal{D} \in \mathfrak{G}$, is contained in bg-res A. As we shall see, the part of the spectrum of $A_{\mathcal{D}}$ in bg-res A is amenable to detailed study. The set bg-spec A has the same generic structure as a spectrum:

Lemma 5.5. *The set* bg-spec A *is either* $\mathbb{C}$, *or closed and discrete.*

Indeed, bg-spec$A$ is an intersection of closed sets, so itself closed, and either all spectra are $\mathbb{C}$ or there is one extension with discrete spectrum.

Thus bg-res A is open. A useful description of bg-res A is as follows.

Lemma 5.6.

$$\text{bg-res}\, A = \{\lambda \in \mathbb{C} : A_{\mathcal{D}_{\min}} - \lambda \textit{ is injective and } A_{\mathcal{D}_{\max}} - \lambda \textit{ is surjective}\}.$$

Proof. If $\lambda \in \text{bg-res}\, A$, let $\mathcal{D} \in \mathfrak{D}$ be such that $\lambda \notin \operatorname{spec} A_{\mathcal{D}}$. Since $\mathcal{D}_{\min} \subset \mathcal{D}$, $A_{\mathcal{D}_{\min}} - \lambda$ is injective, and since $\mathcal{D} \subset \mathcal{D}_{\max}$, $A_{\mathcal{D}_{\max}} - \lambda$ is surjective. Thus $\lambda \in \text{bg-res}\, A$.

Conversely, suppose that λ belongs to the set on the right in the statement of the lemma. Let $R \subset x^{-m/2}L^2_b(M;E)$ be the range of $A_{\mathcal{D}_{\min}} - \lambda$, and let $R^\perp$ be its orthogonal. Since $A_{\mathcal{D}_{\min}} - \lambda$ is injective, $\dim R^\perp = -\operatorname{ind} A_{\mathcal{D}_{\min}} = d''$. Choose a basis $f_1, \dots, f_{d''}$ of $R^\perp$. Since $A_{\mathcal{D}_{\max}} - \lambda$ is surjective, we may choose $u_1, \dots, u_{d''} \in \mathcal{D}_{\max}$ such that $(A-\lambda)u_j = f_j$ for all j. The u_j are independent modulo $\mathcal{D}_{\min}$. Let

$$\mathcal{D} = \mathcal{D}_{\min} \oplus \operatorname{span}\{u_1, \dots, u_{d''}\}.$$

Then $\mathcal{D} \in \mathfrak{D}$ and $A_{\mathcal{D}} - \lambda$ is invertible, since R is closed. □

Proposition 5.7. *Suppose that* (5.2) *holds and that* $\dim \mathfrak{G} > 0$. *Then, for every* $\lambda \in \mathbb{C}$, *there is* $\mathcal{D} \in \mathfrak{G}$ *such that* $\lambda \in \operatorname{spec} A_{\mathcal{D}}$.

Proof. Let $\lambda \in \mathbb{C}$. If $\lambda \in \text{bg-spec}\, A$ then in fact $\lambda \in \operatorname{spec} A_{\mathcal{D}}$ for any $\mathcal{D} \in \mathfrak{G}$. Suppose then that $\lambda \notin \text{bg-spec}\, A$, so by Lemma 5.6 there is $\mathcal{D}_0 \in \mathfrak{G}$ such that $A_{\mathcal{D}_0} - \lambda$ is invertible. The hypothesis that $\dim \mathfrak{G} > 0$ is equivalent to the statement that the two numbers $d'' = -\operatorname{ind} A_{\mathcal{D}_{\min}}$ and $d' = \operatorname{ind} A_{\mathcal{D}_{\max}}$ are strictly positive; recall that their sum is d, the dimension of $\mathcal{D}_{\max}/\mathcal{D}_{\min}$. Let $w \in \mathcal{D}_{\max} \backslash \mathcal{D}_0$. Such w exists because $d'' < d$. Let $f = (A-\lambda)w$, and let $v \in \mathcal{D}_0$ be such that $(A-\lambda)v = f$. Then $w - v \neq 0$ modulo $\mathcal{D}_{\min}$, and thus is an eigenvector of A. Let $\mathcal{D} \in \mathfrak{G}$ contain $w - v$; such $\mathcal{D}$ exists because $d'' > 0$. Then $A_{\mathcal{D}} - \lambda$ has nontrivial kernel. □

We will write $\mathcal{K}_\lambda$ for the kernel of $A_{\mathcal{D}_{\max}} - \lambda$, $\lambda \in \text{bg-res}\, A$. For such λ,

$$\dim \mathcal{K}_\lambda = \operatorname{ind} A_{\mathcal{D}_{\max}},$$

since $A_{\mathcal{D}_{\max}} - \lambda$ is surjective and its index is independent of λ.

Proposition 5.8. *Let* $\mathcal{K} = \bigsqcup_{\lambda \in \text{bg-res}\, A} \mathcal{K}_\lambda$ *and let* $\rho : \mathcal{K} \to \text{bg-res}\, A$ *be the natural map. Then* $\mathcal{K} \to \text{bg-res}\, A$ *is a locally trivial Hermitian holomorphic vector bundle.*

Proof. Let $\lambda_0 \in \text{bg-res}\, A$, let $\mathcal{K}^\perp_{\lambda_0}$ be the orthogonal of $\mathcal{K}_{\lambda_0}$ in $\mathcal{D}_{\max}$. The operators

$$A_1(\lambda) = (A-\lambda)|_{\mathcal{K}_{\lambda_0}} \quad A_2(\lambda) = (A-\lambda)|_{\mathcal{K}^\perp_{\lambda_0}}$$

are continuous as operators into $x^{-m/2}L^2_b(M;E)$ when the domains are given the graph norm of A, and depend holomorphically on λ. Since $A_2(\lambda_0)$ is invertible, the inverse $A_2(\lambda)^{-1}$ exists for λ close to λ_0. It is easy to verify that if $u_0 \in \mathcal{K}_{\lambda_0}$, then

$$u(\lambda) = u_0 - A_2(\lambda)^{-1}A_1(\lambda)u_0 \in \mathcal{K}_\lambda$$

for λ close to λ_0. These are, by definition, holomorphic local sections of $\mathcal{K}$. The statement that $\mathcal{K} \to \text{bg-spec}\, A$ is a locally trivial holomorphic vector bundle follows by taking local frames near λ_0 of the form $u_j(\lambda)$ where the u_j form a basis of $\mathcal{K}_{\lambda_0}$. The Hermitian form in $\mathcal{K}$ is the one whose restriction to $\mathcal{K}_\lambda$ is the restriction of the inner product of $\mathcal{D}_{\max}$ to $\mathcal{K}_\lambda$. □

Note that if u, $v \in \mathcal{K}_\lambda$, then

$$(u,v)_A = (Au, Av) + (u,v) = (1+|\lambda|^2)(u,v). \tag{5.9}$$

Lemma 5.10. *Let* $\mathcal{D} \in \mathfrak{G}$. *The following are equivalent:*

(*i*) $\lambda \in \operatorname{res} A_{\mathcal{D}}$;
(*ii*) $\lambda \in \text{bg-res}\, A$ *and* $\mathcal{K}_\lambda \cap \mathcal{D} = 0$;

(iii) $\lambda \in \text{bg-res}\, A$ *and* $\pi_{\max}\mathcal{K}_\lambda \cap \pi_{\max}\mathcal{D} = 0$.

Moreover, if $\lambda \in \operatorname{res} A_{\mathcal{D}}$*, then*

$$\mathcal{K}_\lambda \oplus \mathcal{D} = \mathcal{D}_{\max} \text{ and } \pi_{\max}\mathcal{K}_\lambda \oplus \pi_{\max}\mathcal{D} = \mathcal{E}_{\max}. \tag{5.11}$$

Proof. To prove the equivalence of (i) and (ii), we recall first that $\operatorname{res} A_{\mathcal{D}} \subset \text{bg-res}\, A$. A point $\lambda \in \mathbb{C}$ belongs to $\operatorname{res} A_{\mathcal{D}}$ if and only if $\ker(A_{\mathcal{D}} - \lambda) = 0$, because A is Fredholm of index 0. But for $\lambda \in \text{bg-res}\, A$, $\ker(A_{\mathcal{D}} - \lambda) = \mathcal{K}_\lambda \cap \mathcal{D}$. Thus (i) and (ii) are equivalent.

Suppose that $\lambda \in \text{bg-res}\, A$. If $u \in \pi_{\max}\mathcal{K}_\lambda \cap \pi_{\max}\mathcal{D}$, then $u = \phi - v$ with $\phi \in \mathcal{K}_\lambda$ and $v \in \mathcal{D}_{\min}$. Thus $\phi = u + v \in \pi_{\max}\mathcal{D} + \mathcal{D}_{\min} = \mathcal{D}$, and so $\phi \in \mathcal{K}_\lambda \cap \mathcal{D}$. If $\pi_{\max}\mathcal{K}_\lambda \cap \pi_{\max}\mathcal{D} \neq 0$, pick $u \neq 0$. Then $\phi \neq 0$, so $\mathcal{K}_\lambda \cap \mathcal{D} \neq 0$. Thus (ii) implies (iii).

Again suppose that $\lambda \in \text{bg-res}\, A$. To prove that (iii) implies (ii) we will first show that $\pi_{\max}|_{\mathcal{K}_\lambda} : \mathcal{K}_\lambda \to \mathcal{E}_{\max}$ is injective. Let $\phi \in \mathcal{K}_\lambda$. If $\pi_{\max}\phi = 0$ then $\phi \in \mathcal{D}_{\min}$. But $A - \lambda$ is injective on $\mathcal{D}_{\min}$, so $\phi = 0$. Thus if $\mathcal{K}_\lambda \cap \mathcal{D} \neq 0$, then $\pi_{\max}\mathcal{K}_\lambda \cap \pi_{\max}\mathcal{D} \neq 0$.

To prove the last statement we first observe that

$$\dim \mathcal{K}_\lambda + \dim \mathcal{D}/\mathcal{D}_{\min} = \operatorname{ind} A_{\mathcal{D}_{\max}} - \operatorname{ind} A_{\mathcal{D}_{\min}} = \dim \mathcal{E}_{\max}.$$

This gives, in view of (iii), that $\pi_{\max}\mathcal{K}_\lambda \oplus \pi_{\max}\mathcal{D} = \mathcal{E}_{\max}$. Adding $\mathcal{D}_{\min}$ to both sides of this formula gives $\mathcal{K}_\lambda + \mathcal{D} = \mathcal{D}_{\max}$, but this sum is direct in view of (ii). $\square$

The lemma gives

$$\operatorname{spec} A_{\mathcal{D}} = \text{bg-spec}\, A \cup \{\lambda \in \text{bg-res}\, A : \mathcal{K}_\lambda \cap \mathcal{D} \neq 0\} \tag{5.12}$$

for any $\mathcal{D} \in \mathfrak{G}$. Since $\mathcal{K}_\lambda \cap \mathcal{D} = 0$ if and only if $\pi_{\max}\mathcal{K}_\lambda \cap \pi_{\max}\mathcal{D} = 0$, the presence of spectrum in $\text{bg-res}\, A$ for a given extension $A_{\mathcal{D}}$ is a purely finite dimensional phenomenon. We will exploit this in Section 9 to give estimates for the resolvent $B_{\mathcal{D}}(\lambda)$ of $A_{\mathcal{D}} - \lambda$ in terms of a canonical right inverse of $A_{\mathcal{D}_{\max}} - \lambda$, a canonical left inverse for $A_{\mathcal{D}_{\min}} - \lambda$, $\lambda \in \text{bg-res}\, A$, and a finite dimensional projection.

If $\lambda \in \text{bg-res}\, A$, more generally, if $A_{\mathcal{D}_{\max}} - \lambda$ is surjective, then $A_{\mathcal{D}_{\max}} - \lambda$ admits a continuous right inverse $B_{\max}(\lambda)$, namely, let $\mathcal{K}_\lambda^\perp \subset \mathcal{D}_{\max}$ be the orthogonal of $\mathcal{K}_\lambda$ with respect to the inner product (4.3) ($\mathcal{K}_\lambda^\perp$ may not be, and does not need to be, an element of $\mathfrak{G}$). The operator

$$(A - \lambda)|_{\mathcal{K}_\lambda^\perp} : \mathcal{K}_\lambda^\perp \to x^{-m/2} L^2_b(M; E)$$

is continuous and bijective. Then the inverse, $B_{\max}(\lambda)$, of $(A-\lambda)|_{\mathcal{K}_\lambda^\perp}$ is continuous. For each $\lambda \in \text{bg-res}\, A$, the operator $B_{\max}(\lambda)$ has the smallest norm among all continuous right inverses of $A_{\mathcal{D}_{\max}} - \lambda$.

The operators $B_{\max}(\lambda)$ can be obtained from any family

$$B'_{\max}(\lambda) : x^{-m/2} L^2_b(M; E) \to \mathcal{D}_{\max}$$

of continuous right inverses for $A_{\mathcal{D}_{\max}} - \lambda$ by means of the formula

$$B_{\max}(\lambda) = B'_{\max}(\lambda) - \pi_{\mathcal{K}_\lambda} B'_{\max}(\lambda) \tag{5.13}$$

in which $\pi_{\mathcal{K}_\lambda} : \mathcal{D}_{\max} \to \mathcal{K}_\lambda$ is the orthogonal projection on $\mathcal{K}_\lambda$ (with respect to (4.3)).

The $B_{\max}(\lambda)$, as operators $x^{-m/2} L^2_b(M; E) \to \mathcal{D}_{\max}$, depend continuously, even smoothly, on λ. To see this, let λ, $\lambda_0 \in \text{bg-res}\, A$. Then

$$(A - \lambda) B_{\max}(\lambda_0) = \big((A - \lambda_0) + (\lambda_0 - \lambda)\big) B_{\max}(\lambda_0) = I + (\lambda_0 - \lambda) B_{\max}(\lambda_0).$$

Since both $B_{\max}(\lambda_0) : x^{-m/2}L^2_b(M;E) \to \mathcal{D}_{\max}$ and the inclusion $\iota : \mathcal{D}_{\max} \hookrightarrow x^{-m/2}L^2_b(M;E)$ are continuous,

$$\iota B_{\max}(\lambda_0) : x^{-m/2}L^2_b(M;E) \to x^{-m/2}L^2_b(M;E)$$

is continuous. So if λ is close enough to λ_0, then

$$B'_{\max}(\lambda) = B_{\max}(\lambda_0)\big(I + (\lambda_0 - \lambda)\iota B_{\max}(\lambda_0)\big)^{-1}$$

is a right inverse for $A_{\mathcal{D}_{\max}} - \lambda$ depending smoothly on λ. Since the $\pi_{\mathcal{K}_\lambda}$, as operators $\mathcal{D}_{\max} \to \mathcal{D}_{\max}$, also depend smoothly on λ, the correction (5.13) gives the smoothness of $\lambda \mapsto B_{\max}(\lambda)$.

The operators $B_{\max}(\lambda)$ can be used to construct the resolvent of $A_{\mathcal{D}} - \lambda$ for any $\mathcal{D} \in \mathfrak{G}$, as follows. For each $\lambda \in \text{bg-res}\, A$ such that $\mathcal{K}_\lambda \cap \mathcal{D} = 0$ let

$$\pi_{\mathcal{K}_\lambda,\mathcal{D}} : \mathcal{D}_{\max} \to \mathcal{K}_\lambda$$

be the projection according to the decomposition (5.11); this is a continuous operator. Noting that $\lambda \in \text{res}\, A$ if and only if $\lambda \in \text{bg-res}\, A$ and $\mathcal{K}_\lambda \cap \mathcal{D} = 0$, define

$$B_{\mathcal{D}}(\lambda) = B_{\max}(\lambda) - \pi_{\mathcal{K}_\lambda,\mathcal{D}} B_{\max}(\lambda). \tag{5.14}$$

Then $\pi_{\mathcal{K}_\lambda,\mathcal{D}} B_{\mathcal{D}}(\lambda) = 0$, so $B_{\mathcal{D}}(\lambda)$ maps into $\mathcal{D}$. Since $(A - \lambda)\pi_{\mathcal{K}_\lambda,\mathcal{D}} = 0$, $B_{\mathcal{D}}(\lambda)$ is a right inverse for $A_{\mathcal{D}} - \lambda$, which must also be the left inverse because $A_{\mathcal{D}} - \lambda$ is invertible.

The canonical left inverse for $A_{\mathcal{D}_{\min}} - \lambda$ is constructed in an analogous manner. Let $\mathcal{R}_\lambda$ be the range of $A_{\mathcal{D}_{\min}} - \lambda$, $\lambda \in \text{bg-res}\, A$ (more generally, one can let λ belong to the set where $A_{\mathcal{D}_{\min}} - \lambda$ is injective). Since $A_{\mathcal{D}_{\min}} - \lambda$ is injective if $\lambda \in \text{bg-res}\, A$, $A_{\mathcal{D}_{\min}} - \lambda : \mathcal{D}_{\min} \to \mathcal{R}_\lambda$ has a continuous left inverse $B^0_{\min}(\lambda)$. The orthogonal $\mathcal{R}_\lambda^\perp$ has dimension $-\operatorname{ind} A_{\mathcal{D}_{\min}}$. Let $B_{\min}(\lambda)$ be the composition of the orthogonal projection on $\mathcal{R}_\lambda$ followed by $B^0_{\min}(\lambda)$. Viewing $\mathcal{R}_\lambda^\perp$ as the kernel of $A^\star - \overline{\lambda}$ on $\mathcal{D}_{\max}(A^\star)$, we see that $\bigsqcup_{\lambda \in \text{bg-spec}\, A} \mathcal{R}_\lambda$ is a smooth (anti-holomorphic) vector bundle over bg-res A. An analysis similar to that done for $B_{\max}(\lambda)$ gives that $B_{\min}(\lambda)$ depends smoothly on $\lambda \in \text{bg-res}\, A$.

If $B'_{\min}(\lambda)$ is a left inverse for $A_{\mathcal{D}_{\min}} - \lambda$, $\lambda \in \text{bg-res}\, A$, and $\pi_{\mathcal{R}_\lambda}$ is the orthogonal projection on $\mathcal{R}_\lambda$ (in $x^{-m/2}L^2_b(M;E)$), then

$$B_{\min}(\lambda) = B'_{\min}(\lambda)\pi_{\mathcal{R}_\lambda}, \tag{5.15}$$

and so

$$\|B_{\min}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\max})} \le \|B'_{\min}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\max})}.$$

Let $\mathcal{D} \in \mathfrak{G}$ and let $B_{\mathcal{D}}(\lambda)$ be the resolvent of $A_{\mathcal{D}} - \lambda$. It is immediate that the formula

$$B_{\mathcal{D}}(\lambda) = B_{\min}(\lambda) + \big(I - B_{\min}(\lambda)(A - \lambda)\big)B_{\mathcal{D}}(\lambda) \tag{5.16}$$

holds. Replacing (5.14) in this formula we get

$$B_{\mathcal{D}}(\lambda) = B_{\max}(\lambda) - \big(I - B_{\min}(\lambda)(A - \lambda)\big)\pi_{\mathcal{K}_\lambda,\mathcal{D}} B_{\max}(\lambda). \tag{5.17}$$

Letting $\pi_{\min} = I - \pi_{\max}$ we see that that

$$\pi_{\mathcal{K}_\lambda,\mathcal{D}} = \pi_{\mathcal{K}_\lambda,\mathcal{D}}(\pi_{\max} + \pi_{\min}) = \pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\max}. \tag{5.18}$$

The operator $I - B_{\min}(\lambda)(A - \lambda)$ is a projection with kernel $\mathcal{D}_{\min}$ so

$$I - B_{\min}(\lambda)(A - \lambda) = \big(I - B_{\min}(\lambda)(A - \lambda)\big)\pi_{\max}.$$

Thus we arrive at

$$B_{\mathcal{D}}(\lambda) = B_{\max}(\lambda) - \big(I - B_{\min}(\lambda)(A-\lambda)\big)\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\max}B_{\max}(\lambda), \tag{5.19}$$

a formula which will prove to be very useful.

Remark 5.20. The range of the projector $I - B_{\min}(\lambda)(A-\lambda)$ contains $\mathcal{K}_\lambda$ so there is no difference between (5.17) and (5.14). Writing $B_{\mathcal{D}}(\lambda)$ in the form (5.19) separates the geometric information, in $\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\max}$, from analytic contributions.

We now focus on $\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\max}$, in particular its norm as a map $\mathcal{E}_{\max} \to \mathcal{E}_{\max}$.

Lemma 5.21. *Let $\mathcal{D} \in \mathfrak{G}$. Suppose $\lambda \in \operatorname{bg-res} A$ and $\mathcal{K}_\lambda \cap \mathcal{D} = 0$. Then $\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}}$ is the projection on $\pi_{\max}\mathcal{K}_\lambda$ according to the decomposition $\mathcal{E}_{\max} = \pi_{\max}\mathcal{K}_\lambda \oplus \pi_{\max}\mathcal{D}$.*

Proof. The map $\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}}$ is a projection. Indeed, in view of (5.18),

$$\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}} = \pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}} = \pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}},$$

The operator $\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}}$ has kernel containing $\pi_{\max}\mathcal{D}$, since the latter space is contained in $\mathcal{D}$, and range contained in $\pi_{\max}\mathcal{K}_\lambda$. To complete the proof we only need to show that $\ker \pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}} = \pi_{\max}\mathcal{D}$. Suppose that $u \in \ker \pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\big|_{\mathcal{E}_{\max}}$. Then $\phi = \pi_{\mathcal{K}_\lambda,\mathcal{D}}u \in \mathcal{K}_\lambda$ has the property that $\pi_{\max}\phi = 0$. Thus $\phi \in \mathcal{D}_{\min}$. But since $A-\lambda$ is injective on $\mathcal{D}_{\min}$ (since $\lambda \in \operatorname{bg-res} A$), $\phi = 0$. That is, $u \in \ker \pi_{\mathcal{K}_\lambda,\mathcal{D}}$. Since u is already in $\mathcal{E}_{\max}$, this gives $u \in \mathcal{D} \cap \mathcal{E}_{\max}$. But the latter space is $\pi_{\max}\mathcal{D}$. ☐

In the course of the proof of Lemma 5.10 we showed that if $\lambda \in \operatorname{bg-res} A$, then $\pi_{\max}|_{\mathcal{K}_\lambda} : \mathcal{K}_\lambda \to \mathcal{E}_{\max}$ is injective. Thus, since the spaces $\mathcal{K}_\lambda$ have dimension $d' = \operatorname{ind} A_{\mathcal{D}_{\max}}$, we have a map

$$\operatorname{bg-res} A \ni \lambda \mapsto \pi_{\max}\mathcal{K}_\lambda \in \operatorname{Gr}_{d'}(\mathcal{E}_{\max}).$$

Write $\mathcal{K}_{\max}$ for this map, so $\mathcal{K}_{\max}(\lambda) = \pi_{\max}\mathcal{K}_\lambda$. If $\lambda_0 \in \operatorname{bg-res} A$, let $\phi_1, \dots, \phi_{d'}$ be a holomorphic frame of $\mathcal{K}$, cf. Proposition 5.8, near λ_0. Thus, in addition to independence, the maps $\lambda \mapsto \phi_j(\lambda)$ are holomorphic for λ near λ_0. If $u_1, \dots, u_d$ is an orthonormal basis of $\mathcal{E}_{\max}$, then $\mathcal{K}_{\max}(\lambda)$ is spanned by the vectors

$$\pi_{\max}\phi_j(\lambda) = \sum_k (\phi_j(\lambda), u_k)u_k,$$

which depend holomorphically on λ. Thus $\mathcal{K}_{\max} : \operatorname{bg-res} A \to \operatorname{Gr}_{d'}(\mathcal{E}_{\max})$ is holomorphic.

If $\mathcal{D} \in \mathfrak{G}$, then Lemma 5.10 asserts that $\lambda \in \operatorname{bg-res} A \cap \operatorname{spec} A_{\mathcal{D}}$ if and only if $\pi_{\max}\mathcal{K}_\lambda \cap \pi_{\max}\mathcal{D} \neq 0$. Writing $\mathcal{W} = \pi_{\max}\mathcal{D}$, let

$$\mathfrak{V}_{\mathcal{W}} = \{\mathcal{V} \in \operatorname{Gr}_{d'}(\mathcal{E}_{\max}) : \mathcal{V} \cap \mathcal{W} \neq 0\}.$$

Then

$$\lambda \in \operatorname{bg-res} A \cap \operatorname{spec} A_{\mathcal{D}} \iff \mathcal{K}_{\max}(\lambda) \in \mathfrak{V}_{\mathcal{W}}.$$

Definition 5.22. For any nonnegative integer $d_0 \le d$ and $\mathcal{W} \in \operatorname{Gr}_{d_0}(\mathcal{E}_{\max})$ let

$$\mathfrak{V}_{\mathcal{W}} = \{\mathcal{V} \in \operatorname{Gr}_{d-d_0}(\mathcal{E}_{\max}) : \mathcal{V} \cap \mathcal{W} \neq 0\}.$$

If $\mathcal{D} \in \mathfrak{D}$, we write $\mathfrak{V}_{\mathcal{D}}$ for $\mathfrak{V}_{\pi_{\max}\mathcal{D}}$.

Thus $\operatorname{spec} A_{\mathcal{D}}$ is the union of $\operatorname{bg-spec} A$ and the pre-image of $\mathfrak{V}_{\mathcal{D}}$ under the map $\mathcal{K}_{\max}$.

Proposition 5.23. *The set $\mathfrak{V}_{\mathcal{W}} \subset \mathrm{Gr}_{d'}(\mathcal{E}_{\max})$ is a variety of (complex) codimension 1. For each $\mathcal{D} \in \mathfrak{G}$,*

$$\operatorname{spec} A_{\mathcal{D}} = \text{bg-spec}\, A \cup \mathcal{K}_{\max}^{-1}(\mathfrak{V}_{\mathcal{D}}).$$

This is a disjoint union.

Proof. We already showed the second statement. To prove the first statement, fix an ordered basis $\mathbf{u} = [u_1, \dots, u_d]$ for $\mathcal{E}_{\max}$. Pick some point $\mathcal{V}_0 \in \mathrm{Gr}_{d'}(\mathcal{E}_{\max})$ and let $\mathbf{\Phi} = [\phi_1, \dots, \phi_{d'}]$ be a holomorphic local section defined near $\mathcal{V}_0$ of the bundle of ordered bases of the canonical bundle over $\mathrm{Gr}_{d'}(\mathcal{E}_{\max})$. Thus

$$\mathbf{\Phi}(\mathcal{V}) = \mathbf{u} \cdot Z(\mathcal{V})$$

for some matrix $Z(\mathcal{V}) \in M^{d\times d'}(\mathbb{C})$ depending holomorphically on $\mathcal{V}$. Let $\mathbf{\Psi} = [\psi_1, \dots, \psi_{d''}]$ be a basis of $\mathcal{W}$, so $\mathbf{\Psi} = \mathbf{u} \cdot W$ with $W \in M^{d\times d''}(\mathbb{C})$. Then

$$f(\mathcal{V}) = \det[Z(\mathcal{V})|W]$$

is holomorphic in $\mathcal{V}$. Since $[\mathbf{\Phi}(\mathcal{V}), \mathbf{\Psi}]$ fails to be a basis of $\mathcal{E}_{\max}$ if and only if $\mathcal{V} \cap \mathcal{W} \neq 0$, $f(\mathcal{V})$ vanishes if and only if $\mathcal{V} \cap \mathcal{W} \neq 0$. Thus $\mathfrak{V}_{\mathcal{W}}$ is a complex variety of codimension 1. □

The norm of the factor $\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\max}$ in (5.19), defined for $\lambda \in \text{bg-res}\, A \setminus \operatorname{spec} A$, can be estimated in quite simple terms. Using Lemma 5.21, the problem is generally to estimate, for any $\mathcal{W} \in \mathrm{Gr}_{d_0}(\mathcal{E}_{\max})$ and $\mathcal{V} \in \mathrm{Gr}_{d-d_0}(\mathcal{E}_{\max}) \setminus \mathfrak{V}_{\mathcal{W}}$, the norm of the projection

$$\pi_{\mathcal{V},\mathcal{W}} : \mathcal{E}_{\max} \to \mathcal{E}_{\max} \tag{5.24}$$

on $\mathcal{V}$ according to the decomposition $\mathcal{E}_{\max} = \mathcal{V} \oplus \mathcal{W}$. We assume that the integer d_0 satisfies $0 < d_0 < d$.

Let then $\mathcal{W} \in \mathrm{Gr}_{d_0}(\mathcal{E}_{\max})$. Fix ordered orthonormal bases $\mathbf{u} = [u_1, \dots, u_d]$ for $\mathcal{E}_{\max}$ and $\mathbf{\Psi} = [\psi_1, \dots, \psi_{d_0}]$ for $\mathcal{W}$. Let $\mathcal{V} \in \mathrm{Gr}_{d-d_0}(\mathcal{E}_{\max})$ and let $\mathbf{\Phi} = [\phi_1, \dots, \phi_{d-d_0}]$ be an ordered orthonormal basis of $\mathcal{V}$. There are unique matrices $V \in M^{d\times(d-d_0)}(\mathbb{C})$, $W \in M^{d\times d_0}(\mathbb{C})$ such that

$$\mathbf{\Phi} = \mathbf{u} \cdot V, \quad \mathbf{\Psi} = \mathbf{u} \cdot W$$

Define

$$\delta(\mathcal{V}, \mathcal{W}) = |\det[V|W]|.$$

The columns of V, likewise the columns of W, form an orthonormal set of vectors in $\mathbb{C}^d$. We claim that $\delta(\mathcal{V}, \mathcal{W})$ is independent of the choices of orthonormal bases $\mathbf{\Phi}$ and $\mathbf{\Psi}$. Indeed, if $\mathbf{\Phi}'$ and $\mathbf{\Psi}'$ are other choices of ordered orthonormal bases of, respectively, $\mathcal{V}$ and $\mathcal{W}$, then $\mathbf{\Phi}' = \mathbf{\Phi} \cdot U_1$, $\mathbf{\Psi}' = \mathbf{\Psi} \cdot U_2$ with unitary matrices U_1 and U_2. Thus $\mathbf{\Phi}' = \mathbf{u} \cdot VU_1$ and $\mathbf{\Psi}' = \mathbf{u} \cdot WU_2$. But

$$[VU_1, WU_2] = [V|W] \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

so $|\det[VU_1|WU_2]| = |\det[V|W] \det U_1 \det U_2| = |\det[V|W]|$ since unitary matrices have determinant of modulus 1. Thus we get a globally defined function

$$\delta : \mathrm{Gr}_{d-d_0}(\mathcal{E}_{\max}) \times \mathrm{Gr}_{d_0}(\mathcal{E}_{\max}) \to \mathbb{R}.$$

This function is clearly continuous, and $\mathfrak{V}_{\mathcal{W}}$ is the set of zeros of $\mathcal{V} \mapsto \delta(\mathcal{V}, \mathcal{W})$. Suppose $\mathcal{V} \notin \mathfrak{V}_{\mathcal{W}}$ and let $\pi_{\mathcal{V},\mathcal{W}} : \mathcal{E}_{\max} \to \mathcal{V}$ be the projection (5.24) on $\mathcal{V}$. The basis $\mathbf{u}$ can be written in terms of the basis $[\mathbf{\Phi}, \mathbf{\Psi}]$, as

$$\mathbf{u} = [\mathbf{\Phi}, \mathbf{\Psi}] \cdot Q$$

where Q is the inverse of $P = [V|W]$. Let $\tilde{P}$ be the matrix of minors of P, so that $Q = (\det P)^{-1}\tilde{P}$. The entries of $\tilde{P}$ are polynomials of degree $d-1$ in the entries of P. Since the columns of the latter matrix are vectors in the unit sphere in $\mathbb{C}^d$, the entries $\tilde{p}^j_k$ of $\tilde{P}$ are bounded by a constant independent of P. If $u = \sum a^\ell u_\ell \in \mathcal{E}_{\max}$, then the two terms in

$$u = \left[\frac{1}{\det P} \sum_{k=1}^{d-d_0} \phi_k \sum_{\ell=1}^{d} \tilde{p}^k_\ell a^\ell\right] + \left[\frac{1}{\det P} \sum_{k=1}^{d_0} \psi_k \sum_{\ell=1}^{d} \tilde{p}^{k+d-d_0}_\ell a^\ell\right]$$

correspond to the decomposition $\mathcal{E}_{\max} = \mathcal{V} \oplus \mathcal{W}$. Thus

$$\pi_{\mathcal{V},\mathcal{W}} u = \frac{1}{\det P} \sum_{k=1}^{d-d_0} \phi_k \sum_{\ell=1}^{d} \tilde{p}^k_\ell a^\ell.$$

This gives:

Lemma 5.25. *Let $\mathcal{W} \in \mathrm{Gr}_{d_0}(\mathcal{E}_{\max})$ and let $\mathcal{V} \in \mathrm{Gr}_{d-d_0}(\mathcal{E}_{\max})\backslash\mathfrak{V}_{\mathcal{W}}$. Then*

$$\|\pi_{\mathcal{V},\mathcal{W}}\| \leq \frac{C}{\delta(\mathcal{V}, \mathcal{W})}.$$

The constant C is independent of $\mathcal{V}$.

The question arises as to whether there is $\mathcal{D} \in \mathfrak{G}$ such that $\operatorname{spec} A_{\mathcal{D}}$ is discrete. The following proposition shows that if there is one such domain, then the set of such domains is open and connected, and its complement is a set with empty interior.

Proposition 5.26. *The set*

$$\mathfrak{V} = \{\mathcal{D} \in \mathfrak{G} : \operatorname{spec} A_{\mathcal{D}} = \mathbb{C}\}$$

is a variety. Thus, since $\mathfrak{G}$ is connected, $\mathfrak{V} \neq \mathfrak{G}$ if and only if $\mathfrak{V}$ has empty interior.

Proof. We identify $\mathfrak{G}$ with $\mathrm{Gr}_{d''}(\mathcal{E}_{\max})$ using the map (5.3). Let $\gamma \to \mathrm{Gr}_{d''}(\mathcal{E}_{\max})$ be the canonical vector bundle. This is a holomorphic vector bundle. Let $\mathcal{D}_0 \in \mathfrak{G}$ and let $u_1, \dots, u_{d''}$ be a holomorphic frame for γ in a neighborhood U of $\mathcal{D}_0$. Thus, if $u^0_1, \dots, u^0_d$ is a basis of $\mathcal{E}_{\max}$, then

$$u_j(\mathcal{D}) = \sum_{\ell=1}^{d} g^\ell_j(\mathcal{D}) u^0_\ell, \quad j = 1, \dots, d''$$

with holomorphic functions $g^\ell_j : U \to \mathbb{C}$. Any $u \in \mathcal{D} \in U$ can be written uniquely as

$$u = v + \sum_{j=1}^{d''} \alpha^j u_j(\mathcal{D})$$

with $v \in \mathcal{D}_{\min}$. For $\mathcal{D} \in U$ define $F(\mathcal{D}) : \mathcal{D}_0 \to \mathcal{D}$ by

$$F(\mathcal{D})(v + \sum_{j=1}^{d''} \alpha^j u_j(\mathcal{D}_0)) = v + \sum_{j=1}^{d''} \alpha^j u_j(\mathcal{D}), \quad v \in \mathcal{D}_{\min}.$$

This operator is bijective, and continuous in the graph norm of A. The operators

$$A(\mathcal{D},\lambda) = (A-\lambda)\circ F(\mathcal{D}) : \mathcal{D}_0 \to x^{-m/2}L^2_b(M;E)$$

depend holomorphically on $(\mathcal{D},\lambda) \in U \times \mathbb{C}$, and the invertibility of $A_{\mathcal{D}} - \lambda$ is equivalent to the invertibility of $A(\mathcal{D},\lambda)$.

If $\mathcal{D}_0 \notin \mathfrak{V}$, then there is $\lambda_0 \notin \operatorname{spec} A_{\mathcal{D}_0}$, and therefore, there is a neighborhood $U' \subset U$ of $\mathcal{D}_0$ and $\varepsilon > 0$ such that $A(\mathcal{D},\lambda)$ is invertible for $(\mathcal{D},\lambda) \in U' \times B(\lambda_0,\varepsilon)$, where $B(\lambda_0,\varepsilon)$ is the open disc in $\mathbb{C}$ with center λ_0 and radius ε. Thus U' is disjoint from $\mathfrak{V}$, which proves that $\mathfrak{V}$ is closed.

Suppose that $\lambda_0 \in \operatorname{spec} A_{\mathcal{D}_0}$, let $K = \ker(A_{\mathcal{D}_0} - \lambda_0)$, and let $K^\perp$ be the orthogonal of K in $\mathcal{D}_0$. Let $R = (A-\lambda_0)(\mathcal{D}_0)$, let $R^\perp$ be the orthogonal in $x^{-m/2}L^2_b(M;E)$ of R, and let π_R and $\pi_{R^\perp}$ be the respective orthogonal projections. Define

$$\begin{aligned} A_{11} &= \pi_{R^\perp} A(\mathcal{D},\lambda)|_K & \qquad A_{12} &= \pi_{R^\perp} A(\mathcal{D},\lambda)|_{K^\perp} \\ A_{21} &= \pi_R A(\mathcal{D},\lambda)|_K & \qquad A_{22} &= \pi_R A(\mathcal{D},\lambda)|_{K^\perp} \end{aligned}$$

so that

$$A(\mathcal{D},\lambda) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{matrix} K \\ \oplus \\ K^\perp \end{matrix} \to \begin{matrix} R^\perp \\ \oplus \\ R \end{matrix}.$$

The operators A_{ij} are continuous as operators into their target spaces as subspaces of $x^{-m/2}L^2_b(M;E)$ when their domains are given the graph norm of A, and depend holomorphically on $(\mathcal{D},\lambda)$ for $\mathcal{D} \in U$ and λ close to λ_0. Since $A_{22}(\mathcal{D}_0,\lambda_0)$ is invertible, we can, perhaps after shrinking U, find $\varepsilon > 0$ such that $A_{22}(\mathcal{D},\lambda)$ is invertible if $\mathcal{D} \in U \times B(\lambda_0,\varepsilon)$. If $(\mathcal{D},\lambda) \in U \times B(\lambda_0,\varepsilon)$, the elements of the kernel of $A_{\mathcal{D}} - \lambda$ are in one-to-one correspondence with the elements in the kernel of

$$\mathcal{A} = A_{11} - A_{12}A_{22}^{-1}A_{21} : K \to R^\perp$$

via the map

$$\ker \mathcal{A} \ni u \mapsto u - A_{22}^{-1}A_{21}u \in \ker A(\mathcal{D},\lambda) \cong \ker A_{\mathcal{D}} - \lambda.$$

Since $A_{\mathcal{D}}$ has index 0, K and $R^\perp$ have the same dimension. Picking bases of K and $R^\perp$ we can define a determinant $f(\mathcal{D},\lambda)$ for $\mathcal{A}$. Since $\mathcal{A}$ depends holomorphically on $(\mathcal{D},\lambda)$, so does $f(\mathcal{D},\lambda)$. The set

$$\{(\mathcal{D},\lambda) \in U \times B(\lambda_0,\varepsilon) : f(\mathcal{D},\lambda) = 0\}$$

is the intersection of

$$\mathfrak{spec} A = \{(\mathcal{D},\lambda) \in \mathfrak{G} \times \mathbb{C} : \lambda \in \operatorname{spec} A_{\mathcal{D}}\}$$

with $U \times B(\lambda_0,\varepsilon)$ (thus $\mathfrak{spec} A$ is a variety). Write f as

$$f(\mathcal{D},\lambda) = \sum_{\ell=0}^{\infty} f_\ell(\mathcal{D})(\lambda-\lambda_0)^\ell;$$

the functions f_ℓ are holomorphic in U. If $\mathcal{D} \in U \cap \mathfrak{V}$, then $f(\mathcal{D},\lambda) = 0$ for all $\lambda \in B(\lambda_0,\varepsilon)$, so $f_\ell(\mathcal{D}) = 0$. And if this condition holds for $\mathcal{D}$, then $f(\mathcal{D},\lambda) = 0$. So $\mathfrak{V} \cap U$ is the set of common zeros of the functions $f_\ell : U \to \mathbb{C}$, and $\mathfrak{V}$ is a variety. □

The following gives examples where $\mathfrak{V}$ is not empty.

Example 5.27. Let $A = e^{-i\rho x} D_x$ on the interval $[-1, 1]$, with $\rho \in \mathbb{C}$, $\rho \neq 0$. This is a cone operator:

$$A = (1 - x^2)^{-1} e^{-i\rho x}(1 - x^2) D_x$$

and $(1 - x^2)$ vanishes simply at $x = \pm 1$. We consider this operator initially as an unbounded operator

$$C_0^\infty(-1, 1) \subset (1 - x^2)^{-1/2} L_b^2(-1, 1) \to (1 - x^2)^{-1/2} L_b^2(-1, 1)$$

with the measure $\mathfrak{m} = (1 - x^2)^{-1} dx$. The space $(1 - x^2)^{-1/2} L_b^2(-1, 1)$ is just $L^2(-1, 1)$ with the measure dx, so the domains of the minimal and maximal extensions are, respectively, the standard Sobolev spaces $H_0^1[-1, 1]$ and $H^1(-1, 1)$. Since $H^1(-1, 1)$ consists of continuous functions on $[-1, 1]$, the elements in $\mathcal{D}_{\max}$ can be evaluated at $x = -1$ and at $x = 1$. The Mellin transforms at either boundary of elements of $(1 - x^2)^{-1/2} L_b^2(-1, 1)$ are holomorphic in $\Im\sigma > 1/2$, of course. To compute the conormal symbol of $P = e^{-i\rho x}(1 - x^2) D_x$ at $x = -1$ let $x_{\mathrm{L}} = 1 + x$. Then

$$P = (2 - x_{\mathrm{L}}) e^{-i\rho(x_L - 1)} x_{\mathrm{L}} D_{x_{\mathrm{L}}} = 2e^{-i\rho(x_L - 1)} x_{\mathrm{L}} D_{x_{\mathrm{L}}} - x_{\mathrm{L}} e^{-i\rho(x_L - 1)} x_{\mathrm{L}} D_{x_{\mathrm{L}}}$$

so the conormal symbol of P at $x = -1$ with respect to x_{L} is $2\sigma e^{i\rho}$, giving a simple pole at $\sigma = 0$ for the inverse of the conormal symbol. If $u \in \mathcal{D}_{\max}$, its value at $x = -1$ is essentially the residue at $\sigma = 0$ of the Mellin transform of u. Using $x_{\mathrm{R}} = 1 - x$ as defining function for $\{x = 1\}$ we get

$$P = -(2 - x_{\mathrm{R}}) e^{-i\rho(1 - x_R)} x_{\mathrm{R}} D_{x_{\mathrm{R}}} = -2e^{-i\rho(1 - x_R)} x_{\mathrm{R}} D_{x_{\mathrm{R}}} + x_{\mathrm{R}} e^{-i\rho(1 - x_R)} x_{\mathrm{R}} D_{x_{\mathrm{R}}}$$

and the conormal symbol at that boundary is $-2\sigma e^{-i\rho}$. Since the only point in $\operatorname{spec}_b(P)$ is 0, we deduce that $\mathcal{D}_{\min} = (1 - x^2)^{1/2} H_b^1$ and that

$$\mathcal{D}_{\min} = \{u \in \mathcal{D}_{\max} : u(-1) = u(1) = 0\}.$$

The operator A with the minimal domain is injective. The formal adjoint of A is $A^\star = e^{i\overline{\rho}x}(D_x + \overline{\rho})$, and the Hilbert space adjoint of $A_{\mathcal{D}_{\min}}$ is $A^\star$ with its maximal domain, $\mathcal{D}^\star_{\max}$. The latter contains the function $e^{-i\overline{\rho}x}$, which spans the kernel of $A^\star$, so the index of $A_{\mathcal{D}_{\min}}$ is -1. This also gives that the index of $A_{\mathcal{D}_{\max}}$ is $+1$.

The domains on which A has index 0 are of the form

$$\mathcal{D}_{\alpha_-, \alpha_+} = \{u \in \mathcal{D}_{\max} : \alpha_- u(-1) + \alpha_+ u(1) = 0\}$$

with $(\alpha_-, \alpha_+) \in \mathbb{C}^2 \backslash 0$. If $z \neq 0$ then $(z\alpha_-, z\alpha_+)$ determines the same domain as (α_-, α_+), so $\mathfrak{G}$, the manifold of domains where A has index 0, is $\mathbb{CP}^1 = S^2$.

Fix some $(\alpha_-, \alpha_+) \in \mathbb{C}^2 \backslash 0$. The kernel of $(A - \lambda)$ on $\mathcal{D}_{\max}$ is spanned by

$$h_\lambda(x) = e^{\lambda e^{i\rho x}/\rho}.$$

The condition that $h_\lambda \in \mathcal{D}_{\alpha_-, \alpha_+}$ is

$$\alpha_- e^{\lambda e^{-i\rho}/\rho} + \alpha_+ e^{\lambda e^{i\rho}/\rho} = 0,$$

equivalently

$$\alpha_- + \alpha_+ e^{2i\lambda\rho^{-1}\sin\rho} = \alpha_- + \alpha_+ e^{\lambda[e^{i\rho} - e^{-i\rho}]/\rho} = 0.$$

Thus, if $\rho \in \pi\mathbb{Z}$ ($\rho \neq 0$), then $\alpha_- + \alpha_+ = 0$ implies $\operatorname{spec} A_{\mathcal{D}_{\alpha_-, \alpha_+}} = \mathbb{C}$ while $\alpha_- + \alpha_+ \neq 0$ implies $\operatorname{spec} A_{\mathcal{D}_{\alpha_-, \alpha_+}} = \varnothing$. And if $\rho \notin \pi\mathbb{Z}$, then for any $(\alpha_+, \alpha_-) \in \mathbb{C}^2 \backslash 0$, the spectrum of $A_{\mathcal{D}_{\alpha_-, \alpha_+}}$ is discrete, and empty if either α_- or $\alpha_+ = 0$.

6. Selfadjointness

We now discuss the important case where A is symmetric on $\mathcal{D}_{\min}$ from the perspective of Section 5. The selfadjoint extensions of such operators were studied by Lesch [5]. Suppose A is such a c-elliptic symmetric operator. Since

$$\|(A-\lambda)u\| \geq |\Im\lambda|\|u\| \quad \text{if } u \in \mathcal{D}_{\min},$$

$A_{\mathcal{D}_{\min}} - \lambda$ is injective when $\Im\lambda \neq 0$. Since A is Fredholm and the Hilbert space adjoint of $A_{\mathcal{D}_{\min}}$ is A with domain $\mathcal{D}_{\max}$, $A_{\mathcal{D}_{\max}} - \lambda$ is surjective if $\Im\lambda \neq 0$. Since the operators $A_{\mathcal{D}_{\min}} - \lambda$ are Fredholm and depend continuously on λ, the indices at $\lambda = i$ and $\lambda = -i$ are equal. So the deficiency indices are the same, and A admits selfadjoint extensions. If $A_{\mathcal{D}}$ is one such extension, then $\operatorname{spec} A_{\mathcal{D}} \subset \mathbb{R}$, therefore $\operatorname{bg-spec} A$ is a discrete subset of $\mathbb{R}$.

The Dirichlet form of a general cone operator A is the sesquilinear form

$$[u,v]_A = (Au,v) - (u,A^\star v), \quad u \in \mathcal{D}_{\max}(A),\ v \in \mathcal{D}_{\max}(A^\star). \tag{6.1}$$

It has the property that

$$[u,v]_A = [\pi_{\max}u, \pi_{\max}v]_A,$$

because $[\pi_{\max}u, \pi_{\min}v]_A = [\pi_{\min}u, \pi_{\min}v]_A = 0$ for any u and v. Moreover, the induced sesquilinear pairing

$$\mathcal{E}_{\max}(A) \times \mathcal{E}_{\max}(A^\star) \to \mathbb{C} \text{ is nonsingular} \tag{6.2}$$

(cf. Theorems 7.11 and 7.17 in [4]). If $\mathcal{D} \in \mathfrak{D}(A)$, let $\mathcal{J}\mathcal{D} \in \mathfrak{D}(A^\star)$ be the annihilator of $\mathcal{D}$ with respect to the pairing (6.1). Thus if $\mathcal{D} \in \mathfrak{D}$, then the Hilbert space adjoint of $A_{\mathcal{D}}$ is $A^\star_{\mathcal{J}\mathcal{D}}$. We will prove in a moment that the mapping $\mathcal{J} : \mathfrak{D}(A) \to \mathfrak{D}(A^\star)$ is real-analytic. Let $\mathcal{J}^\star : \mathfrak{D}(A^\star) \to \mathfrak{D}(A)$ be the analogously defined map. Clearly $\mathcal{J}^\star\mathcal{J}$ is the identity. If A is symmetric on $\mathcal{D}_{\min}$ and $\mathcal{D} \in \mathfrak{G}$, then $\mathcal{J} : \mathfrak{G} \to \mathfrak{G}$, $\mathcal{J}^\star = \mathcal{J}$, and $A_{\mathcal{D}}$ is selfadjoint if and only if $\mathcal{D}$ is a fixed point of $\mathcal{J}$. Such domains will be called selfadjoint.

Lemma 6.3. *Let A be an arbitrary c-elliptic cone operator.*

(i) If $u \in \mathcal{E}_{\max}(A)$, then $Au \in \mathcal{E}_{\max}(A^\star)$. The map

$$\mathcal{E}_{\max}(A) \ni u \mapsto Au \in \mathcal{E}_{\max}(A^\star)$$

is an isometry.

(ii) If $u \in \mathcal{E}_{\max}(A)$ and $v \in \mathcal{D}_{\max}(A^\star)$, then

$$(Au,v)_{A^\star} = [u,v]_A.$$

(iii) Let $\mathcal{D} \in \mathfrak{D}(A)$. If $u \in \mathcal{D} \cap \mathcal{E}_{\max}(A)$, then Au is orthogonal to $\mathcal{J}\mathcal{D}$ with respect to the inner product defined by $A^\star$.

Proof. To prove the first assertion in (i), suppose $u \in \mathcal{E}_{\max}(A)$. Then $Au \in \mathcal{D}_{\max}(A^\star)$. In addition, $A^\star Au = -u$, so $AA^\star(Au) = -Au$, that is, $Au \in \mathcal{E}_{\max}(A^\star)$. If u, $v \in \mathcal{E}_{\max}(A)$, then

$$(Au,Av)_{A^\star} = (A^\star Au, A^\star Av) + (Au,Av) = (u,v) + (Au,Av) = (u,v)_A.$$

For part (ii), suppose u and v are as indicated. Then

$$(Au,v)_{A^\star} = (A^\star Au, A^\star v) + (Au,v) = (-u, A^\star v) + (Au,v) = [u,v]_A.$$

For part (iii) we observe that if $u \in \mathcal{D}$ and $v \in \mathcal{J}\mathcal{D}$, then $[u,v]_A = 0$, and use part (ii). ∎

Proposition 6.4. *The mapping* $\mathcal{J} : \mathfrak{D}(A) \to \mathfrak{D}(A^\star)$ *is real-analytic.*

Proof. Let $\mathcal{D}_0 \in \mathfrak{D}(A)$, and let $\phi_1, \dots, \phi_d$ be an A-orthonormal basis of $\mathcal{E}_{\max}(A)$ whose first d_0 elements form a basis of $\mathcal{D}_0 \cap \mathcal{E}_{\max}(A)$. Let $\psi_j = A\phi_j$, $j = 1, \dots, d$. The ψ_j form an orthonormal basis of $\mathcal{E}_{\max}(A^\star)$, by part (i) of Lemma 6.3. Therefore, by part (ii) of the same lemma,

$$[\phi_j, \psi_k]_A = \delta_{jk}. \tag{6.5}$$

We deduce that

$$\mathcal{J}\mathcal{D}_0 = \operatorname{span}\{\psi_{d_0+1}, \dots, \psi_d\} \oplus \mathcal{D}_{\min}(A^\star).$$

Write $\mathbf{\Phi}_1 = [\phi_1, \dots, \phi_{d_0}]$, $\mathbf{\Phi}_2 = [\phi_{d_0+1}, \dots, \phi_d]$, and analogously $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$. The set

$$\{\operatorname{span}(\mathbf{\Phi}_1 + \mathbf{\Phi}_2 \cdot Z) : Z \in M^{(d-d_0)\times d_0}(\mathbb{C})\}$$

is a neighborhood of $\mathcal{D}_0$ in (a component of) $\mathfrak{D}(A)$, and the map

$$Z \mapsto \operatorname{span}(\mathbf{\Phi}_1 + \mathbf{\Phi}_2 \cdot Z)$$

is the inverse of a holomorphic chart. Likewise, parametrize the $(d-d_0)$-dimensional subspaces of $\mathcal{E}_{\max}(A^\star)$ in a neighborhood of $\mathcal{J}\mathcal{D}_0$ by

$$W \mapsto \operatorname{span}(\mathbf{\Psi}_2 + \mathbf{\Psi}_1 \cdot W)$$

with $W \in M^{d_0\times(d-d_0)}(\mathbb{C})$. The condition that the vector space spanned by

$$\psi_{k+d_0} + \sum_{j=1}^{d_0} W^j_k \psi_j, \quad k = 1, \dots, d - d_0$$

is $[\cdot,\cdot]_A$-orthogonal to

$$\phi_j + \sum_{k=1}^{d-d_0} Z^k_j \phi_{k+d_0}, \quad j = 1, \dots, d_0$$

is $Z^k_j + \overline{W}^j_k = 0$ because of (6.5). Thus, in coordinates, $\mathcal{J}$ maps the space determined by Z to the space determined by $W = -Z^*$. We conclude that $\mathcal{J}$ is real-analytic. □

Since $\mathcal{J}$ is real-analytic, its set of fixed points is a real-analytic variety. In fact:

Proposition 6.6. *Let A be symmetric on $\mathcal{D}_{\min}$. The set $\mathfrak{SA}$ of domains $\mathcal{D} \in \mathfrak{G}$ such that $A_{\mathcal{D}}$ is selfadjoint is a real-analytic (smooth) submanifold of $\mathfrak{G}$ of codimension $(d')^2$, $d' = \operatorname{ind} A_{\mathcal{D}_{\max}}$.*

Proof. If $\operatorname{ind} A_{\mathcal{D}_{\max}} = 0$ then also $\operatorname{ind} A_{\mathcal{D}_{\min}} = 0$, $\mathcal{D}_{\min} = \mathcal{D}_{\max}$, and $A_{\mathcal{D}_{\min}}$ is the only selfadjoint extension of A. Assume then that $\operatorname{ind} A_{\mathcal{D}_{\max}} > 0$ and pick a selfadjoint domain $\mathcal{D}_0$. Let $\phi_1, \dots, \phi_{d'}$ be an orthonormal basis of $\mathcal{D}_0 \cap \mathcal{E}_{\max}$. Then $[\phi_j, \phi_k]_A = 0$. By part (i) of Lemma 6.3, $(A\phi_j, A\phi_k)_A = \delta_{jk}$. Thus by part (ii), $[\phi_j, A\phi_k]_A = (A\phi_j, A\phi_k)_A = \delta_{jk}$. Also by part (ii), $[A\phi_j, A\phi_k]_A = -(\phi_j, A\phi_j)_A$, which vanishes by part (iii). As above, write $\mathbf{\Phi}_1 = [\phi_1, \dots, \phi_{d'}]$ and let $\mathbf{\Phi}_2 = [A\phi_1, \dots, A\phi_{d'}]$. So a neighborhood $U \subset \mathfrak{G}$ of $\mathcal{D}_0$ is parametrized by the vector spaces associated with the bases $\mathbf{\Phi}_1 + \mathbf{\Phi}_2 \cdot Z$, $Z \in \mathfrak{gl}(\mathbb{C}, d')$. Writing the components of $\mathbf{\Phi}_1 + \mathbf{\Phi}_2 \cdot Z$ as

$$\phi_j + \sum_{k=1}^{d'} Z^k_j A\phi_k, \quad j = 1, \dots, d'$$

we see that the selfadjoint domains in U are those that satisfy

$$Z_k^j - \overline{Z}_j^k = 0$$

(i.e., Z is a selfadjoint matrix). These equations represent $(d')^2$ real-analytic conditions. □

Proposition 6.7. *Let A be symmetric on $\mathcal{D}_{\min}$ and assume that $-\operatorname{ind} A_{\mathcal{D}_{\min}} > 0$. For any $\lambda \in \mathbb{R}$ there is $\mathcal{D} \in \mathfrak{S}\mathfrak{A}$ such that $\lambda \in \operatorname{spec} A_{\mathcal{D}}$.*

Proof. If λ belongs to bg-spec A, then λ already belongs to the spectrum of any extension, selfadjoint or not, of A. Suppose $\lambda \in$ bg-res A. If $u,\ v \in \mathcal{K}_\lambda$, cf. Proposition 5.8, then

$$[u, v]_A = 2i\Im(\lambda)(u, v),$$

so if λ is real, then the Dirichlet form of A vanishes on $\mathcal{K}_\lambda$. Since

$$\dim \mathcal{K}_\lambda = \operatorname{ind} A_{\mathcal{D}_{\max}} = -\operatorname{ind} A_{\mathcal{D}_{\min}},$$

and since $\pi_{\max}$ is injective on $\mathcal{K}_\lambda$,

$$\mathcal{D}_\lambda = \mathcal{K}_\lambda + \mathcal{D}_{\min} \tag{6.8}$$

is an element of $\mathfrak{G}$ on which the Dirichlet form vanishes. Thus $A_{\mathcal{D}}$ is selfadjoint, and $\lambda \in \operatorname{spec} A_{\mathcal{D}}$. □

Note that there is no assumption on semiboundedness of A.

Proposition 6.9. *Let A be symmetric on $\mathcal{D}_{\min}$. Then*

$$\text{bg-spec}\, A = \bigcap_{\mathcal{D} \in \mathfrak{S}\mathfrak{A}} \operatorname{spec} A_{\mathcal{D}}. \tag{6.10}$$

Proof. If $\operatorname{ind} A_{\mathcal{D}_{\min}} = 0$, then $\mathfrak{G} = \{\mathcal{D}_{\min}\}$ and bg-spec $A = \operatorname{spec} A_{\mathcal{D}_{\min}}$. Since $A_{\mathcal{D}_{\min}}$ is already selfadjoint, (6.10) is an identity.

Suppose then that $-\operatorname{ind} A_{\mathcal{D}_{\min}} > 0$. Denote the set on the right in (6.10) by S. From the definition of bg-spec A we get bg-spec $A \subset S$.

To prove the opposite inclusion suppose that $\lambda_0 \in$ bg-res A. If $\Im\lambda_0 \neq 0$, then $\lambda_0 \notin S$, since $S \subset \mathbb{R}$. If $\lambda_0 \in \mathbb{R} \cap$ bg-res A, consider $A_{\mathcal{D}_{\lambda_0}}$, where $\mathcal{D}_{\lambda_0}$ is as in (6.8). From the proof of Proposition 6.7 we know that $A_{\mathcal{D}_{\lambda_0}}$ is selfadjoint. Since $A_{\mathcal{D}_{\lambda_0}}$ is Fredholm and $\operatorname{spec} A_{\mathcal{D}_{\lambda_0}} \neq \mathbb{C}$, this spectrum is discrete. Since $\mathcal{K}_{\lambda_0} \subset \mathcal{D}_{\lambda_0}$, $\lambda_0 \in \operatorname{spec} A_{\mathcal{D}_{\lambda_0}}$. We can therefore find a neighborhood $U \subset$ bg-res A of λ_0 with the property that $U \cap \operatorname{spec} A_{\mathcal{D}_{\lambda_0}} = \{\lambda_0\}$. We claim that if $\lambda \in U\backslash\{\lambda_0\}$, then $\lambda_0 \notin \operatorname{spec} A_{\mathcal{D}_\lambda}$. To see this, let $\lambda \in U$ and assume that $\lambda_0 \in \operatorname{spec} A_{\mathcal{D}_\lambda}$. Then $\mathcal{K}_{\lambda_0} \cap \mathcal{D}_\lambda \neq 0$. Thus there are $\phi \in \mathcal{K}_{\lambda_0}$ with $\phi \neq 0$, and $\psi \in \mathcal{K}_\lambda$, $v \in \mathcal{D}_{\min}$ such that $\phi = \psi + v$. The element $\phi - v \in \mathcal{D}_{\lambda_0}$ is equal to ψ, so $\mathcal{D}_{\lambda_0} \cap \mathcal{K}_\lambda \neq 0$. Necessarily $\psi \neq 0$, since $\pi_{\max}\phi = \pi_{\max}\psi$ and $\phi \neq 0$. Thus $\lambda \in U \cap \operatorname{spec} A_{\mathcal{D}_{\lambda_0}}$, which implies $\lambda = \lambda_0$. It follows that if $\lambda \in U \cap \mathbb{R}\backslash\lambda_0$, then $\mathcal{D}_\lambda \in \mathfrak{S}\mathfrak{A}$ and $\lambda_0 \notin \operatorname{spec} A_{\mathcal{D}_\lambda}$, hence $\lambda_0 \notin S$. Therefore $S \subset$ bg-spec A. □

7. The model operator

In this section we focus on the spectra of closed extensions of the operator $A_\wedge$, cf (3.8). We continue to assume that the operator $A \in x^{-m}\operatorname{Diff}_b^m(M; E)$ is c-elliptic. We will usually write $\mathcal{D}_{\wedge,\min}$ for $\mathcal{D}_{\min}(A_\wedge)$ and $\mathcal{D}_{\wedge,\max}$ for $\mathcal{D}_{\max}(A_\wedge)$. Recall that the inner product on $\mathcal{D}_{\wedge,\max}$ is given by (4.10). The nature of $\mathcal{D}_{\wedge,\min}$ was described in Proposition 4.9. We also noted there that $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ is finite dimensional.

Because of the finite dimensionality of this quotient, many of the results concerning the closed extensions of A find their analogue in the situation at hand, despite the fact that neither of the operators

$$A_\wedge : \mathcal{D}_{\wedge,\min} \subset x^{-m/2}L^2_b(Y^\wedge;E) \to x^{-m/2}L^2_b(Y^\wedge;E)$$

nor

$$A_\wedge : \mathcal{D}_{\wedge,\max} \subset x^{-m/2}L^2_b(Y^\wedge;E) \to x^{-m/2}L^2_b(Y^\wedge;E)$$

needs to be Fredholm. On the other hand, the homogeneity property

$$A_\wedge - \lambda = \varrho^m \kappa_\varrho (A_\wedge - \lambda/\varrho^m)\kappa_\varrho^{-1} \text{ for every } \varrho > 0 \tag{7.1}$$

of $A_\wedge - \lambda$, $\lambda \in \mathbb{C}$, cf. (3.11), not available in such simple form in the case of A, permits an essentially complete understanding of the spectra and resolvents for the closed extensions of $A_\wedge$.

We begin our analysis with:

Definition 7.2. The background spectrum of $A_\wedge$ is the set

$$\text{bg-spec}\, A_\wedge = \{\lambda \in \mathbb{C} : \lambda \in \operatorname{spec} A_{\wedge,\mathcal{D}}\ \forall \mathcal{D} \in \mathfrak{D}_\wedge\}.$$

The complement of this set, bg-res $A_\wedge$, is the background resolvent set.

The analogue

$$\text{bg-res}\, A_\wedge = \{\lambda \in \mathbb{C} : A_{\wedge,\mathcal{D}_{\min}} - \lambda \text{ is injective and } A_{\wedge,\mathcal{D}_{\max}} - \lambda \text{ is surjective}\}$$

of Lemma 5.6 holds for $A_\wedge$ in place of A, with the same proof.

Lemma 7.3. *If $\lambda \in$ bg-res $A_\wedge$ and $\mathcal{D} \in \mathfrak{D}_\wedge$, then $A_{\wedge,\mathcal{D}} - \lambda$ is Fredholm. The set* bg-res $A_\wedge$ *is a union of open sectors.*

Proof. Let $\lambda \in$ bg-res $A_\wedge$. Since $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ is finite dimensional and $A_\wedge - \lambda$ is injective on $\mathcal{D}_{\wedge,\min}$, $A_{\wedge,\mathcal{D}_{\max}} - \lambda$ has finite dimensional kernel. Thus $A_{\wedge,\mathcal{D}_{\max}} - \lambda$ is Fredholm, and so is its restriction to any subspace of $\mathcal{D}_{\wedge,\max}$ of finite codimension.

Next, suppose that $\lambda_0 \in$ bg-res $A_\wedge$ and let $\lambda = \varrho^m\lambda_0$. Since κ_ϱ is invertible and

$$A_\wedge - \lambda = \varrho^m\kappa_\varrho(A_\wedge - \lambda_0)\kappa_\varrho^{-1},$$

$A_\wedge - \lambda$ is injective on $\mathcal{D}_{\wedge,\min}$ and surjective on $\mathcal{D}_{\wedge,\max}$. Thus the ray $\{r\lambda_0 : r > 0\}$ is contained in bg-res $A_\wedge$. Since $\lambda_0 \in$ bg-res $A_\wedge$, $A_\wedge - \lambda_0$ admits a continuous left inverse $B(\lambda_0) : x^{-m/2}L^2_b(Y^\wedge;E) \to \mathcal{D}_{\wedge,\min}$. Since the inclusion $\mathcal{D}_{\wedge,\min} \hookrightarrow x^{-m/2}L^2_b(Y^\wedge;E)$ is continuous, the formula

$$B(\lambda_0)(A_\wedge - \lambda) = I + (\lambda_0 - \lambda)B(\lambda_0)$$

gives that $(A_{\wedge,\min} - \lambda)$ admits a left inverse if λ is close to λ_0. Likewise $(A_{\wedge,\max} - \lambda)$ admits a right inverse if λ is close to λ_0. So bg-res $A_\wedge$ is open. Therefore its connected components are open sectors. □

Label the connected components of bg-res $A_\wedge$ by $\mathring{\Lambda}_\alpha$, $\alpha \in \mathfrak{I} \subset \mathbb{N}$. Since the inclusion map $\mathcal{D} \hookrightarrow x^{-m/2}L^2_b(Y^\wedge;E)$ is continuous for any $\mathcal{D} \in \mathfrak{D}_\wedge$,

$$\mathring{\Lambda}_\alpha \ni \lambda \mapsto \operatorname{ind}(A_{\wedge,\mathcal{D}} - \lambda)$$

is constant, and

$$\operatorname{ind}(A_{\wedge,\mathcal{D}} - \lambda) = \operatorname{ind}(A_{\wedge,\mathcal{D}_{\min}} - \lambda) + \dim \mathcal{D}/\mathcal{D}_{\wedge,\min}, \quad \lambda \in \text{bg-res}\, A_\wedge.$$

Let

$$d'_\alpha = \operatorname{ind}(A_{\wedge,\mathcal{D}_{\max}} - \lambda), \quad d''_\alpha = -\operatorname{ind}(A_{\wedge,\mathcal{D}_{\min}} - \lambda), \quad \lambda \in \mathring{\Lambda}_\alpha$$

and let

$$\mathfrak{G}_{\wedge,\alpha} = \{\mathcal{D} \in \mathfrak{D}_\wedge : \dim \mathcal{D}/\mathcal{D}_{\wedge,\min} = d''_\alpha\}, \quad \alpha \in \mathfrak{I}.$$

The elements of $\mathfrak{G}_{\wedge,\alpha}$ are thus the domains $\mathcal{D}$ for which $A_{\wedge,\mathcal{D}} - \lambda$ has index 0 when $\lambda \in \mathring{\Lambda}_\alpha$. Write $\mathcal{E}_{\wedge,\max}$ for the orthogonal of $\mathcal{D}_{\wedge,\min}$ in $\mathcal{D}_{\wedge,\max}$. Using that

$$\mathfrak{D}_\wedge \ni \mathcal{D} \mapsto \mathcal{D} \cap \mathcal{E}_{\wedge,\max}$$

is a bijection onto the set of finite dimensional subspaces of $\mathcal{E}_{\wedge,\max}$ we give each of the $\mathfrak{G}_{\wedge,\alpha}$ the structure of a complex manifold.

The proofs of the following lemma and proposition parallel the arguments in the proofs of Propositions 5.7 and 5.26, respectively.

Lemma 7.4. *For every $\alpha \in \mathfrak{I}$ such that $\dim \mathfrak{G}_{\wedge,\alpha,0} > 0$ and every $\lambda \in \mathring{\Lambda}_\alpha$ there is $\mathcal{D}_0 \in \mathfrak{G}_{\wedge,\alpha}$ such that $\lambda \in \operatorname{spec} A_{\wedge,\mathcal{D}_0}$.*

Proposition 7.5. *For every $\alpha \in \mathfrak{I}$ the set*

$$\mathfrak{V}_\alpha = \{\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha} : \mathring{\Lambda}_\alpha \subset \operatorname{spec} A_{\wedge,\mathcal{D}}\}$$

is a variety.

If $\mathcal{D} \in \mathfrak{D}_\wedge$, then $\kappa_\varrho \mathcal{D}$ is again an element of $\mathfrak{D}_\wedge$. Indeed,

$$\mathcal{D}_{\wedge,\min} \subset \kappa_\varrho \mathcal{D} \subset \mathcal{D}_{\wedge,\max}$$

since both $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$ are κ-invariant. Define

$$\boldsymbol{\kappa}_\varrho : \mathfrak{D}_\wedge \to \mathfrak{D}_\wedge, \quad \boldsymbol{\kappa}_\varrho(\mathcal{D}) = \kappa_\varrho \mathcal{D}.$$

Since $\kappa_\varrho \mathcal{D}_{\wedge,\min} = \mathcal{D}_{\wedge,\min}$,

$$\pi_{\wedge,\max}\kappa_\varrho = \pi_{\wedge,\max}\kappa_\varrho\pi_{\wedge,\max},$$

and therefore the map

$$\mathbb{R} \ni \xi \mapsto \pi_{\wedge,\max}\kappa_{e^\xi}\big|_{\mathcal{E}_{\wedge,\max}} : \mathcal{E}_{\wedge,\max} \to \mathcal{E}_{\wedge,\max} \tag{7.6}$$

is a (continuous) one-parameter group of isomorphisms of $\mathcal{E}_{\wedge,\max}$, necessarily given by exponentiation of its infinitesimal generator. So (7.6) extends to a holomorphic action of $\mathbb{C}$ on $\mathcal{E}_{\wedge,\max}$. We will use the notation $\boldsymbol{\kappa}_\varrho(\mathcal{V})$ for $\pi_{\wedge,\max}\kappa_\varrho(\mathcal{V})$ when $\mathcal{V} \subset \mathcal{E}_{\wedge,\max}$ is a subspace.

Proposition 7.7. *Let $d_0 < d = \dim \mathcal{E}_{\wedge,\max}$ be a nonnegative integer. The map*

$$\mathbb{R} \times \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max}) \ni (\xi, \mathcal{V}) \mapsto \boldsymbol{\kappa}_{e^\xi}\mathcal{V} \in \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$$

extends to a holomorphic map

$$\boldsymbol{\kappa}_{\exp} : \mathbb{C} \times \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max}) \to \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$$

with the property that

$$\boldsymbol{\kappa}_{\exp}(\zeta + \zeta', \mathcal{V}) = \boldsymbol{\kappa}_{\exp}(\zeta, \boldsymbol{\kappa}_{\exp}(\zeta', \mathcal{V}))$$

for all ζ, $\zeta' \in \mathbb{C}$ and $\mathcal{V} \in \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$. In particular, for each $\mathcal{V} \in \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$, the curve

$$\mathbb{R} \ni \xi \mapsto \boldsymbol{\kappa}_{e^\xi}\mathcal{V} \in \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$$

is real-analytic, and the infinitesimal generator of the group action $\boldsymbol{\kappa}_{\exp}$ is the real part of a holomorphic vector field.

Proof. The proof is an elementary argument on Grassmannian varieties. Let $\mathcal{V}_0 \in \mathrm{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$ and pick a basis $\mathbf{\Phi} = [\phi_1, \dots, \phi_d]$ of $\mathcal{E}_{\wedge,\max}$ whose first d_0 elements form a basis of $\mathcal{V}_0$. Then $\pi_{\wedge,\max}\kappa_{e^\zeta}\big|_{\mathcal{E}_{\wedge,\max}}$ sends the basis $\mathbf{\Phi}$ to the basis $\mathbf{\Phi}\cdot\boldsymbol{\kappa}(\zeta)$ whose j-th component is

$$\sum_{k=1}^{d} \phi_k \boldsymbol{\kappa}_j^k(\zeta);$$

the matrix $\boldsymbol{\kappa}(\zeta) = [\kappa_j^k(\zeta)]$ depends holomorphically on ζ. Let $\mathbf{\Phi}_1 = [\phi_1, \dots, \phi_{d_0}]$, $\mathbf{\Phi}_2 = [\phi_{d_0+1}, \dots, \phi_d]$. If $Z \in M^{(d-d_0)\times d_0}(\mathbb{C})$ is a $(d-d_0)\times d_0$ matrix with complex entries, then $\mathbf{\Phi}_2 \cdot Z$ is defined, the entries of $\mathbf{\Phi}_1 + \mathbf{\Phi}_2 \cdot Z$ are independent, and

$$\mathcal{V}(Z) = \mathrm{span}(\mathbf{\Phi}_1 + \mathbf{\Phi}_2 \cdot Z) \subset \mathcal{E}_{\wedge,\max}$$

defines an element of the Grassmannian $\mathrm{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$. For a fixed basis $\mathbf{\Phi}$ the collection of elements $\mathcal{V}(Z)$ is a neighborhood U of $\mathcal{V}_0$ and $Z \mapsto \mathcal{V}(Z)$ is the inverse of a holomorphic chart of $\mathrm{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$. Write the $d\times d$ matrix $\boldsymbol{\kappa}(\zeta)$ in block form,

$$\boldsymbol{\kappa}(\zeta) = \begin{bmatrix} \boldsymbol{\kappa}_1^1(\zeta) & \boldsymbol{\kappa}_2^1(\zeta) \\ \boldsymbol{\kappa}_1^2(\zeta) & \boldsymbol{\kappa}_2^2(\zeta) \end{bmatrix},$$

with $\kappa_1^1(\zeta) \in M^{d_0\times d_0}(\mathbb{C})$. With this notation, $\pi_{\wedge,\max}\kappa_{e^\zeta}\big|_{\mathcal{E}_{\max}}$ maps the components of $\mathbf{\Phi}_1 + \mathbf{\Phi}_2 \cdot Z$ to the components of

$$\mathbf{\Phi}_1 \cdot \big(\boldsymbol{\kappa}_1^1(\zeta) + \boldsymbol{\kappa}_2^1(\zeta)Z\big) + \mathbf{\Phi}_2 \cdot \big(\boldsymbol{\kappa}_1^2(\zeta) + \boldsymbol{\kappa}_2^2(\zeta)Z\big).$$

If Z belongs to a bounded set in $M^{(d-d_0)\times d_0}(\mathbb{C})$ then for ζ small enough the matrix $\boldsymbol{\kappa}_1^1(\zeta) + \boldsymbol{\kappa}_2^1(\zeta)Z$ is invertible, since $\boldsymbol{\kappa}(0) = I$, and we get from

$$\{\mathbf{\Phi}_1 + \mathbf{\Phi}_2 \cdot \big(\boldsymbol{\kappa}_1^2(\zeta) + \boldsymbol{\kappa}_2^2(\zeta)Z\big)\big(\boldsymbol{\kappa}_1^1(\zeta) + \boldsymbol{\kappa}_2^1(\zeta)Z\big)^{-1}\} \cdot \big(\boldsymbol{\kappa}_1^1(\zeta) + \boldsymbol{\kappa}_2^1(\zeta)Z\big)$$

that κ_{e^ζ} maps the point in U of coordinates Z to the point in U of coordinates $\big(\boldsymbol{\kappa}_1^2(\zeta) + \boldsymbol{\kappa}_2^2(\zeta)Z\big)\big(\boldsymbol{\kappa}_1^1(\zeta) + \boldsymbol{\kappa}_2^1(\zeta)Z\big)^{-1}$. The latter is a holomorphic function of ζ and Z. ∎

If $\mathcal{D} \in \mathfrak{D}_\wedge$, then $\mathcal{D} = \pi_{\wedge,\max}\mathcal{D} \oplus \mathcal{D}_{\wedge,\min}$. Therefore

$$\kappa_{e^\xi}\mathcal{D} = (\pi_{\wedge,\max}\kappa_{e^\xi}\pi_{\wedge,\max}\mathcal{D}) \oplus \mathcal{D}_{\wedge,\min}$$

for real ξ. For $\zeta \in \mathbb{C}$, define

$$\kappa_{e^\zeta}\mathcal{D} = (\pi_{\wedge,\max}\kappa_{e^\zeta}\pi_{\wedge,\max}\mathcal{D}) \oplus \mathcal{D}_{\wedge,\min}.$$

The $\boldsymbol{\kappa}_{e^\xi} : \mathrm{Gr}_{d_0}(\mathcal{E}_{\wedge,\max}) \to \mathrm{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$ with $\xi \in \mathbb{R}$ form a one-parameter group of biholomorphisms. Let $\mathcal{T}_\wedge$ be the infinitesimal generator. The points where $\mathcal{T}_\wedge$ vanishes are the fixed points of $\boldsymbol{\kappa}_{e^\xi}$. The vector field $\mathcal{T}_\wedge$ is the real part of a holomorphic vector field $\mathcal{T}'_\wedge$ (a holomorphic section of $T^{1,0}\mathfrak{D}_\wedge$). Since $\mathcal{T}_\wedge$ vanishes at a point if and only if $\mathcal{T}'_\wedge$ vanishes at that point, we have that the set of fixed points of $\boldsymbol{\kappa}_{e^\xi}$ in each $\mathrm{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$ is an analytic variety.

Corollary 7.8. *The set of κ-invariant domains in $\mathfrak{D}_\wedge$ is an analytic variety.*

Thus the set of κ-invariant domains is a small set.

Remark 7.9. By Lemma 5.12 of [4], a subspace of $\mathcal{E}_{\wedge,\max}$ is κ-invariant if and only if it is a direct sum of subspaces $\tilde{\mathcal{E}}_j \subset \mathcal{E}_{\sigma_j}(A_\wedge)$, each of which is itself κ-invariant. The set of κ-invariant subspaces of $\mathcal{E}_{\sigma_j}(A_\wedge)$ of a given dimension needs not be a discrete subset of the corresponding Grassmannian.

Again as in Section 5, let

$$\mathcal{K}_{\wedge,\lambda} = \ker(A_{\wedge,\mathcal{D}_{\max}} - \lambda), \quad \lambda \in \text{bg-res}\, A_\wedge.$$

The proof of Proposition 5.8 gives that the $\mathcal{K}_{\wedge,\lambda}$ are the fibers of a Hermitian holomorphic vector bundle

$$\mathcal{K}_\wedge \to \text{bg-res}\, A_\wedge \tag{7.10}$$

over bg-res $A_\wedge$; the rank of $\mathcal{K}_\wedge|_{\mathring{\Lambda}_\alpha}$ is the number d'_α, which may change with α.

Lemma 7.11. *Let $\lambda \in \text{bg-res}\, A_\wedge$. The map κ_ϱ sends $\mathcal{K}_{\wedge,\lambda}$ to $\mathcal{K}_{\wedge,\varrho^m\lambda}$, and so gives a vector bundle morphism $\mathcal{K}_\wedge \to \mathcal{K}_\wedge$.*

Proof. Writing (7.1) in the form

$$A_\wedge - \lambda = \varrho^{-m}\kappa_\varrho^{-1}(A_\wedge - \varrho^m\lambda)\kappa_\varrho \text{ for every } \varrho > 0, \tag{7.12}$$

and letting each member of this identity act on $\phi \in \mathcal{K}_{\wedge,\lambda}$, we see that $\kappa_\varrho\phi \in \mathcal{K}_{\wedge,\varrho^m\lambda}$. □

Lemma 5.10 has a word by word translation to the situation at hand and if $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$, then

$$\lambda \in \text{res}\, A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha \iff \mathcal{K}_{\wedge,\lambda} \cap \mathcal{D} = 0. \tag{7.13}$$

For such λ,

$$\mathcal{K}_{\wedge,\lambda} \oplus \mathcal{D} = \mathcal{D}_{\wedge,\max}. \tag{7.14}$$

Let

$$\mathcal{K}_{\wedge,\max}(\lambda) = \pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda}.$$

Then

$$\mathring{\Lambda}_\alpha \ni \lambda \to \mathcal{K}_{\wedge,\max}(\lambda) \in \text{Gr}_{d'_\alpha}(\mathcal{E}_{\wedge,\max})$$

is holomorphic.

Suppose that $\lambda_0 \in \mathring{\Lambda}_\alpha$ and let $\Gamma = \{r\lambda_0 : r > 0\}$ be the ray through λ_0. In view of Lemma 7.11, the set spec $A_{\wedge,\mathcal{D}} \cap \Gamma$ will not contain points λ with $|\lambda|$ large if and only if $\kappa_\varrho\mathcal{K}_{\wedge,\lambda_0} \cap \mathcal{D} = 0$ for ϱ large. With the notation introduced in Definition 5.22 (of course with $\mathcal{E}_{\max}$ replaced by $\mathcal{E}_{\wedge,\max}$), this will happen if and only if $\boldsymbol{\kappa}_\varrho\mathcal{K}_{\wedge,\max}(\lambda_0) \notin \mathfrak{V}_{\pi_{\wedge,\max}\mathcal{D}}$ for large ϱ. Since $\mathfrak{V}_{\pi_{\wedge,\max}\mathcal{D}} \subset \text{Gr}_{d'_\alpha}(\mathcal{E}_{\wedge,\max})$ is of complex codimension 1 and $\varrho \mapsto \boldsymbol{\kappa}_\varrho\mathcal{K}_{\wedge,\max}(\lambda_0)$ is a real curve, these curves generically do not intersect $\mathfrak{V}_{\pi_{\wedge,\max}\mathcal{D}}$. However, it can happen that $\boldsymbol{\kappa}_\varrho\mathcal{K}_{\wedge,\max}(\lambda_0) \in \mathfrak{V}_{\pi_{\wedge,\max}\mathcal{D}}$ for all ϱ, for instance if $\mathcal{K}_{\wedge,\lambda_0} \cap \mathcal{D}$ contains a nontrivial κ-invariant subspace. It can also happen that $\boldsymbol{\kappa}_\varrho\mathcal{K}_{\wedge,\max}(\lambda_0) \in \mathfrak{V}_{\pi_{\wedge,\max}\mathcal{D}}$ infinitely often. For example, suppose that $\mathcal{E}_{\wedge,\max}$ is two-dimensional and that the infinitesimal generator of the action $\boldsymbol{\kappa}_\varrho$ has two distinct eigenvalues $i\sigma_1$ and $i\sigma_2$ with $\Im\sigma_1 = \Im\sigma_2$. Let u_1, u_2 be eigenvectors for these eigenvalues. If $a_1u_1 + a_2u_2$ is a basis element for $\pi_{\wedge,\max}\mathcal{D}$ and $\mathcal{K}_{\wedge,\max}(\lambda_0)$ is spanned by the same vector, then $\boldsymbol{\kappa}_\varrho\mathcal{K}_{\wedge,\max}(\lambda_0)$ is spanned by $a_1\varrho^{i\sigma_1}u_1 + a_2\varrho^{i\sigma_2}u_2$, and $\boldsymbol{\kappa}_\varrho\mathcal{K}_{\wedge,\max}(\lambda_0) \cap \pi_{\wedge,\max}\mathcal{D} \neq 0$ whenever $\varrho = e^{2\pi k/(\Re\sigma_2 - \Re\sigma_1)}$ with $k \in \mathbb{Z}$.

We will show that the spaces $\mathcal{K}_{\wedge,\lambda}$, $\lambda \in \mathring{\Lambda}_\alpha$, can be obtained directly from a single space $\mathcal{K}_{\wedge,\lambda_0}$, $\lambda_0 \in \mathring{\Lambda}_\alpha$ via the action of κ and $B_{\wedge,\min}(\lambda)$, the left inverse of $A_{\wedge,\mathcal{D}_{\min}} - \lambda$ with kernel equal to the orthogonal of $\mathcal{R}_{\wedge,\lambda} = \text{rg}(A_{\wedge,\mathcal{D}_{\min}} - \lambda)$. The family $B_{\wedge,\min}(\lambda)$ depends smoothly on $\lambda \in \text{bg-res}\, A_\wedge$, cf. Section 5.

Fix some sector $\mathring{\Lambda}_\alpha$ and for the sake of simplicity let $\lambda_0 \in \mathring{\Lambda}_\alpha$ lie in the axis of symmetry Γ_α of $\mathring{\Lambda}_\alpha$. So

$$\mathring{\Lambda}_\alpha = \{\lambda : |\arg(\lambda/\lambda_0)| < \theta_\alpha\}$$

where arg is the principal branch of the argument function on $\mathbb{C}\backslash\overline{\mathbb{R}}_-$. Let log be the principal branch of the logarithm on the same set. Then

$$\mathfrak{P}_{\max}(\lambda)\phi = \pi_{\wedge,\max}\kappa_{e^{\log(\lambda/\lambda_0)/m}}\pi_{\wedge,\max}\phi$$

is well defined for $\phi \in \mathcal{K}_{\wedge,\lambda_0}$ and is holomorphic in λ for $\lambda \notin -\Gamma_\alpha$. Thus we have a map

$$\mathfrak{P}_{\max}(\lambda) : \mathcal{K}_{\wedge,\lambda_0} \to \mathcal{E}_{\wedge,\max}$$

depending holomorphically on λ for $\lambda \notin -\Gamma_\alpha$. In general, if $\phi \in \mathcal{K}_{\wedge,\lambda}$, then

$$(A_\wedge - \lambda)\pi_{\wedge,\min}\phi = -(A_\wedge - \lambda)\pi_{\wedge,\max}\phi.$$

Thus the right hand side belongs to the range $\mathcal{R}_{\wedge,\lambda}$ of $A_{\wedge,\mathcal{D}_{\min}} - \lambda$, and

$$\pi_{\wedge,\min}\phi = -B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\pi_{\wedge,\max}\phi$$

Conversely, if $u \in \mathcal{E}_{\wedge,\max}$ and $(A_\wedge - \lambda)u \in \mathcal{R}_{\wedge,\lambda}$, then

$$u - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)u \in \mathcal{K}_{\wedge,\lambda}.$$

Define

$$\mathfrak{P}_{\min}(\lambda) : \mathcal{K}_{\wedge,\lambda_0} \to \mathcal{D}_{\wedge,\min}, \quad \lambda \in \mathring{\Lambda}_\alpha$$

by

$$\mathfrak{P}_{\min}(\lambda)\phi = -B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\mathfrak{P}_{\max}(\lambda)\phi, \quad \phi \in \mathcal{K}_{\wedge,\lambda_0}.$$

Let also

$$\mathfrak{P}(\lambda) = \mathfrak{P}_{\max}(\lambda) + \mathfrak{P}_{\min}(\lambda) : \mathcal{K}_{\wedge,\lambda_0} \to \mathcal{D}_{\wedge,\max} \tag{7.15}$$

The operators $B_{\wedge,\min}(\lambda)$ depend smoothly, but not holomorphically, on λ (unless $\operatorname{ind}(A_{\wedge,\mathcal{D}_{\min}} - \lambda) = 0$ for $\lambda \in \mathring{\Lambda}_\alpha$). So it is not obvious that $\mathfrak{P}_{\min}(\lambda)$ depends holomorphically on λ.

Proposition 7.16. *The map $\mathfrak{P}_{\min}(\lambda) : \mathcal{K}_{\wedge,\lambda_0} \to \mathcal{D}_{\wedge,\min}$ depends holomorphically on $\lambda \in \mathring{\Lambda}_\alpha$, and*

$$\mathfrak{P}(\lambda)\phi \in \mathcal{K}_{\wedge,\lambda} \quad \text{for } \lambda \in \mathring{\Lambda}_\alpha \text{ and } \phi \in \mathcal{K}_{\wedge,\lambda_0}. \tag{7.17}$$

Proof. Since $\lambda_0 \in$ bg-res $A_\wedge$, there is $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$ such that $\lambda_0 \in \operatorname{res} A_{\wedge,\mathcal{D}}$, so $\operatorname{res} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha \neq \varnothing$. Let then $B_{\wedge,\mathcal{D}}(\lambda)$ denote the resolvent of $A_{\wedge,\mathcal{D}}$ on $\operatorname{res} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha$. In particular $B_{\wedge,\mathcal{D}}(\lambda)$ depends holomorphically on $\lambda \in \operatorname{res} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha$. Let $\phi \in \mathcal{K}_{\wedge,\lambda_0}$ and define

$$\mathfrak{P}_{\min,\mathcal{D}}(\lambda)\phi = -\pi_{\wedge,\min}B_{\wedge,\mathcal{D}}(\lambda)(A_\wedge - \lambda)\mathfrak{P}_{\max}(\lambda)\phi$$

for $\lambda \in \operatorname{res} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha$. Thus $\mathfrak{P}_{\min,\mathcal{D}}(\lambda)$ is holomorphic on the set where we defined it. Note that $\pi_{\wedge,\min}B_{\wedge,\mathcal{D}}(\lambda)$ is a holomorphic left inverse for $A_{\wedge,\mathcal{D}_{\min}} - \lambda$ when $\lambda \in \operatorname{res} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha$.

If $\phi \in \mathcal{K}_{\wedge,\lambda_0}$ and $\varrho \in \mathbb{R}_+$ then $\kappa_\varrho\phi \in \mathcal{K}_{\wedge,\varrho^m\lambda_0}$ and

$$\mathfrak{P}_{\max}(\varrho^m\lambda_0)\phi = \pi_{\wedge,\max}\kappa_\varrho\phi.$$

Thus if $\varrho^m\lambda_0 \in \operatorname{res} A_{\wedge,\mathcal{D}}$ then

$$\begin{aligned}\pi_{\wedge,\min}\kappa_\varrho\phi &= -\pi_{\wedge,\min}B_{\wedge,\mathcal{D}}(\varrho^m\lambda_0)(A_\wedge - \varrho^m\lambda_0)\pi_{\wedge,\max}\kappa_\varrho\phi\\ &= -\pi_{\wedge,\min}B_{\wedge,\mathcal{D}}(\varrho^m\lambda_0)(A_\wedge - \varrho^m\lambda_0)\mathfrak{P}_{\max}(\varrho^m\lambda_0)\phi.\end{aligned}$$

Consequently,

$$\mathfrak{P}_{\max}(\varrho^m\lambda_0)\phi + \mathfrak{P}_{\min,\mathcal{D}}(\varrho^m\lambda_0)\phi = \kappa_\varrho\phi \in \mathcal{K}_{\wedge,\varrho^m\lambda_0}.$$

This implies that the equation

$$(A_\wedge - \lambda)[\mathfrak{P}_{\max}(\lambda)\phi + \mathfrak{P}_{\min,\mathcal{D}}(\lambda)\phi] = 0$$

is satisfied when $\lambda \in \Gamma_\alpha \cap \operatorname{res} A_{\wedge,\mathcal{D}}$. By unique continuation, it is satisfied for any $\lambda \in \mathring{\Lambda}_\alpha \cap \operatorname{res} A_{\wedge,\mathcal{D}}$. Thus

$$\mathfrak{P}_{\max}(\lambda)\phi + \mathfrak{P}_{\min,\mathcal{D}}(\lambda)\phi \in \mathcal{K}_{\wedge,\lambda} \quad \text{if } \lambda \in \operatorname{res} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha. \tag{7.18}$$

For such λ we therefore have

$$(A_\wedge - \lambda)\mathfrak{P}_{\max}(\lambda)\phi = -(A_\wedge - \lambda)\mathfrak{P}_{\min,\mathcal{D}}(\lambda)\phi$$

so

$$\mathfrak{P}_{\min,\mathcal{D}}(\lambda)\phi = -B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\mathfrak{P}_{\max}(\lambda)\phi \tag{7.19}$$

that is,

$$\mathfrak{P}_{\min,\mathcal{D}}(\lambda)\phi = \mathfrak{P}_{\min}(\lambda)\phi. \tag{7.20}$$

Replacing this in (7.18) we see that formula (7.17) holds where $\mathfrak{P}_{\min,\mathcal{D}}(\lambda)\phi$ is defined.

Since the left hand side of (7.20) is holomorphic where defined, so is the right hand side. Since the right hand side is continuous on $\mathring{\Lambda}_\alpha$, the singularities of the left hand side, i.e. the elements of the discrete set $\operatorname{spec} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha$, are removable. Thus $\mathfrak{P}_{\min}(\lambda)\phi$ is holomorphic for $\lambda \in \mathring{\Lambda}_\alpha$ and by continuity (7.17) holds. □

The sets

$$\mathcal{L}_\alpha = \{\mathcal{K}_{\wedge,\max}(\lambda) : \lambda \in \mathring{\Lambda}_\alpha\} \subset \operatorname{Gr}_{d'_\alpha}(\mathcal{E}_{\wedge,\max})$$

play an important role, particularly their intersection with the varieties $\mathfrak{V}_{\pi_{\wedge,\max}\mathcal{D}}$, $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$. These sets are invariant under the action of $\boldsymbol{\kappa}_{e^\xi}$ for ξ real. If $\mathcal{V} \in \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$, then $\mathbb{C} \ni \zeta \mapsto \boldsymbol{\kappa}_{e^\zeta}\mathcal{V} \in \operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})$ is a holomorphic map, cf. Proposition 7.7. The generator of the one-parameter group $(\xi, \mathcal{V}) \mapsto \kappa_{e^\xi}\mathcal{V}$, the vector field $\mathcal{T}_\wedge$, is the real part of a holomorphic vector field $\mathcal{T}'_\wedge$, cf. the paragraph following the proof of Proposition 7.7, which at $\mathcal{V}$ is the image of the Cauchy-Riemann vector field at $\zeta = 0$, $\partial_\zeta|_0$, under the differential of the map $\zeta \mapsto \boldsymbol{\kappa}_{e^\zeta}\mathcal{V}$. If $\mathcal{V}$ is not κ-invariant, i.e., $\mathcal{T}_\wedge \neq 0$ at $\mathcal{V}$, then also the imaginary part of $\mathcal{T}'_\wedge$ is different from 0 at $\mathcal{V}$; thus $\zeta \mapsto \boldsymbol{\kappa}_{e^\zeta}\mathcal{V}$ is a local embedding near $\zeta = 0$ if $\mathcal{T}_\wedge \neq 0$ at $\mathcal{V}$. As a consequence we get that the real and imaginary parts of $\mathcal{T}'_\wedge$ commute at the noninvariant points. Since the set of invariant points is closed with empty interior, the real and imaginary parts of $\mathcal{T}'_\wedge$ commute everywhere. We can view the images of the maps $\mathbb{C} \ni \zeta \mapsto \boldsymbol{\kappa}_{e^\zeta}\mathcal{V}$ as a point (if $\mathcal{V}$ is invariant) or as an integral manifold of the involutive Frobenius distribution generated by $\Re\mathcal{T}'_\wedge$, $\Im\mathcal{T}'_\wedge$ on $\operatorname{Gr}_{d_0}(\mathcal{E}_{\wedge,\max})\backslash\{\mathcal{V} \in \operatorname{Gr}_{d_0} : \mathcal{V} \text{ is } \kappa\text{-invariant}\}$.

Theorem 7.21. *The set $\mathcal{L}_\alpha$ is contained in one orbit of $\mathcal{T}'_\wedge$.*

In fact, the set $\mathcal{L}_\alpha$ is identical to the set

$$\{\kappa_{e^{\log(\lambda/\lambda_0)/m}}\pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda_0} : \lambda \in \mathring{\Lambda}_\alpha\},$$

a subset of the orbit of $\mathcal{T}'_\wedge$ containing $\pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda_0}$. Thus, if $\dim_{\mathbb{R}} \operatorname{Gr}_{d'_\alpha}(\mathcal{E}_{\wedge,\max}) > 2$, then $\mathcal{L}_\alpha$ is in principle a small set (nevertheless it could be dense).

The following lemma completes our description of the vector bundle $\mathcal{K}_\wedge$ introduced in (7.10).

Lemma 7.22. *If $\phi \in \mathcal{K}_{\wedge,\lambda_0}$, then*

$$\kappa_\varrho\mathfrak{P}(\lambda)\phi = \mathfrak{P}(\varrho^m\lambda)(\phi), \quad \varrho \in \mathbb{R}_+. \tag{7.23}$$

Proof. Write $(\lambda/\lambda_0)^{1/m} = e^{[\log(|\lambda/\lambda_0|)+i\arg(\lambda/\lambda_0)]/m}$, $\lambda \notin -\Gamma_\alpha$. For real ϱ and $\lambda \notin -\Gamma_\alpha$ we have

$$\pi_{\wedge,\max}\kappa_\varrho\pi_{\wedge,\max}\kappa_{(\lambda/\lambda_0)^{1/m}}\pi_{\wedge,\max} = \pi_{\wedge,\max}\kappa_{(\varrho^m\lambda/\lambda_0)^{1/m}}\pi_{\wedge,\max}$$

Thus $\pi_{\wedge,\max}\kappa_\varrho\mathfrak{P}_{\max}(\lambda)\phi = \mathfrak{P}_{\max}(\varrho^m\lambda)\phi$. But if $\lambda \in \mathring{\Lambda}_\alpha$, then $\kappa_\varrho\mathfrak{P}(\lambda)\phi \in \mathcal{K}_{\wedge,\varrho^m\lambda}$ and $\mathfrak{P}_{\min}(\varrho^m\lambda)\phi$ is the unique element of $\mathcal{D}_{\wedge,\min}$ such that

$$\mathfrak{P}_{\max}(\varrho^m\lambda)\phi + \mathfrak{P}_{\min}(\varrho^m\lambda)\phi \in \mathcal{K}_{\wedge,\varrho^m\lambda}$$

so we have

$$\kappa_\varrho\mathfrak{P}(\lambda)\phi = \mathfrak{P}_{\max}(\varrho^m\lambda)\phi + \mathfrak{P}_{\min}(\varrho^m\lambda)\phi.$$

This is (7.23). □

Suppose $\phi \in \mathcal{K}_{\wedge,\lambda_0}$. If $\phi(\lambda) = \mathfrak{P}(\lambda)\phi$ vanishes at some λ_1, then $\phi(\lambda)$ vanishes along the ray through λ_1. Therefore it vanishes identically, since $\phi(\lambda)$ is holomorphic. Thus, if we pick a basis $\{\phi_j\}$ of $\mathcal{K}_{\wedge,\lambda_0}$, then the sections $\mathfrak{P}(\lambda)\phi_j$ form a frame over $\mathring{\Lambda}_\alpha$ for the bundle $\mathcal{K}_\wedge$.

8. Resolvents for the model operator

We now turn our attention to determining the existence of sectors of minimal growth for extensions of $A_\wedge$.

If $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$, we write $B_{\wedge,\mathcal{D}}(\lambda)$ for the inverse of $A_{\wedge,\mathcal{D}} - \lambda$, $\lambda \in \operatorname{res} A_{\wedge,\mathcal{D}}$. If $\lambda \in \mathbb{C}\backslash 0$, then $\hat{\lambda} = \lambda/|\lambda|$. By a closed sector we shall mean a set of the form

$$\Lambda = \{z \in \mathbb{C} : z = re^{i\theta} \text{ for } r \geq 0,\ \theta \in \mathbb{R},\ |\theta - \theta_0| \leq a\}.$$

If $R > 0$ and Λ is a closed sector, then $\Lambda_R = \{\lambda \in \Lambda : |\lambda| \geq R\}$. Let $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$. Let Λ be a closed sector with $\Lambda\backslash 0 \subset \mathring{\Lambda}_\alpha$. Then Λ is called a sector of minimal growth for $A_{\wedge,\mathcal{D}}$ if there is $R > 0$ such that $A_{\wedge,\mathcal{D}} - \lambda$ is invertible if $\lambda \in \Lambda_R$, and either of the equivalent estimates

$$\|B_{\wedge,\mathcal{D}}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \leq C/|\lambda|, \quad \|B_{\wedge,\mathcal{D}}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\wedge,\max})} \leq C \tag{8.1}$$

holds for some $C > 0$ when $\lambda \in \Lambda_R$.

The following lemma is immediate, in view of (7.1) and the fact that κ_ϱ is an isometry on $x^{-m/2}L^2_b(Y^\wedge;E)$.

Lemma 8.2. *If $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$, then $\operatorname{res} A_{\wedge,\kappa_\varrho^{-1}\mathcal{D}} = \varrho^{-m}\operatorname{res} A_{\wedge,\mathcal{D}}$. If $\operatorname{res} A_{\wedge,\mathcal{D}} \neq \varnothing$, then*

$$B_{\wedge,\kappa_\varrho^{-1}\mathcal{D}}(\lambda) = \varrho^m\kappa_\varrho^{-1}B_{\wedge,\mathcal{D}}(\varrho^m\lambda)\kappa_\varrho, \quad \varrho > 0. \tag{8.3}$$

Thus, if the closed sector Λ, $\Lambda\backslash 0 \subset \mathring{\Lambda}_\alpha$, is a sector of minimal growth for $A_{\wedge,\mathcal{D}}$, then Λ is a sector of minimal growth for $A_{\wedge,\kappa_\varrho^{-1}\mathcal{D}}$.

In fact, if the first estimate in (8.1) holds when $\lambda \in \Lambda_R$, then

$$\|B_{\wedge,\kappa_\varrho^{-1}\mathcal{D}}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \leq C/|\lambda|, \quad \lambda \in \Lambda_{R/\varrho^m}$$

with the same constant C.

The simplest domains are those that are κ-invariant:

Proposition 8.4. *Suppose $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$ is κ-invariant. Then either $\mathring{\Lambda}_\alpha \cap \operatorname{res} A_{\wedge,\mathcal{D}} = \varnothing$, or $\mathring{\Lambda}_\alpha \subset \operatorname{res} A_{\wedge,\mathcal{D}}$. In the latter case the resolvent $B_{\wedge,\mathcal{D}}(\lambda)$ of $A_{\wedge,\mathcal{D}}$ satisfies*

$$\|B_{\wedge,\mathcal{D}}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \leq C/|\lambda| \tag{8.5}$$

for some $C > 0$ when $\lambda \in \Lambda\backslash 0$, Λ a closed sector with $\Lambda\backslash 0 \subset \mathring{\Lambda}_\alpha$.

Proof. Suppose that $\lambda_0 \in \mathring{\Lambda}_\alpha \cap \operatorname{spec} A_{\wedge,\mathcal{D}}$. The homogeneity property (7.1) and the assumption that $\mathcal{D}$ is κ-invariant give that $\lambda/\varrho^m \in \operatorname{spec} A_{\wedge,\mathcal{D}}$ for every $\varrho > 0$. Thus $\operatorname{spec} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha$ is not discrete. On the other hand, if $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}\backslash\mathring{\mathfrak{V}}_\alpha$, cf. Proposition 7.5, then $\operatorname{spec} A_{\wedge,\mathcal{D}} \cap \mathring{\Lambda}_\alpha$ is a discrete closed subset of $\mathring{\Lambda}_\alpha$. In particular, for every ray

$$\Gamma = \{z \in \mathbb{C} : z = re^{i\theta_0} \text{ for } r > 0\} \tag{8.6}$$

contained in $\mathring{\Lambda}_\alpha$, $\Gamma \cap \operatorname{spec} A_{\wedge,\mathcal{D}}$ is closed and discrete. Thus, if $\mathcal{D}$ is κ-invariant, then $\mathring{\Lambda}_\alpha \cap \operatorname{spec} A_{\wedge,\mathcal{D}} \neq \varnothing$ implies $\mathring{\Lambda}_\alpha \subset \operatorname{spec} A_{\wedge,\mathcal{D}}$.

Suppose Λ is a closed sector with $\Lambda\backslash 0 \subset \operatorname{res} A_{\wedge,\mathcal{D}}$. Since $\mathcal{D}$ is κ-invariant, (8.3) reads

$$B_{\wedge,\mathcal{D}}(\lambda) = \varrho^m \kappa_\varrho^{-1} B_{\wedge,\mathcal{D}}(\varrho^m\lambda)\kappa_\varrho.$$

Setting $\varrho = |\lambda|^{-1/m}$ gives

$$B_{\wedge,\mathcal{D}}(\lambda) = |\lambda|^{-1}\kappa_{|\lambda|^{1/m}} B_{\wedge,\mathcal{D}}(\hat\lambda)\kappa^{-1}_{|\lambda|^{1/m}}.$$

For $\hat\lambda \in \Lambda$ ($|\hat\lambda| = 1$) we have a uniform estimate for $\|B_{\wedge,\mathcal{D}}(\hat\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)}$, and (8.5) follows immediately, since κ_ϱ is an isometry on $x^{-m/2}L^2_b(Y^\wedge;E)$. $\square$

If the domain $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$ is not κ-invariant, the existence of a ray or sector of minimal growth for $B_{\wedge,\mathcal{D}}(\lambda)$ is more complicated:

Theorem 8.7. *Let $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$, let Λ be a closed sector with $\Lambda\backslash 0 \subset \mathring{\Lambda}_\alpha$. Then Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}}$ if and only if there are C, $R > 0$ such that $\Lambda_R \subset \operatorname{res} A_{\wedge,\mathcal{D}}$ and*

$$\|\pi_{\wedge,\max}\pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\big|_{\mathcal{E}_{\wedge,\max}}\|_{\mathscr{L}(\mathcal{D}_{\wedge,\max})} \leq C, \quad \lambda \in \Lambda_R. \tag{8.8}$$

If $\mathcal{D}$ is κ-invariant, then

$$\pi_{\wedge,\max}\pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\big|_{\mathcal{E}_{\wedge,\max}} = \pi_{\wedge,\max}\pi_{\mathcal{K}_{\wedge,\hat\lambda},\mathcal{D}}\big|_{\mathcal{E}_{\wedge,\max}},$$

and the theorem reduces to the trivial situation of Proposition 8.4.

The proof of the theorem requires some preparation. Define $B_{\wedge,\max}(\lambda)$ for $\lambda \in \operatorname{bg-res} A_\wedge$ as the right inverse of

$$A_\wedge - \lambda : \mathcal{D}_{\wedge,\max} \subset x^{-m/2}L^2_b(Y^\wedge;E) \to x^{-m/2}L^2_b(Y^\wedge;E)$$

with range in $\mathcal{K}^\perp_{\wedge,\lambda}$. Thus $B_{\wedge,\max}(\lambda)$ has the virtue of being the right inverse of $A_{\wedge,\max} - \lambda$ with the smallest operator norm. It depends smoothly on $\lambda \in \operatorname{bg-res} A_\wedge$; this is proved in the same way as the corresponding statement for $B_{\max}(\lambda)$ in Section 5. Recall that $B_{\wedge,\min}(\lambda)$ is the left inverse of $A_{\wedge,\mathcal{D}_{\min}} - \lambda$ with kernel equal to the orthogonal of $\mathcal{R}_{\wedge,\lambda} = \operatorname{rg}(A_{\wedge,\mathcal{D}_{\min}} - \lambda)$. For $\lambda \in \operatorname{res} A_{\wedge,\mathcal{D}}$ let

$$\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}} : \mathcal{D}_{\wedge,\max} \to \mathcal{D}_{\wedge,\max}$$

be the projection on $\mathcal{K}_{\wedge,\lambda}$ according to the decomposition $\mathcal{D}_{\wedge,\max} = \mathcal{K}_{\wedge,\lambda} \oplus \mathcal{D}$, cf. (7.14). Then the resolvent of $A_{\wedge,\mathcal{D}}$ is

$$B_{\wedge,\mathcal{D}}(\lambda) = B_{\wedge,\max}(\lambda) - \big(I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\big)\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}} B_{\wedge,\max}(\lambda), \tag{8.9}$$

cf. (5.17). We will take advantage of this formula by using the group action κ. We begin with estimates for $B_{\wedge,\min}(\lambda)$ and $B_{\wedge,\max}(\lambda)$.

Lemma 8.10. *The operator $B_{\wedge,\min}(\lambda)$ is κ-homogeneous of degree $-m$,*

$$B_{\wedge,\min}(\lambda) = \varrho^{-m}\kappa_\varrho B_{\wedge,\min}(\lambda/\varrho^m)\kappa_\varrho^{-1}. \tag{8.11}$$

Therefore, if Λ is a closed sector with $\Lambda\backslash 0 \subset \text{bg-res}\, A_\wedge$, then

$$\|B_{\wedge,\min}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \leq C/|\lambda| \tag{8.12}$$

for some $C > 0$ when $\lambda \in \Lambda\backslash 0$.

Proof. Let $B'_{\wedge,\min}(\lambda) = \varrho^{-m}\kappa_\varrho B_{\wedge,\min}(\lambda/\varrho^m)\kappa_\varrho^{-1}$. The operator $B'_{\wedge,\min}(\lambda)$ maps into $\mathcal{D}_{\wedge,\min}$ because the latter space is κ-invariant. Using the κ-homogeneity of $A_\wedge$ one verifies that the operator $B'_{\wedge,\min}(\lambda)$ is a left inverse for $A_{\wedge,\min} - \lambda$. Also because of the κ-invariance of $\mathcal{D}_{\wedge,\min}$ and the κ-homogeneity of $A_\wedge - \lambda$, $\mathcal{R}_\lambda = \kappa_\varrho \mathcal{R}_{\lambda/\varrho^m}$. The kernel of $B'_{\wedge,\min}(\lambda)$ is $\kappa_\varrho \mathcal{R}^\perp_{\lambda/\varrho^m}$. Since κ_ϱ is an isometry on $x^{-m/2}L^2_b(Y^\wedge;E)$, κ_ϱ preserves the orthogonality of the decomposition $\mathcal{R}_{\lambda/\varrho^m} \oplus \mathcal{R}^\perp_{\lambda/\varrho^m}$. Hence, the kernel of $B'_{\wedge,\min}(\lambda)$ is orthogonal to $\mathcal{R}_\lambda$. Thus $B'_{\wedge,\min}(\lambda) = B_{\wedge,\min}(\lambda)$, and (8.11) holds.

The estimate in (8.12) follows from setting $\varrho^m = |\lambda|$ in (8.11). □

The operator family $B_{\wedge,\max}(\lambda)$ is not κ-homogeneous. Nevertheless its norm satisfies good estimates.

Proposition 8.13. *Let $\pi_{\mathcal{K}_{\wedge,\lambda}} : \mathcal{D}_{\wedge,\max} \to \mathcal{D}_{\wedge,\max}$ be the orthogonal projection on $\mathcal{K}_{\wedge,\lambda}$. Regard the finite dimensional space $\mathcal{K}_{\wedge,\lambda}$ as a subspace of $x^{-m/2}L^2_b(Y^\wedge;E)$ and let*

$$\mathfrak{p}_{\mathcal{K}_{\wedge,\lambda}} : x^{-m/2}L^2_b(Y^\wedge;E) \to x^{-m/2}L^2_b(Y^\wedge;E)$$

be the orthogonal projection on $\mathcal{K}_{\wedge,\lambda}$. Then

$$B_{\wedge,\max}(\lambda) = |\lambda|^{-1}\kappa_{|\lambda|^{1/m}}\big(I - \frac{1-|\lambda|^2}{1+|\lambda|^2}\mathfrak{p}_{\mathcal{K}_{\wedge,\hat\lambda}}\big)B_{\wedge,\max}(\hat\lambda)\kappa^{-1}_{|\lambda|^{1/m}}. \tag{8.14}$$

Therefore, if Λ is a closed sector such that $\Lambda\backslash 0 \subset \text{bg-res}\, A_\wedge$, then

$$\|B_{\wedge,\max}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \leq C/|\lambda| \tag{8.15}$$

for some $C > 0$ when $\lambda \in \Lambda\backslash 0$.

The proof will require:

Lemma 8.16. *For any $\lambda \in \text{bg-res}\, A_\wedge$ and $\varrho > 0$,*

$$\kappa_\varrho^{-1}\pi_{\mathcal{K}_{\wedge,\varrho^m\lambda}}\kappa_\varrho = \frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\varrho^{2m}\pi_{\mathcal{K}_{\wedge,\lambda}} + \frac{1-\varrho^{2m}}{1+|\varrho^m\lambda|^2}\mathfrak{p}_{\mathcal{K}_{\wedge,\lambda}}. \tag{8.17}$$

We will prove the lemma later.

Proof of Proposition 8.13. Suppose $f \in x^{-m/2}L^2_b(Y^\wedge;E)$ and let $u = B_{\wedge,\max}(\lambda)f$. Then $(A_\wedge - \varrho^m\lambda)\kappa_\varrho u = \varrho^m\kappa_\varrho(A_\wedge - \lambda)u = \varrho^m\kappa_\varrho f$, and consequently

$$B_{\wedge,\max}(\varrho^m\lambda)\varrho^m\kappa_\varrho f = \kappa_\varrho u - \pi_{\mathcal{K}_{\wedge,\varrho^m\lambda}}\kappa_\varrho u.$$

This gives the formula

$$\varrho^m B_{\wedge,\max}(\varrho^m\lambda)\kappa_\varrho f = \kappa_\varrho B_{\wedge,\max}(\lambda)f - \pi_{\mathcal{K}_{\wedge,\varrho^m\lambda}}\kappa_\varrho B_{\wedge,\max}(\lambda)f$$

which, in view of (8.17) and the fact that the range of $B_{\wedge,\max}(\lambda)$ is orthogonal to $\mathcal{K}_{\wedge,\lambda}$, reduces to

$$\varrho^m B_{\wedge,\max}(\varrho^m\lambda)\kappa_\varrho f = \kappa_\varrho B_{\wedge,\max}(\lambda)f - \frac{1-\varrho^{2m}}{1+|\varrho^m\lambda|^2}\kappa_\varrho\mathfrak{p}_{\mathcal{K}_{\wedge,\lambda}}B_{\wedge,\max}(\lambda)f.$$

Thus

$$B_{\wedge,\max}(\varrho^m\lambda) = \varrho^{-m}\kappa_\varrho\big(I - \frac{1-\varrho^{2m}}{1+|\varrho^m\lambda|^2}\mathfrak{p}_{\mathcal{K}_{\wedge,\lambda}}\big)B_{\wedge,\max}(\lambda)\kappa_\varrho^{-1}.$$

The formula (8.14) is obtained from this by replacing ϱ^m by $|\lambda|$ and λ by $\hat\lambda$. The estimate (8.15) is evident given the formula (8.14). □

The operator

$$B^h_{\wedge,\max}(\lambda) = |\lambda|^{-1}\kappa_{|\lambda|^{1/m}}B_{\wedge,\max}(\hat\lambda)\kappa^{-1}_{|\lambda|^{1/m}}, \quad \lambda \in \text{bg-res}\, A_\wedge$$

is a κ-homogeneous right inverse of $A_{\wedge,\max} - \lambda$ of degree $-m$ that coincides with $B_{\wedge,\max}(\lambda)$ when $|\lambda| = 1$. For any closed sector Λ with $\Lambda\backslash 0 \subset \text{bg-res}\, A_\wedge$ there is C such that

$$\|B^h_{\wedge,\max}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \le C/|\lambda|, \quad \lambda \in \Lambda\backslash 0,$$

and for any closed sector Λ as above and $R > 0$,

$$\|B_{\wedge,\max}(\lambda) - B^h_{\wedge,\max}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \le C/|\lambda|$$

for $\lambda \in \Lambda_R$. So in some estimates below, it makes little difference whether the correction term involving $\mathfrak{p}_{\mathcal{K}_{\wedge,\hat\lambda}}$ is present or not. However, we will keep on using $B_{\wedge,\max}(\lambda)$ instead of $B^h_{\wedge,\max}(\lambda)$, as the former family is in some sense more natural than the latter.

Proof of Lemma 8.16. Let $\phi_1, \dots, \phi_{d'_\alpha}$ be an $A_\wedge$-orthonormal basis of $\mathcal{K}_{\wedge,\lambda}$. Then

$$\delta_{jk} = (\phi_j, \phi_k)_{A_\wedge} = (1+|\lambda|^2)(\phi_j, \phi_k).$$

In particular, the $\sqrt{1+|\lambda|^2}\phi_j \in \mathcal{K}_{\wedge,\lambda}$ are orthonormal in $x^{-m/2}L^2_b(Y^\wedge; E)$. On the other hand, using that κ_ϱ is an isometry on $x^{-m/2}L^2_b(Y^\wedge; E)$,

$$\begin{aligned}(\kappa_\varrho\phi_j, \kappa_\varrho\phi_k)_{A_\wedge} &= \varrho^{2m}(\phi_j,\phi_k)_{A_\wedge} + (1-\varrho^{2m})(\phi_j,\phi_k)\\ &= \Big(\varrho^{2m} + \frac{1-\varrho^{2m}}{1+|\lambda|^2}\Big)\delta_{jk} = \frac{1+|\varrho^m\lambda|^2}{1+|\lambda|^2}\,\delta_{jk}.\end{aligned}$$

Thus the $\sqrt{(1+|\lambda|^2)/(1+|\varrho^m\lambda|^2)}\kappa_\varrho\phi_j \in \mathcal{K}_{\wedge,\varrho^m\lambda}$ are $A_\wedge$-orthonormal, and if $u \in \mathcal{K}_{\wedge,\lambda}$, then

$$\begin{aligned}\pi_{\mathcal{K}_{\wedge,\varrho^m\lambda}}\kappa_\varrho u &= \frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\sum_j(\kappa_\varrho u, \kappa_\varrho\phi_j)_{A_\wedge}\kappa_\varrho\phi_j\\ &= \frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\sum_j\big[\varrho^{2m}(u,\phi_j)_{A_\wedge} + (1-\varrho^{2m})(u,\phi_j)\big]\kappa_\varrho\phi_j\\ &= \frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\kappa_\varrho\sum_j\big[\varrho^{2m}(u,\phi_j)_{A_\wedge}\phi_j + (1-\varrho^{2m})(u,\phi_j)\phi_j\big]\\ &= \kappa_\varrho\big(\frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\varrho^{2m}\pi_{\mathcal{K}_{\wedge,\lambda}}u + \frac{1-\varrho^{2m}}{1+|\varrho^m\lambda|^2}\sum_j(1+|\lambda|^2)(u,\phi_j)\phi_j\big)\\ &= \kappa_\varrho\big(\frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\varrho^{2m}\pi_{\mathcal{K}_{\wedge,\lambda}}u + \frac{1-\varrho^{2m}}{1+|\varrho^m\lambda|^2}\mathfrak{p}_{\mathcal{K}_{\wedge,\lambda}}u\big).\end{aligned}$$

This gives the formula in the statement of the lemma. □

Note that if $u \in \mathcal{D}_{\wedge,\max}$, then

$$\|\mathfrak{p}_{\mathcal{K}_{\wedge,\lambda}} u\| \le \|u\| \le \|u\|_{A_\wedge}, \quad \|\mathfrak{p}_{\mathcal{K}_{\wedge,\lambda}} u\|_{A_\wedge} \le \sqrt{1+|\lambda|^2}\|u\|_{A_\wedge}. \tag{8.18}$$

Lemma 8.19. *Let Λ be some closed sector, let $R>0$, and let*

$$P(\lambda) : x^{-m/2}L^2_b(Y^\wedge;E) \to \mathcal{D}_{\wedge,\max}$$

be a family of operators defined for $\lambda \in \Lambda_R$. Then

$$\|P(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \le C/|\lambda| \quad \textit{and} \quad \|P(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\wedge,\max})} \le C \tag{8.20}$$

hold for some $C>0$ and all $\lambda\in\Lambda_R$ if and only if

$$\|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\wedge,\max})} \le C/|\lambda| \tag{8.21}$$

holds for some $C>0$ and all $\lambda\in\Lambda_R$.

Proof. Using that $A_\wedge \kappa^{-1}_{|\lambda|^{1/m}} P(\lambda) = |\lambda|^{-1}\kappa^{-1}_{|\lambda|^{1/m}} A_\wedge P(\lambda)$, and that $\kappa^{-1}_{|\lambda|^{1/m}}$ is an isometry in $x^{-m/2}L^2_b(Y^\wedge;E)$, we obtain

$$\begin{aligned}
\|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|^2_{A_\wedge} &= \|A_\wedge\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|^2 + \|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|^2 \\
&= |\lambda|^{-2}\|\kappa^{-1}_{|\lambda|^{1/m}} A_\wedge P(\lambda)f\|^2 + \|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|^2 \\
&= |\lambda|^{-2}\|A_\wedge P(\lambda)f\|^2 + \|P(\lambda)f\|^2
\end{aligned}$$

if $f\in x^{-m/2}L^2_b(Y^\wedge;E)$. Thus (8.21) follows from (8.20).

Assume now that (8.21) holds and let $f\in x^{-m/2}L^2_b(Y^\wedge;E)$. Then

$$\|P(\lambda)f\| = \|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\| \le \|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|_{A_\wedge}$$

gives the first estimate in (8.20). To obtain the second, write $\|P(\lambda)f\|^2_{A_\wedge}$ as

$$\|A_\wedge P(\lambda)f\|^2 + \|P(\lambda)f\|^2 = \|\kappa^{-1}_{|\lambda|^{1/m}} A_\wedge P(\lambda)f\|^2 + \|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|^2$$

and use the κ-homogeneity of $A_\wedge$ to conclude that

$$\begin{aligned}
\|P(\lambda)f\|^2_{A_\wedge} &= |\lambda|^2\|A_\wedge\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|^2 + \|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|^2 \\
&\le (|\lambda|^2+1)\|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)f\|^2_{A_\wedge}.
\end{aligned}$$

The second estimate in (8.20) follows from this. ☐

Corollary 8.22. *Let $\mathcal{D}\in\mathfrak{G}_{\wedge,\alpha}$, let Λ be a closed sector. Then Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}}$ if and only if there are C, $R>0$ such that*

$$\|\kappa^{-1}_{|\lambda|^{1/m}} B_{\wedge,\mathcal{D}}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\wedge,\max})} \le C/|\lambda|, \quad \lambda\in\Lambda_R. \tag{8.23}$$

Both $B_{\wedge,\min}(\lambda)$ and $B_{\wedge,\max}(\lambda)$ satisfy (8.20), therefore (8.21) for any closed sector Λ with $\Lambda\backslash 0 \subset \mathring{\Lambda}_\alpha$. In the case of $B_{\wedge,\min}(\lambda)$, the first of the estimates in (8.20) is (8.12). To prove the second we note that

$$A_\wedge B_{\wedge,\min}(\lambda) = \pi_{\mathcal{R}_{\wedge,\lambda}} + \lambda B_{\wedge,\min}(\lambda)$$

where $\pi_{\mathcal{R}_{\wedge,\lambda}} : x^{-m/2}L^2_b(Y^\wedge;E) \to x^{-m/2}L^2_b(Y^\wedge;E)$ is the orthogonal projection on $\mathcal{R}_{\wedge,\lambda}$. The norm of this operator is 1, and $\|\lambda B_{\wedge,\min}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)}$ is bounded independently of λ when $\lambda\in\Lambda$ and $|\lambda|$ is large. The argument for $B_{\wedge,\max}(\lambda)$ is analogous, using (8.15) and the fact that this operator is a right inverse for $A_\wedge-\lambda$.

Proof of Theorem 8.7. We will prove that (8.8) is equivalent to (8.23). Recalling the formula (8.9) for $B_{\wedge,\mathcal{D}}(\lambda)$ and that $B_{\wedge,\max}(\lambda)$ satisfies (8.21), we see that $B_{\wedge,\mathcal{D}}(\lambda)$ satisfies (8.23) if and only if

$$\|\kappa^{-1}_{|\lambda|^{1/m}}\big(I - B_{\wedge,\min}(\lambda)(A_\wedge-\lambda)\big)\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}B_{\wedge,\max}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\max})} \leq C/|\lambda|$$

for $\lambda\in\Lambda$, $|\lambda|$ large. We have

$$\kappa^{-1}_{|\lambda|^{1/m}}\big(I - B_{\wedge,\min}(\lambda)(A_\wedge-\lambda)\big)\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}B_{\wedge,\max}(\lambda) = \\ \big(I - B_{\wedge,\min}(\hat\lambda)(A_\wedge-\hat\lambda)\big)\kappa^{-1}_{|\lambda|^{1/m}}\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}\kappa_{|\lambda|^{1/m}}\kappa^{-1}_{|\lambda|^{1/m}}B_{\wedge,\max}(\lambda)$$

Evidently $\mathcal{K}_{\wedge,\lambda}\cap\mathcal{D}=0$ if and only if $\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{K}_{\wedge,\lambda}\cap\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}=0$. By Lemma 7.11, $\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{K}_{\wedge,\lambda}=\mathcal{K}_{\wedge,\hat\lambda}$, and it is not hard to see that

$$\kappa^{-1}_{|\lambda|^{1/m}}\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}\kappa_{|\lambda|^{1/m}} = \pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}.$$

Using that $I-B_{\wedge,\min}(\lambda)(A_\wedge-\lambda)$ and $\pi_{\mathcal{K}_{\wedge,\lambda},\mathcal{D}}$ both vanish on $\mathcal{D}_{\wedge,\min}$ regardless of λ and $\mathcal{D}$, we arrive at the conclusion that $B_{\wedge,\mathcal{D}}(\lambda)$ satisfies (8.23) if and only if the norm of

$$\big(I - B_{\wedge,\min}(\hat\lambda)(A_\wedge-\hat\lambda)\big)\pi_{\wedge,\max}\,\pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\,\pi_{\wedge,\max}\kappa^{-1}_{|\lambda|^{1/m}}B_{\wedge,\max}(\lambda) \tag{8.24}$$

as an operator $x^{-m/2}L^2_b(Y^\wedge;E)\to\mathcal{D}_{\wedge,\max}$ is bounded by $C/|\lambda|$ for some C if $\lambda\in\Lambda$, $|\lambda|$ large. By Lemma 8.19,

$$\|\kappa^{-1}_{|\lambda|^{1/m}}B_{\wedge,\max}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\wedge,\max})}\leq C/|\lambda|$$

for $\lambda\in\Lambda$, $|\lambda|$ large. Evidently

$$\|I - B_{\wedge,\min}(\hat\lambda)(A_\wedge-\hat\lambda)\|_{\mathscr{L}(\mathcal{D}_{\wedge,\max})}$$

is bounded independently of λ, $\lambda\in\Lambda\backslash 0$. Thus if (8.8) holds, then the norm of the operator (8.24) is bounded by $C/|\lambda|$ for some C when $\lambda\in\Lambda$, $|\lambda|$ large.

Conversely, suppose that the norm of the operator (8.24) is bounded by $C/|\lambda|$ for some C when $\lambda\in\Lambda$, $|\lambda|$ large. Composing with $\pi_{\wedge,\max}$ on the left we get that the norm of

$$\pi_{\wedge,\max}\,\pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\,\pi_{\wedge,\max}\kappa^{-1}_{|\lambda|^{1/m}}B_{\wedge,\max}(\lambda)$$

as an operator $x^{-m/2}L^2_b(Y^\wedge;E)\to\mathcal{E}_{\wedge,\max}$ satisfies the same estimate. Using the formula (8.14) for $B_{\wedge,\max}(\lambda)$ we get

$$\pi_{\wedge,\max}\,\pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\kappa^{-1}_{|\lambda|^{1/m}}B_{\wedge,\max}(\lambda) = \\ |\lambda|^{-1}\pi_{\wedge,\max}\,\pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\big(I-\frac{1-|\lambda|^2}{1+|\lambda|^2}\mathfrak{p}_{\mathcal{K}_{\wedge,\hat\lambda}}\big)B_{\wedge,\max}(\hat\lambda)\kappa^{-1}_{|\lambda|^{1/m}}.$$

We dismiss the factor $\kappa^{-1}_{|\lambda|^{1/m}}$ at the end of the last formula, since this is an isometry on $x^{-m/2}L^2_b(Y^\wedge;E)$. Since $\mathfrak{p}_{\mathcal{K}_{\wedge,\hat\lambda}}$ has range in $\mathcal{K}_{\wedge,\hat\lambda}$,

$$|\lambda|^{-1}\pi_{\wedge,\max}\,\pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\left(\frac{1-|\lambda|^2}{1+|\lambda|^2}\right)\mathfrak{p}_{\mathcal{K}_{\wedge,\hat\lambda}}B_{\wedge,\max}(\hat\lambda) \\ = |\lambda|^{-1}\frac{1-|\lambda|^2}{1+|\lambda|^2}\pi_{\wedge,\max}\mathfrak{p}_{\mathcal{K}_{\wedge,\hat\lambda}}B_{\wedge,\max}(\hat\lambda).$$

This operator $x^{-m/2}L^2_b(Y^\wedge;E) \to \mathcal{D}_{\wedge,\max}$ evidently has norm $O(|\lambda|^{-1})$ if $\lambda \in \Lambda$, $|\lambda| \to \infty$ (cf. (8.18)). We conclude that if the norm of (8.24) is bounded as indicated, then the norm of

$$\pi_{\wedge,\max}\, \pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}} B_{\wedge,\max}(\hat\lambda) \tag{8.25}$$

is bounded by a constant when $\lambda \in \Lambda$, $|\lambda|$ large. The operator $A_\wedge - \hat\lambda$ on $\mathcal{D}_{\wedge,\max}$ satisfies $\|A_\wedge - \hat\lambda\|_{\mathscr{L}(\mathcal{D}_{\wedge,\max},x^{-m/2}L^2_b)} \le 1$. So composing the operator (8.25) with $A_\wedge - \hat\lambda$ on the right we get that the norm of

$$\pi_{\wedge,\max}\, \pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}(I - \pi_{\mathcal{K}_{\wedge,\hat\lambda}}) = \pi_{\wedge,\max}\, \pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\, \pi_{\wedge,\max} - \pi_{\wedge,\max}\, \pi_{\mathcal{K}_{\wedge,\hat\lambda}}$$

satisfies the same estimate. Since $\|\pi_{\wedge,\max}\pi_{\mathcal{K}_{\wedge,\hat\lambda}}\|_{\mathscr{L}(\mathcal{D}_{\wedge,\max})} \le 1$, and using that $\pi_{\mathcal{K}_{\wedge,\hat\lambda},\mathcal{D}} = \pi_{\mathcal{K}_{\wedge,\hat\lambda},\mathcal{D}}\pi_{\wedge,\max}$, we obtain that if Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}}$, then

$$\|\pi_{\wedge,\max}\, \pi_{\mathcal{K}_{\wedge,\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\, \pi_{\wedge,\max}\|_{\mathscr{L}(\mathcal{D}_{\wedge,\max})}$$

is bounded for $\lambda \in \Lambda$, $|\lambda|$ large. This completes the proof of the theorem. □

Let $\mathcal{K}_{\wedge,\max}(\lambda) = \pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda}$. Let $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$, let $\lambda_0 \in \mathring{\Lambda}_\alpha$ be such that $|\lambda_0| = 1$, and let $R > 0$. The condition that

$$\varrho^m\lambda_0 \in \operatorname{res} A_{\wedge,\mathcal{D}} \text{ for } \varrho \ge R \tag{8.26}$$

is equivalent to the statement that $\mathcal{K}_{\wedge,\varrho^m\lambda_0} \cap \mathcal{D} = 0$ for $\varrho \ge R$, which in turn is equivalent to the condition that $\mathcal{K}_{\wedge,\lambda_0} \cap \kappa_\varrho^{-1}\mathcal{D} = 0$ for $\varrho \ge R$. Thus, since $\mathcal{K}_{\wedge,\lambda_0} \cap \kappa_\varrho^{-1}\mathcal{D} = 0$ if and only if $\pi_{\wedge,\max}\mathcal{K}_{\wedge,\lambda_0} \cap \pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} = 0$, the condition in (8.26) is equivalent to the statement that the curve γ defined by

$$[R,\infty) \ni \varrho \mapsto \gamma(\varrho) = \pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} \in \operatorname{Gr}_{d''_\alpha}(\mathcal{E}_{\wedge,\max}) \tag{8.27}$$

does not intersect the variety $\mathfrak{V}_{\mathcal{K}_{\wedge,\max}(\lambda_0)}$ introduced in Definition 5.22 (with $\mathcal{E}_{\wedge,\max}$ in place of $\mathcal{E}_{\max}$). With the proof of Lemma 5.21, $\pi_{\wedge,\max}\, \pi_{\mathcal{K}_{\wedge,\lambda_0},\kappa_\varrho^{-1}\mathcal{D}}\big|_{\mathcal{E}_{\wedge,\max}}$ is the projection on $\mathcal{K}_{\wedge,\max}(\lambda_0)$ according to the decomposition

$$\mathcal{K}_{\wedge,\max}(\lambda_0) \oplus \pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} = \mathcal{E}_{\wedge,\max}.$$

Thus if there is a neighborhood U of $\mathfrak{V}_{\mathcal{K}_{\wedge,\max}(\lambda_0)}$ in $\operatorname{Gr}_{d''_\alpha}(\mathcal{E}_{\wedge,\max})$ such that $\gamma(\varrho) \notin U$ if ϱ is sufficiently large, then Lemma 5.25 gives that

$$\|\pi_{\mathcal{K}_{\wedge,\max}(\lambda_0),\gamma(\varrho)}\| \le \frac{C}{\delta(\mathcal{K}_{\wedge,\max}(\lambda_0),\gamma(\varrho))}$$

is bounded as $\varrho \to \infty$. Therefore the necessary condition of Theorem 8.7 is satisfied, and we get:

Theorem 8.28. *Let $\lambda_0 \in \text{bg-res}\, A_\wedge$ belong to $\mathring{\Lambda}_\alpha$. Let $\mathcal{D} \in \mathfrak{G}_{\wedge,\alpha}$ and suppose that there is a neighborhood $U \subset \operatorname{Gr}_{d''_\alpha}(\mathcal{E}_{\wedge,\max})$ of $\mathfrak{V}_{\mathcal{K}_{\wedge,\max}(\lambda_0)}$ such that $\pi_{\wedge,\max}\kappa_\varrho^{-1}\mathcal{D} \notin U$ for all sufficiently large ϱ. Then there is a closed sector Λ containing λ_0 which is a sector of minimal growth for $A_{\wedge,\mathcal{D}}$.*

9. Resolvents

We will now prove the analogue of Theorem 8.7 for $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$.

Define $A_\varrho = \varrho^{-m}\kappa_\varrho^{-1}A\kappa_\varrho$. Then A_ϱ is c-elliptic, since A is assumed to be c-elliptic, cf. (3.5). Using that $A_\varrho - A$ belongs to $x^{-m+1}\operatorname{Diff}_b^m(M;E)$, [4, Proposition 4.1(1)] gives the first formula in

$$\mathcal{D}_{\min}(A_\varrho) = \mathcal{D}_{\min}(A), \quad \mathcal{D}_{\max}(A_\varrho) = \kappa_\varrho^{-1}\mathcal{D}_{\max}(A).$$

The second is obtained using that κ_ϱ preserves $C_0^\infty(\mathring{M};E)$ and $x^{-m/2}L_b^2(M;E)$. We will write $\mathcal{D}_{\varrho,\max}$ instead of $\mathcal{D}_{\max}(A_\varrho)$, and $\mathcal{D}_{\min}$ instead of $\mathcal{D}_{\min}(A_\varrho)$. Generally we prepend the symbol ϱ to subindices of objects associated with A_ϱ originally associated with A. In particular,

$$\mathcal{E}_{\varrho,\max} = \mathcal{D}_{\min}^\perp$$

with the orthogonal computed in $\mathcal{D}_{\varrho,\max}$ using the inner product

$$(u,v)_{A_\varrho} = (A_\varrho u, A_\varrho v) + (u,v)$$

of $\mathcal{D}_{\varrho,\max}$, and $\pi_{\varrho,\max} : \mathcal{D}_{\varrho,\max} \to \mathcal{D}_{\varrho,\max}$ is the orthogonal projection on $\mathcal{E}_{\varrho,\max}$. It is not hard to verify that

$$\mathcal{E}_{\varrho,\max} = \kappa_\varrho^{-1}[\ker(A^\star A + \varrho^{2m}) \cap \mathcal{D}_{\max}], \tag{9.1}$$

cf. Lemma 4.5.

Using

$$\varrho^{-m}\kappa_\varrho^{-1}(A - \varrho^m\lambda)\kappa_\varrho = A_\varrho - \lambda \tag{9.2}$$

we see that

$$\text{bg-res}\, A_\varrho = \varrho^{-m}\,\text{bg-res}\, A.$$

For $\lambda \in \text{bg-res}\, A_\varrho$ let $\mathcal{K}_{\varrho,\lambda} = \ker(A_{\varrho,\mathcal{D}_{\max}} - \lambda)$. Then $\mathcal{K}_{\varrho,\lambda/\varrho^m} = \kappa_\varrho^{-1}\mathcal{K}_\lambda$. If $\mathcal{D} \in \mathfrak{G}$, then $\kappa_\varrho^{-1}\mathcal{D} \in \mathfrak{G}_\varrho$, and if $\lambda \in \operatorname{res} A_{\mathcal{D}}$, then $\lambda/\varrho^m \in \operatorname{res} A_{\varrho,\kappa_\varrho^{-1}\mathcal{D}}$. It is easy to verify that $\mathcal{D}_{\varrho,\max} = \mathcal{K}_{\varrho,\lambda/\varrho^m} \oplus \kappa_\varrho^{-1}\mathcal{D}$ and that

$$\kappa_\varrho^{-1}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\kappa_\varrho = \pi_{\mathcal{K}_{\varrho,\lambda/\varrho^m},\kappa_\varrho^{-1}\mathcal{D}}. \tag{9.3}$$

Let $B_{\varrho,\min}(\lambda) = \varrho^m\kappa_\varrho^{-1}B_{\min}(\varrho^m\lambda)\kappa_\varrho$. This is a left inverse of $A_{\varrho,\mathcal{D}_{\min}} - \lambda$. The operator $B_{\varrho,\min}(\lambda)$ has range in $\mathcal{D}_{\min}$ since this subspace is κ-invariant and the range of $B_{\min}(\varrho^m\lambda)$ is $\mathcal{D}_{\min}$.

Theorem 9.4. *Let $\mathcal{D} \in \mathfrak{G}$ and let Λ be a closed sector. Then Λ is a sector of minimal growth for $A_{\mathcal{D}}$ if and only if there are positive constants C, R such that $\Lambda_R \subset \operatorname{res} A_{\mathcal{D}}$,*

$$\|B_{\min}(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2)} \le C/|\lambda|, \quad \|B_{\max}(\lambda)\|_{\mathscr{L}(x^{-m/2}L_b^2)} \le C/|\lambda|, \tag{9.5}$$

and

$$\|\pi_{|\lambda|^{1/m},\max}\pi_{\mathcal{K}_{|\lambda|^{1/m},\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\big|_{\mathcal{E}_{|\lambda|^{1/m},\max}}\|_{\mathscr{L}(\mathcal{D}_{|\lambda|^{1/m},\max})} \le C, \quad \lambda \in \Lambda_R. \tag{9.6}$$

The proof requires a number of analogues of results obtained in the previous section. Their proofs parallel those in that section.

Lemma 9.7. *Let Λ be some closed sector, let $R > 0$, and let*

$$P(\lambda) : x^{-m/2}L^2_b(M;E) \to \mathcal{D}_{\max}$$

be a family of operators defined for $\lambda \in \Lambda_R$. Then

$$\|P(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \leq C/|\lambda| \quad \textit{and} \quad \|P(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{\max})} \leq C \tag{9.8}$$

hold for some $C > 0$ and all $\lambda \in \Lambda_R$ if and only if

$$\|\kappa^{-1}_{|\lambda|^{1/m}} P(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{|\lambda|^{1/m},\max})} \leq C/|\lambda| \tag{9.9}$$

holds for some $C > 0$ and all $\lambda \in \Lambda_R$.

Proof. Using that $A_{|\lambda|^{1/m}}\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda) = |\lambda|^{-1}\kappa^{-1}_{|\lambda|^{1/m}}AP(\lambda)$, and that $\kappa^{-1}_{|\lambda|^{1/m}}$ is an isometry in $x^{-m/2}L^2_b(M;E)$, we obtain

$$\begin{aligned}\|\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|^2_{A_{|\lambda|^{1/m}}} &= \|A_{|\lambda|^{1/m}}\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|^2 + \|\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|^2\\ &= |\lambda|^{-2}\|\kappa^{-1}_{|\lambda|^{1/m}}A_{|\lambda|^{1/m}}P(\lambda)f\|^2 + \|\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|^2\\ &= |\lambda|^{-2}\|A_{|\lambda|^{1/m}}P(\lambda)f\|^2 + \|P(\lambda)f\|^2\end{aligned}$$

if $f \in x^{-m/2}L^2_b(M;E)$. Thus (9.9) follows from (9.8).

Assume now that (9.9) holds and let $f \in x^{-m/2}L^2_b(M;E)$. Then

$$\|P(\lambda)f\| = \|\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\| \leq \|\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|_{A_{|\lambda|^{1/m}}}$$

gives the first estimate in (9.8). To obtain the second, write $\|P(\lambda)f\|^2_A$ as

$$\|AP(\lambda)f\|^2 + \|P(\lambda)f\|^2 = \|\kappa^{-1}_{|\lambda|^{1/m}}AP(\lambda)f\|^2 + \|\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|^2$$

and use the definition of $A_{|\lambda|^{1/m}}$ to conclude that

$$\begin{aligned}\|P(\lambda)f\|^2_A = |\lambda|^2\|A_{|\lambda|^{1/m}}\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|^2 + \|\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|^2&\\ \leq (|\lambda|^2+1)\|\kappa^{-1}_{|\lambda|^{1/m}}P(\lambda)f\|^2_{A_{|\lambda|^{1/m}}}&.\end{aligned}$$

The second estimate in (9.8) follows from this. ☐

Corollary 9.10. *Let $\mathcal{D} \in \mathfrak{G}$, let Λ be a closed sector. Then Λ is a sector of minimal growth for $A_{\mathcal{D}}$ if and only if there are positive constants C, R such that $\Lambda_R \subset \operatorname{res} A_{\mathcal{D}}$ and*

$$\|\kappa^{-1}_{|\lambda|^{1/m}}B_{\mathcal{D}}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{|\lambda|^{1/m},\max})} \leq C/|\lambda|, \quad \lambda \in \Lambda_R. \tag{9.11}$$

Proof of Theorem 9.4: Sufficiency of the condition. We will show that (9.5) and (9.6) imply (9.11). Since $B_{\min}(\lambda)$ and $B_{\max}(\lambda)$ satisfy the estimate in (9.5), and since these estimates imply for each of them the second estimate in (9.8), we obtain that $\kappa^{-1}_{|\lambda|^{1/m}}B_{\min}(\lambda)$ and $\kappa^{-1}_{|\lambda|^{1/m}}B_{\max}(\lambda)$ both satisfy (9.9). In particular, to prove (9.11) we only need to prove that for some C,

$$\|\kappa^{-1}_{|\lambda|^{1/m}}\big(B_{\mathcal{D}}(\lambda) - B_{\max}(\lambda)\big)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{|\lambda|^{1/m},\max})} \leq C/|\lambda|, \quad \lambda \in \Lambda_R.$$

Writing $B_{\mathcal{D}}(\lambda)$ as in (5.19) we get

$$\kappa^{-1}_{|\lambda|^{1/m}}\big(B_{\max}(\lambda) - B_{\mathcal{D}}(\lambda)\big) = \kappa^{-1}_{|\lambda|^{1/m}}\big(I - B_{\min}(\lambda)(A-\lambda)\big)\pi_{\mathcal{K}_\lambda,\mathcal{D}}B_{\max}(\lambda).$$

We rewrite the right hand side as

$$\big(I - B_{|\lambda|^{1/m},\min}(\hat\lambda)(A_{|\lambda|^{1/m}} - \hat\lambda)\big)\kappa^{-1}_{|\lambda|^{1/m}}\,\pi_{\mathcal{K}_\lambda,\mathcal{D}}\,\kappa_{|\lambda|^{1/m}}\kappa^{-1}_{|\lambda|^{1/m}}B_{\max}(\lambda). \tag{9.12}$$

Using that $I - B_{|\lambda|^{1/m},\min}(\hat\lambda)(A_{|\lambda|^{1/m}} - \hat\lambda)$ vanishes on $\mathcal{D}_{\min}$, the identity (9.3), and that $\mathcal{D}_{\min} \subset \kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}$ we replace the factor $\kappa^{-1}_{|\lambda|^{1/m}}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\kappa_{|\lambda|^{1/m}}$ in (9.12) by

$$\pi_{|\lambda|^{1/m},\max}\,\pi_{\mathcal{K}_{|\lambda|^{1/m},\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\,\pi_{|\lambda|^{1/m},\max}.$$

By hypothesis the norms of these operators $\mathcal{D}_{|\lambda|^{1/m},\max} \to \mathcal{E}_{|\lambda|^{1/m},\max}$ are uniformly bounded when $\lambda \in \Lambda_R$. It is easy to verify that the norm of

$$I - B_{|\lambda|^{1/m},\min}(\hat\lambda)(A_{|\lambda|^{1/m}} - \hat\lambda) : \mathcal{D}_{|\lambda|^{1/m},\max} \to \mathcal{D}_{|\lambda|^{1/m},\max}$$

is bounded independently of λ, $\lambda \in \Lambda_R$. Finally, as already discussed,

$$\|\kappa^{-1}_{|\lambda|^{1/m}} B_{\max}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{|\lambda|^{1/m},\max})} \leq C/|\lambda|$$

holds for some $C > 0$ and all $\lambda \in \Lambda_R$. Altogether these estimates give (9.11). □

To prove the necessity of the condition in Theorem 9.4 we will need two lemmas.

Lemma 9.13. *Suppose that $\mathcal{D} \in \mathfrak{G}$ and that the closed sector Λ is a sector of minimal growth for $A_\mathcal{D}$. Then there are positive constants R and C such that*

$$\|B_{\min}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \leq C/|\lambda|, \quad \|B_{\max}(\lambda)\|_{\mathscr{L}(x^{-m/2}L^2_b)} \leq C/|\lambda|$$

for $\lambda \in \Lambda_R$.

This is a direct consequence of the formulas

$$B_{\min}(\lambda) = B_\mathcal{D}(\lambda)\pi_{\mathcal{R}_\lambda}, \quad B_{\max}(\lambda) = B_\mathcal{D}(\lambda) - \pi_{\mathcal{K}_\lambda}B_\mathcal{D}(\lambda),$$

cf. (5.13) and (5.15) valid for $\lambda \in \operatorname{res} A_\mathcal{D}$.

Lemma 9.14. *Let $\mathfrak{p}_{\mathcal{K}_{\varrho,\lambda}} : x^{-m/2}L^2_b(M;E) \to x^{-m/2}L^2_b(M;E)$ be the orthogonal projection on $\mathcal{K}_{\varrho,\lambda}$ regarded as a subspace of $x^{-m/2}L^2_b(M;E)$. Then*

$$\kappa_\varrho^{-1}\pi_{\mathcal{K}_{\varrho^m\lambda}}\kappa_\varrho = \frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\varrho^{2m}\pi_{\mathcal{K}_{\varrho,\lambda}} + \frac{1-\varrho^{2m}}{1+|\varrho^m\lambda|^2}\mathfrak{p}_{\mathcal{K}_{\varrho,\lambda}}. \tag{9.15}$$

Moreover, $\|\mathfrak{p}_{\mathcal{K}_{\varrho,\lambda}}\|_{\mathscr{L}(\mathcal{D}_{\varrho,\max})} \leq \sqrt{1+|\lambda|^2}$.

Proof. The proof of (9.15) parallels that of Lemma 8.16. Let $\phi_1,\dots,\phi_{d'}$ be an A_ϱ-orthonormal basis of $\mathcal{K}_{\varrho,\lambda} = \kappa_\varrho^{-1}\mathcal{K}_{\varrho^m\lambda}$. Then

$$\delta_{jk} = (\phi_j,\phi_k)_{A_\varrho} = (1+|\lambda|^2)(\phi_j,\phi_k).$$

In particular, the $\sqrt{1+|\lambda|^2}\phi_j \in \mathcal{K}_{\varrho,\lambda}$ are orthonormal in $x^{-m/2}L^2_b(M;E)$. On the other hand, using that κ_ϱ is an isometry on $x^{-m/2}L^2_b(M;E)$,

$$(\kappa_\varrho\phi_j,\kappa_\varrho\phi_k)_A = \varrho^{2m}(\phi_j,\phi_k)_{A_\varrho} + (1-\varrho^{2m})(\phi_j,\phi_k) = \frac{1+|\varrho^m\lambda|^2}{1+|\lambda|^2}\,\delta_{jk}.$$

This gives an A-orthonormal basis of $\mathcal{K}_{\varrho^m\lambda}$, and if $u \in \mathcal{K}_{\varrho,\lambda}$, then

$$\begin{aligned}
\pi_{\mathcal{K}_{\varrho^m\lambda}}\kappa_\varrho u &= \frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\sum_j(\kappa_\varrho u,\kappa_\varrho\phi_j)_A\kappa_\varrho\phi_j \\
&= \frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\sum_j\left[\varrho^{2m}(u,\phi_j)_{A_\varrho} + (1-\varrho^{2m})(u,\phi_j)\right]\kappa_\varrho\phi_j \\
&= \kappa_\varrho\big(\frac{1+|\lambda|^2}{1+|\varrho^m\lambda|^2}\varrho^{2m}\pi_{\mathcal{K}_{\varrho,\lambda}}u + \frac{1-\varrho^{2m}}{1+|\varrho^m\lambda|^2}\mathfrak{p}_{\mathcal{K}_{\varrho,\lambda}}u\big).
\end{aligned}$$

Thus (9.15) follows. The estimate of the norm of $\mathfrak{p}_{\mathcal{K}_{\varrho,\lambda}}$ is elementary. □

Proof of Theorem 9.4: Necessity of the condition. Suppose that Λ is a sector of minimal growth for $A_{\mathcal{D}}$. By Lemma 9.13, (9.5) holds. In particular there are C, R such that the operator

$$\kappa^{-1}_{|\lambda|^{1/m}}(B_{\max}(\lambda)-B_{\mathcal{D}}(\lambda))=\kappa^{-1}_{|\lambda|^{1/m}}\big(I-B_{\min}(\lambda)(A-\lambda)\big)\pi_{\mathcal{K}_\lambda,\mathcal{D}}B_{\max}(\lambda)$$

as an element of $\mathscr{L}(x^{-m/2}L^2_b,\mathcal{D}_{|\lambda|^{1/m},\max})$ has norm bounded by $C/|\lambda|$ if $\lambda\in\Lambda_R$. Composing with $\pi_{|\lambda|^{1/m},\max}$ on the left, and using that $\kappa^{-1}_{|\lambda|^{1/m}}$ preserves $\mathcal{D}_{\min}$, we conclude that

$$\pi_{|\lambda|^{1/m},\max}\,\kappa^{-1}_{|\lambda|^{1/m}}\big(I-B_{\min}(\lambda)(A-\lambda)\big)\pi_{\mathcal{K}_\lambda,\mathcal{D}}B_{\max}(\lambda)\\ =\pi_{|\lambda|^{1/m},\max}\,\kappa^{-1}_{|\lambda|^{1/m}}\,\pi_{\mathcal{K}_\lambda,\mathcal{D}}B_{\max}(\lambda)$$

satisfies the same estimate. The operator

$$(A-\lambda)\kappa_{|\lambda|^{1/m}}=|\lambda|\kappa_{|\lambda|^{1/m}}(A_{|\lambda|^{1/m}}-\hat\lambda),$$

as an element of $\mathscr{L}(\mathcal{D}_{|\lambda|^{1/m},\max},x^{-m/2}L^2_b)$, has norm bounded by $2|\lambda|$, $\lambda\neq 0$. Thus the operator

$$\pi_{|\lambda|^{1/m},\max}\,\kappa^{-1}_{|\lambda|^{1/m}}\,\pi_{\mathcal{K}_\lambda,\mathcal{D}}B_{\max}(\lambda)(A-\lambda)\kappa_{|\lambda|^{1/m}}\\ =\pi_{|\lambda|^{1/m},\max}\,\kappa^{-1}_{|\lambda|^{1/m}}\,\pi_{\mathcal{K}_\lambda,\mathcal{D}}(I-\pi_{\mathcal{K}_\lambda})\kappa_{|\lambda|^{1/m}},$$

as an element of $\mathscr{L}(\mathcal{D}_{|\lambda|^{1/m},\max})$, has norm bounded by a constant independent of $\lambda\in\Lambda_R$. Since $\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\mathcal{K}_\lambda}=\pi_{\mathcal{K}_\lambda}$,

$$\kappa^{-1}_{|\lambda|^{1/m}}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\mathcal{K}_\lambda}\kappa_{|\lambda|^{1/m}}=\frac{2|\lambda|^2}{1+|\lambda|^2}\pi_{\mathcal{K}_{|\lambda|^{1/m},\hat\lambda}}+\frac{1-|\lambda|^2}{1+|\lambda|^2}\mathfrak{p}_{\mathcal{K}_{|\lambda|^{1/m},\hat\lambda}}$$

using (9.15). Thus

$$\|\pi_{|\lambda|^{1/m},\max}\kappa^{-1}_{|\lambda|^{1/m}}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\mathcal{K}_\lambda}\kappa_{|\lambda|^{1/m}}\|_{\mathscr{L}(\mathcal{D}_{|\lambda|^{1/m},\max})}\le C,\quad \lambda\in\Lambda_R,$$

for some C and consequently also

$$\|\pi_{|\lambda|^{1/m},\max}\kappa^{-1}_{|\lambda|^{1/m}}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\kappa_{|\lambda|^{1/m}}\|_{\mathscr{L}(\mathcal{D}_{|\lambda|^{1/m},\max})}\le C,\quad \lambda\in\Lambda_R,$$

for some other C. Using (9.3) we conclude that in particular

$$\|\pi_{|\lambda|^{1/m},\max}\pi_{\mathcal{K}_{|\lambda|^{1/m},\hat\lambda},\kappa^{-1}_{|\lambda|^{1/m}}\mathcal{D}}\big|_{\mathcal{E}_{|\lambda|^{1/m},\max}}\|_{\mathscr{L}(\mathcal{D}_{|\lambda|^{1/m},\max})}\le C,\quad \lambda\in\Lambda_R,$$

This completes the proof of the necessity of the condition. ◻

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