Lorentz invariance relations between parton distributions and the Wandzura–Wilczek approximation

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1. Introduction

Parton distribution functions (PDFs) which are of higher twist and/or which are transverse momentum dependent (p T-dependent) contain important information on the partonic structure of the nucleon being complementary to that encoded in the usual twist-2 distributions. These PDFs also become more important because of the increasing accuracy of recent and planned high energy scattering experiments. The forward twist-3 PDFs are accessible through certain spin asymmetries in polarized inclusive deep inelastic lepton-nucleon scattering (DIS) and Drell–Yan processes (integrated upon the transverse momentum of the dilepton pair) [1–8]. On the other hand the p T-dependent PDFs typically give rise to spin and azimuthal asymmetries in, for instance, semi-inclusive DIS [9–13] and Drell–Yan [14–17], and significant effort has already been devoted to measure such observables [18–30].

Several relations between (forward) twist-3 and (moments of) p T-dependent PDFs have been proposed in the literature [2,9,10]. The derivation of these relations is based upon the general, Lorentz invariant decomposition of the fully unintegrated correlator of two quark-fields, where the fields are located at arbitrary space–time positions. These so-called Lorentz invariance relations (LIRs) impose important constraints on the PDFs which may allow one to eliminate unknown PDFs in favor of the known ones whenever applicable. However, in Refs. [31,32] it was demonstrated by an explicit model calculation that two specific LIRs are actually violated. In [33] it was shown in a model independent way that the violation can be traced back to the path-ordered exponential in the unintegrated quark–quark correlator. For completeness we will repeat the argument below.

In the present Letter we address the question to what extent the LIRs may be violated numerically. To this end it is outlined in a model independent way that the LIRs are actually not violated in a generalized Wandzura–Wilczek approximation, indicating that numerically their violation may be small.

2. Lorentz invariance relations and their violation

In order to discuss the LIRs and their violation we start with the fully unintegrated quark–quark correlation function of a spin-1/2 hadron defined by

\[ \Phi_{ij}(P, p, S | n_−) = \int \frac{d^4ξ}{(2π)^4} e^{ip_ξ} \langle P, S | \bar{ψ}_j(0) V(0, ξ| n_−) ψ_i(ξ) | P, S \rangle. \]  

(1)

The target state is characterized by its four-momentum \( P = P^+ n_+ + (M^2/2P^+) n_− \) and the covariant spin vector \( S (P^2 = M^2, S^2 = −1, P \cdot S = 0) \), with the two light-like vectors \( n_+ \) and \( n_− \) satisfying \( n_+^2 = n_−^2 = 0 \) and \( n_+ \cdot n_− = 1 \). The momentum of the quark is denoted by \( p \). The Wilson line \( V(0, ξ| n_−) \) ensures color gauge invariance of the correlator, where the specific path of the gauge link will be given below. The knowledge of the correlator in

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Footnote: Recent work on how the unintegrated quark–quark correlator may enter observables can be found in [34,35].
Eq. (1) is particularly useful for obtaining the general form of the $p_T$-dependent correlator $\Phi(x, p_T, S)$, which appears in the QCD-description of hard scattering processes like transverse momentum dependent semi-inclusive DIS and the Drell–Yan reaction. The connection between both objects is given by the relation
\begin{equation}
\Phi(x, p_T, S) = \int dp^- \Phi(P, p, S|n-),
\end{equation}
with $x$ defining the plus-momentum of the quark via $p^+ = xp^+$. The Wilson line in Eq. (1) can be fixed according to
\[ \Lambda \{0, \xi [n-] = [0, 0, 0; 0, \infty, 0, \infty, 0, \infty, \xi_T] \times [\xi^+, \infty, \xi_T, \xi^+, \infty, \xi_T] \],\]
where $[a^+, a^-, a_T; b^+, b_T, b_T]$ denotes a gauge link connecting the points $a^\mu = (a^+, a^-, a_T)$ and $b^\mu = (b^+, b^-, b_T)$ along a straight line. It is important to note that the Wilson contour in Eq. (3) not only depends on the coordinates of the initial and final points but also on the light-cone direction $n_\perp$. The path is chosen such that, upon integration over the minus-momentum of the quark, it leads to a proper definition of the $p_T$-dependent correlator in (2) [36–38]. The choice of the contour depends on the process under consideration [39]. Here we restrict ourselves to the case of semi-inclusive DIS, but all arguments hold as well for other processes like Drell–Yan. It is also worthwhile to mention that Wilson lines that are near the light-cone rather than exactly light-like (as those in (3)) are generally more appropriate in connection with unintegrated parton correlation functions. (More details on this important issue can be found in the recent work [40] and references therein.) Our general reasoning here remains valid if one uses a near light-cone direction instead of $n_\perp$.

The general structure of the correlator in (1) was derived in [41], and the result is also given in Eq. (28) in Appendix A. One ends up with 32 matrix structures multiplied by scalar functions that were denoted by $A_i$ and $B_i$ in [41]. In turn, the $p_T$-dependent PDFs can be defined through Dirac traces of the $p_T$-dependent correlator in (2) by [9,12,41]
\[ \phi^{[\Gamma]}(x, p_T, S) = \frac{1}{2} \text{Tr} \{ \Phi(x, p_T, S) \Gamma \}. \]

The results containing all the twist-2 and twist-3 PDFs are repeated in Appendix A, Eqs. (29)–(37). On the basis of the relation (2) one can now express the $p_T$-dependent PDFs through $p_T$ integrals upon the scalar functions $A_i$ and $B_i$. The results are listed in Appendix A, Eqs. (40)–(63).

In total there exist 32 $p_T$-dependent PDFs which exactly agrees with the number of independent amplitudes $A_i$ and $B_i$. If one neglects the dependence on the light-cone vector $n_\perp$, which is induced by the Wilson line, the correlator (1) merely consists of 12 matrix structures — those which are multiplyed by the functions $A_i$. In that case the number of $p_T$-dependent PDFs is larger than the number of the amplitudes $A_i$. This feature, in particular, gives rise to LIRs between certain $p_T$-integrated PDFs and (moments of) $p_T$-dependent PDFs [2,9,10]. Here we list the most important four LIRs on which we focus in this work:

\begin{equation}
to be small [57] such that

\begin{equation}
g_{1T}(x) = \int \frac{dy}{y^2} g_1(x) + \tilde{g}_{1T}(x),
\end{equation}

where $\tilde{g}_{1T}(x)$ and $\tilde{h}_{1T}(x)$ denote (purely interaction dependent) quark–gluon–quark correlations and terms proportional to current quark masses. An explicit representation of these terms can be found, e.g., in [44] and partly also in [45], Eqs. (10) and (11) isolate “pure twist-3 terms” in the PDFs $g_1(x)$ and $h_1(x)$. Here the underlying “working definition” of twist [46] (a PDF is of “twist t” if its contribution to the cross section is suppressed, in addition to kinematic factors, by $1/Q^{t-2}$ with $Q$ denoting the hard scale of the process) differs from the strict definition of twist (mass dimension of the operator minus its spin).

The remarkable experimental observation is that $\tilde{g}_{1T}(x)$ is consistent with zero within the error bars [3–8] and to good accuracy one has

\begin{equation}
g_{1T}(x) \approx \int \frac{dy}{y} g_1(x),
\end{equation}

which is the Wandzura–Wilczek (WW) approximation. Lattice QCD [47,48] and the instanton model of the QCD vacuum [49] support this observation. (Further discussions of the WW approximation in related and other contexts can be found in Refs. [50–56].) Interestingly, the latter predicts also $\tilde{h}_{1T}(x)$ to be small [57] such that
An experimental test of this approximation relation does not yet exist. (In this context see also the recent theoretical study in Ref. [58].)

Now it is possible to show that the LIRs in Eqs. (5) and (6) are not violated if one generalizes the WW approximation. For this purpose we consider the following exact relations [9,12] originating from the QCD equations of motion (EOM):

\[ g_{1T}^{(1)}(x) \approx 2x \int_y^1 \frac{dy}{y^2} g_1(y), \]

(13)

\[ h_{1L}^{\perp(1)}(x) \approx - x^2 \int_y^1 \frac{dy}{y^2} h_1(y), \]

(21)

and the WW-type approximation one finds

\[ \Delta g(x) \approx - g_1 - x \frac{d}{dx} \int_y^1 \frac{dy}{y} g_1(y) = 0, \]

(18)

\[ \Delta h(x) \approx - h_1 - x^2 \frac{d}{dx} \int_y^1 \frac{dy}{y^2} h_1(y) = 0. \]

(19)

Eqs. (18) and (19) show that the LIRs (7) and (8) which contain T-odd PDFs the situation is slightly different and in principle even simpler. Due to time-reversal invariance the \( p_T \)-integrated T-odd PDFs \( f_T(x) \) and \( h(x) \) vanish [12,41],

\[ f_T(x) = \int d^2 p_T f_T(x, p_T^2) = 0, \]

(22)

\[ h(x) = \int d^2 p_T h(x, p_T^2) = 0. \]

(23)

which implies, considering the LIRs in Eqs. (7) and (8), that

\[ \frac{d}{dx} f_{1T}^{\perp(1)}(x) \rightarrow 0, \]

(24)

\[ \frac{d}{dx} h_{1L}^{\perp(1)}(x) \rightarrow 0. \]

(25)

This means that \( f_{1T}^{\perp(1)}(x) \) and \( h_{1L}^{\perp(1)}(x) \) are constants. In fact, since these moments have to vanish for \( x = 1 \), one can conclude that they should vanish for the entire x-range. So far we did not use any approximation and only assumed that the LIRs (7) and (8) are not violated. Now let us explore the EOMs [12] which, keeping in mind (22) and (23), imply (see also Refs. [38,67,68])

\[ f_{1T}^{\perp(1)}(x) \approx x f_T(x), \]

(26)

\[ h_{1L}^{\perp(1)}(x) \approx - \frac{1}{2} h_1(x). \]

(27)

In the WW-type approximation the tilde-functions are set to zero. It then follows directly from Eqs. (26) and (27) that \( f_{1T}^{\perp(1)}(x) \) and \( h_{1L}^{\perp(1)}(x) \)
are zero (see also [12]). This is consistent with the results following from the LIRs (24) and (25). So the LIRs (7) and (8) are also not violated in the WW-type approximation.

Also for the $T$-odd functions we already have some phenomenological input on the status of the WW-type approximation. Since a nonzero asymmetry, typically attributed to the Sivers effect, was found in the HERMES experiment [20,22,24], in the case of $T$-odd PDFs this approximation seems to be violated. On the other hand the observed effect is not very large (of the order of few percent), and one should not expect WW-type approximations to work to a much better accuracy than that. Moreover, the Sivers effect studied at COMPASS is compatible with zero both for a deuteron as well as a proton target [21,23,25,26]. Therefore, the current experimental situation is not in conflict with a rather small $f_1$ in Eq. (26).

4. Summary

We have studied LIRs between parton distributions, known to be violated in general, with the aim to understand how strong this violation might be. It was found that LIRs are satisfied in a generalized WW approximation in which one systematically neglects certain quark–gluon–quark correlations as well as quark mass terms. That would mean that LIRs could provide useful approximations for unknown PDFs whenever applicable. Our approximation goes beyond the successful “standard WW approximation” quoted in Eqs. (12) and (13). In particular, we also neglected purely interaction dependent terms which show up in relations originating from the QCD equations of motion (see also Ref. [12]). We argued that there exists experimental evidence for the validity of the generalized WW approximation. On the other hand more (precise) data and tests are needed before a final conclusion can be reached. Only forthcoming data analyses and experiments at COMPASS, HERMES, and Jefferson Lab can ultimately reveal to what extent the generalized WW approximation (and the LIRs) provide useful approximations. Eventually, it is likely that the quality of the approximation depends on the particular case (function) under consideration.

Note added

After completion of our work the manuscript [45] appeared where, on the basis of the present data for the DIS structure function $g_2$, a violation of the WW-relation in Eq. (12) of the order 15–40% has been reported. However, the authors of [45] also point out that more data are needed to ultimately settle the situation. In any case, if generalized WW-relations were valid within a similar accuracy, they would constitute helpful tools at the present stage for phenomenological studies of first data.

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Appendix A. Quark–quark correlators and the Lorentz invariance relations

For completeness we give here the general structure of the fully unintegrated quark–quark correlator in Eq. (1), the $p_T$-dependent quark–quark correlator (expressed in terms of $p_T$-dependent PDFs) in Eq. (2), the relations between PDFs and the amplitudes $A_i$, $B_i$ which parameterize the correlator in (1), as well as a brief account on how to derive the LIRs in the case that the amplitudes $B_i$ are absent (like in a non-gauge theory).

The fully unintegrated quark–quark correlator (1) can be decomposed according to [41]

$$
\Phi(P, p, S|n_\perp) = MA_1 + iPA_2 + pA_3 + \frac{i}{2M}[P, p]A_4 + i(p \cdot S)\gamma_5 A_5 + M \frac{\gamma_5 A_6}{2} + \frac{p \cdot S}{M} \gamma_5 A_7 + \frac{p \cdot S}{M} \gamma_5 A_8 + \frac{[p, S]}{2M^2}\gamma_5 A_{11} + \frac{1}{M} \epsilon^{\mu \nu \rho \sigma} \gamma_\mu p_\nu p_\rho S_\sigma A_{12} + \frac{M^2}{p \cdot n_\perp} n_\perp B_1 + iM \frac{\gamma_5 A_{10} + \frac{p \cdot S}{M^2}[P, p] \gamma_5 A_{11}}{2} + \frac{1}{M} \frac{\epsilon^{\mu \nu \rho \sigma} \gamma_\mu p_\nu p_\rho n_\perp \sigma B_4}{2p \cdot n_\perp} + \frac{1}{M} \frac{\epsilon^{\mu \nu \rho \sigma} \gamma_\mu p_\nu p_\rho n_\perp \sigma B_5}{2p \cdot n_\perp} + \frac{1}{M} \frac{\epsilon^{\mu \nu \rho \sigma} \gamma_\mu p_\nu p_\rho n_\perp \sigma B_7}{2p \cdot n_\perp} + \frac{1}{M} \frac{\epsilon^{\mu \nu \rho \sigma} \gamma_\mu p_\nu p_\rho n_\perp \sigma B_8}{2p \cdot n_\perp} + \frac{p \cdot S}{M} \frac{\gamma_5 A_{10}}{P \cdot n_\perp} + \frac{M(n \cdot S)}{2P \cdot n_\perp} \gamma_5 B_{11} + \frac{M}{2P \cdot n_\perp} \frac{(n \cdot S)p \gamma_5 B_{12}}{2P \cdot n_\perp} + \frac{M^2}{2P \cdot n_\perp} \frac{(n \cdot S)\gamma_5 B_{13}}{2P \cdot n_\perp} + \frac{M^2}{2P \cdot n_\perp} \frac{(n \cdot S)\gamma_5 B_{14}}{2P \cdot n_\perp} + \frac{M^2}{2P \cdot n_\perp} \frac{(n \cdot S)\gamma_5 B_{15}}{2P \cdot n_\perp} + \frac{p \cdot S}{2P \cdot n_\perp} \frac{[P, n_\perp] \gamma_5 B_{16}}{2P \cdot n_\perp} + \frac{p \cdot S}{2P \cdot n_\perp} \frac{[P, n_\perp] \gamma_5 B_{17}}{2P \cdot n_\perp} + \frac{p \cdot S}{2P \cdot n_\perp} \frac{[P, n_\perp] \gamma_5 B_{18}}{2P \cdot n_\perp} + \frac{M^2}{2P \cdot n_\perp} \frac{(n \cdot S)\gamma_5 B_{19}}{2P \cdot n_\perp} + \frac{M^2}{2P \cdot n_\perp} \frac{(n \cdot S)\gamma_5 B_{20}}{2P \cdot n_\perp}.
$$

where the convention $\epsilon^{0123} = 1$ is understood. The scalar amplitudes depend on the available kinematical invariants. Note that all the $B_i$‘s are associated with matrix structures containing the light-like vector $n_\perp$.

The $p_T$-dependent correlator in Eq. (2) can be specified by all possible Dirac traces $\phi^{1/2}$ defined in Eq. (4), which in turn are parameterized through $p_T$-dependent PDFs. A list of all traces was given in Refs. [12,41]. Limiting oneself to twist-2 and twist-3 effects one has (in the conventions of [12]):

$$
\phi[\gamma^1](x, p_T, S) = f_1(x, p_T^2) - \frac{\epsilon^{\mu \nu \rho \sigma}}{M} p_\mu T^\rho_1 S_\sigma T^\nu_1 f_1(x, p_T^2),
$$

$$
\phi[\gamma^1 \gamma_5](x, p_T, S) = \lambda^\mu \gamma_5(x, p_T^2) + \frac{p_T \cdot S}{M} g_{11}(x, p_T^2),
$$

which is (29) and (30).
\[ \phi^{[\alpha^+\lambda^+]}(x, p_T, S) = S_1^+ h_1(x, p_T^2) + \lambda \frac{p_T^+}{M} h_{1\perp}(x, p_T^2) \]
\[ - \frac{p_T^+ p_T^\perp + \frac{1}{2} p_T^2 g_{ij} P_T^i S_T^j h_1(x, p_T^2)}{M^2} - \frac{\epsilon_T P_T h_1(x, p_T^2)}{M}. \]

\[ \phi^{[\alpha^+\lambda^+]}(x, p_T, S) = M \frac{p_T^+}{P_T^+} \left[ - \frac{e_T^j S_T f_T^i(x, p_T^2)}{M} - \frac{e_T^j P_T f_T^i(x, p_T^2)}{M} + \frac{1}{2} p_T^2 g_{ij} S_T^j f_T^i(x, p_T^2) \right]. \]

\[ \phi^{[\lambda^+\lambda^+]}(x, p_T, S) = M \frac{p_T^+}{P_T^+} \left[ \frac{S_T^i g_T(x, p_T^2)}{M} + \frac{p_T^+}{M} g_{i\perp}(x, p_T^2) \right] - \frac{p_T^+ p_T^\perp + \frac{1}{2} p_T^2 g_{ij} S_T^j g_T^i(x, p_T^2)}{M^2} - \frac{\epsilon_T P_T g_{i\perp}(x, p_T^2)}{M}. \]

\[ \phi^{[\alpha^+\lambda^+\lambda^+]}(x, p_T, S) = M \frac{p_T^+}{P_T^+} \left[ \lambda h_1(x, p_T^2) + \frac{p_T^+}{M} S_T h_1(x, p_T^2) \right]. \]

Here use has been made of the Sudakov decompositions

\[ S = \lambda \frac{P^+}{M} n_+ - \lambda \frac{M}{2P^+} n_- + S_T, \]

\[ p = x P^+ n_+ + p n_- + p_T, \]

and of \( \epsilon_T^j = e^{-\epsilon_T^j}. \)

On the basis of the relation in (2) it is straightforward to express the \( p_T^\perp \)-dependent PDFs through the amplitudes \( A_i \) and \( B_i \). We again restrict ourselves to the twist-2 and twist-3 case for which we obtain

\[ f_1(x, p_T^2) = 2P^+ \int dp^- \left(-A_2 + x A_3 \right), \]

\[ f_1^\perp(x, p_T^2) = 2P^+ \int dp^- A_{12}. \]

\[ g_{11}(x, p_T^2) = 2P^+ \int dp^- \left(-A_6 + \frac{P \cdot p - M^2 x}{M^2} (A_7 + x A_8) \right) \]
\[ - B_{11} - x B_{12} \right), \]

\[ g_{17}(x, p_T^2) = 2P^+ \int dp^- \left(-A_7 + x A_8 \right). \]

\[ h_1(x, p_T^2) = 2P^+ \int dp^- \left(-A_9 + x A_{10} + \frac{p_T^2}{2M^2} A_{11} \right), \]

\[ h_{1\perp}(x, p_T^2) = 2P^+ \int dp^- \left(A_{10} - \frac{P \cdot p - M^2 x}{M^2} A_{11} - B_{18} \right), \]

\[ h_{1\perp}^\perp(x, p_T^2) = 2P^+ \int dp^- A_{11}. \]

\[ h_1^\perp(x, p_T^2) = 2P^+ \int dp^- \left(-A_9 - \frac{p_T^2}{2M^2} A_{11} + \left(\frac{P \cdot p - M^2 x}{M^2}\right)^2 A_{11} \right) \]
\[ - B_{15} + \frac{P \cdot p - M^2 x}{M^2} B_{16} + \frac{P \cdot p - M^2 x}{M^2} x B_{17} \]
\[ + \left(\frac{P \cdot p - M^2 x}{M^2}\right)^2 B_{18} + B_{19} + x B_{20} \right). \]

\[ h_{1\perp}(x, p_T^2) = 2P^+ \int dp^- \left(-A_9 - \frac{p_T^2}{2M^2} A_{11} - B_{16} - x B_{17} \right). \]

In the following we give a brief account on how to derive the LIRs in the case that the amplitudes \( B_i \) are absent using as an example the LIR in Eq. (8). (See also the original references [2,9, 10].) Starting from (60) one gets

\[ h(x) = 2P^+ \int dp^- d^2 p_T \left(\frac{P \cdot p - M^2 x}{M^2} A_4 + B_2 + x B_3 \right) \]
\[ = \int d\sigma d\tau d^2 p_T \delta(\tau - x\sigma + M^2 x^2 + p_T^2) \]
where $\sigma = 2p \cdot P$, $\tau = P^2$, $A_i = A_i(\sigma, \tau)$ and $B_i = B_i(\sigma, \tau)$. Using (9) and (47) allows one to write

\[ h_{1}^{(+)}(x) = 2p^+ \int dp^- d^2p_T \frac{p_T^2}{2M^2} (-A_4) \]

\[ = \int d\sigma \, d\tau \, d^2p_T \frac{p_T^2}{2M^2} (\delta(\tau - \sigma + M^2 x^2 + p_T^2) - (A_4)). \]  

To solve the integral in (65) in the same way as in (64) one can use the relation [2]

\[ \int d\sigma \, d\tau \, d^2p_T \frac{p_T^2}{2M^2} (\delta(\tau - \sigma + M^2 x^2 + p_T^2)) \mathcal{F}(x, \sigma, \tau) \]
\[ = -\pi \int dy \int d\sigma \, d\tau \left( \delta(\tau - y\sigma + M^2 y^2) - \frac{\sigma - 2M^2 y}{2M^2} \right) \]
\[ \times (\mathcal{F}(y, \sigma, \tau) - \left(\frac{\sigma - 2M^2 y}{2M^2} \frac{\partial}{\partial y} (\mathcal{F}(y, \sigma, \tau)) \right)). \]  

where $\mathcal{F}(x, \sigma, \tau)$ is a generic function representing a linear combination of the amplitudes $A_i, B_i$. This relation can be proved by differentiating both sides with respect to $x$ and using the fact that the integration area vanishes at $x = 1$. Applying relation (66) to (65) one obtains

\[ h_{1}^{(+)}(x) = -\pi \int dy \int d\sigma \, d\tau \left( \delta(\tau - y\sigma + M^2 y^2) \right) \]
\[ \times \left(\frac{\sigma - 2M^2 y}{2M^2} \right), \]  

\[ \frac{d}{dx} h_{1}^{(+)}(x) = -\pi \int d\sigma \, d\tau \left( \delta(\tau - \sigma + M^2 x^2) \right) \]
\[ \times \left(\frac{\sigma - 2M^2 x}{2M^2} \right). \]  

Comparing now (64) and (68) it is clear that the LIR in (8) is fulfilled if the amplitudes $B_i$ (here $B_2$ and $B_3$) are absent.

References