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Momentum sum rules for fragmentation functions

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ABSTRACT

Momentum sum rules for fragmentation functions are considered. In particular, we give a general proof of the Schäfer–Teryaev sum rule for the transverse momentum dependent Collins function. We also argue that corresponding sum rules for related fragmentation functions do not exist. Our model-independent analysis is supplemented by calculations in a simple field-theoretical model.

1. Introduction

Fragmentation functions (FFs) contain important information about strong interaction dynamics in the non-perturbative regime. It turns out that a realistic modeling of FFs is nontrivial. Moreover, as a matter of principle, FFs cannot be computed in lattice gauge theory. In this situation, it is desirable to obtain as many model-independent constraints on these objects as possible. Momentum sum rules do provide such constraints, with the momentum sum rule for the (collinear) unpolarized fragmentation function D_1 representing the best known example [1]. Any phenomenological parameterization of D_1 must obey this sum rule [2–9]. Intuitively, the D_1 sum rule follows from conservation of the longitudinal momentum of the fragmenting parton. Though intuitive, a rigorous proof in QCD is nontrivial [1], and we also address this issue in the present Letter.

In recent years, there has been an increased interest in transverse momentum dependent FFs, which not only contain information on the longitudinal momentum of the final state hadron but also on its transverse motion relative to the parton (see, e.g., Refs. [10–14]). In this context, the Collins fragmentation function $H_1^\perp$ [10], which describes the fragmentation of a transversely polarized quark into an unpolarized hadron, plays an important role. It belongs to the class of (naive) time-reversal odd (T-odd) FFs, which implies that it is nonzero only if there exists a nontrivial phase for the decay $q^* \to hX$ of the (virtual) quark into a hadron. Model calculations of $H_1^\perp$ can be found in Refs. [15–22].

The Collins function is of particular interest since it can serve as a tool for addressing the transversity parton distribution in semi-inclusive deep-inelastic scattering (DIS) [10]. The relevant observable — the so-called Collins asymmetry — has already been measured by the HERMES and COMPASS Collaborations for a proton and a deuteron target [23–27]. In the case of a proton target, clearly nonzero effects have been found [23,26,27]. Information about the Collins function is also available through a particular azimuthal asymmetry in $e^+e^- \to h_1 h_2 X$ [28–30] for which data from the Belle Collaboration exist [31,32]. Analyses of the data on the Collins asymmetry in semi-inclusive DIS and on the azimuthal asymmetry in $e^+e^- \to h_1 h_2 X$ not only provided information about the Collins function [33–36] but also about the transversity distribution [35,36], which represented a milestone in transverse spin physics.

The primary purpose of our paper is to address the so-called Schäfer–Teryaev sum rule (ST sum rule) for the Collins function [37]. This sum rule states that a particular moment of $H_1^\perp$ for a fragmenting quark vanishes when summing over all final state hadrons. It was obtained on the basis of intuitive arguments about conservation of transverse momentum in the fragmentation process [37], yet a general proof of the ST sum rule in QCD did not exist. Here we provide such a proof and also argue that related transverse momentum

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dependent FFs do not obey a corresponding sum rule, which is at variance with some statements in the literature [37–39]. In addition to the model-independent analysis, we compute the relevant FFs in a simple self-consistent quark–pion coupling model, and this study confirms the model-independent results.

2. Derivation of sum rules

In order to provide a model-independent derivation of the momentum sum rules for FFs, we start from the basic correlator defining the fragmentation of a quark into a single hadron [1,11,13,40],[1]

$$\Delta^{[\Gamma]}(z,\vec{k}_T,S_h)=\frac{1}{4z}\sum_X\int\frac{d\xi^+\,d^2\vec{\xi}_T}{(2\pi)^3}e^{ik\cdot\xi}\,\mathrm{Tr}\big[\langle 0|\mathcal{W}_1(\infty,\xi)\psi(\xi)|P_h,S_h;X\rangle\langle P_h,S_h;X|\bar{\psi}(0)\mathcal{W}_2(0,\infty)|0\rangle\Gamma\big]_{\xi^-=0}. \tag{1}$$

In this definition a color-average for the fragmenting quark is implicit, a flavor index is suppressed, and the trace acts in Dirac space with Γ representing a Dirac matrix. The final state hadron is specified through its 4-momentum P_h and the covariant spin vector S_h, which satisfy $P_h^2=M_h^2$, $S_h^2=-1$, and $P_h\cdot S_h=0$. The correlator (1) is understood in a frame in which the transverse momentum of the hadron vanishes, while $\vec{k}_T$ is the transverse momentum of the quark. The (large) minus-component of the hadron momentum[2] is given by $P_h^-=zk^-$. Color gauge invariance is ensured by means of the two Wilson lines

$$\mathcal{W}_1(\infty,\xi)=\mathcal{W}\big(\infty^+,0^-,\vec{\infty}_T;\infty^+,0^-,\vec{\xi}_T\big)\mathcal{W}\big(\infty^+,0^-,\vec{\xi}_T;\xi^+,0^-,\vec{\xi}_T\big), \tag{2}$$

$$\mathcal{W}_2(0,\infty)=\mathcal{W}\big(0^+,0^-,\vec{0}_T;\infty^+,0^-,\vec{0}_T\big)\mathcal{W}\big(\infty^+,0^-,\vec{0}_T;\infty^+,0^-,\vec{\infty}_T\big), \tag{3}$$

where, in general, $\mathcal{W}(a^+,a^-,\vec{a}_T;b^+,b^-,\vec{b}_T)$ indicates a Wilson line running from $(a^+,a^-,\vec{a}_T)$ to $(b^+,b^-,\vec{b}_T)$. In connection with k_T-dependent parton correlators, the importance of transversely running gauge links at the light-cone infinity, like the ones showing up in (2) and (3), has been pointed out only relatively recently [40–42]. These links do not disappear in the light-cone gauge $A^-=0$. Nevertheless, as we will argue, their presence does not spoil the longitudinal momentum sum rule for D_1. We also note that the path of the Wilson lines for transverse momentum dependent FFs is not entirely unique. Information on this topic can be found in various articles [43–50]. Our derivation of the momentum sum rules goes through for any allowed path.

The (eight) leading twist transverse momentum dependent FFs (for fragmentation of a quark q into a spin-$\frac{1}{2}$ hadron h) are defined through the correlator in (1) according to [11]

$$\Delta^{[\gamma^-]}(z,\vec{k}_T,S_h)=D_1^{h/q}\big(z,z^2\vec{k}_T^2\big)+\frac{\epsilon_T^{ij}k_T^iS_{hT}^j}{M_h}D_{1T}^{\perp h/q}\big(z,z^2\vec{k}_T^2\big), \tag{4}$$

$$\Delta^{[\gamma^-\gamma_5]}(z,\vec{k}_T,S_h)=\lambda_h G_{1L}^{h/q}\big(z,z^2\vec{k}_T^2\big)+\frac{\vec{k}_T\cdot\vec{S}_{hT}}{M_h}G_{1T}^{h/q}\big(z,z^2\vec{k}_T^2\big), \tag{5}$$

$$\begin{aligned}\Delta^{[i\sigma^{i-}\gamma_5]}(z,\vec{k}_T,S_h)={}&S_{hT}^i\left(H_{1T}^{h/q}\big(z,z^2\vec{k}_T^2\big)+\frac{\vec{k}_T^2}{2M_h^2}H_{1T}^{\perp h/q}\big(z,z^2\vec{k}_T^2\big)\right)-\frac{\epsilon_T^{ij}k_T^j}{M_h}H_1^{\perp h/q}\big(z,z^2\vec{k}_T^2\big)\\&+\frac{\lambda_h k_T^i}{M_h}H_{1L}^{\perp h/q}\big(z,z^2\vec{k}_T^2\big)+\frac{2k_T^i\vec{k}_T\cdot\vec{S}_{hT}-S_{hT}^i\vec{k}_T^2}{2M_h^2}H_{1T}^{\perp h/q}\big(z,z^2\vec{k}_T^2\big).\end{aligned} \tag{6}$$

In these definitions we use $\epsilon_T^{ij}=\epsilon^{-+ij}$ (with the convention $\epsilon_T^{12}=1$) and the representation

$$S_h=\big(S_h^+,S_h^-,\vec{S}_{hT}\big)=\left(-\lambda_h\frac{M_h}{2P_h^-},\lambda_h\frac{P_h^-}{M_h},\vec{S}_{hT}\right) \tag{7}$$

of the covariant spin vector.

It is now convenient to switch to a reference frame in which the fragmenting quark has no transverse momentum. This implies a nonzero transverse momentum of the hadron, and, if one wants to keep the minus-component of 4-momenta fixed, this transverse momentum is given by $\vec{P}_{h\perp}=-z\vec{k}_T$ [1]. One can therefore write the correlator in (1) as (see also, e.g., Ref. [1])

$$\Delta^{[\Gamma]}(z,\vec{P}_{h\perp},S_h)=\frac{1}{4z}\int\frac{d\xi^+\,d^2\vec{\xi}_T}{(2\pi)^3}e^{ik^-\xi^+}\,\mathrm{Tr}\big[\langle 0|\mathcal{W}_1(\infty,\xi)\psi(\xi)\hat{a}_h^\dagger(P_h,S_h)\hat{a}_h(P_h,S_h)\bar{\psi}(0)\mathcal{W}_2(0,\infty)|0\rangle\Gamma\big]_{\xi^-=0}, \tag{8}$$

where we have expressed the final state hadron through the particle creation operator $\hat{a}_h^\dagger(P_h,S_h)$. This leads to

$$\sum_{S_h}\int_0^1 dz\int d^2\vec{P}_{h\perp}P_h^\mu\Delta^{[\Gamma]}(z,\vec{P}_{h\perp},S_h)=\frac{1}{2}\int d\xi^+\,d^2\vec{\xi}_T\,e^{ik^-\xi^+}\,\mathrm{Tr}\big[\langle 0|\mathcal{W}_1(\infty,\xi)\psi(\xi)\hat{P}_h^\mu\bar{\psi}(0)\mathcal{W}_2(0,\infty)|0\rangle\Gamma\big]_{\xi^-=0}, \tag{9}$$

[1] Note that, in general, integrals for which no integration limits are written explicitly run from $-\infty$ to $+\infty$.

[2] For a generic 4-vector v, we define light-cone coordinates according to $v^\pm=(v^0\pm v^3)/\sqrt{2}$ and $\vec{v}_T=(v^1,v^2)$.

with the momentum operator (in light-cone quantization) [1]

$$\hat{P}_h^\mu = \sum_{S_h} \int \frac{dP_h^- \, d^2\vec{P}_{h\perp}}{(2\pi)^3 2P_h^-} \hat{a}_h^\dagger(P_h, S_h) P_h^\mu \hat{a}_h(P_h, S_h). \tag{10}$$

By summing over all hadrons h, we obtain the momentum operator of the theory expressed through hadronic field operators [1],

$$\sum_h \hat{P}_h^\mu = \hat{P}^\mu. \tag{11}$$

We emphasize that, according to (10), the relation (11) also involves a summation over particle spins. As we discuss below in a bit more detail, this is the main reason why momentum sum rules for FFs describing hadron polarization do not exist. Using the properties of the momentum operator in (11) one finds

$$\sum_h \sum_{S_h} \int_0^1 dz \int d^2\vec{P}_{h\perp} \, P_h^\mu \, \Delta^{[\Gamma]}(z, \vec{P}_{h\perp}, S_h) = \frac{1}{2} \int d\xi^+ \, d^2\vec{\xi}_T \, e^{ik^-\xi^+} i\partial^\mu \Big[\mathrm{Tr}\big[\langle 0|\mathcal{W}_1(\infty, \xi)\psi(\xi)\bar{\psi}(0)\mathcal{W}_2(0, \infty)|0\rangle \Gamma \big]\Big]_{\xi^- = 0}, \tag{12}$$

where we exploited the identity

$$\langle 0|\mathcal{W}_1(\infty, \xi)\psi(\xi)\hat{P}^\mu = i\partial^\mu \big[\langle 0|\mathcal{W}_1(\infty, \xi)\psi(\xi) \big]. \tag{13}$$

On the basis of (12) one can now derive both the momentum sum rule for D_1 as well as the ST sum rule for the Collins function.

We begin with the sum rule for D_1. The essential elements of a complete proof of this sum rule in QCD were already indicated in [1]. (For a proper treatment of ultraviolet divergences in k_T-integrated FFs we also refer to [1].) For completeness, and also because of the potential complications arising from the transversely running gauge links in (2) and (3), we consider it worthwhile to write out some details of a proof in the light-cone gauge $A^- = 0$. To this end we choose $\mu = -$ in Eq. (12) and use integration by parts, leading to

$$\sum_h \sum_{S_h} \int_0^1 dz \int d^2\vec{P}_{h\perp} \, P_h^- \, \Delta^{[\Gamma]}(z, \vec{P}_{h\perp}, S_h) = \frac{k^-}{2} \int d\xi^+ \, d^2\vec{\xi}_T \, e^{ik^-\xi^+} \, \mathrm{Tr}\big[\langle 0|\mathcal{W}_1(\infty, \xi)\psi(\xi)\bar{\psi}(0)\mathcal{W}_2(0, \infty)|0\rangle \Gamma \big]_{\xi^- = 0}. \tag{14}$$

Now we consider (14) for $\Gamma = \gamma^-$ and introduce the so-called "good" quark field $\psi_- = \frac{1}{2}\gamma^+\gamma^-\psi$ [51,52] providing

$$\begin{aligned}
&\sum_h \sum_{S_h} \int_0^1 dz \int d^2\vec{P}_{h\perp} \, P_h^- \, \Delta^{[\gamma^-]}(z, \vec{P}_{h\perp}, S_h) \\
&= \frac{k^-}{\sqrt{2}} \int d\xi^+ \, d^2\vec{\xi}_T \, e^{ik^-\xi^+} \, \mathrm{Tr}\big[\langle 0|\mathcal{W}_1(\infty, \xi)\psi_-(\xi)\psi_-^\dagger(0)\mathcal{W}_2(0, \infty)|0\rangle \big]_{\xi^- = 0} \\
&= \frac{k^-}{\sqrt{2}} \int d\xi^+ \, d^2\vec{\xi}_T \, e^{ik^-\xi^+} \, \mathrm{Tr}\big[\langle 0|\mathcal{W}\big(\infty^+, 0^-, \vec{\infty}_T; \infty^+, 0^-, \vec{\xi}_T\big) \\
&\quad \times \psi_-\big(\xi^+, 0^-, \vec{\xi}_T\big)\psi_-^\dagger\big(0^+, 0^-, \vec{0}_T\big)\mathcal{W}\big(\infty^+, 0^-, \vec{0}_T; \infty^+, 0^-, \vec{\infty}_T\big)|0\rangle \big] \\
&= \frac{k^-}{\sqrt{2}} \int d\xi^+ \, d^2\vec{\xi}_T \, e^{ik^-\xi^+} \, \mathrm{Tr}\big[\langle 0|\mathcal{W}\big(\infty^+, 0^-, \vec{\infty}_T; \infty^+, 0^-, \vec{\xi}_T\big) \\
&\quad \times \big\{ \psi_-\big(\xi^+, 0^-, \vec{\xi}_T\big), \psi_-^\dagger\big(0^+, 0^-, \vec{0}_T\big) \big\} \mathcal{W}\big(\infty^+, 0^-, \vec{0}_T; \infty^+, 0^-, \vec{\infty}_T\big)|0\rangle \big].
\end{aligned} \tag{15}$$

In the second step in (15) we made use of the light-cone gauge $A^- = 0$, for which the Wilson lines in (2) and (3) that run along the light-cone reduce to unity. In the last step the anti-commutator of the two quark fields was introduced, which is justified because of

$$\begin{aligned}
&\int d\xi^+ \, d^2\vec{\xi}_T \, e^{ik^-\xi^+} \, \mathrm{Tr}\big[\langle 0|\mathcal{W}\big(\infty^+, 0^-, \vec{\infty}_T; \infty^+, 0^-, \vec{\xi}_T\big) \\
&\quad \times \psi_-^\dagger\big(0^+, 0^-, \vec{0}_T\big)\psi_-\big(\xi^+, 0^-, \vec{\xi}_T\big)\mathcal{W}\big(\infty^+, 0^-, \vec{0}_T; \infty^+, 0^-, \vec{\infty}_T\big)|0\rangle \big] \\
&= \sum_X \int d\xi^+ \, d^2\vec{\xi}_T \, e^{ik^-\xi^+} e^{i\sum_j p_j^- \xi^+} \, \mathrm{Tr}\big[\langle 0|\mathcal{W}\big(\infty^+, 0^-, \vec{\infty}_T; \infty^+, 0^-, \vec{\xi}_T\big) \\
&\quad \times \psi_-^\dagger\big(0^+, 0^-, \vec{0}_T\big)|X\rangle\langle X|\psi_-\big(0^+, 0^-, \vec{\xi}_T\big)\mathcal{W}\big(\infty^+, 0^-, \vec{0}_T; \infty^+, 0^-, \vec{\infty}_T\big)|0\rangle \big] \\
&= (2\pi) \sum_X \int d^2\vec{\xi}_T \, \delta\Big(k^- + \sum_j p_j^-\Big) \, \mathrm{Tr}\big[\langle 0|\mathcal{W}\big(\infty^+, 0^-, \vec{\infty}_T; \infty^+, 0^-, \vec{\xi}_T\big) \\
&\quad \times \psi_-^\dagger\big(0^+, 0^-, \vec{0}_T\big)|X\rangle\langle X|\psi_-\big(0^+, 0^-, \vec{\xi}_T\big)\mathcal{W}\big(\infty^+, 0^-, \vec{0}_T; \infty^+, 0^-, \vec{\infty}_T\big)|0\rangle \big] = 0.
\end{aligned} \tag{16}$$

In Eq. (16), p_j are the 4-momenta of the particles in the intermediate states $|X\rangle$. The expression vanishes since $k^- > 0$ and $p_j^- \geqslant 0$.

In order to proceed with (15), one can use the anti-commutator for the "good" quark fields [1,51],

$$\left\{\psi_-\left(\xi^+,0^-,\vec{\xi}_T\right),\psi_-^\dagger\left(0^+,0^-,\vec{0}_T\right)\right\}=\frac{1}{2\sqrt{2}}\gamma^+\gamma^-\delta\left(\xi^+\right)\delta^{(2)}(\vec{\xi}_T), \tag{17}$$

which immediately gives

$$\sum_h\sum_{S_h}\int_0^1 dz\int d^2\vec{P}_{h\perp}\,P_h^-\Delta^{[\gamma^-]}(z,\vec{P}_{h\perp},S_h)=k^-. \tag{18}$$

On the other hand, because of (4), one also has

$$\sum_h\sum_{S_h}\int_0^1 dz\int d^2\vec{P}_{h\perp}\,P_h^-\Delta^{[\gamma^-]}(z,\vec{P}_{h\perp},S_h)=\sum_h\sum_{S_h}\int_0^1 dz\int d^2\vec{P}_{h\perp}\,zk^-D_1^{h/q}\left(z,\vec{P}_{h\perp}^2\right). \tag{19}$$

Comparing Eqs. (18) and (19), and going back to the original reference frame in which $\vec{P}_{h\perp}=0$, then leads to the momentum sum rule for D_1,

$$\sum_h\sum_{S_h}\int_0^1 dz\,zD_1^{h/q}(z)=1,\quad\text{with} \tag{20}$$

$$D_1^{h/q}(z)=z^2\int d^2\vec{k}_T\,D_1^{h/q}\left(z,z^2\vec{k}_T^2\right). \tag{21}$$

According to (4), D_1 for a spin-$\frac{1}{2}$ hadron is defined by a spin average rather than a spin summation. Therefore, in the sum rule a summation over hadron spins shows up, which implies that one has to multiply FFs for a spin-$\frac{1}{2}$ particle by 2. Also note that a corresponding sum rule for the two collinear FFs

$$G_1^{h/q}(z)=z^2\int d^2\vec{k}_T\,G_{1L}^{h/q}\left(z,z^2\vec{k}_T^2\right), \tag{22}$$

$$H_1^{h/q}(z)=z^2\int d^2\vec{k}_T\left(H_{1T}^{h/q}\left(z,z^2\vec{k}_T^2\right)+\frac{\vec{k}_T^2}{2M_h^2}H_{1T}^{\perp h/q}\left(z,z^2\vec{k}_T^2\right)\right) \tag{23}$$

cannot be derived along the lines described above. These functions drop out when summing the fragmentation correlators (5) and (6) over the hadron polarizations. However, as we pointed out after (11), this summation is a crucial element in the proof of momentum sum rules for FFs. Since one also finds that for G_1 and H_1 the respective traces vanish, i.e., the right-hand side of the formulas corresponding to (18) vanishes, one arrives at the consistent though useless situation $0=0$.

Now we turn to the (simpler) derivation of the ST sum rule. Starting again from Eq. (12) and choosing $\mu=j$, with j being a transverse index, one readily finds

$$\sum_h\sum_{S_h}\int_0^1 dz\int d^2\vec{P}_{h\perp}\,P_h^j\Delta^{[\Gamma]}(z,\vec{P}_{h\perp},S_h)=0. \tag{24}$$

This result holds because of

$$\int d\xi^j\,\partial^j\mathcal{W}_1(\infty,\xi)\psi(\xi)=\mathcal{W}_1(\infty,\xi)\psi(\xi)|_{\xi_T^j=\infty}-\mathcal{W}_1(\infty,\xi)\psi(\xi)|_{\xi_T^j=-\infty}, \tag{25}$$

and the vanishing of the quark field at $\xi_T^j=\pm\infty$. On the other hand, from Eq. (6) one obtains

$$\sum_h\sum_{S_h}\int_0^1 dz\int d^2\vec{P}_{h\perp}\,P_h^j\Delta^{[i\sigma^{i-}\gamma_5]}(z,\vec{P}_{h\perp},S_h)=\epsilon_T^{ij}\sum_h\sum_{S_h}\int_0^1 dz\int d^2\vec{P}_{h\perp}\frac{\vec{P}_{h\perp}^2}{2zM_h}H_1^{\perp h/q}\left(z,\vec{P}_{h\perp}^2\right). \tag{26}$$

Comparing Eqs. (24) and (26), and going back to the original reference frame in which $\vec{P}_{h\perp}=0$, then leads to the ST sum rule for the Collins function [37] in the form

$$\sum_h\sum_{S_h}\int_0^1 dz\,zM_hH_1^{\perp(1)h/q}(z)=0,\quad\text{with} \tag{27}$$

$$H_1^{\perp(1)h/q}(z)=z^2\int d^2\vec{k}_T\,\frac{\vec{k}_T^2}{2M_h^2}H_1^{\perp h/q}\left(z,z^2\vec{k}_T^2\right). \tag{28}$$

Since we stick to the conventions of Ref. [11], the hadron mass M_h appears in (27). This factor would not show up if in (6) and (28) a common mass scale for each hadron was used. As mentioned earlier, this sum rule was already obtained in Ref. [37] (with slightly different

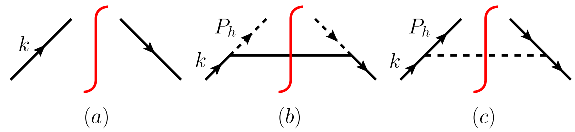

Fig. 1. Cut-diagrams describing the fragmentation of a quark (solid line) into a quark or a pion (dashed line) through $\mathcal{O}(g^2)$ in a quark–pion coupling model as specified in Eq. (29).

conventions) on the basis of intuitive arguments about conservation of transverse momentum in the fragmentation process. However, a general field-theoretical proof was not yet available. In fact, the same argument about conservation of transverse momentum led to the conclusion that sum rules corresponding to the one in (27) should also hold for other transverse momentum dependent FFs [37]. More precisely, sum rules of the type (27) were expected for $D_{1T}^{\perp}$, G_{1T}, and $H_{1L}^{\perp}$ since in Eqs. (4)–(6) those FFs, like the Collins function, are accompanied by a term linear in k_T. Basically by repeating the reasoning we used above in connection with the collinear FFs G_1 and H_1 in (22) and (23), one finds that the proof of the ST sum rule cannot be extended to other transverse momentum dependent FFs. One rather ends up again with the situation $0=0$. Below we will explicitly show by model calculations that $D_{1T}^{\perp}$, G_{1T}, and $H_{1L}^{\perp}$ do not obey a sum rule like the ST sum rule in (27).

3. Model calculations

In this section we explore the momentum sum rules for the FFs in a simple though self-consistent field-theoretical model. To describe the matrix elements in the fragmentation correlator, we use a pseudoscalar coupling between quarks and pions given by the interaction Lagrangian

$$\mathcal{L}_I(x) = -ig\bar{\psi}(x)\gamma_5\psi(x)\pi(x), \tag{29}$$

which is in the spirit of the Manohar–Georgi model [53]. For simplicity we do not take a flavor degree of freedom into account, which is sufficient for our purpose. This model was already exploited in Ref. [16] in order to get an explicit realization of a nonzero Collins function for a pion. A slightly modified/extended version of this model was also studied recently with the main aim of obtaining a reasonable phenomenology for $D_1^{\pi/q}$ by taking into account multiple pion emission [54].

In the case of D_1, we compute all the contributions through $\mathcal{O}(g^2)$. In our model, the summation over all hadrons in the sum rule (20) implies a summation both over pions and quarks in the final state. One obtains (see also Refs. [16,54])

$$D_1^{q/q}\big(z, z^2\vec{k}_T^2\big) = \frac{1}{2}\delta(1-z)\delta^{(2)}(\vec{k}_T)Z_\psi + \frac{g^2}{32\pi^3}\frac{(1-z)\big(\vec{k}_T^2 + \frac{(1-z)^2}{z^2}m^2\big)}{z^2\big(\vec{k}_T^2 + \frac{(1-z)^2}{z^2}m^2 + \frac{m_\pi^2}{z}\big)^2}, \tag{30}$$

$$D_1^{\pi/q}\big(z, z^2\vec{k}_T^2\big) = \frac{g^2}{16\pi^3}\frac{\vec{k}_T^2 + m^2}{z\big(\vec{k}_T^2 + m^2 + \frac{1-z}{z^2}m_\pi^2\big)^2}, \tag{31}$$

with m denoting the quark mass and m_π the pion mass. The first term in (30) arises from diagram (a) in Fig. 1, which represents the lowest order contribution from a vacuum intermediate state. Note that this term must also include the wave function renormalization factor [1,54]

$$Z_\psi = 1 + \frac{\partial\Sigma}{\partial \not{k}}\bigg|_{\not{k}=m}, \tag{32}$$

which in our case is given by the quark self-energy Σ to one loop. The second term in (30) describes the contribution from diagram (c), while diagram (b) in Fig. 1 leads to the result in Eq. (31). A potential contribution to $D_1^{q/q}$ at $\mathcal{O}(g^2)$ from diagram (a) in Fig. 2 is canceled by the counter term diagram (b).

Now we consider the momentum sum rule (20), for which we find

$$\begin{aligned}\int_0^1 dz\, z\big(2D_1^{q/q}(z) + D_1^{\pi/q}(z)\big) &= 1 + \frac{g^2}{16\pi^3}\int_0^1 dz \int d^2\vec{l}_T\, \frac{\vec{l}_T^2(2z-1) + m^2z^2 - m_\pi^2(1-z)^2}{(\vec{l}_T^2 + m^2z^2 + m_\pi^2(1-z))^2} \\ &= 1 + \frac{g^2}{16\pi^2}\int_0^1 dz\left((1-2z)\ln\big(z^2+\mu^2(1-z)\big) + \frac{z^2-\mu^2(1-z)^2}{z^2+\mu^2(1-z)}\right) = 1,\end{aligned} \tag{33}$$

i.e., the sum rule holds in the model (29) through $\mathcal{O}(g^2)$. Note that we introduced the mass ratio $\mu = m_\pi/m$. In the case of the contribution from Z_ψ, z is a Feynman parameter and l_T is the transverse part of the loop momentum. To carry out the ultraviolet divergent l_T-integral, one can use dimensional regularization or a cutoff, with both methods leading to the same result. The vanishing of the remaining z-integral is an exact analytical result. As an independent check, we have computed all the contributions right from the beginning in $4-\epsilon$ dimensions. Then the sum rule can also be verified if one keeps in mind that [1]

$$D_1(z) = \int d^{2-\epsilon}P_{h\perp}\, D_1\big(z, \vec{P}_{h\perp}^2\big) = z^{2-\epsilon}\int d^{2-\epsilon}k_T\, D_1\big(z, z^2\vec{k}_T^2\big). \tag{34}$$

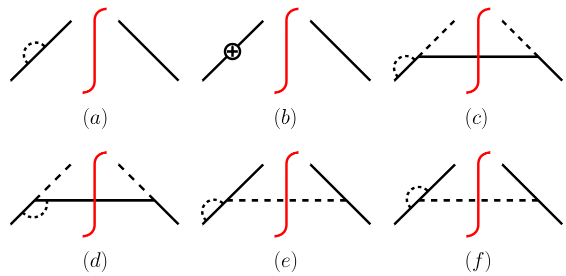

Fig. 2. Loop contributions describing the fragmentation of a quark (solid line) into a quark or a pion (dashed line) in a quark–pion coupling model as specified in Eq. (29). Hermitian conjugate graphs are not shown. Diagram (a) is canceled when adding it to the respective counter term contribution in (b). Diagrams (c)–(f) generate nonzero results for the T-odd FFs $H_1^{\perp}$ and $D_{1T}^{\perp}$.

In particular, the non-integral exponent in the factor $z^{2-\epsilon}$ is crucial for getting the desired result. It is perhaps worth mentioning that for either way we had to carry out all the integrations to the very end in order to establish the momentum sum rule in (20). In this respect our discussion of the D_1 sum rule differs from the corresponding one given in [54].

Next, we turn our attention to the Collins function and the ST sum rule in Eq. (27). The Collins function receives contributions from diagrams (c)–(f) in Fig. 2, and after some algebra one obtains[3]

$$H_1^{\perp q/q}(z, z^2\vec{k}_T^2) = \frac{g^2}{16\pi^3}\frac{m}{1-z}\left(\frac{m\,\mathrm{Im}\,\tilde{\Sigma}(k^2)}{(k^2-m^2)^2} + \frac{\mathrm{Im}\,\tilde{\Gamma}_q(k^2)}{k^2-m^2}\right)\bigg|_{k^2=\frac{z}{1-z}\vec{k}_T^2+\frac{m^2}{z}+\frac{m_\pi^2}{1-z}}, \tag{35}$$

$$H_1^{\perp \pi/q}(z, z^2\vec{k}_T^2) = -\frac{g^2}{8\pi^3}\frac{m_\pi}{1-z}\left(\frac{m\,\mathrm{Im}\,\tilde{\Sigma}(k^2)}{(k^2-m^2)^2} + \frac{\mathrm{Im}\,\tilde{\Gamma}_\pi(k^2)}{k^2-m^2}\right)\bigg|_{k^2=\frac{z}{1-z}\vec{k}_T^2+\frac{m^2}{1-z}+\frac{m_\pi^2}{z}}. \tag{36}$$

In these expressions $\mathrm{Im}\,\tilde{\Sigma}$ arises from the self-energy insertions in the diagrams (c) and (e) in Fig. 2, while $\mathrm{Im}\,\tilde{\Gamma}_q$ and $\mathrm{Im}\,\tilde{\Gamma}_\pi$, respectively, are due to the vertex corrections in the diagram (f) and (d). Note that for the expressions in Eqs. (35) and (36) the virtuality k^2 of the fragmenting quark has a different value. If one actually evaluates them at the same k^2, one can show that

$$\mathrm{Im}\,\tilde{\Gamma}_\pi(k^2) = \mathrm{Im}\,\tilde{\Gamma}_q(k^2), \tag{37}$$

which is quite essential for verifying the ST sum rule. Though the explicit results of the imaginary parts turn out to be irrelevant for the discussion of the ST sum rule, we include them here for completeness [16]:

$$\mathrm{Im}\,\tilde{\Sigma}(k^2) = \frac{g^2}{16\pi^2}\left(1 - \frac{m^2-m_\pi^2}{k^2}\right)I_1, \tag{38}$$

$$\mathrm{Im}\,\tilde{\Gamma}_\pi(k^2) = -\frac{g^2}{8\pi^2}m\frac{k^2-m^2+m_\pi^2}{\lambda(k^2,m^2,m_\pi^2)}\left(I_1 + (k^2-m^2-2m_\pi^2)I_2\right), \tag{39}$$

where we used $\lambda(k^2,m^2,m_\pi^2) = [k^2-(m+m_\pi)^2][k^2-(m-m_\pi)^2]$ and the integrals

$$I_1 = \int d^4l\,\delta(l^2-m_\pi^2)\delta\left((k-l)^2-m^2\right) = \frac{\pi}{2k^2}\sqrt{\lambda(k^2,m^2,m_\pi^2)}\,\theta\left(k^2-(m+m_\pi)^2\right), \tag{40}$$

$$I_2 = \int d^4l\,\frac{\delta(l^2-m_\pi^2)\delta((k-l)^2-m^2)}{(k-P_h-l)^2-m^2} = -\frac{\pi}{2\sqrt{\lambda(k^2,m^2,m_\pi^2)}}\ln\left(1+\frac{\lambda(k^2,m^2,m_\pi^2)}{k^2m^2-(m^2-m_\pi^2)^2}\right)\theta\left(k^2-(m+m_\pi)^2\right). \tag{41}$$

We note that the integral I_2 is evaluated for $P_h^2 = m_\pi^2$ and $(k-P_h)^2 = m^2$.

We are now in a position to check the ST sum rule (27), which in our model takes the form

$$\int_0^1 dz\,z\left(2mH_1^{\perp(1)q/q}(z) + m_\pi H_1^{\perp(1)\pi/q}(z)\right) = 0. \tag{42}$$

By making use of (37), one readily verifies that this sum rule is indeed satisfied if in either of the two results in (35) and (36) one makes the substitutions $z \to z' = 1-z$ and $\vec{k}_T \to \vec{k}_T' = \frac{1-z}{z}\vec{k}_T$. This means that neither the z-integration nor the k_T-integration has to be performed explicitly. One rather finds a cancellation of the contributions from (35) and (36) on the level of the integrand.

Finally, we want to explore if a result like the ST sum rule also holds for the three transverse momentum dependent FFs $D_{1T}^{\perp}$, G_{1T}, and $H_{1L}^{\perp}$ as was suggested in [37]. In the model-independent part of our study we have only shown that the proof we gave for the ST sum rule does not apply to those FFs. These three functions have in common that the final state hadron is polarized, which implies that in our model we only receive contributions from fragmentation into a quark. We begin with the T-odd function $D_{1T}^{\perp}$, which is quite relevant

[3] The result for $H_1^{\perp\pi/q}$ was already given in [16], but the overall sign was wrong as pointed out previously in Ref. [19].

for fragmentation into transversely polarized hyperons (see, e.g., Ref. [39]). It receives nonzero contributions from diagrams (e) and (f) in Fig. 2, and the result reads

$$D_{1T}^{\perp q/q}\big(z, z^2\vec{k}_T^2\big) = -H_1^{\perp q/q}\big(z, z^2\vec{k}_T^2\big), \tag{43}$$

with $H_1^{\perp q/q}$ as given in Eq. (35). One finds that for $D_{1T}^{\perp}$ a sum rule of the type (27) does not hold. This can, for instance, be shown by focusing on the ultraviolet divergent part of the k_T-integral. To be more specific, one finds

$$\int_0^1 dz\, z D_{1T}^{\perp(1)q/q}(z) = -\frac{g^4}{3\times 2^{11}\pi^3}\ln^2\frac{\Lambda^2}{m^2} + \text{less singular}, \tag{44}$$

where Λ^2 is an upper cutoff for the k_T^2-integration. In contrast to the Collins function, for which fragmentation into a quark and fragmentation into a pion show up, for $D_{1T}^{\perp}$ the fragmentation into a quark is not compensated by another term. The T-even functions G_{1T} and $H_{1L}^{\perp}$ can be computed to $\mathcal{O}(g^2)$ on the basis of diagram (c) in Fig. 1 leading to

$$G_{1T}^{q/q}\big(z, z^2\vec{k}_T^2\big) = H_{1L}^{\perp q/q}\big(z, z^2\vec{k}_T^2\big) = \frac{g^2}{16\pi^3}\frac{(1-z)^2 m^2}{z^3\big(\vec{k}_T^2 + \frac{(1-z)^2}{z^2}m^2 + \frac{m_\pi^2}{z}\big)^2}. \tag{45}$$

Again, by just focusing on the ultraviolet divergent part of the k_T-integral, one also readily verifies that for these two FFs a sum rule of the type (27) cannot exist. Explicit calculation provides

$$\int_0^1 dz\, z G_{1T}^{(1)q/q}(z) = \int_0^1 dz\, z H_{1L}^{\perp(1)q/q}(z) = \frac{g^2}{96\pi^2}\ln\frac{\Lambda^2}{m^2} + \text{ultraviolet finite}. \tag{46}$$

4. Summary

In this Letter, momentum sum rules for fragmentation functions have been studied by performing both a model-independent analysis as well as explicit model calculations. In particular, we have provided a general field-theoretical proof of the ST sum rule [37] for the Collins function $H_1^{\perp}$ [10] in QCD. The existing derivation of the ST sum rule was merely based on intuitive arguments about conservation of transverse momentum in the fragmentation process [37]. In this respect, there is a strong similarity between the ST sum rule and the longitudinal momentum sum rule for the unpolarized fragmentation function D_1: they are both intuitive, but their general proof in QCD is more involved [1]. The same statement also applies to the so-called Burkardt sum rule [55,56] for the transverse momentum dependent Sivers parton distribution [57,58].

In the literature it was suggested that the ST sum rule should also hold for the three additional transverse momentum dependent FFs $D_{1T}^{\perp}$, G_{1T}, and $H_{1L}^{\perp}$ [37–39]. However, here we have shown that the general proof of the ST sum rule cannot be extended to these cases. We have also demonstrated that, in the light-cone gauge, the proof of the longitudinal momentum sum rule for D_1 is not spoiled by the relatively recently discovered transversely running Wilson lines in the fragmentation correlator.

We have exploited a simple self-consistent quark–pion coupling model in order to explicitly verify/falsify the momentum sum rules. Though the model does not know about all the complexities of QCD, it nevertheless can be used for interesting cross checks. We have been able to verify the sum rule for D_1 as well as the ST sum rule for the Collins function $H_1^{\perp}$ to lowest nontrivial order in the coupling constant. On the other hand, we have shown explicitly that the ST sum rule does not hold for the aforementioned additional three FFs.

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