PRIME IDEALS OF q-COMMUTATIVE POWER SERIES RINGS

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ABSTRACT. We study the "q-commutative" power series ring $R := k_q[[x_1, \ldots, x_n]]$, defined by the relations $x_i x_j = q_{ij} x_j x_i$, for mulitiplicatively antisymmetric scalars q_{ij} in a field k. Our results provide a detailed account of prime ideal structure for a class of noncommutative, complete, local, noetherian domains having arbitrarily high (but finite) Krull, global, and classical Krull dimension. In particular, we prove that the prime spectrum of R is normally separated and is finitely stratified by commutative noetherian spectra. Combining this normal separation with results of Chan, Wu, Yekutieli, and Zhang, we are able to conclude that R is catenary. Following the approach of Brown and Goodearl, we also show that links between prime ideals are provided by canonical automorphisms. Moreover, for sufficiently generic q_{ij}, we find that R has only finitely many prime ideals and is a UFD (in the sense of Chatters).

1. INTRODUCTION

Given a field k and an $n \times n$ matrix $q = (q_{ij})$, with $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1} \in k^\times$, we can construct the "$q$-commutative" power series ring $R = k_q[[x_1, \ldots, x_n]]$. Multiplication is determined by the commutation relations $x_i x_j = q_{ij} x_j x_i$, leading to a rich combinatorial structure (see, e.g., [16]). Moreover, it follows from longstanding ring-theoretic results and techniques (cf. [20],[28]) that R is a complete, local, Auslander regular, noetherian, zariskian (in the sense of Li and Van Oystaeyen [20]) domain with Krull dimension, classical Krull dimension, and global dimension all equal to n. In this paper, the goal is to provide a detailed account of the prime ideal theory of R. Our approach builds on earlier studies (particularly [4, §4], [8, §2], and [10]) on q-commutative polynomial rings (also known as "quantum affine spaces" and "twisted polynomial algebras"). However, it should be no surprise that our work in this paper frequently involves topological considerations not needed in the earlier studies.

1.1. Our results depend on a careful examination of the two-sided ideal structure of the q-commutative Laurent series ring $L = k_q[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]]$. This analysis can be briefly described as follows:

To start, there is an obvious action of the n-torus $H = (k^\times)^n$ on L (and R) by automorphisms. In (3.3) we prove, when k is infinite, that L is H-simple (i.e., that the only H-stable ideals of L are the zero ideal and L itself). Consequently, every H-orbit of prime ideals of L is Zariski dense in $\operatorname{Spec} L$ (when k is infinite).

Next, in (3.11) we prove that extension and contraction of ideals produces a bijection

$$\{\text{ideals of } L\} \longleftrightarrow \{\text{ideals of the center } Z = Z(L)\}$$

for any choice of the field k. This bijection produces a homeomorphism between the prime spectrum $\operatorname{Spec} L$, equipped with the Zariski topology, and $\operatorname{Spec} Z$; see (3.14). It follows that Z is a noetherian domain.

Our results for L parallel, in part, those found in [4, §4], [8, §2], and [10, §1] for q-commutative Laurent polynomial rings (also known as "quantum tori" and "McConnell-Pettit algebras"). Additional references are given in (3.10). However, in contrast to our own work below, the q-commutative Laurent polynomial case permits changes of variables not available for q-commutative Laurent series; see (3.9). In particular, while the center of a q-commutative Laurent polynomial ring is isomorphic to a commutative Laurent polynomial ring, it is possible (following an observation of K. R. Goodearl) that Z as above is not a commutative Laurent series ring; see (3.8).

1.2. The analysis of L can be applied to R as follows. First of all, the $x_1, \ldots, x_n$ provide a stratification

$$\operatorname{Spec} R = \bigsqcup_{w \in W} \operatorname{Spec}_w R,$$

where each w is a subset of $\{1, \ldots, n\}$, and where

$$\operatorname{Spec}_w R = \{P \in \operatorname{Spec} R \mid x_i \in P \Leftrightarrow i \in w\}.$$

Each $\operatorname{Spec}_w R$ is naturally homeomorphic to $\operatorname{Spec} L_w$, where L_w is a q-commutative Laurent series ring in $n - |w|$ variables, for a suitable replacement of the original matrix q; see (4.1).

Furthermore, each $\operatorname{Spec}_w R$ is a union of H-orbits in $\operatorname{Spec} R$, and every H-orbit in $\operatorname{Spec}_w R$ is dense in $\operatorname{Spec}_w R$ (when k is infinite); see (4.2). We also see, in (4.3), that the H-prime ideals of R are exactly those ideals of the form $\langle x_i \mid i \in w \rangle$, for subsets w of $\{1, \ldots, n\}$.

This stratification of $\operatorname{Spec} R$ parallels that found in [4, §4], [8, §2], and [10, §2] for q-commutative polynomial rings.

1.3. Noetherian unique factorization domains were defined by Chatters [6] (cf. [7]) to be noetherian domains such that each height-one prime ideal is completely prime and generated by a normal element. In [27], Venjakob presented the first known examples of noetherian, noncommutative, complete, local, unique factorization domains of global, Krull, and classical Krull dimension $d > 1$; these examples are completed

group algebras of certain uniform pro-p groups of rank 2, and for these examples $d = 2$.

In (4.10) we note, for sufficiently generic choices of the q_{ij}, that the prime ideals of R are precisely the H-prime ideals mentioned above: $\langle x_i \mid i \in w \rangle$, for subsets w of $\{1, \ldots, n\}$. It then follows, in this special case, that R is a noetherian unique factorization domain, in Chatters' sense, having global, Krull, and classical Krull dimension n.

1.4. Following the approach of Brown and Goodearl in [4], we also consider issues related to the localization and representation theory of R (see [2] and [14]). First, in (4.5i) we show that $\operatorname{Spec} R$ is normally separated. Consequently, R satisfies the strong second layer condition. Second, letting G denote the group of automorphisms of R generated by $r \mapsto x_i r x_i^{-1}$, for $r \in R$ and $1 \leq i \leq n$, we show in (4.5ii) that if $P \rightsquigarrow Q$ in $\operatorname{Spec} R$ then there exists an automorphism $\tau \in G$ such that $\tau(P) = Q$. It then follows from well-known results of Stafford [25] and Warfield [29] that R has a classical localization theory, in the sense of Jategaonkar [14], when k is uncountable.

1.5. Combining the normal separation for R established in (4.5i) with results of Chan, Wu, Yekutieli, and Zhang [5], [30], [31], we are able to conclude that R is catenary.

1.6. In addition to similarities with q-commutative polynomial rings, the behavior of q-commutative power series rings also has deep analogues in the theory of quantum semisimple groups and other quantum function algebras; see (e.g.) [3],[4],[13],[15], [17].

1.7. As noted above, R is zariskian (with respect to the $\langle x_1, \ldots, x_n \rangle$-adic filtration), and so it follows from [20] that the ideals of R are closed in the $\langle x_1, \ldots, x_n \rangle$-adic topology; see (2.3ii) and (2.6). This fact plays a key role throughout this paper.

1.8. The Goldie ranks of the prime factors of a given q-commutative polynomial ring have a single finite upper bound (see [10]). At this time we can only ask whether a similar upper bound exists, in general, for the Goldie ranks of the prime factors of R.

1.9. The paper is organized as follows: Section 2 reviews some well known results and develops some of the preliminary theory. Section 3 is a thorough analysis of the ideal structure of L. Section 4 gives the main results for R.

2. q-Commutative Power and Laurent Series Rings: Preliminaries

We assume basic familiarity with noetherian rings (see, e.g., [11],[22]) and filtered rings (see, e.g., [20], [23]).

2.1. **Setup.** The following notation will remain in effect throughout this paper.

(i) To start, k will denote a field, $k^\times$ will denote the multiplicative group of units in k, n will denote a positive integer, and $q = (q_{ij})$ will denote a multiplicatively antisymmetric $n \times n$ matrix (i.e., $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$). We will use $\mathbb{N}$ to denote the set of non-negative integers. We will use Spec to denote the set of prime ideals of a designated ring, and we will always equip such sets with the Zariski topology.

(ii) $R := k_q[[x_1, \ldots, x_n]]$ will denote the associative unital k-algebra of formal skew power series in the indeterminates $x_1, \ldots, x_n$, subject only to the commutation relations $x_i x_j = q_{ij} x_j x_i$. We will refer to R, in general, as a *q-commutative power series ring* (allowing q and n to vary in this usage).

The elements of R are power series

$$\sum_{s \in \mathbb{N}^n} c_s x^s,$$

for $c_s \in k$, for $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$, and for $x^s = x_1^{s_1} \cdots x_n^{s_n}$. We will use x^s to refer to a general monic monomial in R, and we can write $R = k_q[[x]]$.

(iii) Contained within R is the k-subalgebra $k_q[x] = k_q[x_1, \ldots, x_n]$ of q-commuting polynomials in $x_1, \ldots, x_n$. This algebra has been extensively studied; see, for example, [4, §4], [8, §2], [10]. As noted above, our analysis builds on these studies. Observe that R is the completion of $k_q[x]$ at the ideal generated by $x_1, \ldots, x_n$.

(iv) J will denote the augmentation ideal $\langle x_1, \ldots, x_n \rangle$ of R.

(v) L will denote the q-commutative Laurent series ring $k_q[[x^{\pm 1}]] = k_q[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]]$, defined later in (2.9).

Of course, when $k = \mathbb{F}_2$ it follows immediately that R is the commutative power series ring in n variables. However, we will allow this case unless indicated otherwise. We will also include the possibility that $n = 1$, in which case R is the commutative power series ring over k in one variable.

2.2. Much of our analysis depends on standard results concerning filtered rings, which we now briefly review. Again, the reader is referred to [20] and [23] for background.

(i) To start, let A be an (associative unital) ring, and suppose further there exist additive subgroups

$$\cdots \supseteq A_{-2} \supseteq A_{-1} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

of A with $A_i A_j \subseteq A_{i+j}$, for all integers i and j. We refer to the preceding as a *filtration* of A. This filtration turns A into a topological additive group by letting the cosets of the A_i, for all i, form a fundamental system of neighborhoods in A.

(ii) The above filtration on A is *exhaustive* if

$$A = \bigcup_{j \in \mathbb{Z}} A_j,$$

is *separated* if the intersection of the A_j is equal to zero (or equivalently, if the corresponding topology is Hausdorff), and is *complete* if Cauchy sequences converge in the corresponding topology.

(iii) We have

$$\text{gr } A = \cdots \oplus A_{-2}/A_{-1} \oplus A_{-1}/A_0 \oplus A_0/A_1 \oplus A_1/A_2 \oplus \cdots ,$$

the *associated graded ring* corresponding to the given filtration.

(iv) Suppose that the filtration on A is exhaustive, separated, and complete. It follows from standard arguments that A is right noetherian if gr A is right noetherian and that A is left noetherian if gr A is left noetherian.

2.3. Let A be a ring, and let I be an ideal of A.

(i) Keeping (2.2) in mind, we have the *I-adic filtration* of A:

$$A = I^0 \supseteq I^1 \supseteq \cdots$$

We also have the corresponding *I-adic topology*, making A a topological ring. Note that the I-adic filtration of A is exhaustive.

(ii) We also have the *Rees ring* associated to the I-adic filtration on A:

$$\widetilde{A} := A \oplus I \oplus I^2 \oplus \cdots$$

Following the definition in [20, p. 83], when I is contained in the Jacobson radical of A, and when $\widetilde{A}$ is right noetherian, we say that A is *(right) zariskian* with respect to the I-adic filtration.

In [20, p. 87, Proposition] it is proved that A is (right) zariskian with respect to the I-adic filtration if the I-adic filtration is complete and if the associated graded ring corresponding to the I-adic filtration is right noetherian.

The only consequence of the zariskian property necessary for our analysis is the following: If A is zariskian with respect to the I-adic filtration, then every right ideal of A is closed in the I-adic topology [20, p. 85, Corollary].

Before turning to R, we first consider skew power series, in a single variable, over more general coefficient rings.

2.4. Let A be a ring, let α be an automorphism of A, and let $A[[y; \alpha]]$ denote the ring of skew power series

$$\sum_{i=0}^{\infty} a_i y^i = a_0 + a_1 y + a_2 y^2 + \cdots ,$$

for $a_0, a_1, \ldots \in A$, with multiplication determined by $ya = \alpha(a)y$, for $a \in A$. Of course, since α is an automorphism, we can just as well write the coefficients on the right. Set $B = A[[y; \alpha]]$. As either a left or right A-module, we can view B as a direct product of copies of A, indexed by $\mathbb{N}$. Note that y is normal in B (i.e., $By = yB$), and $B/\langle y \rangle$ is naturally isomorphic to A.

(i) Set $f = 1 + b_1 y + b_2 y^2 + \cdots \in B$, for $b_1, b_2, \ldots \in A$, and set $g = 1 + c_1 y + c_2 y^2 + \cdots \in B$, for $c_1, c_2, \ldots \in A$. Then

$$fg = 1 + (b_1 + c_1)y + (b_2 + c_2 + p_2(b_1, c_1))y^2 + (b_3 + c_3 + p_3(b_1, b_2, c_1, c_2))y^3 + \cdots,$$

where $p_i(b_1, \ldots, b_{i-1}, c_1, \ldots, c_{i-1}) \in A$ depends only on $b_1, \ldots, b_{i-1}$ and $c_1, \ldots, c_{i-1}$, for integers $i \geq 2$. If $b_1, b_2, \ldots$ are arbitrary then we can choose $c_1, c_2, \ldots$ such that $fg = 1$, and if $c_1, c_2, \ldots$ are arbitrary then we can choose $b_1, b_2, \ldots$ such that $fg = 1$.

We can deduce that a power series in B is invertible if and only if its degree-zero term is an invertible element of A.

(ii) It follows from (i) that $1 + yh$ is invertible for all $h \in B$. Hence y is contained in the Jacobson radical $J(B)$. Furthermore, it now follows from the natural isomorphism of A onto $B/\langle y \rangle$ that $J(B) = J(A) + \langle y \rangle$.

(iii) The $\langle y \rangle$-adic filtration on B is exhaustive, separated, and complete. The associated graded ring $\mathrm{gr}\, B$ is isomorphic to $A[z; \alpha]$, and so B is right noetherian, by (2.2iv), if A is right noetherian.

(iv) We will refer to the localization of B at the powers of the normal element y as the *skew Laurent series ring* $A[[y^{\pm 1}; \alpha]]$.

2.5. We now briefly summarize some of the readily available information on R.

(i) Writing

$$R = k[[x_1]][[x_2; \tau_2]] \cdots [[x_n; \tau_n]],$$

with $\tau_j(x_i) = q_{ji}x_i$ for all $1 \leq i < j \leq n$, we see from (2.4ii) that $J = \langle x_1, \ldots, x_n \rangle$ is the Jacobson radical of R. It follows that J is the unique (left or right) primitive ideal of R.

(ii) The J-adic filtration on R is exhaustive, separated, and complete. Combining this data with (i), we see that R is a complete local ring with unique primitive factor isomorphic to k.

(iii) The associated graded ring of R corresponding to the J-adic filtration is isomorphic to the noetherian domain $k_q[x]$, and it therefore follows that R is a noetherian domain.

(iv) It follows from [20, p. 174, 6. Theorem (3)] that R is Auslander regular. It follows from [28] that the right Krull dimension, classical Krull dimension, and global dimension of R are all equal to n.

2.6. In view of (2.3ii) and (2.5ii, iii), we see that R is zariskian with respect to the J-adic filtration, since the J-adic filtration is complete and since the associated graded ring of R corresponding to the J-adic filtration is noetherian.

Consequently, the right ideals of R are closed in the J-adic topology, following (2.3ii).

2.7. Also required in our later analysis (particularly in the proofs of (3.3) and (3.11)) is the *graded lexicographic ordering* on $\mathbb{N}^n$. To review, let $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ be n-tuples in $\mathbb{N}^n$. The *total degree* $|s|$ is the sum $s_1 + \cdots + s_n$. Then

$s <_{\mathrm{grlex}} t$ either when $|s| < |t|$ or when $|s| = |t|$ and s precedes t in the lexicographic ordering.

To see the usefulness of this ordering for our purposes, let N denote an infinite collection of pairwise distinct n-tuples in $\mathbb{N}^n$. Then we can write $N = \{j_1, j_2, \ldots\}$ such that $j_\ell <_{\mathrm{grlex}} j_{\ell+1}$, for all positive integers ℓ, and such that

$$\lim_{\ell \to \infty} x^{j_\ell} = 0$$

in the J-adic topology on R.

Also, given a power series $f \in R$, we can choose $s_1, s_2, \ldots \in \mathbb{N}^n$ and $c_{s_1}, c_{s_2}, \ldots \in k$ such that

$$f = \sum_{i=1}^{\infty} c_{s_i} x^{s_i},$$

such that $s_\ell <_{\mathrm{grlex}} s_{\ell+1}$ for all positive integers ℓ, and such that $c_{s_1} \neq 0$. We will refer to s_1 as the *graded lexicographic degree* of f.

When the meaning is clear we will simply use "$<$" to denote " $<_{\mathrm{grlex}}$."

The tools developed so far allow the following observation:

2.8. **Proposition.** *Let P be a prime ideal of R. Then $P \cap k_q[x]$ is prime.*

Proof. Set $Q = P \cap k_q[x]$. Let $a, b \in k_q[x]$, and suppose that $a(k_q[x])b \subseteq Q$. Choose $f \in R$. For each non-negative integer i, let f_i denote the sum of the monomials in f of total degree no greater than i. Since P is closed in the J-adic topology, and since $a f_i b \in Q \subseteq P$ for all i, we see that

$$afb = \lim_{i \to \infty} a f_i b \in P.$$

Since f was arbitrary, we see that $aRb \subseteq P$. Therefore, since P is prime, either a or b is contained in P. But then either a or b is contained in $Q = P \cap k_q[x]$, and so Q is a prime ideal of $k_q[x]$. $\square$

2.9. Now let X denote the multiplicatively closed subset of R generated by 1 and the indeterminates $x_1, \ldots, x_n$. Since each x_i is normal, we see that X is an Ore set in R. For the remainder we will let L denote the *q-commutative Laurent series ring* $k_q[[x^{\pm 1}]] = k_q[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]]$, obtained via the Ore localization of R at X. (See, e.g., [11] or [22] for details on Ore localizations.)

Each Laurent series in L will have the form

$$\sum_{s \in \mathbb{Z}^n} c_s x^s,$$

for $c_s \in k$, for $s = (s_1, \ldots, s_n) \in \mathbb{Z}^n$, and for $x^s = x^{s_1} \cdots x^{s_n}$, but with $c_s = 0$ for $\min\{s_1, \ldots, s_n\} \ll 0$.

In particular, for $n > 1$ it is *not* the case that L can be written as an iterated skew Laurent series ring

$$k[[x_1^{\pm 1}]][[x_2^{\pm 1}; \tau_2]] \cdots [[x_n^{\pm 1}; \tau_n]],$$

which is well known to be a division ring.

We always view R as a subalgebra of L. Also inside L is the q-commutative Laurent polynomial ring, $k_q[x^{\pm 1}] = k_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the localization of $k_q[x]$ at the products of the $x_1, \ldots, x_n$.

3. Prime Ideals in q-Commutative Laurent Series Rings

We now give a detailed description of the ideals, and in particular the prime ideals, of q-commutative Laurent series rings. Retain the notation of the preceding section, continuing to let $R = k_q[[x]]$ and $L = k_q[[x^{\pm 1}]]$. Henceforth, we will let Z denote the center of L.

We begin by considering the "obvious" n-torus action on L.

3.1. Let H denote the algebraic torus $(k^\times)^n$, and set

$$h(s) = h_1^{s_1} \cdots h_n^{s_n},$$

for $h = (h_1, \ldots, h_n) \in H$ and $s = (s_1, \ldots, s_n) \in \mathbb{Z}^n$. For each $h \in H$, it is not hard to check that the assignment

$$h.x^s = h(s)x^s,$$

for $s \in \mathbb{N}^n$, extends linearly and continuously to a k-algebra automorphism of R. It also is not hard to check, then, that we thereby obtain an action of H on R by k-algebra automorphisms. In turn, this action by H on R extends to an action by automorphisms on the localization L of R.

Observe that the H-actions on R and L induce H-actions on $\operatorname{Spec} R$ and $\operatorname{Spec} L$. Also note, when k is infinite, that the set of H-eigenvectors in R and the set of monomials in R coincide. The analogous statement for L also holds true.

3.2. Let A be a ring and let Γ be a group of automorphisms of A. We will refer to ideals of A stable under the Γ-action as Γ-*ideals*, and we will say that A is Γ-*simple* if the only Γ-ideals of A are the zero ideal and A itself. We will say that a Γ-ideal I of A (other than A itself) is Γ-*prime* if whenever the product of two Γ-ideals of A is contained in I, one of these Γ-ideals must itself be contained in I. In particular, if P is a prime ideal of A that is also a Γ-ideal, then it immediately follows that P is also Γ-prime. If A is right or left noetherian, recall that an ideal I of A is a Γ-ideal exactly when $g(I) = I$ for all $g \in \Gamma$.

Suppose A is right or left noetherian, and suppose that I is a Γ-prime ideal of A. Then I is a semiprime ideal of A and is the intersection of a finite Γ-orbit $I_1, \ldots, I_t$ of prime ideals in A all minimal over I; see (e.g.) [3, II.1.10].

3.3. **Proposition.** *Assume that k is infinite.* (i) *Every nonzero H-ideal of R is generated by monomials.* (ii) *L is H-simple.*

Proof. (i) Let I be a nonzero H-ideal of R, and choose

$$0 \neq f = \sum_{i \in \mathbb{N}^n} a_i x^i \in I.$$

The first step of our proof is to show, as follows, that each x^i appearing nontrivially in f is contained in I.

To start, let $\Psi = \{i \in \mathbb{N}^n \mid a_i \neq 0\}$. For notational convenience we will assume that Ψ is infinite; the case when Ψ is finite can be handled similarly. Equip Ψ with the graded lexicographic ordering (see (2.7)), and write $\Psi = \{j_1, j_2, j_3, \ldots\}$ such that $j_\ell < j_{\ell+1}$ for all positive integers ℓ. So

$$f = \sum_{\ell=1}^{\infty} a_{j_\ell} x^{j_\ell}.$$

Replacing f with $a_{j_1}^{-1} f$, we can assume without loss that $a_{j_1} = 1$. Set $f_1 = f$ and $a_{j_\ell}(1) = a_{j_\ell}$, for all ℓ. Since k is infinite, there exists an $h \in H$ such that $h(j_1) \neq h(j_2)$. Set $f_2 =$

$$\frac{h.f - h(j_2)f}{h(j_1) - h(j_2)} = x^{j_1} + \sum_{\ell=3}^{\infty} \left(\frac{h(j_\ell) - h(j_2)}{h(j_1) - h(j_2)} \right) a_{j_\ell} x^{j_\ell} = x^{j_1} + \sum_{\ell=3}^{\infty} a_{j_\ell}(2) x^{j_\ell} \in I,$$

where

$$a_{j_\ell}(2) = \left(\frac{h(j_\ell) - h(j_2)}{h(j_1) - h(j_2)} \right) . a_{j_\ell}(1).$$

Continuing, we get

$$f_m = x^{j_1} + \sum_{\ell=m+1}^{\infty} a_{j_\ell}(m) x^{j_\ell} \in I,$$

for all integers $m \geq 2$. Note that $f_m \to x^{j_1}$ in the J-adic topology (again see (2.7)). By (2.6), it then follows that $x^{j_1} \in I$.

Replacing f with $f - x^{j_1} \in I$, we can repeat the above procedure to show that $x^{j_2} \in I$. Continuing, we see that $x^{j_1}, x^{j_2}, x^{j_3}, \ldots \in I$. This completes the first step of our proof.

Now let M be the set of all monic monomials appearing nontrivially in power series contained in I. It follows from the above that I is the smallest closed ideal of R containing M. However, by (2.6), it then follows that I is the smallest ideal of R containing M. In other words, I is generated by M, and part (i) follows.

(ii) Let I be a nonzero H-ideal of L. Then $I \cap R$ is a nonzero H-ideal of R. Therefore, $I \cap R$ contains a nonzero monomial x^s for some $s \in \mathbb{N}^n$. But x^s is invertible in L, and so $I = L$. $\square$

3.4. **Corollary.** *Assume that k is infinite.* (i) *Every H-orbit of prime ideals in L is (Zariski) dense in $\operatorname{Spec} L$.* (ii) *$J(L) = 0$.*

Proof. Let P be a prime ideal of L, let O be the H-orbit of P in $\operatorname{Spec} L$, and let I be the intersection of all of the ideals in O. Then I is an H-ideal of L, and I is not equal to L itself. Therefore, by (3.3ii), $I = 0$. It follows that the closure of O in $\operatorname{Spec} L$ is $\operatorname{Spec} L$, and so O is dense in $\operatorname{Spec} L$. Part (i) follows. Part (ii) also follows, since the intersection of the orbit of a primitive ideal must equal the zero ideal. $\square$

3.5. Part (ii) of the preceding corollary also follows from [26], a study of the Jacobson radical in general skew Laurent series rings.

3.6. Aside from the last corollary, we are unable at this time to provide information on the primitive spectrum of L. In particular, we ask: Must every primitive ideal of L be maximal?

3.7. Next, following [10, §1] (cf. [4, §4]), we describe Z.
  (i) To start, consider the alternating bicharacter $\sigma \colon \mathbb{Z}^n \times \mathbb{Z}^n \to k^\times$ defined by

$$\sigma(s,t) = \prod_{i,j=1}^{n} q_{ij}^{s_i t_j},$$

for $s = (s_1, \ldots, s_n), t = (t_1, \ldots, t_n) \in \mathbb{Z}^n$. As noted in [10, 1.1],

$$x^s x^t = \sigma(s,t) x^t x^s.$$

  (ii) Set

$$S = \{ s \in \mathbb{Z}^n \mid \sigma(s,t) = 1 \text{ for all } t \in \mathbb{Z}^n \}.$$

As noted in [10, 1.2], a monomial x^s, for $s \in \mathbb{Z}^n$, commutes with all monomials x^t, for all $t \in \mathbb{Z}^n$, if and only if $s \in S$. It follows that

$$Z = \left\{ \sum_{s \in S} c_s x^s \in L \;\middle|\; c_s \in k, \, c_s = 0 \text{ for } \min\{s_1, \ldots, s_n\} \ll 0 \right\}.$$

It will follow from (3.13) that Z is noetherian. However, it is easy to see that Z is a filtered commutative domain.
  (iii) Let

$$f = \sum_{s \in N} c_s x^s$$

be a Laurent series in L, for a suitable index set $N \subseteq \mathbb{Z}^n$. We can write

$$f = \sum_{s \in N \cap S} c_s x^s + \sum_{t \in N \setminus S} c_t x^t.$$

It now follows from (ii) that Z is a right (and left) Z-module direct summand of L.
  (iv) It is not hard to check that L is finite over Z if and only if each q_{ij} is a root of unity, if and only if $\operatorname{rank} S = n$.

3.8. Let $b_1, \ldots, b_r$ be a $\mathbb{Z}$-basis for S. As noted in [10, 1.3i], the center of $k_q[x^{\pm 1}]$ is generated as a k-algebra by $(x^{b_1})^{\pm 1}, \ldots, (x^{b_r})^{\pm 1}$ and so is isomorphic to a commutative Laurent polynomial ring over k in r many variables. One might expect, similarly, that the center of L is a commutative Laurent series ring.

However, the situation for L is more subtle than for $k_q[x^{\pm 1}]$. Let γ be a nonzero, non-root-of-unity in k, and consider the q-commutative Laurent series ring in variables x_1, x_2, x_3, such that $x_1 x_2 = x_2 x_1$, $x_1 x_3 = \gamma x_3 x_1$, and $x_2 x_3 = \gamma x_3 x_2$. We see, in this case, that the central monic monomials all have the form $(x_1 x_2^{-1})^i$, for integers

i. It is now not hard to see that the center of this q-commutative Laurent series ring is $k[(x_1 x_2^{-1})^{\pm 1}]$ and is certainly not a commutative Laurent series ring. Indeed, the center in this case is isomorphic to a commutative Laurent *polynomial* ring. (Our gratitude to Ken Goodearl for this observation.)

3.9. More generally, let $\mu_1, \ldots, \mu_n$ be any $\mathbb{Z}$-basis for $\mathbb{Z}^n$. Key to the approach used throughout [8] and [10] (see e.g. [10, §1.6]) is that the assignment $x^{\mu_i} \mapsto y_i$, for $1 \leq i \leq n$, produces an isomorphism from $k_q[x^{\pm 1}]$ onto $k_r[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$, where $r = (r_{ij})$ is the multiplicatively antisymmetric $n \times n$ matrix defined by $r_{ij} = \sigma(\mu_i, \mu_j)$. (Here, σ is as in (3.7).) In this way, for $k_q[x^{\pm 1}]$, one can often reduce to the case where q has a particularly desirable form.

However, changes of variables of this type do not in general extend to L. To illustrate, consider the commutative Laurent polynomial ring $k[X^{\pm 1}]$, in the single variable X; note that $X^{-1} \mapsto Y$ induces an isomorphism of this algebra onto the Laurent polynomial ring $k[Y^{\pm 1}]$, in the single variable Y. However, this automorphism of Laurent polynomial rings does not extend to an automorphism from the Laurent series ring $k[[X^{\pm 1}]]$ onto $k[[Y^{\pm 1}]]$.

3.10. We now present the "main lemma" of this paper. Analogous conclusions for q-commutative Laurent polynomial and related rings have appeared in various forms, including [1, 5.1], [4, 4.4], [8, §2], [9, 2.2], [10, 1.4], [12, 2.1], and [24, Chapter 11, §3].

3.11. **Lemma.** (i) *Let I be an ideal of L. Then $I = L(I \cap Z)$.*
  (ii) *Let K be an ideal of Z. Then $K = (LK) \cap Z$.*

Proof. (i) If $I = 0$ there is nothing to prove, and so we assume otherwise. Let g be a nonzero element of I, and choose $u \in \mathbb{N}^n$ such that $x^u g \in I \cap R$. Set $f = x^u g$. Let S be as in (3.7ii), and fix a transversal T for S in $\mathbb{Z}^n$. By (3.7ii) we can write

$$f = \sum_{t \in T} x^t z_t,$$

where each z_t is a skew Laurent series contained in Z, and where each $x^t z_t \in R$.
  Set

$$T_f = \{t \in T \mid x^t z_t \neq 0\}.$$

Then

$$f = \sum_{t \in T_f} x^t z_t.$$

Choose an arbitrary element t_0 in the set T_f. The main work of the proof is to prove the following claim: $x^{t_0} z_{t_0} \in I \cap R$. Of course, if $f = x^{t_0} z_{t_0}$ there is nothing to prove, and so we assume otherwise.

To start, we can choose subsets S_t of S, for $t \in T_f \setminus \{t_0\}$, such that

$$
f \;=\; x^{t_0} z_{t_0} + \sum_{t \in T_f \setminus \{t_0\}} x^t z_t \;=\; x^{t_0} z_{t_0} + \sum_{t \in T_f \setminus \{t_0\}} x^t \sum_{s \in S_t} c_s x^s
$$

$$
=\; x^{t_0} z_{t_0} + \sum_{t \in T_f \setminus \{t_0\}} \sum_{s \in S_t} c_s x^t x^s,
$$

for suitable choices of $c_s \in k$.

Next, regrouping the preceding monomials $c_s x^t x^s$ according to their total degree, we can write

$$
f \;=\; x^{t_0} z_{t_0} + \sum_{i \in N} \sum_{t \in T_i} \sum_{s \in S_t^i} c_s x^t x^s,
$$

for sets N, $T_i \subseteq T_f \setminus \{t_0\}$, and $S_t^i \subseteq S$ satisfying the following conditions:

(1) $N = \{1, 2, \ldots\}$ is a (possibly finite) index set.
(2) $\{d_i \mid i \in N\}$ is a set of positive integers such that $d_i < d_{i+1}$ for all $i \in N$ (if $|N| > 1$).
(3) If $t \in T_i$ and $s \in S_t^i$, then the total degree of the monomial $c_s x^t x^s$ is d_i.

Note that the T_i need not be pairwise disjoint. However, $T_f \setminus \{t_0\} = \cup T_i$. We now have

$$
f = x^{t_0} z_{t_0} + \sum_{i \in N} \sum_{t \in T_i} x^t w_{t,i},
$$

where

$$
w_{t,i} = \sum_{s \in S_t^i} c_s x^s,
$$

for $i \in N$ and $t \in T_i$. Note that each $c_s x^s$ is a central monomial, and so each $w_{t,i}$ is contained in Z.

To prove the claim, set

$$
u_1 = f - x^{t_0} z_{t_0} = \sum_{i \in N} \sum_{t \in T_i} x^t w_{t,i}
$$

and

$$
f_1 = f = x^{t_0} z_{t_0} + u_1 \in I \cap R.
$$

Recall that $J = \langle x_1, \ldots, x_n \rangle$ is the augmentation ideal of R and that each $\sum_{t \in T_i} x^t w_{t,i}$, for $i \in N$, is a sum of monomials of total degree d_i ($\geq d_1$). Then $u_1 \in J^{d_1}$. Next, choose some $r \in T_1$. Since $r \neq t_0$, and since r and t_0 are contained in the transversal T for S in $\mathbb{Z}^n$, it follows that $r - t_0 \notin S$. Hence, there exists $v \in \mathbb{Z}^n$ such that

$$
\sigma(v, t_0) \neq \sigma(v, r),
$$

where σ is as defined in (3.7i). Consider

$$
\rho_v(f_1) = \frac{x^v f_1 x^{-v} - \sigma(v, r) f_1}{\sigma(v, t_0) - \sigma(v, r)}.
$$

Note that $\rho_v(f_1)$ is contained in $I \cap R$. Also,

$$\rho_v(f_1) = x^{t_0} z_{t_0} + \sum_{i \in N} \sum_{\substack{t \in T_i, \\ t \neq r}} \frac{\sigma(v,t) - \sigma(v,r)}{\sigma(v,t_0) - \sigma(v,r)} x^t w_{t,i}.$$

Repeating the preceding process, at most $|T_1|$-many times, we obtain

$$f_2 = x^{t_0} z_{t_0} + u_2 \in I \cap R,$$

with

$$u_2 = \sum_{\substack{i \in N \\ i \geq 2}} \sum_{t \in T_i \setminus T_1} x^t w'_{t,i},$$

for suitable $w'_{t,i} \in Z$. Note that each $w'_{t,i}$ appearing in the preceding will be a nonzero scalar multiple of $w_{t,i}$ and that $u_2 \in J^{d_2}$. If $u_2 = 0$ then $x^{t_0} z_{t_0} \in I \cap R$, and the claim follows; so assume otherwise.

Continuing as above, we obtain either $x^{t_0} z_{t_0} \in I \cap R$ (proving the claim) or an infinite sequence

$$f_i = x^{t_0} z_{t_0} + u_i \in I \cap R, \quad i = 1, 2, \ldots$$

converging to $x^{t_0} z_{t_0}$ in the J-adic topology. Since the ideals in R are closed in the J-adic topology, as noted in (2.6), we see that $x^{t_0} z_{t_0} \in I \cap R$. The claim follows.

Next, it follows from the claim that $z_{t_0} \in I \cap Z$, and so

$$x^{t_0} z_{t_0} \in (L(I \cap Z)) \cap R.$$

Recall, however, that t_0 was arbitrarily chosen from T_f. Consequently,

$$x^t z_t \in (L(I \cap Z)) \cap R$$

for all $t \in T_f$.

Following (2.7), and recalling that T is a transversal for S in $\mathbb{Z}^n$, we can write

$$\{x^t z_t \mid t \in T_f\}$$

as a (possibly finite) set

$$\{x^{t_1} z_{t_1}, x^{t_2} z_{t_2}, \ldots\},$$

for integers $1, 2, \ldots$, with the graded lexicographic degree of $x^{t_j} z_{t_j}$ less than that of $x^{t_{j'}} z_{t_{j'}}$ whenever $j < j'$. Again following (2.7), the sequence of partial sums

$$x^{t_1} z_{t_1} + \cdots + x^{t_j} z_{t_j}$$

converges to f in the J-adic topology on R. Therefore

$$f \in L(I \cap Z) \cap R,$$

since the ideal $L(I \cap Z) \cap R$ is closed in R.

Now note that

$$g = x^{-u} f \in L(L(I \cap Z) \cap R) \subseteq L(I \cap Z).$$

Therefore, $I \subseteq L(I \cap Z)$. Of course $L(I \cap Z) \subseteq I$, and (i) follows.

(ii) This follows from (3.7iii). □

We now record some applications.

3.12. **Corollary.** Z *is a noetherian domain.*

Proof. We know that L is a noetherian domain, and from (3.11) we know that there is an inclusion preserving bijection between the ideals of Z and the ideals of L. □

The second application is a partial analogue to [21, 1.3].

3.13. **Corollary.** *If S is trivial then L is simple.*

Proof. If S is trivial then $Z = k$, and it then follows from (3.11i) that L is simple. □

The following is analogous to [10, 1.5].

3.14. **Proposition.** *The assignments $P \mapsto P \cap Z$, for $P \in \operatorname{Spec} L$, and $Q \mapsto LQ$, for $Q \in \operatorname{Spec} Z$, provide mutually inverse homeomorphisms between $\operatorname{Spec} L$ and $\operatorname{Spec} Z$.*

Proof. It follows from (3.11) that the assignments $I \mapsto I \cap Z$, for ideals I of L, and $K \mapsto LK$, for ideals K of Z, provide mutually inverse, inclusion preserving bijections between the sets of ideals of L and Z.

Next, it is a well known elementary fact that a prime ideal in any ring intersects with the center at a prime ideal, and so $P \cap Z$ is a prime ideal of Z for all prime ideals P of L. Thus $P \mapsto P \cap Z$ produces a map from $\operatorname{Spec} L$ to $\operatorname{Spec} Z$.

Now let Q be a prime ideal of Z. To show that LQ is a prime ideal of L, suppose that I_1 and I_2 are ideals of L such that $I_1 I_2 \subseteq LQ$. By (3.11i), there exist ideals K_1 and K_2 of Z such that $I_1 = LK_1$ and $I_2 = LK_2$. So

$$L(K_1 K_2) = (LK_1)(LK_2) = I_1 I_2 \subseteq LQ.$$

Then, by (3.11ii),

$$K_1 K_2 = (L(K_1 K_2)) \cap Z \subseteq (LQ) \cap Z = Q.$$

Since Q is prime, it follows that either K_1 or K_2 is contained in Q. Therefore, either $I_1 = LK_1$ or $I_2 = LK_2$ is contained in LQ, and we see that LQ is a prime ideal of L. Hence, $Q \mapsto LQ$ produces a map from $\operatorname{Spec} Z$ to $\operatorname{Spec} L$.

It now follows from the first paragraph that the assignments $P \mapsto P \cap Z$, for prime ideals P of L, and $Q \mapsto LQ$, for prime ideals Q of Z, provide mutually inverse, Zariski continuous bijections between $\operatorname{Spec} L$ and $\operatorname{Spec} Z$. The proposition follows. □

4. Prime Ideals in q-Commutative Power Series Rings

Our aim now is to systematically develop a detailed description of the prime spectrum of R. Using the results we obtained in the preceding sections for q-commutative Laurent series rings, our approach now largely follows – and in many cases mimics – the studies of q-commutative polynomial rings found in [4, §4] and [10]. Moreover, much of the theory developed in this section is also analogous to that for various

more complicated (finitely generated) quantum function algebras; see (e.g.) [3] for details.

4.1. We start by giving an account of the "obvious" stratification of $\operatorname{Spec} R$. Let W be the set of subsets of $\{1, \ldots, n\}$.

(i) For each $w \in W$, let J_w be the ideal of R generated by the indeterminates x_i for $i \in w$, and let $R_w = R/J_w$. Set $n_w = n - |w|$, and observe that R_w is isomorphic to a q-commutative power series ring (for a replacement of q by a suitable $n_w \times n_w$ matrix) in n_w many variables. In particular, R_w is a domain, and so each J_w is completely prime. Next, let X_w be the multiplicatively closed subset of R_w generated by 1 and the images of the x_i for $i \notin w$. Set

$$L_w = R_w X_w^{-1},$$

the Ore localization of R_w at the set X_w (which consists of normal regular elements of R_w). Then L_w is isomorphic to a q-commutative Laurent series ring in n_w many variables (again using a suitable replacement of q).

(ii) For each $w \in W$, let

$$\operatorname{Spec}_w R = \{P \in \operatorname{Spec} R \mid x_i \in P \Leftrightarrow i \in w\}.$$

Then

$$\operatorname{Spec} R = \bigsqcup_{w \in W} \operatorname{Spec}_w R.$$

Notice that each $\operatorname{Spec}_w R$ is a locally closed subset of $\operatorname{Spec} R$; specifically, $\operatorname{Spec}_w R$ is equal to the intersection of the closed set of prime ideals containing x_i, for $i \in w$, with the open set of prime ideals not containing x_i, for $i \notin w$. Also, the closure in $\operatorname{Spec} R$ of each $\operatorname{Spec}_w R$ is a union of subsets $\operatorname{Spec}_{w'} R$, for suitable $w' \in W$, and so this partition of $\operatorname{Spec} R$ is in fact a stratification. Equip each $\operatorname{Spec}_w R$ with the subspace topology inherited from $\operatorname{Spec} R$.

It follows from [11, Theorem 10.18] that there is a natural homeomorphism

$$\operatorname{Spec}_w R \xrightarrow{\Theta_w} \operatorname{Spec} L_w$$

obtained via the assignment

$$P \longmapsto (P/J_w)L_w,$$

for $P \in \operatorname{Spec}_w R$. Note that $\Theta_w(J_w) = 0$, for all $w \in W$.

(iii) Let $w \in W$, and let Z_w denote the center of L_w. By (3.12), Z_w is a commutative noetherian domain, and it follows from (3.14) that $\operatorname{Spec}_w R$ is homeomorphic to $\operatorname{Spec} Z_w$.

Recall the H-actions on R and $\operatorname{Spec} R$ discussed in (3.1).

4.2. Let $w \in W$, and assume that k is infinite.

(i) Since each x_i is an H-eigenvector, it follows that $\operatorname{Spec}_w R$ is a union of H-orbits of prime ideals in $\operatorname{Spec} R$, and the H-action on $\operatorname{Spec} R$ restricts to an H-action on $\operatorname{Spec}_w R$.

(ii) Again set $n_w = n - |w|$, and let H_w denote the n_w-torus $(k^\times)^{n_w}$. Then H_w acts on L_w as in (3.1), and so H_w also acts on $\operatorname{Spec} L_w$. By (3.3ii), each L_w is H_w-simple.

(iii) Letting H act on $\operatorname{Spec}_w R$ as in (i), there is an "obvious" surjection of H onto H_w such that the homeomorphism $\Theta_w : \operatorname{Spec}_w R \to \operatorname{Spec} L_w$, and its inverse, are both H-equivariant. In particular, by (ii), each L_w is H-simple.

(iv) It follows from (iii) or (3.4) that each H-orbit of prime ideals in $\operatorname{Spec}_w R$ is dense in $\operatorname{Spec}_w R$. (Compare e.g. with [4, §4.3].)

4.3. **Proposition.** *Assume that k is infinite. Then $\{J_w \mid w \in W\}$ is the set of H-prime ideals of R.*

Proof. To start, we saw in (4.1i) that each J_w is a (completely) prime ideal of R. Moreover, each J_w is H-stable. It therefore follows from (3.2) that each J_w is an H-prime ideal of R.

Conversely, let I be an arbitrary H-prime ideal of R. Then, by (3.2), $I = I_1 \cap \cdots \cap I_t$ for some finite H-orbit $I_1, \ldots, I_t$ of prime ideals of R. As noted in (4.2i), each stratum in the stratification described in (4.1) is a union of H-orbits, and so $I_1, \ldots, I_t \in \operatorname{Spec}_w R$ for some $w \in W$. Therefore,

$$\Theta_w(I_1), \ldots, \Theta_w(I_t)$$

is a single H-orbit of prime ideals of L_w, by (4.2iii). Again following (4.2iii), we see that L_w is H-simple, and so

$$\Theta_w(I_1) \cap \cdots \cap \Theta_w(I_t) = 0$$

in L_w. Since L_w is prime, it now follows that $\Theta_w(I_1), \ldots, \Theta_w(I_t)$ (which comprise a single H-orbit) must all equal 0. Consequently,

$$I = I_1 = \cdots = I_t = J_w,$$

by (4.1ii). The proposition follows.   □

4.4. Next, we consider localization and representation theoretic issues, in the sense of [2] and [14]. Retaining the notation of (4.1), recall that the prime spectrum of R is *normally separated* if for each inclusion of prime ideals $P_0 \subsetneq P_1$ in $\operatorname{Spec} R$ there exists an element $y \in P_1 \setminus P_0$ such that $Ry = yR + P_0$. Also, for prime ideals P and Q of R, there is a *(second layer) link* $P \rightsquigarrow Q$ provided $P \cap Q/PQ$ has a nonzero R-R-bimodule factor that is torsionfree as both a left R/P-module and a right R/Q-module; we can thus view $\operatorname{Spec} R$ as a directed graph, the connected components of which are termed the *cliques* in $\operatorname{Spec} R$.

A definition of the *right strong second layer condition* can be found, for example, in [11, p. 206]. Normal separation implies the strong second layer condition; see, for instance, [11, 12.17]. We will use "rank( )" to denote Goldie rank.

Our approach in the following theorem is largely based on [4].

4.5. **Theorem.** (i) *The prime spectrum of R is normally separated, and consequently, R satisfies the strong second layer condition.* (ii) *Let G denote the group of k-algebra automorphisms of R generated by the maps $r \mapsto x_i r x_i^{-1}$ for $r \in R$ and $1 \leq i \leq n$. If P and Q are prime ideals of R such that $P \rightsquigarrow Q$, then $\tau(P) = Q$ for some $\tau \in G$. Consequently, if X is a clique in $\mathrm{Spec}\, R$, then $\mathrm{rank}(R/P) = \mathrm{rank}(R/P')$ for all $P, P' \in X$.*

Proof. (i) Suppose that $P_0 \subsetneq P_1$ are prime ideals of R, and choose w such that $P_0 \in \mathrm{Spec}_w R$. In particular, $J_w \subseteq P_0 \subsetneq P_1$. If $P_1 \notin \mathrm{Spec}_w R$ then there exists some $x_i \in P_1 \setminus P_0$; since $R x_i = x_i R$ we see that normal separation holds in this case. Now suppose that $P_1 \in \mathrm{Spec}_w R$. Set $P_0' = \Theta_w(P_0)$ and $P_1' = \Theta_w(P_1)$, as in (4.1ii). By (4.1ii), $P_0' \subsetneq P_1'$. Therefore, by (3.11i), there exists an element z in the center Z_w of L_w such that $z \in P_1' \setminus P_0'$. Moreover, in L_w, z is regular modulo P_0'. Now recall from (4.1ii) that L_w is the localization of R_w at the set X_w (consisting of normal regular elements of R_w) and so there exists some $u \in X_w$ such that $uz \in R_w$. Note that uz is a normal element of R_w, since u is normal and z is central. Also, it is easy to see that

$$uz \in (P_1' \cap R_w) \setminus (P_0' \cap R_w).$$

But $(P_0' \cap R_w) = P_0/J_w$ and $(P_1' \cap R_w) = P_1/J_w$. So let y be the preimage in R of uz. Then $y \in P_1 \setminus P_0$, and $Ry = yR + P_0$. We can now conclude that the prime spectrum of R is normally separated. Part (i) follows.

(ii) Suppose that P and Q are prime ideals of R such that $P \rightsquigarrow Q$. It follows (e.g.) from [11, 12.15], for all $1 \leq i \leq n$, that $x_i \in P$ if and only if $x_i \in Q$. Hence, there exists $w \in W$ such that $P, Q \in \mathrm{Spec}_w R$. In particular, $J_w \subseteq P \cap Q$.

Set $P' = P/J_w$ and $Q' = Q/J_w$. It follows directly from [4, 2.7] that either $P' \rightsquigarrow Q'$ in $\mathrm{Spec}\, R_w$ or that $\tau(P) = Q$ for some $\tau \in G$.

So assume that $P' \rightsquigarrow Q'$ in R_w. Set $P'' = P'.L_w$ and $Q'' = Q'.L_w$. It is left as an exercise, then, to prove that $P'' \rightsquigarrow Q''$ in L_w. However, since every ideal of L_w is centrally generated, as proved in (3.11i), it follows that $P'' = Q''$. Hence $P = Q$. Part (ii) follows. $\qquad \square$

A more detailed description of the link structure of $\mathrm{Spec}\, R$, following the methods of [4], is left to the interested reader.

4.6. Next, we combine (4.5) with [25],[29] to obtain a specific application to the localization theory of R. Recall, when P is a prime ideal of R, that $\mathcal{C}(P)$ denotes the set of elements of R regular modulo P. If X is a set of prime ideals of R, then $\mathcal{C}(X)$ denotes the intersection of all of the $\mathcal{C}(P)$ for $P \in X$, and if $\mathcal{C}(X)$ is a right Ore set then we use R_X to denote the right Ore localization of R at $\mathcal{C}(X)$.

Now suppose that X is a clique of prime ideals of R, and suppose that $\mathcal{C}(X)$ is a right Ore set in R. Following [14, §7.1], we say that X is *(right) classically localizable* provided R_X has the following properties: (1) R_X/PR_X is a simple artinian ring for all $P \in X$. (2) Every right primitive ideal of R_X has the form PR_X for some $P \in X$.

(3) The right R_X-injective hull of an arbitrary simple right R_X-module is the union of its socle series.

4.7. **Corollary.** *Suppose that k is uncountable. Then the cliques in* $\operatorname{Spec} R$ *are classically localizable.*

Proof. By [25, 4.5] and [29, Theorem 8], cliques X for which there is an upper bound on the Goldie ranks modulo prime ideals in X, in rings with the strong second layer condition, are classically localizable. The corollary follows, then, from (4.5). $\square$

We do not know, in general, whether the cliques in $\operatorname{Spec} R$ are classically localizable.

4.8. **Catenarity.** We obtain a second corollary to (4.5) by combining it with the results and techniques of [5], [30], and [31]. (Our gratitude to James Zhang both for bringing this corollary to our attention and for suggesting to us its proof.)

Recall in a given ring that a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_u$, connecting P_0 to P_u, is *saturated* if for all $1 \leq i \leq u$ there does not exist a prime ideal P' such that $P_{i-1} \subsetneq P' \subsetneq P_i$. Further recall that a ring is *catenary* if for each of its comparable pairs of prime ideals $P \subsetneq Q$, every saturated chain of prime ideals connecting P to Q has the same length.

4.9. **Corollary.** *R is catenary.*

Proof. (We require familiarity with the terminology of [5], [30], and [31].) To start, as noted in (2.5iv), R is Auslander regular of global dimension n and so in particular is Auslander Gorenstein. Therefore, since R is noetherian and local, it follows from [5, 4.3] (cf. [19, 6.3]) that R is AS-Gorenstein (Artin-Schelter Gorenstein).

Next, from [5, Proof of Corollary 0.3, p. 302], it follows that there exists an Auslander (in the sense of [31, §2]), pre-balanced dualizing complex over R. Moreover, by [5, Theorem 0.1], this dualizing complex must be Cdim-symmetric. Since R is normally separated, by (4.5i), we see that R satisfies all of the hypotheses of [30, Theorem 6.5], whose conclusion now ensures that R is catenary. $\square$

4.10. **A special case.** We now consider the following situation: Assume that the abelian subgroup $\langle q_{ij} \rangle$ of $k^\times$ is free of rank $n(n-1)/2$. (This condition will hold, for instance, if the q_{ij}, for $1 \leq i < j \leq n$, are algebraically independent over k.) Of course, in this case k is infinite.

(i) Using (3.13) it is not hard to show that each L_w, for $w \in W$, is simple. Consequently, $\operatorname{Spec} R = \{J_w \mid w \in W\}$. In particular, in this special case the prime ideals of R are all completely prime.

(ii) It follows from (i) and (4.5ii) that each J_w can only be linked to itself. Hence, by (4.5i) and [4, 2.5], each J_w is linked to itself and only to itself. It then follows from (4.5i) and (e.g.) [11, 14.20] that each J_w is classically localizable.

(iii) It follows from (i) that $\langle x_1 \rangle, \ldots, \langle x_n \rangle$ are exactly the height-one prime ideals of R. Moreover, each $\langle x_i \rangle$ is generated by the normal element x_i of R. In [6] (cf. [7]),

Chatters defined a noncommutative noetherian *unique factorization domain (UFD)* to be a noetherian domain in which every height-one prime ideal is completely prime and generated by a normal element. Hence R is a UFD in this sense. (Recent results on unique factorization in quantum function algebras can be found in [17],[18].)

In [27], the first known examples of noetherian, noncommutative, complete, local, unique factorization domains of global, Krull, and classical Krull dimension $d > 1$ were presented; these examples are completed group algebras of certain uniform pro-p groups of rank 2, and for these algebras $d = 2$. However, $R = k_q[[x]]$, in the present (suitably generic) special case, provides examples of noncommutative, complete, local, unique factorization domains of global, Krull, and classical Krull dimension equal to n.

4.11. **Example.** To see that prime factors of L (and of R) need not be completely prime (in contrast to the preceding special case), consider the situation when $n = 2$, when $k \neq \mathbb{F}_2$, when ε is a primitive ℓth root of 1 in k for some integer $\ell \geq 2$, and when

$$q = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon^{-1} & 1 \end{bmatrix}.$$

In other words, L now is the k-algebra of Laurent series in $x = x_1$ and $y = x_2$, subject (only) to the relation $yx = \varepsilon xy$. By (3.7ii), the center Z of L is $k[[x^{\pm\ell}, y^{\pm\ell}]]$. Hence, L is generated as a right (or left) module over Z by $\{x^i y^j \mid 0 \leq i, j \leq \ell - 1\}$. It follows that the Goldie rank of a prime factor of L can be no greater than ℓ.

Now consider the prime ideal $Q = Z.(x^\ell + y^\ell)$ of Z. Then $P = LQ$ is a prime ideal of L, by (3.14). It is not hard to check that $(x + y)^i \notin P$, for all $1 \leq i \leq \ell - 1$. However, it is well known that

$$(x + y)^\ell = x^\ell + y^\ell,$$

and so $(x + y)^\ell$ is contained in P. In other words, $x + y$ is nilpotent modulo P of index ℓ. In view of the preceding paragraph, we see that the Goldie rank of L/P must be ℓ.

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