

# Convergence of quantum random walks with decoherence

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Shimao Fan\*, Zhiyong Feng\*, Sheng Xiong\*\* and Wei-Shih Yang\*

\*Department of Mathematics

Temple University, Philadelphia, PA 19122

\*\*Department of Mathematics and Sciences

Edward Waters College, Jacksonville, FL 32209

Email: shimao.fan@temple.edu, zhiyong.feng@temple.edu  
sheng.xiong@ewc.edu, yang@temple.edu

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## Abstract

In this paper, we study the discrete-time quantum random walks on a line subject to decoherence. The convergence of the rescaled position probability distribution  $p(x, t)$  depends mainly on the spectrum of the superoperator  $\mathcal{L}_{kk}$ . We show that if 1 is an eigenvalue of the superoperator with multiplicity one and there is no other eigenvalue whose modulus equals to 1, then  $\hat{P}(\frac{\nu}{\sqrt{t}}, t)$  converges to a convex combination of normal distributions. In terms of position space, the rescaled probability mass function  $p_t(x, t) \equiv p(\sqrt{t}x, t)$ ,  $x \in Z/\sqrt{t}$ , converges in distribution to a continuous convex combination of normal distributions. We give an necessary and sufficient condition for a  $U(2)$  decoherent quantum walk that satisfies the eigenvalue conditions. We also give a complete description of the behavior of quantum walks whose eigenvalues do not satisfy these assumptions. Specific examples such as the Hadamard walk, walks

under real and complex rotations are illustrated. For the  $O(2)$  quantum random walks, an explicit formula is provided for the scaling limit of  $p(x, t)$  and their moments. We also obtain exact critical exponents for their moments at the critical point and show universality classes with respect to these critical exponents.

## 1 Introduction

In recent years quantum walks (QWs), as the quantum analog of the classical random walks (CRWs), have attracted great attention from mathematicians, computer scientists, physicists and engineers. Two forms of QWs, continuous-time QWs (CTQW) [7] and discrete-time QW (DTQW)[1, 10, 3, 11, 4], are widely studied. In this work, we restrict our discussion to DTQW. In this case, an extra “coin” degree of freedom is introduced into the system. Unlike the classical random walk, where the direction of the particle moves is determined by the outcome of a “coin flip”, for quantum random walks both the “flip” of the coin and the conditional motion of the particle given by unitary transformations are needed.

In 2003, Brun, Carteret and Ambainis [4] discussed the decoherent Hadamard walk on 1-dimensional integer lattice  $Z$ , and found the expressions for the first and second moments of the position and showed that in the long time limit the variance grows linearly with time with the diffusive character. However, they did not provide an exact expression for the higher order moments nor the limiting distribution, due to the complicated forms of the eigenvalues of the superoperator  $\mathcal{L}_{kk}$  and the difficulty to evaluate the position probability analytically. In 2004, Grimmett, Janson and Scudo [8] obtained the scaling limit of quantum random walks without decoherence. Their scaling factor is  $\frac{1}{t}$ . Recently in [2], the scaling limit of a quantum random walk with a Markov controlled coin process converges either with scaling factor  $\frac{1}{t}$  or  $\frac{1}{\sqrt{t}}$ , depending on eigenvalue conditions of the walk operator.

In this paper, we consider the model with decoherence operators given by [4]. We overcome the difficulties in there to obtain the scaling limit of decoherent quantum random walks, by analyzing  $\hat{P}(\nu, t)$ , the characteristic function (Fourier transformation) of the position probability distribution  $p(x, t)$ . Here  $p(x, t)$ ,  $x \in Z$ , denotes the probability that the particle is found at position  $x$  at time  $t$ . It turns out that the convergence of  $\hat{P}(\frac{\nu}{\sqrt{t}}, t)$  depends on the spectrum of the superoperator  $\mathcal{L}_{kk}$ . We show that if 1 is an eigenvalue of the superoperator with multiplicity one and there is no other eigenvalue whose modulus equals to 1, then  $\hat{P}(\frac{\nu}{\sqrt{t}}, t)$  converges to a convex combination of normal distributions. In terms of position space, the rescaled prob-

ability mass function  $p_t(x, t) \equiv p(\sqrt{t}x, t)$ ,  $x \in Z/\sqrt{t}$ , converges in distribution to a continuous convex combination of normal distributions. We give an necessary and sufficient condition for a  $U(2)$  decoherent quantum walk that satisfies the eigenvalue conditions. For the  $O(2)$  quantum walks such as the Hadamard walk, and walks under real or complex rotations are discussed. An explicit limiting distribution formula is provided for these walks.

Our article is organized as follows. In Section 2, we present basic concepts of quantum random walks. In Section 3, we present our main result about the limit of  $p_t(x, t)$ . In Section 4, we give examples that illustrate our results. We obtain the scaling limit and the exact critical exponents for their moments at the critical point. We show that the decoherent quantum random walks, with coin space unitary transformation  $U \in O(2)$ ,  $\theta \neq \frac{n\pi}{2}$ ,  $n = 0, 1, 2, 3$ , belong to the same universality class with respected to the critical exponents of all moments at their critical points.

## 2 The unitary walk on the line and decoherence

Consider a general quantum random walk on the 1-dimensional integer lattices  $Z$ . To be consistent, we adapt analogous definitions and notations as those outlined in [4]. We denote the state space by a Hilbert space  $H_p \otimes H_2$ , where  $H_p$  denotes the position space and  $H_2$  denotes the coin space. The basis of the position space are  $|x \rangle$ , where  $x \in Z$  and, the basis of the coin space are  $|R \rangle$  and  $|L \rangle$ . We will assume that the walk starts at the origin. The shift operators in  $H_p$  are defined as follows

$$S^+|x \rangle = |x + 1 \rangle, \quad (2.1)$$

$$S^-|x \rangle = |x - 1 \rangle, \quad (2.2)$$

where  $S^-$  and  $S^+$  are unitary shift operators on the particle position. Let  $P_R, P_L$  be two orthogonal projections on the coin space  $H_2$  spanned by  $|R \rangle$  or  $|L \rangle$ , where  $P_R + P_L = I$ . Let  $U$  be a unitary transformation on  $H_2$ , that acts as the "flipping" of the coin. Then the evolution operator of the quantum random walk is given by

$$E = (S^+ \otimes P_R + S^- \otimes P_L)(I \otimes U). \quad (2.3)$$

The eigenvectors  $|k \rangle$  of  $S^-, S^+$  are

$$|k \rangle = \sum_x e^{ikx} |x \rangle, \quad k \in [0, 2\pi], \quad (2.4)$$

with eigenvalues

$$\begin{aligned} S^+|k\rangle &= e^{-ik}|k\rangle, \\ S^-|k\rangle &= e^{ik}|k\rangle. \end{aligned} \quad (2.5)$$

Therefore, in  $|k\rangle$  basis, the evolution operator is

$$E(|k\rangle \otimes |\Phi\rangle) = |k\rangle \otimes (e^{-ik}P_R + e^{ik}P_L)U|\Phi\rangle \equiv |k\rangle \otimes U_k|\Phi\rangle, \quad (2.6)$$

where  $U_k = (e^{-ik}P_R + e^{ik}P_L)U$  is also a unitary operator.

The decoherence on the coin space is defined as follows. Suppose before each unitary transformation acting on the coin, a measurement is performed on the coin. This measurement is given by a set of operators  $\{A_n\}$  on  $H_2$  which satisfy

$$\sum_n A_n^* A_n = I. \quad (2.7)$$

Through out this paper, we also assume that the measurement is unital, i.e., it satisfies

$$\sum_n A_n A_n^* = I. \quad (2.8)$$

After the measurement, a density operator  $\chi$  on  $H_2$  is transformed by

$$\chi \rightarrow \chi' = \sum_n A_n \chi A_n^*. \quad (2.9)$$

The general density operator of quantum random walk is then given by

$$\rho = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} |k\rangle\langle k'| \otimes \chi_{kk'}, \quad (2.10)$$

where  $\chi_{kk'} \in L(H_2)$ , and  $L(H_2)$  is a vector space of linear operators on  $H_2$ . Then after one step of the evolution under coin space decoherence, the density operator becomes

$$\rho' = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} |k\rangle\langle k'| \otimes \sum_n U_k A_n \chi_{kk'} A_n^* U_k^* \quad (2.11)$$

Suppose the quantum walk starts at the state  $|0\rangle \otimes |\Phi_0\rangle$ , then the initial state is given by the density operator

$$\rho_0 = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} |k\rangle\langle k'| \otimes |\Phi_0\rangle\langle\Phi_0|. \quad (2.12)$$

After  $t$  steps, the state evolves to

$$\rho_t = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} |k\rangle\langle k'| \otimes \sum_{n_1, \dots, n_t} U_k A_{n_t} \cdots U_k A_{n_1} |\Phi_0\rangle\langle\Phi_0| A_{n_1}^* U_{k'}^* \cdots A_{n_t}^* U_{k'}^*. \quad (2.13)$$

If we define the superoperator  $\mathcal{L}_{kk'}$  to be an operator which maps  $L(H_2)$  to  $L(H_2)$ :

$$\mathcal{L}_{kk'} B \equiv \sum_n U_k A_n B A_n^* U_{k'}^*, \quad \forall B \in L(H_2), \quad (2.14)$$

then

$$\rho_t = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} |k\rangle\langle k'| \otimes \mathcal{L}_{kk'}^t |\Phi_0\rangle\langle\Phi_0|. \quad (2.15)$$

The probability of reaching a point  $x$  at time  $t$  is

$$\begin{aligned} p(x, t) &= \text{Tr}\{(|x\rangle\langle x| \otimes I) \rho_t\} \\ &= \frac{1}{(2\pi)^2} \int dk \int dk' \langle x|k\rangle\langle k'|x\rangle \text{Tr}\{\mathcal{L}_{kk'}^t |\Phi_0\rangle\langle\Phi_0|\} \\ &= \frac{1}{(2\pi)^2} \int dk \int dk' e^{ix(k-k')} \text{Tr}\{\mathcal{L}_{kk'}^t |\Phi_0\rangle\langle\Phi_0|\}. \end{aligned} \quad (2.16)$$

### 3 The limiting distributions of quantum walks with decoherence coin

Let

$$\hat{P}(\nu, t) \equiv \langle e^{i\nu x} \rangle_t = \sum_x e^{i\nu x} p(x, t) \quad (3.1)$$

be the characteristic function of  $p(x, t)$ . By the property of the  $\delta$  function

$$\frac{1}{2\pi} \sum_x x^m e^{-ix(k-k')} = (-i)^m \delta^{(m)}(k - k'), \quad (3.2)$$

and (2.16), we have

$$\begin{aligned}
\langle e^{i\nu x} \rangle_t &= \sum_x e^{i\nu x} p(x, t) \\
&= \sum_x e^{i\nu x} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} e^{ix(k-k')} \text{Tr}\{\mathcal{L}_{kk'}^t | \Phi_0 \rangle \langle \Phi_0 | \} \\
&= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \sum_x e^{-ix(k'-k-\nu)} \text{Tr}\{\mathcal{L}_{kk'}^t | \Phi_0 \rangle \langle \Phi_0 | \} \\
&= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} 2\pi \delta(k' - k - \nu) \text{Tr}\{\mathcal{L}_{kk'}^t | \Phi_0 \rangle \langle \Phi_0 | \} \\
&= \frac{1}{2\pi} \int dk \text{Tr}\{\mathcal{L}_{k, k+\nu}^t | \Phi_0 \rangle \langle \Phi_0 | \}. \tag{3.3}
\end{aligned}$$

For any initial state  $\hat{O} \in L(H_2)$ , the generating function of  $\langle e^{i\nu x} \rangle_t$  is given by

$$G(z, \nu) = \sum_{t=0}^{\infty} z^t \langle e^{i\nu x} \rangle_t \tag{3.4}$$

$$= \frac{1}{2\pi} \int dk \sum_{t=0}^{\infty} z^t \text{Tr}\{\mathcal{L}_{k, k+\nu}^t \hat{O}\} \tag{3.5}$$

$$= \frac{1}{2\pi} \int dk \text{Tr}\left\{ \frac{1}{I - z\mathcal{L}_{k, k+\nu}} \hat{O} \right\}, \tag{3.6}$$

where  $|z| < 1$  and  $\hat{O} \in L(H_2)$ . Note that the generating function is well defined since the spectrum of  $\mathcal{L}_{k, k+\nu}$  is less than or equal to 1 by the following lemma.

**Lemma 3.1.** *Suppose  $U \in U_2(\mathbb{C})$  and the set of operators  $\{A_n\}$  is unital. Let  $\lambda$  be an eigenvalue of  $\mathcal{L}_{k, k+\nu}$ , then  $|\lambda| \leq 1$ .*

**Proof:** Define  $\|\mathcal{L}_{k, k}\| = \sup_{0 \neq \hat{O} \in L(H_2)} \frac{\|\mathcal{L}_{k, k} \hat{O}\|}{\|\hat{O}\|}$  with  $\|\hat{O}\|^2 = \text{Tr}(\hat{O}^* \hat{O})$ , then we have

$\|\mathcal{L}_{k, k}\| \leq 1$  (see ref.[5]).

$$\begin{aligned}
\|\mathcal{L}_{k, k+\nu} \hat{O}\|^2 &= \text{Tr}\left( \left( \sum_n U_k A_n \hat{O} A_n^* U_{k+\nu}^* \right)^* \left( \sum_n U_k A_n \hat{O} A_n^* U_{k+\nu}^* \right) \right) \\
&= \text{Tr}\left( U_{k+\nu} \left( \sum_n A_n \hat{O} A_n^* \right)^* U_k^* U_k \left( \sum_n A_n \hat{O} A_n^* \right) U_{k+\nu}^* \right) \\
&= \text{Tr}\left( \sum_n A_n \hat{O} A_n^* \right)^* \left( \sum_n A_n \hat{O} A_n^* \right) \\
&= \|\mathcal{L}_{k, k} \hat{O}\|^2.
\end{aligned}$$

This shows that  $\|\mathcal{L}_{k,k+\nu}\| = \|\mathcal{L}_{k,k}\| \leq 1$  and therefore  $|\lambda| \leq 1$ .

It follows from the above lemma that  $\sum_{t=0}^{\infty} (z\mathcal{L}_{k,k+\nu})^t \hat{O}$  converges in  $|z| < 1$  and thus  $I - z\mathcal{L}_{k,k+\nu}$  does not have any pole inside the disk  $|z| < 1$ .

It is well known that any  $\hat{O} \in L(H_2)$  can be written as a linear combination of Pauli matrices:

$$\hat{O} = r_0 I + r_1 \sigma_1 + r_2 \sigma_2 + r_3 \sigma_3, \quad (3.7)$$

where  $\sigma_{1,2,3} = \sigma_{x,y,z}$  are usual Pauli matrices. Hence  $\hat{O}$  can be represented by a column vector

$$\hat{O} = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}. \quad (3.8)$$

By Lemma 3.1 and  $1 - z\mathcal{L}_{k,k+\nu} \in L(L(H_2))$ , we have

$$\langle e^{i\nu x} \rangle_t = \frac{1}{2\pi i} \oint_{|z|=r < 1} \frac{G(z, \nu)}{z^{t+1}} dz, \quad (3.9)$$

for some  $0 < r < 1$ . Let  $A$  be the matrix associated with  $1 - z\mathcal{L}_{k,k+\nu}$ , with respect to the Pauli matrices, then

$$\frac{1}{1 - z\mathcal{L}_{k,k+\nu}} \hat{O} = A^{-1} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix},$$

where  $A_{ij}$  is the cofactor of  $A$ .

Note that  $Tr(\sigma_i) = 0$  for  $i = 1, 2, 3$ , and  $Tr(\sigma_0) = 2$ . So when taking the trace in (3.6) only the first row action  $h(z, \nu) = A_{11}r_0 + A_{21}r_1 + A_{31}r_2 + A_{41}r_3$  remains. Therefore

$$G(z, \nu) = \frac{1}{2\pi} \int dk \frac{2h(z, \nu)}{\det A}. \quad (3.10)$$

Let  $L = (l_{ij}(\nu))$  be the matrix representation of  $\mathcal{L}_{k,k+\nu}$  in terms of Pauli matrices. Then we have the following lemma.

**Lemma 3.2.** *Suppose  $U \in U_2(\mathbb{C})$  and  $\{A_n\}$  is unital. Then  $\mathcal{L}_{k,k+\nu}$  has the following representation*

$$\begin{pmatrix} \cos \nu & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \\ i \sin \nu & \times & \times & \times \end{pmatrix}.$$

Moreover, if  $\nu = 0$ , then  $\mathcal{L}_{k,k}$  has the following representation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}.$$

**Proof:** Let  $U \in U_2(\mathbb{C})$ . Then  $|\det U| = 1$ . Let  $\det U = e^{i\gamma}$ . We consider the normalized operator

$$W = e^{-i\frac{\gamma}{2}}U. \quad (3.11)$$

Then  $W \in SU_2(\mathbb{C})$ . By (2.14),  $\mathcal{L}_{kk'}$  is the same for  $U$  and  $W$ . Therefore, without loss of generality, we may assume that  $U \in SU_2(\mathbb{C})$  with the following form

$$U = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

where  $\alpha, \beta \in \mathbb{C}$  and  $|\alpha|^2 + |\beta|^2 = 1$ . Then

$$U_k = \begin{pmatrix} e^{-ik}\alpha & -e^{-ik}\bar{\beta} \\ e^{ik}\beta & e^{ik}\bar{\alpha} \end{pmatrix}, \quad (3.12)$$

and

$$\mathcal{L}_{k,k+\nu}\sigma_i = l_{1i}(\nu)\sigma_0 + l_{2i}(\nu)\sigma_1 + l_{3i}(\nu)\sigma_2 + l_{4i}(\nu)\sigma_3 = \begin{pmatrix} l_{1i}(\nu) + l_{4i}(\nu) & -il_{2i}(\nu) + l_{3i}(\nu) \\ il_{2i}(\nu) + l_{3i}(\nu) & l_{1i}(\nu) - l_{4i}(\nu) \end{pmatrix},$$

where  $i = 0, 1, 2, 3$ . On the other hand,

$$\mathcal{L}_{k,k+\nu}\sigma_0 = \sum_n U_k A_n I A_n^* U_{k+\nu}^* = U_k \left( \sum_n A_n I A_n^* \right) U_{k+\nu}^* = U_k U_{k+\nu}^* = \begin{pmatrix} e^{i\nu} & 0 \\ 0 & e^{-i\nu} \end{pmatrix}. \quad (3.13)$$

Hence

$$\begin{aligned} l_{11}(\nu) + l_{41}(\nu) &= e^{i\nu}, \\ l_{11}(\nu) - l_{41}(\nu) &= e^{-i\nu}, \\ -il_{21}(\nu) + l_{31}(\nu) &= 0, \\ il_{21}(\nu) + l_{31}(\nu) &= 0, \end{aligned}$$



which implies that  $l_{11}(\nu) = \cos(\nu)$ ,  $l_{21}(\nu) = l_{31}(\nu) = 0$  and  $l_{41}(\nu) = i \sin(\nu)$ . In particular, if  $\nu = 0$ , then the first column of  $\mathcal{L}_{k,k}$  is  $(1, 0, 0, 0)^T$ .

Next we finish our proof by showing  $l_{1i}(0) = 0$ , for  $i = 2, 3, 4$ . Suppose

$$\sum_n U_k A_n \sigma_{i-1} A_n^* U_k^* = \begin{pmatrix} \tau_1^i & \tau_2^i \\ \tau_3^i & \tau_4^i \end{pmatrix},$$

then

$$\begin{aligned} l_{1i}(0) + l_{4i}(0) &= \tau_1^i(0), \\ l_{1i}(0) - l_{4i}(0) &= \tau_4^i(0), \end{aligned}$$

and hence

$$l_{1i}(0) = \frac{1}{2}(\tau_1^i(0) + \tau_4^i(0)) = \frac{1}{2}Tr(\mathcal{L}_{k,k}\sigma_{i-1}) = \frac{1}{2}Tr(\sigma_{i-1}) = 0, \quad (3.14)$$

for  $i = 2, 3, 4$ . The third equality in the above holds because  $\mathcal{L}_{k,k}$  preserves the trace.

**Theorem 3.1.** *Suppose  $U \in U_2(\mathbb{C})$ , the set of operators  $\{A_n\}$  is unital, 1 is an eigenvalue of  $\mathcal{L}_{k,k}$  with multiplicity one, and  $|\lambda| < 1$  for any other eigenvalue  $\lambda$  of  $\mathcal{L}_{k,k}$ . Then*

$$\lim_{t \rightarrow \infty} \hat{P}\left(\frac{\nu}{\sqrt{t}}, t\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2}z_0''(0)\nu^2} dk, \quad \forall \nu \in [0, 2\pi],$$

where  $z_0(\nu)$  is the root of  $\det(1 - z\mathcal{L}_{k,k+\nu}) = 0$  such that  $z_0(0) = 1$ .

By the Cramer-Levy Theorem (see e.g. Theorem 6.3.2 [6]), the above theorem implies that under the conditions, the distribution of the scaling limit is a continuous convex combination of normal distributions with variance  $z_0''(0)$ . In other words, if we define the rescaled probability mass function on  $\frac{Z}{\sqrt{t}}$  by

$$p_t(x, t) \equiv p(\sqrt{t}x, t), x \in \frac{Z}{\sqrt{t}}, \quad (3.15)$$

then  $p_t$  converges in distribution to the continuous convex combination of normal distributions whose density function is given by

$$F(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{2\pi z_0''(0)}} e^{-\frac{1}{2z_0''(0)}x^2} dk, x \in R.$$

**Proof:** By (3.11),  $U$  can be normalized to an  $SU_2$  operator  $W$ . Note that by (2.14) and (2.16),  $\mathcal{L}_{kk+\nu}$  and  $p(x, t)$  are the same for  $U$  and  $W$ . Therefore, without loss of generality, we may assume that  $U \in SU_2(\mathbb{C})$ . By (3.10) and Cauchy's integral formula,

$$\langle e^{i\nu x} \rangle_t = \frac{1}{2\pi i} \oint_{|z|=r<1} \frac{G(z, \nu)}{z^{t+1}} dz = \frac{1}{2\pi} \int dk \frac{1}{2\pi i} \oint_{|z|=r<1} \frac{2h(z, \nu)}{z^{t+1} \det A} dz.$$

Let  $z_0(\nu), z_1(\nu), z_2(\nu), z_3(\nu)$  be four roots of  $\det A = 0$ . Then  $\frac{1}{z_0(\nu)}, \frac{1}{z_1(\nu)}, \frac{1}{z_2(\nu)}, \frac{1}{z_3(\nu)}$  are four eigenvalues of  $\mathcal{L}_{k, k+\nu}$ . By the assumptions that 1 is an eigenvalue of  $\mathcal{L}_{k, k}$  with multiplicity one, and  $|\lambda| < 1$  for any other eigenvalue  $\lambda$  of  $\mathcal{L}_{k, k}$ , we may make the following ordering  $1 = |z_0(0)| < |z_1(0)| \leq |z_2(0)| \leq |z_3(0)|$ . Let  $l(z, \nu) = \frac{2h(z, \nu)}{z^{t+1} \det A}$ . By continuity of  $z_i(\nu), i = 0, 1, 2, 3$ , in both variables  $\nu$  and  $k$ , there exist a constant  $R > 1$ , and a small neighborhood  $V$  of  $\nu = 0$  such that  $|z_0(\nu)| < R < |z_i(\nu)|$  for any  $\nu \in \bar{V}$  and  $k \in [0, 2\pi]$ , for all  $i = 1, 2, 3$ . By Cauchy's Residue Theorem,

$$\frac{1}{2\pi i} \oint_{|z|=R} \frac{2h(z, \nu)}{z^{t+1} \det A} dz = \text{Res}(l, z=0) + \text{Res}(l, z=z_0(\nu)).$$

Note that there exists  $t_0$  such that  $\frac{\nu}{\sqrt{t}} \in \bar{V}$  for all  $\nu \in (0, 2\pi)$  if  $t \geq t_0$ . Let  $g(z, \nu) = \det A$ . By compactness of  $\{(z, \nu, k); |z| = R, \nu \in \bar{V}, k \in [0, 2\pi]\}$ , there exists a constant  $C$  such that  $|\frac{2h(z, \frac{\nu}{\sqrt{t}})}{g(z, \frac{\nu}{\sqrt{t}})}| \leq C$  on  $\{(z, \nu); |z| = R, \nu \in [0, 2\pi], k \in [0, 2\pi]\}$ , for all  $t \geq t_0$ . Therefore,

$$|\lim_{t \rightarrow \infty} \frac{1}{2\pi i} \oint_{|z|=R} \frac{1}{z^{t+1}} \frac{2h(z, \frac{\nu}{\sqrt{t}})}{g(z, \frac{\nu}{\sqrt{t}})} dz| \leq \lim_{t \rightarrow \infty} \frac{1}{R^{t+1}} C = 0.$$

Hence

$$\lim_{t \rightarrow \infty} \text{Res}(l, z=0) = - \lim_{t \rightarrow \infty} \text{Res}(l, z=z_0(\frac{\nu}{\sqrt{t}})).$$

For any fixed  $\nu$ , we have

$$\text{Res}(l, z=z_0(\frac{\nu}{\sqrt{t}})) = \frac{2h(z_0(\frac{\nu}{\sqrt{t}}), \frac{\nu}{\sqrt{t}})}{z_0^{t+1}(\frac{\nu}{\sqrt{t}}) \frac{\partial g}{\partial z}(z_0(\frac{\nu}{\sqrt{t}}), \frac{\nu}{\sqrt{t}})}.$$

Let  $t \rightarrow \infty$ , we have

$$\lim_{t \rightarrow \infty} \text{Res}(l, z=z_0(\frac{\nu}{\sqrt{t}})) = \frac{2h(1, 0)}{\frac{\partial g}{\partial z}(1, 0)} \lim_{t \rightarrow \infty} z_0 \left( \frac{\nu}{\sqrt{t}} \right)^{-t-1}$$

We claim that

$$\frac{2h(1, 0)}{\frac{\partial g}{\partial z}(1, 0)} = -1; \quad z'(0) = 0. \quad (3.16)$$

By the Dominated Convergence Theorem, we have

$$\lim_{t \rightarrow \infty} \hat{P}\left(\frac{\nu}{\sqrt{t}}, t\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2}z_0''(0)\nu^2} dk. \quad (3.17)$$

We will finish our proof by proving the claim (3.16). Let  $M$  denote the 3 by 3 submatrix of  $L = (l_{ij})$ :

$$M(\nu) = \begin{pmatrix} l_{22} & l_{23} & l_{24} \\ l_{32} & l_{33} & l_{34} \\ l_{42} & l_{43} & l_{44} \end{pmatrix}.$$

Then the matrix

$$A|_{\nu=0} = \begin{pmatrix} 1 - z & \mathbf{0} \\ \mathbf{0} & I_3 - zM(0) \end{pmatrix}.$$

The cofactor  $A_{21} = A_{31} = A_{41} = 0$  and  $A_{11} = \det(I_3 - zM(0))$ . By Lemma 3.2,  $\frac{1}{z_i(0)}$  are eigenvalues of  $\mathcal{L}_{k,k}$  for  $i = 0, 1, 2, 3$ . Therefore  $\frac{1}{z_1(0)}, \frac{1}{z_2(0)}, \frac{1}{z_3(0)}$  are eigenvalues of  $M(0)$ . Hence

$$\det(I_3 - zM(0)) = \left(1 - \frac{z}{z_1(0)}\right)\left(1 - \frac{z}{z_2(0)}\right)\left(1 - \frac{z}{z_3(0)}\right) \quad (3.18)$$

$$= -\frac{1}{z_1(0)z_2(0)z_3(0)}(z - z_1(0))(z - z_2(0))(z - z_3(0)). \quad (3.19)$$

On the other hand,

$$\frac{\partial g}{\partial z}(1, 0) = \frac{1}{z_1(0)z_2(0)z_3(0)}(1 - z_1(0))(1 - z_2(0))(1 - z_3(0)). \quad (3.20)$$

Hence

$$\frac{2h(1, 0)}{\frac{\partial g}{\partial \nu}(1, 0)} = 2r_0 \frac{-\frac{1}{z_1(0)z_2(0)z_3(0)}(1 - z_1(0))(1 - z_2(0))(1 - z_3(0))}{\frac{1}{z_1(0)z_2(0)z_3(0)}(1 - z_1(0))(1 - z_2(0))(1 - z_3(0))} = -2r_0 = -Tr(\hat{O}) = -1,$$

since  $\hat{O}$  is a density operator. Next we will show that  $z'_0(0) = 0$ . Since  $g(z_0(\nu), \nu) = 0$ , we have

$$0 = \frac{dg(z_0(\nu), z)}{d\nu} = \frac{\partial g(z, \nu)}{\partial \nu}\Big|_{z=z_0(\nu)} + \frac{\partial g(z, \nu)}{\partial z}\Big|_{z=z_0(\nu)} z'_0(\nu).$$

When  $\nu = 0$ ,  $z_0(0) = 1$ , then the above equation becomes

$$0 = \frac{\partial g(1, \nu)}{\partial \nu} \Big|_{\nu=0} + \frac{\partial g(z, 0)}{\partial z} \Big|_{z=1} z'_0(0). \quad (3.21)$$

Consider the matrix  $A$  at  $z = 1$ , and note that  $l_{21}(\nu) = l_{31}(\nu) = 0$ , we have

$$\begin{aligned} g(1, \nu) &= (1 - l_{11}(\nu))A_{11}(\nu) + l_{21}(\nu)A_{21}(\nu) - l_{31}(\nu)A_{31}(\nu) + l_{41}(\nu)A_{41}(\nu) \\ &= (1 - \cos(\nu))A_{11}(\nu) + i \sin \nu A_{41}(\nu). \end{aligned}$$

By Lemma 3.2, the cofactor  $A_{41}(0) = 0$ . It follows that

$$\frac{\partial g(1, \nu)}{\partial \nu} \Big|_{\nu=0} = 0. \quad (3.22)$$

By (3.20),  $\frac{\partial g(z_0, \nu)}{\partial z} \Big|_{z=z_0} \neq 0$ , since  $z_i(0) \neq 1$ , for all  $i = 1, 2, 3$ , by the assumptions of the theorem. By (3.21) and (3.22) we have  $z'_0(0) = 0$ .

**Q.E.D**

## 4 Applications

The assumptions that 1 is the largest eigenvalue of  $\mathcal{L}_{kk}$  with algebraic multiplicity 1 and that there is no other eigenvalues whose modulus equals to 1, are crucial in determining the convergence of  $\hat{P}(\frac{\nu}{\sqrt{t}}, t)$ . In this section, we consider the measurements given by

$$\begin{aligned} A_0 &= \sqrt{1-p}I, \\ A_1 &= \sqrt{p}|R \rangle \langle R|, \\ A_2 &= \sqrt{p}|L \rangle \langle L|. \end{aligned}$$

By analyzing the spectrum of  $\mathcal{L}_{kk}$ , we obtain a necessary and sufficient conditions in which the assumptions of Theorem 3.1 are satisfied. Specific examples such as the Hadamard walk, walks under real and complex rotations are illustrated. For certain class of convergent quantum walks, explicit formulas are obtained for the limits of the characteristic functions of properly scaled  $p(x, t)$ . In addition, we will also give a complete description of the behavior of those walks that do not satisfy the conditions.

**Lemma 4.1.** *Let  $\mathcal{L}_{k,k'}$  be a superoperator on the Hilbert space  $L(\mathbb{C}^2)$ , defined by*

$$\mathcal{L}_{k,k'}(\hat{O}) = \sum_{n=0}^2 U_k A_n \hat{O} A_n^* U_{k'}^*,$$

where  $U_k$  and  $U_{k'}$  are  $2 \times 2$  unitary matrices and  $\hat{O} \in L(\mathbb{C}^2)$ . Then

$$\langle \mathcal{L}_{k,k'} \hat{O}, \mathcal{L}_{k,k'} \hat{O} \rangle \leq \langle \hat{O}, \hat{O} \rangle.$$

In particular,  $\langle \mathcal{L}_{k,k'} \hat{O}, \mathcal{L}_{k,k'} \hat{O} \rangle = \langle \hat{O}, \hat{O} \rangle$  if and only if the decoherence rate  $p = 0$  or  $\hat{o}_{12} = \hat{o}_{21} = 0$ .

Part of the above lemma has been obtained by Liu and Petulante in [9], but our lemma extends the equality part of their lemma to a wider scope with more applications.

**Proof:** The inequality follows from Lemma 1 in [9]. By the proof of Lemma 1 in [9], the equality holds if and only if  $(2p - p^2)(\hat{o}_{12}^2 + \hat{o}_{21}^2) = 0$ . That is,  $p = 0$  or  $\hat{o}_{12} = \hat{o}_{21} = 0$ .

If  $p \neq 0$  and  $\hat{o}_{12} = \hat{o}_{21} = 0$ , then 1 may not be the unique eigenvalue of  $\mathcal{L}_{k,k}$  with largest modulus. In this case, we have the following theorem. Let  $\dim(\lambda)$  denote the dimensions of the eigenspace associated with the eigenvalue  $\lambda$ .

**Theorem 4.1.** *Let  $0 < p < 1$ . Let  $U \in U(2)$ . Suppose  $\lambda$  is an eigenvalue of  $\mathcal{L}_{kk}$ , then we have*

- (a)  $\sum_{|\lambda|=1} \dim(\lambda) \leq 2$ .
- (b) 1 is an eigenvalue of  $\mathcal{L}_{k,k}$  and  $\dim(1) \geq 1$ .
- (c) If  $|\lambda| = 1$ , then  $\lambda = 1$  or  $-1$ .
- (d)  $\dim(1) = 2$  if and only if  $u_{12} = u_{21} = 0$  and  $|u_{11}| = |u_{22}| = 1$ . In this case its multiplicity is 2.
- (e) There exists eigenvalue  $\lambda = -1$  if and only if  $u_{11} = u_{22} = 0$  and  $|u_{12}| = |u_{21}| = 1$ . In this case its multiplicity is 1.

**Proof:** By considering the normalized operator  $W$  as in (3.11), and noting that the statements (a)-(e) do not depend on whether it is  $U$  or  $W$ , we may assume without loss of generality that  $U$  is in  $SU_2$  with the form

$$U = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix},$$

where  $\alpha, \beta \in \mathbb{C}$  and  $|\alpha|^2 + |\beta|^2 = 1$ .

a) Let  $\lambda$  be an eigenvalue of  $\mathcal{L}_{kk}$  with eigenvector  $\hat{O} = \begin{pmatrix} \hat{o}_{11} & \hat{o}_{12} \\ \hat{o}_{21} & \hat{o}_{22} \end{pmatrix}$ . By Lemma 3.2,  $\mathcal{L}_{k,k}$  has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{22} & l_{23} & l_{24} \\ 0 & l_{32} & l_{33} & l_{34} \\ 0 & l_{42} & l_{43} & l_{44} \end{pmatrix}, \quad (4.23)$$

where

$$l_{24} = -2 \cos(2k) \operatorname{Re}(\beta \bar{\alpha}) + 2 \sin(2k) \operatorname{Im}(\beta \bar{\alpha}), \quad (4.24)$$

$$l_{34} = -2 \cos(2k) \operatorname{Im}(\beta \bar{\alpha}) - 2 \sin(2k) \operatorname{Re}(\beta \bar{\alpha}), \quad (4.25)$$

$$l_{42} = 2q \operatorname{Re}(\alpha \beta), \quad (4.26)$$

$$l_{43} = 2iq \operatorname{Im}(\alpha \beta), \quad (4.27)$$

$$l_{44} = |\alpha|^2 - |\beta|^2. \quad (4.28)$$

Since  $|\lambda| = 1$ , by Lemma 4.1, we have  $\hat{o}_{21} = \hat{o}_{12} = 0$ . This implies that the dimension of the space spanned by the eigenspace for all eigenvalues with modulus 1 is at most 2. Moreover, the intersections of eigenspace corresponding to different eigenvalues is  $\{0\}$ . Therefore a) holds.

b) By (4.23), 1 is an eigenvalue of  $\mathcal{L}_{k,k}$ . Furthermore,  $(1, 0, 0, 0)^T$  is one of its eigenvectors. Therefore  $\dim(1) \geq 1$ .

c) Note that

$$\hat{O} = \frac{1}{2}(\hat{o}_{11} + \hat{o}_{22})\sigma_0 + \frac{i}{2}(\hat{o}_{12} - \hat{o}_{21})\sigma_1 + \frac{1}{2}(\hat{o}_{12} + \hat{o}_{21})\sigma_2 + \frac{1}{2}(\hat{o}_{11} - \hat{o}_{22})\sigma_3, \quad (4.29)$$

so if  $L_{k,k}\hat{O} = \lambda\hat{O}$ , with  $|\lambda| = 1$ , then by Lemma 4.1,

$$\frac{1}{2}(\hat{o}_{11} + \hat{o}_{22}) = \frac{\lambda}{2}(\hat{o}_{11} + \hat{o}_{22}), \quad (4.30)$$

$$(2 \cos(2k) \operatorname{Re}(\beta \bar{\alpha}) - 2 \sin(2k) \operatorname{Im}(\beta \bar{\alpha})) \frac{1}{2}(\hat{o}_{11} - \hat{o}_{22}) = 0, \quad (4.31)$$

$$(2 \cos(2k) \operatorname{Im}(\beta \bar{\alpha}) + 2 \sin(2k) \operatorname{Re}(\beta \bar{\alpha})) \frac{1}{2}(\hat{o}_{11} - \hat{o}_{22}) = 0, \quad (4.32)$$

$$(|\alpha|^2 - |\beta|^2) \frac{1}{2}(\hat{o}_{11} - \hat{o}_{22}) = \frac{\lambda}{2}(\hat{o}_{11} - \hat{o}_{22}). \quad (4.33)$$

(4.31) and (4.32) can be written as the following matrix form

$$\begin{pmatrix} \cos 2k & -\sin 2k \\ \sin 2k & \cos 2k \end{pmatrix} (Re\beta\bar{\alpha}, Im\beta\bar{\alpha})^T (\hat{o}_{11} - \hat{o}_{22}) = 0. \quad (4.34)$$

Note that the matrix on the left hand side of (4.34) has determinant 1, for all  $k$ , hence invertible. Therefore, if  $\hat{o}_{11} - \hat{o}_{22} \neq 0$ , then  $|\beta\bar{\alpha}| = 0$ , or equivalently,  $\alpha\beta = 0$ . Consequently, if  $\beta = 0$ , then  $\lambda = 1$ . If  $\alpha = 0$ , then  $\lambda = -1$ . On the other hand, if  $\hat{o}_{11} - \hat{o}_{22} = 0$ , then  $\hat{o}_{11} + \hat{o}_{22} \neq 0$  (otherwise  $\hat{O} = 0$ ). In this case,  $\lambda = 1$  by (4.30). So  $\lambda = 1$  or  $-1$ .

d) We first assume  $dim(1) = 2$ . This implies that there exists an eigenvector of the form  $(a, 0, 0, b)^T$ , with  $b \neq 0$ . Since  $(1, 0, 0, 0)^T$  is an eigenvector,  $(0, 0, 0, 1)^T$  is also an eigenvector. By (4.29),  $(1, 0, 0, 0)^T$  and  $(0, 0, 0, 1)^T$  are two eigenvectors corresponding to  $\hat{o}_{11} - \hat{o}_{22} = 0$  and  $\hat{o}_{11} + \hat{o}_{22} = 0$ , respectively. Hence  $|\alpha|^2 - |\beta|^2 = 1$  by (4.33). Therefore  $\beta = 0$  and  $|\alpha| = 1$ .

Conversely, if  $\beta = 0$  and  $|\alpha| = 1$ , then  $\mathcal{L}_{k,k}$  has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{22} & l_{23} & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & l_{42} & l_{43} & 1 \end{pmatrix}. \quad (4.35)$$

It follows from (4.26) and (4.27) that  $l_{42} = l_{43} = 0$ . Therefore,  $dim(1) \geq 2$ . So  $dim(1) = 2$  by part a). Therefore, by (4.35) again, the multiplicity of 1 is also two, otherwise  $dim(1) > 2$  which contradicts a).

e) If  $-1$  is an eigenvalue, then its dimension must be 1 by part a). Suppose  $(a, 0, 0, 1)^T$  is the associated eigenvector, then  $a = 0$  by (4.30). Therefore  $|\alpha|^2 - |\beta|^2 = -1$  by (4.33). This implies  $\alpha = 0$  and  $|\beta| = 1$ .

Conversely, if  $\alpha = 0$  and  $|\beta| = 1$ , then  $\mathcal{L}_{k,k}$  has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_{22} & l_{23} & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (4.36)$$

So  $dim(-1)=1$  by part a) and b). Therefore, by (4.36) again, the multiplicity of  $-1$  is also one, otherwise  $dim(-1) \geq 2$  which contradicts a).

**Corollary 4.1.** *If  $U \in O(2)$ , i.e.  $U = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  or  $\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$  for some  $\theta \in [0, 2\pi]$ , then*

a) 1 is an eigenvalue of  $\mathcal{L}_{k,k}$  with multiplicity one, and  $|\lambda| < 1$  for any other eigenvalue holds if and only if  $\theta \neq \frac{n\pi}{2}$  where  $n = 0, 1, 2, 3$ .

b) If  $\theta = 0, \pi$ , 1 is an eigenvalue of  $\mathcal{L}_{k,k}$  with multiplicity 2.

c) If  $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ ,  $\mathcal{L}_{k,k}$  has eigenvalues 1 and -1, each has multiplicity 1.

**Proof:** Note that  $\theta = 0, \pi$  if and only if  $u_{12} = u_{21} = 0$  and  $|u_{11}| = |u_{22}| = 1$ , and  $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$  if and only if  $u_{12} = u_{21} = 0$  and  $|u_{11}| = |u_{22}| = 1$ . Therefore corollary follows.

We are now ready to discuss examples according to different values of  $\theta$ .

In the Hadamard walk, the evolution operator is given by

$$U_k = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ik} & e^{-ik} \\ e^{ik} & -e^{-ik} \end{pmatrix}, \quad (4.37)$$

where  $U = \begin{pmatrix} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & -\cos \frac{\pi}{4} \end{pmatrix}$ . By Theorem 4.1, the associated superoperator  $\mathcal{L}_{k,k+\nu}$  satisfies the assumptions in Theorem 3.1, and we have

$$z'_0(0) = 0; \quad z''_0(0) = \frac{1 + q^2 + 2q \cos 2k}{1 - q^2},$$

where  $q = 1 - p$ . By Theorem 3.1, we have

$$\lim_{t \rightarrow \infty} \hat{P}\left(\frac{\nu}{\sqrt{t}}, t\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2} \frac{1+q^2+2q \cos 2k}{1-q^2} \nu^2} dk \quad (4.38)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2} \frac{1+q^2+2q \cos k}{1-q^2} \nu^2} dk, \quad (4.39)$$

by change of variables and periodicity of cosine.

In general, if  $U \in O(2)$  with  $\det U = -1$  and  $\theta \neq \frac{n\pi}{2}$  where  $n = 0, 1, 2, 3$ . Then

$$U_k = \begin{pmatrix} e^{-ik} \cos(\theta) & e^{-ik} \sin(\theta) \\ e^{ik} \sin(\theta) & -e^{ik} \cos(\theta) \end{pmatrix}, \quad (4.40)$$

and

$$\mathcal{L}_{k,k+\nu} = \begin{pmatrix} \cos(\nu) & (1-p)i \sin(\nu) \sin(2\theta) & 0 & i \sin(\nu) \cos(2\theta) \\ 0 & -(1-p) \cos(2k+\nu) \cos(2\theta) & (1-p) \sin(2k+\nu) & \cos(2k+\nu) \sin(2\theta) \\ 0 & -(1-p) \sin(2k+\nu) \cos(2\theta) & -(1-p) \cos(2k+\nu) & \sin(2k+\nu) \sin(2\theta) \\ i \sin(\nu) & (1-p) \cos(\nu) \sin(2\theta) & 0 & \cos(\nu) \cos(2\theta) \end{pmatrix}$$



which satisfies all the assumptions of Theorem 3.1. Direct computation gives

$$z'_0(0) = 0; \quad z''_0(0) = \frac{1 + 2q \cos(2k) + q^2}{1 - q^2} \cot^2(\theta).$$

where  $q = 1 - p$ . Hence by Theorem 3.1,

$$\lim_{t \rightarrow \infty} \hat{P}\left(\frac{\nu}{\sqrt{t}}, t\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2} \frac{1+2q \cos(2k)+q^2}{1-q^2} \cot^2(\theta) \nu^2} dk \quad (4.41)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2} \frac{1+2q \cos(k)+q^2}{1-q^2} \cot^2(\theta) \nu^2} dk, \quad (4.42)$$

by change of variables and periodicity of cosine.

Similar calculations also show that (4.42) holds for  $U \in O(2)$  with  $\det U = 1$  and  $\theta \neq \frac{n\pi}{2}$ ,  $n = 0, 1, 2, 3$ .

The  $n$ -th moments  $M_n$  of the limiting distribution can be calculated from (4.42) by using moment generation functions. Let

$$\varphi(\nu) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{1}{2} \frac{1+2q \cos(k)+q^2}{1-q^2} \cot^2(\theta) \nu^2} dk.$$

Then

$$\sum_{n=0}^{\infty} \frac{1}{n!} i^n M_n \nu^n = \varphi(\nu) \quad (4.43)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)^n \cot^{2n}(\theta) \nu^{2n} \frac{1}{(1-q^2)^n} \frac{1}{2\pi} \int_0^{2\pi} (1 + 2q \cos(k) + q^2)^n dk \quad (4.44)$$

It follows that for  $U \in O(2)$  with  $\theta \neq \frac{n\pi}{2}$ ,  $n = 0, 1, 2, 3$ , we have

$$M_{2n+1} = 0, \quad n = 0, 1, 2, 3, \dots, \quad (4.45)$$

and for even moments,

$$M_{2n} = \frac{(2n)!}{n!} \cot^{2n}(\theta) \frac{1}{2^n(1-q^2)^n} \frac{1}{2\pi} \int_0^{2\pi} (1 + 2q \cos(2k) + q^2)^n dk \quad (4.46)$$

$$= \frac{(2n)!}{n!} \cot^{2n}(\theta) \frac{1}{2^n(1-q^2)^n} \frac{1}{2\pi} \int_0^{2\pi} (e^{i2k} + q)^n (e^{-i2k} + q)^n dk \quad (4.47)$$

$$= \frac{(2n)!}{n!} \cot^{2n}(\theta) \frac{1}{2^n(1-q^2)^n} \frac{1}{2\pi} \int_0^{2\pi} \sum_{l=0}^n \sum_{l'=0}^n \binom{n}{l} e^{il2k} q^{n-l} \binom{n}{l'} e^{-il'2k} q^{n-l'} dk \quad (4.48)$$

$$= \frac{(2n)!}{n!} \cot^{2n}(\theta) \frac{1}{2^n(1-q^2)^n} \sum_{l=0}^n \binom{n}{l}^2 q^{2(n-l)}, \quad n = 0, 1, 2, \dots \quad (4.49)$$

Therefore we have

$$M_{2n} = \frac{(2n)!}{n!2^n} \left( \frac{\cot^2 \theta}{1 - q^2} \right)^n T_n(q), \quad n = 0, 1, 2, \dots \quad (4.50)$$

where  $T_n(q)$  is a polynomial of  $q$  of order  $2n$  given by

$$T_n(q) = \sum_{l=0}^n \binom{n}{l}^2 q^{2l}. \quad (4.51)$$

In particular, for Hadamard walk, the second moment of the limiting distribution is given in terms of

$$T_1(q) = 1 + q^2. \quad (4.52)$$

This result agrees with the results given in [4].

Comparing to the well known  $2n$ -th moment  $N_{2n}$  for the normal distribution with mean 0 and variance  $\sigma^2 = \frac{\cot^2 \theta}{1 - q^2}$ ,

$$N_{2n} = \frac{(2n)!}{n!2^n} \left( \frac{\cot^2 \theta}{1 - q^2} \right)^n, \quad n = 0, 1, 2, \dots, \quad (4.53)$$

we see that the scaling limits of the decoherent quantum random walks are not normally distributed if  $q \neq 0$ . The deviation from the normal distribution gets larger as the even moments gets larger. However the deviations of the  $2n$ -th moment are by the same factor  $T_n(q)$  for all  $\theta \neq \frac{j\pi}{2}$ , where  $j = 0, 1, 2, 3$ .

From (4.50), we also obtain the exact critical exponents for  $M_{2n}$  at  $p = 0$ :

$$\gamma_{2n} \equiv \lim_{p \rightarrow 0} -\frac{\ln M_{2n}}{\ln p} = n. \quad (4.54)$$

This result shows universality in which the critical exponents do not depend on  $\theta$  as long as it converges. In other words, the coin-space decoherent quantum random walks, with coin space unitary transformation  $U \in O(2)$ ,  $\theta \neq \frac{n\pi}{2}$ ,  $n = 0, 1, 2, 3$ , belong to the same universality class with respected to the critical exponents of all moments as  $p \rightarrow 0$ .

Now we discuss the limiting behavior of the special cases.

a)  $\theta = 0, \pi$ . If the initial state is  $|0\rangle \otimes |R\rangle$ , then  $E(|0\rangle \otimes |R\rangle) = |1\rangle \otimes |R\rangle$ . If the initial state is  $|0\rangle \otimes |L\rangle$ , then  $E(|0\rangle \otimes |R\rangle) = -| -1\rangle \otimes |L\rangle$ . That is, the walk goes either left or right forever. Hence

$$\hat{P}(\nu, t) = e^{\pm it\nu}$$

depending on the initial conditions. In general, if the initial state is  $\phi_0 = c_R|R\rangle + c_L|L\rangle$ , with  $|c_R|^2 + |c_L|^2 = 1$ , then

$$\hat{P}\left(\frac{\nu}{t}, t\right) = |c_R|^2 e^{i\nu} + |c_L|^2 e^{-i\nu}.$$

b)  $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ . If the initial state is  $|0\rangle \otimes |R\rangle$ , then  $E(|0\rangle \otimes |R\rangle) = -|1\rangle \otimes |L\rangle$ . If the initial state is  $|0\rangle \otimes |L\rangle$ , then  $E(|0\rangle \otimes |R\rangle) = -|-1\rangle \otimes |R\rangle$ . That is, the walk switches back and forth between two positions, which is trivial and

$$\lim_{t \rightarrow \infty} \hat{P}\left(\frac{\nu}{t^\kappa}, t\right) = 1,$$

for any  $\kappa > 0$ .

## 5 Concluding Remarks

In this paper we consider coin space decoherent quantum random walks with coin space unitary transformation  $U$ . We prove that under the eigenvalue conditions, the scaling limit of the probability distribution converges in distribution to a continuous convex combination of normal distributions. An necessary and sufficient condition is obtained for  $U$  to satisfy the eigenvalue conditions. For  $U$  in  $O(2)$ , an exact form of the limiting distribution is given and the moments of all orders are obtained. For this case, the critical exponents are obtained and we show that all  $U$  with rotation angles  $\theta \neq \frac{n\pi}{2}$ ,  $n = 0, 1, 2, 3$ , belong to the same universality class with respect to the critical exponents of all moments as  $p \rightarrow 0$ . Our analysis is based on the characteristic functions of the position distribution and the analysis of eigenvalues, Theorem 4.1 and its corollary, which plays an important role in the applications of our main convergence theorem, Theorem 4.1. We believe that a wider class of universality should hold for general quantum random walks in general  $d$ -dimensional lattices with general rotation  $U \in U(n)$ . For the future research, it would be very interesting to explore and classify their universality classes with respect to their critical points. On the other hand, we have fixed a measurement in our applications, while our general convergence theorem does not depend on the special form as that in our applications. An interesting problem would be to understand how the general measurements affect the limiting distributions and, especially their universality classes.

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