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ON PRIME IDEALS OF NOETHERIAN SKEW POWER SERIES RINGS

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ABSTRACT. We study prime ideals in skew power series rings $T := R[[y;\tau,\delta]]$, for suitably conditioned complete right noetherian rings R, automorphisms τ of R, and τ-derivations δ of R. Such rings were introduced by Venjakob, motivated by issues in noncommutative Iwasawa theory. Our main results concern "Cutting Down" and "Lying Over." In particular, assuming that τ extends to a compatible automorphsim of T, we prove: If I is an ideal of R, then there exists a τ-prime ideal P of T contracting to I if and only if I is a τ-δ-prime ideal of R. Consequently, under the more specialized assumption that $\delta = \tau - \text{id}$ (a basic feature of the Iwasawa-theoretic context), we can conclude: If I is an ideal of R, then there exists a prime ideal P of T contracting to I if and only if I is a τ-prime ideal of R. Our approach depends essentially on two key ingredients: First, the algebras considered are zariskian (in the sense of Li and Van Oystaeyen), and so the ideals are all topologically closed. Second, topological arguments can be used to apply previous results of Goodearl and the author on skew polynomial rings.

1. INTRODUCTION

Given a commutative ring C, the following two properties of the formal power series ring $C[[x]]$ are obvious and fundamental: (1) If P is a prime ideal of $C[[x]]$ then $P \cap C$ is prime. (2) If Q is a prime ideal of C then there exists a prime ideal of $C[[x]]$ contracting to Q. As elementary as these facts are, their analogues for noncommutative skew power series rings are not immediately clear. Our aim in this paper is to establish such analogues in the setting of the skew power series algebras $R[[y;\tau,\delta]]$, for suitably conditioned complete right noetherian rings R, automorphisms τ of R, and τ-derivations δ of R. Skew power series rings of this type were introduced by Venjakob in [14], motivated by issues in noncommutative Iwasawa theory. Further results were then established in [12],[13], and several questions on the ideal theory of these rings were set forth in [1]. In [15], completions of certain quantum coordinate rings were also shown to be iterated skew power series rings in the above sense.

1.1. To briefly describe the objects of our study, let R be a ring containing an ideal $\mathfrak{i}$ such that the $\mathfrak{i}$-adic filtration of R is separated and complete. Also suppose that the

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graded ring gr$R$ associated to the $\mathfrak{i}$-adic filtration is noetherian; it follows that R is noetherian and that the filtration is *zariskian* in the sense of [10].

Now assume that τ is an automorphism of R, that $\tau(\mathfrak{i}) \subseteq \mathfrak{i}$, that δ is a left τ-derivation of R, and that τ induces a compatible automorphism $\overline{\tau}$ of gr$R$. Further assuming that $\delta(R) \subseteq \mathfrak{i}$ and $\delta(\mathfrak{i}) \subseteq \mathfrak{i}^2$, we can adapt [13],[14] to construct the right noetherian skew power series ring $T := R[[y;\tau,\delta]]$, which will also be complete, separated, and zariskian with respect to the $\mathfrak{j}$-adic filtration, where $\mathfrak{j} = \mathfrak{i} + \langle y \rangle$ is contained in the Jacobson radical of T. (In this paper we work in a slightly different context than in [13] or [14]. In [13] it is assumed that R is not necessarily noetherian but is pseudocompact, and in [14] it is assumed that R is local and $\mathfrak{i}$ is the Jacobson radical.)

1.2. The motivating examples for the above and related constructions, in [1],[12],[13], and [14], are the noncommutative Iwasawa algebras $\Lambda(G)$, for compact p-adic Lie groups G containing closed normal subgroups H such that $G/H \cong \mathbb{Z}_p$. In this case, $\Lambda(G) \cong \Lambda(H)[[t;\sigma,\delta]]$, for an automorphism σ and σ-derivation δ of $\Lambda(H)$. We also have $\Omega(G) \cong \Omega(H)[[t;\sigma,\delta]]$; again see [1],[12],[13], and [14].

1.3. Following [15], another class of examples arises from the well-known quantized coordinate rings that can be realized as iterated skew polynomial rings

$$k[y_1][y_2;\tau_2,\delta_2]\cdots[y_n;\tau_n,\delta_n]$$

over a field k, for suitable automorphisms τ_i and τ_i-derivations δ_i. (Constructions of these rings can be found, e.g., in [4].) In [15] it is shown, under addtional assumptions generalizing Venjakob's original approach in [14], that the completion at $\langle y_1,\dots,y_n\rangle$ is an iterated skew power series ring

$$k[[\widehat{y}_1]][[\widehat{y}_2;\widehat{\tau}_2,\widehat{\delta}_2]]\cdots[[\widehat{y}_n;\widehat{\tau}_n,\widehat{\delta}_n]],$$

where each $\widehat{y}_i$ represents the image of y_i in the completion. Applicable examples include the quantized coordinate rings of $n{\times}n$ matrices, of symplectic spaces, and of euclidean spaces; see [15] for further details.

1.4. To sketch our results, assume for the remainder of this introduction that τ extends to an automorphism of the skew polynomial ring $R[y;\tau,\delta]$ and further extends to an automorphism of $T = R[[y;\tau,\delta]]$. This assumption is satisfied by the examples cited above; see (3.15). Next, recall the definitions of *τ-prime* and *τ-δ-prime* ideals, reviewed in (2.2) and (3.13). In particular, the presence of notherianity ensures that a τ-prime ideal will be the intersection of a finite τ-orbit of prime ideals; see (2.2ii).

1.5. In (3.17) we prove: (i) *If Q is a τ-δ-prime ideal of R then there exists a τ-prime ideal P of T such that $P \cap R = Q$.* (ii) *If P is a τ-prime ideal of T then $P \cap R$ is a τ-δ-prime ideal of R.*

1.6. Next, assume further that $\delta = \tau - \mathrm{id}$, a condition present for the Iwasawa algebras noted above. Under this assumption, an ideal I of R is a δ-ideal if and only if I is a τ-ideal, if and only if I is a τ-δ-ideal. In (3.18) we prove: (i) *If P is a prime ideal of T then $P \cap R$ is a τ-prime ideal of R.* (ii) *If Q is a τ-prime ideal of R then there exists a prime ideal P of T such that $P \cap R = Q$.*

1.7. Notation and Terminology. All rings mentioned will be assumed to be associative and to posess a multiplicative identity 1. We will use Spec( ) to denote the prime spectrum of a ring.

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2. Complete Noetherian Rings

2.1. **Setup.** Throughout, A will denote a nonzero right noetherian topological ring. We will further assume:

(1) The topology for A is determined by a filtration of ideals

$$A = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots .$$

(In other words, this filtration forms a fundamental system of open neighborhoods of 0. We use this fundamental system to define limits in A.)

(2) Under the given topology, A is Hausdorff (equivalently, $\langle 0 \rangle$ is closed in A; equivalently, $\cap_i \mathfrak{a}_i = \langle 0 \rangle$). We say that the filtration is *separated*.

(3) A is complete as a topological space (i.e., Cauchy sequences converge). We also say that the filtration is complete.

(4) A contains a dense subring A', equipped with the subspace topology.

(5) α is an automorphism of A restricting to an automorphism of A'. We include the case when α is the identity map.

When $\mathfrak{a}_i = \mathfrak{a}^i$ for some ideal $\mathfrak{a}$ of A, then the preceding topology on A is the well-known $\mathfrak{a}$-*adic topology*. We include the case when $\mathfrak{a}_i = \langle 0 \rangle$ for all i, in which case we obtain the discrete topology on A.

2.2. (i) An ideal I of A is an α-*ideal* provided $\alpha(I) \subseteq I$; since A is right noetherian, note that $\alpha(I) \subseteq I$ if and only if $\alpha(I) = I$. We say that A (assumed to be nonzero) is α-*prime* if the product of two nonzero α-ideals of A is always nonzero, and we say that a proper (i.e., not equal to A) α-ideal P of A is α-*prime* provided A/P is α-prime (under the induced α-action on A/P).

(ii) Since A is right noetherian, an ideal P of A is α-prime if and only if $P = P_1 \cap \cdots \cap P_t$, for some finite α-orbit $P_1, \ldots, P_t$, under the naturally induced α-action on Spec$A$; see [5, Remarks 4*, 5*, p. 338]. Furthermore, $P_1, \ldots, P_t$ is a complete list of the prime ideals of A minimal over P.

2.3. (See e.g. [2, 3.2.29].) Let B denote a complete, Hausdorff topological ring containing a dense subring B'. If $\varphi : A' \to B'$ is a bicontinuous ring isomorphism, then φ extends to a bicontinuous ring isomorphism from A onto B. In particular, a bicontinuous automorphism of A' extends to a bicontinuous automorphism of A.

Given a subset X of A, we will use $\overline{X}$ to denote the closure in A.

2.4. **Lemma.** (i) *Let P be a closed α-prime ideal of A. Then $P \cap A'$ is an α-prime ideal of A'.*

(ii) *If I is an ideal of A' then $\overline{I}$ is an ideal of A.*

Proof. (i) Saying that a proper ideal P of A is α-prime is equivalent to saying, for all $x, y \in A$, and all integers n, that $\alpha^n(x)Ay \subseteq P$ only if x or y is contained in P. Now choose arbitrary $r, s \in A'$, and an arbitary integer n. Suppose that $\alpha^n(r)A's \subseteq P \cap A'$. To prove $P \cap A'$ is α-prime it suffices to prove that r or s is contained in $P \cap A'$.

So let a be an arbitrary element of A, and choose a sequence $a_i \in A'$ converging to a. Then

$$\alpha^n(r)as \;=\; \lim_{i\to\infty} \alpha^n(r)a_i s \;\in\; \overline{P \cap A'} \;\subseteq\; P,$$

because P is closed. Next, since P is α-prime and n was chosen arbitrarily, it follows that one of r or s is contained in P. Therefore, one of r or s is contained in $P \cap A'$, and so $P \cap A'$ is α-prime.

(ii) Straightforward. □

2.5. We next discuss zariskian filtrations, following [10].

(i) To start, we have the *associated graded ring*,

$$\mathrm{gr}A \;:=\; A/\mathfrak{a}_1 \oplus \mathfrak{a}_1/\mathfrak{a}_2 \oplus \mathfrak{a}_2/\mathfrak{a}_3 \oplus \cdots ,$$

and the corresponding function

$$\mathrm{gr}(a) \;:=\; (a_i)_{i=0}^{\infty}, \quad \text{with} \quad a_i \;=\; \begin{cases} a + \mathfrak{a}_i/\mathfrak{a}_{i+1} & \text{when } a \in \mathfrak{a}_i \setminus \mathfrak{a}_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

We also have the *Rees ring*:

$$\widetilde{A} \;:=\; A \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots$$

(ii) Following [10, p. 83, Definition], we say that A (or more precisely, the given filtration of A) is *right zariskian* provided that $\mathfrak{a}_1$ is contained in the Jacobson radical of A, and provided that $\widetilde{A}$ is right noetherian. Since the filtration is assumed to be complete, it follows from [10, p. 87, Proposition] that A is right zariskian if gr$A$ is right noetherian.

(iii) The consequence of the zariskian property we will require most is the following: If A is right zariskian then every right ideal of A is closed with respect to the topology defined by the corresponding filtration; see [10, p. 85, Corollary 5].

(iv) Assume that gr$A$ is right noetherian, and let I be an ideal of A. By (iii), the induced filtration on A/I,

$$A/I = \mathfrak{a}_0 + I \supset \mathfrak{a}_1 + I \supset \cdots ,$$

is separated and complete. Moreover, gr(A/I) must be right noetherian. It follows that A/I is right zariskian with respect to the induced filtration.

(v) More generally, let R be any ring equipped with a complete filtration of right ideals. If gr$R$ is right noetherian then it follows from well-known arguments that R is also right noetherian.

(vi) Of course, all of the preceding remains true when "left" is substituted for "right."

3. Noetherian Skew Power Series Rings

We now turn to skew power series rings. Our treatment is adapted from [13],[14]. However, we do not assume that the coefficient ring is either local (as in [13]) or pseudocompact (as in [14]).

3.1. **Setup.** In this section we assume:

(1) R is a (nonzero) ring containing the proper ideal $\mathfrak{i}$.

(2) The $\mathfrak{i}$-adic filtration of R is separated and complete.

(3) The associated graded ring gr$R$ corresponding to the $\mathfrak{i}$-adic filtration is noetherian. Consequently, R is (right and left) noetherian, as noted in (2.5v). Also, by (2.5ii), we know that R is right (and left) zariskian, and so all of the right and left ideals of R are closed in the $\mathfrak{i}$-adic topology, by (2.5iii).

(4) R is equipped with an automorphism τ and a left τ-derivation δ (i.e., $\delta(ab) = \tau(a)\delta(b) + \delta(a)b$ for all $a, b \in R$). We assume further that $\tau(\mathfrak{i}) = \mathfrak{i}$ and that τ induces an automorphism $\overline{\tau}$ of gr$R$, with $\overline{\tau}\text{gr}(a) = \text{gr}\tau(a)$, for $a \in A$.

(5) (Following [14, §2].) $\delta(R) \subseteq \mathfrak{i}$, and $\delta(\mathfrak{i}) \subseteq \mathfrak{i}^2$.

3.2. All further references to filtrations and topologies on R will refer to the $\mathfrak{i}$-adic filtration and $\mathfrak{i}$-adic topology. Note, since $\tau(\mathfrak{i}^i) = \mathfrak{i}^i$ for all non-negative integers i, that τ is a homeomorphism in the $\mathfrak{i}$-adic topology.

3.3. We will let S denote the skew polynomial ring $R[y; \tau, \delta]$.

(i) The elements of S are (skew) polynomials

$$r_0 + r_1 y + r_2 y^2 + \cdots + r_n y^n,$$

for $r_0, \ldots, r_n \in R$, and with multiplication determined by $yr = \tau(r)y + \delta(r)$, for $r \in R$; more details are given in (3.4ii).

(ii) Set $\tau' = \tau^{-1}$ (recalling that τ is an automorphsims), and set $\delta' = -\delta\tau^{-1}$. Then δ is a *right* τ'-derivation of R, and using a process symmetric to the preceeding we can construct a skew polynomial ring S' over R, with coefficients on the right, such that multiplication is determined by

$$ry = y\tau'(r) + \delta'(r),$$

for $r \in R$. It is well known that S' is isomorphic to S. By (5) of (3.1), $\delta'(R) \subseteq \mathfrak{i}$ and $\delta'(\mathfrak{i}) \subseteq \mathfrak{i}^2$.

(iii) When δ is the zero derivation, it is customary to denote the skew polynomial ring by $R[y; \tau]$.

(iv) See, for instance, [7],[8], or [11] for detailed background on skew polynomial rings. Recall, in particular, that S is noetherian since R is noetherian.

3.4. (Cf. [13, §1], [14, §2].)

(i) To start, let T denote the set of (formal skew power) series

$$\sum r_i y^i \;=\; \sum_{i=0}^{\infty} r_i y^i,$$

for $r_0, r_1, \ldots \in R$.

For now, regard T as a left R-module, isomorphic to a direct product of infinitely many copies of R. Identify S, as a left R-module, with the left R-submodule of T of series having only finitely many nonzero coefficients.

(ii) For non-negative integers i and all integers k, define $\theta_{ik}(r)$, for $r \in R$, via

$$y^i r = \theta_{i,i}(r) y^i + \theta_{i,\, i-1}(r) y^{i-1} + \cdots \theta_{i,0}(r),$$

with $\theta_{i,k}(r) = 0$ when $k \leq 0$ or $k > i$. Note that multiplication in S is given by

$$\left(\sum_{i=0}^{p} a_i y^i\right)\left(\sum_{j=0}^{q} b_j y^j\right) \;=\; \sum_{i=0}^{p}\sum_{j=0}^{q} a_i \left(\sum_{k=0}^{i} \theta_{i,k}(b_j) y^k\right) y^j \;=$$

$$\sum_{k=0}^{\infty}\sum_{i=0}^{p}\sum_{j=0}^{q} a_i \theta_{i,k}(b_j) y^{k+j} \;=\; \sum_{n=0}^{\infty}\sum_{i=0}^{p}\sum_{j=0}^{q} a_i \theta_{i,n-j}(b_j) y^n,$$

for $a_0, \ldots, a_p, b_0, \ldots, b_q \in R$.

(iii) By (4) and (5) of (3.1), for $\ell = 2, 3, \ldots$,

$$\delta(\mathfrak{i}^{\ell}) \;\subseteq\; \mathfrak{i}^{\ell-1}\delta(\mathfrak{i}) + \delta(\mathfrak{i}^{\ell-1})\mathfrak{i} \;\subseteq\; \mathfrak{i}^{\ell+1}.$$

Also, $\theta_{i,k}(R) \subseteq \mathfrak{i}^{i-k}$, for all $i \geq k$. Therefore,

$$\sum_{i=0}^{\infty} a_i \theta_{ik}(b)$$

converges in R, for all $a_0, a_1, \ldots$ and b in R.

(iv) By (iii), we now can (and will) define multiplication in T via

$$\left(\sum a_i y^i\right)\left(\sum b_j y^j\right) \;=\; \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left(\sum_{i=0}^{\infty} a_i \theta_{i,n-j}(b_j)\right)\right) y^n,$$

for $a_0, b_0, a_1, b_1, \ldots \in R$. It can then be verified that T is an associative unital ring, and we set $T = R[[y;\tau,\delta]]$. Observe that T contains S as a subring. (Compare with [13, §1],[14, §2].)

(v) In view of (3.3ii), using τ' and δ' instead of τ and δ, we can use a process symmetric to the preceding to construct an associative skew power series ring T' with coefficients on the right. In particular, as a right R-module, T' is isomorphic to a direct product of infinitely many copies of R. It will follow from (3.9) that T and T' are isomorphic.

(vi) We let $\mathfrak{j}$ denote the ideal of T generated by $\mathfrak{i}$ and y, and we let $\mathfrak{j}'$ denote the ideal of T' generated by $\mathfrak{i}$ and y.

3.5. Examples of skew power series rings fitting the above framework include:

(i) (Following [1, 3.3],[13, §4],[14].) Let G be a compact p-adic Lie group containing a closed normal subgroup H such that $G/H \cong \mathbb{Z}_p$. Then the Iwasawa algebra $\Lambda(G)$ has the form $\Lambda(H)[[t;\sigma,\delta]]$, where σ is an automorphism of the ring $\Lambda(H)$, and where δ is a left σ-derivation of $\Lambda(H)$. The related algebra $\Omega(G)$ can similarly be written as the skew power series ring $\Omega(H)[[t;\sigma,\delta]]$.

(ii) Following [3] (cf., e.g., [4] for further background and context), let n be a positive integer, let k be a field, let $p = (p_{ij})$ be a multiplicatively antisymmetric $n{\times}n$ matrix (i.e., $p_{ij} = p_{ji}^{-1}$ and $p_{ii} = 1$) with entries in k, and let $\mathcal{O}_{\lambda,p} = \mathcal{O}_{\lambda,p}(M_n(k))$ denote the multiparameter quantized coordinate ring of $n{\times}n$ matrices. Then $\mathcal{O}_{\lambda,p}$ is the k-algebra generated by $y_{11}, y_{12}, \ldots, y_{nn}$, with the following defining relations: $y_{ij}y_{rs} = p_{ir}p_{sj}y_{rs}y_{ij} + (\lambda-1)p_{ir}y_{rj}y_{rs}$, for $i > r$ and $j > s$; $y_{ij}y_{rs} = \lambda p_{ir}p_{sj}y_{rs}y_{ij}$, for $i > r$ and $j \leq s$; $y_{ij}y_{rs} = p_{sj}y_{rs}y_{ij}$, for $i = r$ and $j > s$.

In [15], the completion $\widehat{\mathcal{O}}_{\lambda,p}$ of $\mathcal{O}_{\lambda,p}$ at $\langle y_{11}, y_{12}, \ldots, y_{nn}\rangle$ is studied, building on the approach of [13],[14]. It is shown that $\widehat{\mathcal{O}}_{\lambda,p}$ can be written as an iterated skew power series ring,

$$k[[\widehat{y}_{11}]][[\widehat{y}_{12};\widehat{\tau}_{12},\widehat{\delta}_{12}]]\cdots[[\widehat{y}_{nn};\widehat{\tau}_{nn},\widehat{\delta}_{nn}]],$$

for suitable automorphisms $\widehat{\tau}_{ij}$ and left $\widehat{\tau}_{ij}$-derivations $\widehat{\delta}_{ij}$. Similar results for completions of other quantum coordinate rings can also be found in [15].

3.6. **Lemma.** *For all positive integers ℓ, $\mathfrak{j}^\ell = \mathfrak{i}^\ell + \mathfrak{i}^{\ell-1}y + \mathfrak{i}^{\ell-2}y^2 + \cdots + \mathfrak{i}y^{\ell-1} + Ty^\ell$.*

Proof. We prove $\mathfrak{j}^\ell \subseteq \mathfrak{i}^\ell + \mathfrak{i}^{\ell-1}y + \cdots + \mathfrak{i}y^{\ell-1} + Ty^\ell$; the reverse inclusion is immediate. To start, recall that $yR \subseteq \mathfrak{i} + Ry$. Also, $T = R \oplus Ty$. Therefore,

$$yT \;=\; y(R\oplus Ty) \;=\; yR + yTy \;\subseteq\; \mathfrak{i} + Ry + yTy \;\subseteq\; \mathfrak{i} + Ty.$$

So

$$\mathfrak{j} \;=\; \mathfrak{i} + \langle y\rangle \;=\; \mathfrak{i} + TyT \;\subseteq \mathfrak{i} + (\mathfrak{i} + yT)T \;\subseteq\; \mathfrak{i} + \mathfrak{i}T + yT \;\subseteq\; \mathfrak{i} + yT.$$

In particular, the lemma holds for $\ell = 1$.

Now, for all positive integers j, it follows from (3.1(5)) that $y\mathfrak{i}^j \subseteq \mathfrak{i}^j y + \mathfrak{i}^{j+1}$. Therefore, by induction, for $\ell \geq 2$,

$$\begin{aligned}\mathfrak{j}^\ell \;=\; \mathfrak{j}^{\ell-1}\mathfrak{j} \;\subseteq\; (\mathfrak{i}+Ty)\left(\mathfrak{i}^{\ell-1} + \mathfrak{i}^{\ell-2}y + \mathfrak{i}^{\ell-3}y^2 + \cdots + \mathfrak{i}y^{\ell-2} + Ty^{\ell-1}\right)\\ \subseteq\; \mathfrak{i}^\ell + \mathfrak{i}^{\ell-1}y + \mathfrak{i}^{\ell-2}y^2 + \cdots + \mathfrak{i}y^{\ell-1} + Ty^\ell.\end{aligned}$$

The lemma follows. □

3.7. **Proposition.** (Cf. [13, §1],[14, §2].) *The $\mathfrak{j}$-adic filtration on T is separated and complete.*

Proof. Let $\sum a_i y^i$ be a series in T contained in the intersection of all of the powers of $\mathfrak{j}$. It then follows from (3.6) that each a_i is contained in the intersection of all of the powers of $\mathfrak{i}$. Therefore, each $a_i = 0$, since R is separated with respect to the $\mathfrak{i}$-adic filtration. Hence T is separated.

Now let $\sum a_{i1} y^i, \sum a_{i2} y^i, \ldots$ be a Cauchy sequence in T with respect to the $\mathfrak{j}$-adic topology. For each i, it follows that the sequence $\{a_{ij}\}_{j=0}^{\infty}$ is a Cauchy sequence in R, by (3.6), and so we can set

$$a_i = \lim_{j\to\infty} a_{ij},$$

since R is complete with respect to the $\mathfrak{i}$-adic topology. Again using (3.6), it is not hard to check that

$$\lim_{j\to\infty} \sum a_{ij} \ = \ \sum a_i,$$

under the $\mathfrak{j}$-adic topology, and so T is complete with respect to the $\mathfrak{j}$-adic topology. □

3.8. Henceforth, references to topologies and filtrations on T will refer to the $\mathfrak{j}$-adic topology and the $\mathfrak{j}$-adic filtration. By (3.6), the relative topology on R, viewed as a subspace of T, coincides with the $\mathfrak{i}$-adic topology. Note that S is a dense subring of T, and equip S with the relative topology from T.

3.9. Consider again T', as described in (3.4v), and $\mathfrak{j}'$ as defined in (3.4vi). By symmetry, (3.7) tells us that the $\mathfrak{j}'$-adic filtration of T' is separated and complete, and we henceforth equip T' with the $\mathfrak{j}'$-adic topology. Next, S', as described in (3.3ii), is a dense subring of T', and we equip S' with the relative topology. Now observe that the identification of S with S' noted in (3.3iii) is bicontinuous. It then follows from (2.3) and (3.8) that the ring isomorphism from S onto S' noted in (3.3ii) extends to a bicontinuous ring isomorphism from T onto T'. We therefore identify T with T', and we therefore can write power series in T with coefficients on either the left or right.

3.10. Let $\mathrm{gr}T$ denote the associated graded ring of T corresponding to the $\mathfrak{j}$-adic filtration.

(i) It follows from (3.6) that $\mathrm{gr}T \cong (\mathrm{gr}R)[y;\overline{\tau}]$, where $\overline{\tau}$ is the automorphism of $\mathrm{gr}R$ from (3.1(4)). In particular, $\mathrm{gr}T$ is noetherian.

(ii) Since $\mathrm{gr}T$ is noetherian, and since the $\mathfrak{j}$-adic filtration of T is complete, by (3.7), it follows that the $\mathfrak{j}$-adic filtration of T is right and left zariskian, by (2.5ii). Consequently the right and the left ideals of T are closed in the $\mathfrak{j}$-adic topology, by (2.5iii).

3.11. **Lemma.** *Let $\sum a_{i1} y^i, \sum a_{i2} y^i, \ldots$ be a convergent sequence in T, for $a_{ij} \in R$, with*

$$\lim_{j\to\infty} \left(\sum_{i=0}^{\infty} a_{ij} y^i \right) = \sum a_i y^i,$$

for $a_i \in R$. Then $\lim_{j\to\infty} a_{ij} = a_i$.

Proof. Choose a positive integer ℓ. Then, for some positive integer m,

$$\sum(a_i - a_{ij})y^i \; = \; \sum a_i y^i - \sum a_{ij} y^i \; \in \mathfrak{j}^\ell,$$

for all $j \geq m$. Next, fix i and choose a positive integer k. If ℓ was chosen large enough, it now follows from (3.6) that

$$a_i - a_{ij} \in \mathfrak{i}^k,$$

for all $j \geq m$. The lemma follows. □

3.12. **Proposition.** (cf. [7, 5.12]) *Let P be a prime ideal of T. Then there exists a non-negative integer n and a prime ideal Q of R such that $Q, \tau(Q), \ldots, \tau^n(Q)$ are exactly the prime ideals of R minimal over $P \cap R$.*

Proof. It follows from (2.4i) that $P \cap S$ is a prime ideal of S. Of course, the prime ideals of R minimal over $P \cap R$ will be precisely the prime ideals of R minimal over $(P \cap S) \cap R$. The proposition now follows from [7, 5.12]. □

We now turn to induced ideals.

3.13. Recall: (i) Following (2.2), we have τ-*ideals* and τ-*prime* ideals of R.

(ii) An ideal I of R is a τ-δ-*ideal* if $\tau(I) = I$ and $\delta(I) \subseteq I$.

(iii) If the product of nonzero τ-δ-ideals of R is always nonzero, we say that R is τ-δ-*prime*. We say that a proper τ-δ-ideal Q of R is τ-δ-*prime* provided R/Q is τ-δ-prime (under the induced actions of τ and δ on R/Q). Observe that a δ-stable τ-prime ideal must be τ-δ-prime.

The following lemma is similar to [9, 2.6–2.8]; however, the proof is somewhat different.

3.14. **Lemma.** *Let I be a τ-δ-ideal of R. Then:*

(i)

$$I[[y;\tau,\delta]] \; := \; \left\{ \sum a_i y^i \; : \; a_i \in I \right\} \; = \left\{ \sum y^i b_i \; : \; b_i \in I \right\}.$$

(ii) *$I[[y;\tau,\delta]]$ is is an ideal of T and is the topological closure in T of $IS = SI$.*

(iii) *$IT = I[[y;\tau,\delta]] = TI$.*

(iv) *$T/IT \cong (R/I)[[y;\tau,\delta]]$, under the induced actions of τ and δ on R/I.*

Proof. (i) Recall θ_{ik} from (3.4ii). Since I is τ-δ-stable, it follows that $\theta_{ik}(I) \subseteq I$ for all non-negative integers i and k. Next, for $b_i \in I$, it follows from (3.4iii) that

$$\sum_{i=0}^{\infty} \theta_{ik}(b_i)$$

is well defined for all k; the closure of I in R, established in (3) of (3.1), further ensures that this series is contained in I. Therefore,

$$\sum y^i b_i \;=\; \left(\sum_i \theta_{i0}(b_i)\right) + \left(\sum_i \theta_{i1}(b_i)\right) y + \left(\sum_i \theta_{i2}(b_i)\right) y^2 + \cdots \\ \in \left\{ \sum b'_k y^k \;:\; b'_k \in I \right\}.$$

A symmetric argument, beginning with $\sum a_i y^i$, for $a_i \in I$, establishes (i); see (3.4v).

(ii) Since I is a τ-δ-ideal and τ is an automorphism,

$$IS = \{a_0 + a_1 y + \cdots + a_m y^m : a_i \in I\} \;=\; \{b_0 + y b_1 + \cdots + y^n b_n : b_i \in I\} \;=\; SI.$$

However, since I is a closed subset of R, it follows from (3.11) that $I[[y;\tau,\delta]]$ is the topological closure in T of the ideal $IS = SI$ of S. Furthermore, it now follows from (2.4iv) that $I[[y;\tau,\delta]]$ is an ideal of T.

(iii) First, it is easy to see that IT and TI are contained in $I[[y;\tau,\delta]]$. On the other hand, IT and TI are closed subsets of T, by (3.10ii), since IT is a right ideal of T and TI is a left ideal of T. We learned in (ii) that $I[[y;\tau,\delta]]$ is the topological closure of $IS = SI$, and so $I[[y;\tau,\delta]]$ is contained in IT and TI, since $IS \subseteq IT$ and $SI \subseteq TI$. Part (iii) follows.

(iv) Straightforward, given (ii) and (iii). □

3.15. A well known special case occurs when τ can be extended to compatible automorphisms of S and T. (Here, compatible means that τ extends to an automorphism of T that restricts to an automorphism of S). This situation occurs, for instance, when $\delta\tau = \tau\delta$ (as operators on R); in this case we can extend τ to S and T by setting $\tau(y) = y$. (A proof of this assertion will follow from the next paragraph, setting $q = 1$.) In particular, δ and τ will satisfy the equation $\delta\tau = \tau\delta$ when $\delta = \tau - \text{id}$. For the examples $\Lambda(H)[[t;\sigma,\delta]]$ and $\Omega(H)[[t;\sigma,\delta]]$ mentioned in (3.5i), it is a basic feature of their construction that $\delta = \sigma - \text{id}$; see [13, §4] and [14, 2.2].

More generally, suppose for the moment that $\delta\tau = q\tau\delta$ for some central unit q of R such that $\tau(q) = q$ and $\delta(q) = 0$. Following [7, 2.4ii], τ extends to an automorphism of S such that $\tau(y) = q^{-1}y$. We see that this automorphism of S is bicontinuous, and so τ extends to an automorphism of T, by (2.3). The condition $\delta\tau = q\tau\delta$ holds for the skew power series discussed in (3.5ii).

When τ extends to compatible automorphisms of S and T, we can refer to τ-prime ideals of these rings, following (2.2).

3.16. Assume that τ extends to compatible automorphisms of S and T. Let I be a τ-ideal of S or T. Then, for all $a \in I$, $ya - \tau(a)y \in I$. It follows that $I \cap R$ is a τ-δ-ideal of R.

Part (i) of the following, a weakened analogue to [7, 3.3i], establishes "Lying Over," and part (ii) provides another "Cutting Down."

3.17. **Theorem.** *Assume that τ extends to compatible automorphisms of S and T.*

(i) *Suppose that Q is a τ-δ-prime ideal of R. Then there exists a τ-prime ideal P of T such that $P \cap R = Q$.*

(ii) *Suppose that P is a τ-prime ideal of T. Then $P \cap R$ is a τ-δ-prime ideal of R.*

Proof. (i) It follows from (3.14) that $QT = Q[[y; \tau, \delta]] = TQ$, from which it is easy to deduce that $QT \cap R = Q$. Also, $\tau(QT) = \tau(Q)\tau(T) = QT$, and so QT is a τ-ideal of T. We see, then, that there exists at least one τ-ideal of T whose intersection with R is equal to Q. Therefore, we can choose a τ-ideal P of T maximal such that $P \cap R = Q$. (Note that $P \neq T$.)

We can prove as follows that P is τ-prime: Let I and J be τ-ideals of T, both containing P, such that $IJ \subseteq P$. By (3.16), $I \cap R$ and $J \cap R$ are τ-δ-ideals of R. Also, $(I \cap R)(J \cap R) \subseteq P \cap R = Q$. Now, since Q is τ-δ-prime, at least one of $I \cap R$ or $J \cap R$ is contained in Q; say $I \cap R \subseteq Q$. But $I \supseteq P$, and so $I \cap R = Q$. The maximality of P now ensures that $I = P$.

We conclude that P is a τ-prime ideal of T contracting to Q.

(ii) Let $I = P \cap R$. It follows from (3.16) that I is a τ-δ-ideal of R.

Now suppose that J and K are τ-δ-ideals of R such that $JK \subseteq I$. By (3.14iii), $TJ = JT$ and $TK = KT$ are ideals of T. Moreover, JT and KT are τ-ideals of T. Now, $(JT)(KT) = (JK)T \subseteq (P \cap R)T \subseteq P$, and so one of JT or KT is contained in P. By (3.14iii), $JT \cap R = J[[y; \tau, \delta]] \cap R = J$ and $KT \cap R = K[[y; \tau, \delta]] \cap R = K$. We see that one of J or K is contained in I, and (ii) follows. □

We can obtain even more precise results in the case where $\delta = \tau - \text{id}$. As noted in (3.15), the skew polynomial rings $\Lambda(H)[[t; \sigma, \delta]]$ and $\Omega(H)[[t; \sigma, \delta]]$ satisfy $\delta = \sigma - \text{id}$. Note, when $\delta = \tau - \text{id}$, that an ideal I of R is a δ-ideal if and only if I is a τ-ideal, if and only if I is a τ-δ-ideal.

3.18. **Theorem.** *Assume that $\delta = \tau - id$.*

(i) *Let P be a prime ideal of T. Then $P \cap R$ is a τ-prime ideal of R. In particular, $P \cap R = Q \cap \tau(Q) \cap \cdots \cap \tau^{n-1}(Q)$, for some prime ideal Q of R, and some positive integer n, such that $\tau^n(Q) = Q$.*

(ii) *Let Q be a τ-prime ideal of R. Then there exists a prime ideal P of T such that $P \cap R = Q$.*

Proof. Following [14, 4.1], if we set $z = 1 + y$, then

$$zr = (1+y)r = r + yr = r + \tau(r)y + \delta(r) = r + \tau(r)y + \tau(r) - r = \tau(r)(1+y) = \tau(r)z,$$

for $r \in R$. Also, $z = 1 + y$ is a unit in T, with inverse $z^{-1} = 1 - y + y^2 - y^3 + \cdots$. We thus obtain an inner automorphism of T, via $z(\ \ )z^{-1}$.

This inner automorphism maps y to itself and restricts to τ on R. Setting $\tau(\ \ ) = z(\ \ )z^{-1}$ on T, we obtain exactly the extension of τ to T discussed in (3.15). Moreover, $\tau : T \to T$ restricts to the unique automorphism of S that maps y to itself and maps each $r \in R$ to $\tau(r)$.

Note, for all ideals I of T, that $\tau(I) = zIz^{-1} = I$. In other words, every ideal of T is a τ-ideal. Consequently, it follows directly from the definition (3.13ii) that an ideal of T is prime if and only if it is τ-prime. Also, by (3.16), every ideal of T contracts to a τ-δ-ideal of R.

Next, observe that $S = R[y;\tau,\delta] = R[z;\tau]$. Let P be a prime ideal of T. We know from (2.4i) that $P \cap S$ is a prime ideal of S. Also, since z is a unit in T, we know that $z \notin P \cap S \subseteq P$. It now follows from the standard theory that $P \cap R = (P \cap S) \cap R$ is τ-prime; see, for example, [11, 10.6.4iii]. Recalling (2.2ii), part (i) follows.

Part (ii) now follows from (3.17i), since an ideal of R is τ-prime if and only if it is τ-δ-prime. □

3.19. Assume that τ extends to compatible automorphisms of S and T, and let Q be a τ-δ-prime ideal of R. In [7, 3.3i] it is proved that $QS = SQ$ is τ-prime. We ask: Is $TQ = QT$ a τ-prime ideal of T?

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