Exhausting formal quantization procedures

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Abstract. In paper [5] the author introduced stable formality quasi-isomorphisms and described the set of its homotopy classes. This result can be interpreted as a complete description of formal quantization procedures. In this note we give a brief exposition of stable formality quasi-isomorphisms and prove that every homotopy class of stable formality quasi-isomorphisms contains a representative which admits globalization. This note is loosely based on the talk given by the author at XXX Workshop on Geometric Methods in Physics in Bialowieza, Poland.

1. Introduction

In seminal paper [12] M. Kontsevich constructed an L_∞ quasi-isomorphism from the graded Lie algebra of polyvector fields on the affine space $\mathbb{R}^d$ to the dg Lie algebra of Hochschild cochains $C^\bullet(A)$ for the polynomial algebra $A = \mathbb{R}[x^1, x^2, \dots, x^d]$. This result implies that equivalence classes of star-products on $\mathbb{R}^d$ are in bijection with the equivalence classes of formal Poisson structures on $\mathbb{R}^d$. This theorem also implies that Hochschild cohomology of a deformation quantization algebra is isomorphic to the Poisson cohomology of the corresponding formal Poisson structure.

In the view of these consequences, we will think about L_∞ quasi-isomorphisms from the graded Lie algebra of polyvector fields on the affine space $\mathbb{R}^d$ to the dg Lie algebra of Hochschild cochains $C^\bullet(A)$ as *formal quantization procedures.*

Following [2] one can define a natural notion of homotopy equivalence on the set of L_∞-morphisms between dg Lie algebras (or even L_∞-algebras). Furthermore, according to Lemma B.5 from [1], homotopy equivalent L_∞ quasi-morphisms for $C^\bullet(A)$ give the same bijection between the set of equivalence classes of star-products and the set of equivalence classes of formal Poisson structures. Thus, for the purposes of applications, we should only be interested in homotopy classes of formality quasi-isomorphisms.

In paper [5] the author developed a framework of what he calls *stable formality quasi-isomorphisms (SFQ)* and showed that homotopy classes of such SFQ's form a torsor for the group which is obtained by exponentiating the Lie algebra $H^0(\mathsf{GC})$ where GC is the graph complex introduced by M. Kontsevich in [11, Section 5]. Any SFQ gives us an L_∞ quasi-isomorphism for the Hochschild cochains of $A = \mathbb{R}[x^1, x^2, \dots, x^d]$ in all[1] dimensions d simultaneously. Moreover, homotopy equivalent SFQ's give homotopy equivalent L_∞ quasi-isomorphisms for the Hochschild cochains of $A = \mathbb{R}[x^1, x^2, \dots, x^d]$. Thus the main result (Theorem 6.2) of [5] can be interpreted as a complete description of formal quantization procedures in the stable setting.

In the next section we remind the full (directed) graph complex and its relation to Kontsevich's graph complex GC [11, Section 5]. In Section 3 we give a brief exposition of stable formality quasi-isomorphisms (SFQ). Finally, in Section 4 we prove that every SFQ is homotopy equivalent to an SFQ which admits globalization.

Notation and conventions. In this note we assume that the ground field $\mathbb{K}$ contains the field of reals. For most of algebraic structures considered in this note, the underlying symmetric monoidal category is the category of unbounded cochain complexes of $\mathbb{K}$-vector spaces. For a cochain complex $\mathcal{V}$ we denote by $\mathbf{s}\mathcal{V}$ (resp. by $\mathbf{s}^{-1}\mathcal{V}$) the suspension (resp. the desuspension) of $\mathcal{V}$. In other words,

$$\big(\mathbf{s}\mathcal{V}\big)^{\bullet} = \mathcal{V}^{\bullet-1}\,, \qquad \big(\mathbf{s}^{-1}\mathcal{V}\big)^{\bullet} = \mathcal{V}^{\bullet+1}\,.$$

$C^{\bullet}(A)$ denotes the Hochschild cochain complex of an associative algebra (or more generally an A_∞-algebra) A with coefficients in A. For a commutative ring R and an R-module V we denote by $S_R(V)$ the symmetric algebra of V over R.

Given an operad $\mathcal{O}$, we denote by $\circ_i$ the elementary operadic insertions:

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(k) \to \mathcal{O}(n+k-1)\,, \qquad 1 \le i \le n\,.$$

The notation $\mathrm{Sh}_{p,q}$ is reserved for the set of (p,q)-shuffles in S_{p+q}. A graph is *directed* if each edge carries a chosen direction. A graph Γ with n vertices is called *labeled* if Γ is equipped with a bijection between the set of its vertices and the set $\{1, 2, \dots, n\}$. ε denotes a formal deformation parameter.

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[1] In fact they are also defined for any $\mathbb{Z}$-graded affine space.

2. The full directed graph complex dfGC

In this section we recall from [14] an extended version dfGC of Kontsevich's graph complex GC [11, Section 5]. For this purpose, we first introduce a collection of auxiliary sets $\{\mathrm{dgra}(n)\}_{n\ge 1}$. An element of dgra_n is a directed labelled graph Γ with n vertices and with the additional piece of data: the set of edges of Γ is equipped with a total order. An example of an element in dgra_4 is shown on figure 2.1.

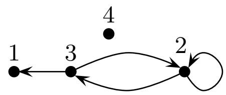


FIG. 2.1. The edges are equipped with the order $(3,1) < (3,2) < (2,3) < (2,2)$

Next, we introduce a collection of graded vector spaces $\{\mathsf{dGra}(n)\}_{n\ge 1}$. The space $\mathsf{dGra}(n)$ is spanned by elements of dgra_n, modulo the relation $\Gamma^\sigma = (-1)^{|\sigma|}\Gamma$ where the graphs Γ^σ and Γ correspond to the same directed labelled graph but differ only by permutation σ of edges. We also declare that the degree of a graph Γ in $\mathsf{dGra}(n)$ equals $-e(\Gamma)$, where $e(\Gamma)$ is the number of edges in Γ. For example, the graph Γ on figure 2.1 has 4 edges. Thus its degree is -4.

Following [14], the collection $\{\mathsf{dGra}(n)\}_{n\ge 1}$ forms an operad. The symmetric group S_n acts on $\mathsf{dGra}(n)$ in the obvious way by rearranging labels and the operadic multiplications are defined in terms of natural operations of erasing vertices and attaching edges to vertices.

The operad dGra can be upgraded to a 2-colored operad KGra whose spaces[2] are formal linear combinations of graphs used by M. Kontsevich in [12].

We define the graded vector space dfGC by setting

$$\mathsf{dfGC} = \prod_{n\ge 1} \mathbf{s}^{2n-2}\Big(\mathsf{dGra}(n)\Big)^{S_n}. \tag{2.1}$$

Next, we observe that the formula

$$\Gamma \bullet \widetilde{\Gamma} = \sum_{\sigma\in \mathrm{Sh}_{k,n-1}} \sigma\big(\Gamma \circ_1 \widetilde{\Gamma}\big) \tag{2.2}$$

$$\Gamma \in \Big(\mathsf{dGra}(n)\Big)^{S_n}, \qquad \widetilde{\Gamma} \in \Big(\mathsf{dGra}(k)\Big)^{S_k}$$

defines a degree zero $\mathbb{K}$-bilinear operation on $\bigoplus_{n\ge 1} \mathbf{s}^{2n-2}\Big(\mathsf{dGra}(n)\Big)^{S_n}$ which extends in the obvious way to the graded vector space dfGC (2.1).

[2]For more details, we refer the reader to [5, Section 3].

It is not hard to show that the operation (2.2) satisfies axioms of the pre-Lie algebra and hence dfGC is naturally a Lie algebra with the bracket give by the formula

$$[\gamma, \widetilde{\gamma}] = \gamma \bullet \widetilde{\gamma} - (-1)^{|\gamma||\widetilde{\gamma}|}\, \widetilde{\gamma} \bullet \gamma\,, \tag{2.3}$$

where γ and $\widetilde{\gamma}$ are homogeneous vectors in dfGC.

A direct computation shows that the degree 1 vector

$\Gamma_{\bullet\!-\!\bullet} =$  (2.4)

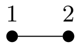
satisfies the Maurer-Cartan equation $[\,\Gamma_{\bullet\!-\!\bullet}, \Gamma_{\bullet\!-\!\bullet}\,] = 0$.

Thus, dfGC forms a dg Lie algebra with the bracket (2.3) and the differential

$$\partial = [\,\Gamma_{\bullet\!-\!\bullet},\ \,]\,. \tag{2.5}$$

Definition 2.1. The cochain complex $(\mathsf{dfGC}, \partial)$ is called the full directed graph complex.

Let us observe that every undirected labeled graph Γ with n vertices and with a chosen order on the set of its edges can be interpreted as the sum of all directed labeled graphs Γ_α in $\mathrm{dgra}(n)$ from which the graph Γ is obtained by forgetting directions on edges. For example,

$\Gamma_{\bullet\!-\!\bullet} =$  (2.6)

Thus, using undirected labeled graphs we may form a suboperad Gra inside dGra and the sub- dg Lie algebra

$$\mathsf{fGC} = \prod_{n \ge 1} \mathbf{s}^{2n-2} \Big(\mathsf{Gra}(n)\Big)^{S_n} \quad \subset \quad \mathsf{dfGC} \tag{2.7}$$

Definition 2.2 (M. Kontsevich, [11]**).** *Kontsevich's graph complex* GC *is the subcomplex*

$$\mathsf{GC} \subset \mathsf{fGC} \tag{2.8}$$

formed by (possibly infinite) linear combinations of connected graphs Γ satisfying these two properties: *each vertex of* Γ *has valency* ≥ 3*, and the complement to any vertex is connected.*

It is easy to see that GC is a sub- dg Lie algebra of fGC. Furthermore, following[3] [14] we have

Theorem 2.3 (T. Willwacher, [14]**).** *The cohomology of* dfGC *can be expressed in terms of cohomology of* GC*. More precisely,*

$$H^\bullet(\mathsf{dfGC}) = \mathbf{s}^{-2}\, S\big(\mathbf{s}^2\, \mathcal{H}\big) \tag{2.9}$$

where

$$\mathcal{H} = H^\bullet(\mathsf{GC}) \ \oplus\ \bigoplus_{m \ge 0} \mathbf{s}^{4m-1}\mathbb{K}\,.$$

[3]See lecture notes [7] for more detailed exposition.

Using decomposition (2.9), it is not hard to see that

$$H^0(\mathsf{dfGC}) \cong H^0(\mathsf{GC}) \qquad (2.10)$$

and the Lie algebra $H^0(\mathsf{dfGC})$ is pro-nilpotent.

3. Stable formality quasi-isomorphisms

Let $A = \mathbb{K}[x^1, x^2, \dots, x^d]$ be the algebra of functions on the affine space $\mathbb{K}^d$ and let $V_A^\bullet$ be the algebra of polyvector fields on $\mathbb{K}^d$

$$V_A^\bullet = S_A\big(\mathbf{s}\,\mathrm{Der}(A)\big)\,. \qquad (3.1)$$

Recall that $V_A^\bullet$ is a free commutative algebra $V_A^\bullet = \mathbb{K}[x^1, x^2, \dots, x^d, \theta_1, \theta_2, \dots, \theta_d]$ over $\mathbb{K}$ in d generators $x^1, x^2, \dots, x^d$ of degree zero and d generators $\theta_1, \theta_2, \dots, \theta_d$ of degree one.

It is know that $V_A^{\bullet+1}$ is a graded Lie algebra. The Lie bracket on $V_A^{\bullet+1}$ is given by the formula:

$$[v, w]_S = (-1)^{|v|} \sum_{i=1}^{d} \frac{\partial v}{\partial \theta_i} \frac{\partial w}{\partial x^i} - (-1)^{|v||w|+|w|} \sum_{i=1}^{d} \frac{\partial w}{\partial \theta_i} \frac{\partial v}{\partial x^i}\,. \qquad (3.2)$$

It is called the *Schouten bracket*.

In plain English an L_∞-morphism U from $V_A^{\bullet+1}$ to $C^{\bullet+1}(A)$ is an infinite collection of maps

$$U_n : \big(V_A^{\bullet+1}\big)^{\otimes n} \to C^{\bullet+1}(A)\,, \qquad n \ge 1 \qquad (3.3)$$

compatible with the action of symmetric groups and satisfying an intricate sequence of quadratic relations. The first relation says that U_1 is a map of cochain complexes, the second relation says that U_1 is compatible with the Lie brackets up to homotopy with U_2 serving as a chain homotopy and so on.

Kontsevich's construction of such a sequence (3.3) is "natural" in the following sense: given polyvector fields $v_1, v_2, \dots, v_n \in V_A^{\bullet+1}$, the value

$$U_n\big(v_1, v_2, \dots, v_n\big)(a_1, a_2, \dots, a_k) \qquad (3.4)$$

of the cochain $U_n(v_1, v_2, \dots, v_n)$ on polynomials $a_1, a_2, \dots, a_k \in A$ is obtained via contracting all indices of derivatives of various orders of $v_1, \dots, v_n, a_1, \dots, a_k$ in such a way that the resulting map

$$(V_A^\bullet)^{\otimes n} \otimes A^{\otimes k} \to A$$

is $\mathfrak{gl}_d(\mathbb{K})$-equivariant. Thus each term in U_n can be encoded by a directed graph with two types of vertices: vertices of one type are reserved for polyvector fields and vertices of another type are reserved for polynomials.

Motivated by this observation, the author introduced in [5] a notion of *stable formality quasi-isomorphism (SFQ)* which formalizes L_∞ quasi-isomorphisms U for Hochschild cochains satisfying this property: *each term in U_n is encoded by a graph with two types of vertices and all the desired relations on U_n's hold universally, i.e. on the level of linear combinations of graphs.*

The precise definition of SFQ is given in terms of 2-colored dg operads OC and KGra. The later operad KGra is a 2-colored extension of the operad dGra which is "assembled" from graphs used by M. Kontsevich in [12]. This operad comes with a natural action on the pair $(V_A^{\bullet+1}, A = \mathbb{K}[x^1, \dots, x^d])$. The operad OC governs open-closed homotopy algebras introduced in [10] by H. Kajiura and J. Stasheff. We recall that an open-closed homotopy algebra is a pair $(\mathcal{V}, \mathcal{A})$ of cochain complexes equipped with the following data:

- An L_∞-structure on $\mathcal{V}$;
- an A_∞-structure on $\mathcal{A}$; and
- an L_∞-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^\bullet(\mathcal{A})$ of the A_∞-algebra $\mathcal{A}$.

Since the operad KGra acts on the pair $(V_A^{\bullet+1}, A = \mathbb{K}[x^1, \dots, x^d])$, any morphism of dg operads

$$F : \mathsf{OC} \to \mathsf{KGra} \tag{3.5}$$

gives us an L_∞-structure on $V_A^{\bullet+1}$, an A_∞-structure on A and an L_∞ morphism from $V_A^{\bullet+1}$ to $C^\bullet(A)$.

An SFQ is defined as a morphism (3.5) of dg operads satisfying three boundary conditions. The first condition guarantees that the L_∞-algebra structure on $V_A^{\bullet+1}$ induced by F coincides with the Lie algebra structure given by the Schouten bracket (3.2). The second condition implies that the A_∞-algebra structure on A coincides with the usual associative (and commutative) algebra structure on polynomials. Finally, the third condition ensures that the L_∞-morphism

$$U : V_A^{\bullet+1} \leadsto C^{\bullet+1}(A)$$

induced by F starts with the Hochschild-Kostant-Rosenberg embedding. In particular, the last condition implies that U is an L_∞ quasi-isomorphism.

Kontsevich's construction [12] provides us with an example of an SFQ over any extension of the field of reals[4]

In paper [5] the author also defined the notion of homotopy equivalence for SFQ's. This notion is motivated by the property that L_∞ quasi-isomorphisms

$$U, \widetilde{U} : V_A^{\bullet+1} \leadsto C^{\bullet+1}(A)$$

corresponding to homotopy equivalent SFQ's F and $\widetilde{F}$ are connected by a homotopy which "admits a graphical expansion" in the above sense.

Following [11] we have a chain map Θ from the full (directed) graph complex dfGC to the deformation complex of the dg Lie algebra $V_A^{\bullet+1}$ of polyvector fields. In particular, every degree zero cocycle in dfGC produces an L_∞-derivation of $V_A^{\bullet+1}$. Exponentiating these L_∞-derivations we get an action of the (pro-unipotent) group

$$\exp\left(\mathsf{dfGC}^0 \cap \ker \partial\right)$$

[4]The existence of an SFQ over rationals is proved in papers [4] and [6].

on the set of L_∞ quasi-isomorphisms

$$U : V_A^{\bullet+1} \rightsquigarrow C^{\bullet+1}(A) \tag{3.6}$$

for $A = \mathbb{K}[x^1, \dots, x^d]$. Namely, given a cocycle $\gamma \in \mathsf{dfGC}^0$, the action of $\exp(\gamma)$ is defined by the formula

$$U \mapsto U \circ \exp\big(-\Theta(\gamma)\big)\,, \tag{3.7}$$

where Θ is the chain map from dfGC to the deformation complex of $V_A^{\bullet+1}$.

In [5], it was proved that the action (3.7) descends to an action of the (pro-unipotent) group

$$\exp\big(H^0(\mathsf{dfGC})\big) \tag{3.8}$$

on the set of homotopy classes of SFQ's. Moreover,

Theorem 3.1 (Theorem 6.2, [5]**).** *The group* (3.8) *acts simply transitively on the set of homotopy classes of SFQ's.*

In the view of philosophy outlined in the Introduction, this result can be interpreted as a complete description of formal quantization procedures.

Remark 3.2. According to the recent result [14, Theorem 1] of T. Willwacher, $\exp\big(H^0(\mathsf{GC})\big)$ is isomorphic to the Grothendieck-Teichmueller group GRT introduced by V. Drinfeld in [9]. Thus, combining this result with Theorem 3.1, we conclude that formal quantization procedures are "governed" by the group GRT.

Remark 3.3. In recent preprint [15] Thomas Willwacher computes stable cohomology of the graded Lie algebra of polyvector fields with coefficients in the adjoint representation. His computations partially justify the name "stable formality quasi-isomorphism" chosen by the author in [5]. In particular, Thomas Willwacher mentions in [15] a possibility to deduce the part about transitivity from Theorem 3.1 in a more conceptual way.

4. Globalization of stable formality quasi-isomorphisms

Given an L_∞ quasi-isomorphism (3.6) for $A = \mathbb{K}[x^1, \dots, x^d]$ we can ask the question of whether we can use it to construct a sequence of L_∞ quasi-isomorphisms which connects the sheaf $V_X^{\bullet+1}$ of polyvector fields to the sheaf $\mathcal{D}_X^{\bullet+1}$ of polydifferential operators on a smooth algebraic variety X over $\mathbb{K}$. There are several similar constructions [3], [13], [16] which allow us to produce such a sequence under the assumption that the L_∞ quasi-isomorphism (3.6) satisfies the following properties:

- **A)** One can replace $A = \mathbb{K}[x^1, \dots, x^d]$ in (3.6) by its completion $A_{formal} = \mathbb{K}[[x^1, \dots, x^d]]$;
- **B)** the structure maps U_n of U are $\mathfrak{gl}_d(\mathbb{K})$-equivariant;
- **C)** if $n > 1$ then

$$U_n(v_1, v_2, \dots, v_n) = 0 \tag{4.1}$$

for every set of vector fields $v_1, v_2, \dots, v_n \in \mathrm{Der}(A_{formal})$;

D) if $n \ge 2$ and $v \in \mathrm{Der}(A_{formal})$ has the form

$$v = \sum_{i,j=1}^{d} v^i_j x^j \frac{\partial}{\partial x^i}\,, \qquad v^i_j \in \mathbb{K}$$

then for every set $w_2, \dots, w_n \in V^{\bullet+1}_{A_{formal}}$

$$U_n(v, w_2, \dots, w_n) = 0\,. \tag{4.2}$$

In paper [8] it was shown that for every degree zero cocycle $\gamma \in \mathsf{GC}$ the structure maps $\Theta(\gamma)_n$ of the L_∞-derivation $\Theta(\gamma)$ satisfy these properties:

a) $\Theta(\gamma)$ can be viewed as an L_∞-derivation of $V^{\bullet+1}_{A_{formal}}$ with $A_{formal} = \mathbb{K}[[x^1, \dots, x^d]]$;

b) the structure maps $\Theta(\gamma)_n$ of $\Theta(\gamma)$ are $\mathfrak{gl}_d(\mathbb{K})$-equivariant;

c) if $n > 1$ then

$$\Theta(\gamma)_n(v_1, v_2, \dots, v_n) = 0 \tag{4.3}$$

for every set of vector fields $v_1, v_2, \dots, v_n \in \mathrm{Der}(A_{formal})$;

d) if $n \ge 2$ and $v \in \mathrm{Der}(A_{formal})$ has the form

$$v = \sum_{i,j=1}^{d} v^i_j x^j \frac{\partial}{\partial x^i}\,, \qquad v^i_j \in \mathbb{K}$$

then for every set $w_2, \dots, w_n \in V^{\bullet+1}_{A_{formal}}$

$$\Theta(\gamma)_n(v, w_2, \dots, w_n) = 0\,. \tag{4.4}$$

Properties **a)** and **b)** are obvious, while properties **c)** and **d)** follow from the fact that each graph in the linear combination $\gamma \in \mathsf{GC}$ has only vertices of valencies ≥ 3.

Using these properties of $\Theta(\gamma)$ together with Theorems 2.3 and 3.1 we deduce the main result of this note:

Theorem 4.1. *Every homotopy class of SFQ's contains a representative which can be used to construct a sequence of L_∞ quasi-isomorphisms connecting the sheaf $V^{\bullet+1}_X$ of polyvector fields to the sheaf $\mathcal{D}^{\bullet+1}_X$ of polydifferential operators on a smooth algebraic variety X over $\mathbb{K}$.*

Proof. Let F' be an SFQ. Our goal is to prove that the homotopy class of F' contains a representative F whose corresponding L_∞ quasi-isomorphism (3.6) satisfies Properties **A)** – **D)** listed above.

Let us denote by F_K an SFQ whose corresponding L_∞ quasi-isomorphism

$$U_K : V^{\bullet+1}_A \rightsquigarrow C^{\bullet+1}(A) \tag{4.5}$$

satisfies Properties **A)** – **D)**. (For example, we can choose the SFQ coming from Kontsevich's construction [12].)

Theorem 3.1 implies that there exists a degree zero cocycle $\gamma' \in \mathsf{dfGC}$ such that F' is homotopy equivalent to the SFQ

$$\exp(\gamma')\big(F_K\big)\,. \tag{4.6}$$

On the other hand, we have isomorphism (2.10). Therefore, γ' is cohomologous to a cocycle $\gamma \in \mathsf{GC}$ and hence F' is homotopy equivalent to

$$\exp(\gamma)\big(F_K\big)\,. \tag{4.7}$$

Since the L_∞-derivation $\Theta(\gamma)$ satisfies Properties **a)** – **d)** and the L_∞ quasi-isomorphism (4.5) satisfies Properties **A)** – **D)**, we conclude that the L_∞ quasi-isomorphism corresponding to the SFQ (4.7) also satisfies Properties **A)** – **D)**.

Theorem 4.1 is proved. □

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