

Extending Sobolev Functions with Partially Vanishing Traces from Locally (ε, δ) -Domains and Applications to Mixed Boundary Problems

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Abstract

We prove that given any $k \in \mathbb{N}$, for each open set $\Omega \subseteq \mathbb{R}^n$ and any closed subset D of $\bar{\Omega}$ such that Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$ there exists a linear and bounded extension operator $\mathfrak{E}_{k,D}$ mapping, for each $p \in [1, \infty]$, the space $W_D^{k,p}(\Omega)$ into $W_D^{k,p}(\mathbb{R}^n)$. Here, with \mathcal{O} denoting either Ω or \mathbb{R}^n , the space $W_D^{k,p}(\mathcal{O})$ is defined as the completion in the classical Sobolev space $W^{k,p}(\mathcal{O})$ of (restrictions to \mathcal{O} of) functions from $\mathcal{C}_c^\infty(\mathbb{R}^n)$ whose supports are disjoint from D . In turn, this result is used to develop a functional analytic theory for the class $W_D^{k,p}(\Omega)$ (including intrinsic characterizations, boundary traces and extensions results, interpolation theorems, among other things) which is then employed in the treatment of mixed boundary value problems formulated in locally (ε, δ) -domains.

1 Introduction

Extension results for Sobolev spaces defined on open subsets of the Euclidean ambient are important tools in many branches of mathematics, including harmonic analysis, potential theory, and partial differential equations. Recall that, given $k \in \mathbb{N}$ and $p \in [1, \infty]$, an open set $\Omega \subseteq \mathbb{R}^n$ is called a $W^{k,p}$ -extension domain if there exists a bounded linear operator

$$E : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n) \quad (1.1)$$

with the property that $(Eu)|_\Omega = u$ for each $u \in W^{k,p}(\Omega)$ (for background definitions the reader is referred to §2). Such a condition necessarily imposes restrictions on the underlying set Ω . For example, not all functions from the Sobolev space $W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ and $p \in (n/k, \infty)$ may be extended to $W^{k,p}(\mathbb{R}^n)$, $n \geq 2$, in the case in which

$$\Omega_a = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_1, \dots, x_{n-1} < 1 \text{ and } 0 < x_n < x_{n-1}^a\} \text{ with } a > kp - 1. \quad (1.2)$$

Indeed, the fact that $p > n/k$ ensures that $W^{k,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^0(\mathbb{R}^n)$ and yet if $b > 0$ is small enough so that $a > p(k+b) - 1$, then the function $u(x) := x_{n-1}^{-b}$ for each $x = (x_1, \dots, x_n) \in \Omega_a$ belongs to $W^{k,p}(\Omega_a)$, but obviously has no continuous extension to \mathbb{R}^n . The obstruction in this case is the presence of outward cusps on $\partial\Omega_a$ (caused by the fact that $a > kp - 1$, $p > n/k$, and $n \geq 2$ necessarily entails $a > 1$).

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On the positive side, a classical result in harmonic analysis asserts that any Lipschitz domain is a $W^{k,p}$ -extension domain for all $k \in \mathbb{N}$ and $p \in [1, \infty]$. The first breakthrough came in the work of A.P. Calderón in [6] where, for each given Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ and each given $k \in \mathbb{N}$, a linear extension operator E_k is constructed which maps $W^{k,p}(\Omega)$ boundedly into $W^{k,p}(\mathbb{R}^n)$ for each $p \in (1, \infty)$, and which has the additional property that

$$\text{supp}(E_k u) \subseteq \overline{\Omega} \quad \text{for every } u \in \mathcal{C}_c^\infty(\Omega). \quad (1.3)$$

In the same geometrical setting but via a different approach, E.M. Stein has produced (see the exposition in [53, Theorem 5, p. 181]) an extension operator which, as opposed to Calderón's, is universal in the sense that it does not depend on the order of smoothness (and, of course, the integrability exponent), and which is also bounded in the limiting cases $p = 1$ and $p = \infty$. Nonetheless, Stein's operator no longer enjoys property (1.3) as it scatters the support of the function on which it acts across the boundary.

Both original proofs of Calderón's and Stein's theorems make essential use of the fact that Lipschitz domains satisfy a uniform cone property. The latter property actually characterizes Lipschitzianity, so new ideas must be involved if the goal is to go establish extension results beyond this class of domains. Via a conceptually novel approach, which builds on the seminal work of H. Whitney on his extension theorem for Lipschitz functions in [55], P.W. Jones succeeded (cf. [25, Theorem 1, p. 73]) in generalizing the results of Calderón and Stein to a much larger class of sets, which he called (ε, δ) -domains. Jones also proved that a finitely connected open set $\Omega \subseteq \mathbb{R}^2$ is a $W^{k,p}$ -extension domain for all $k \in \mathbb{N}$ and $p \in [1, \infty]$ if and only if Ω is (ε, δ) -domains for some values $\varepsilon, \delta > 0$ (cf. [25, Theorem 3, p. 74]). Since Jones' class of domains is going to be of basic importance for the goals we have in mind, below we record its actual definition.

Definition 1.1. *Assume that $\varepsilon \in (0, \infty)$ and $\delta \in (0, \infty]$. A nonempty, open, proper subset Ω of \mathbb{R}^n is called an (ε, δ) -domain if for any $x, y \in \Omega$ with $|x - y| < \delta$ there exists a rectifiable curve $\gamma : [0, 1] \rightarrow \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$, and*

$$\text{length}(\gamma) \leq \frac{1}{\varepsilon} |x - y| \quad \text{and} \quad \frac{\varepsilon |z - x| |z - y|}{|x - y|} \leq \text{dist}(z, \partial\Omega), \quad \forall z \in \gamma([0, 1]). \quad (1.4)$$

Informally, the first condition in (1.4) says that Ω is locally connected in some quantitative sense, while the second condition in (1.4) says that there exists some type of "tube" T , with $\gamma([0, 1]) \subset T \subset \Omega$ and the width of T at a point z on the curve is of the order $\min\{|z - x|, |z - y|\}$.

Examples of (ε, δ) -domains include bi-Lipschitz images of Lipschitz domains, open sets whose boundaries are given locally as graphs of functions in the Zygmund class Λ_1 , or of functions with gradients in the John-Nirenberg space BMO, as well as the classical van Koch snowflake domain of conformal mapping theory. The boundary of an (ε, δ) -domain can be highly nonrectifiable and, in general, no regularity condition on $\partial\Omega$ can be inferred from the (ε, δ) property described in Definition 1.1. The fact that, in general, (ε, δ) -domains are not sets of finite perimeter can be seen from the fact that the classical van Koch snowflake domain does not have finite perimeter. In fact, for each $d \in [n - 1, n)$ there exists an open set $\Omega \subseteq \mathbb{R}^n$ such that Ω is an (ε, ∞) -domain for some $\varepsilon = \varepsilon(d) \in (0, \infty)$ and $\partial\Omega$ has Hausdorff dimension d . This being said, it has been shown in [25, Lemma 2.3, p. 77] that

$$\text{any } (\varepsilon, \delta)\text{-domain } \Omega \subseteq \mathbb{R}^n \text{ satisfies } \mathcal{L}^n(\partial\Omega) = 0, \quad (1.5)$$

where \mathcal{L}^n denotes the Lebesgue measure in \mathbb{R}^n .

Jone's (ε, δ) -domains interface tightly with the category of uniform domains considered a little earlier by O. Martio and J. Sarvas in [30]. Recall that a nonempty, proper, open subset Ω of \mathbb{R}^n is said to be a **uniform domain** provided there exists a constant $c = c(\Omega) \in [1, \infty)$ with the property that each pair of points $x_1, x_2 \in \Omega$ can be joined by a rectifiable curve γ in Ω for which

$$\text{length}(\gamma) \leq c |x_1 - x_2| \quad \text{and} \quad \min_{j=1,2} |x_j - x| \leq c \text{dist}(x, \partial\Omega) \quad \text{for each } x \in \gamma. \quad (1.6)$$

Also, call Ω a **locally uniform domain** if there exist $c, r \in (0, \infty)$ with the property that (1.6) holds whenever $|x_1 - x_2| < r$. Then a nonempty, proper, open subset of the Euclidean space is an (ε, δ) -domain for some $\varepsilon, \delta \in (0, \infty)$ if and only if it is a locally uniform domain. Moreover, if Ω is a uniform domain, then Ω satisfies an interior corkscrew condition as well as a Harnack chain condition, in the sense of D. Jerison and C.E. Kenig (cf. [24]). Conversely, if Ω satisfies an interior corkscrew condition, a Harnack chain condition, and has the property that $\partial\Omega$ is bounded, then Ω is a uniform domain. The interested reader is referred to [22, Propositions A.2-A.3] for more details in this regard. As a consequence, here we only wish to note that the class of (ε, δ) -domains with a compact boundary coincides with the category of **one-sided NTA domains** (i.e., domains satisfying an interior corkscrew condition and a Harnack chain condition), from [24].

Returning to the issue of extension results for Sobolev spaces, the following is [25, Theorem 1, p. 73].

Theorem 1.2. *Let Ω be a finitely connected (ε, δ) -domain in \mathbb{R}^n and fix $k \in \mathbb{N}$. Then there exists a linear operator Λ_k mapping $W^{k,p}(\Omega)$ boundedly into $W^{k,p}(\mathbb{R}^n)$ for each $p \in [1, \infty]$, and such that $(\Lambda_k u)|_{\Omega} = u$ for each $u \in W^{k,p}(\Omega)$.*

Since its introduction in the early 80's, Jones' extension operator Λ_k has been the focal point of a considerable amount of work. For example, R. DeVore and R. Sharpley have successfully adapted Jones' ideas as to construct in [14] an extension operator for Besov spaces on (ε, δ) -domains in \mathbb{R}^n . Furthermore, B.L. Fain in [17], A. Seeger in [48], and P.A. Shvartsman in [51] have generalized Jones' theorem to anisotropic Sobolev spaces, while S.-K. Chua has proved weighted versions of Jones' theorem, involving Muckenhoupt weights in [9], and doubling weights satisfying a Poincaré inequality in [10]. Here we also wish to mention the work of N. Garofalo and D.M. Nhieu in [19] where the authors have established extension theorems for Sobolev functions in Carnot-Carathéodory spaces in a suitable analogue of the class of (ε, δ) -domains for this setting. Another significant development appeared in [46] where L.G. Rogers combines the techniques of P. Jones and E. Stein to produce an extension operator for Sobolev functions in (ε, δ) -domains which, as opposed to Jones', is universal. However, Rogers' hybrid operator scatters supports of functions across the boundary even more severely than Jones's extension operator (which already fails to satisfy property (1.3)).

A version of Jones' extension operator in (ε, δ) -domains which is more in line with the original design from [25] is due to M. Christ (cf. [8]), who has shown that a mild alteration renders Jones' operator semi-universal (i.e., it simultaneously extends functions with preservation of class up to any desired, a priori given, threshold of smoothness). Moreover, M. Christ works (cf. also [39]) with a more general scale, which he denotes by $\mathfrak{N}_\alpha^p(\Omega)$, $1 < p \leq \infty$, $\alpha > 0$, originally introduced by R.A. DeVore and R.C. Sharpley in [13]. Indeed, $\mathfrak{N}_\alpha^p(\Omega)$ turns out to be a genuine Sobolev space in the case when α is an integer (cf. [13]), and a Triebel-Lizorkin space otherwise (cf. [27]). The semi-universality character of M. Christ's extension is going to have some degree of significance for our work. This being said, the main issues we are presently concerned with (see below) have, to the best of our knowledge, never been addressed before.

To gain a broader perspective let us now revisit the concept of $W^{k,p}$ -extension domain and introduce an extra nuance. Specifically, fix $k \in \mathbb{N}$ and $p \in [1, \infty]$ then, given an arbitrary nonempty open set $\Omega \subseteq \mathbb{R}^n$ and a closed linear subspace V of $W^{k,p}(\Omega)$, call Ω a *V-extension domain* provided there exists a linear and bounded operator $E : V \rightarrow W^{k,p}(\mathbb{R}^n)$ such that $(Eu)|_{\Omega} = u$ for each $u \in V$. In this terminology, any nonempty open set $\Omega \subseteq \mathbb{R}^n$ is a $\dot{W}^{k,p}(\Omega)$ -extension domain (taking E to be the operator of extension by zero outside Ω) while, at the other end of the spectrum, Theorem 1.2 comes to assert that any finitely connected (ε, δ) -domain Ω in \mathbb{R}^n is a $W^{k,p}(\Omega)$ -extension domain. As an intermediate case, note that given $k \in \mathbb{N}$ and $p \in (n/k, \infty)$, even though the set Ω_a from (1.2) fails to be a $W^{k,p}(\Omega_a)$ -extension domain it is a V -extension domain for any space of the form

$$V := \{u \in W^{k,p}(\Omega_a) : u \equiv 0 \text{ on } O \cap \Omega_a\} \quad (1.7)$$

where O is some neighborhood of the cuspidal edge $C := \{(x_1, \dots, x_{n-2}, 0, 0) : 0 \leq x_1, \dots, x_{n-2} \leq 1\}$ of Ω_a . This is clear from Calderón's extension theorem and the fact that $\partial\Omega_a$ is a Lipschitz surface away

from C . In light of this discussion, it is very much apparent that the geometry of a nonempty open set Ω influences the nature of the linear closed spaces $V \subseteq W^{k,p}(\Omega)$ for which Ω is a V -extension domain.

By way of analogy, suppose now that Ω is an open, nonempty subset of \mathbb{R}^n which satisfies the (ε, δ) condition from Definition 1.1 only near a relatively open portion N of its topological boundary $\partial\Omega$. The question now becomes: *for what closed linear subspaces V of $W^{k,p}(\Omega)$ is Ω a V -extension domain?* Our earlier analysis suggests considering the closure in $W^{k,p}(\Omega)$ of restrictions to Ω of functions from $\mathcal{C}_c^\infty(\mathbb{R}^n)$ whose support is disjoint from the rough portion of the boundary, i.e., $\partial\Omega \setminus N$.

More generally, in the case when Ω is a nonempty open set in \mathbb{R}^n and D (playing the role of $\partial\Omega \setminus N$) is an arbitrary closed subset of $\overline{\Omega}$, introduce

$$W_D^{k,p}(\Omega) := \text{the closure of } \{\varphi|_\Omega : \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ with } D \cap \text{supp } \varphi = \emptyset\} \text{ in } W^{k,p}(\Omega). \quad (1.8)$$

Then our first main result in this paper is the following extension result (for a more general formulation see Theorem 3.7; also, the class of sets which are locally (ε, δ) -domains near a portion of their boundary is introduced in Definition 3.4).

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ be such that D is closed and Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. Then for any $k \in \mathbb{N}$ there exists an operator $\mathfrak{E}_{k,D}$ such that for any $p \in [1, \infty]$ one has*

$$\mathfrak{E}_{k,D} : W_D^{k,p}(\Omega) \longrightarrow W_D^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly, and} \quad (1.9)$$

$$(\mathfrak{E}_{k,D} u)|_\Omega = u, \quad \mathcal{L}^n\text{-a.e. on } \Omega \text{ for every } u \in W_D^{k,p}(\Omega). \quad (1.10)$$

Theorem 1.3 bridges between the “trivial” extension of functions from $\mathring{W}^{k,p}(\Omega)$ by zero outside an arbitrary open set Ω , to which it reduces in the case when $D := \partial\Omega$, and Jones’ extension operator Λ_k , which our $\mathfrak{E}_{k,D}$ becomes in the case when $D := \emptyset$. Indeed, such a choice forces Ω to be a genuine (ε, δ) -domain and, as explained in the proof of Corollary 3.8, our operator $\mathfrak{E}_{k,D}$ automatically reduces to Jones’ extension operator Λ_k for this class of domains (irrespective of the nature of D). As such, Theorem 1.3 brings to light a novel basic feature of Jones’ extension operator, i.e., the property that for any (ε, δ) -domain Ω and any closed set $D \subseteq \Omega$, the operator Λ_k maps the subspace $W_D^{k,p}(\Omega)$ of $W^{k,p}(\Omega)$ into the subspace $W_D^{k,p}(\mathbb{R}^n)$ of $W^{k,p}(\mathbb{R}^n)$ (with $D := \emptyset$ yielding Theorem 1.2, at least when $p \neq \infty$). It should be noted that Theorem 1.2 would readily imply this more refined version of itself if Jones’ extension operator were to be support preserving, i.e., if $\text{supp}(\Lambda_k u) = \text{supp } u$ for each $u \in W^{k,p}(\Omega)$, but this is far from being the case. In fact, as already remarked by P.W. Jones in [25], his extension operator Λ_k lacks even the weaker property that $\text{supp}(\Lambda_k u) \subseteq \overline{\Omega}$ for every $u \in \mathcal{C}_c^\infty(\Omega)$ which, as pointed out earlier, Calderón’s extension operator enjoys (in the setting of Lipschitz domains, of course).

The main ingredients in the proof of Theorem 1.3 are Jones’ extension result stated in Theorem 1.2, augmented with the property that

$$\overline{\Omega} \cap \text{supp}(\Lambda_k u) = \text{supp } u, \quad \forall u \in W^{k,p}(\Omega), \quad (1.11)$$

and the existence of a quantitative partition of unity which is geometrically compatible with our notion of locally (ε, δ) -domain near a portion of its boundary. The key non-expansive support property (1.11) is proved in Theorem 2.4 via a careful inspection of the format of Jones’ extension operator, recounted in (2.19). One may regard (1.11) as a vestigial form of property (1.3), in the more general context of (ε, δ) -domains. In fact, it is possible to prove a version of Theorem 1.3 in which the intervening extension operator is semi-universal, at least if $1 < p \leq \infty$. We do so in Theorem 3.9, whose proof relies on M. Christ’s alteration of Jones’ extension operator, recorded in Theorem 2.3. This extra feature is important in the context of interpolation with change of smoothness, although we shall not pursue this in the present paper.

Given that, in the special cases when $D = \partial\Omega$ and $D = \emptyset$ the space $W_D^{k,p}(\Omega)$ becomes, respectively, $\mathring{W}^{k,p}(\Omega)$ and $W^{k,p}(\Omega)$ (for $p \neq \infty$), we shall refer to $W_D^{k,p}(\Omega)$ as a *Sobolev space with a partially vanishing trace* (on the set D). We stress that, in general, the set $D \subseteq \overline{\Omega}$ is not necessarily assumed to be contained in the boundary of Ω , and that the terminology “vanishing trace” warrants further

clarification. The reader is referred to Theorem 4.2 for a formal statement in which the vanishing of the higher-order restriction of u to D is formulated in an appropriate capacity sense. See also Theorem 4.6 where the aforementioned restriction is interpreted in the sense of \mathcal{H}^d , the d -dimensional Hausdorff measure, in the case when D is a closed subset of $\overline{\Omega}$ which is d -Ahlfors regular for some $d \in (0, n)$ (a piece of terminology explained in (4.25)). Finally, in Theorem 5.2 we are able to describe $W_D^{k,p}(\Omega)$ as the space consisting of those $u \in W^{k,p}(\Omega)$ whose intrinsic restriction to D , as functions defined in Ω , vanishes \mathcal{H}^d -a.e. on D . This is done under the assumption that Ω is an (ε, δ) -domain and D is a closed subset of $\overline{\Omega}$ which is d -Ahlfors regular for some $d \in (0, n)$.

The key ingredients in the proofs of the structure theorems for the spaces $W_D^{k,p}(\Omega)$ from Theorem 4.2 and Theorem 4.6 are the extension result from Theorem 1.3, a deep result of L.I. Hedberg and T.H. Wolff from [23] stating that any closed set in \mathbb{R}^n has the so-called (k, p) -synthesis property, for any $p \in (1, \infty)$ and any $k \in \mathbb{N}$, along with the trace/extension theory on d -Ahlfors regular subsets of \mathbb{R}^n developed by A. Jonsson and H. Wallin in [26]. The intrinsic characterization of the spaces $W_D^{k,p}(\Omega)$ from Theorem 5.2 (reviewed in the earlier paragraph) requires refining the Jonsson-Wallin theory in several important regards. This is accomplished in Theorem 4.4, Theorem 4.9, and Theorem 5.1 which, in turn, are used to study the issue of preservation of Sobolev class under extension by zero, in Theorem 5.10, and under gluing functions with matching traces, in Theorem 5.12.

Collectively, these results amount to a robust functional analytic theory for the category of Sobolev spaces with partially vanishing traces introduced in (1.8). Along the way, we also answer a recent question posed to us by D. Arnold (cf. Theorem 5.7 and Corollary 5.8 for precise statements, in various degrees of generality), and provide a solution to a question raised by J. Nečas in 1967. Specifically, Problem 4.1 on p. 91 of [40] asks whether for any Lipschitz domain Ω in \mathbb{R}^n , any $k \in \mathbb{N}$ and $p \in (1, \infty)$, one has

$$\mathring{W}^{k,p}(\Omega) = \left\{ u \in W^{k,p}(\Omega) : \frac{\partial^j u}{\partial \nu^j} = 0 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \text{ for } 0 \leq j \leq k-1 \right\}, \quad (1.12)$$

where $\frac{\partial^j}{\partial \nu^j}$ denotes the j -th iterated directional derivative with respect to the outward unit normal ν to Ω (suitably defined). In Theorem 5.13 we prove that this is the case even in the considerably more general setting when Ω is an (ε, δ) -domain in \mathbb{R}^n with the property that $\partial\Omega$ is $(n-1)$ -Ahlfors regular and such that its measure theoretic boundary, $\partial_*\Omega$, has full \mathcal{H}^{n-1} -measure in $\partial\Omega$. The latter condition, i.e. that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, merely ensures that the geometric measure theoretic outward unit normal ν to Ω is defined \mathcal{H}^{n-1} -a.e. on $\partial\Omega$. In this vein, it is worth noting that, by Rademacher's a.e. differentiability theorem for Lipschitz functions, this is always the case in the category of Lipschitz domains.

While the functional analytic study of the spaces $W_D^{k,p}(\Omega)$ undertaken in the first part of the paper is of independent interest, the principal motivation for such an endeavor remains its impact on the study of partial differential equations. In particular, the specific nature of the spaces $W_D^{k,p}(\Omega)$ from (1.8) naturally makes the body of results established here particularly well-suited for the treatment of boundary value problems of mixed type in very general classes of Euclidean domains. Mixed boundary value problems arises naturally in connection to a series of important problems in mathematical physics and engineering, dealing with conductivity, heat transfer, wave phenomena, electrostatics, metallurgical melting, stamp problems in elasticity and hydrodynamics, among many other applications. Specific references can be found in [3], [11], [16], [20], [21], [29], [34], [37], [44], [45], [47], [49], [50], [52], to cite just a fraction of a vast literature on this topic.

In the last section of our paper we formulate and solve such mixed boundary problems for strongly elliptic higher-order systems in bounded open subsets Ω of \mathbb{R}^n which are locally (ε, δ) -domains near $\partial\Omega \setminus D$ with D closed subset of $\partial\Omega$ which is d -Ahlfors regular for some $d \in (n-2, n)$. In such a scenario, D is the portion of the boundary on which a homogeneous higher-order Dirichlet condition is imposed, while a homogeneous Neumann condition is assigned on $\partial\Omega \setminus D$. In this connection, we wish to note that the class of domains just described is much more general than those previously considered in the literature. The following is a slightly sanitized version of our main well-posedness result proved in Theorem 7.3.

Theorem 1.4. *Let Ω be a bounded, connected, open, nonempty, subset of \mathbb{R}^n , $n \geq 2$, and suppose that D is a nonempty closed subset of $\partial\Omega$ which is d -Ahlfors regular for some $d \in (n-2, n)$. In addition, assume that Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$, and consider a strongly elliptic, divergence-form system \mathcal{L} of order $2m$, whose tensor coefficient A consists of bounded measurable functions in Ω .*

Then there exists $p_ \in (2, \infty)$ with the following significance. If*

$$\frac{p_*}{p_* - 1} < p < p_* \quad (1.13)$$

then the mixed boundary value problem

$$\begin{cases} \mathcal{L}u = f|_{\Omega} & \text{in } \mathcal{D}'(\Omega), \\ u \in W_D^{m,p}(\Omega), \\ \partial_{\nu}^A(u, f) = 0 & \text{on } \partial\Omega \setminus D, \end{cases} \quad (1.14)$$

is uniquely solvable for each functional $f \in (W_D^{m,p'}(\Omega))^$, where $1/p + 1/p' = 1$.*

Above, $f|_{\Omega}$ denotes the distribution in Ω canonically associated with the functional $f \in (W_D^{m,p'}(\Omega))^*$, and the homogeneous Neumann boundary condition $\partial_{\nu}^A(u, f) = 0$ on $\partial\Omega \setminus D$ is understood in a variational sense, made clear in Definition 7.1. We also remark that the membership of u to $W_D^{m,p}(\Omega)$ automatically implies (by virtue of results mentioned earlier) that the higher-order restriction of u to D vanishes at \mathcal{H}^d -a.e. point on D . Thus a homogeneous Dirichlet boundary condition on D is implicit, making it clear that problem (1.14) has a mixed character.

Specializing (1.14) to the particular case when $D = \partial\Omega$ yields a well-posedness result for the inhomogeneous Dirichlet problem. For the sake of this introduction, we choose to formulate this corollary in a way which emphasizes the traditional Dirichlet boundary condition in the higher-order setting (i.e., using iterated normal derivatives). Compared with Theorem 1.4, this requires upgrading the underlying geometrical assumptions in order to make this type of boundary condition meaningful. Specifically, the following is a particular case of Theorem 7.4 from the body of the paper.

Theorem 1.5. *Let Ω be a bounded (ε, δ) -domain in \mathbb{R}^n whose boundary is $(n-1)$ -Ahlfors regular. In addition, assume that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ and denote by ν the geometric measure theoretic outward unit normal to Ω . Also, consider a strongly elliptic, divergence-form system \mathcal{L} of order $2m$, with bounded measurable coefficients in Ω . Then there exists $p_* \in (2, \infty)$, depending only on Ω as well as the bounds on the coefficients and the ellipticity constant of \mathcal{L} , with the property that the classical inhomogeneous Dirichlet boundary value problem*

$$\begin{cases} \mathcal{L}u = f \in W^{-m,p}(\Omega), \\ u \in W^{m,p}(\Omega), \\ \frac{\partial^j u}{\partial \nu^j} = 0 & \text{on } \partial\Omega \text{ for } 0 \leq j \leq m-1, \end{cases} \quad (1.15)$$

is well-posed whenever $\frac{p_}{p_* - 1} < p < p_*$.*

By means of counterexamples (based on classical constructions due to N. Meyers [36], E. De Giorgi [12], and V. Maz'ya [31]), in the last part of § 7 we prove that the restriction of p to a small interval near 2 is actually necessary for the well-posedness of the inhomogeneous Dirichlet boundary value problem (1.15). A key ingredient in the proof of Theorems 1.4-1.5 is an interpolation result (cf. Theorem 6.4 for a more complete statement) to the effect that, for each fixed $k \in \mathbb{N}$,

$$\begin{aligned} & \text{the scale } \{W_D^{k,p}(\Omega)\}_{\max\{1, n-d\} < p < \infty} \text{ is stable} \\ & \text{both under the complex and the real method,} \end{aligned} \quad (1.16)$$

whenever $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ are such that D is closed and d -Ahlfors regular for some $d \in (0, n)$, and Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$.

The layout of the paper is as follows. In §2 we collect background definitions and results, clarify terminology, and record a detailed statement of Jones' extension result, expanding on the succinct presentation from Theorem 1.2. In turn, this is used in Theorem 2.4 to establish the fact that Jones' extension operator Λ_k does not enlarge the support of a function $u \in W^{k,p}(\Omega)$ in $\overline{\Omega}$ (assuming, of course, that Ω is an (ε, δ) -domain). This aspect plays a basic role in the proof of Theorem 1.3 (reformulated more generally as Theorem 3.7) in §3. In Theorem 2.3 we record M. Christ's version of Jones' extension theorem, specialized to Sobolev spaces, and subsequently note that Christ's semi-universal extension operator also enjoys the non-expansive support property alluded to above. In particular, this allows us to construct a semi-universal extension operator for Sobolev spaces with partially vanishing traces in Theorem 3.9.

The task of elucidating the structure of Sobolev spaces with partially vanishing traces from (1.8) is taken up in §4. Here, characterizations involving the vanishing of traces on D is proved in Theorem 4.2 in a capacity quasieverywhere sense, and in Theorem 4.6 where such a vanishing condition is formulated using the Hausdorff measure in place of Bessel capacities. The key ingredient in the proof of the latter result is the intrinsic characterization of the null-space of the higher-order boundary trace operator (in the sense considered by A. Jonsson and H. Wallin in [26]) in Theorem 4.4 when this trace operator is acting from Sobolev spaces defined in \mathbb{R}^n . Subsequently, in Theorem 4.9 we refine the Jonsson-Wallin theory from Theorem 4.3 in a manner which allows considering extension/restriction operators preserving certain types of vanishing conditions.

The main result in §5 is Theorem 5.1, containing a trace/extension theory on locally (ε, δ) -domains onto/from Ahlfors regular subsets in \mathbb{R}^n . One of the basic consequences of this theory is the intrinsic description of Sobolev spaces with partially vanishing traces from Theorem 5.2. We then proceed to deduce several important properties of this scale of spaces, including the hereditary property from Theorem 5.7, the issue of preservation of Sobolev class under extension by zero in Theorem 5.10, and under gluing Sobolev functions with matching traces in Theorem 5.12. The last result in this section is Theorem 5.13, which establishes the characterization of the null-space of the higher-order Dirichlet trace operator from (1.12).

The main goal in §6 is proving interpolation results in the spirit of (1.16). See Theorem 6.4 in this regard. Finally, §6 is devoted to applications to boundary value problems in a very general geometric measure theoretic setting. More specifically, Theorem 7.3 deals with the higher-order mixed boundary value problem in locally (ε, δ) -domains, Theorem 7.4 treats the higher-order inhomogeneous Dirichlet problem in arbitrary bounded open sets with d -Ahlfors regular boundaries, Theorem 7.5 addresses the fully inhomogeneous higher-order Poisson problem in bounded (ε, δ) -domains with d -Ahlfors regular boundaries, while the higher-order Neumann problem is considered in Theorem 7.6 in the context of bounded (ε, δ) -domains.

2 Sobolev spaces and Jones' extension operator

We begin by discussing some background definitions and results. Fix a space dimension $n \in \mathbb{N}$, $n \geq 2$, and denote by \mathcal{L}^n the n -dimensional Lebesgue measure in \mathbb{R}^n . Given a Lebesgue measurable set \mathcal{O} in \mathbb{R}^n , we let $L^p(\mathcal{O}, \mathcal{L}^n)$, $0 < p \leq \infty$, stand for the scale of (equivalent classes of) Lebesgue-measurable functions which are p -th power \mathcal{L}^n -integrable in \mathcal{O} . Also, given an open set $\Omega \subseteq \mathbb{R}^n$, for each $p \in (0, \infty]$ denote by $L^p_{loc}(\Omega, \mathcal{L}^n)$ the space of Lebesgue-measurable functions u in Ω with the property that $u|_K \in L^p(K, \mathcal{L}^n)$ for every compact subset K of Ω .

With \mathbb{N} denoting the collection of all (strictly) positive integers, we shall abbreviate $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, \mathbb{N}_0^n may be regarded as the set of all multi-indices $\{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathbb{N}_0, 1 \leq i \leq n\}$. As usual, for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we denote by $|\alpha| := \alpha_1 + \dots + \alpha_n$ its length, and define $\alpha! := \alpha_1! \dots \alpha_n!$ (with the usual convention that $0! := 1$). Also, write $\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ and, given $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, by $\beta \leq \alpha$ it is understood that $\beta_j \leq \alpha_j$ for each $j \in \{1, \dots, n\}$. For an arbitrary set $E \subseteq \mathbb{R}^n$ we shall denote by E° , \overline{E} , $\text{diam } E$, and E^c , respectively the

interior, closure, diameter, distance to and complement of E in \mathbb{R}^n . In addition, $\text{dist}(F, E)$, denotes the distance from F to E . As usual, $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ for each $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, where $|\cdot|$ denotes the standard Euclidean norm. Next, given a measurable space $(\mathcal{X}, \mathfrak{M}, \mu)$, for any subset E of the ambient \mathcal{X} which belongs to the sigma-algebra \mathfrak{M} we denote by $\mu|_E$ the restriction of the measure μ to E . Throughout, $\#E$ and $\mathbf{1}_E$ stand, respectively, for the cardinality and the characteristic function of a given set E . Given an open subset \mathcal{O} of \mathbb{R}^n and $k \in \mathbb{N}_0 \cup \{\infty\}$, we shall denote by $\mathcal{C}^k(\mathcal{O})$ the collection of all k -times continuously differentiable functions in \mathcal{O} , and by $\mathcal{C}_c^\infty(\mathcal{O})$ the collection of all indefinitely differentiable functions which are compactly supported in \mathcal{O} . Finally, define $\mathcal{C}^\infty(\overline{\mathcal{O}}) := \{\varphi|_{\mathcal{O}} : \varphi \in \mathcal{C}^\infty(\mathbb{R}^n)\}$.

Assume next that $\Omega \subseteq \mathbb{R}^n$ is an arbitrary nonempty open set. Then for each integer $k \in \mathbb{N}_0$ and integrability exponent $p \in [1, \infty]$, the L^p -based Sobolev space of order k in Ω is defined intrinsically by

$$W^{k,p}(\Omega) := \{u \in L^1_{loc}(\Omega, \mathcal{L}^n) : \partial^\alpha u \in L^p(\Omega, \mathcal{L}^n) \text{ for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k\}, \quad (2.1)$$

where all derivatives are taken in the sense of distributions in the open set Ω . As is well-known (cf., e.g., [1, Theorem 3.3, p. 60]), $W^{k,p}(\Omega)$ becomes a Banach space when equipped with the natural norm

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega, \mathcal{L}^n)} \text{ for each } u \in W^{k,p}(\Omega). \quad (2.2)$$

As is well-known, $W^{k,p}(\Omega)$ is separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$. Since taking distributional derivatives commutes with the operation of restricting distributions to open subsets of their domain, and does not increase the Lebesgue norm, it is clear that the restriction operator

$$W^{k,p}(\Omega) \ni u \mapsto u|_{\mathcal{O}} \in W^{k,p}(\mathcal{O}) \quad (2.3)$$

is well-defined, linear and bounded, whenever Ω is an open subset of \mathbb{R}^n , \mathcal{O} is a nonempty subset of Ω , $k \in \mathbb{N}$, and $p \in [1, \infty]$.

For future purposes let us also define

$$\mathring{W}^{k,p}(\Omega) := \text{the closure of } \mathcal{C}_c^\infty(\Omega) \text{ in } (W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)}) \quad (2.4)$$

and, assuming that $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, set

$$W^{-k,p}(\Omega) := (\mathring{W}^{k,p'}(\Omega))^*. \quad (2.5)$$

As is well-known, for each $p \in (1, \infty)$, an alternative description of the the above Sobolev space of negative order is

$$W^{-k,p}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) : u = \sum_{|\alpha| \leq k} \partial^\alpha v_\alpha, \text{ where each } v_\alpha \in L^p(\Omega, \mathcal{L}^n) \right\}, \quad (2.6)$$

where $\mathcal{D}'(\Omega)$ is the space of distributions in Ω . For this, as well as other related matters, the interested reader is referred to, e.g., [1, pp. 62-65]. Here we only wish to note that if $k \in \mathbb{N}$, $p \in [1, \infty]$ and Ω is an arbitrary nonempty open subset of \mathbb{R}^n then

$$\mathring{W}^{k,p}(\Omega) \ni u \mapsto \tilde{u} \in W^{k,p}(\mathbb{R}^n) \text{ isometrically,} \quad (2.7)$$

where tilde denotes the extension by zero outside Ω , to the entire \mathbb{R}^n . Indeed, this rests on the observation that $\partial^\alpha \tilde{u} = \partial^\alpha u$ for every $u \in \mathring{W}^{k,p}(\Omega)$ and any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k - 1$ (which, in turn, is readily seen from (2.4) and a simple limiting argument).

Moving on to the topic of extensions of functions from Sobolev spaces, we recall that the idea underpinning the construction of Jones' extension operator from [25, Theorem 1, p. 73] is to glue together, via a scale-sensitive partition of unity associated with a Whitney decomposition of the interior of the complement of the domain, certain polynomials which best fit the given function on the corresponding

reflected cube across the boundary. While all this is made precise in Theorem 2.1 below, for the time being we briefly digress in order to clarify terminology and review some standard results.

According to Whitney's decomposition lemma (cf. [53, Theorem 1, p. 167]), one can associate to any open, nonempty, proper subset \mathcal{O} of \mathbb{R}^n a family $\mathcal{W}(\mathcal{O}) = \{Q_j\}_{j \in \mathbb{N}}$ of countably many closed dyadic cubes from \mathbb{R}^n such that

$$\mathcal{O} = \bigcup_{j \in \mathbb{N}} Q_j, \quad (2.8)$$

$$\sqrt{n}\ell(Q_j) \leq \text{dist}(Q_j, \partial\mathcal{O}) \leq 4\sqrt{n}\ell(Q_j), \quad \text{for all } j \in \mathbb{N}, \quad (2.9)$$

$$Q_j^\circ \cap Q_k^\circ = \emptyset, \quad \text{for all } j, k \in \mathbb{N} \text{ with } j \neq k, \quad (2.10)$$

$$\frac{1}{4}\ell(Q_j) \leq \ell(Q_k) \leq 4\ell(Q_j), \quad \text{for all } j, k \in \mathbb{N} \text{ with } Q_j \cap Q_k \neq \emptyset. \quad (2.11)$$

Above, $\ell(Q)$ denotes the side-length of the cube Q , and Q° stands for the interior of Q . Also, given a positive number λ and a cube Q , we denote by λQ the cube with the same center x_Q as Q , and side-length $\lambda\ell(Q)$. With this convention, it is then straightforward to check that (2.9) implies

$$\text{if } \lambda \in (0, 3) \text{ then } \lambda Q_j \subseteq \mathcal{O} \text{ for all } j \in \mathbb{N}. \quad (2.12)$$

In fact, for each $\lambda \in (0, 3)$ there exists $c_\lambda \in (0, 1)$ such that

$$c_\lambda \leq \frac{\text{dist}(\lambda Q_j, \partial\mathcal{O})}{\ell(Q_j)} \leq c_\lambda^{-1}, \quad \text{for all } j \in \mathbb{N}. \quad (2.13)$$

Finally, given an arbitrary nonempty open set $\Omega \subseteq \mathbb{R}^n$, define

$$\text{rad}(\Omega) := \inf_m \inf_{x \in \Omega_m} \sup_{y \in \Omega_m} |x - y|, \quad (2.14)$$

where $\{\Omega_m\}_m$ are the connected components of Ω .

Unraveling definitions yields the following geometric characterization in the class of connected open sets in \mathbb{R}^n :

$$\text{rad}(\Omega) = \inf \{r > 0 : \exists x \in \Omega \text{ such that } \Omega \subseteq B(x, r)\}, \quad (2.15)$$

for any connected open set $\Omega \subseteq \mathbb{R}^n$,

i.e., $\text{rad}(\Omega)$ is, loosely speaking, the smallest of all radii of balls centered in Ω which envelop this connected set. In particular, it is clear that $\text{rad}(\Omega) > 0$ for every finitely connected open set Ω in \mathbb{R}^n .

Here is the result advertised earlier. It is a detailed statement of [25, Theorem 1, p. 73] (which should also be compared with the statement of [9, Theorem B, p. 1029]).

Theorem 2.1 (P.W. Jones's Extension Theorem). *Let Ω be an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$, and fix $k \in \mathbb{N}$. Also, pick a Whitney decomposition $\mathcal{W}(\Omega)$ of Ω along with a Whitney decomposition $\mathcal{W}((\Omega^c)^\circ)$ of $(\Omega^c)^\circ$, and consider the collection of all small cube in the latter, i.e., define*

$$\mathcal{W}_s((\Omega^c)^\circ) := \{Q \in \mathcal{W}((\Omega^c)^\circ) : \ell(Q) \leq \varepsilon\delta/(16n)\}. \quad (2.16)$$

For any function $u \in L^1_{loc}(\Omega, \mathcal{L}^n)$ and any dyadic cube $Q \in \mathcal{W}(\Omega)$ let $P_Q(u)$ denote the unique polynomial of degree $k - 1$ which best fits u on Q in the sense that

$$\int_Q \partial^\alpha (u - P_Q(u)) d\mathcal{L}^n = 0 \text{ for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k - 1. \quad (2.17)$$

To each $Q \in \mathcal{W}_s((\Omega^c)^\circ)$ assign a cube $Q^ \in \mathcal{W}(\Omega)$ satisfying (cf. [25, Lemma 2.4, p. 77])*

$$\ell(Q) \leq \ell(Q^*) \leq 4\ell(Q) \text{ and } \text{dist}(Q, Q^*) \leq C_{n,\varepsilon}\ell(Q), \quad \forall Q \in \mathcal{W}_s((\Omega^c)^\circ), \quad (2.18)$$

Finally, to each $u \in L^1_{loc}(\Omega, \mathcal{L}^n)$ associate the function $\Lambda_k u$ defined \mathcal{L}^n -a.e. in \mathbb{R}^n by

$$\Lambda_k u := \begin{cases} u & \text{in } \Omega, \\ \sum_{Q \in \mathcal{W}_s((\Omega^c)^\circ)} P_{Q^*}(u) \varphi_Q & \text{in } (\Omega^c)^\circ, \end{cases} \quad (2.19)$$

where the family $\{\varphi_Q\}_{Q \in \mathcal{W}_s((\Omega^c)^\circ)}$ consists of functions satisfying

$$\varphi_Q \in \mathcal{C}_c^\infty(\mathbb{R}^n), \quad \text{supp } \varphi_Q \subseteq \frac{17}{16}Q, \quad 0 \leq \varphi_Q \leq 1, \quad |\partial^\alpha \varphi_Q| \leq C_\alpha \ell(Q)^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n, \quad (2.20)$$

for every $Q \in \mathcal{W}_s((\Omega^c)^\circ)$, as well as

$$\sum_{Q \in \mathcal{W}_s((\Omega^c)^\circ)} \varphi_Q \equiv 1 \quad \text{on} \quad \bigcup_{Q \in \mathcal{W}_s((\Omega^c)^\circ)} Q. \quad (2.21)$$

Then for every $p \in [1, \infty]$ the operator Λ_k satisfies

$$\Lambda_k : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly,} \quad (2.22)$$

$$\Lambda_k u|_\Omega = u, \quad \mathcal{L}^n\text{-a.e. on } \Omega \text{ for every } u \in W^{k,p}(\Omega), \quad (2.23)$$

and with operator norm which may be controlled solely in terms of $\varepsilon, \delta, n, p, k, \text{rad}(\Omega)$ in the following fashion:

$$\forall p \in [1, \infty], \quad \forall k \in \mathbb{N}, \quad \forall M \in (0, \infty), \quad \exists C(n, k, p, M) \in (0, \infty) \quad \text{such that} \\ \|\Lambda_k\|_{W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)} \leq C(n, k, p, M) \quad \text{whenever} \quad (2.24)$$

$$\Omega \text{ is an } (\varepsilon, \delta)\text{-domain in } \mathbb{R}^n \text{ such that } \max\{\varepsilon^{-1}, \delta^{-1}, \text{rad}(\Omega)^{-1}\} \leq M.$$

We continue by describing an alteration of Jones' extension operator, following the work of M. Christ in [8]. In preparation, we first record the statement of [8, Proposition 2.5, p. 67] (cf. also [5]).

Lemma 2.2. *Fix a number $\Upsilon \in \mathbb{N}$ and let $Q_o := (-\frac{1}{2}, \frac{1}{2})^n$ be the standard unit cube centered at the origin in \mathbb{R}^n . Then there exists a linear projection operator assigning*

$$L^1(Q_o, \mathcal{L}^n) \ni u \longmapsto \widehat{P}_o(u) \in \{P|_{Q_o} : P \text{ polynomial of degree } \leq \Upsilon \text{ in } \mathbb{R}^n\} \quad (2.25)$$

with the property that, given any $M \in \mathbb{N}$ such that $M \leq \Upsilon$, then for any $p \in [1, \infty]$ and any $r \in (0, 1]$,

$$\sum_{|\alpha| \leq M-1} \|\partial^\alpha(u - \widehat{P}_o(u))\|_{L^p(rQ_o, \mathcal{L}^n)} \leq C(\Upsilon, r) \sum_{|\beta|=M} \|\partial^\beta u\|_{L^p(rQ_o, \mathcal{L}^n)}, \quad (2.26)$$

for every $u \in W^{M,p}(Q_o)$, and such that

$$\|\partial^\alpha \widehat{P}_o(u)\|_{L^p(rQ_o, \mathcal{L}^n)} \leq C(\Upsilon, r) \sum_{|\beta|=|\alpha|} \|\partial^\beta u\|_{L^p(rQ_o, \mathcal{L}^n)}, \quad \forall \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq \Upsilon - 1, \quad (2.27)$$

for every $u \in W^{\Upsilon-1,p}(Q_o)$.

In general, given any cube Q in \mathbb{R}^n , define the operator \widehat{P}_Q by translating and dilating Q as to coincide with the unit cube Q_o , apply \widehat{P}_o from Lemma 2.2, and then reverse the dilation and translation.

Theorem 2.3 below is a restatement of [8, Theorem 1.1, p. 64] (complemented with [8, Remark, p. 66]) specialized to the case when the smoothness index is a positive integer, in which scenario the DeVore-Sharpely spaces (in terms of which Christ's theorem is originally stated) coincide with Sobolev

spaces (see [13, Theorem 6.2, p.37] in this regard). Compared to Λ_k from Theorem 2.1, the novel feature of the operator $\widehat{\Lambda}$ constructed here is the fact that the latter *simultaneously* extends functions from $W^{k,p}(\Omega)$ to \mathbb{R}^n , with preservation of class, for all k 's up to an a priori upper bound Υ . It is therefore natural to refer to such an extension operator as being **semi-universal**. There is a price to pay, however, namely the exclusion of the case when $p = 1$.

Theorem 2.3 (M. Christ's version of Jones' Extension Theorem). *Let Ω be an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$, and fix $\Upsilon \in \mathbb{N}$. In this context, retain the notation introduced in the statement of Theorem 2.1 and, to each $u \in L^1_{loc}(\Omega, \mathcal{L}^n)$ associate the function $\widehat{\Lambda}u$ defined \mathcal{L}^n -a.e. in \mathbb{R}^n by*

$$\widehat{\Lambda}u := \begin{cases} u & \text{in } \Omega, \\ \sum_{Q \in \mathcal{W}_s((\Omega^c)^\circ)} \widehat{P}_{Q^*}(u) \varphi_Q & \text{in } (\Omega^c)^\circ, \end{cases} \quad (2.28)$$

where \widehat{P}_{Q^*} is the polynomial projection associated with the reflected cube Q^* (cf. (2.18)) as in Lemma 2.2 and the subsequent comment.

Then for every $p \in (1, \infty]$ and every $k \in \mathbb{N}$ such that $k < \Upsilon$, the operator $\widehat{\Lambda}$ satisfies

$$\widehat{\Lambda} : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly,} \quad (2.29)$$

$$\widehat{\Lambda}u|_{\Omega} = u, \quad \mathcal{L}^n\text{-a.e. on } \Omega \text{ for every } u \in W^{k,p}(\Omega), \quad (2.30)$$

and with operator norm which may be controlled solely in terms of $\varepsilon, \delta, n, p, \text{rad}(\Omega)$ and Υ in a similar manner to (2.24).

Our first result in this paper brings to the forefront a salient feature of Jones' extension operator Λ_k from Theorem 2.1, namely the property that for any function $u \in W^{k,p}(\Omega)$ the support of its extension $\Lambda_k u \in W^{k,p}(\mathbb{R}^n)$ does not touch $\partial\Omega$ outside the region where the support of u itself makes contact with $\partial\Omega$. As we shall see momentarily, the same type of property is enjoyed by Christ's version of Jones' operator. In order to make this precise, we need to introduce some notation.

Generally speaking, given an open set $\mathcal{O} \subseteq \mathbb{R}^n$ and an \mathcal{L}^n -measurable function v on \mathcal{O} , define

$$\text{supp } v := \{x \in \overline{\mathcal{O}} : \text{there is no } r > 0 \text{ such that } v \equiv 0 \text{ } \mathcal{L}^n\text{-a.e. in } B(x, r) \cap \mathcal{O}\}. \quad (2.31)$$

Note that while the function v is known to be defined only in \mathcal{O} , the set $\text{supp } v$ (itself a closed subset of \mathbb{R}^n) is contained in $\overline{\mathcal{O}}$. It is also clear from the above definition that if $\mathcal{O} \subseteq \mathbb{R}^n$ is an open set and v is an \mathcal{L}^n -measurable function defined on \mathcal{O} , then

$$\text{supp}(v|_U) \subseteq \overline{U} \cap \text{supp } v, \quad \text{for any open subset } U \text{ of } \mathcal{O}. \quad (2.32)$$

Moreover, since every open cover of $\mathcal{O} \setminus \text{supp } v$ has a countable subcover (given that the open set in question is σ -compact), it follows that

$$v \text{ vanishes } \mathcal{L}^n\text{-a.e. on } \mathcal{O} \setminus \text{supp } v. \quad (2.33)$$

Here is the precise formulation of the result announced earlier.

Theorem 2.4 (Preservation of support in the closure of domain). *Let Ω be an (ε, δ) -domain in \mathbb{R}^n and fix some $k \in \mathbb{N}$. Then the Jones' extension operator Λ_k from Theorem 2.1 has the property that, given any $p \in [1, \infty]$, one has*

$$\overline{\Omega} \cap \text{supp}(\Lambda_k u) = \text{supp } u, \quad \forall u \in W^{k,p}(\Omega). \quad (2.34)$$

Moreover, there exists a finite number $R = R(n, \varepsilon, \delta) > 0$ with the property that

$$\text{supp}(\Lambda_k u) \subseteq \{x \in \mathbb{R}^n : \text{dist}(x, \text{supp } u) \leq R\} \quad \text{for every } u \in W^{k,p}(\Omega). \quad (2.35)$$

In particular, for every $u \in W^{k,p}(\Omega)$, one has

$$\text{supp } u \text{ compact} \implies \text{supp } (\Lambda_k u) \text{ compact.} \quad (2.36)$$

Finally, for any given $\Upsilon \in \mathbb{N}$, the operator $\widehat{\Lambda}$ from Theorem 2.3 satisfies similar properties, provided $k < \Upsilon$ and $1 < p \leq \infty$.

Proof. Fix $k \in \mathbb{N}$, $p \in [1, \infty]$, and select an arbitrary $u \in W^{k,p}(\Omega)$. Then combining (2.23) with (2.32) (used here with $v := \Lambda_k u$, $\mathcal{O} := \mathbb{R}^n$, and $U := \Omega$) yields

$$\text{supp } u = \text{supp } (\Lambda_k u|_{\Omega}) \subseteq \overline{\Omega} \cap \text{supp } \Lambda_k u, \quad (2.37)$$

proving the right-to-left inclusion in (2.34). Proceeding in the opposite direction, we first claim that

$$\text{supp } \Lambda_k u \subseteq \text{supp } u \cup \overline{F_u}, \quad (2.38)$$

where

$$F_u := \bigcup_{\substack{Q \in \mathcal{W}_s((\Omega^c)^\circ), \\ Q^* \cap \text{supp } u \neq \emptyset}} \frac{17}{16}Q. \quad (2.39)$$

To justify this claim, assume that $x_o \in \mathbb{R}^n \setminus (\text{supp } u \cup \overline{F_u})$. Then there exists $r > 0$ with the property that

$$u \text{ vanishes } \mathcal{L}^n\text{-a.e. in } B(x_o, r) \cap \Omega, \quad (2.40)$$

and such that $B(x_o, r) \cap F_u = \emptyset$. The latter condition further entails (in concert with (2.20)) that, on the one hand,

$$\sum_{\substack{Q \in \mathcal{W}_s((\Omega^c)^\circ), \\ Q^* \cap \text{supp } u \neq \emptyset}} P_{Q^*}(u) \varphi_Q \equiv 0 \quad \text{in } B(x_o, r) \cap (\Omega^c)^\circ. \quad (2.41)$$

On the other hand, we claim that we also have

$$\sum_{\substack{Q \in \mathcal{W}_s((\Omega^c)^\circ), \\ Q^* \cap \text{supp } u = \emptyset}} P_{Q^*}(u) \varphi_Q \equiv 0 \quad \text{in } B(x_o, r) \cap (\Omega^c)^\circ. \quad (2.42)$$

Indeed, since for every $Q \in \mathcal{W}_s((\Omega^c)^\circ)$ the best fit polynomial $P_{Q^*}(u)$ has degree $k-1$, condition (2.17) entails

$$P_{Q^*}(u)(x) = \sum_{|\alpha| \leq k-1} \frac{x^\alpha}{\alpha!} \int_{Q^*} \partial^\alpha u d\mathcal{L}^n, \quad \forall x \in \mathbb{R}^n. \quad (2.43)$$

Here and elsewhere, \int stands for integral average. In particular, this and (2.33) show that

$$P_{Q^*}(u) \equiv 0 \quad \text{for every } Q \in \mathcal{W}_s((\Omega^c)^\circ) \text{ with } Q^* \cap \text{supp } u = \emptyset, \quad (2.44)$$

and (2.42) readily follows from (2.44). Together, (2.41) and (2.42) imply that

$$\sum_{Q \in \mathcal{W}_s((\Omega^c)^\circ)} P_{Q^*}(u) \varphi_Q \equiv 0 \quad \text{in } B(x_o, r) \cap (\Omega^c)^\circ. \quad (2.45)$$

From (2.19), (2.40), (2.45), and (1.5), we may then deduce that

$$\Lambda_k u \text{ vanishes } \mathcal{L}^n\text{-a.e. in } B(x_o, r). \quad (2.46)$$

Hence, $x_o \notin \text{supp } \Lambda_k u$ which finishes the proof of (2.38).

Having established (2.38), we next claim that

$$\overline{\Omega} \cap \overline{F_u} \subseteq \text{supp } u. \quad (2.47)$$

To justify this, reason by contradiction and assume that there exists a point $x_o \in \overline{\Omega} \cap \overline{F_u}$ such that $x_o \notin \text{supp } u$. In particular, the latter condition entails the existence of some $r > 0$ for which

$$u \text{ vanishes } \mathcal{L}^n\text{-a.e. in } B(x_o, r) \cap \Omega. \quad (2.48)$$

Let us take a closer look at the fact that $x_o \in \overline{\Omega} \cap \overline{F_u}$. For starters, the fact that $F_u \subseteq (\Omega^c)^\circ$ (as seen from (2.12) and (2.39)) forces

$$\overline{\Omega} \cap \overline{F_u} \subseteq \overline{\Omega} \cap (\overline{\Omega^c})^\circ = \overline{\Omega} \setminus (\overline{\Omega})^\circ \subseteq \overline{\Omega} \setminus \Omega = \partial\Omega, \quad (2.49)$$

whereupon

$$x_o \in \partial\Omega. \quad (2.50)$$

Next, the membership of x_o to the closure of the set F_u defined in (2.39) entails the existence of a sequence of dyadic cubes $\{Q_j\}_j \subseteq \mathcal{W}_s((\Omega^c)^\circ)$ and a sequence $\{x_j\}_j$ of points in \mathbb{R}^n satisfying

$$x_j \in \frac{17}{16}Q_j \quad \text{for every } j, \quad (2.51)$$

$$Q_j^* \cap \text{supp } u \neq \emptyset \quad \text{for every } j, \quad (2.52)$$

$$\lim_j x_j = x_o. \quad (2.53)$$

Now, (2.51) forces

$$|x_j - x_{Q_j}| \leq \sqrt{n} \frac{17}{16} \ell(Q_j) \quad \text{for every } j, \quad (2.54)$$

while from (2.13) we conclude that there exists $c \in (0, 1)$ such that

$$c \ell(Q_j) \leq \text{dist}(\frac{17}{16}Q_j, \partial\Omega) \leq \text{dist}(x_j, \partial\Omega) \leq |x_j - x_o|, \quad \text{for all } j, \quad (2.55)$$

where the last inequality uses (2.50). From (2.18) we also deduce that

$$\ell(Q_j) \leq \ell(Q_j^*) \leq 4\ell(Q_j) \quad \text{and} \quad |x_{Q_j} - x_{Q_j^*}| \leq C_{n,\varepsilon} \ell(Q_j), \quad \text{for all } j. \quad (2.56)$$

Combining now (2.53)-(2.56) yields

$$\lim_j x_{Q_j^*} = x_o \quad \text{and} \quad \lim_j \ell(Q_j^*) = 0. \quad (2.57)$$

In turn, from (2.57) we deduce that

$$\text{there exists } j \text{ such that } Q_j^* \subseteq B(x_o, r) \cap \Omega \quad (2.58)$$

which, in light of (2.48), implies that

$$\text{there exists } j \text{ such that } u \text{ vanishes } \mathcal{L}^n\text{-a.e. in } Q_j^*. \quad (2.59)$$

This, however, contradicts (2.52). The proof of (2.47) is therefore complete.

With (2.47) in hand, and availing ourselves of (2.38) we may write

$$\overline{\Omega} \cap \text{supp } \Lambda_k u \subseteq (\overline{\Omega} \cap \text{supp } u) \cup (\overline{\Omega} \cap \overline{F_u}) \subseteq \text{supp } u, \quad (2.60)$$

which proves the left-to-right inclusion in (2.34).

As far as (2.36) is concerned, assume some $u \in W^{k,p}(\Omega)$ has been given. Consider an arbitrary point $x_1 \in F_u$. Then there exists some cube $Q \in \mathcal{W}_s((\Omega^c)^\circ)$ such that $x_1 \in \frac{17}{16}Q$ and $Q^* \cap \text{supp } u \neq \emptyset$. Pick some $x_2 \in Q^* \cap \text{supp } u$ and note that, thanks to (2.16)-(2.18), we may estimate

$$|x_1 - x_Q| \leq 2\sqrt{n} \frac{17}{16} \ell(Q) \leq C(n, \varepsilon, \delta), \quad (2.61)$$

$$|x_Q - x_{Q^*}| \leq \text{dist}(Q, Q^*) \leq C_{n,\varepsilon} \ell(Q) \leq C(n, \varepsilon, \delta), \quad (2.62)$$

$$|x_{Q^*} - x_2| \leq 2\sqrt{n} \ell(Q^*) \leq 8\sqrt{n} \ell(Q) \leq C(n, \varepsilon, \delta), \quad (2.63)$$

for some finite constant $C(n, \varepsilon, \delta) > 0$. Collectively, (2.61)-(2.63) imply that

$$\text{dist}(x_1, \text{supp } u) \leq |x_1 - x_2| \leq |x_1 - x_Q| + |x_Q - x_{Q^*}| + |x_{Q^*} - x_2| \leq R := 3C(n, \varepsilon, \delta). \quad (2.64)$$

Since x_1 has been arbitrarily chosen in F_u , this proves that

$$F_u \subseteq \{x \in \mathbb{R}^n : \text{dist}(x, \text{supp } u) \leq R\}. \quad (2.65)$$

At this point, (2.35) readily follows from (2.65) and (2.38). Finally, if $\text{supp } u$ is compact, this shows that $\text{supp } \Lambda_k u$ is a bounded set, hence also compact, as desired.

Finally, to see that similar properties may be established for the operator $\widehat{\Lambda}$ from Theorem 2.3 (assuming that $k < \Upsilon$ and $1 < p \leq \infty$), it suffices to observe that (2.44) holds for $\widehat{P}_{Q^*}(u)$ in place of $P_{Q^*}(u)$; the rest of the proof is virtually identical to the one carried out for Λ_k . \square

3 Extension of Sobolev functions with partially vanishing traces defined in locally (ε, δ) -domains

The class of domains alluded to in the title of this section is going to be introduced a little later, in Definition 3.4. To set the stage, we shall make the following definition which plays an important role throughout the proceedings.

Definition 3.1. *Given a nonempty open set Ω in \mathbb{R}^n and a closed subset D of $\overline{\Omega}$, consider*

$$\mathcal{C}_D^\infty(\Omega) := \{\widetilde{\psi}|_\Omega : \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus D)\} = \{\varphi|_\Omega : \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ with } D \cap \text{supp } \varphi = \emptyset\} \quad (3.1)$$

where tilde denotes the extension by zero outside the support to \mathbb{R}^n , and for each $k \in \mathbb{N}$, $p \in [1, \infty]$, define

$$W_D^{k,p}(\Omega) := \text{the closure of } \mathcal{C}_D^\infty(\Omega) \text{ in } \left(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)}\right). \quad (3.2)$$

In particular, given a closed subset D of \mathbb{R}^n , for each $k \in \mathbb{N}$, $p \in [1, \infty]$, corresponding to the case when $\Omega = \mathbb{R}^n$ in Definition 3.1 we have

$$W_D^{k,p}(\mathbb{R}^n) := \text{the closure of } \{\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n) : D \cap \text{supp } \varphi = \emptyset\} \text{ in } \left(W^{k,p}(\mathbb{R}^n), \|\cdot\|_{W^{k,p}(\mathbb{R}^n)}\right). \quad (3.3)$$

In particular, corresponding to the case when $D = \emptyset$ we have $W_\emptyset^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

A number of other useful elementary properties of the spaces considered above are collected in the next lemma (whose routine proof is omitted).

Lemma 3.2. *Let Ω be a nonempty open subset of \mathbb{R}^n and consider a closed subset D of $\overline{\Omega}$. Also, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then the following hold:*

- (1) $\mathcal{C}_D^\infty(\Omega)$ is a dense linear subspace of $W_D^{k,p}(\Omega)$;
- (2) $W_D^{k,p}(\Omega)$ is a closed linear subspace of $W^{k,p}(\Omega)$;

- (3) $(W_D^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ is a Banach space, which is separable if $1 \leq p < \infty$, and reflexive if $1 < p < \infty$;
- (4) the inclusion $\{u|_\Omega : u \in W_D^{k,p}(\mathbb{R}^n)\} \hookrightarrow W_D^{k,p}(\Omega)$ is well-defined and continuous, and the restriction operator $W_D^{k,p}(\mathbb{R}^n) \ni u \mapsto u|_\Omega \in W_D^{k,p}(\Omega)$ is well-defined, linear and bounded;
- (5) corresponding to the case when $D = \partial\Omega$, there holds $W_{\partial\Omega}^{k,p}(\Omega) = \dot{W}^{k,p}(\Omega)$;
- (6) if $\partial\Omega \subseteq D$ then the inclusion $W_D^{k,p}(\Omega) \hookrightarrow \dot{W}^{k,p}(\Omega)$ is well-defined and isometric;
- (7) $\mathcal{C}_{\partial\Omega}^\infty(\Omega) = \mathcal{C}_c^\infty(\Omega)$, and if $\partial\Omega \subseteq D$ then $\mathcal{C}_D^\infty(\Omega) \subseteq \mathcal{C}_c^\infty(\Omega)$;
- (8) if Σ is a closed subset of Ω which contains the given set D , then $\mathcal{C}_\Sigma^\infty(\Omega) \subseteq \mathcal{C}_D^\infty(\Omega)$, and the inclusion $W_\Sigma^{k,p}(\Omega) \hookrightarrow W_D^{k,p}(\Omega)$ is well-defined, linear and isometric;
- (9) if $v \in W_D^{k,p}(\Omega)$ then $v|_\mathcal{O} \in W_{\mathcal{O} \cap D}^{k,p}(\mathcal{O})$ for any open subset \mathcal{O} of Ω .

The special case of the scale (3.2) with $D = \emptyset$ is considered below, in the context of (ε, δ) -domains and for $p < \infty$.

Lemma 3.3. *Let Ω be an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$, and fix some $k \in \mathbb{N}$ along with some $p \in [1, \infty)$. Then, corresponding to the case when $D = \emptyset$, one has*

$$W_\emptyset^{k,p}(\Omega) = W^{k,p}(\Omega). \quad (3.4)$$

Proof. The approximation result proved in [25, § 4, pp. 83-85] shows that any given $u \in W^{k,p}(\Omega)$ may be approximated arbitrarily well by functions of the form $\psi|_\Omega$ where $\psi \in \mathcal{C}^\infty(\mathbb{R}^n) \cap W^{k,\infty}(\mathbb{R}^n)$ with the property that $\psi|_\Omega \in W^{k,p}(\Omega)$. Fix now such a function ψ and consider $\eta \in \mathcal{C}_c^\infty(B(0, 2))$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $B(0, 1)$. For each $j \in \mathbb{N}$ then set $\eta_j := \eta(\cdot/j)$ and consider $\varphi_j := \psi\eta_j$. Then clearly $\varphi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and it is straightforward to check that $\varphi_j|_\Omega \rightarrow \psi|_\Omega$ in $W^{k,p}(\Omega)$ as $j \rightarrow \infty$. This proves the right-to-left inclusion in (3.4). Since the left-to-right inclusion is contained in part (2) of Lemma 3.2, formula (3.4) follows. \square

We are now in a ready to introduce a class of domains for which, roughly speaking, the (ε, δ) -property only holds near a (possibly proper) subset of the boundary. The following definition is central to the work carried out in this paper.

Definition 3.4. *Let $\varepsilon, \delta > 0$ be given. In addition, suppose that Ω is an open, nonempty, proper subset of \mathbb{R}^n , and that N is an arbitrary subset of $\partial\Omega$. Then Ω is said to be **locally an (ε, δ) -domain near N** provided there exist a number $\varkappa > 0$ and an at most countable family $\{O_j\}_{j \in J}$ of open subsets of \mathbb{R}^n satisfying*

$$\{O_j\}_{j \in J} \text{ is locally finite and has bounded overlap,} \quad (3.5)$$

$$\forall j \in J \exists \Omega_j \text{ } (\varepsilon, \delta)\text{-domain in } \mathbb{R}^n \text{ with } \text{rad}(\Omega_j) > \varkappa \text{ and } O_j \cap \Omega = O_j \cap \Omega_j, \quad (3.6)$$

$$\exists r \in (0, \infty] \text{ such that } \forall x \in N \exists j \in J \text{ for which } B(x, r) \subseteq O_j. \quad (3.7)$$

Occasionally, when the nature of the set N is not important, or it is clear from the context, we shall slightly abuse language and refer to a domain Ω as in Definition 3.4 simply as being **locally an (ε, δ) -domain**.

It is obvious from Definition 3.4 that the following hereditary property holds: if Ω is an open, nonempty, proper subset of \mathbb{R}^n which is locally an (ε, δ) -domain near a subset N of $\partial\Omega$, then Ω continues to be locally an (ε, δ) -domain near any subset N_o of N . Other features of the class of domains considered in Definition 3.4 are reviewed below.

Remark 3.5. *Suppose that $\varepsilon, \delta > 0$ are given. Then as a consequence of Lebesgue's Number Lemma, one may readily verify that an open, nonempty, proper subset Ω of \mathbb{R}^n is locally an (ε, δ) -domain near a bounded subset N of $\partial\Omega$ if and only if there exists a finite open cover $\{O_j\}_{j \in J}$ of \overline{N} with the property that*

$$\forall j \in J \exists \Omega_j \text{ } (\varepsilon, \delta)\text{-domain in } \mathbb{R}^n \text{ with } \text{rad}(\Omega_j) > 0 \text{ and } O_j \cap \Omega = O_j \cap \Omega_j. \quad (3.8)$$

In particular, this more streamlined characterization is valid for the class of locally (ε, δ) -domains with compact boundaries.

Clearly, when N is a proper subset of the topological boundary allows a locally (ε, δ) -domain near N to be quite different than an ordinary (ε, δ) -domain in the sense of Definition 1.1 since no condition is imposed on the portion of the boundary outside N . In particular, the class of domains described in Definition 1.1 corresponding to the case in which $N = \emptyset$ is the collection of all open, nonempty, proper subsets of \mathbb{R}^n (since in this situation conditions (3.5)-(3.7) are satisfied with $J = \emptyset$). However, in the case in which N coincides with the topological boundary of the underlying domain, the classes of domains introduced in Definition 3.4 and Definition 1.1 relate to one another in the manner made precise in the lemma below.

Lemma 3.6. *Assume that Ω is a nonempty proper open set in \mathbb{R}^n and let $\varepsilon, \delta > 0$. Then the following implications hold:*

$$\Omega \text{ is an } (\varepsilon, \delta)\text{-domain with } \text{rad}(\Omega) > 0 \implies \Omega \text{ is a locally } (\varepsilon, \delta)\text{-domain near } \partial\Omega, \quad (3.9)$$

$$\Omega \text{ is a locally } (\varepsilon, \delta)\text{-domain near } \partial\Omega \implies \exists \delta' > 0 \text{ such that } \Omega \text{ is an } (\varepsilon, \delta')\text{-domain.} \quad (3.10)$$

Proof. Obviously, any (ε, δ) -domain Ω in \mathbb{R}^n is a locally (ε, δ) -domain near $\partial\Omega$ (since (3.5)-(3.7) are verified if we take $J := \{1\}$, $O_1 := \mathbb{R}^n$, $\Omega_1 := \Omega$ and $r > 0$ arbitrary). Conversely, we claim that if $\Omega \subseteq \mathbb{R}^n$ is a locally (ε, δ) -domain near $\partial\Omega$ then there exists a small number $c = c(\varepsilon, r) > 0$, with $r > 0$ as in (3.7), such that Ω is actually a $(\varepsilon, c\delta)$ -domain in \mathbb{R}^n . To see that this is the case, fix two points $x, y \in \Omega$ such that $|x - y| < c\delta$, with $c \in (0, 1)$ small to be determined later. Then the existence of a rectifiable curve $\gamma : [0, 1] \rightarrow \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$, and (1.4) holds is readily verified when $B(x, 100c\delta) \cap \partial\Omega = \emptyset$ (by simply taking γ to be the line segment joining x and y), whereas in the case when $B(x, 100c\delta) \cap \partial\Omega$ contains a point x_o one may reason as follows. First, from (3.7) (with $N = \partial\Omega$) there exists some $j \in J$ such that $B(x_o, r) \subseteq O_j$. Second, since Ω_j is an (ε, δ) -domain, Definition 1.1 guarantees the existence of a rectifiable curve $\gamma : [0, 1] \rightarrow \Omega_j$ with $\gamma(0) = x$, $\gamma(1) = y$, and such that

$$\text{length}(\gamma) \leq \frac{1}{\varepsilon}|x - y| \quad \text{and} \quad \frac{\varepsilon|z - x||z - y|}{|x - y|} \leq \text{dist}(z, \partial\Omega_j), \quad \forall z \in \gamma([0, 1]). \quad (3.11)$$

In particular, $\text{length}(\gamma) \leq c\delta/\varepsilon$. Also, starting from $O_j \cap \Omega = O_j \cap \Omega_j$, elementary topology gives that

$$O_j \cap \partial\Omega = O_j \cap \partial\Omega_j. \quad (3.12)$$

Combining the above facts it is now not difficult to show that by choosing $c = c(\varepsilon, r) \in (0, 1)$ small enough then $\gamma([0, 1]) \subseteq B(x_o, r) \cap \Omega$ and for every $z \in \gamma([0, 1])$

$$\text{dist}(z, \partial\Omega) = \text{dist}(z, B(x_o, r) \cap \partial\Omega) = \text{dist}(z, B(x_o, r) \cap \partial\Omega_j) = \text{dist}(z, \partial\Omega_j). \quad (3.13)$$

With this in hand, the desired conclusion follows. \square

In view of Lemma 3.6 and the comments preceding it, the class of domains introduced in Definition 1.1 bridges between arbitrary open, nonempty, proper subsets of \mathbb{R}^n , on the one hand, and Jones' class of (ε, δ) -domains (with the set N playing the role of a fine-tuning parameter). The most significant feature of the category of locally (ε, δ) -domains is the fact that they are extension domains relative to the scale of Sobolev spaces introduced in Definition 3.1, in a sense made precise in our next theorem. The main ingredients in its proof are Theorem 2.1 (including the nature of the quantitative bound from (2.24)), Theorem 2.4, and a geometrically compatible quantitative partition of unity.

Theorem 3.7 (Extension Theorem for locally (ε, δ) -domains). *Suppose that $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ are such that D is closed and Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. Then for any $k \in \mathbb{N}$ there exists a linear operator $\mathfrak{E}_{k,D}$, mapping locally integrable functions in Ω into Lebesgue measurable functions in \mathbb{R}^n , such that for each $p \in [1, \infty]$ and each closed subset Σ of $\overline{\Omega}$ satisfying*

$$D \cap \partial\Omega \subseteq \Sigma \cap \partial\Omega, \quad (3.14)$$

one has

$$\mathfrak{E}_{k,D} : W_{\Sigma}^{k,p}(\Omega) \longrightarrow W_{\Sigma}^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly,} \quad (3.15)$$

(with operator norm controlled in terms of $\varepsilon, \delta, n, k, p$, and the quantitative aspects of (3.5)-(3.7)), and

$$(\mathfrak{E}_{k,D} u)|_{\Omega} = u, \quad \mathcal{L}^n\text{-a.e. on } \Omega \text{ for every } u \in W_{\Sigma}^{k,p}(\Omega). \quad (3.16)$$

In particular, corresponding to the case when $\Sigma := D$, one has

$$\mathfrak{E}_{k,D} : W_D^{k,p}(\Omega) \longrightarrow W_D^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly, and} \quad (3.17)$$

$$(\mathfrak{E}_{k,D} u)|_{\Omega} = u, \quad \mathcal{L}^n\text{-a.e. on } \Omega \text{ for every } u \in W_D^{k,p}(\Omega). \quad (3.18)$$

Proof. Introduce $N := \partial\Omega \setminus D$. We shall consider first the case when $N \neq \emptyset$, then indicate the alterations needed if $N = \emptyset$ in the last part of the proof. As a preliminary matter, we shall construct a quantitative partition of unity which is then used to glue together Jones' extension operators acting from genuine (ε, δ) -domains which agree with Ω near points in N . Turning to specifics, since Ω is assumed to be locally an (ε, δ) -domain near N , Definition 3.4 ensures that there exist a number $\varkappa > 0$ and a collection $\{O_j\}_{j \in J}$, where J is an at most countable set, of open subsets of \mathbb{R}^n satisfying

$$\exists L \in \mathbb{N} \text{ such that } \sum_{j \in J} \mathbf{1}_{O_j} \leq L \text{ in } \mathbb{R}^n, \quad (3.19)$$

$$\forall K \text{ compact in } \mathbb{R}^n, \#\{j \in J : O_j \cap K \neq \emptyset\} < +\infty, \quad (3.20)$$

$$\forall j \in J \exists \Omega_j \text{ } (\varepsilon, \delta)\text{-domain in } \mathbb{R}^n \text{ with } \text{rad}(\Omega_j) > \varkappa \text{ and } O_j \cap \Omega = O_j \cap \Omega_j, \quad (3.21)$$

$$\exists r \in (0, \infty) \text{ such that } \forall x \in N \exists j \in J \text{ for which } B(x, r) \subseteq O_j. \quad (3.22)$$

Now, generally speaking, for each nonempty $E \subseteq \mathbb{R}^n$ and each $\rho \in (0, \infty)$ define its ρ -contraction by

$$[E]_{\rho} := \{x \in \mathbb{R}^n : B(x, \rho) \subseteq E\} = \{x \in \mathbb{R}^n : \text{dist}(x, E^c) \geq \rho\}. \quad (3.23)$$

This and (3.22) imply that

$$N \subseteq \bigcup_{j \in J} [O_j]_r. \quad (3.24)$$

Choose a function $\theta \in \mathcal{C}_c^{\infty}(\mathbb{R}^n)$ such that $0 \leq \theta \leq 1$, $\text{supp } \theta \subseteq B(0, 1)$, and $\int_{\mathbb{R}^n} \theta d\mathcal{L}^n = 1$. As usual, for each $t > 0$, define $\theta_t(x) := t^{-n}\theta(x/t)$ for every $x \in \mathbb{R}^n$. Next, for each $j \in J$, consider (with $r > 0$ as in (3.22))

$$\psi_j := \theta_{r/4} * \mathbf{1}_{[O_j]_{r/4}} \text{ in } \mathbb{R}^n. \quad (3.25)$$

Then for each $j \in J$,

$$\begin{aligned} \psi_j &\in \mathcal{C}^{\infty}(\mathbb{R}^n), \quad 0 \leq \psi_j \leq 1, \quad \text{supp } \psi_j \subseteq O_j, \quad \psi_j \equiv 1 \text{ in } [O_j]_{r/2}, \\ &\text{and } |\partial^{\alpha} \psi_j| \leq 4^{|\alpha|} r^{-|\alpha|} \|\partial^{\alpha} \theta\|_{L^1(\mathbb{R}^n, \mathcal{L}^n)} \quad \text{for each } \alpha \in \mathbb{N}_0^n. \end{aligned} \quad (3.26)$$

Let us also note that $\sum_{j \in J} \psi_j^2$ is a well-defined function belonging to $\mathcal{C}^\infty(\mathbb{R}^n)$, thanks to (3.26) and (3.21), and that

$$\sum_{j \in J} \psi_j^2 \geq 1 \quad \text{on} \quad \{x \in \mathbb{R}^n : \text{dist}(x, N) < r/2\}. \quad (3.27)$$

Indeed, if $x \in \mathbb{R}^n$ is such that $\text{dist}(x, N) < r/2$, then there exists $x_0 \in N$ such that $x \in B(x_0, r/2)$. Thanks to (3.22) we know that there exists $j_0 \in J$ such that $B(x_0, r) \subseteq O_{j_0}$. Combining these facts it follows that $x \in [O_{j_0}]_{r/2}$, whereupon $\psi_{j_0}(x) = 1$ by (3.26). Consequently, $\sum_{j \in J} \psi_j(x)^2 \geq 1$, as wanted.

Next, introduce

$$U := \{x \in \mathbb{R}^n : \text{dist}(x, N) < r/4\}. \quad (3.28)$$

If we now define

$$\eta := \theta_{r/8} * \mathbf{1}_U \quad \text{in} \quad \mathbb{R}^n, \quad (3.29)$$

then

$$\begin{aligned} \eta &\in \mathcal{C}^\infty(\mathbb{R}^n), \quad 0 \leq \eta \leq 1, \\ \text{supp } \eta &\subseteq \{x \in \mathbb{R}^n : \text{dist}(x, N) < r/2\}, \\ \eta &\equiv 1 \quad \text{in} \quad \{x \in \mathbb{R}^n : \text{dist}(x, N) < r/8\}, \\ \text{and } |\partial^\alpha \eta| &\leq 8^{|\alpha|} r^{-|\alpha|} \|\partial^\alpha \theta\|_{L^1(\mathbb{R}^n, \mathcal{L}^n)} \quad \text{for each } \alpha \in \mathbb{N}_0^n. \end{aligned} \quad (3.30)$$

Given that by (3.27) and (3.30) we have $\sum_{j \in J} \psi_j^2 \geq 1$ in a neighborhood of $\text{supp } \eta$, it follows that if for each $j \in J$ we now set

$$\varphi_j := \frac{\eta \psi_j}{\sum_{i \in J} \psi_i^2}, \quad (3.31)$$

then for every $j \in J$

$$\begin{aligned} \varphi_j &\in \mathcal{C}^\infty(\mathbb{R}^n), \quad \text{supp } \varphi_j \subseteq O_j, \quad 0 \leq \varphi_j \leq 1, \\ \text{and } |\partial^\alpha \varphi_j| &\leq C_{\theta, \alpha} r^{-|\alpha|}, \quad \text{for each } \alpha \in \mathbb{N}_0^n, \end{aligned} \quad (3.32)$$

where $C_{\theta, \alpha} > 0$ is a finite constant independent of j . Moreover,

$$\sum_{j \in J} \psi_j \varphi_j = \eta \quad \text{in} \quad \mathbb{R}^n. \quad (3.33)$$

Fix now a closed set Σ satisfying $D \subseteq \Sigma \subseteq \overline{\Omega}$, along with a number $k \in \mathbb{N}$ and, for each $j \in J$, denote by $\Lambda_{k, j}$ Jones' extension operator for the (ε, δ) -domain Ω_j . In this context, we now define the operator

$$\mathfrak{E}_{k, D} u := (\widetilde{1 - \eta})u + \sum_{j \in J} \varphi_j \Lambda_{k, j}(E_j(\psi_j u)), \quad \text{for every } u \in \mathcal{C}_\Sigma^\infty(\Omega), \quad (3.34)$$

where tilde is the operation of extending functions from $\mathcal{C}_c^\infty(\Omega)$ by zero outside of their support to the entire \mathbb{R}^n . Also, for each $j \in J$ we have denoted by E_j the operator mapping each function v from $\mathcal{C}^\infty(\overline{\Omega})$ with the property that there exists a compact subset K of O_j such that $v \equiv 0$ on $\Omega \setminus K$ into

$$E_j v := \begin{cases} v & \text{in } \Omega \cap O_j = \Omega_j \cap O_j, \\ 0 & \text{in } \Omega_j \setminus O_j. \end{cases} \quad (3.35)$$

In particular, it is clear that for any v as above,

$$\begin{aligned} E_j v &\in \mathcal{C}^\infty(\overline{\Omega_j}), \quad \text{supp}(E_j v) \subseteq \text{supp } v, \quad \text{and} \\ \partial^\alpha(E_j v) &= E_j(\partial^\alpha v) \quad \text{in } \Omega_j \quad \text{for each } \alpha \in \mathbb{N}_0^n. \end{aligned} \tag{3.36}$$

With these conventions we first claim that, given any $u \in \mathcal{C}_\Sigma^\infty(\Omega)$, the right-hand side of (3.34) is well-defined. To justify this claim note that if $u = \varphi|_\Omega$ for some $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with the property that $\Sigma \cap \text{supp } \varphi = \emptyset$, then

$$\mathfrak{K} := \text{supp } \varphi \cap \{x \in \mathbb{R}^n : \text{dist}(x, N) \geq r/8\} \tag{3.37}$$

is a compact subset of \mathbb{R}^n which satisfies $\mathfrak{K} \cap N = \emptyset$ and $\mathfrak{K} \cap \partial\Omega \cap D \subseteq \mathfrak{K} \cap \partial\Omega \cap \Sigma = \emptyset$, by (3.14). Hence,

$$\mathfrak{K} \cap \partial\Omega = (\mathfrak{K} \cap N) \cup (\mathfrak{K} \cap \partial\Omega \cap D) = \emptyset. \tag{3.38}$$

As such, $\mathfrak{K} \cap \Omega$ is a compact subset of Ω outside of which the function $(1 - \eta)u$ vanishes identically. This shows that $(1 - \eta)u \in \mathcal{C}_c^\infty(\Omega)$, hence the first term in the right-hand side of (3.34) is meaningful. The sum in the right-hand side of (3.34) is also meaningful, thanks to the support condition in (3.32) and (3.20). In fact, (3.32)-(3.20) may also be used to justify that, given an arbitrary function $u \in \mathcal{C}_\Sigma^\infty(\Omega)$, for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$, we have

$$\partial^\alpha \mathfrak{E}_{k,D} u = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} (-1)^{|\beta|} \widetilde{\partial^\beta \eta} \partial^\gamma u + \sum_{j \in J} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} \partial^\beta \varphi_j \partial^\gamma \Lambda_{k,j}(E_j(\psi_j u)). \tag{3.39}$$

Consequently, given any $p \in [1, \infty)$, for every $u \in \mathcal{C}_\Sigma^\infty(\Omega)$ we may estimate

$$\begin{aligned} |\partial^\alpha \mathfrak{E}_{k,D} u|^p &\leq C_k \left(\sum_{\beta+\gamma=\alpha} |\widetilde{\partial^\beta \eta}| |\partial^\gamma u| + \sum_{j \in J} \sum_{\beta+\gamma=\alpha} |\partial^\beta \varphi_j| |\partial^\gamma \Lambda_{k,j}(E_j(\psi_j u))| \right)^p \\ &\leq C_{k,\theta,r} \left(\sum_{\gamma \leq \alpha} |\widetilde{\partial^\gamma u}| + \sum_{j \in J} \sum_{\gamma \leq \alpha} \mathbf{1}_{O_j} |\partial^\gamma \Lambda_{k,j}(E_j(\psi_j u))| \right)^p \\ &\leq C_{k,\theta,r,L,p} \left(\sum_{\gamma \leq \alpha} |\widetilde{\partial^\gamma u}|^p + \sum_{j \in J} \sum_{\gamma \leq \alpha} |\partial^\gamma \Lambda_{k,j}(E_j(\psi_j u))|^p \right), \end{aligned} \tag{3.40}$$

where the last inequality uses the support condition in (3.32) and (3.19). Thus, further,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\partial^\alpha \mathfrak{E}_{k,D} u|^p d\mathcal{L}^n &\leq C_{k,\theta,r,L,p} \left(\int_{\mathbb{R}^n} \sum_{\gamma \leq \alpha} |\widetilde{\partial^\gamma u}|^p d\mathcal{L}^n + \sum_{j \in J} \sum_{\gamma \leq \alpha} \int_{\mathbb{R}^n} |\partial^\gamma \Lambda_{k,j}(E_j(\psi_j u))|^p d\mathcal{L}^n \right) \\
&\leq C_{k,\theta,r,L,p} \left(\int_{\Omega} \sum_{\gamma \leq \alpha} |\partial^\gamma u|^p d\mathcal{L}^n + \sum_{j \in J} \|\Lambda_{k,j}(E_j(\psi_j u))\|_{W^{k,p}(\mathbb{R}^n)}^p \right) \\
&\leq C_{k,\theta,r,L,p,\varepsilon,\delta,n,\varkappa} \left(\|u\|_{W^{k,p}(\Omega)}^p + \sum_{j \in J} \|E_j(\psi_j u)\|_{W^{k,p}(\Omega_j)}^p \right) \\
&= C_{k,\theta,r,L,p,\varepsilon,\delta,n,\varkappa} \left(\|u\|_{W^{k,p}(\Omega)}^p + \sum_{j \in J} \|\psi_j u\|_{W^{k,p}(\Omega)}^p \right) \\
&\leq C_{k,\theta,r,L,p,\varepsilon,\delta,n,\varkappa} \left(\|u\|_{W^{k,p}(\Omega)}^p + \sum_{j \in J} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \int_{\Omega} |\partial^\beta \psi_j|^p |\partial^\gamma u|^p d\mathcal{L}^n \right) \\
&\leq C_{k,\theta,r,L,p,\varepsilon,\delta,n,\varkappa} \left(\|u\|_{W^{k,p}(\Omega)}^p + \sum_{j \in J} \sum_{\gamma \leq \alpha} \int_{\Omega} \mathbf{1}_{O_j} |\partial^\gamma u|^p d\mathcal{L}^n \right) \\
&= C_{k,\theta,r,L,p,\varepsilon,\delta,n,\varkappa} \left(\|u\|_{W^{k,p}(\Omega)}^p + \sum_{\gamma \leq \alpha} \int_{\Omega} \left(\sum_{j \in J} \mathbf{1}_{O_j} \right) |\partial^\gamma u|^p d\mathcal{L}^n \right) \\
&\leq C_{k,\theta,r,L,p,\varepsilon,\delta,n,\varkappa} \left(\|u\|_{W^{k,p}(\Omega)}^p + L \sum_{\gamma \leq \alpha} \int_{\Omega} |\partial^\gamma u|^p d\mathcal{L}^n \right) \\
&\leq C_{k,\theta,r,L,p,\varepsilon,\delta,n,\varkappa} \|u\|_{W^{k,p}(\Omega)}^p. \tag{3.41}
\end{aligned}$$

Above, the first inequality is implied by (3.40), the second inequality is obvious, the third inequality is based on Theorem 2.1 and the fact that each Ω_j is an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega_j) \geq \varkappa$, the subsequent equality is readily seen from (3.35)-(3.36), the fourth inequality uses Leibniz's formula, the fifth inequality follows from (3.26), the next equality is trivial, the sixth inequality is clear from (3.19), while the last inequality is obvious. In turn, (3.41) goes to show that

$$\|\mathfrak{E}_{k,D} u\|_{W^{k,p}(\mathbb{R}^n)} \leq C_{k,\theta,r,L,p,\varepsilon,\delta,n,\varkappa} \|u\|_{W^{k,p}(\Omega)}, \quad \forall u \in \mathcal{C}_\Sigma^\infty(\Omega), \tag{3.42}$$

at least if $p \in [1, \infty)$. In fact, a minor variation of the above argument shows that estimate (3.42) is actually valid for $p = \infty$ as well. This is based on the fact that for any family of $[0, \infty]$ -valued Lebesgue-measurable functions $\{\xi_j\}_{j \in J}$ in \mathbb{R}^n with the property that there exists $M \in \mathbb{N}$ such that $\#\{j \in J : \xi_j(x) \neq 0\} \leq M$ for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$, there holds

$$\left\| \sum_{j \in J} \xi_j \right\|_{L^\infty(\mathbb{R}^n, \mathcal{L}^n)} \leq M \sup_{j \in J} \|\xi_j\|_{L^\infty(\mathbb{R}^n, \mathcal{L}^n)}. \tag{3.43}$$

Based on (3.42) and (3.2), we may therefore conclude that

$$\mathfrak{E}_{k,D} : W_\Sigma^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly,} \tag{3.44}$$

with operator norm controlled in terms of $n, \varepsilon, \delta, k, p$. On account of this and (3.3), the claim in (3.15) will follow as soon as we show that

$$\mathfrak{E}_{k,D} [\mathcal{C}_\Sigma^\infty(\Omega)] \subseteq W_\Sigma^{k,p}(\mathbb{R}^n). \tag{3.45}$$

To this end, we first remark that

$$\overline{\Omega} \cap O_j \subseteq \overline{\Omega \cap O_j} \quad \text{for every } j \in \{1, \dots, J\}. \tag{3.46}$$

Indeed, if $j \in J$ and $x \in \overline{\Omega} \cap O_j$ are arbitrary, then there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subseteq \Omega$ such that $\lim_{i \rightarrow \infty} x_i = x \in O_j$. Given that O_j is open, there is no loss of generality in assuming that $\{x_i\}_{i \in \mathbb{N}} \subseteq \Omega \cap O_j$ which, in turn, goes to show that the limit point x belongs to $\overline{\Omega \cap O_j}$. This proves (3.46). Now, if $u \in \mathcal{C}_\Sigma^\infty(\Omega)$ then for every $j \in J$ we may write

$$\begin{aligned}
\overline{\Omega} \cap \text{supp} \left(\varphi_j \Lambda_{k,j} (E_j(\psi_j u)) \right) &\subseteq \overline{\Omega} \cap O_j \cap \text{supp} \left(\Lambda_{k,j} (E_j(\psi_j u)) \right) \\
&\subseteq (\overline{\Omega \cap O_j}) \cap \text{supp} \left(\Lambda_{k,j} (E_j(\psi_j u)) \right) \\
&= (\overline{\Omega_j \cap O_j}) \cap \text{supp} \left(\Lambda_{k,j} (E_j(\psi_j u)) \right) \\
&\subseteq \overline{\Omega_j} \cap \text{supp} \left(\Lambda_{k,j} (E_j(\psi_j u)) \right) \\
&= \text{supp} (E_j(\psi_j u)) \\
&\subseteq \text{supp} (\psi_j u) \\
&\subseteq \text{supp} u.
\end{aligned} \tag{3.47}$$

The first inclusion above is a consequence of the fact that $\text{supp} \varphi_j \subseteq O_j$, the second inclusion is based on (3.46), the first equality is guaranteed by (3.21), the third inclusion is obvious, the second equality follows from Theorem 2.4, the penultimate inclusion is implied by (3.36), while the last one is obvious. From (3.34) and (3.47) we then deduce that

$$\overline{\Omega} \cap \text{supp} \mathfrak{E}_{k,D} u \subseteq \text{supp} u \quad \text{for each } u \in \mathcal{C}_\Sigma^\infty(\Omega). \tag{3.48}$$

As a consequence, from (3.48) and the fact that $\Sigma \subseteq \overline{\Omega}$, for each $u \in \mathcal{C}_\Sigma^\infty(\Omega)$ we have

$$\Sigma \cap \text{supp} \mathfrak{E}_{k,D} u = (\Sigma \cap \overline{\Omega}) \cap \text{supp} \mathfrak{E}_{k,D} u = \Sigma \cap (\overline{\Omega} \cap \text{supp} \mathfrak{E}_{k,D} u) \subseteq \Sigma \cap \text{supp} u = \emptyset. \tag{3.49}$$

Hence, $\Sigma \cap \text{supp} \mathfrak{E}_{k,D} u = \emptyset$ for each $u \in \mathcal{C}_\Sigma^\infty(\Omega)$. Given that for each $u \in \mathcal{C}_\Sigma^\infty(\Omega)$ the set $\text{supp} \mathfrak{E}_{k,D} u$ is compact (by (3.34) and (2.35) in Theorem 2.4), and since Σ is closed, it follows that

$$\text{dist}(\Sigma, \text{supp} \mathfrak{E}_{k,D} u) > 0 \quad \text{for each } u \in \mathcal{C}_\Sigma^\infty(\Omega). \tag{3.50}$$

At this stage, for each $i \in \mathbb{N}$ define $\theta_i(x) := i^n \theta(ix)$ for every $x \in \mathbb{R}^n$ and, having fixed some $u \in \mathcal{C}_\Sigma^\infty(\Omega)$, set

$$\xi_i := \theta_i * \mathfrak{E}_{k,D} u \quad \text{in } \mathbb{R}^n, \quad \text{for each } i \in \mathbb{N}. \tag{3.51}$$

Then

$$\xi_i \in \mathcal{C}_c^\infty(\mathbb{R}^n), \quad \text{and } \Sigma \cap \text{supp} \xi_i = \emptyset \quad \text{if } i \text{ is large enough,} \tag{3.52}$$

$$\text{and } \xi_i \rightarrow \mathfrak{E}_{k,D} u \quad \text{in } W^{k,p}(\mathbb{R}^n) \quad \text{as } i \rightarrow \infty,$$

by virtue of (3.50) and the fact that $\mathfrak{E}_{k,D} u$ has compact support and belongs to $W^{k,p}(\mathbb{R}^n)$. In light of (3.3), the approximation result in (3.52) implies that actually

$$\mathfrak{E}_{k,D} u \in W_\Sigma^{k,p}(\mathbb{R}^n) \quad \text{for each } u \in \mathcal{C}_\Sigma^\infty(\Omega). \tag{3.53}$$

From (3.53), (3.44), (3.2), and the fact that $W_\Sigma^{k,p}(\mathbb{R}^n)$ is a closed subspace of $W^{k,p}(\mathbb{R}^n)$, it now follows that the operator (3.34) extends to a linear and bounded mapping as in (3.15).

There remains to show that the mapping just defined also satisfies (3.16). To this end, for each $j \in J$ denote by F_j the operator mapping functions w defined in $O_j \cap \Omega$ into functions defined in Ω according to

$$F_j w := \begin{cases} w & \text{in } \Omega \cap O_j = \Omega_j \cap O_j, \\ 0 & \text{in } \Omega \setminus O_j. \end{cases} \tag{3.54}$$

For any function $u \in \mathcal{C}_\Sigma^\infty(\Omega)$ we then have

$$\begin{aligned}
\mathfrak{E}_{k,D} u \Big|_\Omega &= \widetilde{(1-\eta)u} \Big|_\Omega + \sum_{j \in J} \left[\varphi_j \Lambda_{k,j} (E_j(\psi_j u)) \right] \Big|_\Omega \\
&= (1-\eta)u + \sum_{j \in J} (\varphi_j|_\Omega) F_j \left(\left[\Lambda_{k,j} (E_j(\psi_j u)) \right] \Big|_{O_j \cap \Omega} \right) \\
&= (1-\eta)u + \sum_{j \in J} (\varphi_j|_\Omega) F_j \left(\left[\Lambda_{k,j} (E_j(\psi_j u)) \right] \Big|_{O_j \cap \Omega_j} \right) \\
&= (1-\eta)u + \sum_{j \in J} (\varphi_j|_\Omega) F_j \left(\left[E_j(\psi_j u) \right] \Big|_{O_j \cap \Omega_j} \right) \\
&= (1-\eta)u + \sum_{j \in J} \varphi_j F_j \left((\psi_j u) \Big|_{O_j \cap \Omega} \right) \\
&= (1-\eta)u + \sum_{j \in J} \varphi_j \psi_j u \\
&= (1-\eta)u + \eta u = u,
\end{aligned} \tag{3.55}$$

thanks to (3.34), (3.20), (3.21), (3.54), the fact that $\Lambda_{k,j}$ is an extension operator for $O_j \cap \Omega_j$ for each $j \in J$, as well as (3.35) and (3.36). Having established this, (3.16) now follows from (3.55), part (4) in Lemma 3.2, (3.15), and (3.2). This concludes the construction and analysis of the operator $\mathfrak{E}_{k,D}$ in the case when $N \neq \emptyset$.

Consider now the case when $N = \emptyset$, i.e., when $\partial\Omega \subseteq D$; in particular, $\partial\Omega \subseteq \Sigma$ by (3.14). In this scenario, Ω is just an arbitrary open, nonempty, proper subset of \mathbb{R}^n (as noted in the comments before the statement of Lemma 3.6), and we define the operator

$$\mathfrak{E}_{k,D} : W_\Sigma^{k,p}(\Omega) \longrightarrow W_\Sigma^{k,p}(\mathbb{R}^n), \quad \mathfrak{E}_{k,D} u := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \Omega^c := \mathbb{R}^n \setminus \Omega, \end{cases} \quad \forall u \in W_\Sigma^{k,p}(\Omega), \tag{3.56}$$

which formally corresponds to choosing $\eta \equiv 0$ and $J := \emptyset$ in (3.34). Property (3.16) is now a simple feature of the design of $\mathfrak{E}_{k,D}$ and we are left with checking that this operator maps $W_\Sigma^{k,p}(\Omega)$ boundedly into $W_\Sigma^{k,p}(\mathbb{R}^n)$. With this goal in mind, consider first the operator $\mathfrak{E}_{k,D} : W_\Sigma^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$ defined by the same formula as in the last part of (3.56), and observe that this operator may be viewed as a composition between the isometric inclusion $W_\Sigma^{k,p}(\Omega) \hookrightarrow W_{\partial\Omega}^{k,p}(\Omega)$ (cf. (8) in Lemma 3.2) and the bounded linear mapping from (2.7). That this composition is meaningful is ensured by part (5) in Lemma 3.2. Given the goals we have in mind, there remains to prove that $\tilde{u} \in W_\Sigma^{k,p}(\mathbb{R}^n)$ for every $u \in W_\Sigma^{k,p}(\Omega)$ (where tilde denotes the extension by zero from Ω to \mathbb{R}^n). To justify this, note that if $u \in W_\Sigma^{k,p}(\Omega)$ then $u \in \dot{W}^{k,p}(\Omega)$ by (5) in Lemma 3.2, and also there exists a sequence $\{\varphi_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\Sigma \cap \text{supp } \varphi_j = \emptyset$ for each $j \in \mathbb{N}$ and $\varphi_j|_\Omega \rightarrow u$ in $W^{k,p}(\Omega)$ as $j \rightarrow \infty$, by (3.2). In particular, $\varphi_j|_\Omega \in \mathcal{C}_c^\infty(\Omega) \subseteq \dot{W}^{k,p}(\Omega)$ for every $j \in \mathbb{N}$, by the support condition and our assumption on D . In turn, this readily entails that for each $j \in \mathbb{N}$ we have $\widetilde{\varphi_j|_\Omega} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\Sigma \cap \text{supp } (\widetilde{\varphi_j|_\Omega}) = \emptyset$ which, in light of (2.7), implies that $\widetilde{\varphi_j|_\Omega} \rightarrow \tilde{u}$ in $W^{k,p}(\mathbb{R}^n)$ as $j \rightarrow \infty$. By (3.3), the latter convergence may be interpreted as saying that $\tilde{u} \in W_\Sigma^{k,p}(\mathbb{R}^n)$, as wanted. Hence, the operator (3.56) has all desired properties in this case as well, and this completes the proof of the theorem. \square

It is instructive to note that Theorem 3.7 contains as a particular case the fact that Jones' mapping Λ_k continues to be a well-defined and bounded extension operator on the scale of Sobolev spaces introduced in Definition 3.1 considered on (ε, δ) -domains. This is made precise in the following corollary.

Corollary 3.8. *Let Ω be an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$, and fix an arbitrary number $k \in \mathbb{N}$. Then Jones' extension operator Λ_k (from Theorem 2.1) has the property that, for each closed subset D of $\overline{\Omega}$ and each $p \in [1, \infty]$,*

$$\Lambda_k : W_D^{k,p}(\Omega) \longrightarrow W_D^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly,} \quad (3.57)$$

with operator norm controlled solely in terms of $n, \varepsilon, \delta, k, p$.

Proof. Let Ω be an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$ and fix a closed subset D of $\overline{\Omega}$. Then the hypotheses of Theorem 3.7 are satisfied if we choose Σ to be the current D and take D (in the statement of Theorem 3.7) to be the empty set. Indeed, conditions (3.5)-(3.7) are presently satisfied if we take $J := \{1\}$, $O_1 := \mathbb{R}^n$, $\Omega_1 := \Omega$ and $r := \infty$. In such a scenario, the choice $\eta = \varphi_1 = \psi_1 \equiv 1$ in \mathbb{R}^n is permissible and formula (3.34) reduces to $\mathfrak{E}_{k,\emptyset} = \Lambda_k$, Jones' extension operator for the domain Ω . As such, all desired conclusions follow from Theorem 3.7. \square

It should be noted that, at least if $1 \leq p < \infty$, specializing Corollary 3.8 to the particular case $D := \emptyset$ yields (in light of Lemma 3.3) Jones' extension result recorded in Theorem 2.1. Corollary 3.8 may be regarded as a suitable analogue of the property of Calderón's extension operator in Lipschitz domains of not increasing the support of functions $u \in W^{k,p}(\Omega)$ which vanish near $\partial\Omega$, in the case of Jones' extension operator in (ε, δ) -domains.

Substituting Theorem 2.3 for Theorem 2.1 in the proof of Theorem 3.7 yields a semi-universal extension operator for locally (ε, δ) -domains. Specifically, we have the following result.

Theorem 3.9 (A semi-universal extension operator for locally (ε, δ) -domains). *Assume that $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ are such that D is closed and Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. Then for any $\Upsilon \in \mathbb{N}$ there exists a linear operator \mathfrak{E}_D , mapping locally integrable functions in Ω into Lebesgue measurable functions in \mathbb{R}^n , such that for each $p \in (1, \infty]$, each $k \in \mathbb{N}$ such that $k < \Upsilon$, and each closed subset Σ of $\overline{\Omega}$ satisfying $D \cap \partial\Omega \subseteq \Sigma \cap \partial\Omega$, one has*

$$\mathfrak{E}_D : W_\Sigma^{k,p}(\Omega) \longrightarrow W_\Sigma^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly,} \quad (3.58)$$

(with operator norm controlled in terms of $\varepsilon, \delta, n, p, \Upsilon$, and the quantitative aspects of (3.5)-(3.7)), and

$$(\mathfrak{E}_D u)|_\Omega = u, \quad \mathcal{L}^n\text{-a.e. on } \Omega \text{ for every } u \in W_\Sigma^{k,p}(\Omega). \quad (3.59)$$

As a consequence, corresponding to the case when $\Sigma := D$, one has

$$\mathfrak{E}_D : W_D^{k,p}(\Omega) \longrightarrow W_D^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly, and} \quad (3.60)$$

$$(\mathfrak{E}_D u)|_\Omega = u, \quad \mathcal{L}^n\text{-a.e. on } \Omega \text{ for every } u \in W_D^{k,p}(\Omega). \quad (3.61)$$

Proof. Construct \mathfrak{E}_D as in the proof of Theorem 3.7 (cf. (3.34) and (3.56)) replacing in (3.34) the operators $\Lambda_{k,j}$ by $\widehat{\Lambda}_j$, naturally associated with Ω_j as in Theorem 2.3 for the given Υ . Then the same argument as in the proof of Theorem 3.7 yields all desired conclusions, by substituting Theorem 2.3 for Theorem 2.1. \square

It is also worth noting that the semi-universal extension operator $\widehat{\Lambda}$ constructed in Theorem 2.3 in relation to a given (ε, δ) -domain Ω in \mathbb{R}^n with $\text{rad}(\Omega) > 0$ and a given $\Upsilon \in \mathbb{N}$, has the property that for each closed subset D of $\overline{\Omega}$, each $p \in (1, \infty]$, and each $k \in \mathbb{N}$ with $k < \Upsilon$,

$$\widehat{\Lambda} : W_D^{k,p}(\Omega) \longrightarrow W_D^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly,} \quad (3.62)$$

with operator norm controlled solely in terms of $n, \varepsilon, \delta, p, \Upsilon$. This can be seen by reasoning as in the proof of Corollary 3.8.

We conclude this section by recording the following useful consequence of Theorem 3.7.

Corollary 3.10. *Suppose that $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ are such that D is closed and Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. Then for any $k \in \mathbb{N}$, $p \in [1, \infty]$, and any closed subset Σ of $\overline{\Omega}$ satisfying $D \cap \partial\Omega \subseteq \Sigma \cap \partial\Omega$, there holds*

$$W_{\Sigma}^{k,p}(\Omega) = \left\{ u|_{\Omega} : u \in W_{\Sigma}^{k,p}(\mathbb{R}^n) \right\}. \quad (3.63)$$

As a consequence,

$$W_D^{k,p}(\Omega) = \left\{ u|_{\Omega} : u \in W_D^{k,p}(\mathbb{R}^n) \right\}. \quad (3.64)$$

In particular, formula (3.64) holds for any $k \in \mathbb{N}$ and $p \in [1, \infty]$ whenever Ω is an (ε, δ) -domain in \mathbb{R}^n and D is a closed subset of $\overline{\Omega}$.

Proof. The right-to-left inclusion in (3.63) is implied by (4) in Lemma 3.2, whereas the left-to-right inclusion in (3.63) is a consequence of Theorem 3.7. Finally, (3.64) follows by specializing (3.63) to the case when $\Sigma = D$. \square

4 The structure of Sobolev spaces with partially vanishing traces

In this section we shall make use of the extension result established in Theorem 3.7 in order to further shed light on the nature of the spaces introduced in (3.2). To set the stage, we first record some useful capacity results. For an authoritative extensive discussion on this topic see the monographs [2], [32], [33], and [56]. Given $\alpha > 0$ and $p \in (1, \infty)$, denote by $C_{\alpha,p}(\cdot)$ the L^p -based Bessel capacity of order α in \mathbb{R}^n . When $K \subseteq \mathbb{R}^n$ is a compact set, this is defined by

$$C_{\alpha,p}(K) := \inf \left\{ \int_{\mathbb{R}^n} f^p d\mathcal{L}^n : f \text{ nonnegative, measurable, and } G_{\alpha} * f \geq 1 \text{ on } K \right\}, \quad (4.1)$$

where the Bessel kernel G_{α} is defined as the function whose Fourier transform is given by

$$\widehat{G_{\alpha}}(\xi) = (2\pi)^{-n/2} (1 + |\xi|^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}^n. \quad (4.2)$$

When $\mathcal{O} \subseteq \mathbb{R}^n$ is open, we define

$$C_{\alpha,p}(\mathcal{O}) := \sup \{ C_{\alpha,p}(K) : K \subseteq \mathcal{O}, K \text{ compact} \}, \quad (4.3)$$

and, finally, when $E \subseteq \mathbb{R}^n$ is an arbitrary set,

$$C_{\alpha,p}(E) := \inf \{ C_{\alpha,p}(\mathcal{O}) : \mathcal{O} \supseteq E, \mathcal{O} \text{ open} \}. \quad (4.4)$$

As is customary, generic properties which hold with the possible exception of a set $A \subseteq \mathbb{R}^n$ satisfying $C_{\alpha,p}(A) = 0$ are said to be true (α, p) -quasieverywhere (or, briefly, (α, p) -q.e.).

Given a function $u \in L_{loc}^q(\mathbb{R}^n, \mathcal{L}^n)$ for some $q \in [1, \infty)$, denote by $\overline{L}_{u,q} \subseteq \mathbb{R}^n$ the set of points $x \in \mathbb{R}^n$ with the property that

$$\overline{u}(x) := \lim_{r \rightarrow 0^+} \int_{B(x,r)} u d\mathcal{L}^n \text{ exists and } \lim_{r \rightarrow 0^+} \int_{B(x,r)} |u - \overline{u}(x)|^q d\mathcal{L}^n = 0. \quad (4.5)$$

It is then clear that for every $u \in L_{loc}^q(\mathbb{R}^n, \mathcal{L}^n)$, $1 \leq q < \infty$, one has

$$L_{u,q} := \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0^+} \int_{B(x,r)} |u - u(x)|^q d\mathcal{L}^n = 0 \right\} \subseteq \overline{L}_{u,q}. \quad (4.6)$$

Based on this and the classical Lebesgue Differentiation Theorem, it follows that for every function $u \in L_{loc}^q(\mathbb{R}^n, \mathcal{L}^n)$ with $1 \leq q < \infty$,

$$\mathcal{L}^n(\mathbb{R}^n \setminus L_{u,q}) = \mathcal{L}^n(\mathbb{R}^n \setminus \overline{L}_{u,q}) = 0 \text{ and } \overline{u}(x) = u(x) \text{ for every } x \in L_{u,q}. \quad (4.7)$$

Fix now $p \in (1, \infty)$, $k \in \mathbb{N}$, and consider the Sobolev space $W^{k,p}(\mathbb{R}^n)$. If $p > n/k$ then classical embedding results ensure that in the equivalence class of any $u \in W^{k,p}(\mathbb{R}^n)$ there is a continuous representative. It is then immediate from definitions that in this case

$$\overline{L}_{u,q} = \mathbb{R}^n \quad \text{for every } q \in [1, \infty), \quad (4.8)$$

and that \overline{u} (defined in (4.5)) actually equals the aforementioned continuous representative of u everywhere in \mathbb{R}^n .

Consider next the case when $p \leq n/k$, in which scenario also fix some

$$q \in \left[1, \frac{np}{n-kp}\right] \text{ if } kp < n \text{ and } q \in [1, \infty) \text{ if } kp = n. \quad (4.9)$$

Then $W^{k,p}(\mathbb{R}^n) \subseteq L_{loc}^q(\mathbb{R}^n, \mathcal{L}^n)$ and, according to a classical result in potential theory, for every $u \in W^{k,p}(\mathbb{R}^n)$ there exist $A \subseteq \mathbb{R}^n$ and some $g \in L^p(\mathbb{R}^n, \mathcal{L}^n)$ such that

$$C_{k,p}(A) = 0, \quad \mathbb{R}^n \setminus A \subseteq \overline{L}_{u,q}, \quad u = \overline{u} \text{ } \mathcal{L}^n\text{-a.e. in } \mathbb{R}^n, \quad (4.10)$$

$$u = G_k * g \text{ } \mathcal{L}^n\text{-a.e. in } \mathbb{R}^n, \quad \text{and } \overline{u} = G_k * g \text{ on } \mathbb{R}^n \setminus A. \quad (4.11)$$

See, e.g., [2, Theorem 6.2.1, p.159]. Moreover, based on the fact that $G_k * g$ is (k,p) -quasicontinuous in the sense of [2, Definition 6.1.1, p.156] (see [2, Proposition 6.1.2, p.156]) it may be easily checked that \overline{u} is also (k,p) -quasicontinuous. This makes \overline{u} a (k,p) -quasicontinuous representative of u .

Definition 4.1. Assume that $p \in (1, \infty)$, $k \in \mathbb{N}$, and fix an arbitrary set $E \subseteq \mathbb{R}^n$. In this setting, define the operator of restriction to E as the mapping associating to each $u \in W^{k,p}(\mathbb{R}^n)$ the function

$$\mathcal{R}_E u := \begin{cases} \overline{u} & \text{on } E \cap \overline{L}_{u,q}, \\ 0 & \text{on } E \setminus \overline{L}_{u,q}, \end{cases} \quad (4.12)$$

where q is as in (4.9). Moreover, we shall interpret $\mathcal{R}_E u$ as being independent of q , at the price of regarding this function as being defined only (k,p) -q.e. on E .

It is useful to note that, in the context of Definition 4.1,

$$(\mathcal{R}_E u)(x) = \begin{cases} \lim_{r \rightarrow 0^+} \int_{B(x,r)} u \, d\mathcal{L}^n & \text{if } x \in E \cap \overline{L}_{u,q}, \\ 0 & \text{if } x \in E \setminus \overline{L}_{u,q}, \end{cases} \quad \forall x \in E. \quad (4.13)$$

In particular, when interpreting $\mathcal{R}_E u$ as being independent of q (hence, regarding it as being defined only (k,p) -q.e. on E), the fact that $\mathcal{R}_E u = 0$ becomes equivalent to

$$\lim_{r \rightarrow 0^+} \int_{B(x,r)} u \, d\mathcal{L}^n = 0 \quad (k,p)\text{-q.e. on } E. \quad (4.14)$$

We are now in a position to prove, in the case $1 \leq p < \infty$, the following intrinsic characterization result of the spaces (3.2), originally defined via a completion procedure. Given any $d \in [0, n]$, denote by \mathcal{H}^d the d -dimensional Hausdorff measure in \mathbb{R}^n .

Theorem 4.2 (Structure Theorem for spaces on subsets of \mathbb{R}^n : Version 1). *Let Ω be an open subset of \mathbb{R}^n and let D be a closed subset D of $\overline{\Omega}$. In addition, assume that Ω is either the entire \mathbb{R}^n , or locally an (ε, δ) -domain near $\partial\Omega \setminus D$. Finally, fix an arbitrary $k \in \mathbb{N}$. Then for every $p \in (1, \infty)$ one has*

$$W_D^{k,p}(\Omega) = \left\{ u|_{\Omega} : u \in W^{k,p}(\mathbb{R}^n) \text{ and } \mathcal{R}_D(\partial^\alpha u) = 0 \quad (k - |\alpha|, p)\text{-q.e. on } D, \right. \\ \left. \text{for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k - 1 \right\}, \quad (4.15)$$

whereas corresponding to $p = 1$ one has

$$W_D^{k,1}(\Omega) = \left\{ u|_{\Omega} : u \in W^{k,1}(\mathbb{R}^n) \text{ and } \mathcal{R}_D(\partial^\alpha u) = 0 \text{ } \mathcal{H}^{\max\{0, n-k+|\alpha|\}}\text{-a.e. on } D, \right. \\ \left. \text{for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1 \right\}. \quad (4.16)$$

As a consequence, if Ω is an arbitrary nonempty open set in \mathbb{R}^n and $k \in \mathbb{N}$, then for any $p \in (1, \infty)$ one has

$$\mathring{W}^{k,p}(\Omega) = \left\{ u|_{\Omega} : u \in W^{k,p}(\mathbb{R}^n) \text{ and } \mathcal{R}_{\partial\Omega}(\partial^\alpha u) = 0 \text{ } (k-|\alpha|, p)\text{-q.e. on } \partial\Omega, \right. \\ \left. \text{for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1 \right\}, \quad (4.17)$$

and, corresponding to $p = 1$,

$$\mathring{W}_D^{k,1}(\Omega) = \left\{ u|_{\Omega} : u \in W^{k,1}(\mathbb{R}^n) \text{ and } \mathcal{R}_{\partial\Omega}(\partial^\alpha u) = 0 \text{ } \mathcal{H}^{\max\{0, n-k+|\alpha|\}}\text{-a.e. on } D, \right. \\ \left. \text{for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1 \right\}. \quad (4.18)$$

Proof. We begin by treating the case $\Omega = \mathbb{R}^n$. In this scenario, for each $p \in (1, \infty)$ formula (4.15) becomes

$$W_D^{k,p}(\mathbb{R}^n) = \left\{ u \in W^{k,p}(\mathbb{R}^n) : \mathcal{R}_D(\partial^\alpha u) = 0 \text{ } (k-|\alpha|, p)\text{-q.e. on } D, \right. \\ \left. \text{for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1 \right\}. \quad (4.19)$$

In turn, this is a consequence of a remarkable result of L.I. Hedberg and T.H. Wolff to the effect that any closed set in \mathbb{R}^n admits what has become known as (k, p) -synthesis, for any $p \in (1, \infty)$ and any $k \in \mathbb{N}$. See [23, Theorem 5, p. 166], as well as [2, Theorem 9.1.3, p. 234]. The end-point case $p = 1$ of this result, namely

$$W_D^{k,1}(\mathbb{R}^n) = \left\{ u \in W^{k,1}(\mathbb{R}^n) : \mathcal{R}_D(\partial^\alpha u) = 0 \text{ } \mathcal{H}^{\max\{0, n-k+|\alpha|\}}\text{-a.e. on } D, \right. \\ \left. \text{for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1 \right\}, \quad (4.20)$$

has been obtained by Yu. Netrusov in [43]. With (4.19)-(4.20) in hand, (4.15)-(4.16) follow, granted the assumptions made on Ω and D in the statement of the theorem, by appealing to Corollary 3.10. Finally, (4.17)-(4.18) are immediate from (4.15)-(4.16) with $D = \partial\Omega$ and part (5) in Lemma 3.2. \square

Theorem 4.2 suggests considering higher-order restriction operators of the following nature. Assuming that $p \in (1, \infty)$, $k \in \mathbb{N}$, and $E \subseteq \mathbb{R}^n$ is arbitrary, for each $u \in W^{k,p}(\mathbb{R}^n)$ define

$$\mathcal{R}_E^{(k)} u := \left\{ \mathcal{R}_E(\partial^\alpha u) \right\}_{|\alpha| \leq k-1}. \quad (4.21)$$

In particular, $\mathcal{R}_E^{(1)} = \mathcal{R}_E$. With this piece of notation, given a function $u \in W^{k,p}(\mathbb{R}^n)$, where $p \in (1, \infty)$ and $k \in \mathbb{N}$, along with an arbitrary set $E \subseteq \mathbb{R}^n$, we agree to interpret the condition

$$\mathcal{R}_E^{(k)} u = 0 \text{ quasi-everywhere on } E \quad (4.22)$$

as an abbreviation of the fact that

$$\mathcal{R}_E(\partial^\alpha u) = 0 \text{ } (k-|\alpha|, p)\text{-q.e. on } E, \text{ for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1. \quad (4.23)$$

With this interpretation, in the context of Theorem 4.2, formula (4.15) becomes

$$W_D^{k,p}(\Omega) = \left\{ u|_{\Omega} : u \in W^{k,p}(\mathbb{R}^n) \text{ and } \mathcal{R}_D^{(k)} u = 0 \text{ quasi-everywhere on } D \right\}. \quad (4.24)$$

We are now interested in the case in which the vanishing trace condition intervening in the right-hand side of (4.24) may be reformulated using the Hausdorff measure in lieu of Bessel capacities. This requires some preparations. To set the stage, recall that a subset D of \mathbb{R}^n is said to be d -Ahlfors regular provided there exists some finite constant $C \geq 1$ with the property that

$$C^{-1}r^d \leq \mathcal{H}^d(B(x,r) \cap D) \leq Cr^d, \quad \forall x \in D, \quad 0 < r \leq \text{diam}(D) \quad (4.25)$$

(where, as before, \mathcal{H}^d is the d -dimensional Hausdorff measure in \mathbb{R}^n). For example, the boundary of the Koch's snowflake in \mathbb{R}^2 is d -Ahlfors regular for $d = \frac{\ln 4}{\ln 3}$.

Assuming that $D \subseteq \mathbb{R}^n$ is closed and d -Ahlfors regular for some $0 < d \leq n$, we shall define the measure

$$\sigma := \mathcal{H}^d \llcorner D. \quad (4.26)$$

In this context, a brand of Besov spaces have been introduced by A. Jonsson and H. Wallin in [26] as follows. Given $p, q \in [1, \infty]$ and $s \in (0, \infty) \setminus \mathbb{N}$, define the Besov space $B_s^{p,q}(D)$ as the collection of families $\dot{f} := \{f_\alpha\}_{|\alpha| \leq [s]}$ (where $[s]$ denotes the integer part of s), whose components are functions from $L^p(D, \sigma)$, with the property that if for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq [s]$ we set

$$R_\alpha(x, y) := f_\alpha(x) - \sum_{|\beta| \leq [s] - |\alpha|} \frac{(x-y)^\beta}{\beta!} f_{\alpha+\beta}(y) \quad \text{for } \sigma\text{-a.e. } x, y \in D, \quad (4.27)$$

then

$$\begin{aligned} \|\dot{f}\|_{B_s^{p,q}(D)} &:= \sum_{|\alpha| \leq [s]} \|f_\alpha\|_{L^p(D, \sigma)} \\ &+ \left(\sum_{j=0}^{\infty} \sum_{|\alpha| \leq [s]} 2^{j(s-|\alpha|)q} \left(2^{jd} \iint_{|x-y| < 2^{-j}} |R_\alpha(x, y)|^p d\sigma(x) d\sigma(y) \right)^{q/p} \right)^{1/q} < +\infty, \end{aligned} \quad (4.28)$$

with a natural interpretation when $\max\{p, q\} = \infty$. Hereafter, we shall always understand that $B_s^{p,q}(D)$ is equipped with the norm (4.28), in which case this becomes a Banach space. In fact, a suitable definition of $B_s^{p,q}(D)$ may also be given when $s \in \mathbb{N}$ as the collection of families $\dot{f} = \{f_\alpha\}_{|\alpha| \leq s-1}$ whose components satisfy a certain approximation property, though we shall not be needing this case here (this being said, the interested reader is refer to [26, Definition 2, p. 123] for details).

The following theorem is a particular case of more general results, regarding traces and extensions on (and from) d -Ahlfors regular closed subsets of \mathbb{R}^n proved by A. Jonsson and H. Wallin in [26, Theorem 1, p. 182], [26, Theorem 2, p. 183], and [26, Theorem 3, p. 197].

Theorem 4.3 (Jonsson-Wallin trace/extension theory from/into \mathbb{R}^n). *Let $D \subseteq \mathbb{R}^n$ be a closed set which is d -Ahlfors regular for some $d \in (0, n)$. Also, fix a number $k \in \mathbb{N}$ and assume that*

$$\max\{1, n-d\} < p < \infty. \quad (4.29)$$

Then for every $v \in W^{k,p}(\mathbb{R}^n)$ the vector-valued limit

$$\left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha v d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} \quad \text{exists at } \mathcal{H}^d\text{-a.e. } x \in D, \quad (4.30)$$

the higher-order trace operator from (4.21) satisfies

$$(\mathcal{R}_D^{(k)} v)(x) = \left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha v d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} \quad \text{at } \mathcal{H}^d\text{-a.e. } x \in D, \quad (4.31)$$

and induces a well-defined, linear, and bounded mapping

$$\mathcal{R}_D^{(k)} : W^{k,p}(\mathbb{R}^n) \longrightarrow B_{k-(n-d)/p}^{p,p}(D). \quad (4.32)$$

Conversely, to each $\dot{f} = \{f_\alpha\}_{|\alpha| \leq k-1} \in B_{k-(n-d)/p}^{p,p}(D)$ associate the polynomial

$$P_{\dot{f}}(x, y) := \sum_{|\alpha| \leq k-1} \frac{(x-y)^\alpha}{\alpha!} f_\alpha(y), \quad x \in \mathbb{R}^n, \quad y \in D, \quad (4.33)$$

and introduce the function $\mathcal{E}_D^{(k)} \dot{f}$ defined \mathcal{L}^n -a.e. in \mathbb{R}^n (since $\mathcal{L}^n(D) = 0$ given that D is d -Ahlfors regular with $d < n$) according to

$$(\mathcal{E}_D^{(k)} \dot{f})(x) := \sum_{\substack{Q \in \mathcal{W}(\mathbb{R}^n \setminus D) \\ \ell(Q) \leq 1}} \varphi_Q(x) \int_{D \cap B(x_Q, 6 \operatorname{diam}(Q))} P_{\dot{f}}(x, y) d\mathcal{H}^d(y), \quad \forall x \in \mathbb{R}^n \setminus D, \quad (4.34)$$

where the family $\{\varphi_Q\}_{Q \in \mathcal{W}(\mathbb{R}^n \setminus D)}$ consists of functions satisfying

$$\varphi_Q \in \mathcal{C}_c^\infty(\mathbb{R}^n), \quad \operatorname{supp} \varphi_Q \subseteq \frac{17}{16}Q, \quad 0 \leq \varphi_Q \leq 1, \quad |\partial^\alpha \varphi_Q| \leq C_\alpha \ell(Q)^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n, \quad (4.35)$$

for every $Q \in \mathcal{W}(\mathbb{R}^n \setminus D)$, as well as

$$\sum_{\substack{Q \in \mathcal{W}(\mathbb{R}^n \setminus D) \\ \ell(Q) \leq 1}} \varphi_Q \equiv 1 \quad \text{on} \quad \bigcup_{\substack{Q \in \mathcal{W}(\mathbb{R}^n \setminus D) \\ \ell(Q) \leq 1}} Q. \quad (4.36)$$

Then the operator $\mathcal{E}_D^{(k)}$ (whose action depends only on D and k) has the property that, for each p as in (4.29),

$$\mathcal{E}_D^{(k)} : B_{k-(n-d)/p}^{p,p}(D) \longrightarrow W^{k,p}(\mathbb{R}^n) \quad \text{linearly and boundedly,} \quad (4.37)$$

and

$$\mathcal{R}_D^{(k)} \circ \mathcal{E}_D^{(k)} = I, \quad \text{the identity on } B_{k-(n-d)/p}^{p,p}(D). \quad (4.38)$$

Clearly, the existence of a right-inverse makes the higher-order trace operator in (4.32) surjective whenever (4.29) holds. However, one result conspicuously absent from the above theorem is an intrinsic description of the null-space of (4.32). Our next theorem addresses this aspect.

Theorem 4.4 (Characterization of the null-space of the trace operator acting from \mathbb{R}^n). *Suppose that $D \subseteq \mathbb{R}^n$ is a closed set which is d -Ahlfors regular for some $d \in (0, n)$, and fix a number $k \in \mathbb{N}$. Then*

$$W_D^{k,p}(\mathbb{R}^n) = \left\{ u \in W^{k,p}(\mathbb{R}^n) : \mathcal{R}_D^{(k)} u = 0 \text{ at } \mathcal{H}^d\text{-a.e. point on } D \right\} \quad (4.39)$$

whenever

$$\max\{1, n-d\} < p < \infty. \quad (4.40)$$

Proof. Suppose that p is as in (4.40) and assume that $u \in W^{k,p}(\mathbb{R}^n)$ is such that $\mathcal{R}_D^{(k)} u = 0$ at σ -a.e. point on D or, equivalently, in $B_{k-(n-d)/p}^{p,p}(D)$. Since $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$, it is possible to select a sequence

$$\{\varphi_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n) \quad \text{such that } \varphi_j \longrightarrow u \text{ in } W^{k,p}(\mathbb{R}^n) \text{ as } j \rightarrow \infty. \quad (4.41)$$

In particular, by our assumptions on u and the continuity of (4.32), we have

$$\mathcal{R}_D^{(k)} \varphi_j \rightarrow \mathcal{R}_D^{(k)} u = 0 \quad \text{in } B_{k-(n-d)/p}^{p,p}(D) \quad \text{as } j \rightarrow \infty. \quad (4.42)$$

Going further, for each $j \in \mathbb{N}$ define

$$v_j := \varphi_j - \mathcal{E}_D^{(k)}(\mathcal{R}_D^{(k)} \varphi_j) \quad \text{in } \mathbb{R}^n, \quad (4.43)$$

and note that, in light of (4.43), (4.41), and Theorem 4.3, we have

$$v_j \in W^{k,q}(\mathbb{R}^n) \quad \text{whenever } \max\{1, n-d\} < q < \infty. \quad (4.44)$$

In concert with standard embedding results, this further entails

$$v_j \in \mathcal{C}^{k-1}(\mathbb{R}^n) \quad \text{for each } j \in \mathbb{N}. \quad (4.45)$$

Next, thanks to (4.37), for every $j \in \mathbb{N}$ we may estimate

$$\begin{aligned} \|u - v_j\|_{W^{k,p}(\mathbb{R}^n)} &\leq \|u - \varphi_j\|_{W^{k,p}(\mathbb{R}^n)} + \|\mathcal{E}_D^{(k)}(\mathcal{R}_D^{(k)} \varphi_j)\|_{W^{k,p}(\mathbb{R}^n)} \\ &\leq \|u - \varphi_j\|_{W^{k,p}(\mathbb{R}^n)} + C \|\mathcal{R}_D^{(k)} \varphi_j\|_{B_{k-(n-d)/p}^{p,p}(D)}. \end{aligned} \quad (4.46)$$

Consequently,

$$v_j \rightarrow u \quad \text{in } W^{k,p}(\mathbb{R}^n) \quad \text{as } j \rightarrow \infty, \quad (4.47)$$

by (4.46), (4.41), and (4.42). Moreover, from (4.43) and (4.38) we deduce that for every $j \in \mathbb{N}$,

$$\mathcal{R}_D^{(k)} v_j = 0 \quad \sigma\text{-a.e. on } D. \quad (4.48)$$

From (4.48), (4.21), (4.13), (4.45), and (4.25) we may now conclude that, for each $j \in \mathbb{N}$,

$$(\partial^\alpha v_j)|_D = 0 \quad \text{everywhere on } D, \quad \text{for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1. \quad (4.49)$$

When used together with (4.19), (4.44) and (4.40), the everywhere vanishing trace condition from (4.49) implies that

$$v_j \in W_D^{k,p}(\mathbb{R}^n) \quad \text{for every } j \in \mathbb{N}. \quad (4.50)$$

Finally, from (4.50), (4.47) and the fact that $W_D^{k,p}(\mathbb{R}^n)$ is a closed subspace of $W^{k,p}(\mathbb{R}^n)$ we deduce that

$$u \in W_D^{k,p}(\mathbb{R}^n). \quad (4.51)$$

The membership in (4.51) proves the right-to-left inclusion in (4.39).

Conversely, if $u \in W_D^{k,p}(\mathbb{R}^n)$ then there exists a sequence

$$\begin{aligned} \{\varphi_j\}_{j \in \mathbb{N}} &\subseteq \mathcal{C}_c^\infty(\mathbb{R}^n) \quad \text{such that } D \cap \text{supp } \varphi_j = \emptyset \quad \text{for each } j \in \mathbb{N}, \\ &\text{and } \varphi_j \rightarrow u \quad \text{in } W^{k,p}(\mathbb{R}^n) \quad \text{as } j \rightarrow \infty. \end{aligned} \quad (4.52)$$

In particular, from (4.52), (4.21) and (4.13) we see that

$$\mathcal{R}_D^{(k)} \varphi_j = 0 \quad \text{everywhere on } D, \quad \text{for each } j \in \mathbb{N}. \quad (4.53)$$

Collectively, (4.52), the continuity of (4.32), and (4.53) imply that $\mathcal{R}_D^{(k)} u = 0$ in $B_{k-(n-d)/p}^{p,p}(D)$ or, equivalently, at \mathcal{H}^d -a.e. point on D . This establishes the left-to-right inclusion in (4.39), and finishes the proof of the theorem. \square

Corollary 4.5. *Assume that D is a closed subset of \mathbb{R}^n which is d -Ahlfors regular for some $d \in (0, n)$, and fix $k \in \mathbb{N}$ and p such that $\max\{1, n - d\} < p < \infty$. Then for any function $v \in W^{k,p}(\mathbb{R}^n)$,*

$$\mathcal{R}_D^{(k)} v = 0 \text{ at } \mathcal{H}^d\text{-a.e. point on } D \quad (4.54)$$

if and only if

$$\mathcal{R}_D^{(k)} v = 0 \text{ quasi-everywhere on } D. \quad (4.55)$$

Proof. This is an immediate consequence of formulas (4.24), (4.19). \square

We shall now provide an alternative description of the space $W_D^{k,p}(\Omega)$, which should be contrasted to that provided in Theorem 4.2. The new feature is that the vanishing trace condition is now formulated using the Hausdorff measure in place of Bessel capacities.

Theorem 4.6 (Structure Theorem for spaces on subdomains of \mathbb{R}^n : Version 2). *Suppose that $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ are such that D is closed and d -Ahlfors regular for some $d \in (0, n)$, while Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. In addition, fix a number $k \in \mathbb{N}$ and assume that $\max\{1, n - d\} < p < \infty$. Then*

$$W_D^{k,p}(\Omega) = \left\{ u|_{\Omega} : u \in W^{k,p}(\mathbb{R}^n) \text{ and } \mathcal{R}_D^{(k)} u = 0 \text{ at } \mathcal{H}^d\text{-a.e. point on } D \right\}. \quad (4.56)$$

In particular, if Ω is a nonempty open subset of \mathbb{R}^n with the property that $\partial\Omega$ is d -Ahlfors regular for some $d \in (0, n)$, then

$$\mathring{W}^{k,p}(\Omega) = \left\{ u|_{\Omega} : u \in W^{k,p}(\mathbb{R}^n) \text{ and } \mathcal{R}_{\partial\Omega}^{(k)} u = 0 \text{ at } \mathcal{H}^d\text{-a.e. point on } \partial\Omega \right\}, \quad (4.57)$$

whenever $k \in \mathbb{N}$ and $\max\{1, n - d\} < p < \infty$.

Proof. Formula (4.56) follows from Theorem 4.4 in combination with Corollary 3.10, while formula (4.57) is a direct consequence of (4.56) and (5) in Lemma 3.2. \square

In the last part of this section we shall revisit the Jonsson-Wallin extension operator from Theorem 4.3, the main goal being establishing the refinement of property (4.37) presented in Theorem 4.9. This requires a number of preliminaries to which we now turn.

Assume that $D \subseteq \mathbb{R}^n$ is a given closed set which is d -Ahlfors regular for some $d \in (0, n)$, and consider $\sigma := \mathcal{H}^d \llcorner D$. For any σ -measurable function f on D , define

$$\text{supp } f := \{x \in D : \text{there is no } r > 0 \text{ such that } f \equiv 0 \text{ } \sigma\text{-a.e. in } B(x, r) \cap D\}. \quad (4.58)$$

In particular, $\text{supp } f$ is a closed subset of D and f vanishes σ -a.e. on $D \setminus \text{supp } f$. If, in addition, two numbers k, p are given such that $k \in \mathbb{N}$ and $\max\{1, n - d\} < p < \infty$, then for every $\dot{f} = \{f_{\alpha}\}_{|\alpha| \leq k-1} \in B_{k-(n-d)/p}^{p,p}(D)$ we define

$$\text{supp } \dot{f} := \bigcup_{|\alpha| \leq k-1} \text{supp } f_{\alpha}. \quad (4.59)$$

In a first stage, we wish to augment Theorem 4.3 with the following result.

Proposition 4.7. *Let $D \subseteq \mathbb{R}^n$ be a closed set which is d -Ahlfors regular for some $d \in (0, n)$, and define $\sigma := \mathcal{H}^d \llcorner D$. Also, assume that $k \in \mathbb{N}$ and $\max\{1, n - d\} < p < \infty$. Then the extension operator $\mathcal{E}_D^{(k)}$ from Theorem 4.3 has the property that*

$$D \cap \text{supp } (\mathcal{E}_D^{(k)} \dot{f}) = \text{supp } \dot{f}, \quad \forall \dot{f} \in B_{k-(n-d)/p}^{p,p}(D). \quad (4.60)$$

Furthermore,

$$\dot{f} \in B_{k-(n-d)/p}^{p,p}(D) \text{ with } \text{supp } \dot{f} \text{ compact} \implies \text{supp } (\mathcal{E}_D^{(k)} \dot{f}) \text{ compact}. \quad (4.61)$$

Proof. Pick an arbitrary $\dot{f} \in B_{k-(n-d)/p}^{p,p}(D)$ and note that, thanks to (4.33), we have

$$\text{supp}(P_{\dot{f}}(x, \cdot)) \subseteq \text{supp} \dot{f}, \quad \forall x \in \mathbb{R}^n. \quad (4.62)$$

Consequently, for every $x \in \mathbb{R}^n \setminus D$, formula (4.34) may be re-written in the form

$$(\mathcal{E}_D^{(k)} \dot{f})(x) = \sum_{\substack{Q \in \mathcal{W}(\mathbb{R}^n \setminus D) \\ \ell(Q) \leq 1}} \frac{\varphi_Q(x)}{\mathcal{H}^d(D \cap B(x_Q, 6 \text{diam}(Q)))} \int_{B(x_Q, 6 \text{diam}(Q)) \cap \text{supp} \dot{f}} P_{\dot{f}}(x, y) d\mathcal{H}^d(y). \quad (4.63)$$

From this and the support condition on the φ_Q 's from (4.35), we may then conclude that

$$\text{supp}(\mathcal{E}_D^{(k)} \dot{f}) \subseteq \overline{G_{\dot{f}}} \quad (4.64)$$

where

$$G_{\dot{f}} := \bigcup_{\substack{Q \in \mathcal{W}(\mathbb{R}^n \setminus D), \ell(Q) \leq 1 \\ B(x_Q, 6 \text{diam}(Q)) \cap \text{supp} \dot{f} \neq \emptyset}} \frac{17}{16} Q. \quad (4.65)$$

As such, the left-to-right inclusion in (4.60) follows as soon as we establish that

$$D \cap \overline{G_{\dot{f}}} \subseteq \text{supp} \dot{f}. \quad (4.66)$$

To justify (4.66), select an arbitrary point $x_o \in D \cap \overline{G_{\dot{f}}}$. The membership of x_o to $\overline{G_{\dot{f}}}$ entails the existence of a sequence of dyadic cubes $\{Q_j\}_j \subseteq \mathcal{W}(\mathbb{R}^n \setminus D)$ with $\ell(Q_j) \leq 1$ for every j , along with a sequence $\{x_j\}_j$ of points in \mathbb{R}^n , satisfying

$$x_j \in \frac{17}{16} Q_j \quad \text{for every } j, \quad (4.67)$$

$$B(x_{Q_j}, 6 \text{diam}(Q_j)) \cap \text{supp} \dot{f} \neq \emptyset \quad \text{for every } j, \quad (4.68)$$

$$\lim_j x_j = x_o. \quad (4.69)$$

Now, from (2.13) we conclude that there exists $c \in (0, 1)$ such that

$$c \ell(Q_j) \leq \text{dist}\left(\frac{17}{16} Q_j, D\right) \leq \text{dist}(x_j, D) \leq |x_j - x_o|, \quad \text{for all } j, \quad (4.70)$$

where the last inequality uses the fact that $x_o \in D$. In concert with (4.69) and (4.67), this forces

$$\lim_j \ell(Q_j) = 0 \quad \text{and} \quad \lim_j x_{Q_j} = x_o. \quad (4.71)$$

On the other hand, from (4.68) we deduce that for each j there exists $y_j \in \text{supp} \dot{f}$ such that

$$|x_{Q_j} - y_j| < 6 \text{diam}(Q_j) = 6\sqrt{n} \ell(Q_j). \quad (4.72)$$

Consequently,

$$x_o = \lim_j x_{Q_j} = \lim_j y_j \in \text{supp} \dot{f} \quad (4.73)$$

by (4.71), (4.72), and the fact that $\text{supp} \dot{f}$ is a closed set. This justifies (4.66) and finishes the proof of the left-to-right inclusion in (4.60).

To proceed in the opposite direction, assume now that $x_o \in D$ is such that $x_o \notin \text{supp}(\mathcal{E}_D^{(k)} \dot{f})$. Then there exists $r > 0$ with the property that $\mathcal{E}_D^{(k)} \dot{f} = 0$ at \mathcal{L}^n -a.e. point in $B(x_o, r)$. As a consequence of this, (4.38), and (4.31), at \mathcal{H}^d -a.e. $x \in D \cap B(x_o, r)$ we may write

$$\dot{f}(x) = \mathcal{R}_D^{(k)}(\mathcal{E}_D^{(k)} \dot{f})(x) = \left\{ \lim_{\rho \rightarrow 0^+} \int_{B(x, \rho)} \partial^\alpha (\mathcal{E}_D^{(k)} \dot{f}) d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} = 0. \quad (4.74)$$

Thus, $\dot{f} = 0$ at \mathcal{H}^d -a.e. point on $D \cap B(x_o, r)$, which shows that $x_o \notin \text{supp } \dot{f}$, by (4.58)-(4.59). Altogether, this argument shows that $\text{supp } \dot{f} \subseteq D \cap \text{supp } (\mathcal{E}_D^{(k)} \dot{f})$, hence the right-to-left inclusion in (4.60) holds as well. This concludes the proof of (4.60).

Turning our attention to (4.61), assume that some $\dot{f} \in B_{k-(n-d)/p}^{p,p}(D)$ such that $\text{supp } \dot{f}$ is a compact set has been fixed. In view of (4.64), it suffices to show that $G_{\dot{f}}$ is a bounded set. However, a simple calculation based on (4.65) reveals that

$$G_{\dot{f}} \subseteq \left\{ x \in \mathbb{R}^n : \text{dist}(x, \text{supp } \dot{f}) < \sqrt{n} \left(6 + \frac{17}{32} \right) \right\}, \quad (4.75)$$

hence the desired conclusion follows. \square

In addition to the Besov space naturally associated with the quantitative condition (4.28), we shall now bring into focus a related (closed) subspace of it, whose distinguished feature is the requirement that the Besov functions vanish, in an appropriate sense, on a given subset of the (Ahlfors regular) ambient. This class is formally introduced in the following definition.

Definition 4.8. *Let $D \subseteq \mathbb{R}^n$ be a closed set which is d -Ahlfors regular for some $d \in (0, n)$, and fix some $k \in \mathbb{N}$. For each p such that $\max\{1, n-d\} < p < \infty$ and each closed subset F of D define*

$$B_{k-(n-d)/p}^{p,p}(D; F) := \text{the closure in } B_{k-(n-d)/p}^{p,p}(D) \text{ of the space} \\ \left\{ \{(\partial^\alpha \varphi)|_D\}_{|\alpha| \leq k-1} : \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ and } F \cap \left(\bigcup_{|\alpha| \leq k-1} \text{supp } (\partial^\alpha \varphi|_D) \right) = \emptyset \right\}, \quad (4.76)$$

where each $\text{supp } (\partial^\alpha \varphi|_D)$ is interpreted in the sense of (4.58), regarding $\partial^\alpha \varphi|_D$ as a \mathcal{H}^d [D -measurable function on D].

Obviously, in the context of the above definition,

$$B_{k-(n-d)/p}^{p,p}(D; F) \text{ is a closed subspace of } B_{k-(n-d)/p}^{p,p}(D), \quad (4.77)$$

$$\text{the class } B_{k-(n-d)/p}^{p,p}(D; F) \text{ is monotonic with respect to } F, \text{ and} \quad (4.78)$$

$$B_{k-(n-d)/p}^{p,p}(D; D) = \{0\}. \quad (4.79)$$

Moreover, from Theorem 4.3 it is also clear that

$$B_{k-(n-d)/p}^{p,p}(D; \emptyset) = B_{k-(n-d)/p}^{p,p}(D). \quad (4.80)$$

The relevance of the category of Besov spaces considered in Definition 4.8 is most apparent in the context of the theorem below, refining the Jonsson-Wallin trace/extension results from Theorem 4.3.

Theorem 4.9 (Extending/restricting partially vanishing functions). *Suppose that $D \subseteq \mathbb{R}^n$ is a closed set which is d -Ahlfors regular for some $d \in (0, n)$, and assume that F is a closed subset of D . Also, fix $k \in \mathbb{N}$ and some p such that $\max\{1, n-d\} < p < \infty$.*

Then the extension and restriction operators, $\mathcal{E}_D^{(k)}$, $\mathcal{R}_D^{(k)}$, from Theorem 4.3 have the property that

$$\mathcal{E}_D^{(k)} : B_{k-(n-d)/p}^{p,p}(D; F) \longrightarrow W_F^{k,p}(\mathbb{R}^n), \quad (4.81)$$

$$\mathcal{R}_D^{(k)} : W_F^{k,p}(\mathbb{R}^n) \longrightarrow B_{k-(n-d)/p}^{p,p}(D; F), \quad (4.82)$$

are well-defined, linear and bounded mappings satisfying

$$\mathcal{R}_D^{(k)} \circ \mathcal{E}_D^{(k)} = I, \quad \text{the identity on } B_{k-(n-d)/p}^{p,p}(D; F). \quad (4.83)$$

In particular, the restriction operator in (4.82) is onto.

We wish to note that the functional analytic properties of the Jonsson-Wallin trace and extension operators recorded in Theorem 4.3 are particular manifestations of the above result, as seen by specializing Theorem 4.9 to the case when $F := \emptyset$ (cf. (4.80) and (3.3) in this regard).

Proof of Theorem 4.9. Since $B_{k-(n-d)/p}^{p,p}(D; F) \hookrightarrow B_{k-(n-d)/p}^{p,p}(D)$ isometrically, it follows from Theorem 4.3 that $\mathcal{E}_D^{(k)}$ maps $B_{k-(n-d)/p}^{p,p}(D; F)$ linearly and boundedly into $W_F^{k,p}(\mathbb{R}^n)$. Given that $W_F^{k,p}(\mathbb{R}^n)$ is a closed subspace of $W^{k,p}(\mathbb{R}^n)$ (cf. part (2) in Lemma 3.2), the fact that $\mathcal{E}_D^{(k)}$ is well-defined, linear and bounded mapping in the context of (4.81) follows as soon as we prove that

$$\mathcal{E}_D^{(k)} \dot{\varphi} \in W_F^{k,p}(\mathbb{R}^n) \quad (4.84)$$

where $\dot{\varphi} := \{(\partial^\alpha \varphi)|_D\}_{|\alpha| \leq k-1}$, whenever $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is such that

$$F \cap \left(\bigcup_{|\alpha| \leq k-1} \text{supp}(\partial^\alpha \varphi|_D) \right) = \emptyset. \quad (4.85)$$

With this goal in mind, fix some $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that (4.85) holds, and note that this condition amounts to (cf. (4.59))

$$F \cap \text{supp} \dot{\varphi} = \emptyset, \quad (4.86)$$

with $\dot{\varphi} = \{(\partial^\alpha \varphi)|_D\}_{|\alpha| \leq k-1}$ is regarded as an element in $B_{k-(n-d)/p}^{p,p}(D)$. Since $F \subseteq D$, Proposition 4.7 and (4.86) imply that

$$\begin{aligned} F \cap \text{supp}(\mathcal{E}_D^{(k)} \dot{\varphi}) &= (F \cap D) \cap \text{supp}(\mathcal{E}_D^{(k)} \dot{\varphi}) \\ &= F \cap \left(D \cap \text{supp}(\mathcal{E}_D^{(k)} \dot{\varphi}) \right) = F \cap \text{supp} \dot{\varphi} = \emptyset. \end{aligned} \quad (4.87)$$

Furthermore, given that $\text{supp} \dot{\varphi}$ is compact, (4.61) ensures that $\text{supp}(\mathcal{E}_D^{(k)} \dot{\varphi})$ is a compact subset of \mathbb{R}^n . As such, (4.87) yields $\text{dist}(F, \text{supp}(\mathcal{E}_D^{(k)} \dot{\varphi})) > 0$. Having proved this, mollifying the function $\mathcal{E}_D^{(k)} \dot{\varphi} \in W^{k,p}(\mathbb{R}^n)$ (much as in the past) yields a sequence of functions $\{\psi_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n)$ with the property that $F \cap \text{supp} \psi_j = \emptyset$ for each $j \in \mathbb{N}$ and such that $\psi_j \rightarrow \mathcal{E}_D^{(k)} \dot{\varphi}$ in $W^{k,p}(\mathbb{R}^n)$ as $j \rightarrow \infty$. In light of (3.3), we may therefore conclude that (4.84) holds, finishing the proof of the fact that $\mathcal{E}_D^{(k)}$ is a well-defined, linear and bounded operator in the context of (4.81).

As regards (4.82), the starting point is the observation that whenever $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is such that $F \cap \text{supp} \varphi = \emptyset$ then

$$\mathcal{R}_D^{(k)} \varphi = \{(\partial^\alpha \varphi)|_D\}_{|\alpha| \leq k-1} \quad \text{and} \quad F \cap \left(\bigcup_{|\alpha| \leq k-1} \text{supp}(\partial^\alpha \varphi|_D) \right) = \emptyset. \quad (4.88)$$

In view of (4.76), this proves that $\mathcal{R}_D^{(k)}$ maps $\{\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n) : F \cap \text{supp} \varphi = \emptyset\}$ into $B_{k-(n-d)/p}^{p,p}(D; F)$, and since the former space is dense in $W_F^{k,p}(\mathbb{R}^n)$ which is mapped by $\mathcal{R}_D^{(k)}$ boundedly into $B_{k-(n-d)/p}^{p,p}(D)$ (by Theorem 4.3), we conclude that the restriction operator $\mathcal{R}_D^{(k)}$ maps $W_F^{k,p}(\mathbb{R}^n)$ boundedly into the closed subspace $B_{k-(n-d)/p}^{p,p}(D; F)$ of $B_{k-(n-d)/p}^{p,p}(D)$. Hence, $\mathcal{R}_D^{(k)}$ is indeed a well-defined, linear and bounded operator in the context of (4.82). Finally, (4.83) is a direct consequence of (4.38), finishing the proof of the theorem. \square

Further information about the version of Besov spaces introduced in Definition 4.8 is contained in our next result.

Proposition 4.10. *Let $D \subseteq \mathbb{R}^n$ be a closed set which is d -Ahlfors regular for some $d \in (0, n)$, and let F be a closed subset of D . Also, fix $k \in \mathbb{N}$ and assume that $\max\{1, n-d\} < p < \infty$. Then*

$$B_{k-(n-d)/p}^{p,p}(D; F) \hookrightarrow \{\dot{f} \in B_{k-(n-d)/p}^{p,p}(D) : \dot{f} = 0 \text{ } \mathcal{H}^d\text{-a.e. on } F\} \quad (4.89)$$

and, moreover,

$$\begin{aligned} & B_{k-(n-d)/p}^{p,p}(D; F) \text{ coincides with the space} \\ & \{\dot{f} \in B_{k-(n-d)/p}^{p,p}(D) : \dot{f} = 0 \text{ } \mathcal{H}^d\text{-a.e. on } F\} \end{aligned} \quad (4.90)$$

whenever the set F is d -Ahlfors regular.

Proof. Let $\dot{f} \in B_{k-(n-d)/p}^{p,p}(D; F)$ be arbitrary. Then $\dot{f} \in B_{k-(n-d)/p}^{p,p}(D)$ and there exists a sequence $\{\varphi_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that if $\dot{\varphi}_j := \{(\partial^\alpha \varphi_j)|_D\}_{|\alpha| \leq k-1}$ for each $j \in \mathbb{N}$ then

$$F \cap \text{supp } \dot{\varphi}_j = \emptyset \text{ for each } j \in \mathbb{N}, \text{ and } \dot{\varphi}_j \longrightarrow \dot{f} \text{ in } B_{k-(n-d)/p}^{p,p}(D) \text{ as } j \rightarrow \infty. \quad (4.91)$$

In particular, for each $j \in \mathbb{N}$ we have $\dot{\varphi}_j = 0$ at \mathcal{H}^d -a.e. point on F which, in concert with $\lim_{j \rightarrow \infty} \dot{\varphi}_j = \dot{f}$ in $B_{k-(n-d)/p}^{p,p}(D)$ and the fact that $B_{k-(n-d)/p}^{p,p}(D) \hookrightarrow L^p(D, \mathcal{H}^d \llcorner D)$ continuously, implies that $\dot{f} = 0$ at \mathcal{H}^d -a.e. point on $F \subseteq D$. This proves (4.89).

Moving on, make the additional assumption that the set F is d -Ahlfors regular, and consider an arbitrary $\dot{f} \in B_{k-(n-d)/p}^{p,p}(D)$ with the property that $\dot{f} = 0$ at \mathcal{H}^d -a.e. point on F . If we now define $u := \mathcal{E}_D^{(k)} \dot{f}$, then $u \in W^{k,p}(\mathbb{R}^n)$ and $\mathcal{R}_D^{(k)} u = \dot{f}$ at \mathcal{H}^d -a.e. point on D by Theorem 4.3. As a consequence, $\mathcal{R}_F^{(k)} u = 0$ at \mathcal{H}^d -a.e. point on F , hence $u \in W_F^{k,p}(\mathbb{R}^n)$ by Theorem 4.4, given that F is d -Ahlfors regular. In turn, the membership of u to $W_F^{k,p}(\mathbb{R}^n)$ entails, by virtue of (4.82) that $\dot{f} = \mathcal{R}_D^{(k)} u$ belongs to $B_{k-(n-d)/p}^{p,p}(D; F)$. In concert with (4.89), this proves that the claim made in (4.90) holds. \square

5 Trace/Extension theory on locally (ε, δ) -domains onto/from Ahlfors regular subsets

The first goal in this section is to extend the scope of Theorems 4.3-4.4 by proving results similar in spirit but formulated in a domain Ω in place of the entire ambient \mathbb{R}^n . This is done in Theorem 5.1, where appropriate versions of the Jonsson-Wallin restriction and extension operators in locally (ε, δ) -domains are introduced and studied. To facilitate the reading of this result, the reader is advised to recall the version of Besov spaces introduced in Definition 4.8.

Theorem 5.1 (Trace/Extension theory on locally (ε, δ) -domains). *Assume that $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ are such that D is closed and Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. In addition, suppose that Σ is a closed subset of $\overline{\Omega}$ which is d -Ahlfors regular, for some $d \in (0, n)$. Finally, set $\sigma := \mathcal{H}^d \llcorner \Sigma$ and fix $k \in \mathbb{N}$ along with some p satisfying $\max\{1, n-d\} < p < \infty$.*

Then for every $u \in W_D^{k,p}(\Omega)$ the function given by

$$(\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} u)(x) := \left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha v d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} \text{ at } \sigma\text{-a.e. } x \in \Sigma, \quad (5.1)$$

is meaningfully and unambiguously defined whenever

$$v \in W_D^{k,p}(\mathbb{R}^n) \text{ is such that } v|_\Omega = u \quad (5.2)$$

(the existence of such functions being guaranteed by (3.64) in Corollary 3.10). Moreover,

$$\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} : W_D^{k,p}(\Omega) \longrightarrow \{\dot{f} \in B_{k-(n-d)/p}^{p,p}(\Sigma) : \dot{f} = 0 \text{ } \sigma\text{-a.e. on } \Sigma \cap D\} \quad (5.3)$$

is a well-defined, linear and bounded operator, whose range is contained in $B_{k-(n-d)/p}^{p,p}(\Sigma; D \cap \Sigma)$, and whose null-space is precisely $W_{D \cup \Sigma}^{k,p}(\Omega)$, i.e.,

$$\left\{ u \in W_D^{k,p}(\Omega) : \mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} u = 0 \text{ at } \mathcal{H}^d\text{-a.e. point on } \Sigma \right\} = W_{D \cup \Sigma}^{k,p}(\Omega). \quad (5.4)$$

Moreover, under the additional assumption that

$$D \subseteq \Sigma, \quad (5.5)$$

if one defines

$$\mathcal{E}_{\Sigma \rightarrow \Omega}^{(k)} \dot{f} := \left(\mathcal{E}_{\Sigma}^{(k)} \dot{f} \right) \Big|_{\Omega} \text{ for each } \dot{f} \in B_{k-(n-d)/p}^{p,p}(\Sigma) \quad (5.6)$$

(where $\mathcal{E}_{\Sigma}^{(k)}$ denotes the operator $\mathcal{E}_D^{(k)}$ from Theorem 4.3 corresponding to $D := \Sigma$), then

$$\mathcal{E}_{\Sigma \rightarrow \Omega}^{(k)} : B_{k-(n-d)/p}^{p,p}(\Sigma; D) \longrightarrow W_D^{k,p}(\Omega) \text{ linearly and boundedly,} \quad (5.7)$$

and

$$\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} \circ \mathcal{E}_{\Sigma \rightarrow \Omega}^{(k)} = I, \text{ the identity on } B_{k-(n-d)/p}^{p,p}(\Sigma; D). \quad (5.8)$$

As a corollary, whenever (5.5) holds, the operator

$$\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} : W_D^{k,p}(\Omega) \longrightarrow B_{k-(n-d)/p}^{p,p}(\Sigma; D) \text{ is surjective.} \quad (5.9)$$

Finally, if actually D is a d -Ahlfors regular subset of Σ , then $B_{k-(n-d)/p}^{p,p}(\Sigma; D)$ may be replaced everywhere above by $\{\dot{f} \in B_{k-(n-d)/p}^{p,p}(\Sigma) : \dot{f} = 0 \text{ } \sigma\text{-a.e. on } D\}$.

Proof. The fact that for every $v \in W_D^{k,p}(\mathbb{R}^n) \subseteq W^{k,p}(\mathbb{R}^n)$ the vector-valued limit in (5.1) exists at σ -a.e. point in Σ is contained in Theorem 4.3. Consider now the task of proving that the definition of the higher-order restriction operator from (5.1) does not depend on the extension of $u \in W_D^{k,p}(\Omega)$ to a function v in $W_D^{k,p}(\mathbb{R}^n)$. With this goal in mind assume that $u \in W_D^{k,p}(\Omega)$ has been given and suppose that $v_1, v_2 \in W_D^{k,p}(\mathbb{R}^n)$ are such that $v_1|_{\Omega} = v_2|_{\Omega} = u$. Then the function $v := v_1 - v_2$ satisfies

$$v \in W_D^{k,p}(\mathbb{R}^n) \text{ and } v|_{\Omega} = 0. \quad (5.10)$$

Since by design $\{\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n) : D \cap \text{supp } \varphi = \emptyset\}$ is dense in $W_D^{k,p}(\mathbb{R}^n)$, it is possible to select a sequence

$$\begin{aligned} \{\varphi_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ such that } D \cap \text{supp } \varphi_j = \emptyset \text{ for every } j \in \mathbb{N}, \\ \text{and } \varphi_j \longrightarrow v \text{ in } W^{k,p}(\mathbb{R}^n) \text{ as } j \rightarrow \infty. \end{aligned} \quad (5.11)$$

Then, by (5.10)-(5.11) and part (4) in Lemma 3.2, we have

$$\begin{aligned} \varphi_j|_{\Omega} \in W_D^{k,q}(\Omega) \text{ for each } j \in \mathbb{N} \text{ and } q \in [1, \infty], \\ \text{and } \varphi_j|_{\Omega} \longrightarrow 0 \text{ in } W^{k,p}(\Omega) \text{ as } j \rightarrow \infty. \end{aligned} \quad (5.12)$$

Next, recall the extension operator $\mathfrak{E}_{k,D}$ from Theorem 3.7 (relative to Ω) and, for each $j \in \mathbb{N}$, introduce

$$w_j := \varphi_j - \mathfrak{E}_{k,D}(\varphi_j|_{\Omega}) \text{ in } \mathbb{R}^n. \quad (5.13)$$

Thanks to (5.13), (5.12), and Theorem 3.7, for each $j \in \mathbb{N}$ we have

$$w_j \in W_D^{k,q}(\mathbb{R}^n) \hookrightarrow W^{k,q}(\mathbb{R}^n) \text{ for every } q \in [1, \infty], \text{ and } w_j|_{\Omega} = 0. \quad (5.14)$$

In concert with standard embedding results, this further entails

$$w_j \in \mathcal{C}^{k-1}(\mathbb{R}^n) \quad \text{for each } j \in \mathbb{N}. \quad (5.15)$$

Thus, for each $j \in \mathbb{N}$ we may compute

$$\partial^\alpha w_j = 0 \quad \text{everywhere on } \overline{\Omega}, \quad \text{for all } \alpha \in \mathbb{N}_0^n \quad \text{with } |\alpha| \leq k-1, \quad (5.16)$$

using (4.31), (5.16), and (5.15). In particular, given that $\Sigma \subseteq \overline{\Omega}$, for each $j \in \mathbb{N}$ we have

$$\begin{aligned} (\mathcal{R}_\Sigma^{(k)} w_j)(x) &= \left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha w_j d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} \\ &= \{(\partial^\alpha w_j)(x)\}_{|\alpha| \leq k-1} \\ &= (0, \dots, 0) \quad \text{at every } x \in \Sigma. \end{aligned} \quad (5.17)$$

Next, for every $j \in \mathbb{N}$ we may estimate

$$\begin{aligned} \|v - w_j\|_{W^{k,p}(\mathbb{R}^n)} &\leq \|v - \varphi_j\|_{W^{k,p}(\mathbb{R}^n)} + \|\mathfrak{E}_{k,D}(\varphi_j|_\Omega)\|_{W^{k,p}(\mathbb{R}^n)} \\ &\leq \|v - \varphi_j\|_{W^{k,p}(\mathbb{R}^n)} + C\|\varphi_j|_\Omega\|_{W^{k,p}(\Omega)}. \end{aligned} \quad (5.18)$$

Consequently, by (5.18), (5.11), and (5.12),

$$w_j \longrightarrow v \quad \text{in } W^{k,p}(\mathbb{R}^n) \quad \text{as } j \rightarrow \infty, \quad (5.19)$$

hence, further, using the boundedness of (4.32) and (5.17),

$$\mathcal{R}_\Sigma^{(k)} v = \lim_{j \rightarrow \infty} \mathcal{R}_\Sigma^{(k)} w_j = 0 \quad \text{in } B_{k-(n-d)/p}^{p,p}(\Sigma). \quad (5.20)$$

In turn, the fact that $\mathcal{R}_\Sigma^{(k)} v = 0$ in $B_{k-(n-d)/p}^{p,p}(\Sigma)$ forces $\mathcal{R}_\Sigma^{(k)} v_1 = \mathcal{R}_\Sigma^{(k)} v_2$ in $B_{k-(n-d)/p}^{p,p}(\Sigma) \hookrightarrow L^p(\Sigma, \sigma)$, hence σ -a.e. on Σ . In light of (4.31), this implies

$$\left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha v_1 d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} = \left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha v_2 d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} \quad \text{at } \sigma\text{-a.e. } x \in \Sigma. \quad (5.21)$$

This finishes the proof of the fact that the higher-order restriction operator (5.1)-(5.2) is meaningfully and unambiguously defined for each $u \in W^{k,p}(\Omega)$. Subsequently, this shows that it is also linear.

To prove that this operator is bounded in the context of (5.3), recall the extension operator $\mathfrak{E}_{k,D}$ from Theorem 3.7 (relative to Ω). Then given any $u \in W_D^{k,p}(\Omega)$ we have $\mathfrak{E}_{k,D} u \in W_D^{k,p}(\mathbb{R}^n) \subseteq W_{D \cap \Sigma}^{k,p}(\mathbb{R}^n)$ and $(\mathfrak{E}_{k,D} u)|_\Omega = u$. Since $D \cap \Sigma$ is a closed subset of the d -Ahlfors regular set Σ , these conditions and Theorem 4.9 imply that $\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} u = \mathcal{R}_\Sigma^{(k)}(\mathfrak{E}_{k,D} u) \in B_{k-(n-d)/p}^{p,p}(\Sigma; D \cap \Sigma)$, plus a naturally accompanying estimate. Granted this, a reference to (4.89) then proves that the restriction operator in the context of (5.3) is a well-defined, linear and bounded operator, whose range is contained in $B_{k-(n-d)/p}^{p,p}(\Sigma; D \cap \Sigma)$.

Turning to the task of justifying the right-to-left inclusion in (5.4), consider an arbitrary function $u \in W_{D \cup \Sigma}^{k,p}(\Omega)$. From part (8) in Lemma 3.2 we know that $u \in W_D^{k,p}(\Omega)$. Moreover, there exists a sequence $\{\varphi_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} (D \cup \Sigma) \cap \text{supp } \varphi_j &= \emptyset \quad \text{for every } j \in \mathbb{N}, \\ \text{and } \varphi_j|_\Omega &\longrightarrow u \quad \text{in } W^{k,p}(\Omega) \quad \text{as } j \rightarrow \infty. \end{aligned} \quad (5.22)$$

As a consequence of this and the boundedness of (5.3), we have

$$\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} u = \lim_{j \rightarrow \infty} \mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)}(\varphi_j|_\Omega) \quad \text{in } B_{k-(n-d)/p}^{p,p}(\Sigma) \hookrightarrow L^p(\Sigma, \sigma). \quad (5.23)$$

Given that for each $j \in \mathbb{N}$ we have $\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)}(\varphi_j|_{\Omega}) = \{\partial^{\alpha} \varphi_j\}_{|\alpha| \leq k-1} = (0, \dots, 0)$ everywhere on Σ , it follows that $\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} u = 0$ at σ -a.e. point on Σ . This places u in the left-hand side of (5.4), as desired.

Consider next the left-to-right inclusion in (5.4). In this regard, suppose that $u \in W_D^{k,p}(\Omega)$ is such that $\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} u = 0$ at \mathcal{H}^d -a.e. point on Σ . With $\mathfrak{E}_{k,D}$ denoting the extension operator from Theorem 3.7 relative to Ω , define $v := \mathfrak{E}_{k,D} u$ and note that, thanks to Theorem 3.7,

$$v \in W_D^{k,p}(\mathbb{R}^n) \quad \text{and} \quad v|_{\Omega} = u. \quad (5.24)$$

Furthermore, $(\mathcal{R}_{\Sigma}^{(k)} v)(x) = (\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} u)(x) = 0$ at σ -a.e. $x \in \Sigma$, by (5.1)-(5.2), (4.31), and our assumptions on u . Based on this and Corollary 4.5, we may then conclude that, on the one hand,

$$\mathcal{R}_{\Sigma}^{(k)} v = 0 \quad \text{quasi-everywhere on } \Sigma. \quad (5.25)$$

On the other hand, the membership of v to $W_D^{k,p}(\mathbb{R}^n)$ entails, in light of (4.19), that

$$\mathcal{R}_D^{(k)} v = 0 \quad \text{quasi-everywhere on } D. \quad (5.26)$$

Collectively, (5.25)-(5.26) now imply that the function $v \in W^{k,p}(\mathbb{R}^n)$ satisfies

$$\mathcal{R}_{D \cup \Sigma}^{(k)} v = 0 \quad \text{quasi-everywhere on } D \cup \Sigma. \quad (5.27)$$

As such, $u = v|_{\Omega}$ belongs to $W_{D \cup \Sigma}^{k,p}(\Omega)$, by (4.15). This finishes the justification of (5.4).

For the remainder of the proof make the additional assumption that (5.5) holds. To proceed, pick an arbitrary $\dot{f} \in B_{k-(n-d)/p}^{p,p}(\Sigma; D)$ and set $v := \mathcal{E}_{\Sigma}^{(k)} \dot{f}$ in \mathbb{R}^n . Then Theorem 4.9 gives

$$v \in W_D^{k,p}(\mathbb{R}^n), \quad \mathcal{R}_{\Sigma}^{(k)} v = \dot{f}, \quad \text{and} \quad \|v\|_{W^{k,p}(\mathbb{R}^n)} \leq C \|\dot{f}\|_{B_{k-(n-d)/p}^{p,p}(\Sigma)}, \quad (5.28)$$

for some finite constant $C > 0$ independent of \dot{f} . Based on this and part (4) in Lemma 3.2 we deduce that $v|_{\Omega} \in W_D^{k,p}(\Omega)$ and $\|v|_{\Omega}\|_{W^{k,p}(\Omega)} \leq \|v\|_{W^{k,p}(\mathbb{R}^n)}$. The above argument shows that $\mathcal{E}_{\Sigma \rightarrow \Omega}^{(k)} \dot{f} := (\mathcal{E}_{\Sigma}^{(k)} \dot{f})|_{\Omega}$ belongs to $W_D^{k,p}(\Omega)$ and $\|\mathcal{E}_{\Sigma \rightarrow \Omega}^{(k)} \dot{f}\|_{W^{k,p}(\Omega)} \leq C \|\dot{f}\|_{B_{k-(n-d)/p}^{p,p}(\Sigma)}$ for some constant independent of \dot{f} . Hence, the operator in (5.7) is well-defined, linear, and bounded.

To show that (5.8) holds, for every $\dot{f} \in B_{k-(n-d)/p}^{p,p}(\Sigma; D)$ we write

$$\mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} \left(\mathcal{E}_{\Sigma \rightarrow \Omega}^{(k)} \dot{f} \right) = \mathcal{R}_{\Omega \rightarrow \Sigma}^{(k)} \left((\mathcal{E}_{\Sigma}^{(k)} \dot{f})|_{\Omega} \right) = \mathcal{R}_{\Sigma}^{(k)} \left(\mathcal{E}_{\Sigma}^{(k)} \dot{f} \right) = \dot{f}, \quad (5.29)$$

by (5.6), (5.1)-(5.2) and (4.83).

Finally, the claim in (5.9) is a direct consequence of (5.8), while the very last claim in the statement of the theorem follows from Proposition 4.10. \square

We now proceed to record several basic consequences of Theorem 5.1, starting with the following result which provides an intrinsic characterization of the Sobolev spaces from Definition 3.1 considered in (ε, δ) -domains.

Theorem 5.2 (Intrinsic description of spaces on domains). *Assume that Ω is an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$, and that D is a closed subset of $\overline{\Omega}$ which is d -Ahlfors regular for some $d \in (0, n)$. In addition, fix $k \in \mathbb{N}$ and suppose that $\max\{1, n-d\} < p < \infty$. Then*

$$W_D^{k,p}(\Omega) = \left\{ u \in W^{k,p}(\Omega) : \mathcal{R}_{\Omega \rightarrow D}^{(k)} u = 0 \quad \text{at } \mathcal{H}^d\text{-a.e. point on } D \right\}. \quad (5.30)$$

Proof. This follows from (5.4), specialized to the case when $D := \emptyset$ and Σ playing the role of the current set D , in combination with (3.9) and Lemma 3.3. \square

Another useful application of Theorem 5.1 is presented in the next corollary.

Corollary 5.3. *Assume that $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ are such that D is closed and d -Ahlfors regular for some $d \in (0, n)$, while Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. Then, whenever $k \in \mathbb{N}$ and $\max\{1, n - d\} < p < \infty$, one has*

$$\mathcal{R}_{\Omega \rightarrow D}^{(k)} u = 0 \quad \text{at } \mathcal{H}^d\text{-a.e. point on } D, \quad \text{for each } u \in W_D^{k,p}(\Omega). \quad (5.31)$$

Proof. This is a particular case of (5.3), considered here with $\Sigma = D$. \square

Finally, it is of independent interest to state the version of Theorem 5.1 corresponding to the case of genuinely (ε, δ) -domains, given its potential for applications and since in such a scenario the conclusions have a more streamlined format. We do so in the corollary below.

Corollary 5.4. *Let Ω be an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$ and such that $\partial\Omega$ is d -Ahlfors regular for some $d \in (0, n)$. Also, fix some $k \in \mathbb{N}$ along with p satisfying $\max\{1, n - d\} < p < \infty$. Then for every $u \in W^{k,p}(\Omega)$ one has*

$$\left(\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} u \right)(x) = \left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha v \, d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} \quad \text{at } \mathcal{H}^d\text{-a.e. } x \in \partial\Omega, \quad (5.32)$$

for every function $v \in W^{k,p}(\mathbb{R}^n)$ satisfying $v|_\Omega = u$. Moreover,

$$\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} : W^{k,p}(\Omega) \longrightarrow B_{k-(n-d)/p}^{p,p}(\partial\Omega) \quad (5.33)$$

is a well-defined, linear and bounded operator. Also, if

$$\mathcal{E}_{\partial\Omega \rightarrow \Omega}^{(k)} \dot{f} := \left(\mathcal{E}_{\partial\Omega}^{(k)} \dot{f} \right)|_\Omega \quad \text{for each } \dot{f} \in B_{k-(n-d)/p}^{p,p}(\partial\Omega), \quad (5.34)$$

(where $\mathcal{E}_{\partial\Omega}^{(k)}$ denotes the operator $\mathcal{E}_D^{(k)}$ from Theorem 4.3 corresponding to $D := \partial\Omega$), then

$$\mathcal{E}_{\partial\Omega \rightarrow \Omega}^{(k)} : B_{k-(n-d)/p}^{p,p}(\partial\Omega) \longrightarrow W^{k,p}(\Omega) \quad \text{linearly and boundedly,} \quad (5.35)$$

and

$$\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} \circ \mathcal{E}_{\partial\Omega \rightarrow \Omega}^{(k)} = I, \quad \text{the identity on } B_{k-(n-d)/p}^{p,p}(\partial\Omega). \quad (5.36)$$

Finally, the restriction operator $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)}$ from (5.33) is surjective, and its null-space is $\mathring{W}^{k,p}(\Omega)$, i.e.,

$$\left\{ u \in W^{k,p}(\Omega) : \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} u = 0 \quad \text{at } \mathcal{H}^d\text{-a.e. point on } \partial\Omega \right\} = \mathring{W}^{k,p}(\Omega). \quad (5.37)$$

In particular,

$$\mathcal{C}_c^\infty(\Omega) \hookrightarrow \left\{ u \in W^{k,p}(\Omega) : \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} u = 0 \quad \text{at } \mathcal{H}^d\text{-a.e. point on } \partial\Omega \right\} \quad \text{densely.} \quad (5.38)$$

Proof. All claims up to, and including, (5.37) are direct consequences of Theorem 5.1 specialized to the case when $\Sigma := \partial\Omega$ and $D := \emptyset$, keeping in mind (3.9) and (5) in Lemma 3.2. Finally, (5.38) is immediate from (5.37) and (2.4). \square

Remark 5.5. *Suppose that Ω be an (ε, δ) -domain in \mathbb{R}^n with the property that $\partial\Omega$ is $(n-1)$ -Ahlfors regular, and set $\sigma := \mathcal{H}^{n-1} \llcorner \partial\Omega$. Also, fix $k \in \mathbb{N}$ and $p \in (1, \infty)$. Then for every $u \in W^{k,p}(\Omega)$ the vector-valued limit*

$$\left\{ \lim_{r \rightarrow 0^+} \frac{1}{\mathcal{L}^n(\Omega \cap B(x,r))} \int_{\Omega \cap B(x,r)} \partial^\alpha u \, d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} \quad (5.39)$$

exists and equals $(\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)}u)(x)$ at σ -a.e. $x \in \partial\Omega$. This is a consequence of [26, Proposition 2, p. 206]. In turn, the applicability of the latter result in the present context is ensured by Theorem 2.1 and the observation that the set $\overline{\Omega}$ is n -Ahlfors regular (as seen from an inspection of the proof of [25, Lemma 2.3, p. 77] which actually reveals that Ω has the interior corkscrew property, in the sense of Jerison-Kenig [24]).

Remark 5.6. Theorem 5.2 and Corollary 5.4 deal with the class of (ε, δ) -domains in \mathbb{R}^n whose boundaries are d -Ahlfors regular for some $d \in (0, n)$. While there are many examples of such domains when $d \in [n-1, n)$ (for example Lipschitz domains, in which case $d = n-1$, and certain fractal sets like a multi-dimensional analogue of the von Koch snowflake, in which case matters can be arranged for d to be any desired number in $(n-1, n)$) we wish to note that (ε, δ) -domains having a d -Ahlfors regular boundary with $d \in (0, n-1)$ also occur naturally. For example, one may readily verify that for any affine subspace H of \mathbb{R}^n of dimension $d \leq n-2$ the set $\Omega := \mathbb{R}^n \setminus H$ is a (ε, ∞) -domain for some $\varepsilon > 0$ whose boundary, H , is d -Ahlfors regular (indeed, given any two points $x, y \in \mathbb{R}^n \setminus H$, the semi-circular path γ joining them, having $|x-y|$ as diameter, and which is contained in a plane perpendicular on the affine variety spanned by H and the line passing through x, y , satisfies (1.6) for some $c = c(n) > 0$).

In the second part of this section we shall employ the trace/extension theory on locally (ε, δ) -domains onto/from Ahlfors regular subsets developed in Theorem 5.1 and its corollaries in order to derive several important properties of the Sobolev spaces with partially vanishing traces considered in Definition 3.1. First, we shall use the characterization (5.37) as the key ingredient in the proof of the following theorem.

Theorem 5.7 (Hereditary property). *Let Ω be an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$, and consider a closed subset D of $\overline{\Omega}$ which is d -Ahlfors regular, for some $d \in (0, n)$. Then for each $k, m \in \mathbb{N}$ and p such that $\max\{1, n-d\} < p < \infty$ one has*

$$W_D^{k+m,p}(\Omega) = \left\{ u \in W^{k+m,p}(\Omega) \cap W_D^{k,p}(\Omega) : \partial^\gamma u \in W_D^{m,p}(\Omega), \forall \gamma \in \mathbb{N}_0^n, |\gamma| = k \right\}. \quad (5.40)$$

Proof. Let $u \in W_D^{k+m,p}(\Omega)$. Then clearly

$$u \in W^{k+m,p}(\Omega) \cap W_D^{k,p}(\Omega). \quad (5.41)$$

In addition, using the definition of the space $W_D^{k+m,p}(\Omega)$ it follows that there exists a sequence of functions $\{\varphi_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $D \cap \text{supp} \varphi_j = \emptyset$ for each $j \in \mathbb{N}$, and $\varphi_j|_\Omega \rightarrow u$ in $W^{k+m,p}(\Omega)$ as $j \rightarrow \infty$. In particular, for every $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = k$ there holds

$$(\partial^\gamma \varphi_j)|_\Omega = \partial^\gamma(\varphi_j|_\Omega) \longrightarrow \partial^\gamma u \text{ in } W^{m,p}(\Omega) \text{ as } j \rightarrow \infty. \quad (5.42)$$

Since for each $j \in \mathbb{N}$ and each $\gamma \in \mathbb{N}_0^n$ we have $\partial^\gamma \varphi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $D \cap \text{supp}(\partial^\gamma \varphi_j) = \emptyset$, (5.42) and the definition of $W_D^{m,p}(\Omega)$ guarantee that

$$\partial^\gamma u \in W_D^{m,p}(\Omega), \quad \forall \gamma \in \mathbb{N}_0^n \text{ such that } |\gamma| = k. \quad (5.43)$$

Combining (5.43) and (5.41) we obtain that the left-to-right inclusion in (5.40) holds. Parenthetically, we wish to note that this portion of the proof works for any nonempty open subset Ω of \mathbb{R}^n and any closed set $D \subseteq \overline{\Omega}$.

There remains to establish the right-to-left inclusion in (5.40), which makes full use of the assumptions on Ω and D stipulated in the statement of the theorem. To this end, pick a function u such that

$$u \in W^{m+k,p}(\Omega) \cap W_D^{k,p}(\Omega) \text{ and } \partial^\gamma u \in W_D^{m,p}(\Omega), \forall \gamma \in \mathbb{N}_0^n, |\gamma| = k. \quad (5.44)$$

Keeping in mind that $u \in W^{k+m,p}(\Omega)$, it follows from (4.21) and (5.30) that the membership of u to $W_D^{k+m,p}(\Omega)$ is equivalent to

$$\mathcal{R}_{\Omega \rightarrow D}^{(1)}[\partial^\alpha u] = 0 \text{ } \mathcal{H}^d\text{-a.e. on } D, \forall \alpha \in \mathbb{N}_0^n \text{ such that } |\alpha| \leq m+k-1. \quad (5.45)$$

With the goal of proving (5.45), first notice that, on the one hand, the last condition in (5.44) implies (thanks to (4.21) and (5.30)) that

$$\mathcal{R}_{\Omega \rightarrow D}^{(1)}[\partial^\beta(\partial^\gamma u)] = 0 \quad \mathcal{H}^d\text{-a.e. on } D, \quad \forall \beta, \gamma \in \mathbb{N}_0^n \text{ such that } |\beta| \leq m-1 \text{ and } |\gamma| = k, \quad (5.46)$$

whereupon

$$\mathcal{R}_{\Omega \rightarrow D}^{(1)}[\partial^\alpha u] = 0 \quad \mathcal{H}^d\text{-a.e. on } D, \quad \forall \alpha \in \mathbb{N}_0^n \text{ such that } |\alpha| \in \{k, \dots, m+k-1\}. \quad (5.47)$$

On the other hand, the first condition in (5.44) ensures that $u \in W_D^{k,p}(\Omega)$, and thus, by once again appealing to (4.21) and (5.30),

$$\mathcal{R}_{\Omega \rightarrow D}^{(1)}[\partial^\alpha u] = 0 \quad \mathcal{H}^d\text{-a.e. on } D, \quad \forall \alpha \in \mathbb{N}_0^n \text{ such that } |\alpha| \in \{0, \dots, k-1\}. \quad (5.48)$$

Altogether, (5.47) and (5.48) prove that (5.45) holds, as desired. This shows that if u is as in (5.44) then $u \in W_D^{k+m,p}(\Omega)$. Thus, the right-to-left inclusion in (5.40) holds as well, finishing the proof of the theorem. \square

The following consequence of Theorem 5.7 answers a question posed to us by D. Arnold (in the more specialized setting of Lipschitz domains a solution has been given in [38]).

Corollary 5.8. *Suppose Ω is an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$ and such that $\partial\Omega$ is d -Ahlfors regular for some $d \in (0, n)$. Then*

$$\mathring{W}^{k+m,p}(\Omega) = \left\{ u \in W^{k+m,p}(\Omega) \cap \mathring{W}^{k,p}(\Omega) : \partial^\gamma u \in \mathring{W}^{m,p}(\Omega), \quad \forall \gamma \in \mathbb{N}_0^n, \quad |\gamma| = k \right\}, \quad (5.49)$$

whenever $k, m \in \mathbb{N}$ and $\max\{1, n-d\} < p < \infty$.

Proof. Formula (5.49) is a direct consequence of (5.40) (with $D := \partial\Omega$) and part (5) in Lemma 3.2 \square

Moving on to a different, yet related topic, for every nonempty open subset Ω of \mathbb{R}^n and any $k \in \mathbb{N}$, $p \in [1, \infty]$, let us now introduce the following brand of Sobolev space,

$$\widetilde{W}^{k,p}(\Omega) := \{u \in W^{k,p}(\Omega) : \tilde{u} \in W^{k,p}(\mathbb{R}^n)\}, \quad (5.50)$$

where, as in the past, for any function u defined in Ω we have set

$$\tilde{u} := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \Omega^c := \mathbb{R}^n \setminus \Omega. \end{cases} \quad (5.51)$$

Also, equip the space $\widetilde{W}^{k,p}(\Omega)$ with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$. For the time being, we note the following elementary lemma.

Lemma 5.9. *Suppose that Ω is an arbitrary nonempty open subset of \mathbb{R}^n , and fix $k \in \mathbb{N}$ along with $p \in [1, \infty]$. Then*

$$\mathring{W}^{k,p}(\Omega) \hookrightarrow \widetilde{W}^{k,p}(\Omega) \hookrightarrow W^{k,p}(\Omega) \quad \text{isometrically}, \quad (5.52)$$

and

$$\mathcal{L}^n(\partial\Omega) = 0 \implies \widetilde{W}^{k,p}(\Omega) = \{v|_\Omega : v \in W^{k,p}(\mathbb{R}^n) \text{ with } \text{supp } v \subseteq \overline{\Omega}\}. \quad (5.53)$$

Moreover,

$$\begin{aligned} \mathring{W}^{k,p}(\Omega) = \widetilde{W}^{k,p}(\Omega) &= \{v|_\Omega : v \in W^{k,p}(\mathbb{R}^n) \text{ with } \text{supp } v \subseteq \overline{\Omega}\} \\ &\text{whenever } \partial\Omega = \partial(\overline{\Omega}) \text{ and } p > n. \end{aligned} \quad (5.54)$$

Proof. The first inclusion in (5.52) follows from (2.7) and (5.50), whereas the second one is clear from definitions. With the goal of proving (5.53), first observe that if $u \in \widetilde{W}^{k,p}(\Omega)$ then, by design, $\tilde{u} \in W^{k,p}(\mathbb{R}^n)$, $\tilde{u}|_{\Omega} = u$, and $\text{supp } \tilde{u} \subseteq \overline{\Omega}$. This shows that

$$\text{the inclusion } \widetilde{W}^{k,p}(\Omega) \subseteq \{v|_{\Omega} : v \in W^{k,p}(\mathbb{R}^n) \text{ with } \text{supp } v \subseteq \overline{\Omega}\} \text{ always holds.} \quad (5.55)$$

For the opposite inclusion it is useful to have $\mathcal{L}^n(\partial\Omega) = 0$, a condition we now assume. In this context, suppose that $v \in W^{k,p}(\mathbb{R}^n)$ satisfies $\text{supp } v \subseteq \overline{\Omega}$ and set $u := v|_{\Omega}$. Then clearly $u \in W^{k,p}(\Omega)$ and \tilde{u} coincides with v pointwise \mathcal{L}^n -a.e. in $\mathbb{R}^n \setminus \partial\Omega$, thus ultimately \mathcal{L}^n -a.e. in \mathbb{R}^n , granted the assumption on $\partial\Omega$. As a consequence, \tilde{u} also belongs to $W^{k,p}(\mathbb{R}^n)$, which puts $u = v|_{\Omega}$ in $\widetilde{W}^{k,p}(\Omega)$, as desired.

As far as (5.54) is concerned, assume that $\partial\Omega = \partial(\overline{\Omega})$ and $p > n$. Select an arbitrary $v \in W^{k,p}(\mathbb{R}^n)$ with $\text{supp } v \subseteq \overline{\Omega}$ and note that $v \in \mathcal{C}^{k-1}(\mathbb{R}^n)$ by standard embeddings. In addition, since $v \equiv 0$ on $(\Omega^c)^{\circ}$ it follows that

$$(\partial^{\alpha} v)|_{(\Omega^c)^{\circ}} = 0 \text{ everywhere on } \overline{(\Omega^c)^{\circ}} \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1. \quad (5.56)$$

Let us momentarily digress in order to note that the assumption $\partial\Omega = \partial(\overline{\Omega})$ forces

$$(\overline{\Omega})^{\circ} = \overline{\Omega} \setminus \partial(\overline{\Omega}) = \overline{\Omega} \setminus \partial\Omega = \Omega. \quad (5.57)$$

As such,

$$\overline{(\Omega^c)^{\circ}} = \overline{(\overline{\Omega})^c} = ((\overline{\Omega})^{\circ})^c = \Omega^c \quad (5.58)$$

thus (5.56) becomes

$$(\partial^{\alpha} v)|_{\Omega^c} = 0 \text{ everywhere on } \Omega^c \text{ for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1. \quad (5.59)$$

In particular (cf. (4.22)),

$$\mathcal{R}_{\Omega^c}^{(k)} v = 0 \text{ quasi-everywhere on } \Omega^c, \quad (5.60)$$

therefore $v \in W_{\Omega^c}^{k,p}(\mathbb{R}^n)$ by (4.19). With the help of (9) and (5) in Lemma 3.2 we then further deduce from this that $v|_{\Omega} \in W_{\partial\Omega}^{k,p}(\Omega) = \mathring{W}^{k,p}(\Omega)$. All in all, this argument shows that in the current setting $\{v|_{\Omega} : v \in W^{k,p}(\mathbb{R}^n) \text{ with } \text{supp } v \subseteq \overline{\Omega}\} \subseteq \mathring{W}^{k,p}(\Omega)$. Based on this, the first inclusion in (5.52), and (5.55), it follows that the double equality in (5.54) holds, finishing the proof of the lemma. \square

The issue of the coincidence of the spaces displayed in (5.54) for *all* values of $p \in (1, \infty)$ is addressed in the theorem below. This is accomplished under the assumptions that $\Omega \subseteq \mathbb{R}^n$ is an open set which sits on only one side of its topological boundary and such that the interior of its complement is an (ε, δ) -domain. In particular, these conditions are satisfied if $(\Omega^c)^{\circ}$ is an NTA domain in the sense of [24].

Theorem 5.10 (Extension of Sobolev functions by zero). *Let Ω be a nonempty proper open subset of \mathbb{R}^n with the property that $\partial\Omega = \partial(\overline{\Omega})$ and such that $(\Omega^c)^{\circ}$ is an (ε, δ) -domain with $\text{rad}((\Omega^c)^{\circ}) > 0$. Then for every $k \in \mathbb{N}$ and $p \in (1, \infty)$,*

$$\widetilde{W}^{k,p}(\Omega) = \mathring{W}^{k,p}(\Omega). \quad (5.61)$$

As a corollary, in the current setting the following properties hold:

$$\left(\widetilde{W}^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)} \right) \text{ is a Banach space,} \quad (5.62)$$

$$\widetilde{W}^{k,p}(\Omega) \ni u \longmapsto \tilde{u} \in W^{k,p}(\mathbb{R}^n) \text{ isometrically, and} \quad (5.63)$$

$$\widetilde{W}^{k,p}(\Omega) = \{v|_{\Omega} : v \in W^{k,p}(\mathbb{R}^n) \text{ with } \text{supp } v \subseteq \overline{\Omega}\}. \quad (5.64)$$

Proof. The inclusion $\mathring{W}^{k,p}(\Omega) \subseteq \widetilde{W}^{k,p}(\Omega)$ is contained in (5.52), so the crux of the matter is establishing the opposite one. To this end, let $u \in \widetilde{W}^{k,p}(\Omega)$ be an arbitrary function. Then $\tilde{u} \in W^{k,p}(\mathbb{R}^n)$ and, hence, there exists a sequence $\{v_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n)$ with the property that

$$v_j \longrightarrow \tilde{u} \text{ in } W^{k,p}(\mathbb{R}^n) \text{ as } j \rightarrow \infty. \quad (5.65)$$

In particular,

$$v_j|_{(\Omega^c)^\circ} \longrightarrow \tilde{u}|_{(\Omega^c)^\circ} = 0 \text{ in } W^{k,p}((\Omega^c)^\circ) \text{ as } j \rightarrow \infty. \quad (5.66)$$

To proceed, for each $j \in \mathbb{N}$ consider

$$w_j := v_j - \Lambda_k^c(v_j|_{(\Omega^c)^\circ}) \text{ in } \mathbb{R}^n, \quad (5.67)$$

where Λ_k^c denotes Jones' extension operator for the (ε, δ) -domain $(\Omega^c)^\circ$, which, by assumption, satisfies $\text{rad}((\Omega^c)^\circ) > 0$. Based on (5.67) and Theorem 2.1, for each $j \in \mathbb{N}$ we then have

$$w_j \in W^{k,q}(\mathbb{R}^n) \text{ for each } q \in [1, \infty]. \quad (5.68)$$

Together with standard embedding results, this implies

$$w_j \in \mathcal{C}^{k-1}(\mathbb{R}^n) \text{ for each } j \in \mathbb{N}. \quad (5.69)$$

Furthermore, in light of (5.67) and Theorem 2.1, for every $j \in \mathbb{N}$ we may estimate

$$\begin{aligned} \|\tilde{u} - w_j\|_{W^{k,p}(\mathbb{R}^n)} &\leq \|\tilde{u} - v_j\|_{W^{k,p}(\mathbb{R}^n)} + \left\| \Lambda_k^c(v_j|_{(\Omega^c)^\circ}) \right\|_{W^{k,p}(\mathbb{R}^n)} \\ &\leq \|\tilde{u} - v_j\|_{W^{k,p}(\mathbb{R}^n)} + C \|v_j|_{(\Omega^c)^\circ}\|_{W^{k,p}((\Omega^c)^\circ)}. \end{aligned} \quad (5.70)$$

In turn, this forces

$$w_j \longrightarrow \tilde{u} \text{ in } W^{k,p}(\mathbb{R}^n) \text{ as } j \rightarrow \infty, \quad (5.71)$$

by (5.65) and (5.66). In addition, from (5.67) and the analogue of (2.23) for the domain $(\Omega^c)^\circ$, we conclude that for every $j \in \mathbb{N}$ we have

$$w_j|_{(\Omega^c)^\circ} = 0 \text{ } \mathcal{L}^n\text{-a.e. on } (\Omega^c)^\circ. \quad (5.72)$$

From (5.72) and (5.69) we may now conclude that, for each $j \in \mathbb{N}$,

$$(\partial^\alpha w_j)|_{\overline{(\Omega^c)^\circ}} = 0 \text{ everywhere on } \overline{(\Omega^c)^\circ} \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1. \quad (5.73)$$

Based on this and the fact that $\partial\Omega = \partial(\overline{\Omega})$ implies (5.58), we may ultimately conclude that

$$(\partial^\alpha w_j)|_{\Omega^c} = 0 \text{ everywhere on } \Omega^c \text{ for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1. \quad (5.74)$$

When combined with (4.19) and (5.68), the everywhere vanishing trace condition from (5.74) implies that

$$w_j \in W_{\Omega^c}^{k,p}(\mathbb{R}^n) \text{ for every } j \in \mathbb{N}. \quad (5.75)$$

Finally, from (5.75), (5.71) and the fact that $W_{\Omega^c}^{k,p}(\mathbb{R}^n)$ is a closed subspace of $W^{k,p}(\mathbb{R}^n)$ we deduce that

$$\tilde{u} \in W_{\Omega^c}^{k,p}(\mathbb{R}^n). \quad (5.76)$$

From this and the definition of $W_{\Omega^c}^{k,p}(\mathbb{R}^n)$ it follows that there exists a sequence $\{\varphi_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\Omega)$ with the property that

$$\widetilde{\varphi}_j \longrightarrow \widetilde{u} \text{ in } W^{k,p}(\mathbb{R}^n) \text{ as } j \rightarrow \infty. \quad (5.77)$$

Consequently,

$$\varphi_j = \widetilde{\varphi}_j|_{\Omega} \longrightarrow \widetilde{u}|_{\Omega} = u \text{ in } W^{k,p}(\Omega) \text{ as } j \rightarrow \infty, \quad (5.78)$$

hence, further, $u \in \mathring{W}^{k,p}(\Omega)$ by (2.4). Since the function $u \in \widetilde{W}^{k,p}(\Omega)$ has been arbitrarily chosen, this shows that $\widetilde{W}^{k,p}(\Omega) \subseteq \mathring{W}^{k,p}(\Omega)$ and finishes the proof of (5.61).

Moving on, (5.62) is a direct consequence of (5.61) and (2.4), whereas (5.63) is immediate from (5.61) and (2.7). Finally, as regards (5.64), first note that since $(\Omega^c)^\circ$ is assumed to be an (ε, δ) -domain, (1.5) implies that $\mathcal{L}^n(\partial((\Omega^c)^\circ)) = 0$. On the other hand,

$$\partial((\Omega^c)^\circ) = \partial((\overline{\Omega})^c) = \partial(\overline{\Omega}) = \partial\Omega. \quad (5.79)$$

Hence, $\mathcal{L}^n(\partial\Omega) = 0$, so (5.64) now follows from (5.53). \square

We now propose to study the issue as to whether Sobolev functions defined on either side of the boundary of a domain may be “glued” together with preservation of smoothness. In order to introduce a natural geometrical setting for this type of question we make the following definition.

Definition 5.11. *Call an open, nonempty, proper subset Ω of \mathbb{R}^n a two-sided (ε, δ) -domain provided both Ω and $(\Omega^c)^\circ$ are (ε, δ) -domains, $\text{rad}(\Omega) > 0$, $\text{rad}((\Omega^c)^\circ) > 0$, and $\partial\Omega = \partial(\overline{\Omega})$.*

Note that, as seen from (5.79),

$$\text{any two-sided } (\varepsilon, \delta)\text{-domain } \Omega \text{ satisfies } \partial((\Omega^c)^\circ) = \partial\Omega. \quad (5.80)$$

Examples of two-sided (ε, δ) -domains include the class of bounded Lipschitz domains and, more generally, the class of two-sided NTA domains (by which we mean connected sets which are NTA and whose interior of their complement is also connected and NTA).

Theorem 5.12 below states that gluing Sobolev functions defined inside and outside of a two-sided (ε, δ) -domain preserves Sobolev smoothness if and only if the functions in question have matching traces across the boundary. To facilitate the reading of its statement, the reader is advised to recall the definition and properties of the higher-order restriction map $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)}$ from Corollary 5.4.

Theorem 5.12 (Gluing Sobolev functions with matching traces). *Let Ω be a two-sided (ε, δ) -domain in \mathbb{R}^n with the property that $\partial\Omega$ is d -Ahlfors regular for some $d \in [n-1, n)$. Also, fix $k \in \mathbb{N}$ along with some p such that $\max\{1, n-d\} < p < \infty$. Then for any $u \in W^{k,p}(\Omega)$ and $v \in W^{k,p}((\Omega^c)^\circ)$ the following two conditions are equivalent:*

(i) *the functions u, v have matching traces, i.e.,*

$$\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} u = \mathcal{R}_{(\Omega^c)^\circ \rightarrow \partial\Omega}^{(k)} v \text{ at } \mathcal{H}^d\text{-a.e. point on } \partial\Omega; \quad (5.81)$$

(ii) *the function*

$$w := \begin{cases} u & \text{in } \Omega, \\ v & \text{in } (\Omega^c)^\circ, \end{cases} \quad (5.82)$$

(which is defined \mathcal{L}^n -a.e. in \mathbb{R}^n) belongs to $W^{k,p}(\mathbb{R}^n)$.

Moreover, whenever condition (i) holds and w is defined as in (5.82), one has

$$\|w\|_{W^{k,p}(\mathbb{R}^n)} \leq C \left(\|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}((\Omega^c)^\circ)} \right), \quad (5.83)$$

where $C > 0$ is a finite constant depending only on $n, \varepsilon, \delta, k, p$.

Proof. For starters, observe that thanks to property (5.80) it makes sense to talk about $\mathcal{R}_{(\Omega^c)^\circ \rightarrow \partial\Omega}^{(k)}$.

Consider the implication (i) \Rightarrow (ii). In this regard, pick two functions $u \in W^{k,p}(\Omega)$, $v \in W^{k,p}((\Omega^c)^\circ)$ satisfying (5.81), and consider w as in (5.82). To begin with, this function is defined \mathcal{L}^n -a.e. in

$$\Omega \cup (\Omega^c)^\circ = \Omega \cup \left(\Omega^c \setminus \partial((\Omega^c)^\circ) \right) = \Omega \cup \left(\Omega^c \setminus \partial\Omega \right) = \mathbb{R}^n \setminus \partial\Omega \quad (5.84)$$

thus, ultimately, \mathcal{L}^n -a.e. in \mathbb{R}^n by (1.5) and the fact that Ω is an (ε, δ) -domain in \mathbb{R}^n . Going further, define

$$u_* := (\Lambda_k^c v)|_\Omega \quad \text{in } \Omega, \quad (5.85)$$

where Λ_k^c denotes Jones' extension operator for the (ε, δ) -domain $(\Omega^c)^\circ$. Then there exists a finite constant $C > 0$ depending only on $n, \varepsilon, \delta, k, p$ such that

$$u_* \in W^{k,p}(\Omega) \quad \text{and} \quad \|u_*\|_{W^{k,p}(\Omega)} \leq C \|v\|_{W^{k,p}((\Omega^c)^\circ)}, \quad (5.86)$$

thanks to (2.3) and Theorem 2.1. Furthermore, at \mathcal{H}^d -a.e. $x \in \partial\Omega = \partial((\Omega^c)^\circ)$ we have

$$\begin{aligned} (\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} u_*)(x) &= \left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha (\Lambda_k^c v) d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} \\ &= \left(\mathcal{R}_{(\Omega^c)^\circ \rightarrow \partial\Omega}^{(k)} \left((\Lambda_k^c v)|_{(\Omega^c)^\circ} \right) \right)(x) \\ &= \left(\mathcal{R}_{(\Omega^c)^\circ \rightarrow \partial\Omega}^{(k)} v \right)(x) \\ &= \left(\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} u \right)(x), \end{aligned} \quad (5.87)$$

by (5.32) in Corollary 5.4 (used twice), the analogue of (2.23) for $(\Omega^c)^\circ$, and (5.81). As a result, the function $u - u_* \in W^{k,p}(\Omega)$ satisfies $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)}(u - u_*) = 0$ at \mathcal{H}^d -a.e. point in $\partial\Omega$. Hence, $u - u_* \in \mathring{W}^{k,p}(\Omega)$ by (5.37). Consequently, from this, (5.61) and (5.86) we deduce that

$$\widetilde{u - u_*} \in W^{k,p}(\mathbb{R}^n) \quad \text{and} \quad \|\widetilde{u - u_*}\|_{W^{k,p}(\mathbb{R}^n)} \leq C \left(\|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}((\Omega^c)^\circ)} \right), \quad (5.88)$$

where $C > 0$ is a finite constant depending only on $n, \varepsilon, \delta, k, p$. Thus, if we now introduce

$$w_* := \widetilde{u - u_*} + \Lambda_k^c v \quad \text{in } \mathbb{R}^n, \quad (5.89)$$

it follows from (5.89), (5.88) and Theorem 2.1 that

$$w_* \in W^{k,p}(\mathbb{R}^n) \quad \text{and} \quad \|w_*\|_{W^{k,p}(\mathbb{R}^n)} \leq C \left(\|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}((\Omega^c)^\circ)} \right), \quad (5.90)$$

where $C > 0$ is a finite constant depending only on $n, \varepsilon, \delta, k, p$. Moreover, from (5.89) and (5.85) we have

$$w_*|_\Omega = u - u_* + (\Lambda_k^c v)|_\Omega = u - u_* + u_* = u \quad \mathcal{L}^n\text{-a.e. in } \Omega, \quad (5.91)$$

whereas (5.89) and the analogue of (2.23) for $(\Omega^c)^\circ$ we obtain

$$w_*|_{(\Omega^c)^\circ} = 0 + (\Lambda_k^c v)|_{(\Omega^c)^\circ} = v \quad \mathcal{L}^n\text{-a.e. in } (\Omega^c)^\circ. \quad (5.92)$$

Thus, $w_* = w$ \mathcal{L}^n -a.e. in \mathbb{R}^n , by (5.91)-(5.92), (5.82), (5.84), and (1.5). With this in hand, the fact that w belongs to $W^{k,p}(\mathbb{R}^n)$ and satisfies (5.83) follows from (5.90). This concludes the proof of the implication (i) \Rightarrow (ii) and also justifies the last claim in the statement of the theorem.

There remains to show that the implication (ii) \Rightarrow (i) also holds. To this end, suppose that $u \in W^{k,p}(\Omega)$, $v \in W^{k,p}((\Omega^c)^\circ)$ are such that the function w defined as in (5.82) belongs to $W^{k,p}(\mathbb{R}^n)$. Since, by design, $w|_\Omega = u$ and $w|_{(\Omega^c)^\circ} = v$, condition (5.81) follows by writing

$$(\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(k)} u)(x) = \left\{ \lim_{r \rightarrow 0^+} \int_{B(x,r)} \partial^\alpha w d\mathcal{L}^n \right\}_{|\alpha| \leq k-1} = (\mathcal{R}_{(\Omega^c)^\circ \rightarrow \partial\Omega}^{(k)} v)(x) \quad (5.93)$$

at \mathcal{H}^d -a.e. point x in $\partial\Omega = \partial((\Omega^c)^\circ)$, thanks to (a two-fold application of) Corollary 5.4. Hence (ii) \Rightarrow (i) also holds, completing the proof of the theorem. \square

A word of clarification regarding the statement of the above theorem is in order. A cursory inspection of the proof of Theorem 5.12 reveals that, in principle, the argument carries through under the less stringent demand that the Ahlfors regularity dimension d of $\partial\Omega$ belongs to the interval $(0, n)$. However, if $d \in (0, n-1)$ then the d -Ahlfors regular set $\partial\Omega$ has $(n-1)$ -dimensional Hausdorff measure zero and, as such, well-known removability results (cf., e.g., [2, Lemma 9.1.10, p. 237]) give that the function w defined as in (5.82) automatically belongs to $W^{k,p}(\mathbb{R}^n)$ irrespective of the choice of the functions $u \in W^{k,p}(\Omega)$ and $v \in W^{k,p}((\Omega^c)^\circ)$. Now, on the one hand, such functions may be constructed with arbitrary traces on $B_{k-(n-d)/p}^{p,p}(\partial\Omega)$ (by Corollary 5.4), and this Besov space is nontrivial. On the other hand, as already mentioned, the equivalence (i) \Leftrightarrow (ii) in Theorem 5.12 continues to hold for $d \in (0, n-1)$ as well. This contradiction shows that there is no two-sided (ε, δ) -domain Ω in \mathbb{R}^n whose boundary is d -Ahlfors regular for some $d \in (0, n-1)$.

The last result in this section requires certain notions from geometric measure theory (for standard terminology and basic results in this area the reader is referred to the informative discussion in [15]). Specifically, assume now that Ω is an open subset of \mathbb{R}^n which is of locally finite perimeter. Recall that the measure-theoretic boundary $\partial_*\Omega$ of the set Ω is defined by

$$\partial_*\Omega := \left\{ x \in \partial\Omega : \limsup_{r \rightarrow 0} \frac{\mathcal{L}^n(B(x,r) \cap \Omega)}{r^n} > 0 \text{ and } \limsup_{r \rightarrow 0} \frac{\mathcal{L}^n(B(x,r) \setminus \Omega)}{r^n} > 0 \right\}. \quad (5.94)$$

The condition that the set Ω has locally finite perimeter allows us to define an outward unit normal $\nu = (\nu_j)_{1 \leq j \leq n}$ at \mathcal{H}^{n-1} -a.e. point on $\partial_*\Omega$, in the sense of H. Federer. In particular, if $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ then ν is defined \mathcal{H}^{n-1} -a.e. on $\partial\Omega$. In such a context, given $m \in \mathbb{N}$ it is natural to consider a related version of the higher-order restriction operator (5.39), namely the higher-order Dirichlet trace

$$\text{TR}^{(m)} u := \left\{ \frac{\partial^k u}{\partial \nu^k} \right\}_{0 \leq k \leq m-1}, \quad (5.95)$$

which has been traditionally employed in the formulation of the classical Dirichlet boundary value problem for higher-order operators. A word of caution is in order here. Specifically, in general the unit normal ν has only bounded, measurable components, hence taking iterated normal derivatives requires attention. Concretely, we define for each $k \in \{0, \dots, m-1\}$

$$\frac{\partial^k}{\partial \nu^k} := \left(\sum_{j=1}^n \xi_j \partial / \partial x_j \right)^k \Big|_{\xi = \nu} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha \partial^\alpha, \quad (5.96)$$

which suggests setting (in an appropriate context)

$$\frac{\partial^k u}{\partial \nu^k} := \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(1)} [\partial^\alpha u], \quad \forall k \in \{0, 1, \dots, m-1\}, \quad (5.97)$$

where $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(1)}$ is the boundary trace operator of order one from Theorem 5.1.

Compared to (5.39), a distinguished feature of (5.95) is that the latter has fewer components. More specifically, while $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)} u$ has

$$\sum_{k=0}^{m-1} \binom{n+k-1}{n-1} \quad (5.98)$$

components, $\text{TR}^{(m)} u$ has only m components. It is then remarkable that the two trace mappings have the same null-space. This is made precise in the theorem stated below, which answers the question raised by J. Nečas in [40, Problem 4.1, p. 91], [41, Problem 4.1, p. 86], in a considerably more general setting than the class of Lipschitz domains, as originally asked (for the latter setting see also [35], [38]).

Theorem 5.13 (The null-space of the higher-order Dirichlet trace operator). *Let Ω be an (ε, δ) -domain in \mathbb{R}^n with $\text{rad}(\Omega) > 0$, and such that $\partial\Omega$ is $(n-1)$ -Ahlfors regular and satisfies*

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \quad (5.99)$$

Denote by ν the geometric measure theoretic outward unit normal to Ω .

Then for every $m \in \mathbb{N}$ and $p \in (1, \infty)$ one has

$$\mathring{W}^{m,p}(\Omega) = \left\{ u \in W^{m,p}(\Omega) : \frac{\partial^k u}{\partial \nu^k} = 0 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \text{ for } 0 \leq k \leq m-1 \right\}. \quad (5.100)$$

Proof. To get started, set $\sigma := \mathcal{H}^{n-1} \llcorner \partial\Omega$. The long term goal is to show that the assignment

$$B_{m-1/p}^{p,p}(\partial\Omega) \ni f = \{f_\alpha\}_{|\alpha| \leq m-1} \mapsto \left\{ \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha f_\alpha \right\}_{0 \leq k \leq m-1} \in L^p(\partial\Omega, \sigma) \quad (5.101)$$

is one-to-one. To justify this, we shall make use of the fact that for each multi-index $\alpha \in \mathbb{N}_0^n$ there exist polynomial functions $\{p_{jk}^{\alpha\beta}\}_{\substack{1 \leq j, k \leq n \\ |\beta|=|\alpha|-1}}$ of n variables with the property that

$$\partial^\alpha = \nu^\alpha \frac{\partial^{|\alpha|}}{\partial \nu^{|\alpha|}} + \sum_{|\beta|=|\alpha|-1} \sum_{j,k=1}^n p_{jk}^{\alpha\beta}(\nu) (\nu_j \partial_k - \nu_k \partial_j) \partial^\beta. \quad (5.102)$$

To prove identity (5.102), assume that some $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ has been fixed and write

$$\begin{aligned} \partial^\alpha &= \prod_{j=1}^n \left(\frac{\partial}{\partial x_j} \right)^{\alpha_j} = \prod_{j=1}^n \left[\sum_{k=1}^n \xi_k \left(\xi_k \frac{\partial}{\partial x_j} - \xi_j \frac{\partial}{\partial x_k} \right) + \sum_{k=1}^n \xi_j \xi_k \frac{\partial}{\partial x_k} \right]^{\alpha_j} \Big|_{\xi=\nu} \\ &= \prod_{j=1}^n \left[\sum_{\ell=0}^{\alpha_j} \frac{\alpha_j!}{\ell!(\alpha_j-\ell)!} \left(\sum_{k=1}^n \xi_k \left(\xi_k \frac{\partial}{\partial x_j} - \xi_j \frac{\partial}{\partial x_k} \right) \right)^{\alpha_j-\ell} \left(\sum_{k=1}^n \xi_j \xi_k \frac{\partial}{\partial x_k} \right)^\ell \right] \Big|_{\xi=\nu} \\ &= \prod_{j=1}^n \left[\left(\sum_{k=1}^n \xi_j \xi_k \frac{\partial}{\partial x_k} \right)^{\alpha_j} + \sum_{\ell=0}^{\alpha_j-1} \frac{\alpha_j!}{\ell!(\alpha_j-\ell)!} \left(\sum_{k=1}^n \xi_k \left(\xi_k \frac{\partial}{\partial x_j} - \xi_j \frac{\partial}{\partial x_k} \right) \right)^{\alpha_j-\ell} \left(\sum_{k=1}^n \xi_j \xi_k \frac{\partial}{\partial x_k} \right)^\ell \right] \Big|_{\xi=\nu}. \end{aligned} \quad (5.103)$$

Upon noticing that

$$\prod_{j=1}^n \left[\left(\sum_{k=1}^n \xi_j \xi_k \frac{\partial}{\partial x_k} \right)^{\alpha_j} \right] \Big|_{\xi=\nu} = \prod_{j=1}^n \nu_j^{\alpha_j} \frac{\partial^{\alpha_j}}{\partial \nu^{\alpha_j}} = \nu^\alpha \frac{\partial^{|\alpha|}}{\partial \nu^{|\alpha|}}, \quad (5.104)$$

and $(\xi_k \partial / \partial x_j - \xi_j \partial / \partial x_k) \Big|_{\xi=\nu} = -(\nu_j \partial_k - \nu_k \partial_j)$, formula (5.102) follows.

Assume next that $\dot{f} \in B_{m-1/p}^{p,p}(\partial\Omega)$ is mapped to zero by the assignment (5.101) and consider the function $u := \mathcal{E}_{\partial\Omega \rightarrow \Omega}^{(m)} \dot{f}$ in Ω , with $\mathcal{E}_{\partial\Omega \rightarrow \Omega}^{(m)}$ as in Theorem 5.1. Then Theorem 5.1 ensures that

$$u \in W^{m,p}(\Omega) \quad \text{and} \quad f_\alpha = \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(1)}(\partial^\alpha u) \quad \sigma\text{-a.e. on } \partial\Omega \quad (5.105)$$

for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m-1$.

Also, granted the current assumptions on \dot{f} ,

$$\frac{\partial^k u}{\partial \nu^k} = 0 \quad \sigma\text{-a.e. on } \partial\Omega \quad \text{for } k = 0, 1, \dots, m-1. \quad (5.106)$$

To proceed, observe that since $\mathcal{C}_c^\infty(\mathbb{R}^n) \hookrightarrow W^{m,p}(\mathbb{R}^n)$ densely and since the restriction operator $W^{m,p}(\mathbb{R}^n) \ni v \mapsto v|_\Omega \in W^{m,p}(\Omega)$ is well-defined, linear, continuous and surjective (as seen from Theorem 2.1), it follows that

$$\{\varphi|_\Omega : \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)\} \hookrightarrow W^{m,p}(\Omega) \quad \text{densely.} \quad (5.107)$$

Consequently, it is possible to select a sequence $\{\varphi_i\}_{i \in \mathbb{N}} \subseteq \mathcal{C}_c^\infty(\mathbb{R}^n)$ with the property that if $v_i := \varphi_i|_\Omega$ for each $i \in \mathbb{N}$ then $v_i \rightarrow u$ in $W^{m,p}(\Omega)$ as $i \rightarrow \infty$. In view of Theorem 5.1 and (5.105), this implies

$$\begin{aligned} (\partial^\alpha v_i)|_{\partial\Omega} &\longrightarrow f_\alpha \quad \text{in } L^p(\partial\Omega, \sigma) \quad \text{as } i \rightarrow \infty, \\ &\text{for every multi-index } \alpha \in \mathbb{N}_0^n \quad \text{with } |\alpha| \leq m-1. \end{aligned} \quad (5.108)$$

In particular, from (5.108), (5.97), and (5.106), we obtain

$$\frac{\partial^\ell v_i}{\partial \nu^\ell} \longrightarrow 0 \quad \text{in } L^p(\partial\Omega, \sigma) \quad \text{as } i \rightarrow \infty, \quad \text{for every } \ell \in \{0, 1, \dots, m-1\}. \quad (5.109)$$

We shall now prove by induction on the number $\ell \in \{0, 1, \dots, m-1\}$ that the sequence

$$\begin{aligned} \{(\partial^\alpha v_i)|_{\partial\Omega}\}_{i \in \mathbb{N}} &\text{ converges to zero weakly in } L^p(\partial\Omega, \sigma) \\ &\text{for every multi-index } \alpha \in \mathbb{N}_0^n \quad \text{with } |\alpha| = \ell, \end{aligned} \quad (5.110)$$

and that the sequence

$$\begin{aligned} \left\{ \nu_j(\partial_k \partial^\beta v_i)|_{\partial\Omega} - \nu_k(\partial_j \partial^\beta v_i)|_{\partial\Omega} \right\}_{i \in \mathbb{N}} &\text{ converges to zero weakly in } L^p(\partial\Omega, \sigma) \\ &\text{for every multi-index } \beta \in \mathbb{N}_0^n \quad \text{with } |\beta| = \ell-1 \quad \text{and any } j, k \in \{1, \dots, n\}. \end{aligned} \quad (5.111)$$

When $\ell = 0$ the second condition is void, whereas (5.110) follows from by combining (5.108) and (5.105) (written for $\alpha = (0, \dots, 0) \in \mathbb{N}_0^n$) with (5.106) (used with $k = 0$; cf. also (5.97) in this regard). Assume next that both (5.110) and (5.111) hold for some number $\ell \in \{0, 1, \dots, m-2\}$ and fix $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \ell+1$, $\beta \in \mathbb{N}_0^n$ with $|\beta| = \ell$, as well as some $j, k \in \{1, \dots, n\}$. Now, (5.108) implies that $\left\{ \nu_j(\partial_k \partial^\beta v_i)|_{\partial\Omega} \right\}_{i \in \mathbb{N}}$ and $\left\{ \nu_k(\partial_j \partial^\beta v_i)|_{\partial\Omega} \right\}_{i \in \mathbb{N}}$ are bounded sequences in $L^p(\partial\Omega, \sigma)$. Granted this and given that

$$\{\psi|_{\partial\Omega} : \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)\} \hookrightarrow L^p(\partial\Omega, \sigma) \quad \text{densely,} \quad (5.112)$$

the fact that the sequence $\left\{ \nu_j(\partial_k \partial^\beta v_i)|_{\partial\Omega} - \nu_k(\partial_j \partial^\beta v_i)|_{\partial\Omega} \right\}_{i \in \mathbb{N}}$ converges to zero weakly in $L^p(\partial\Omega, \sigma)$ will follow as soon as we show that for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ there holds

$$\int_{\partial\Omega} \left(\nu_j(\partial_k \partial^\beta v_i)|_{\partial\Omega} - \nu_k(\partial_j \partial^\beta v_i)|_{\partial\Omega} \right) (\psi|_{\partial\Omega}) \, d\sigma \longrightarrow 0 \quad \text{as } i \rightarrow \infty. \quad (5.113)$$

To see that this is the case, denote by $\{\mathbf{e}_r\}_{1 \leq r \leq n}$ the standard orthonormal basis in \mathbb{R}^n and, for each fixed $i \in \mathbb{N}$, consider the vector field

$$\vec{F} := \left(\psi \partial_k \partial^\beta v_i + \partial_k \psi \partial^\beta v_i \right) \mathbf{e}_j - \left(\psi \partial_j \partial^\beta v_i + \partial_j \psi \partial^\beta v_i \right) \mathbf{e}_k. \quad (5.114)$$

Note that all components of \vec{F} are Lipschitz functions in Ω with bounded support, and

$$\begin{aligned} \operatorname{div} \vec{F} &= \partial_j \left(\psi \partial_k \partial^\beta v_i + \partial_k \psi \partial^\beta v_i \right) - \partial_k \left(\psi \partial_j \partial^\beta v_i + \partial_j \psi \partial^\beta v_i \right) \\ &= \partial_j \psi \partial_k \partial^\beta v_i + \psi \partial_j \partial_k \partial^\beta v_i + \partial_j \partial_k \psi \partial^\beta v_i + \partial_k \psi \partial_j \partial^\beta v_i \\ &\quad - \partial_k \psi \partial_j \partial^\beta v_i - \psi \partial_k \partial_j \partial^\beta v_i - \partial_k \partial_j \psi \partial^\beta v_i - \partial_j \psi \partial_k \partial^\beta v_i \\ &= 0 \text{ in } \Omega. \end{aligned} \quad (5.115)$$

Based on this, the De Giorgi-Federer Divergence Theorem (cf., e.g., [15]), and (5.99), we deduce that

$$\begin{aligned} 0 = \int_{\partial\Omega} \nu \cdot (\vec{F}|_{\partial\Omega}) \, d\sigma &= \int_{\partial\Omega} \left(\nu_j (\partial_k \partial^\beta v_i)|_{\partial\Omega} - \nu_k (\partial_j \partial^\beta v_i)|_{\partial\Omega} \right) (\psi|_{\partial\Omega}) \, d\sigma \\ &\quad + \int_{\partial\Omega} \left(\nu_j (\partial_k \psi)|_{\partial\Omega} - \nu_k (\partial_j \psi)|_{\partial\Omega} \right) (\partial^\beta v_i)|_{\partial\Omega} \, d\sigma. \end{aligned} \quad (5.116)$$

The bottom line is that for each $i \in \mathbb{N}$ we have

$$\begin{aligned} \int_{\partial\Omega} \left(\nu_j (\partial_k \partial^\beta v_i)|_{\partial\Omega} - \nu_k (\partial_j \partial^\beta v_i)|_{\partial\Omega} \right) (\psi|_{\partial\Omega}) \, d\sigma \\ = - \int_{\partial\Omega} \left(\nu_j (\partial_k \psi)|_{\partial\Omega} - \nu_k (\partial_j \psi)|_{\partial\Omega} \right) (\partial^\beta v_i)|_{\partial\Omega} \, d\sigma. \end{aligned} \quad (5.117)$$

Since the multi-index β has length ℓ , hypothesis (5.110) ensures that the sequence $\{(\partial^\beta v_i)|_{\partial\Omega}\}_{i \in \mathbb{N}}$ converges to zero weakly in $L^p(\partial\Omega, \sigma)$. Now (5.113) follows from this and (5.117). This takes care of the version of (5.111) with $\ell + 1$ in place of ℓ . As regards the version of (5.110) with $\ell + 1$ in place of ℓ , pick a multi-index $\alpha \in \mathbb{N}_0^n$ of length $\ell + 1$ and observe that identity (5.102) gives

$$\partial^\alpha v_i = \nu^\alpha \frac{\partial^{\ell+1} v_i}{\partial \nu^{\ell+1}} + \sum_{|\beta|=|\alpha|-1} \sum_{j,k=1}^n p_{jk}^{\alpha\beta}(\nu) (\nu_j \partial_k - \nu_k \partial_j) \partial^\beta v_i \quad \sigma\text{-a.e. on } \partial\Omega, \quad \forall i \in \mathbb{N}. \quad (5.118)$$

From what we have just proved, the double sum in the right-hand side of (5.118) converges to zero weakly in $L^p(\partial\Omega, \sigma)$ as $i \rightarrow \infty$ (since the length of all multi-indices β involved is ℓ , and each $p_{jk}^{\alpha\beta}(\nu)$ is a bounded function). With this in hand and recalling (5.109), it readily follows from (5.118) that $\{(\partial^\alpha v_i)|_{\partial\Omega}\}_{i \in \mathbb{N}}$ converges to zero weakly in $L^p(\partial\Omega, \sigma)$. This completes the induction scheme, hence (5.110) and (5.111) hold for every $\ell \in \{0, 1, \dots, m-1\}$.

At this stage, by combining (5.108) with (5.110) we deduce that $f_\alpha = 0$ for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m-1$, finishing the proof of the fact that the assignment

$$\Psi : B_{m-1/p}^{p,p}(\partial\Omega) \longrightarrow L^p(\partial\Omega, \sigma), \quad \Psi f := \left\{ \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha f_\alpha \right\}_{0 \leq k \leq m-1}, \quad \forall f = \{f_\alpha\}_{|\alpha| \leq m-1}, \quad (5.119)$$

is one-to-one. Let us also note that thanks to (5.97), (5.119), and (5.32), we have

$$\begin{aligned} \left\{ \frac{\partial^k u}{\partial \nu^k} \right\}_{0 \leq k \leq m-1} &= \left\{ \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(1)} [\partial^\alpha u] \right\}_{0 \leq k \leq m-1} \\ &= \Psi \left(\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)} u \right), \quad \forall u \in W^{m,p}(\Omega). \end{aligned} \quad (5.120)$$

All things considered, formula (5.100) now follows from (5.120), the fact that (5.33) is a well-defined operator (used here with $k := m$), the injectivity of the map (5.119), and (5.37). \square

6 Interpolation results for Sobolev spaces with partially vanishing traces

Here we take up the task of proving that, in an appropriate geometrical context, the scale of spaces introduced in (3.2) is stable under both the complex and the real method of interpolation. In order to facilitate the subsequent discussion, call a family of Banach spaces $(X_p)_{p \in I}$ indexed by an open interval $I \subseteq (1, \infty)$ a **complex interpolation scale** provided

$$[X_{p_0}, X_{p_1}]_\theta = X_p \quad (6.1)$$

whenever $p_0, p_1 \in I$, $0 < \theta < 1$, and $1/p = (1 - \theta)/p_0 + \theta/p_1$, and call $(X_p)_{p \in I}$ a **real interpolation scale** if

$$(X_{p_0}, X_{p_1})_{\theta, p} = X_p \quad (6.2)$$

whenever $p_0, p_1 \in I$, $0 < \theta < 1$, and $1/p = (1 - \theta)/p_0 + \theta/p_1$. Above, $[\cdot, \cdot]_\theta$ and $(\cdot, \cdot)_{\theta, p}$ denote, respectively, the standard complex and real brackets of interpolation (as defined in, e.g., [4]). The starting point is the following consequence of Theorems 4.3-4.4.

We continue by recording an useful result which is essentially folklore (a proof may be found in, e.g., [27]). First, we make a definition. Let X_0, X_1 and Y_0, Y_1 be two compatible pairs of Banach spaces. Call $\{Y_0, Y_1\}$ a **retract** of $\{X_0, X_1\}$ if there exist two bounded, linear operators $E : Y_i \rightarrow X_i$, $R : X_i \rightarrow Y_i$, $i = 0, 1$, such that $R \circ E = I$, the identity map, on each Y_i , $i = 0, 1$.

Lemma 6.1. *Assume that X_0, X_1 and Y_0, Y_1 are two compatible pairs of Banach spaces such that $\{Y_0, Y_1\}$ is a retract of $\{X_0, X_1\}$ (as before, the “extension-restriction” operators are denoted by E and R , respectively). Then for each $\theta \in (0, 1)$ and $0 < q \leq \infty$,*

$$[Y_0, Y_1]_\theta = R\left([X_0, X_1]_\theta\right) \quad \text{and} \quad (Y_0, Y_1)_{\theta, q} = R\left((X_0, X_1)_{\theta, q}\right). \quad (6.3)$$

As a corollary, the following also holds. Assume that X_0, X_1 is a compatible pair of Banach spaces and that P is a common projection (i.e., a linear, bounded operator on X_i , $i = 0, 1$, such that $P^2 = P$). Then the real and complex interpolation brackets commute with the action of P , i.e.,

$$[PX_0, PX_1]_\theta = P\left([X_0, X_1]_\theta\right) \quad \text{and} \quad (PX_0, PX_1)_{\theta, q} = P\left((X_0, X_1)_{\theta, q}\right), \quad (6.4)$$

for each $\theta \in (0, 1)$ and $0 < q \leq \infty$.

A word of clarification is in order here. Specifically, generally speaking, given two normed spaces X, Y and a linear, bounded operator $T : X \rightarrow Y$, by TX we shall denote its range equipped with the graph-norm

$$\|y\|_{TX} := \inf\{\|x\|_X : x \in X \text{ such that } y = Tx\}, \quad y \in TX. \quad (6.5)$$

In particular, this is the sense in which (6.3) and (6.4) should be understood.

Proposition 6.2. *Assume that $D \subseteq \mathbb{R}^n$ is a closed set which is d -Ahlfors regular for some $d \in (0, n)$, and fix a number $k \in \mathbb{N}$. Then the operator*

$$\mathcal{P}_D^{(k)} := I - \mathcal{E}_D^{(k)} \circ \mathcal{R}_D^{(k)} \quad (6.6)$$

is a continuous and linear projection from the classical Sobolev space $W^{k,p}(\mathbb{R}^n)$ onto the space $W_D^{k,p}(\mathbb{R}^n)$ whenever $\max\{1, n - d\} < p < \infty$.

Proof. The claims about the operator (6.6) are readily seen from Theorem 4.4 and Theorem 4.3. \square

In combination with the abstract results from Lemma 6.1, Proposition 6.2 permits us to prove the following interpolation result in the context of \mathbb{R}^n .

Proposition 6.3. *Suppose $D \subseteq \mathbb{R}^n$ is a closed set which is d -Ahlfors regular for some $d \in (0, n)$, and fix a number $k \in \mathbb{N}$. Then, in a natural sense,*

$$\begin{aligned} \{W_D^{k,p}(\mathbb{R}^n)\}_{\max\{1,n-d\} < p < \infty} & \text{ is an interpolation scale} \\ & \text{both for the complex and the real method.} \end{aligned} \quad (6.7)$$

Proof. This follows from Proposition 6.2, the fact that $\{W^{k,p}(\mathbb{R}^n)\}_{1 < p < \infty}$ is an interpolation scale both for the complex and the real method for each fixed $k \in \mathbb{N}$, and the second part in Lemma 6.1. \square

With this in hand, we are ready to prove our main interpolation result, formulated below.

Theorem 6.4 (Interpolation Theorem). *Assume that $\Omega \subseteq \mathbb{R}^n$ and $D \subseteq \overline{\Omega}$ are such that D is closed and d -Ahlfors regular for some $d \in (0, n)$, and Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. In addition, fix a number $k \in \mathbb{N}$. Then*

$$\begin{aligned} \{W_D^{k,p}(\Omega)\}_{\max\{1,n-d\} < p < \infty} & \text{ is an interpolation scale} \\ & \text{both for the complex and the real method,} \end{aligned} \quad (6.8)$$

and, with the convention that $1/p + 1/p' = 1$,

$$\begin{aligned} \{(W_D^{k,p'}(\Omega))^*\}_{\max\{1,n-d\} < p < \infty} & \text{ is an interpolation scale} \\ & \text{both for the complex and the real method.} \end{aligned} \quad (6.9)$$

In particular, if Ω is a nonempty open subset of \mathbb{R}^n with the property that $\partial\Omega$ is d -Ahlfors regular for some $d \in (0, n)$, then

$$\begin{aligned} \{\mathring{W}_D^{k,p}(\Omega)\}_{\max\{1,n-d\} < p < \infty} & \text{ is an interpolation scale} \\ & \text{both for the complex and the real method,} \end{aligned} \quad (6.10)$$

and

$$\begin{aligned} \{W^{-k,p}(\Omega)\}_{\max\{1,n-d\} < p < \infty} & \text{ is an interpolation scale} \\ & \text{both for the complex and the real method.} \end{aligned} \quad (6.11)$$

Proof. Collectively, Theorem 4.6, Theorem 3.8, and Theorem 3.7 prove that, under assumptions made in the statement of the current corollary,

$$\{W_D^{k,p}(\Omega)\}_{\max\{1,n-d\} < p < \infty} \text{ is a retract of } \{W_D^{k,p}(\mathbb{R}^n)\}_{\max\{1,n-d\} < p < \infty}. \quad (6.12)$$

Based on this, (6.7), the first part in Lemma 6.1, and Corollary 3.10, the claim made in (6.8) follows. Then the claim in (6.9) becomes a consequence of (6.8), duality theorems for the complex and real methods of interpolation (cf., e.g., [4, Corollary 4.5.2, p. 98] and [4, Theorem 3.7.1, p. 54]), and part (3) in Lemma 3.2. Finally, the claims in (6.10)-(6.11) are implied by (6.8)-(6.9), part (5) in Lemma 3.2, and (2.5). \square

7 Applications to mixed boundary value problems for higher-order systems

The aim in this section is to illustrate how the functional analytic results proved so far may be used to establish some very general solvability results for mixed boundary value problems. In this endeavor, we shall work with higher-order systems of PDE's in divergence form, with bounded measurable coefficients. This requires a number of preliminaries which we first dispense with. The reader is alerted to the fact

that while here we shall frequently work with vector-valued functions, our notation does not necessarily reflect that (though matters should always be clear from the context).

To start the build-up in earnest, let \mathcal{L} be the differential operator of order $2m$, where $m \in \mathbb{N}$, in divergence form given by

$$\mathcal{L}u := \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u), \quad (7.1)$$

whose tensor coefficient $A = (A_{\alpha\beta})_{|\alpha|=|\beta|=m}$ consists of $M \times M$ matrices $A_{\alpha\beta}$ with bounded, measurable, complex-valued entries, i.e.,

$$A_{\alpha\beta} = (a_{ij}^{\alpha\beta})_{1 \leq i, j \leq M} \quad \text{with each } a_{ij}^{\alpha\beta} \in L^\infty(\Omega, \mathcal{L}^n), \quad (7.2)$$

and where the function $u = (u_1, \dots, u_M)$ is \mathbb{C}^M -valued. The first order of business is to associate a conormal derivative for \mathcal{L} .

In order to motivate the subsequent discussion, consider first the situation when Ω is a bounded \mathcal{C}^∞ domain and denote by $\nu = (\nu_j)_{1 \leq j \leq n}$ its outward unit normal. Also, set $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. In the case when the entries of each $A_{\alpha\beta}$ are functions from $\mathcal{C}^\infty(\overline{\Omega})$, given any \mathbb{C}^M -valued functions u, v with components from $\mathcal{C}^\infty(\overline{\Omega})$, repeated integrations by parts yield

$$\begin{aligned} & \sum_{|\alpha|=|\beta|=m} \int_{\Omega} \langle A_{\alpha\beta} \partial^\beta u, \partial^\alpha v \rangle d\mathcal{L}^n \\ &= (-1)^{m+1} \int_{\partial\Omega} \langle \partial_\nu^A u, \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)} v \rangle d\sigma + (-1)^m \int_{\Omega} \langle \mathcal{L}u, v \rangle d\mathcal{L}^n, \end{aligned} \quad (7.3)$$

where $\langle \cdot, \cdot \rangle$ is the usual (real) pointwise inner product of vector-valued functions, and the vector-valued function $\partial_\nu^A u$ is given by

$$\begin{aligned} \partial_\nu^A u &:= \left\{ (\partial_\nu^A u)_\delta \right\}_{|\delta| \leq m-1} \quad \text{with the } \delta\text{-component given by the formula} \\ (\partial_\nu^A u)_\delta &:= \sum_{|\alpha|=|\beta|=m} \sum_{j=1}^n (-1)^{|\delta|} \frac{\alpha! |\delta|! (m - |\delta| - 1)!}{m! \delta! (\alpha - \delta - e_j)!} \nu_j A_{\alpha\beta} \left(\partial^{\alpha+\beta-\delta-e_j} u \right) \Big|_{\partial\Omega}, \end{aligned} \quad (7.4)$$

with the convention that the sum in α and j is only performed over those α 's and j 's such that $\alpha - \delta - e_j$ does not have any negative components (and with $e_j := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}_0^n$ with the only nonzero component on the j -th slot, $j \in \{1, \dots, n\}$). While retaining the same context as above, suppose we are interested in the case in which the function $\partial_\nu^A u$ vanishes on $\partial\Omega \setminus D$ where D is a given closed subset of $\partial\Omega$. Note that if $v \in \mathcal{C}_D^\infty(\Omega)$, then (7.3) becomes

$$\begin{aligned} & \sum_{|\alpha|=|\beta|=m} \int_{\Omega} \langle A_{\alpha\beta} \partial^\beta u, \partial^\alpha v \rangle d\mathcal{L}^n \\ &= (-1)^{m+1} \int_{\partial\Omega \setminus D} \langle \partial_\nu^A u, \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)} v \rangle d\sigma + (-1)^m \int_{\Omega} \langle \mathcal{L}u, v \rangle d\mathcal{L}^n, \end{aligned} \quad (7.5)$$

since $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)} v = 0$ near D . In fact, for every $p \in (1, \infty)$ we have $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)} v \in B_{m-1/p}^{p,p}(\partial\Omega)$, with compact support contained in the relatively open subset $\partial\Omega \setminus D$ of $\partial\Omega$. Based on this and (7.5), it is not difficult to see that under the current smoothness hypotheses on the objects involved, the fact that $\partial_\nu^A u$ vanishes on $\partial\Omega \setminus D$ is equivalent to having

$$\sum_{|\alpha|=|\beta|=m} \int_{\Omega} \langle A_{\alpha\beta} \partial^\beta u, \partial^\alpha v \rangle d\mathcal{L}^n = (-1)^m \int_{\Omega} \langle \mathcal{L}u, v \rangle d\mathcal{L}^n, \quad \forall v \in \mathcal{C}_D^\infty(\Omega). \quad (7.6)$$

We continue to retain the same setting as above and, in addition, fix some $p' \in (1, \infty)$. Then thanks to (3.2) and (7.6) the condition that $\partial_\nu^A u$ vanishes on $\partial\Omega \setminus D$ may be further equivalently expressed as

$$\sum_{|\alpha|=|\beta|=m} \int_{\Omega} \langle A_{\alpha\beta} \partial^\beta u, \partial^\alpha v \rangle d\mathcal{L}^n = (-1)^m \int_{\Omega} \langle \mathcal{L}u, v \rangle d\mathcal{L}^n, \quad \forall v \in W_D^{m,p'}(\Omega). \quad (7.7)$$

At this stage in the build-up, we wish to assign a meaning of the condition that

$$\partial_\nu^A u \text{ vanishes on } \partial\Omega \setminus D \quad (7.8)$$

in a much more general setting than the smooth one considered so far. To be specific, suppose that Ω is an open subset of \mathbb{R}^n and that \mathcal{L} is as in (7.1)-(7.2), for some $m \in \mathbb{N}$. Also, assume that D is a closed subset of $\partial\Omega$ and that $u \in W_D^{m,p}(\Omega)$ for some $p \in (1, \infty)$. Of course, this renders the pointwise definition of $\partial_\nu^A u$ from (7.4) utterly inadequate so, instead, we shall attempt to make sense of (7.7) (which, as indicated above, is an equivalent way of expressing (7.8) in the smooth case). With this goal in mind, note that if p, p' are Hölder conjugate exponents (something we will assume from now on), then the left-hand side of (7.7) is well-defined for any $v \in W_D^{m,p'}(\Omega)$, given that we are assuming that the tensor coefficient has bounded entries. This being said, making sense of the right-hand side could be problematic since one cannot expect $f := \mathcal{L}u$ to be more regular than a generic distribution in $W^{-m,p}(\Omega)$.

Having identified this issue, the remedy we propose is as follows. As a preamble, let us agree that given a topological vector space X , with dual X^* , the symbol $X^*\langle \cdot, \cdot \rangle_X$ indicates the pairing between functionals from X^* and vectors from X . Also, we shall associate to any functional $f \in (W_D^{m,p'}(\Omega))^*$ the distribution $f|_{\Omega} \in \mathcal{D}'(\Omega)$ defined by

$$\mathcal{D}'(\Omega)\langle f|_{\Omega}, \varphi \rangle_{\mathcal{D}'(\Omega)} := (W_D^{m,p'}(\Omega))^*\langle f, \varphi \rangle_{W_D^{m,p'}(\Omega)}, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega). \quad (7.9)$$

The reader is alerted to the fact that, while linear and continuous,

$$\text{the assignment } (W_D^{m,p'}(\Omega))^* \ni f \mapsto f|_{\Omega} \in \mathcal{D}'(\Omega) \text{ is not injective,} \quad (7.10)$$

generally speaking (a remarkable exception is when $D = \partial\Omega$; cf. (5) in Lemma 3.2, (2.6) and (2.5)). To see this, assume for a moment that Ω is an (ε, δ) -domain in \mathbb{R}^n with the property that its boundary is $(n-1)$ -Ahlfors regular, and set $\sigma := \mathcal{H}^{n-1}|_{\partial\Omega}$. Also, suppose that D is a proper closed subset of $\partial\Omega$. Then there exists a nontrivial function $g \in L^p(\partial\Omega, \sigma)$ with support in $\partial\Omega \setminus D$, and we define the functional $f \in (W_D^{m,p'}(\Omega))^*$ by setting

$$(W_D^{m,p'}(\Omega))^*\langle f, v \rangle_{W_D^{m,p'}(\Omega)} := \int_{\partial\Omega} g \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(1)} v d\sigma, \quad \forall v \in W_D^{m,p'}(\Omega). \quad (7.11)$$

By Theorem 5.1, the above functional is then well-defined and nonzero, but it is clear that $f|_{\Omega}$ is zero as a distribution. Hence, (7.10) holds, as claimed. A heuristic, yet suggestive, way of expressing this is by saying that arbitrary distributions in Ω may have multiple extensions to functionals in $(W_D^{m,p'}(\Omega))^*$.

This discussion sets the stage for making the following definition.

Definition 7.1. *Let Ω be an arbitrary open subset of \mathbb{R}^n and assume that \mathcal{L} is as in (7.1)-(7.2), for some $m \in \mathbb{N}$. Also, suppose that D is a closed subset of $\partial\Omega$ and fix $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$. Then, given $u \in W_D^{m,p}(\Omega)$ and $f \in (W_D^{m,p'}(\Omega))^*$ satisfying the (necessary) compatibility condition*

$$\mathcal{L}u = f|_{\Omega} \text{ in } \mathcal{D}'(\Omega), \quad (7.12)$$

abbreviate by

$$\partial_\nu^A(u, f) = 0 \text{ on } \partial\Omega \setminus D \quad (7.13)$$

the demand that

$$(-1)^m \sum_{|\alpha|=|\beta|=m} \int_{\Omega} \langle A_{\alpha\beta} \partial^\beta u, \partial^\alpha v \rangle d\mathcal{L}^n = (W_D^{m,p'}(\Omega))^*\langle f, v \rangle_{W_D^{m,p'}(\Omega)} \quad \forall v \in \mathcal{C}_D^\infty(\Omega). \quad (7.14)$$

We wish to stress that, in (7.13), the symbol “ $\partial_\nu^A(u, f)$ ” is not defined individually, but rather we assign a meaning to the condition “ $\partial_\nu^A(u, f) = 0$ on $\partial\Omega \setminus D$ ” as a block, through (7.14).

A few comments pertaining to the nature of Definition 7.1 are in order. First, the fact that the compatibility condition (7.12) is necessary if (7.14) is to hold, is readily seen by specializing (7.14) to the case when $v \in \mathcal{C}_c^\infty(\Omega)$ and keeping in mind (7.9). Second, by density, (7.14) is equivalent to the condition that

$$(-1)^m \sum_{|\alpha|=|\beta|=m} \int_{\Omega} \langle A_{\alpha\beta} \partial^\beta u, \partial^\alpha v \rangle d\mathcal{L}^n = (W_D^{m,p'}(\Omega))^* \langle f, v \rangle_{W_D^{m,p'}(\Omega)} \quad \forall v \in W_D^{m,p'}(\Omega). \quad (7.15)$$

Third, if f is more regular than originally assumed, say if $f \in L^p(\Omega, \mathcal{L}^n)$, then (7.15) further becomes equivalent to (7.7). In particular, (7.13) reduces to condition (7.8), interpreted pointwise, in the case when all objects involved are regular enough (as in the earlier discussion, pertaining to (7.3)-(7.7)).

However, in general, condition (7.13) is not an ordinary generalization of the demand that (7.8) holds in a pointwise sense (with $\partial_\nu^A u$ as in (7.4)). In fact, it is more appropriate to regard the former as a “renormalization” of the latter, in a fashion that depends strongly on the choice of an extension of the distribution $\mathcal{L}u \in \mathcal{D}'(\Omega)$ to a functional $f \in (W_D^{m,p'}(\Omega))^*$ (a phenomenon which may be traced back to (7.10) and the subsequent discussion). Indeed, if $f_i \in (W_D^{m,p'}(\Omega))^*$, $i = 1, 2$ are two such extensions of $\mathcal{L}u \in \mathcal{D}'(\Omega)$, in the sense that

$$\mathcal{L}u = f_i|_{\Omega} \text{ in } \mathcal{D}'(\Omega), \text{ for } i = 1, 2. \quad (7.16)$$

then the validity of (7.14) with $f = f_1$ does not necessarily imply the validity of (7.14) with $f = f_2$. It is precisely for this reason that, in contrast to (7.8) interpreted as a pointwise condition in the sense of (7.4), the notation in (7.13) reflects the fact that the functional f plays a basic role in this case.

Definition 7.1 is now employed in the formulation of the mixed boundary value problem in the next proposition.

Proposition 7.2. *Let Ω be an arbitrary open subset of \mathbb{R}^n and assume that \mathcal{L} is as in (7.1)-(7.2), for some $m \in \mathbb{N}$. Also, suppose that D is a closed subset of $\partial\Omega$ and fix $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$. In this context, for an arbitrary $f \in (W_D^{m,p'}(\Omega))^*$ consider the mixed boundary value problem*

$$\begin{cases} \mathcal{L}u = f|_{\Omega} \text{ in } \mathcal{D}'(\Omega), \\ u \in W_D^{m,p}(\Omega), \\ \partial_\nu^A(u, f) = 0 \text{ on } \partial\Omega \setminus D, \end{cases} \quad (7.17)$$

where the last condition is understood in the sense of Definition 7.1. Finally, define the (linear and bounded) operator

$$T_{\mathcal{L}} : W_D^{m,p}(\Omega) \longrightarrow (W_D^{m,p'}(\Omega))^* \quad (7.18)$$

by setting

$$(W_D^{m,p'}(\Omega))^* \langle T_{\mathcal{L}}u, v \rangle_{W_D^{m,p}(\Omega)} := (-1)^m \sum_{|\alpha|=|\beta|=m} \int_{\Omega} \langle A_{\alpha\beta} \partial^\beta u, \partial^\alpha v \rangle d\mathcal{L}^n, \quad (7.19)$$

for every $u \in W_D^{m,p}(\Omega)$ and $v \in W_D^{m,p'}(\Omega)$.

Then the mixed boundary value problem (7.17) has a solution if and only if $f \in \text{Im } T_{\mathcal{L}}$, and the solution is unique up to functions in $\text{Ker } T_{\mathcal{L}}$.

In particular, the mixed boundary value problem (7.17) is well-posed (for arbitrary data in the space $(W_D^{m,p'}(\Omega))^*$) if and only if the operator $T_{\mathcal{L}}$ in (7.18)-(7.19) is invertible.

Proof. After unraveling notation, given an arbitrary functional $f \in (W_D^{m,p'}(\Omega))^*$ the existence of a function $u \in W_D^{m,p}(\Omega)$ solving the mixed boundary value problem (7.17) comes down to finding some $u \in W_D^{m,p}(\Omega)$ such that

$$(W_D^{m,p'}(\Omega))^* \langle T_{\mathcal{L}}u, v \rangle_{W_D^{m,p'}(\Omega)} = (W_D^{m,p'}(\Omega))^* \langle f, v \rangle_{W_D^{m,p'}(\Omega)} \quad \forall v \in \mathcal{C}_D^\infty(\Omega). \quad (7.20)$$

Since, by design, $\mathcal{C}_D^\infty(\Omega)$ is dense in $W_D^{m,p'}(\Omega)$ and since the operator $T_{\mathcal{L}}$ in (7.18)-(7.19) is continuous, this is further equivalent to finding $u \in W_D^{m,p}(\Omega)$ solving

$$T_{\mathcal{L}}u = f \quad \text{in } (W_D^{m,p'}(\Omega))^*. \quad (7.21)$$

From this, all desired conclusions follow. \square

Proposition 7.2 highlights the relevance of the functional analytic properties of $T_{\mathcal{L}}$ in (7.18)-(7.19) from the perspective of the solvability of the mixed boundary value problem (7.17). In order to state our main Fredholm solvability result for the mixed boundary value problem (7.17), let us agree that, for each open set $\Omega \subseteq \mathbb{R}^n$ and each $m \in \mathbb{N}$,

$$\mathcal{P}_m(\Omega) := \{P|_{\Omega} : P \text{ complex polynomial of degree } \leq m \text{ in } \mathbb{R}^n\}. \quad (7.22)$$

Here is the main well-posedness result in this paper.

Theorem 7.3 (Well-posedness of the higher-order mixed boundary problem). *Let Ω be a bounded, connected, open, nonempty, subset of \mathbb{R}^n , $n \geq 2$, and suppose that D is a closed subset of $\partial\Omega$ which is d -Ahlfors regular for some $d \in (n-2, n)$. In addition, assume that Ω is locally an (ε, δ) -domain near $\partial\Omega \setminus D$. Next, consider an $M \times M$ divergence-form system \mathcal{L} of order $2m$, as in (7.1)-(7.2), for some $m \in \mathbb{N}$, and suppose that \mathcal{L} satisfies the strong ellipticity condition asserting that there exists $\kappa > 0$ such that*

$$\operatorname{Re} \left[\sum_{|\alpha|=|\beta|=m} \sum_{i,j=1}^M a_{ij}^{\alpha\beta}(x) \zeta_i^\alpha \overline{\zeta_j^\beta} \right] \geq \kappa \sum_{|\alpha|=m} \sum_{i=1}^M \frac{\alpha!}{m!} |\zeta_i^\alpha|^2 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \quad (7.23)$$

and for all families of complex numbers $(\zeta_i^\alpha)_{\substack{|\alpha|=m \\ 1 \leq i \leq M}}$.

Then there exists $p_* \in (2, \infty)$, depending only on $n, m, M, d, \Omega, D, A, \kappa$, with the property that, whenever

$$\frac{p_*}{p_* - 1} < p < p_*, \quad (7.24)$$

for each functional

$$f \in (W_D^{m,p'}(\Omega))^* \quad \text{where } 1/p + 1/p' = 1, \quad (7.25)$$

the mixed boundary value problem

$$\begin{cases} \mathcal{L}u = f|_{\Omega} \text{ in } \mathcal{D}'(\Omega), \\ u \in W_D^{m,p}(\Omega), \\ \partial_\nu^A(u, f) = 0 \text{ on } \partial\Omega \setminus D, \end{cases} \quad (7.26)$$

(with the last condition understood in the sense of Definition 7.1) is well-posed when $D \neq \emptyset$.

Moreover, in the case when $D = \emptyset$, the problem (7.26) has a solution if and only if

$$(W_D^{m,p'}(\Omega))^* \langle f, v \rangle_{W_D^{m,p'}(\Omega)} = 0, \quad \forall v \in \mathcal{P}_{m-1}(\Omega), \quad (7.27)$$

in which scenario solutions of (7.26) are unique up to functions from $\mathcal{P}_{m-1}(\Omega)$.

Note that the membership $u \in W_D^{m,p}(\Omega)$ entails $\mathcal{R}_{\Omega \rightarrow D}^{(m)} u = 0$ at \mathcal{H}^d -a.e. point on D , by Corollary 5.3 (and, in fact, the latter is equivalent to $u \in W_D^{m,p}(\Omega)$ under the background assumption that u is in $W^{k,p}(\Omega)$ to being with, in the context specified in Theorem 5.2). As such, problem (7.26) imposes the homogeneous Dirichlet boundary condition $\mathcal{R}_{\Omega \rightarrow D}^{(m)} u = 0$ on D and the homogeneous Neumann boundary condition $\partial_\nu^A(u, f) = 0$ on $\partial\Omega \setminus D$, thus justifying the terminology ‘‘mixed boundary value problem’’.

Proof of Theorem 7.3. As a preamble, we first claim that if Ω, D are as in the statement of the theorem then whenever $m \in \mathbb{N}$ and $\max\{1, n - d\} < p < \infty$, we have

$$\mathcal{P}_{m-1}(\Omega) \cap W_D^{m,p}(\Omega) = \begin{cases} \{0\} & \text{if } D \neq \emptyset, \\ \mathcal{P}_{m-1}(\Omega) & \text{if } D = \emptyset. \end{cases} \quad (7.28)$$

To justify this, observe that in the case when $D \neq \emptyset$ we necessarily have $\mathcal{H}^d(D) > 0$, given that D is assumed to be d -Ahlfors regular. On the other hand, from (5.2)-(5.32) and (5.31) we deduce that

$$\begin{aligned} \{\partial^\alpha P|_D\}_{|\alpha| \leq m-1} &= \mathcal{R}_{\Omega \rightarrow D}^{(k)} v = (0, \dots, 0) \text{ at } \mathcal{H}^d\text{-a.e. point on } D, \\ \text{if } v \in W_D^{m,p}(\Omega) \text{ is of the form } v &= P|_\Omega \text{ for some } P \in \mathcal{P}_{m-1}(\mathbb{R}^n). \end{aligned} \quad (7.29)$$

All together, the above analysis show that $\mathcal{P}_{m-1}(\Omega) \cap W_D^{m,p}(\Omega) = \{0\}$ if $D \neq \emptyset$ (given that Ω is assumed to be connected). Finally, if $D = \emptyset$ then the fact that $\mathcal{P}_{m-1}(\Omega) \cap W_D^{m,p}(\Omega) = \mathcal{P}_{m-1}(\Omega)$ is clear from (3.10), Lemma 3.3, and the assumption that Ω is bounded.

To proceed in earnest with the proof of the well-posedness of the mixed boundary problem (7.26), recall first the operator $T_{\mathcal{L}}$ introduced in (7.18)-(7.19). When considered in the context

$$T_{\mathcal{L}} : W_D^{m,2}(\Omega) \longrightarrow (W_D^{m,2}(\Omega))^*, \quad (7.30)$$

the ellipticity condition (7.23) implies that this mapping has the property that, for every function $u = (u_j)_{1 \leq j \leq M} \in W_D^{m,2}(\Omega)$,

$$\begin{aligned} \kappa \sum_{|\alpha|=m} \sum_{i=1}^M \frac{\alpha!}{m!} \int_{\Omega} |\partial^\alpha u_i|^2 d\mathcal{L}^n &\leq \operatorname{Re} \left[\sum_{|\alpha|=|\beta|=m} \sum_{i,j=1}^M \int_{\Omega} a_{ij}^{\alpha\beta} \partial^\beta u_j \overline{\partial^\alpha u_i} d\mathcal{L}^n \right] \\ &\leq \left| (W_D^{m,2}(\Omega))^* \langle T_{\mathcal{L}} u, \bar{u} \rangle_{W_D^{m,2}(\Omega)} \right| \\ &\leq \|T_{\mathcal{L}} u\|_{(W_D^{m,2}(\Omega))^*} \|u\|_{W_D^{m,2}(\Omega)} \\ &\leq (4\theta)^{-1} \|T_{\mathcal{L}} u\|_{(W_D^{m,2}(\Omega))^*}^2 + \theta \|u\|_{W_D^{m,2}(\Omega)}^2, \end{aligned} \quad (7.31)$$

for each $\theta \in (0, \infty)$. Choosing θ small, (7.31) ultimately shows that there exists some finite constant $C > 0$, independent of u , such that

$$\|u\|_{W_D^{m,2}(\Omega)} \leq C \|T_{\mathcal{L}} u\|_{(W_D^{m,2}(\Omega))^*} + C \|u\|_{W^{m-1,2}(\Omega)}, \quad \forall u \in W_D^{m,2}(\Omega). \quad (7.32)$$

Given that the embedding $W_D^{m,2}(\Omega) \hookrightarrow W^{m-1,2}(\Omega)$ is compact, estimate (7.32) implies that $T_{\mathcal{L}}$ in (7.30) is bounded from below modulo compact operators. Granted this, standard functional analysis gives that $T_{\mathcal{L}}$ in (7.30) has closed range and finite dimensional kernel. Since

$$(T_{\mathcal{L}})^* = T_{\mathcal{L}^*} \quad (7.33)$$

and the adjoint \mathcal{L}^* of \mathcal{L} also satisfies the ellipticity condition (7.23) (written for its tensor coefficient), it follows that the adjoint of $T_{\mathcal{L}}$ from (7.30) also enjoys the aforementioned properties. That is,

$$(T_{\mathcal{L}})^* : W_D^{m,2}(\Omega) \longrightarrow (W_D^{m,2}(\Omega))^* \text{ has closed range and finite dimensional kernel,} \quad (7.34)$$

since the spaces (3.2) are reflexive for every $p \in (1, \infty)$. All together, this analysis proves that

$$T_{\mathcal{L}} : W_D^{m,2}(\Omega) \longrightarrow (W_D^{m,2}(\Omega))^* \text{ is a Fredholm operator.} \quad (7.35)$$

It is also implicit in estimate (7.31) (cf. the third inequality there) that if $u \in W_D^{m,2}(\Omega)$ is such that $T_{\mathcal{L}}u = 0 \in (W_D^{m,2}(\Omega))^*$ then necessarily u is a polynomial of degree $\leq m - 1$ in Ω . Conversely, as seen from (7.19), any such function is mapped by $T_{\mathcal{L}}$ to zero. Hence, for the operator (7.30),

$$\text{Ker } T_{\mathcal{L}} = \mathcal{P}_{m-1}(\Omega) \cap W_D^{m,2}(\Omega). \quad (7.36)$$

From this and (7.33) we also deduce that

$$\text{Ker } (T_{\mathcal{L}})^* = \mathcal{P}_{m-1}(\Omega) \cap W_D^{m,2}(\Omega). \quad (7.37)$$

In concert, (7.35)-(7.37) show that

$$T_{\mathcal{L}} : W_D^{m,2}(\Omega) \longrightarrow (W_D^{m,2}(\Omega))^* \text{ is Fredholm with index zero.} \quad (7.38)$$

At this stage, from (7.38), the fact that the operator (7.18) is linear and bounded, (6.8)-(6.9), and the stability of the quality of being Fredholm with index zero, as well as the stability of null-space on complex interpolation scales which are nested (with respect to the scale parameter), and the assumption that $n - d < 2$ (cf., [7], [27], [28], [54]), we deduce that there exists $p_* \in (2, \infty)$, depending only on $n, m, M, d, \Omega, D, A, \kappa$, with the property that whenever (7.24) holds and $1/p + 1/p' = 1$,

$$\begin{aligned} T_{\mathcal{L}} : W_D^{m,p}(\Omega) \longrightarrow (W_D^{m,p'}(\Omega))^* \text{ is a Fredholm operator with index zero, and} \\ \text{both its kernel and the kernel of its adjoint coincide with } \mathcal{P}_{m-1}(\Omega) \cap W_D^{m,p}(\Omega). \end{aligned} \quad (7.39)$$

Based on this, (7.28), and Proposition 7.2, all claims in the statement of the theorem now readily follow. \square

The case $D = \partial\Omega$ of Theorem 7.3 deserves to be stated separately since this corresponds to the well-posedness of the inhomogeneous Dirichlet boundary value problem for higher-order systems in a very general analytic and geometric measure theoretic setting. An artifact of the choice $D = \partial\Omega$ which deserves special mention is the fact that condition $d \in (n - 2, n)$, which would normally carry over from the formulation of Theorem 7.3, naturally readjusts to $d \in [n - 1, n)$. To see this, recall a general version of the classical isoperimetric inequality proved by H. Federer in [18, 3.2.43-3.2.44, p. 278], according to which

$$\mathcal{L}^n(\bar{E}) \leq \frac{1}{n\omega_{n-1}} \mathcal{H}^{n-1}(\partial E)^{n/(n-1)}, \quad \forall E \subset \mathbb{R}^n \text{ with } \mathcal{L}^n(\bar{E}) < +\infty, \quad (7.40)$$

where ω_{n-1} stands for the surface area of the unit sphere in \mathbb{R}^n . Hence, in the case of a bounded, open, nonempty set $\Omega \subseteq \mathbb{R}^n$ such that $\partial\Omega$ is d -Ahlfors regular, we simultaneously have $\mathcal{H}^{n-1}(\partial\Omega) > 0$ and $\mathcal{H}^d(\partial\Omega) < +\infty$. Together, these two conditions imply $d \geq n - 1$, which accounts for the adjustment mentioned earlier.

Theorem 7.4 (Well-posedness of the higher-order inhomogeneous Dirichlet problem). *Assume that Ω is a bounded, open, nonempty subset of \mathbb{R}^n , $n \geq 2$, whose boundary is d -Ahlfors regular for some $d \in [n - 1, n)$. Also, consider an $M \times M$ divergence-form system \mathcal{L} of order $2m$ with bounded measurable coefficients, as in (7.1)-(7.2), for some $m \in \mathbb{N}$, and suppose that \mathcal{L} satisfies the strong ellipticity condition (7.23) for some $\kappa > 0$.*

Then there exists $p_ \in (2, \infty)$, depending only on $n, m, M, d, \Omega, A, \kappa$ with the property that the inhomogeneous Dirichlet boundary value problem*

$$\begin{cases} \mathcal{L}u = f \in W^{-m,p}(\Omega), \\ u \in \mathring{W}^{m,p}(\Omega), \end{cases} \quad (7.41)$$

is well-posed whenever $\frac{p_}{p_* - 1} < p < p_*$.*

Proof. Essentially, this is a consequence of Theorem 7.3 employed with $D = \partial\Omega$, and part (5) in Lemma 3.2. The only matter which requires further clarification, given that we are not assuming in the current case that Ω is necessarily connected, is the fact that

$$\{u \in \dot{W}^{k,p}(\Omega) : u \text{ locally a polynomial of degree } \leq m-1\} = \{0\}. \quad (7.42)$$

This, however, is readily seen from an $(m-1)$ -fold application of Poincaré's inequality (which is valid in the space $\dot{W}^{k,p}(\Omega)$, given its definition in (2.4)). \square

Both Theorem 7.3 and Theorem 7.4 are sharp, in the sense that the membership of p to a small neighborhood of 2 is a necessary condition, even when $\Omega \subseteq \mathbb{R}^n$ is a bounded \mathcal{C}^∞ domain, if the coefficients $A_{\alpha\beta}$ are merely bounded and measurable. To treat both cases simultaneously, consider the (permissible) scenario in which $D = \partial\Omega$. In the case $n = M = 2$ and $m = 1$ a relevant counterexample has been given by N. Meyers in §5 of [36]. Specifically, take $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ and consider the tensor coefficient given by

$$\begin{aligned} A_{11}(x_1, x_2) &= 1 - (1 - \mu^2)x_2^2(x_1^2 + x_2^2)^{-1}, \\ A_{12}(x_1, x_2) &= A_{21}(x_1, x_2) = (1 - \mu^2)x_1x_2(x_1^2 + x_2^2)^{-1}, \quad \forall (x, y) \in \Omega \setminus \{(0, 0)\}, \\ A_{22}(x_1, x_2) &= 1 - (1 - \mu^2)x_1^2(x_1^2 + x_2^2)^{-1}, \end{aligned} \quad (7.43)$$

where $\mu \in (0, 1)$ is a fixed parameter. Define the scalar operator $\mathcal{L}u := \sum_{\alpha, \beta=1,2} \partial_\alpha (A_{\alpha\beta}(x_1, x_2) \partial_\beta u)$ in Ω . Note that the $A_{\alpha\beta}$'s belong to $L^\infty(\Omega, \mathcal{L}^2)$ and a direct calculation shows that

$$\sum_{\alpha, \beta=1,2} A_{\alpha\beta}(x_1, x_2) \zeta^\alpha \zeta^\beta = |\zeta|^2 - (1 - \mu^2) \frac{(x_1 \zeta^2 - x_2 \zeta^1)^2}{x_1^2 + x_2^2} \geq \mu^2 |\zeta|^2, \quad (7.44)$$

for each $\zeta = (\zeta^1, \zeta^2) \in \mathbb{R}^2$ and $(x_1, x_2) \in \Omega \setminus \{0\}$. Hence, \mathcal{L} satisfies the strong ellipticity condition (7.23). To proceed, introduce the function

$$v(x_1, x_2) := x_1(x_2^2 + x_1^2)^{(\mu-1)/2} \in L^\infty(\Omega, \mathcal{L}^2) \cap \mathcal{C}^\infty(\overline{\Omega} \setminus \{0\}). \quad (7.45)$$

A straightforward calculation shows that $\mathcal{L}v = 0$ near origin. Also, fix $\phi \in \mathcal{C}_c^\infty(\Omega)$ so that $\phi \equiv 1$ near origin, and set $u := \phi v$. It follows that

$$u \in \dot{W}^{1,2}(\Omega), \quad f := \mathcal{L}u \in \mathcal{C}_c^\infty(\Omega), \quad |(\nabla u)(x_1, x_2)| \approx (x_1^2 + x_2^2)^{(\mu-1)/2} \text{ near } (0, 0). \quad (7.46)$$

Consequently,

$$u \in W^{1,p}(\Omega) \iff p < \frac{2}{1-\mu}. \quad (7.47)$$

In particular, the fact that $2/(1-\mu) \searrow 2$ as $\mu \searrow 0$ shows that for each $p > 2$ there exists $\mu \in (0, 1)$ with the property that the operator $\mathcal{L} : \dot{W}^{1,p}(\Omega) \rightarrow W^{-1,p}(\Omega)$ fails to be an isomorphism. By duality, (\mathcal{L} is formally self-adjoint), the same type of conclusion holds for $p < 2$.

When $n \geq 3$, $m = 1$, $N = n$, a counterexample may be produced by altering (in the spirit of [42]) a construction of E. De Giorgi from [12]. Specifically, consider $\Omega := \{x \in \mathbb{R}^n : |x| < 1\}$ and, for each $\gamma \in [0, \frac{n}{2})$ and $\alpha, \beta \in \{1, \dots, n\}$, let $A_{\alpha\beta}$ be the $n \times n$ matrix whose (i, j) -entry is

$$a_{ij}^{\alpha\beta}(x) := \delta_{\alpha\beta} \delta_{ij} + \frac{\gamma(n-\gamma)(n-2)^2}{(n-2\gamma)^2(n-1)^2} \left[\delta_{i\alpha} + \frac{n}{n-2} \frac{x_i x_\alpha}{|x|^2} \right] \left[\delta_{j\beta} + \frac{n}{n-2} \frac{x_j x_\beta}{|x|^2} \right], \quad (7.48)$$

for each $x \in \Omega \setminus \{0\}$. Obviously, $a_{ij}^{\alpha\beta} \in L^\infty(\Omega, \mathcal{L}^n)$ and a straightforward calculation shows that

$$\sum_{\alpha, \beta=1}^n \sum_{i, j=1}^n a_{ij}^{\alpha\beta}(x) \zeta_i^\alpha \zeta_j^\beta = |\zeta|^2 + \frac{\gamma(n-\gamma)(n-2)^2}{(n-2\gamma)^2(n-1)^2} \left(\sum_{i=1}^n \zeta_i^i + \frac{n}{n-2} \sum_{i, \alpha=1}^n \zeta_i^\alpha \frac{x_i x_\alpha}{|x|^2} \right)^2 \quad (7.49)$$

for each $\zeta = (\zeta_i^\alpha)_{1 \leq \alpha, i \leq n} \in \mathbb{R}^{n^2}$ and $x \in \Omega \setminus \{0\}$. Given our assumptions on γ , it follows that the strong ellipticity condition holds:

$$\sum_{\alpha, \beta=1}^n \sum_{i, j=1}^n a_{ij}^{\alpha\beta}(x) \zeta_i^\alpha \zeta_j^\beta \geq |\zeta|^2 \quad \mathcal{L}^n\text{-a.e. in } \Omega, \quad \forall \zeta = (\zeta_i^\alpha)_{1 \leq \alpha, i \leq n} \in \mathbb{R}^{n^2}. \quad (7.50)$$

Now, the fact that $\gamma < n/2$ ensure that the function

$$u(x) := \frac{x}{|x|^\gamma} - x, \quad \forall x \in \Omega \setminus \{0\}, \quad (7.51)$$

belongs to $W^{1,2}(\Omega)$. Since by design $u|_{\partial\Omega} = 0$, we deduce that actually $u \in \mathring{W}^{1,2}(\Omega)$. Furthermore, if

$$f := (f_1, \dots, f_n) \quad \text{with} \quad f_i := - \sum_{\alpha=1}^n \sum_{j=1}^n \partial_\alpha a_{ij}^{\alpha j} \quad \text{for} \quad 1 \leq i \leq n, \quad (7.52)$$

then clearly

$$f \in \bigcap_{1 < p < \infty} W^{-1,p}(\Omega), \quad (7.53)$$

while a direct computation shows that

$$\sum_{\alpha, \beta=1}^n \partial_\alpha (A_{\alpha\beta}(x) \partial_\beta u) = f \quad \text{in} \quad \mathcal{D}'(\Omega). \quad (7.54)$$

However, on the one hand $u \in W^{1,p}(\Omega)$ if and only if $p < n/\gamma$, while on the other hand $n/\gamma \searrow 2$ as $\gamma \nearrow n/2$.

For $n \geq 2$ and higher-order operators we make use of an example originally due to V.G. Maz'ya (cf. [31]). Specifically, when $m \in \mathbb{N}$ is even, consider the divergence-form operator of order $2m$

$$\mathcal{L} := \Delta^{\frac{1}{2}m-1} \mathcal{L}_4 \Delta^{\frac{1}{2}m-1} \quad \text{in} \quad \Omega := \{x \in \mathbb{R}^n : |x| < 1\}, \quad (7.55)$$

where \mathcal{L}_4 is the fourth-order operator

$$\begin{aligned} \mathcal{L}_4 u := & a \Delta^2 u + b \sum_{i, j=1}^n \Delta \left(\frac{x_i x_j}{|x|^2} \partial_i \partial_j u \right) + b \sum_{i, j=1}^n \partial_i \partial_j \left(\frac{x_i x_j}{|x|^2} \Delta u \right) \\ & + c \sum_{i, j, k, l=1}^n \partial_k \partial_l \left(\frac{x_i x_j x_k x_l}{|x|^4} \partial_i \partial_j u \right). \end{aligned} \quad (7.56)$$

Obviously, the coefficients of \mathcal{L}_4 are bounded, and if the parameters $a, b, c \in \mathbb{R}$, $a > 0$, are chosen such that $b^2 < ac$ then \mathcal{L}_4 along with $\mathcal{L} = \Delta^{\frac{1}{2}m-1} \mathcal{L}_4 \Delta^{\frac{1}{2}m-1}$ are strongly elliptic. Now, it has been observed in [31] that if

$$\theta := 2 - \frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{(n-1)(bn+c)}{a+2b+c}}, \quad (7.57)$$

then the function $v(x) := |x|^{\theta+m-2}$ for each $x \in \Omega \setminus \{0\}$ belongs to $W^{m,2}(\Omega)$ and satisfies $\mathcal{L}v = 0$ in $\mathcal{D}'(\Omega)$. Furthermore, v is \mathcal{C}^∞ in a neighborhood of $\partial\Omega$ and, as such, there exists a function $w \in \mathcal{C}^\infty(\bar{\Omega})$ with the property that

$$u := v - w \in \mathring{W}^{m,2}(\Omega). \quad (7.58)$$

Note that, by design, $\mathcal{L}u = f$ in $\mathcal{D}'(\Omega)$, where

$$f := -\mathcal{L}w \in \bigcap_{1 < p < \infty} W^{-m,p}(\Omega), \quad (7.59)$$

and $u \in W^{m,p}(\Omega)$ if and only if $v \in W^{m,p}(\Omega)$. In order to focus on the veracity of the latter condition, we find it convenient to specialize matters by taking $a := (n-2)^2 + \varepsilon$, $b := n(n-2)$, $c := n^2$, for some small $\varepsilon > 0$. The strong ellipticity condition is satisfied, and the parameter θ from (7.57) becomes

$$\theta(\varepsilon) = 2 - \frac{n}{2} + \frac{n\varepsilon^{1/2}}{2\sqrt{4(n-1)^2 + \varepsilon}}. \quad (7.60)$$

However, $v \in W^{m,p}(\Omega)$ if and only if $p < n/(2 - \theta(\varepsilon))$, and the bound $n/(2 - \theta(\varepsilon))$ approaches 2 when $\varepsilon \rightarrow 0^+$. The bottom line is that range of p 's in the interval $(2, \infty)$ for which $u \in W^{m,p}(\Omega)$ shrinks to 2 as $\varepsilon \rightarrow 0^+$.

In [31] an analogous example was also constructed when $m > 1$ is odd, starting with the sixth order operator

$$\begin{aligned} \mathcal{L}_6 u := & a \Delta^3 u + b \sum_{i,j,k,l=1}^n \partial_i \partial_j \partial_k \left(\frac{x_i x_j x_k x_l}{|x|^4} \Delta \partial_l u \right) + b \sum_{i,j,k,l=1}^n \Delta \partial_l \left(\frac{x_i x_j x_k x_l}{|x|^4} \partial_i \partial_j \partial_k u \right) \\ & + c \sum_{i,j,k,l,r,t=1}^n \partial_i \partial_j \partial_k \left(\frac{x_i x_j x_k x_l x_r x_t}{|x|^6} \partial_l \partial_r \partial_t u \right) \end{aligned} \quad (7.61)$$

and then considering

$$\mathcal{L} := \Delta^{\frac{m-3}{2}} \mathcal{L}_6 \Delta^{\frac{m-3}{2}} \quad \text{in } \Omega := \{x \in \mathbb{R}^n : |x| < 1\}. \quad (7.62)$$

For the choice $a := (n-4)^2 + \varepsilon$, $b := (n-4)(n+2)$, $c := (n+2)^2$, $\varepsilon > 0$, the operator (7.62) is strongly elliptic and the function $v(x) := |x|^{\mu+m-3}$ for each $x \in \Omega \setminus \{0\}$ belongs to $W^{m,2}(\Omega)$ and satisfies $\mathcal{L}v = 0$ in $\mathcal{D}'(\Omega)$ if $\mu = \mu(\varepsilon)$ given by

$$\mu := 3 - \frac{n}{2} + \frac{(n+2)(n-4)}{2} \sqrt{\frac{\varepsilon}{4(n-1)^2 + \varepsilon}}. \quad (7.63)$$

Moreover, $v \in W^{m,p}(\Omega)$ if and only if $p < n/(3 - \mu(\varepsilon))$, and the bound $n/(3 - \mu(\varepsilon))$ approaches 2 when $\varepsilon \rightarrow 0^+$. With this in hand and proceeding as in the previous case, the same type of conclusion may be derived in the current setting as well.

Moving on, we wish to extend the well-posedness result for the higher-order inhomogeneous Dirichlet problem from Theorem 7.4 by treating its fully inhomogeneous version, albeit in a more resourceful geometrical setting.

Theorem 7.5 (Well-posedness of the fully inhomogeneous higher-order Poisson problem). *Let Ω be a bounded (ε, δ) -domain in \mathbb{R}^n , $n \geq 2$, with $\text{rad}(\Omega) > 0$, and whose boundary is d -Ahlfors regular for some $d \in [n-1, n)$. In addition, consider an $M \times M$ divergence-form system \mathcal{L} of order $2m$ with bounded measurable coefficients, as in (7.1)-(7.2), for some $m \in \mathbb{N}$, and suppose that \mathcal{L} satisfies the strong ellipticity condition (7.23) for some $\kappa > 0$.*

Then there exists $p_ \in (2, \infty)$ sufficiently close to 2, which depends only on $n, m, M, d, \Omega, A, \kappa$, and with the property that the fully inhomogeneous Poisson problem*

$$\begin{cases} \mathcal{L}u = f \in W^{-m,p}(\Omega), \\ u \in W^{m,p}(\Omega), \\ \mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)} u = \dot{g} \in B_{m-(n-d)/p}^{p,p}(\partial\Omega), \end{cases} \quad (7.64)$$

is well-posed whenever $\frac{p_}{p_*-1} < p < p_*$.*

Proof. Let $p_* > 2$ be as in Theorem 7.4 (without loss of generality, it may be assumed that p_* is sufficiently close to 2), and fix some p such that $\frac{p_*}{p_*-1} < p < p_*$. Suppose now that an arbitrary

$f \in W^{-m,p}(\Omega)$ and $\dot{g} \in B_{m-(n-d)/p}^{p,p}(\partial\Omega)$ have been given. Since in the current context Corollary 5.4 guarantees that the restriction operator $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)}$ maps $W^{m,p}(\Omega)$ onto $B_{m-(n-d)/p}^{p,p}(\partial\Omega)$, it follows that there exists $v \in W^{m,p}(\Omega)$ such that $\mathcal{R}_{\Omega \rightarrow \partial\Omega}^{(m)} v = \dot{g}$. Moreover, as a consequence of the Open Mapping Theorem, it may be assumed that $\|v\|_{W^{m,p}(\Omega)} \leq C \|\dot{g}\|_{B_{m-(n-d)/p}^{p,p}(\partial\Omega)}$ for some finite constant $C > 0$ independent of \dot{g} . If we now use the well-posedness statement in Theorem 7.4 in order to solve the inhomogeneous Dirichlet boundary value problem

$$\begin{cases} \mathcal{L}w = f - \mathcal{L}v \in W^{-m,p}(\Omega), \\ w \in \dot{W}^{m,p}(\Omega), \end{cases} \quad (7.65)$$

it follows that $u := v + w$ solves the original problem (7.64) (keeping in mind (5.37)), and also obeys natural estimates. Finally, uniqueness for (7.64) is a consequence of the uniqueness part in Theorem 7.4 and (5.37). \square

The well-posedness of the fully inhomogeneous Poisson problem (7.64) has been established in [35] in the context of weighted Sobolev spaces in Lipschitz domains, for strongly elliptic higher-order systems with bounded measurable coefficients, when the integrability parameter p belongs to a small neighborhood of 2. In [35], the authors have also proved that problem (7.64) continues to be well-posed if, additionally, the outward unit normal ν to the Lipschitz domain Ω belongs to $VMO(\partial\Omega)$, the Sarason space of functions with vanishing mean oscillations and the coefficients of the operator \mathcal{L} are in $VMO(\Omega)$.

We conclude this section with the following solvability result for the inhomogeneous Neumann problem in (ε, δ) -domains.

Theorem 7.6 (Well-posedness of the higher-order Neumann boundary problem). *Let Ω be a bounded, connected (ε, δ) -domain in \mathbb{R}^n , $n \geq 2$, with $\text{rad}(\Omega) > 0$, and suppose that \mathcal{L} is an $M \times M$ divergence-form system of order $2m$, as in (7.1)-(7.2), for some $m \in \mathbb{N}$, which satisfies the strong ellipticity condition (7.23) for some constant $\kappa > 0$.*

Then there exists $p_ \in (2, \infty)$, depending only on $n, m, M, \Omega, D, A, \kappa$, with the following significance. If $\frac{p_*}{p_*-1} < p < p_*$ then for each functional $f \in (W^{m,p'}(\Omega))^*$ (where $1/p + 1/p' = 1$), the inhomogeneous Neumann boundary value problem*

$$\begin{cases} \mathcal{L}u = f|_{\Omega} \text{ in } \mathcal{D}'(\Omega), \\ u \in W^{m,p}(\Omega), \\ \partial_{\nu}^A(u, f) = 0 \text{ on } \partial\Omega, \end{cases} \quad (7.66)$$

(with the last condition understood in the sense of Definition 7.1 specialized to the case when $D = \emptyset$) has a solution if and only if

$$(W^{m,p'}(\Omega))^* \langle f, v \rangle_{W^{m,p'}(\Omega)} = 0, \quad \forall v \in \mathcal{P}_{m-1}(\Omega), \quad (7.67)$$

in which scenario solutions of (7.66) are unique up to functions from $\mathcal{P}_{m-1}(\Omega)$.

Proof. This is an immediate consequence of Theorem 7.3 specialized to the case in which $D := \emptyset$ (cf. also Lemma 3.3 in this regard). \square

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