ON GRADED CHARACTERIZATIONS OF FINITE DIMENSIONALITY FOR ALGEBRAIC ALGEBRAS

EDWARD S. LETZTER

ABSTRACT. We observe that a finitely generated algebraic algebra R (over a field) is finite dimensional if and only if the associated graded ring $\mathrm{gr}\,R$ is right noetherian, if and only if $\mathrm{gr}\,R$ has right Krull dimension, if and only if $\mathrm{gr}\,R$ satisfies a polynomial identity.

1. INTRODUCTION

Examples of infinite dimensional, finitely generated, algebraic algebras (over fields) were first produced by Golod and Shafarevich in 1964 [6], providing a negative answer to the longstanding and famous Kurosh Problem. Since the early 2000s there has been increased interest – and several new significant results – in the study of Kurosh-type and related problems for associative algebras; see, e.g., [3], [16], and [18] for an introduction and overview. A thumbnail sketch of relevant developments could include: Smoktunowicz's 2002 construction of a simple nil ring over an arbitrary countable field [15]; Bell and Small's 2002 construction of a finitely generated primitive algebraic algebra over an arbitrary field [2]; Lenagan and Smoktunowicz's 2007 construction of an infinite dimensional nil algebra, over an arbitrary countable field, of finite Gelfand-Kirillov (GK-) dimension [9]; Lenagan, Smoktunowicz, and Young's 2012 construction of an infinite dimensional nil algebra, over an arbitrary countable field, of GK-dimension at most three [10]; and Bell, Small, and Smoktunowicz's construction of an infinite dimensional primitive algebraic algebra, over an arbitrary countable field, of GK-dimension at most six [3].

Included among prior results specifically focused on associated graded rings are Smoktunowicz's 2010 example of a finitely generated algebaic algebra, over an arbitrary countable field, for which the associated graded ideal spanned by homogeneous elements of positive degree is not nil [17], and Regev's 2010 theorem, for finitely generated algebraic algebras over an uncountable field, that the associated graded ideal spanned by homogeneous elements of positive degree must be nil [14]. In both cases, these ideals are graded nil. See (2.2) below.

Our main result, presented in somewhat more precise form in (2.4), asserts that a finitely generated algebraic algebra R over a field is finite dimensional if and only if the associated graded ring $\mathrm{gr}R$ is right noetherian, if and only if $\mathrm{gr}R$ has right Krull dimension, if and only if $\mathrm{gr}R$ satisfies a polynomial identity.

Numerous naturally arising examples of infinite dimensional, finitely generated algebras whose associated graded rings are noetherian can be found in [4] and [11]. In-depth treatments of graded and filtered rings can be found in [11] and [13].

2. Graded Characterizations of Finite Dimensionality

2.1. Let k be a field, and let R be a k-algebra generated by a finite dimensional k-subspace V of R with $1 \in V$. The following standard setup will remain in effect throughout this note:

(i) Letting $V^{-1} = 0$ and $V^0 = k$, set V^m equal to the k-vector space spanned by products over V of length m.

(ii) For $a \in R$, set $|a|$ equal to the minimum integer m such that $a \in V^m$, and set

$$\widehat{a} = a + V^m \in V^m/V^{m-1}.$$

Multiplication in the associated ($\mathbb{Z}$-)graded ring

$$\mathrm{gr}R = \bigoplus_{i=0}^{\infty} V^i/V^{i-1}$$

is determined via

$$\widehat{a} \cdot \widehat{b} = ab + V^{m+n-1} \in V^{m+n}/V^{m+n-1},$$

for $|a| = m$ and $|b| = n$.

(iii) Set

$$\mathrm{gr}R^+ = \bigoplus_{i=1}^{\infty} V^i/V^{i-1},$$

a graded ideal of $\mathrm{gr}R$ that is k-linearly spanned by the homogeneous elements $\widehat{a}$, for $a \in R$, $|a| \geq 1$. It is easy to check that R is finite dimensional if and only if $\mathrm{gr}R$ is finite dimensional, if and only if $\mathrm{gr}R^+$ is nilpotent.

(iv) A subset S of R is *nil* if each element of S is nilpotent. A graded subring of a group graded ring is *graded nil* if each homogeneous element is nilpotent.

2.2. Assuming R is algebraic over k, Smoktunowicz proved that $\mathrm{gr}R^+$ need not be nil if k is countable [17], and A. Regev proved that $\mathrm{gr}R^+$ must be nil if k is uncountable [14]. However, for arbitrary choices of k, it can easily be checked that $\mathrm{gr}R^+$ is graded nil if R is algebraic over k.

2.3. Next, we give a brief survey of Jacobson's theorem [7] (and some of its applications) concerning the nilpotence of certain subrings of artinian rings.

(i) A subset B of a ring A is *weakly closed* if for each pair of elements $a, b \in B$ there exists an element $\gamma(a, b)$ in the center of A such that $ab + \gamma(a, b)ba \in B$. In [7] it is proved that if A is artinian then subrings of A generated by nil weakly closed subsets are nilpotent. The earlier theorem of Levitski states that a nil one-sided ideal of a right noetherian ring is nilpotent.

(ii) Goldie's Theorem can be employed to obtain the following corollary, also due to Goldie [5, Theorem 6.1]: If A is a right noetherian ring, then a nil weakly closed subset of A generates a nilpotent subring. In fact, the proof of this last result more generally shows: Suppose that A is a ring, that the prime radical J of A is nilpotent, and that A/J embeds in a right artinian ring (e.g., A/J is right Goldie). Then the weakly closed nil subsets of A generate nilpotent subrings of A.

(iii) Montgomery and Small apply Goldie's result to conclude that graded nil subrings of noetherian group graded rings must be nilpotent; see [12, Corollary 1.2]. Their key insight in this situation is that the set of homogeneous elements of a graded subring of a group graded ring is weakly closed. Consequently, their argument can be applied to show that if A is a group graded ring with nilpotent prime radical J, and if A/J embeds in a right artinian ring, then the graded nil subrings of A are nilpotent.

(iv) Assuming that A is a ring with right Krull dimension, then the prime radical J of A is nilpotent and every semiprime factor ring of A is right Goldie; see for example [11, Chapter 6] for details. Therefore, by (iii), if A is a group graded ring with right Krull dimension then graded nil subrings of A are nilpotent.

(v) Assuming that A is a finitely generated algebra, over a field k, satisfying a polynomial identity, then the prime radical of A is nilpotent and every semiprime factor of A is a Goldie ring; see for example [11, Chapter 13]. Therefore, again by (iii), if A is a group graded ring satisfying a polynomial identity, then the graded nil subrings of A are nilpotent.

We now present the main result. Recall R, $\mathrm{gr}R$, and $\mathrm{gr}R^+$ from (2.1).

Theorem 2.4. *If R is algebraic over k then the following conditions are equivalent:* (i) R *is finite dimensional.* (ii) $\mathrm{gr}R$ *is finite dimensional.* (iii) $\mathrm{gr}R^+$ *is nilpotent.* (iv) $\mathrm{gr}R$ *is right noetherian.* (v) $\mathrm{gr}R$ *has right Krull dimension.* (vi) $\mathrm{gr}R$ *satisfies a polynomial identity.*

Proof. To start, it is easy to check that (i), (ii), and (iii) are equivalent, as already noted in (2.1.iii). Next, it is immediate that (ii) implies (iv), (v), and (vi).

That (iv) implies (iii) follows from (2.3.iii), that (v) implies (iii) follows from (2.3.iv), and that (vi) implies (iii) follows from (2.3.iv).

The result follows. □

2.5. Following [11, Chapter 8, §3] (also see [8]), if A is a finitely generated algebra, over a field k, of finite GK-dimension, and if GK-dimension is *finitely partitive* for A, then A has finite right Krull dimension. Consequently, the finitely generated algebraic algebra R of (2.4) is finite dimensional if and only if $\mathrm{gr}\,R$ is finitely partitive for GK-dimension. In [1], Bell gives examples of finitely generated algebras of Gelfand-Kirillov dimension 2, over arbitrary fields, that are not finitely partitive for GK-dimension.

References

[1] J. P. Bell, Examples in finite Gel'fand-Kirillov dimension II, *Comm. Algebra*, 33 (2005), 3323–3334.

[2] J. P. Bell and L. W. Small, A question of Kaplansky, Special issue in celebration of Claudio Procesi's 60th birthday, *J. Algebra*, 258 (2002), 386–388.

[3] J. P. Bell, L. W. Small, A. Smoktunowicz, Primitive algebraic algebras of polynomially bounded growth, in *New trends in noncommutative algebra*, Contemp. Math. 562 (Amer. Math. Soc., Providence, 2012), 41–52.

[4] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics, CRM Barcelona (Birkhäuser Verlag, Basel, 2002).

[5] A. W. Goldie, Semi-prime rings with maximum condition, *Proc. London Math. Soc. (3)*, 10 (1960), 201–220.

[6] E. S. Golod and I. R. Šafarevič, I. R., On the class field tower, *Izv. Akad. Nauk SSSR Ser. Mat.*, 28 (1964), 261–272.

[7] N. Jacobson, Une généralisation du théorème d'Engel, *C. R. Acad. Sci. Paris*, 234 (1952), 579–581.

[8] Günter R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Graduate Studies in Mathematics 22, Revised Edition (American Mathematical Society, Providence, RI, 2000).

[9] T. H. Lenagan and A. Smoktunowicz, An infinite dimensional affine nil algebra with finite Gelfand-Kirillov dimension, *J. Amer. Math. Soc.*, 20 (2007), 989–1001.

[10] T. H. Lenagan, A. Smoktunowicz, and A. Young, Nil algebras with restricted growth, *Proc. Edinb. Math. Soc. (2)*, 55 (2012), 461–475.

[11] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics 30 (American Mathematical Society, Providence, 2000).

[12] S. Montgomery and L. W. Small, Nil subsets of graded algebras, *Proc. Amer. Math. Soc.*, 126 (1998), 653–656.

[13] C. Năstăsescu and F. van Oystaeyen, *Graded ring theory*, North-Holland Mathematical Library 28 (North-Holland Publishing Co., Amsterdam-New York, 1982).

[14] A. Regev, Filtered algebraic algebras, *Proc. Amer. Math. Soc.*, 138, (2010), 1941–1947.

[15] A. Smoktunowicz, A simple nil ring exists, *Comm. Algebra*, 30 (2002), 27–59.

[16] ———, Some results in noncommutative ring theory, in *International Congress of Mathematicians. Vol. II*, (Eur. Math. Soc., Zürich, 2006), 259–269.

[17] ———, Graded algebras associated to algebraic algebras need not be algebraic, in *European Congress of Mathematics* (Eur. Math. Soc., Zürich, 2010), 441–449.

[18] E. Zelmanov, Some open problems in the theory of infinite dimensional algebras, *J. Korean Math. Soc.*, 44 (2007), 1185–1195.

Department of Mathematics, Temple University, Philadelphia, PA 19122-6094
E-mail address: letzter@temple.edu