NOETHERIAN SKEW POWER SERIES RINGS

A Dissertation
Submitted to
the Temple University Graduate Board

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

by
Linhong Wang
August, 2008

ABSTRACT

NOETHERIAN SKEW POWER SERIES RINGS

Linhong Wang
DOCTOR OF PHILOSOPHY

Temple University, August, 2008

Professor Edward S. Letzter, Chair

This dissertation is concerned with noncommutative analogues of formal power series rings in multiple variables. Our motivating examples arises from quantized coordinate rings; the completions of these quantized coordinate rings are iterated noetherian skew power series rings.

Our first focus is on q-commutative power series rings, having the following form: $R = \mathbf{k}_q[[x_1, \ldots, x_n]]$, where $q = (q_{ij})_{n\times n}$ with $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1} \in \mathbf{k}^\times$ and where $x_j x_i = q_{ji} x_i x_j$. The corresponding skew Laurent series ring is $L = \mathbf{k}_q[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]]$. We first study the ideal structure of L. We prove that extension and contraction of ideals produces a bijection between the set of ideals of L and the set of ideals of the center Z of L. This bijection further produces a homeomorphism between $\operatorname{Spec} L$ and $\operatorname{Spec} Z$. Applying the analysis of L to R, we prove that the prime spectrum $\operatorname{Spec} R$ can be partitioned into finitely many strata each homeomorphic to the prime spectrum of a commutative noetherian ring. The rings R and L are completions, respectively, of the quantum coordinate ring of n-space and of the n-torus.

Our second focus is on power series completions of iterated skew polynomial rings with nonzero derivations. Given an iterated skew polynomial ring $C[y_1; \tau_1, \delta_1] \ldots [y_n; \tau_n, \delta_n]$ over a complete local ring C with maximal ideal $\mathfrak{m}$, we prove, under suitable assumptions, that the completion at the ideal $\mathfrak{m} + \langle y_1, y_2, \ldots, y_n \rangle$ is an iterated skew power series ring. Under further conditions, this completion is a local, noetherian, Auslander regular domain. Ap-

plicable examples include the following quantized coordinate rings: quantum matrices, quantum symplectic spaces, and quantum Euclidean spaces.

Results in this dissertation are included in the following two preprints:

1. Prime ideals of q-commutative power series rings (joint with E. S. Letzter), submitted for publication.

2. Completions of quantum coordinate rings, to appear in *Proceedings of the American Mathematical Society*.

ACKNOWLEDGEMENTS

The author is deeply indebted to her thesis advisor, Professor E. S. Letzter, for his constant guidance, generous help, and warmest encouragement during the course of the dissertation research and the writing of the thesis. Gratitude is due as well to Dr. B. Datskovsky and Dr. M. Lorenz for carefully reading preliminary versions of this dissertation and for offering useful comments and helpful suggestions. Lastly, the author would like to express her appreciation for the support and help from Dr. O. Hijab, the Chair of Mathematics Department, Temple University.

To Sheng Xiong

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In this dissertation, we study noncommutative analogues of formal power series rings. The primary motivating examples arise from quantized coordinate rings; the completions of these quantized coordinate rings are iterated noetherian skew power series rings. The skew power series rings discussed in this dissertation are divided into two classes. The first class consists of q-commutative power series rings. The second class consists of iterated skew power series rings with nonzero derivations.

Results in this dissertation are included in [22] and [30].

q-commutative power series rings

Given a field $\mathbf{k}$ and an $n \times n$ matrix $q = (q_{ij})$, with $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1} \in \mathbf{k}^{\times}$, we can construct the "$q$-commutative" power series ring $R = \mathbf{k}_q[[x_1, \ldots, x_n]]$. Multiplication is determined by the commutation relations $x_j x_i = q_{ji} x_i x_j$, leading to a rich combinatorial structure (see, e.g., [21]). Moreover, it follows from longstanding ring-theoretic results and techniques (cf. [23], [29]) that R is a complete, local, Auslander regular, noetherian domain with Krull dimension, classical Krull dimension, and global dimension all equal to n (see 3.1.9 and 3.1.11).

We can also construct $L = \mathbf{k}_q[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]]$, the corresponding skew Laurent series ring. We mainly focus on the ideal theory of L and prime ideal theory of R. Our approach builds on earlier studies (particularly [6, §4], [10, §2], and [13]) on q-commutative polynomial rings (also known as "quantum affine spaces" and "twisted polynomial algebras"). However, our work in this dissertation frequently involves topological considerations not needed in the earlier studies.

Our results depend on a careful examination of the two-sided ideal structure of L. Our analysis in this setting can be briefly described as follows:

To start, there is an obvious action of the n-torus $H = (\mathbf{k}^\times)^n$ on L (and R) by automorphisms. In 3.2.1 we prove, when $\mathbf{k}$ is infinite, that L is H-simple (i.e., that the only H-stable ideals of L are the zero ideal and L itself). Consequently, every H-orbit of prime ideals of L is Zariski dense in $\operatorname{Spec} L$ (when $\mathbf{k}$ is infinite).

Next, in 3.2.7, we prove that extension and contraction of ideals produces a bijection

$$\{\text{ideals of } L\} \longleftrightarrow \{\text{ideals of the center } Z = Z(L)\}$$

for any choice of the field $\mathbf{k}$. This bijection produces a homeomorphism between the prime spectrum $\operatorname{Spec} L$, equipped with the Zariski topology, and $\operatorname{Spec} Z$; see 3.2.10. Moreover, Z is a commutative noetherian domain.

Our results for L parallel those found in [6, §4], [10, §2], and [13, §1] for q-commutative Laurent polynomial rings (also known as "quantum tori" and "McConnell-Pettit algebras"). Additional references are given in 3.2.7.

The analysis of L can be applied to R as follows. First of all, the $x_1, \ldots, x_n$ provide a stratification

$$\operatorname{Spec} R = \bigsqcup_{w \in W} \operatorname{Spec}_w R,$$

where each w is a subset of $\{1, \ldots, n\}$, and where

$$\operatorname{Spec}_w R = \{P \in \operatorname{Spec} R \mid x_i \in P \Leftrightarrow i \in w\}.$$

Each $\operatorname{Spec}_w R$ is naturally homeomorphic to $\operatorname{Spec} L_w$, where L_w is a q-commutative Laurent series ring in $n - |w|$ variables, for a suitable replacement of the original matrix q, and then is homeomorphic to the spectrum of a commutative noetherian domain; see 3.3.1.

Furthermore, each $\operatorname{Spec}_w R$ is a union of H-orbits in $\operatorname{Spec} R$. When $\mathbf{k}$ is infinite, every H-orbit in $\operatorname{Spec}_w R$ is dense in $\operatorname{Spec}_w R$ (see 3.3.5), and the H-prime ideals of R are precisely the ideals generated by some of the $x_1, \ldots, x_n$ (see 3.3.6).

These results parallel those found in [6, §4], [10, §2], and [13, §2] for q-commutative polynomial rings.

Following the approach in [6], we also consider issues related to the localization and representation theory of R, in the sense of [7] and [18]. First, in 3.3.7 (i), we show that $\operatorname{Spec} R$ is normally separated. Consequently, R satisfies the strong second layer condition. Second, letting G denote the group of automorphisms of R generated by $r \mapsto x_i r x_i^{-1}$, for $r \in R$ and $1 \leq i \leq n$, we show in 3.3.7 (ii) that if $P \rightsquigarrow Q$ in $\operatorname{Spec} R$ then there exists an automorphism $\tau \in G$ such that $\tau(P) = Q$.

In addition to similarities with q-commutative polynomial rings, the behavior of q-commutative power series rings also has deep analogues in the theory of quantum semisimple groups and other quantum function algebras; see for example [5], [6], [17], and [19].

Completions of quantum coordinate rings

Let R be a ring equipped with a skew derivation (τ, δ). The skew power series ring $R[[y; \tau]]$, when $\delta = 0$, is a well known, classical object (cf. [9], [24]). The skew power series ring $R[[y; \tau, \delta]]$, when $\delta \neq 0$, has more recently appeared in quantum algebras (cf. [20, §4], [21, §4]) and in noncommutative Iwasawa theory (cf. [26], [28]).

Let S be the additive group of formal power series in y,

$$\sum_{i=0}^{\infty} r_i y^i,$$

with coefficients r_i in R. This additive group always contains a subgroup which is the underlying set of the skew polynomial ring $R[y\ \tau, \delta]$. The multiplication in the ring $R[y\ \tau, \delta]$ is determined by the commutation relation $yr = \tau(r)y + \delta(r)$, for $r \in R$. In [28], Venjakob proved that $S = R[[y; \tau, \delta]]$ is a well-defined skew power series ring (with the same multiplication as that of $R[y\ \tau, \delta]$) under the following hypotheses: R is a complete local ring, $\tau(\mathfrak{m}) \subseteq \mathfrak{m}$, $\delta(R) \subseteq \mathfrak{m}$, and $\delta(\mathfrak{m}) \subseteq \mathfrak{m}^2$.

Our research builds on the work of Venjakob in [28]. We extend the construction in [28] to suitable iterated skew polynomial rings, and then obtain the power series completions of these polynomial rings. Our main result can be stated as follows, see 4.1.7:

Let

$$R_n = C[y_1; \tau_1, \delta_1] \dots [y_l; \tau_l, \delta_l] \dots [y_n; \tau_n, \delta_n]$$

be an iterated skew polynomial ring, where C is a complete local ring with maximal ideal $\mathfrak{m}$, and where C is stable under each skew derivation (τ_l, δ_l). For each $1 \leq l \leq n$, let $I_{l-1} = \mathfrak{m} + \langle y_1, \dots, y_{l-1} \rangle$, and suppose that $\tau_l(I_{l-1}) \subseteq I_{l-1}$, $\delta_l(R_{l-1}) \subseteq I_{l-1}$, and $\delta_l(I_{l-1}) \subseteq I_{l-1}^2$. Then there exists an iterated skew power series ring

$$S_n = C[[y_1; \hat{\tau}_1, \hat{\delta}_1]] \dots [[y_l; \hat{\tau}_l, \hat{\delta}_l]] \dots [[y_n; \hat{\tau}_n, \hat{\delta}_n]],$$

such that $\hat{\tau}_l \mid_{R_{l-1}} = \tau_l$ and $\hat{\delta}_l \mid_{R_{l-1}} = \delta_l$, for $1 \leq l \leq n$. Moreover, S_n is the completion of R_n at the ideal $\mathfrak{m} + \langle y_1, \dots, y_l \rangle$.

By applying the above Theorem, we construct power series completions of the following quantized coordinate rings at the ideal generated by all variables: quantum matrices, quantum symplectic spaces, quantum Euclidean spaces, and more general the class of algebras $K^{P,Q}_{n,\Gamma}(\mathbf{k})$ introduced by Horton [15]. Moreover, it follows again from longstanding ring-theoretic results and techniques (cf. [23], [29]) that completions of $K^{P,Q}_{n,\Gamma}(\mathbf{k})$ are local Auslander regular, noetherian domains with Krull dimension, classical Krull dimension, and global dimension all equal to the number of variables.

Organization

The dissertation is organized as follows: The second chapter provides necessary definitions and background results from noncommutative ring theory and topological ring theory. The third chapter considers prime ideals of q-commutative power series rings and q-commutative Laurent series rings. The fourth chapter considers completions of quantum coordinate rings and iterated skew power series ring over general complete local rings.

The structure of chapter two is as follows: Section one concerns prime spectrum. Section two records the generalities about local rings, graded rings and filtered rings. Section three discusses Ore set and localization. Section four deals with dimensions of noetherian rings including classical Krull dimension, Krull dimension, and global dimension. Section five discusses some known results concerning skew polynomial rings and skew power series rings. Section six provides three classes of quantum groups which are motivating examples of our study.

Chapter three is organized as follows: Section one provides basic properties of q-commutative power series rings. Section two discusses the ideal theory of q-commutative Laurent series rings. Section three applies the results in section two and deals with the prime ideal theory of q-commutative power series rings.

Chapter four consists of two sections: Section one discusses the construction of iterated skew power series rings over general complete local rings. Section two presents the completions of various quantum coordinate rings.

CHAPTER 2

PRELIMINARY

This chapter provides a minimal amount of basic theory of noetherian rings and filtered rings necessary to describe and prove our results. Almost all of the results recorded here are either well known or are easily deduced from well known results. Our exposition has been strongly influenced by [14], [23], and [24], and the reader is encouraged to consult these works for more details on what follows. No attempt has been made here to present results in their fullest generality.

The rings we consider will always be associative and will always contain an identity element. Also, when we denote I to be an *ideal* of a ring, we mean that I is both a left and right ideal. All homomorphisms and modules are unital.

Definition 2.0.1. R is right (left) *noetherian* provided R satisfies the ascending chain condition on right (left) ideals, i.e., there does not exist a properly ascending infinite chain $I_1 < I_2 < \ldots$ of right (left) ideals of R.

Definition 2.0.2. R is right (left) *artinian* provided R satisfies the descending chain condition on right (left) ideals, i.e., there does not exist a properly descending infinite chain $I_1 > I_2 > \ldots$ of right (left) ideals of R.

2.1 Prime spectrum

Throughout this section, let R be a ring.

Definition 2.1.1. A proper (i.e., not equal to R itself) ideal P of R is *prime* if for any two ideals I and J of R the following property holds: If $IJ \subseteq P$, then one of I or J is contained in P. Also, R is said to be *prime* if its zero ideal is prime.

Proposition 2.1.2. *For a proper ideal P in R, the following are equivalent:*

(a) *P is a prime ideal.*

(b) *R/P is a prime ring.*

(c) *If I and J are any right (respectively, left) ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.*

(d)*If $x, y \in R$ with $xRy \subseteq P$, then either $x \in P$ or $y \in P$.*

Proposition 2.1.3. *Let Z be the center of R. If P is a prime ideal of Z, then $P \cap Z$ is a prime ideal of Z.*

Definition 2.1.4. A prime ideal P of R is *completely prime* provided that R/P is an integral domain.

In the special case of noncommutative power series ring, we will study the prime ideal structure. Next we briefly review the Zariski topology on the set of prime ideals of a ring.

Definition 2.1.5. (i) The set of prime ideals of R is called the *prime spectrum* of R, and denoted by $\operatorname{Spec} R$. (ii) For each ideal I of R, set

$$V(I) = \{P \in \operatorname{Spec} R \mid P \supseteq I\}.$$

Subsets of $\operatorname{Spec} R$ of the form $V(I)$ are called *algebraic subsets*.

It is easy to check that the following are true: $\varnothing$ and $\operatorname{Spec} R$ are algebraic subsets of $\operatorname{Spec} R$; if I and J are ideals of R then $V(I) \cup V(J) = V(IJ)$; if A is any set of ideals of R, and if $I(A)$ denotes the ideal generated by the set $\cup_{I \in A} I$ then $\cap_{I \in A} V(I) = V(I(A))$.

Definition 2.1.6. $\operatorname{Spec} R$ is a topological space, with the closed sets taken to be the algebraic subsets, that is, precisely those of the form $V(I) = \{P \in \operatorname{Spec} A \mid P \supseteq I\}$, for ideals I of R. This topology is referred to as the *Zariski topology.*

Definition 2.1.7. A subset of $\operatorname{Spec} R$ is said to be a *locally closed* subset if it is an intersection of an open subset and a closed subset of $\operatorname{Spec} R$.

Next, following [14], we consider normal separation of the prime spectrum of a ring.

Definition 2.1.8. An element x in a ring R is said to be a normal element of R if $xR = Rx$.

Definition 2.1.9. $\operatorname{Spec} R$ is *normally separated* if for each inclusion of prime ideals $P_0 \subsetneq P_1$, the ideal P_1/P_0 of R/P_0 contains a nonzero normal element of R/P_0; that is, there exists an element $y \in P_1 \setminus P_0$ such that $Ry+P_0 = yR+P_0$.

Definition 2.1.10. Let P and Q be prime ideals of a noetherian ring R. There is a *second layer link* $P \rightsquigarrow Q$ provided $P \cap Q/PQ$ has a nonzero R-R-bimodule factor that is torsionfree (i.e., the annihilator of any nonzero element is the zero ideal) as both a left R/P-module and a right R/Q-module.

A definition of *right strong second layer condition* can be found, for example, in [14, p. 206], which is related to the Jategaonkar's Main Lemma.

Theorem 2.1.11. *Any noetherian ring R with normal separation satisfies the right and left strong second layer condition.*

Proof. See, e.g., [14, 12.17]. □

Let Γ be a group of automorphisms of a ring R. The following are standard terminology for ideals satisfying certain conditions related to the group action.

Definition 2.1.12. An ideal I of R is *stable* (or *invariant*) under the Γ-action provided $\gamma(I) = I$ for all $\gamma \in \Gamma$. Ideals of R stable under the Γ-action are referred to as Γ-*ideals*, and we will say that R is Γ-*simple* if the only Γ-ideals of R are the zero ideal and R itself.

Definition 2.1.13. We will say that a proper Γ-ideal I of R is Γ-*prime* if whenever the product of two Γ-ideals of R is contained in I, one of these Γ-ideals must itself be contained in I. Note that any prime Γ-ideal is Γ-prime.

Proposition 2.1.14. *Suppose R is right or left noetherian, and suppose that I is a Γ-prime ideal of R. Then I is a semiprime ideal of R and furthermore is the intersection of a finite Γ-orbit $I_1, \ldots, I_t$ of prime ideals in R all minimal over I.*

Proof. See [11, Remarks 4^*, 5^*, p. 338] □

2.2 Local rings, graded and filtered rings

Definition 2.2.1. An ideal P of a ring R is right (left) *primitive* if it is the annihilator of a simple right (left) R-module. The intersection of all the right (left) primitive ideals of a ring R is termed the *Jacobson radical* of R and denoted by $J(R)$.

It follows easily that a right or left primitive ideal is prime. Also, unlike the commutative case, a right or left primitive ideal need not be maximal.

Definition 2.2.2. Let R be a ring. The following conditions are equivalent:

(a) R is prime and right (left) artinian.

(b) R is simple and right (left) artinian.

(c) R is simple and semisimple.

(d) $R \cong M_n(D)$ for some positive integer n and some division ring D.

Such a ring R is referred to as a *simple artinian* ring.

Definition 2.2.3. By a *local ring*, we will always mean a ring R such that the quotient ring by the Jacobson radical $J(R)$ is simple artinian.

It follows immediately that a local ring has a unique maximal ideal (also the unique primitive ideal) which is equal to the Jacobson radical. The power series rings we will study in this paper are local rings. The topology induced by the Jacobson radical on a local ring will be an important tool in our study.

Next, following [23], we briefly review rings with grading, filtration, and in particular, with I-adic filtration.

Definition 2.2.4. A $\mathbb{Z}$-*graded* ring is a ring T together with a family of additive subgroups

$$\cdots, T_{-2},\ T_{-1},\ T_0,\ T_1,\ T_2, \cdots$$

such that (i) $T_iT_j \subseteq T_{i+j}$, for all integers i and j, and (ii) $T = \oplus_n T_n$, as an abelian group. The family $\{T_n,\ n \in \mathbb{Z}\}$ is called a $\mathbb{Z}$-*grading* of T.

Definition 2.2.5. A ring A is said to be a $\mathbb{Z}$-*filtered ring* if there is an ascending chain of additive subgroups of A,

$$\cdots \subseteq F_{-2}A \subseteq F_{-1}A \subseteq F_0A \subseteq F_1A \subseteq F_2A \subseteq \cdots$$

satisfying (i) $1 \in F_0A$ and (ii) $F_nAF_mA \subseteq F_{n+m}A$ for all $m, n \in \mathbb{Z}$. The family $\{F_nA,\ n \in \mathbb{Z}\}$ is called a $\mathbb{Z}$-*filtration* of A.

Set $A_0 = F_0A,\ A_1 = F_{-1}A,\ A_2 = F_{-2}A,\ \ldots$ Then we have a descending chain of additive subgroups (which forms the negative part of the original filtration),

$$A_0 \supseteq A_1 \supseteq A_2 \cdots,$$

satisfying $1 \in A_0$ and $A_iA_j \subseteq A_{i+j}$, for all positive integers i and j. In this paper, we will only be interested in rings with the "negative" filtration. Throughout this section we will let A be a ring with the filtration $A_0 \supseteq A_1 \supseteq A_2 \cdots$, or simply $\{A_n\}$.

Now, we briefly describe the topology on A with respect to the filtration $\{A_n\}$. First of all, A turns into a *topological additive group* by letting the cosets of the A_i, for all $i = 0, 1, 2, \ldots$, form a fundamental system of neighborhoods of 0 in A, and then A is a *topological ring* since the ring operation are continuous. A *Cauchy sequence* in A is defined to be a sequence $\{x_v\}$ of elements of A such that, for any neighborhood U of 0, there exists an integer $k(U)$ with the property that $x_u - x_v \in U$ for all $u, v \geq k(U)$.

Definition 2.2.6. The filtration $\{A_n\}$ on A is said to be

(i) *exhaustive* if $A = A_0$,

(ii) *separated* if $\cap_i A_i = 0$,

(iii) *complete* if Cauchy sequences converge in the corresponding topology.

Remark. Recall that a topological ring is said to be *Hausdorff* if and only if the intersection of all neighborhoods of 0 is 0. It is a standard fact that the filtration $\{A_n\}$ is separated if the corresponding topology on A is Hausdorff.

Next, we give the definition of completion. Here we adopt an algebraic method which can be found in, e.g., [3]. This definition is equivalent to those approached from the topological point of view; see, e.g., [1] and [23].

Suppose $\{a_v\}$ is a Cauchy sequence in A. The image of a_v in A/A_n is ultimately constant, equal to say x_n. Let θ_{n+1} be the projection from A/A_{n+1} to A/A_n. Then $\theta_{n+1} : \ x_{n+1} \mapsto x_n$. It is clear that equivalent Cauchy sequences define the same coherent sequence $\{x_n\}$. On the other hand, given any sequence $\{y_n\}$ of elements in A/A_n such that $\theta_{n+1}(y_n) = y_{n+1}$, we can construct a Cauchy sequence $\{b_n\}$ giving rise to it by taking b_n simply to be any element in the coset y_n (so that $b_{n+1} - b_n \in A_n$). The set of all coherent sequences $\{x_n\}$ forms a ring which is called the *inverse limit* of the sequence $\{A/A_n\}$.

Definition 2.2.7. The *completion* of A with respect to the filtration $\{A_n\}$ is defined to be the inverse limit of the sequence $\{A/A_n\}$, and denoted by $\widehat{A}$.

We will also need the following:

Theorem 2.2.8. *Let A be a Hausdorff topological ring. Then* (i) *there exists a complete Hausdorff topological ring $\widehat{A}$ that contains A as a dense subring,* (ii) *if A' is a complete Hausdorff topological ring containing A as a dense subring, then there exists a topological isomorphism $\beta : \widehat{A} \to A'$ such that $\beta(a) = a$ for all $a \in A$.*

Proof. See [1, 3.3.5] □

Next, following [23], we introduce the Rees ring and the associated graded ring of A with respect to the filtration $\{A_n\}$.

Definition 2.2.9. Consider the abelian group $\widetilde{A} = \oplus_{n=0}^{\infty} A_n$ given by the filtration. Define the grading and multiplication as follows: for $n \in \mathbb{Z}$, $\widetilde{A}_n = A_n$, $a_i \cdot a_j = (a_i a_j)_{i+j}$ for $a_n \in A_n$, $a_m \in A_m$, and $(a_i a_j)_{i+j}$ is the product $a_i a_j$ in A viewed as an element in $\widetilde{A}_{i+j}$. The graded ring $\widetilde{A}$ defined above is called the *Rees ring* of A associated with the filtration $\{A_n\}$.

Definition 2.2.10. The *associated graded ring* of A with respect to the filtration $\{A_n\}$ is

$$\operatorname{gr} A = A_0/A_1 \oplus A_1/A_2 \oplus \cdots$$

Naturally there is some connection between the properties of a filtered ring A and its associated graded ring $\operatorname{gr} A$.

Proposition 2.2.11. *If* $\operatorname{gr} A$ *is an integral domain then* A *is also an integral domain.*

Proof. See, e.g., [24, 1.6.6] □

Theorem 2.2.12. *Suppose that the filtration* $\{A_n\}$ *on* A *is exhaustive, separated, and complete. Then* A *is right (left) noetherian if* $\operatorname{gr} A$ *is right (left) noetherian.*

Proof. See, e.g., [23, pp. 60–61] □

Appearing in commutative algebra and algebraic geometry (see, e.g., [31]), a *Zariski* ring is a commutative noetherian ring A with an I-adic filtration, where I is an ideal of A contained in the Jacobson radical of A. The idea of Zariski ring has been generalized to the noncommutative case by H. Li, F. Van Oystaeyen, Bjork and others; see, e.g., [23].

Definition 2.2.13. A filtered ring A with filtration $\{A_n\}$ is said to be a *right (left) Zariskian ring*, or, the filtration $\{A_n\}$ is said to be a *right (left) Zariskian filtration*, if the Rees ring $\tilde{A}$ of A associated with this filtration is right (left) noetherian and $A_1 \subseteq J(A)$. A (noncommutative) Zariski ring is a filtered ring which is both right and left Zariskian.

The following provides a large class of (noncommutative) Zariski rings.

Proposition 2.2.14. *Suppose that A is complete with respect to the filtration $\{A_n\}$ and that $\operatorname{gr} A$ is left noetherian. Then A is a left Zariski ring.*

Proof. See [23, II.2.2.1] □

In the commutative case, every ideal in a Zariski ring is closed (see, e.g. [31]). A parallel property also holds in the noncommutative case.

Theorem 2.2.15. *The right (respectively left) ideals of a right (respectively left) Zariski ring are closed in the filtration topology.*

Proof. See [23, Remark 3 b), page 85] □

We end this section with the $\mathfrak{m}$-adic topology on a local ring, where $\mathfrak{m}$ is the maximal ideal.

Definition 2.2.16. Let I be a two-sided ideal of a ring R. Then we have the *I-adic filtration*, $R = I^0 \supseteq I^1 \supseteq \ldots$, and the corresponding *I-adic topology* on R. The fundamental system of neighborhoods of 0 is formed by the cosets of I^i, for all i. Of course, since $R = I^0$, the I-adic filtration on R is always exhaustive. The *I-adic completion* of R at the ideal I is the inverse limit of R/I^n. Also note that an ideal K of R is closed if

$$K = \cap_{n=0}^{\infty}(K + I^n),$$

or, equivalently, any Cauchy sequence in K converges in K.

Recall from 2.2.3 that a local ring has a unique maximal ideal (the Jacobson radical) such that the quotient ring is simple artinian. Let R be a local ring with maximal ideal $\mathfrak{m}$. We will always equip R with the $\mathfrak{m}$-adic topology. By the associated graded ring $\operatorname{gr} R$, we will always mean with respect to the $\mathfrak{m}$-adic filtration

Definition 2.2.17. We will refer to a local ring R as a *complete local ring* if the $\mathfrak{m}$-adic filtration on R is complete (i.e., Cauchy sequences converge in the $\mathfrak{m}$-adic topology) and separated. (i.e., the $\mathfrak{m}$-adic topology is Hausdorff).

Note that the $\mathfrak{m}$-adic filtration on a complete local ring is always exhaustive, separated, and complete.

2.3 Regular elements, Ore set, Localization

Definition 2.3.1. A *multiplicative set* in a ring R is a subset $X \subseteq R$ such that $1 \in X$ and X is closed under multiplication.

Following [14], we describe Ore's method of localization at a given multiplicative set.

Definition 2.3.2. Let X be a multiplicative set in a ring R. Then X satisfies the *right Ore condition* provided that, for each $x \in X$ and $r \in R$, there exist $y \in X$ and $s \in R$ such that $ry = xs$, that is, $rX \cap xR \neq \emptyset$. A multiplicative set satisfying the right Ore condition is said to be a *right Ore set*. The *left Ore condition* and *left Ore sets* are defined symmetrically. An *Ore set* is a multiplicative set which is both a right and left Ore set.

Recall that an element $a \in R$ is normal if $aR = Ra$. It follows immediately that a multiplicative set generated by normal elements is an Ore set.

Definition 2.3.3. Let R be a ring and $X \subseteq R$ a multiplicative set. A *right quotient ring* (or *right Ore localization*) for R with respect to X is a ring homomorphism $\phi : R \longrightarrow S$ such that:

(a) $\phi(x)$ is a unit of S for all $x \in X$.

(b) Every element of S has the form $\phi(a)\phi(x)^{-1}$ for some $a \in R$ and $x \in X$.

(c) $\ker(\phi) = \{r \in R \mid rx = 0 \quad \text{for some } x \in X\}$

Definition 2.3.4. Let X be a multiplicative set in a ring R. Then X is *right reversible* if and only if $xr = 0$, for $x \in X$ and $r \in R$ implies that there exists

$x' \in X$ such that $rx' = 0$. A *right denominator set* is any right reversible right Ore set.

Theorem 2.3.5. *Let X be a multiplicative set in a ring R. Then there exists a right quotient ring of R with respect to X if and only if X is a right denominator set.*

Corollary 2.3.6. *Let X be a right denominator set in a ring R, and suppose that $\phi_1 : R \mapsto S_1$ and $\phi_2 : R \mapsto S_2$ are right quotient rings of R with respect to X. Then there is a unique ring isomorphism $\eta : S_1 \mapsto S_2$ such that $\eta\phi_1 = \phi_2$.*

Given a right denominator set X in R, we now know that there exists a unique (up to isomorphism) right quotient ring S of R. We shall denote S by RX^{-1}. It follows that if X is also a left denominator set and if R has a left quotient ring S' with respect to X, then S and S' must be isomorphic. If this last situation occurs, we will simply refer to the *quotient ring* of R with respect to X, or *localization* of R at X.

Theorem 2.3.7. *Let X be a right denominator set in a ring R. Then the localization RX^{-1} is noetherian if R is noetherian.*

Proof. See [14, 10.16]. □

We next state a result about prime ideals in the localization. We first need:

Definition 2.3.8. An element a of a ring R is said to be *regular* if the left and right annihilator of a in R are both equal to 0. Let I be an ideal of R. An element x in R is said to be *regular modulo* I if the coset $x + I$ is a regular element of the quotient ring R/I. The set of all such x is denoted by $\mathcal{C}_R(I)$.

Theorem 2.3.9. *Let X be a right denominator set in a ring R and assume that RX^{-1} is right noetherian.* (i) *An ideal J of RX^{-1} is prime if and only if $J \cap R$ is prime.* (ii) *Let P be a prime ideal of R. Then $P = Q \cap R$ for some prime ideal Q of RX^{-1} if and only if $X \subseteq \mathcal{C}_R(P)$.*

Proof. See [14, Theorem 10.18]. □

2.4 Dimension Theory

In this section, following [23] and [24], we briefly review the definitions of classical Krull dimension, Krull dimension, and global dimension. Some of the terminologies appearing in those definitions are very technical and have been skipped; the reader is encouraged to consult [23] and [24].

The classical Krull dimension is a natural generalization, due to G. Krause, of Krull dimension for commutative rings.

Definition 2.4.1. Let R be a right noetherian ring. Let $\operatorname{Spec}^0 R$ be the set of maximal ideals of R, and let α be an ordinal such that $\operatorname{Spec}^\beta R$ exists for all ordinals β less than α. Set $\operatorname{Spec}^\alpha R$ to be the set of prime ideals P of R satisfying the following property: If P' is a prime ideal of R properly containing P, then $P' \in \operatorname{Spec}^\beta R$ for some ordinal β less than α. Note that $\operatorname{Spec}^\alpha R = \operatorname{Spec} R$ for some ordinal α. The *classical Krull dimension* of R, denoted by $\operatorname{clKdim} R$, is the minimum ordinal α such that $\operatorname{Spec}^\alpha R = \operatorname{Spec} R$.

The Krull dimension of noncommutative rings and their modules, as defined by Gabriel and Rentschler, is also a generalization of the Krull dimension for commutative noetherian rings.

Definition 2.4.2. Let R be a ring and let M be a right R-module. The Krull dimension of M will be denoted by $\operatorname{Kdim} M$. If M is the zero module, then $\operatorname{Kdim} M = -\infty$. If M is nonzero but satisfies the descending chain condition on right R-submodules, then $\operatorname{Kdim} M = 0$. Let α now be a general ordinal. Then $\operatorname{Kdim} M = \alpha$ provided: (i) If β is an ordinal less than α, then $\operatorname{Kdim} M \neq \beta$. (ii) If $\{M_i : i = 1, 2, \ldots\}$ is a descending chain of R-submodules of M, then $\operatorname{Kdim}([M_i/M_{i+1}])$ is less than α for all but finitely many $i \in \{1, 2, \ldots\}$.

For any ring R, the Krull dimension of the right module R_R (if exists) is called the *right Krull dimension* of R and is denoted $\operatorname{rKdim} R$. Similarly, the *left Krull dimension* of R is the value of the Krull dimension of R as a left R-module.

Theorem 2.4.3. *Let R be a ring with complete, separated filtration. If* $\operatorname{gr} R$ *is right noetherian then* $\operatorname{rKdim} R \leq \operatorname{rKdim} \operatorname{gr} R$.

Proof. See, e.g., [23, I.7.1.2]. □

The global dimension of a ring is a non-negative integer or infinity which is a homological invariant of the ring.

Definition 2.4.4. The *right (left) global dimension* of a ring R, denoted $\operatorname{rgl} R$, is the supremum of the set of projective dimensions (see [24]) of all right (left) R-modules. If R is a noetherian ring, then the right and left global dimension of R are the same, we refer to the *global dimension* of R.

Theorem 2.4.5. *Let R be a a ring with complete, separated filtration. Then* $\operatorname{rgl} R \leq \operatorname{rgl} \operatorname{gr} R$.

Proof. See, e.g., [24, 7.6.18]. □

The following are about Auslander regularity; see [23]) for further details.

Definition 2.4.6. For any R-module M, the *grade number* of M, denoted $J_R(M)$, is the unique smallest integer such that $\operatorname{Ext}_R^{J_R(M)}(M, R) \neq 0$. if such an integer does not exist we write $J_R(M) = \infty$.

Definition 2.4.7. Let R be a ring and M be a R-module. We say that M satisfies the *Auslander condition* if, for any $k \geq 0$ and any R-submodule N of $\operatorname{Ext}_R^k(M, R)$, it holds that $j_R(N) \geq k$.

Definition 2.4.8. Let R be a noetherian ring with finite global dimension. If every finitely generated left and right R-module satisfies the Auslander condition, then we say that R is an *Auslander regular ring.*

Proposition 2.4.9. *Let R be a filtered ring. If* $\operatorname{gr} R$ *is Auslander regular, then so is R.*

We close this section by a theorem due to R. Walker; see [29, Theorem 2.7]. This theorem will be applied to some power series completions in this paper. We first need the following definition:

Definition 2.4.10. Let R be a ring. Let I be an ideal of R. The subset $\{x_1, x_2, \ldots, x_n\}$ is a *normalizing set* of generators of I if

(a) the ideal $\langle x_1, x_2, \ldots, x_n\rangle$ is equal to I,

(b) x_1 is normal in R.

(c) $x_i + \langle x_1, x_2, \ldots, x_{i-1}\rangle$ is normal in the quotient ring $R/\langle x_1, x_2, \ldots, x_{i-1}\rangle$, for each $i = 2, \ldots, n$.

Theorem 2.4.11. *Let R be a right noetherian ring with a maximal ideal M such that* (i) M is the Jacobson radical of R, (ii) R/M is artinian, (iii) M has a normalizing set of generators containing n elements. Then R is prime, and the right Krull dimension, classical Krull dimension, and right global dimension of R are all equal to n.

2.5 Skew polynomial, power series rings

Let R be a ring. The structure of the skew polynomial ring can be found in [14] and [24].

Definition 2.5.1. Let τ be an endomorphism of R. A (left) τ-*derivation* on R is any additive map $\delta : R \to R$ such that $\delta(rs) = \tau(r)\delta(s) + \delta(r)s$ for all r, $s \in R$. The pair of maps (τ, δ) is called a *skew derivation* on R.

Proposition 2.5.2. *Given a ring R and a skew derivation (τ, δ) on R, there exits a skew polynomial ring $R[y; \tau, \delta]$.*

Proof. See, e.g., [14, Proposition 2.3]. □

Theorem 2.5.3. *Let $T = R[y; \tau, \delta]$, where τ is an automorphism of R. If R is right (left) noetherian, then so is T.*

Proof. See, e.g., [14, Theorem 2.6]. □

Theorem 2.5.4. *Let R be a right noetherian ring and $S = R[x; \tau]$, where τ is an automorphism. Then* $\operatorname{rKdim} S = \operatorname{rKdim} R + 1$ *and* $\operatorname{rgl} S = \operatorname{rgl} R + 1$.

Proof. See, e.g., [24, 6.5.4] and [24, 7.5.3]. □

Definition 2.5.5. An *iterated skew polynomial ring* is a polynomial ring in the form of

$$T_n = R[y_1; \tau_1, \delta_1][y_2; \tau_2, \delta_2] \ldots [y_n; \tau_n, \delta_n],$$

where each (τ_l, δ_l), for $1 \leq l \leq n$, is a skew derivation on T_{l-1}.

The structure of the skew power and Laurent series ring with zero derivations can be found in [24, 1.4.2].

Definition 2.5.6. Let R be a ring and τ an endomorphism. The *skew power series ring* $R[[y; \tau]]$ denotes the ring of skew power series $\sum_{i=0}^{\infty} a_i y^i$, for $a_0, a_1, \ldots \in R$, with multiplication determined by the commutation rule $ya = \alpha(a)y$, for any $a \in R$. Suppose that τ is an automorphism. The Ore localization of $R[[y; \tau]]$ at the multiplicative set generated by y is called the *skew Laurent series ring*, denoted $R[[y, y^{-1}; \tau]]$. Each element of $R[[y, y^{-1}; \tau]]$ has a unique representation as $\sum_{i \in \mathbb{Z}} a_i y^i$ with $a_{-n} = 0$ for all but finitely many $n \in \mathbb{N}$ and $ya = \tau(a)y$, for any $a \in R$.

Theorem 2.5.7. *Let R be a ring with an automorphism τ and let S be either $R[[y; \tau]]$ or $R[[y, y^{-1}; \tau]]$. If R is a right (respectively left) noetherian domain, then S is also a right (respectively left) noetherian domain.*

Proof. See Theorem 4.5, [24, 1.4.2] (this is a variation on a standard proof of the Hilbert basis theorem). □

Theorem 2.5.8. *Let R be a ring with an automorphism τ and let S be the skew power series ring $R[[y; \tau]]$. If R is Auslander regular, then so is S.*

Proof. See Theorem 6 (3), [23, III.3.4]. □

Next, we consider skew power series ring (in one variable) with nonzero derivations, following Venjakob [28].

Let R be a ring with a skew derivation (τ, δ). Let S be the additive group of power series of the form $\sum_{i=0}^{\infty} r_i y^i$ with coefficients r_i in R. Using the commutation relation $yr = \tau(r)y + \delta(r)$, for $r \in R$, we wish to write the product of two arbitrary elements in S as

$$\left(\sum_i r_i y^i\right)\left(\sum_j s_j y^j\right) = \sum_n^{\infty}\sum_{j=0}^{n}\sum_{i=n-j}^{\infty} r_i(y^i s_j)_{n-j}y^n, \tag{2.1}$$

where each $(y^n r)_i$, for $0 \leq i \leq n$, denotes an element in R such that

$$y^n r = \sum_{i=0}^{n}(y^n r)_i y^i,$$

for $n \geq 0$. Note that the coefficents of y^n, for $n = 0, 1, \ldots$, in the right hand side of (2.1) may not be well defined in R, in the absence of convergence. If, under some additional restrictions (see 2.5.9), the multiplication formula (2.1) is well defined for any two power series in S, then we say that S is a *well-defined skew power series ring* with (nonzero) τ-derivation δ (or briefly, *skew power series ring*), and write $S = R[[y; \tau, \delta]]$. Note that $S = R[[y; \tau, \delta]]$ always contains the skew polynomial ring $R[y; \tau, \delta]$ as a subring.

Theorem 2.5.9. *[Venjakob] Let R be a complete local ring with maximal ideal $\mathfrak{m}$ and (τ, δ) be a skew derivation on R. Assume that*

$$\tau(\mathfrak{m}) \subseteq \mathfrak{m}, \quad \delta(R) \subseteq \mathfrak{m}, \quad \textit{and} \quad \delta(\mathfrak{m}) \subseteq \mathfrak{m}^2.$$

Then the multiplication formula (2.1) is well defined and so $S = R[[y; \tau, \delta]]$ is a well defined skew power series ring.

The results in the following proposition can be found in or easily deduced from [28, 2.8-2.11] (cf. [23, Chap. III, 2.2.5], [23, Chap. III, 3.4.6 (1)]).

Proposition 2.5.10. *Retaining the notation in 2.5.9.* (i) *Any element $\sum_i r_i y^i$ is a unit (in S) if and only if the constant term r_0 is a unit in R. In particular, S is a local ring.* (ii) *The $\langle\mathfrak{m},\ y\rangle$-adic filtration on S is exhaustive, separated, and complete.* (iii) *There is a canonical isomorphism $\operatorname{gr} S \cong (\operatorname{gr} R)[\bar{y}; \bar{\tau}]$. Assume further that $\bar{\tau}$ is an isomorphism. Then, S is right (left) noetherian if $\operatorname{gr} R$ is right (left) noetherian, S is a domain if $\operatorname{gr} R$ is a domain, and S*

is Auslander regular if the same holds for $\operatorname{gr} R$. (iv) *Suppose that* $\operatorname{gr} R$ *is right noetherian and that* $\bar{\tau}$ *is an automorphisms. Then it holds that* $\operatorname{rgl} S \leq \operatorname{rgl} \operatorname{gr} R + 1$ *and* $\operatorname{rKdim} S = \operatorname{rKdim} R + 1$.

Proposition 2.5.11. *Let* R *and* S *be as in 2.5.9. Then* S *is the completion of the skew polynomial ring* $R[y;\, \tau, \delta]$ *with respect to the* $\langle \mathfrak{m},\, y \rangle$*-adic filtration.*

Proof. Set $T = R[y;\, \tau, \delta]$. It is not hard to see that S and T are Hausdorff topological rings with respect to the ideal filtration induced by $\langle \mathfrak{m},\, y \rangle_S$ and $\langle \mathfrak{m},\, y \rangle_T$, respectively. Of course, T is a dense subring of S under the $\langle \mathfrak{m},\, y \rangle$-adic topology. Therefore, S is the completion of T with respect to the $\langle \mathfrak{m},\, y \rangle$-adic topology, following 2.2.8. □

2.6 Quantum groups

In this section, we give three well known classes of examples of quantum algebras, including quantum affine spaces, quantum matrices, quantum symplectic spaces, and quantum Euclidean spaces.

Example 2.6.1. Let $\mathcal{O}_{\mathbf{q}}(\mathbf{k}^n)$ be the multiparameter quantum coordinate ring of affine n-space over $\mathbf{k}$, where $\mathbf{q} = (q_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix over $\mathbf{k}$; that is, $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. This algebra is generated by elements $x_1, \ldots x_n$ subject only to the relations $x_i x_j = q_{ji} x_j x_i$ for $i, j = 1, \ldots n$, and can be presented as an iterated skew polynomial ring

$$k[[x_1]][[x_2; \tau_2]] \cdots [[x_n; \tau_n]],$$

where the τ_j are automorphisms such that $\tau_j(x_i) = q_{ij} x_i$ for all $1 \leq i < j \leq n$. This algebra is often referred to as *quantum affine* n*-space* and has been extensively studied; see, e.g., [6, §4], [10, §2], and [13].

Example 2.6.2. Let $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbf{k}))$ be the multiparameter quantum coordinate ring of $n \times n$ matrices over $\mathbf{k}$, as studied in [2] (cf. e.g., [5]). Here $\mathbf{p} = (p_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix over $\mathbf{k}$, and λ is a nonzero element of $\mathbf{k}$ not equal to 1. Further information of this algebra can be found in [5]. As shown in [2], $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbf{k}))$ can be presented as follows:

$$\mathbf{k}[y_{11}]\,[y_{12};\tau_{12}]\,\cdots\,[y_{lm};\,\tau_{lm},\delta_{lm}]\,\cdots\,[y_{nn};\tau_{nn},\delta_{nn}].$$

Each (τ_{lm},δ_{lm}) is a skew derivation as follows:

$$\tau_{lm}(y_{ij}) = \begin{cases} p_{li}p_{jm}y_{ij}, & \text{when } l \geq i \text{ and } m > j, \\ \lambda p_{li}p_{jm}y_{ij}, & \text{when } l > i \text{ and } m \leq j, \end{cases}$$

$$\delta_{lm}(y_{ij}) = \begin{cases} (\lambda-1)p_{li}y_{im}y_{lj}, & \text{when } l > i \text{ and } m > j, \\ 0, & \text{otherwise.} \end{cases}$$

This algebra is often referred to as *quantum matrices.*

There are other well known quantum coordinate rings, for example coordinate rings of quantum symplectic space and quantum Euclidean $2n$-space (see, e.g., [5]). Horton introduced a class of algebras, denoted $K^{P,Q}_{n,\Gamma}(\mathbf{k})$, that includes coordinate rings of both quantum symplectic space and quantum Euclidean $2n$-space; see [15]. Next we describe this class of algebras.

Example 2.6.3. Let $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ be elements of $(\mathbf{k}^\times)^n$ such that $p_i \neq q_i$ for each $i \in \{1, \ldots, n\}$. Further, let $\Gamma = (\gamma_{i,j}) \in M_n(\mathbf{k}^\times)$, with $\gamma_{j,i} = \gamma_{i,j}^{-1}$ and $\gamma_{i,i} = 1$ for all $i,\ j$. Then, as in [15, Proposition 3.5], $K^{P,Q}_{n,\Gamma}(\mathbf{k})$ is the iterated skew polynomial ring,

$$\mathbf{k}[x_1][y_1;\,\tau_1][x_2;\,\sigma_2][y_2;\,\tau_2,\,\delta_2]\cdots[x_n;\,\sigma_n][y_n;\,\tau_n,\,\delta_n],$$

with automorphisms $\sigma_i,\ \tau_i$ and τ_i-derivations δ_i defined as follows:

$$\begin{aligned}
\sigma_i(x_j) &= q_j^{-1}p_i\gamma_{i,j}x_j && 1 \leq j \leq i-1, \\
\sigma_i(y_j) &= q_j\gamma_{j,i}y_j && 1 \leq j \leq i-1, \\
\tau_i(x_j) &= p_i^{-1}\gamma_{j,i}x_j && 1 \leq j \leq i-1, \\
\tau_i(y_j) &= \gamma_{i,j}y_j && 1 \leq j \leq i-1, \\
\tau_i(x_i) &= q_i^{-1}x_i, && \\
\delta_i(x_j) &= 0 && 1 \leq j \leq i-1, \\
\delta_i(y_j) &= 0 && 1 \leq j \leq i-1, \\
\delta_i(x_i) &= -q_i^{-1}\textstyle\sum_{l<i}(q_l-p_l)y_lx_l. &&
\end{aligned}$$

CHAPTER 3

q-COMMUTATIVE POWER SERIES RINGS

In this chapter, we study the power series ring, over a field, in finitely many q-commuting variables. We refer to this ring as the *q-commutative power series ring*. This ring is in fact the completion of the quantum affine n-space, or the *q-commutative polynomial ring*.

We study the prime ideal structure of the q-commutative power series ring. Our main results show that much of the theory for the quantum affine spaces carries over into this completion. The prime spectrum of the q-commutative power series ring can be partitioned into finitely many strata each homeomorphic to the prime spectrum of a commutative noetherian ring. The results in this chapter will appear in [22].

3.1 Notation and preliminary results

In this section, we will fix notation and deduce some ring-theoretic properties for q-commutative power series rings.

Setup 3.1.1. (i) To start, $\mathbf{k}$ will denote a field, $\mathbf{k}^\times$ will denote the multiplicative group of units in $\mathbf{k}$, n will denote a positive integer, and $q = (q_{ij})$

will denote a multiplicatively antisymmetric $n \times n$ matrix (i.e., $q_{ij} = q_{ji}^{-1}$ and $q_{ii} = 1$) with entries from $\mathbf{k}^{\times}$. We will use $\mathbb{N}$ to denote the set of non-negative integers.

(ii) $R := \mathbf{k}_q[[x_1, \ldots, x_n]]$ will denote the associative unital $\mathbf{k}$-algebra of formal skew power series in the indeterminates $x_1, \ldots, x_n$, subject only to the commutation relations $x_j x_i = q_{ji} x_i x_j$. We will refer to R, in general, as a *q-commutative power series ring* (allowing q and n to vary in this usage).

(iii) The elements of R are *formal power series*

$$\sum_{s \in \mathbb{N}^n} c_s x^s,$$

for $c_s \in \mathbf{k}$, for $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$, and for $x^s = x_1^{s_1} \cdots x_n^{s_n}$. We will use x^s to refer to a general monic monomial in R, and we can write $R = \mathbf{k}_q[[x]]$.

(iv) L will denote the q-commutative Laurent series ring

$$\mathbf{k}_q[[x^{\pm 1}]] = \mathbf{k}_q[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]],$$

which will be defined later in 3.1.15.

(v) Contained within R is the $\mathbf{k}$-subalgebra $\mathbf{k}_q[x_1, \ldots, x_n]$ of q-commuting polynomials in $x_1, \ldots, x_n$. We will denote this algebra $\mathbf{k}_q[x]$.

Recall from 2.6.1, the algebra $\mathbf{k}_q[x]$ is also known as quantum affine n-space. Both $R = \mathbf{k}_q[[x]]$ and $\mathbf{k}_q[x]$ are endowed with J-adic topology, where $J = \langle x_1, \ldots, x_n \rangle$. Then, by 2.2.8, we have the following:

Proposition 3.1.2. *The q-commutative power series ring $R = \mathbf{k}_q[[x]]$ is the completion of the quantum affine n-space $\mathbf{k}_q[x]$ at the ideal generated by $x_1, \ldots, x_n$.*

Remark. Recalling the construction of iterated skew polynomial rings in 2.5.5, it is clear that R can be written as an iterated skew power series ring as follows:

$$\mathbf{k}[[x_1]][[x_2; \tau_2]] \cdots [[x_n; \tau_n]],$$

with automorphisms τ_j such that $\tau_j(x_i) = q_{ji} x_i$ for all $1 \le i < j \le n$.

Of course, when $\mathbf{k} = \mathbb{F}_2$ it follows immediately that R is the commutative power series ring in n variables. However, we will allow this case unless indicated otherwise. We will also include the possibility that $n = 1$, in which case R is the commutative power series ring over $\mathbf{k}$ in one variable.

Before studying the properties of R, we first consider skew power series, in a single variable, over more general coefficient rings.

Let A be a ring with an automorphism α, and let $B = A[[y;\alpha]]$ be the skew power series defined as in 2.5.6. Then we have the following observation.

Proposition 3.1.3. (i) *The $\langle y\rangle$-adic filtration on B is exhaustive, separated, and complete.* (ii) *The associated graded ring* $\operatorname{gr} B$ *is isomorphic to* $A[z;\alpha]$.

Note that 3.1.3(ii) and 2.2.12 provided an alternative way to prove the first part of Theorem 2.5.7; that is, B is noetherian if A is noetherian. Furthermore, in view of 2.5.8, we have

Proposition 3.1.4. *If A is a noetherian Auslander regular domain, then so is B.*

Next, we point out the form of the invertible elements in B.

Proposition 3.1.5. *A power series in B is a unit if and only if its constant term is a unit of A.*

Proof. Set $f = 1 + b_1y + b_2y^2 + \cdots \in B$, for $b_1, b_2, \ldots \in A$, and set $g = 1 + c_1y + c_2y^2 + \cdots$, for $c_1, c_2, \ldots \in A$. Then

$$fg = 1 + (b_1 + c_1)y + (b_2 + c_2 + p_2(b_1, c_1))y^2 + (b_3 + c_3 + p_3(b_1, b_2, c_1, c_2))y^3 + \cdots,$$

where each $p_i(b_1, \ldots, b_{i-1}, c_1, \ldots, c_{i-1})$ is an element in A depends only on $b_1, \ldots, b_{i-1}$ and $c_1, \ldots, c_{i-1}$, for integers $i \geq 2$. If $b_1, b_2, \ldots$ are arbitrary then we can choose $c_1, c_2, \ldots$ such that $fg = 1$, and if $c_1, c_2, \ldots$ are arbitrary then we can choose $b_1, b_2, \ldots$ such that $fg = 1$. Therefore, the proposition follows. □

Then, we are able to study the Jacobson radical of $B = A[[y;\alpha]]$.

Proposition 3.1.6. *There is a natural isomorphism from $B/\langle y\rangle$ to A, and $J(B) = J(A) + \langle y\rangle$.*

Proof. Since α is an automorphism, we can just as well write the coefficients on the right. As either a left or right A-module, we can view B as a direct product of copies of A, indexed by $\mathbb{N}$. It follows from the commutation rule $ya = \alpha(a)y$ that y is normal in B. Hence $B/\langle y\rangle \cong A$.

It follows from 3.1.5 that $1 + yh$ is invertible for all $h \in B$. Hence y is contained in the Jacobson radical $J(B)$. It now follows from the natural isomorphism of A onto $B/\langle y\rangle$ that $J(B) = J(A) + \langle y\rangle$. □

The proof of the next proposition is adapted directly from the commutative case.

Proposition 3.1.7. *Let A be a ring equipped with an automorphism α, and let $B = A[[y;\alpha]]$. Let I be an ideal of A finitely generated on the right and left, and assume further that $\alpha(I) = I$. Then $BI = IB$, and (abusing the notation slightly)*

$$B/IB \cong (A/I)[[y;\alpha]].$$

Proof. Let $I[[y;\alpha]]$ be the set of power series in B whose coefficients (written on the left) are all contained in I.

Assume first that $I = g_1A + \cdots + g_nA$, for some $g_1, \ldots, g_n \in I$. Choose

$$\sum_{i=0}^{\infty} h_i y^i \ \in \ I[[y;\alpha]],$$

with $h_0, h_1, \ldots \in I$. Then, for suitable choices of $r_{ij} \in A$,

$$\sum_{i=0}^{\infty} h_i y^i = \sum_{i=0}^{\infty}(g_1r_{1i}+\cdots+g_nr_{ni})y^i = g_1\left(\sum_{i=0}^{\infty} r_{1i}y^i\right)+\cdots+g_n\left(\sum_{i=0}^{\infty} r_{ni}y^i\right) \in IB.$$

Hence $I[[y;\alpha]] \subseteq IB$. It is easy to see that $IB \subseteq I[[y;\alpha]]$, and so $I[[y;\alpha]] = IB$ when I is finitely generated on the right.

Since α is an automorphism and $\alpha(I) = I$, we can also view $I[[y;\alpha]]$ to be the set of power series in B whose coefficients, when written on the right, are all contained in I. A mirror-image argument to the preceding one now shows that $I[[y;\alpha]] = BI$ (when I is finitely generated as a left ideal). In particular, $I[[y;\alpha]] = IB = BI$ is an ideal of B.

It is now easy to see that $B/IB \cong (A/I)[[y;\alpha]]$, and the lemma follows. □

We are now back to R, which is an iterated skew power series ring over a field. In view of 3.1.3, 3.1.6, and Remark 3.1, we have some readily available information on R by induction:

Proposition 3.1.8. *Let $R = \mathbf{k}_q[[x]]$ as defined in 3.1.1. Then*

(i) *The Jacobson radical $J(R)$ is the ideal $J = \langle x_1, \dots, x_n \rangle$, and the J-adic filtration on R is exhaustive, separated, and complete. Furthermore, $R/J \cong \mathbf{k}$.*

(ii) *The associated graded ring $\operatorname{gr} R$ is isomorphic to the q-commutative polynomial ring $\mathbf{k}_q[x]$.*

(iii) *A power series in R is a unit if and only if it has nonzero constant term.*

Therefore, the fact that R is a complete local ring follows directly from 3.1.8 (i). Also, in view of 3.1.4, we have:

Corollary 3.1.9. *R is a complete local noetherian Auslander regular domain with unique primitive factor isomorphic to $\mathbf{k}$.*

Recall from 2.2.14 that a filtered ring is a Zariski ring if it is complete with respect to the filtration and the associated graded ring is noetherian. The next proposition, which is crucial to our later analysis, follows from 3.1.8 (i) and (ii).

Proposition 3.1.10. *R is a Zariski ring. The right or left ideals of R are all closed in the J-adic topology.*

We now end our collection of information on R with the following:

Proposition 3.1.11. *The right (left) Krull dimension, classical Krull dimension, and the right (left) global dimension of R are all equal to n.*

Proof. It is clear that $\{x_1, \ldots, x_n\}$ is a set of normal elements. The Jacobson radical $J(R) = \langle x_1, \ldots, x_n \rangle$ and $R/J(R) \cong \mathbf{k}$. Therefore the corollary follows from 2.4.11. □

Also required in our later analysis (particularly in the proof of 3.2.1) is the graded lexicographic ordering on $\mathbb{N}^n$. We review the definition of the ordering as follows.

Definition 3.1.12. Let $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ be n-tuples in $\mathbb{N}^n$. The *total degree* $|s|$ is the sum $s_1 + \cdots + s_n$. Then we say that s precedes t in *graded lexicographic ordering* and write $s <_{\text{grlex}} t$ either when $|s| < |t|$ or when $|s| = |t|$ and s precedes t in the lexicographic ordering.

To see the usefulness of this ordering for our purposes, let N denote an infinite collection of pair-wise distinct n-tuples in $\mathbb{N}^n$. Then we can write $N = \{j_1, j_2, \ldots\}$ such that $j_\ell <_{\text{grlex}} j_{\ell+1}$, for all positive integers ℓ, and such that

$$\lim_{\ell \to \infty} x^{j_\ell} = 0$$

in the J-adic topology on R. When the meaning is clear we will simply use "$<$" to denote "$<_{\text{grlex}}$."

Definition 3.1.13. Given a power series $f \in R$, we can choose $s_1, s_2, \ldots \in \mathbb{N}^n$ and $c_{s_1}, c_{s_2}, \ldots \in \mathbf{k}$ such that

$$f = \sum_{i=1}^{\infty} c_{s_i} x^{s_i},$$

where $s_l <_{\text{grlex}} s_{l+1}$ for all positive integers l, and where $c_{s_1} \neq 0$. We will refer to s_1 as the *graded lexicographic degree* of f.

The tools developed so far allow the following observation:

Proposition 3.1.14. *Let P be a prime ideal of R. Then $P \cap \mathbf{k}_q[x]$ is a prime ideal of $\mathbf{k}_q[x]$.*

Proof. Set $Q = P \cap \mathbf{k}_q[x]$. Let $a, b \in \mathbf{k}_q[x]$, and suppose that $a(\mathbf{k}_q[x])b \subseteq Q$. Choose $f \in R$. For each non-negative integer i, let f_i denote the sum of the monomials appearing in f of total degree no greater than i. Since P is closed in the J-adic topology, and since $af_ib \in Q$ for all i, we see that

$$afb = \lim_{i \to \infty} af_ib \in P.$$

Since f was arbitrary, we see that $aRb \in P$. Therefore, since P is prime, either a or b is contained in P. But then either a or b is contained in $Q = P \cap \mathbf{k}_q[x]$, and so Q is a prime ideal of $\mathbf{k}_q[x]$. □

We end this section by describing an Ore localization of R, the Laurent series ring, constructed as follows:

Setup 3.1.15. Let X denote the multiplicatively closed subset of R generated by 1 and the indeterminates $x_1, \ldots, x_n$. Since each x_i is normal, we see that X is an Ore set in R. We will use L to denote the *q-commutative Laurent series ring*

$$\mathbf{k}_q[[x^{\pm 1}]] = \mathbf{k}_q[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]],$$

obtained via the Ore localization of R at X. We always view R, the q-commutative power series ring $\mathbf{k}_q[[x]] = \mathbf{k}_q[[x_1, \ldots, x_n]]$, as a subalgebra of L. Each Laurent series in L will have the form

$$\sum_{s \in \mathbb{Z}^n} c_s x^s,$$

for $c_s \in \mathbf{k}$, for $s = (s_1, \ldots, s_n) \in \mathbb{Z}^n$, and for $x^s = x^{s_1} \cdots x^{s_n}$, but with $c_s = 0$ for $\min\{s_1, \ldots, s_n\} \ll 0$.

Remark. i) For $n > 1$ it is *not* the case that L can be written as an iterated skew Laurent series ring

$$\mathbf{k}[[x_1^{\pm 1}]][[x_2^{\pm 1}; \tau_2]] \cdots [[x_n^{\pm 1}; \tau_n]],$$

which is well known to be a division ring.

ii) Also inside L is the q-commutative Laurent polynomial ring, $\mathbf{k}_q[x^{\pm 1}] = \mathbf{k}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the localization of $\mathbf{k}_q[x]$ at the products of the $x_1, \ldots, x_n$, also known as McConnell-Pettit algebra.

3.2 q-commutative Laurent series rings

Retain the notation of the preceding sections, continuing to let $R = \mathbf{k}_q[[x]]$ and $L = \mathbf{k}_q[[x^{\pm 1}]]$. In this section, we will give a detailed description of the ideal theory of the q-commutative Laurent series ring L. First we study the ideal theory of L provided by a group action on L.

We begin by considering the "obvious" n-torus action on L. Let H denote the algebraic torus $(\mathbf{k}^{\times})^n$, and set

$$h(s) = h_1^{s_1} \cdots h_n^{s_n},$$

for $h = (h_1, \ldots, h_n) \in H$ and $s = (s_1, \ldots, s_n) \in \mathbb{Z}^n$. For each $h \in H$, the assignment

$$h.x^s = h(s)x^s,$$

for $s \in \mathbb{Z}^n$, extends linearly to a $\mathbf{k}$-algebra automorphism

$$h : \quad \sum_{s \in \mathbb{Z}^n} c_s x^s \longrightarrow \sum_{s \in \mathbb{Z}^n} c_s h(s) x^s$$

of L, where each $c_s \in \mathbf{k}$. Hence we obtain an action of H on L by $\mathbf{k}$-algebra automorphisms. It is clear that this action by H on L restricts to an action by automorphisms on the subalgebra R of L.

For ideals I, J, P, and any $h \in H$, observe that $IJ \subseteq h.P$ implies that $h^{-1}.(IJ) \subseteq P$, and that $h^{-1}.(I)\, h^{-1}.(J) \subseteq P$. Hence the H-actions on R and L induce H-actions on $\operatorname{Spec} R$ and $\operatorname{Spec} L$.

Also note, when k is infinite, that the set of H-eigenvectors in R and the set of monomials in R coincide. The analogous statement for L also holds true.

We will use the terminology introduced in 2.1.13 to describe the ideals satisfying certain conditions related to the H action. For example, H-ideals of R are the ideals of R stable under the H action; L is H-simple if the only H-ideals of L are the zero ideal and L itself.

Theorem 3.2.1. *Assume that* $\mathbf{k}$ *is infinite. Then*

(i) *every nonzero* H*-ideal of* R *is generated by monomials.*

(ii) L *is* H*-simple.*

Proof. (i) Let I be a nonzero H-ideal of R, and choose

$$0 \neq f = \sum_{i \in \mathbb{N}^n} a_i x^i \in I.$$

The first step of our proof is to show, as follows, that each x^i appearing nontrivially in f is contained in I.

To start, let $\Psi = \{i \in \mathbb{N}^n \mid a_i \neq 0\}$. For notational convenience we will assume that Ψ is infinite; the case when Ψ is finite can be handled similarly. Equip Ψ with the graded lexicographic ordering (see 3.1.12), and write $\Psi = \{j_1, j_2, j_3, \ldots\}$ such that $j_\ell < j_{\ell+1}$ for all positive integers ℓ. So

$$f = \sum_{\ell=1}^{\infty} a_{j_\ell} x^{j_\ell}.$$

Replacing f with $a_{j_1}^{-1} f$, we can assume without loss of generality that $a_{j_1} = 1$. Set $f_1 = f$ and $a_{j_\ell}(1) = a_{j_\ell}$, for all ℓ. Since $\mathbf{k}$ is infinite, there exists an $h \in H$ such that $h(j_1) \neq h(j_2)$. Set

$$f_2 = \frac{h.f - h(j_2) f}{h(j_1) - h(j_2)}.$$

Then

$$f_2 = x^{j_1} + \sum_{\ell=3}^{\infty} \left(\frac{h(j_\ell) - h(j_2)}{h(j_1) - h(j_2)} \right) a_{j_\ell} x^{j_\ell} = x^{j_1} + \sum_{\ell=3}^{\infty} a_{j_\ell}(2) x^{j_\ell} \in I,$$

where

$$a_{j_\ell}(2) = \left(\frac{h(j_\ell) - h(j_2)}{h(j_1) - h(j_2)} \right) \cdot a_{j_\ell}(1).$$

Continuing, we get

$$f_m = x^{j_1} + \sum_{\ell=m+1}^{\infty} a_{j_\ell}(m)x^{j_\ell} \in I,$$

for all integers $m \geq 2$. Note that f_m converges to x^{j_1} in the J-adic topology (again see 3.1.12). By 3.1.10, it then follows that $x^{j_1} \in I$.

Replacing f with $f - x^{j_1} \in I$, we can repeat the above procedure to show that $x^{j_2} \in I$. Continuing, we see that $x^{j_1}, x^{j_2}, x^{j_3}, \ldots \in I$. This completes the first step of our proof.

Now let M be the set of all monic monomials appearing nontrivially in power series contained in I. It follows from the above that I is the smallest closed ideal of R containing M. However, by 3.1.10, it then follows that I is the smallest ideal of R containing M. In other words, I is generated by M, and part (i) follows.

(ii) Let K be a nonzero H-ideal of L. Then $K \cap R$ is a nonzero H-ideal of R. Therefore, $K \cap R$ contains a nonzero monomial x^s for some $s \in \mathbb{N}^n$. But x^s is invertible in L, and so $K = L$. □

Corollary 3.2.2. *Assume that* $\mathbf{k}$ *is infinite. Then*

(i) *every H-orbit of prime ideals in L is (Zariski) dense in* $\operatorname{Spec} L$.

(ii) $J(L) = 0$.

Proof. Let P be a prime ideal of L, let O be the H-orbit of P in $\operatorname{Spec} L$, and let I be the intersection of all of the ideals in O. Then I is an H-ideal of L, and I is not equal to L itself. Therefore, by 3.2.1, $I = 0$. It follows that the closure of O in $\operatorname{Spec} L$ is $\operatorname{Spec} L$, and so O is dense in $\operatorname{Spec} L$. Part (i) follows. Part (ii) also follows, since the intersection of the orbit of a primitive ideal must equal the zero ideal. □

Remark. Part (ii) of the preceding corollary also follows from [27], a study of the Jacobson radical in general skew Laurent series rings.

We now give a detailed description of the ideals, and in particular the prime ideals, of the q-commutative Laurent series ring L through its center. Henceforth, we will let Z denote the center of L.

To start, we consider a character on $\mathbb{Z}^n \times \mathbb{Z}^n$, following [13].

Notation 3.2.3. Set $\sigma\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbf{k}^\times$ by

$$\sigma(s,t) = \prod_{i,j=1}^{n} q_{ij}^{s_i t_j},$$

for $s = (s_1, \ldots, s_n),\ t = (t_1, \ldots, t_n) \in \mathbb{Z}^n$. Then it is easy to check that

$$\sigma(s,s) = 1$$
$$\sigma(t,s) = \sigma(s,t)^{-1}$$
$$\sigma(s, t+t') = \sigma(s,t)\sigma(s,t')$$

for $s, t, t' \in \mathbb{Z}^n$, and so σ is an *alternating bicharacter.*

Now, we describe Z, following [13, §1] (cf. [6, §4]).

Lemma 3.2.4. (cf. [13, 1.2]) Set

$$S = \{s \in \mathbb{Z}^n \mid \sigma(s,t) = 1 \text{ for all } t \in \mathbb{Z}^n\},$$

and

$$\mathcal{F}_S = \left\{ \sum_{s\in S} c_s x^s \in L \,\middle|\, c_s \in \mathbf{k},\ \min\{s_1, \ldots s_n\} \ll 0 \right\}.$$

Then

$$Z = \mathcal{F}_S.$$

Proof. It is clear that the commutation rule in L can be given in the form

$$x^s x^t = \sigma(s,t) x^t x^s.$$

Then a monomial x^s, for $s \in \mathbb{Z}^n$, commutes with all monomials x^t, for all $t \in \mathbb{Z}^n$, if and only if $s \in S$. Hence any series $\sum_{s\in S} c_s x^s$ in L is contained in the center; that is, $\mathcal{F}_S \subseteq Z$. Now given $z \in Z$, write $z = \sum_{v \in \mathbb{Z}^n} \alpha_v x^v$ for some $\alpha_v \in \mathbf{k}$. For any $t \in \mathbb{Z}^n$,

$$\sum_{v \in \mathbb{Z}^n} \alpha_v x^t x^v = x^t z = z x^t = \sum_{v \in \mathbb{Z}^n} \alpha_v x^v x^t = \sum_{v \in \mathbb{Z}^n} \alpha_v \sigma(v,t) x^t x^v,$$

and since the products $x^t x^v$ are linearly independent, it follows that $\sigma(v,t) = 1$, whenever $\alpha_v \neq 0$. Therefore the monomials appearing in z are all contained in S, and so $Z \subseteq \mathcal{F}_S$. □

Proposition 3.2.5. *Z is a right (or left) Z-module direct summand of L.*

Proof. Let

$$f = \sum_{s \in N} c_s x^s$$

be a Laurent series in L, for a suitable index set $N \subseteq \mathbb{Z}^n$. Then f can be written as a sum of two parts:

$$f = \sum_{s \in N \cap S} c_s x^s + \sum_{t \in N \setminus S} c_t x^t.$$

Let $s \in S$ and $w, v \in \mathbb{Z}^n$ such that $x^v x^s = q x^w$ for some $q \in \mathbf{k}$. Then $w \in S$ if and only if $v \in S$. Therefore, it is not hard to check that $L = Z \oplus NZ$ where NZ is a right Z-submodule of L in the form of

$$\left\{ \sum_{v \notin S} c_v x^v \in L \,\middle|\, c_v \in k \right\}.$$

That NZ is also a left Z-submodule of L follows similarly. □

Recall that $\mathbb{N}$ denotes the set of all non-negative integers. Next, we provide some further information on Z. However, the general structure of Z is not clear.

Proposition 3.2.6. *Suppose that* $\operatorname{rank} S = r \geq 1$. *If there exists a basis of S with all elements in $\mathbb{N}^n$ then Z is isomorphic to a commutative Laurent series ring in r variables. In particular, Z is a field if and only if S is generated by a single n-tuple in $\mathbb{N}^n$.*

Proof. If S is generated by n-tuples in $\mathbb{N}^n$ say, $b_1, b_2, \dots b_r$ then the Laurent series ring in $x^{b_1}, \dots, x^{b_r}$ is a subring of L and Z is isomorphic to $\mathbf{k}[[z_1^{\pm 1}, \dots, z_r^{\pm 1}]]$. In particular, if $S = \langle b \rangle$ for some $b \in \mathbb{N}^n$ then Z is isomorphic to $k[[x^{\pm b}]]$, which is a field. Now suppose that $\operatorname{rank} S = r \geq 1$ and Z is a field. We claim that $S \subseteq \mathbb{N}^n \cup \mathbb{M}^n$, where $\mathbb{M}$ is the set of non-positive integers. If not, there exists $0 \neq s \in S$ such that $s_i \cdot s_j < 0$, for some $1 \leq i, j \leq n$. Then $1 - x^s$ is an element in Z which is not invertible in L, contradiction. Hence S is generated by n-tuples in $\mathbb{N}^n$, and so, by (i), Z is isomorphic to a commutative Laurent series ring in r variables. However any Laurent series ring in more than one variable can not be a field. Therefore the proposition follows. □

We now present the "main lemma" of this chapter. Analogous conclusions for q-commutative Laurent polynomial and related rings have appeared in various forms, including [4, 5.1], [6, 4.4], [10, §2], [12, 2.2], [13, 1.4], [16, 2.1], and [25, Chapter 11, §3].

Lemma 3.2.7. (i) Let I be an ideal of L. Then $I = L(I \cap Z)$. (ii) Let K be an ideal of Z. Then $K = (LK) \cap Z$.

Proof. (i) If $I = 0$ there is nothing to prove, and so we assume otherwise. Let g be a nonzero element of I, and choose $u \in \mathbb{N}^n$ such that $x^u g \in I \cap R$, and set $f = x^u g$. Let S be as in 3.2.4, and fix a transversal T for S in $\mathbb{Z}^n$. Then we can write

$$f = \sum_{t \in T} x^t z_t,$$

where each z_t is a skew Laurent series contained in Z. Set

$$T_f = \{t \in T \mid x^t z_t \neq 0\}.$$

Then

$$f = \sum_{t \in T_f} x^t z_t.$$

Choose an arbitrary element t_0 in the set T_f. The main work of the proof is to prove the following claim: $x^{t_0} z_{t_0} \in I \cap R$. Of course, if $f = x^{t_0} z_{t_0}$ there is nothing to prove, and so we assume otherwise.

To start, we can choose subsets S_t of S, for $t \in T_f \backslash \{t_0\}$, such that

$$\begin{aligned} f &= x^{t_0} z_{t_0} + \sum_{t \in T_f \backslash \{t_0\}} x^t z_t &= x^{t_0} z_{t_0} + \sum_{t \in T_f \backslash \{t_0\}} x^t \sum_{s \in S_t} c_s x^s \\ & &= x^{t_0} z_{t_0} + \sum_{t \in T_f \backslash \{t_0\}} \sum_{s \in S_t} c_s x^t x^s, \end{aligned}$$

for suitable choices of $c_s \in \mathbf{k}$.

Next, regrouping the preceding monomials $c_s x^t x^s$ according to their total degree, we can write

$$f = x^{t_0} z_{t_0} + \sum_{i \in N} \sum_{t \in T_i} \sum_{s \in S_t^i} c_s x^t x^s,$$

for sets N, T_i, and $S_t^i \subseteq S$ satisfying the following conditions:

1. $N = \{1, 2, \ldots\}$ is a suitable (possibly finite) index set.

2. $\{d_i \,|\, i \in N\}$ is a set of positive integers such that $d_i < d_{i+1}$ for all $i \in N$ (if $|N| > 1$).

3. If $t \in T_i$ and $s \in S_t^i$, then the total degree of the monomial $c_s x^t x^s$ is d_i.

Note that $T_f \backslash t_0 = \cup T_i$, but the subsets T_i need not be pairwise disjoint. In the later analysis, we will mainly focus on the monomial x^t for each $t \in T_i$. We now have

$$f = x^{t_0} z_{t_0} + \sum_{i \in N} \sum_{t \in T_i} x^t w_{t,i},$$

where

$$w_{t,i} = \sum_{s \in S_t^i} c_s x^s,$$

for $i \in N$ and $t \in T_i$. Note that each $c_s x^s$ is a central monomial, and so each $w_{t,i}$ is contained in Z.

To prove the claim, set

$$u_1 = f - x^{t_0} z_{t_0} = \sum_{i \in N} \sum_{t \in T_i} x^t w_{t,i}.$$

and

$$f_1 = f = x^{t_0} z_{t_0} + u_1 \in I \cap R.$$

Recall that $J = \langle x_1, \ldots, x_n \rangle$ is the augmentation ideal of R and that each $\sum_{t \in T_i} x^t w_{t,i}$, for $i \in N$ is a sum of monomials of total degree d_i ($\geq d_1$). Then $u_1 \in J^{d_1}$. Choose some $r \in T_1$. Since $r \neq t_0$, and since r and t_0 are contained in the transversal T for S in $\mathbb{Z}^n$, it follows that $r - t_0 \notin S$. Hence, there exists $v \in \mathbb{Z}^n$ such that

$$\sigma(v, t_0) \neq \sigma(v, r),$$

where σ is as defined in 3.2.3. Next, we consider

$$\rho_v(f_1) = \frac{x^v f_1 x^{-v} - \sigma(v, r) f_1}{\sigma(v, t_0) - \sigma(v, r)}.$$

Note that $\rho_v(f_1)$ is contained in $I \cap R$. Also,

$$\rho_v(f_1) = x^{t_0} z_{t_0} + \sum_{i \in N} \sum_{\substack{t \in T_i, \\ t \neq r}} \frac{\sigma(v, t) - \sigma(v, r)}{\sigma(v, t_0) - \sigma(v, r)} x^t w_{t,i}.$$

Repeating the preceding process, at most $|T_1|$-many times, we obtain a series, which does not have any term of total degree d_1,

$$f_2 = x^{t_0} z_{t_0} + u_2 \in I \cap R,$$

with

$$u_2 = \sum_{\substack{i \in N \\ i \geq 2}} \sum_{t \in T_i \setminus T_1} x^t w'_{t,i},$$

for suitable $w'_{t,i} \in Z$. Note that each $w'_{t,i}$ appearing will be a nonzero scalar multiple of $w_{t,i}$ and that $u_2 \in J^{d_2}$. If $u_2 = 0$ then $x^{t_0} z_{t_0} \in I \cap R$, and the claim follows; so assume otherwise.

Continuing as above, we obtain either $x^{t_0}z_{t_0} \in I \cap R$ (proving the claim) or an infinite sequence

$$f_i = x^{t_0}z_{t_0} + u_i \in I \cap R, \quad i = 1, 2, \dots,$$

converging to $x^{t_0}z_{t_0}$ in the J-adic topology. Since the ideals in R are closed in the J-adic topology, as noted in (3.1.10), we see that $x^{t_0}z_{t_0} \in I \cap R$. The claim follows.

Next, it follows from the claim that $z_{t_0} \in I \cap Z$, and so

$$x^{t_0}z_{t_0} \in (L(I \cap Z)) \cap R.$$

Recall, however, that t_0 was arbitrarily chosen from T_f. Consequently,

$$x^t z_t \in (L(I \cap Z)) \cap R$$

for all $t \in T_f$.

Following 3.1.13, and recalling that T is a transversal for S in $\mathbb{Z}^n$, we can write

$$\{x^t z_t \mid t \in T_f\}$$

as (possibly a finite set)

$$\{x^{t_1}z_{t_1},\ x^{t_2}z_{t_2},\ \dots\},$$

for integers $1, 2, \dots$, with the graded lexicographic degree of $x^{t_j}z_{t_j}$ less than that of $x^{t_{j'}}z_{t_{j'}}$ whenever $j < j'$. Again following 3.1.13, the sequence of partial sums

$$x^{t_1}z_{t_1} + x^{t_2}z_{t_2} + \dots + x^{t_j}z_{t_j}$$

converges to f in the J-adic topology on R. Therefore,

$$f \in (L(I \cap Z)) \cap R,$$

since $(L(I \cap Z)) \cap R$ is a closed (left) ideal in R.

Now note that

$$g = x^{-u}f \in L(L(I \cap Z) \cap R) \subseteq L(I \cap Z).$$

Therefore, $I \subseteq L(I \cap Z)$. Of course $L(I \cap Z) \subseteq I$, and (i) follows.

(ii) This follows from 3.2.5. $\square$

We now record some applications of the preceding lemma.

Corollary 3.2.8. *Z is a noetherian domain.*

Proof. It follows from 2.3.7 that the Laurent series ring L is noetherian. Now, by 3.2.7, we see that Z is noetherian. □

Corollary 3.2.9. *L is simple if either S is trivial or S, isomorphic to $\mathbb{Z}$, is generated by a single n-tuple in $\mathbb{N}^n$.*

Proof. This follows from 3.2.6 and 3.2.7. □

The following is analogous to [13, 1.5], and provides a detailed description of the prime ideals of q-commutative Laurent series rings.

Theorem 3.2.10. *The assignments $P \mapsto P \cap Z$, for $P \in \operatorname{Spec} L$, and $Q \mapsto LQ$, for $Q \in \operatorname{Spec} Z$, provide mutually inverse homeomorphisms between $\operatorname{Spec} L$ and $\operatorname{Spec} Z$.*

Proof. It follows from 3.2.7 that the assignments $I \mapsto I \cap Z$, for ideals I of L, and $K \mapsto LK$, for ideals K of Z, provide mutually inverse, inclusion preserving bijections between the sets of ideals of L and Z.

Next, recall from 2.1.3 that a prime ideal in any ring intersects with the center at a prime ideal, and so $P \cap Z$ is a prime ideal of Z for all prime ideals P of L. Thus $P \mapsto P \cap Z$ produces a map from $\operatorname{Spec} L$ to $\operatorname{Spec} Z$.

Now let Q be a prime ideal of Z. To show that LQ is a prime ideal of L, suppose that I_1 and I_2 are ideals of L such that $I_1 I_2 \subseteq LQ$. By 3.2.7(i), there exist ideals K_1 and K_2 of Z such that $I_1 = LK_1$ and $I_2 = LK_2$. So

$$L(K_1K_2) = (LK_1)(LK_2) = I_1I_2 \subseteq LQ.$$

Then, by 3.2.7(ii),

$$K_1K_2 = (L(K_1K_2)) \cap Z \subseteq (LQ) \cap Z = Q.$$

Since Q is prime, it follows that either K_1 or K_2 is contained in Q. Therefore, either $I_1 = LK_1$ or $I_2 = LK_2$ is contained in LQ, and we see that LQ is a prime ideal of L. Hence, $Q \mapsto LQ$ produces a map from $\operatorname{Spec} Z$ to $\operatorname{Spec} L$.

It now follows from the first paragraph that the assignments $P \mapsto P \cap Z$, for prime ideals P of L, and $Q \mapsto LQ$, for prime ideals Q of Z, provide mutually inverse, Zariski continuous bijections between $\operatorname{Spec} L$ and $\operatorname{Spec} Z$. The proposition follows. □

3.3 Prime ideals in q-commutative power series rings

Our aim now is to systematically develop a detailed description of the prime spectrum of R. Using the results we obtained in the preceding sections for q-commutative Laurent series rings, our approach now largely follows – and in many cases mimics – the studies of q-commutative polynomial rings found in [6, §4] and [13]. Moreover, much of the theory developed in this section is also analogous to that for various more complicated (finitely generated) quantum function algebras; see (e.g.) [5] for details.

We start by giving an account of the "obvious" stratification of $\operatorname{Spec} R$. Let W be the set of subsets of $\{1, \ldots, n\}$. For each $w \in W$, let

$$\operatorname{Spec}_w R = \{P \in \operatorname{Spec} R \mid x_i \in P \Leftrightarrow i \in w\}.$$

Then

$$\operatorname{Spec} R = \bigsqcup_{w \in W} \operatorname{Spec}_w R. \tag{3.1}$$

For each $w \in W$, let J_w be the ideal of R generated by the indeterminates x_i for $i \in w$, let $R_w = R/J_w$, and set $n_w = n - |w|$. Then R_w is isomorphic to a skew power series ring in n_w variables, and hence J_w is completely prime.

Notice that each $\operatorname{Spec}_w R$ is the intersection of the closed set of prime ideals containing x_i, for $i \in w$, with the open set of prime ideals not containing x_j, for $j \notin w$. That is,

$$\operatorname{Spec}_w R = V(J_w) \backslash V(I_1 \cdots I_{n_w}),$$

where each $I_k = \langle x_{j_k} \rangle$ with $j_k \notin w$. Also, the closure in $\operatorname{Spec} R$ of each $\operatorname{Spec}_w R$ is $V(J_w)$, which is a union of subsets $\operatorname{Spec}_{w'} R$, for $w' \in W$ and $w \subseteq w'$. Therefore the partition (3.1) is in fact a *stratification* of $\operatorname{Spec} R$ which meshes nicely with the Zariski topology in the sense that the closure of each stratum is a union of strata, and each stratum is a locally closed subset (see 2.1.7) of $\operatorname{Spec} R$. Equip each $\operatorname{Spec}_w R$ with the subspace topology inherited from $\operatorname{Spec} R$.

Let X_w be the multiplicatively closed subset of R_w generated by 1 and the images of the x_j for $j \notin w$. Since these generators are normal, in viewing of 2.1.2, the prime ideals in R_w that are disjoint from X_w are precisely the prime ideals that do not contain $x_j + J_w$ for any $j \notin w$. These are exactly the prime ideals of the form P/J_w for $P \in \operatorname{Spec}_w R$. On the other hand, note that X_w consists of normal elements of R_w, and so it is an Ore set of R_w. We set

$$L_w = R_w X_w^{-1},$$

the Ore localization of R_w at the set X_w. It follows from 2.3.9 that there is an inclusion preserving bijection between the set of prime ideals in R_w that are disjoint from X_w and the set $\operatorname{Spec} L_w$. Therefore, we now have a natural homeomorphism

$$\operatorname{Spec}_w R \xrightarrow{\Phi_w} \operatorname{Spec} L_w \tag{3.2}$$

obtained via the assignment

$$P \longmapsto (P/J_w) L_w,$$

for $P \in \operatorname{Spec}_w R$. Observe that L_w will be isomorphic to a q-commutative Laurent series ring (for a replacement of q by a suitable $n_w \times n_w$ matrix) in n_w variables. Let Z_w denote the center of L_w. By 3.2.8, Z_w is a commutative noetherian domain, and by 3.2.10, $\operatorname{Spec} L_w$ is homeomorphic to $\operatorname{Spec} Z_w$. Therefore, we have the following:

Theorem 3.3.1. *The indeterminates $x_1, \ldots, x_n$ provide a finite stratification on* $\operatorname{Spec} R$. *Each stratum is naturally homeomorphic to the spectrum of a commutative noetherian domain.*

Recall the H-action discussed in the beginning of Section 3.2. Now we consider the H-action on $\operatorname{Spec} L_w$, for $w \in W$. Again set $n_w = n - |w|$, and let H_w denote the n_w-torus $(\mathbf{k}^{\times})^{n_w}$. Then H_w acts on L_w and so H_w also acts on $\operatorname{Spec} L_w$, in the same way as that H acts on L and $\operatorname{Spec} L$ in Section 3.2. Also, H acts on L_w and $\operatorname{Spec} L_w$ via the obvious surjection of H onto H_w. Hence, by 3.2.1 (ii), we have

Corollary 3.3.2. *Assume that* $\mathbf{k}$ *is infinite. Then each* L_w *is* H*-simple.*

Next, since each x_i is an H-eigenvector, we have:

Proposition 3.3.3. $\operatorname{Spec}_w R$ *is a union of* H*-orbits of prime ideals in* R*, and the* H*-action on* $\operatorname{Spec} R$ *restricts to an* H*-action on* $\operatorname{Spec}_w R$*.*

Let $w \in W$ and Φ_w be the homeomorphism between $\operatorname{Spec}_w R$ and $\operatorname{Spec} L_w$, as in (3.2). Then it is clear that the extension $P \mapsto (P/J_w)L_w$ and the contraction $Q \mapsto \Phi_w^{-1}(Q)$ interwine the H-action; that is,

Theorem 3.3.4. *Let* $w \in W$*. Then the* n*-torus* H *acts on* $\operatorname{Spec}_w R$ *and* $\operatorname{Spec} L_w$*, and the homeomorphism* Φ_w *between* $\operatorname{Spec}_w R$ *and* $\operatorname{Spec} L_w$ *is* H*-equivariant.*

Corollary 3.3.5. *Let* $w \in W$*. Assume that* $\mathbf{k}$ *is infinite. Then each* H*-orbit of prime ideals in* $\operatorname{Spec}_w R$ *is (Zariski) dense in* $\operatorname{Spec}_w R$*.*

Proof. By 3.2.2, when $\mathbf{k}$ is infinite each H-orbit of prime ideals in $\operatorname{Spec} L_w$ is (Zariski) dense in $\operatorname{Spec} L_w$, and so the corollary follows from the preceding theorem. □

Proposition 3.3.6. *Assume that* $\mathbf{k}$ *is infinite. Then* $\{J_w \mid w \in W\}$ *is the set of* H*-prime ideals of* R*.*

Proof. Recall that for any $w \in W$, J_w is a (completely) prime ideal of R. Moreover, each J_w is H-stable. It therefore follows from 2.1.13 that each J_w is an H-prime ideal of R.

Conversely, let I be an arbitrary H-prime ideal of R. Then, by 2.1.14, $I = I_1 \cap \cdots \cap I_t$ for some finite H-orbit $I_1, \ldots, I_t$ of prime ideals of R. As noted in 3.3.3, each stratum in the stratification (3.1) is a union of H-orbits, and so $I_1, \ldots, I_t \in \operatorname{Spec}_w R$ for some $w \in W$. Therefore, by 3.3.4,

$$\Phi_w(I_1), \ldots, \Phi_w(I_t)$$

is a single H-orbit of prime ideals in L_w, and so $\Phi_w(I_1) \cap \cdots \cap \Phi_w(I_t)$ is an H-ideal of L_w. However, since L_w is H-simple, again following 3.3.4, we see that

$$\Phi_w(I_1) \cap \cdots \cap \Phi_w(I_t) = 0$$

in L_w. Since L_w is prime, it now follows that $\Phi_w(I_1), \ldots, \Phi_w(I_t)$ (which comprise a single H-orbit) must all equal 0. Consequently,

$$I = I_1 = \cdots I_t = J_w,$$

by the homeomorphism 3.2 and the assignment $P \mapsto (P/J_w)L_w$. The proposition follows. $\square$

Next, we consider localization and representation theoretic issues, in the sense of [7] and [18]. Retaining the notation of 3.3.1, recall that the prime spectrum of R is normally separated if for each inclusion of prime ideals $P_0 \subsetneq P_1$ in $\operatorname{Spec} R$ there exists an element $y \in P_1 \setminus P_0$ such that $Ry = yR + P_0$. Also, for prime ideals P and Q of R, there is a second layer link $P \rightsquigarrow Q$ provided $P \cap Q/PQ$ has a nonzero R-R-bimodule factor that is torsionfree as both a left R/P-module and a right R/Q-module. Normal separation implies the strong second layer condition. Our approach below is inspired by [6].

Proposition 3.3.7. (i) *The prime spectrum of R is normally separated, and consequently, R satisfies the strong second layer condition.* (ii) *Let G denote the group of $\mathbf{k}$-algebra automorphisms of R generated by the maps $r \mapsto x_i r x_i^{-1}$ for $r \in R$ and $1 \leq i \leq n$. Suppose that P and Q are prime ideals of R such that $P \rightsquigarrow Q$. Then $\tau(P) = Q$ for some $\tau \in G$.*

Proof. (i) Suppose that $P_0 \subsetneq P_1$ are prime ideals of R, and choose w such that $P_0 \in \operatorname{Spec}_w R$. In particular, $J_w \subseteq P_0 \subsetneq P_1$. If $P_1 \notin \operatorname{Spec}_w R$ then there exists some $x_i \in P_1 \setminus P_0$; since $Rx_i = x_i R$ we see that normal separation holds in this case. Now suppose that $P_1 \in \operatorname{Spec}_w R$. Set $P_0' = \Phi_w(P_0)$ and $P_1' = \Phi_w(P_1)$, where Φ_w is the inclusion preserving homeomorphism between $\operatorname{Spec}_w R$ and $\operatorname{Spec} L_w$. Then $P_0' \subsetneq P_1'$. Note that P_0' and P_1' are generated by their intersection with the center Z_w of L_w; see 3.2.7 (i). Therefore, there exists an element z in Z_w such that $z \in P_1' \setminus P_0'$. Moreover, in L_w, z is regular modulo P_0'. Now recall that L_w is the localization of R_w at the set X_w (consisting of normal regular elements of R_w) and so there exists some $u \in X_w$ such that $uz \in R_w$. Note that uz is a normal element of R_w, since u is normal and z is central. Also, it is easy to see that

$$uz \in (P_1' \cap R_w) \setminus (P_0' \cap R_w).$$

But $(P_0' \cap R_w) = P_0/J_w$ and $(P_1' \cap R_w) = P_1/J_w$. So let y be the preimage of uz in R. Then $y \in P_1 \setminus P_0$, and $Ry = yR + P_0$. We can now conclude that the prime spectrum of R is normally separated. Part (i) follows.

(ii) Suppose that P and Q are prime ideals of R such that $P \rightsquigarrow Q$. It follows (e.g.) from [14, 12.15], for all $1 \leq i \leq n$, that $x_i \in P$ if and only if $x_i \in Q$. Hence, there exists $w \in W$ such that $P, Q \in \operatorname{Spec}_w R$. In particular, $J_w \subseteq P \cap Q$.

Set $P' = P/J_w$ and $Q' = Q/J_w$. It follows directly from [6, 2.7] that either $P' \rightsquigarrow Q'$ in $\operatorname{Spec} R_w$ or that $\tau(P) = Q$ for some $\tau \in G$.

So assume that $P' \rightsquigarrow Q'$ in R_w. Set $P'' = P'.L_w$ and $Q'' = Q'.L_w$. Since L_w is noetherian, we have $P'' \rightsquigarrow Q''$ in L_w. Let $T = P'' \cap Q''/A$ be the linked module, which is torsionfree as both a left L_w/P'' module and a right L_w/Q'' module. However, since every ideal of L_w is centrally generated, as proved in 3.2.7 (i), it follows that Q'' left annihilates T, and P'' right annihilates T. Therefore $P'' = Q''$, and so $P = Q$. Part (ii) follows. $\square$

Remark. We close by briefly considering the "generic" case: Assume that the abelian subgroup $\langle q_{ij} \rangle$ of $\mathbf{k}^\times$ is free of rank $n(n-1)/2$. (This condition will hold, for instance, if the q_{ij}, for $1 \le i < j \le n$, are algebraically independent over $\mathbf{k}$.)

(i) Using 3.2.9, it is not hard to show that each L_w, for $w \in W$, is simple. Consequently, $\operatorname{Spec} R = \{J_w \mid w \in W\}$.

(ii) It follows from (i) and 3.3.7 (ii) that each J_w can only be linked to itself. Hence, by 3.3.7 (i) and [6, 2.5], each J_w is linked to itself and only to itself. It then follows from 3.3.7 (i) and (e.g.) [14, 14.20] that each J_w is classically localizable.

A more detailed description of the link structure of $\operatorname{Spec} R$, following the methods of [6], is left for the future.

CHAPTER 4

POWER SERIES COMPLETIONS OF QUANTUM COORDINATE RINGS

Given an iterated skew polynomial ring $C[y_1;\tau_1,\delta_1]\dots[y_n;\tau_n,\delta_n]$ over a complete local ring C with maximal ideal $\mathfrak{m}$, we prove, under suitable assumptions, that the completion at the ideal $\mathfrak{m}+\langle y_1,y_2,\dots,y_n\rangle$ is an iterated skew power series ring. Under further conditions, this completion is a local, noetherian, Auslander regular domain. Applicable examples include quantum matrices, quantum symplectic spaces, and quantum Euclidean spaces. The results in this chapter will appear in [30].

4.1 Iterated skew power series rings

In this section, we construct iterated skew power series rings over general complete local rings. First we set up an iterated skew polynomial ring over a complete local ring with suitable skew derivations. Then we prove, by extending skew derivations, the existence of the corresponding skew power series rings. Our approach builds on Venjakob's study of skew power series ring $R[[y;\tau,\delta]]$ described in Chapter 2 section 5.

Setup 4.1.1. Let C be a complete local ring with maximal ideal $\mathfrak{m}$. Set $R_0 = C$, and let

$$R_n = C[y_1;\tau_1,\delta_1]\dots[y_l;\tau_l,\delta_l]\dots[y_n;\tau_n,\delta_n]$$

be an iterated skew polynomial ring with skew derivations (τ_l,δ_l) of R_{l-1}, for $1\le l\le n$. For each $1\le l\le n$, set

$$I_{l-1} = \mathfrak{m} + \langle y_1,\dots,y_{l-1}\rangle \subseteq R_{l-1},$$

and assume that

$$\tau_l(I_{l-1})\subseteq I_{l-1},\quad \delta_l(R_{l-1})\subseteq I_{l-1},\quad \text{and}\quad \delta_l(I_{l-1})\subseteq I_{l-1}^2.$$

We will also need the following notation.

Definition 4.1.2. Let $1\le l\le n+1$. A *nonzero* monomial

$$c_{i_1,\dots,i_{l-1}}y_1^{i_1}\cdots y_{l-1}^{i_{l-1}}$$

in R_{l-1} is said to be in *normal form.* We will write

$$c_iY_{l-1}^i \qquad \text{for} \qquad c_{i_1,\dots,i_{l-1}}y_1^{i_1}\cdots y_{l-1}^{i_{l-1}},$$

where $i=(i_1,\dots,i_{l-1})\in\mathbb{N}^{l-1}$.

Next we introduce the notion of degree that we will use for monomials in normal form.

Definition 4.1.3. Let $c_iY_{l-1}^i\in R_{l-1}$. Then there exists an integer k largest such that $c_i\in\mathfrak{m}^k$. Set

$$s(c_i,i) = k + i_1 + i_2 + \cdots + i_{l-1}.$$

We will refer to $s(c_i,i)$ as the *degree* of $c_iY_{l-1}^i$.

Definition 4.1.4. Let $1 \leq l \leq n$. By a *formal power series* in $y_1, \ldots, y_l$ over C, we will mean an infinite series

$$f = \sum_i c_i Y_l^i,$$

where the c_i are elements in C and where $i \in \mathbb{N}^l$. Note that each monomial $c_i Y_l^i$ is in normal form. The set of all formal power series in $y_1, \ldots, y_l$ over C forms an abelian group, which we will denote as A_l.

Next, based on the assumptions in 4.1.1, we have the following observation.

Lemma 4.1.5. *Let* $1 \leq l \leq n$, *and let* $c_i Y_{l-1}^i$ *and* $d_j Y_{l-1}^j$ *be two nonzero monomials in* R_{l-1}. *Then*

(i) *The product* $c_i Y_{l-1}^i \cdot d_j Y_{l-1}^j$ *is a sum of monomials each with degree greater than or equal to* $s(c_i, i) + s(d_j, j)$.

(ii) *Both polynomials* $\tau_l\left(c_i Y_{l-1}^i\right)$ *and* $\delta_l\left(c_i Y_{l-1}^i\right)$ *are* 0 *or finite sums of monomials each with degree greater than or equal to* $s(c_i, i)$.

Lemma 4.1.6. *Let* $1 \leq l \leq n$

(i) *Any power series* $f = \sum_i c_i Y_l^i \in A_{l-1}$ *can be written as*

$$f = \sum_{k=0}^{\infty} \sum_{s(c_i,i)=k} c_i Y_l^i,$$

where each sum

$$\sum_{s(c_i,i)=k} c_i Y_l^i$$

is 0 or a finite sum.

(ii) *Let*

$$g = G_0 + G_1 + \ldots + G_k + \ldots,$$

where each G_k *is 0 or a finite sum of monomials in* R_l *all with degree* k. *Then* g *is a well-defined (in the sense of 4.1.4) formal power series in* A_l.

Proof. (i) can be proved by regrouping the monomials appearing in f (if necessary). To prove (ii), suppose that

$$G_k = \sum_{j \in M_k} c_j^{(k)} Y_l^j,$$

where $c_j^{(k)} \in C$ and where $M_k \subseteq \mathbb{N}^l$, for k=0,1,.... We will set $c_i^{(k)} = 0$ when $i \notin M_k$. Now, for a fixed j, the sum

$$c_j^{(0)} + c_j^{(1)} + \ldots + c_j^{(k)} + \ldots$$

might contain infinitely many terms. But each $c_j^{(k)}$ is such that the degree of $c_j^{(k)} Y_l^j$ is equal to k. Hence, the preceding sum is convergent in the $\mathfrak{m}$-adic topology. Therefore,

$$g = G_0 + G_1 + \ldots + G_k + \ldots = \sum_{j \in \cup M_k} \left(c_j^{(0)} + c_j^{(1)} + \ldots + c_j^{(k)} + \ldots \right) Y_l^j$$

is a formal power series in A_l with all coefficients in C well-defined. □

Theorem 4.1.7. *Let C be a complete local ring with maximal ideal $\mathfrak{m}$. Set $R_0 = C$, and let*

$$R_n = C[y_1; \tau_1, \delta_1] \ldots [y_l; \tau_l, \delta_l] \ldots [y_n; \tau_n, \delta_n] \quad (n \geq 1)$$

be an iterated skew polynomial ring with skew derivations (τ_l, δ_l) of R_{l-1}, for $1 \leq l \leq n$. For each $1 \leq l \leq n$, set

$$I_{l-1} = \mathfrak{m} + \langle y_1, \ldots, y_{l-1} \rangle \subseteq R_{l-1},$$

and assume that

$$\tau_l(I_{l-1}) \subseteq I_{l-1}, \quad \delta_l(R_{l-1}) \subseteq I_{l-1}, \quad \textit{and} \quad \delta_l(I_{l-1}) \subseteq I_{l-1}^2.$$

Set $S_0 = C$. Then there exists an iterated skew power series ring

$$S_n = C[[y_1; \widehat{\tau}_1, \widehat{\delta}_1]] \ldots [[y_l; \widehat{\tau}_l, \widehat{\delta}_l]] \ldots [[y_n; \widehat{\tau}_n, \widehat{\delta}_n]],$$

where each $(\widehat{\tau}_l, \widehat{\delta}_l)$ is a skew derivation on S_{l-1} with $\widehat{\tau}_l \mid_{R_{l-1}} = \tau_l$ and $\widehat{\delta}_l \mid_{R_{l-1}} = \delta_l$, for $1 \leq l \leq n$. Moreover, S_n is a complete local ring with maximal ideal $\mathfrak{m}_n = \mathfrak{m} + \langle y_1, \ldots, y_n \rangle$. (We will refer to S_n as the power series extension *of R_n.)*

Proof. Following 2.5.9, the ring $C[[y_1; \tau_1, \delta_1]]$ is well defined and we may take $S_1 = C[[y_1; \tau_1, \delta_1]]$. In the notation of 4.1.4, S_1 is the abelian group A_1 equipped with a well-defined multiplication restricting to the original multiplication in R_1. Our goal is to show that each abelian group A_l becomes an iterated skew power series ring. In the first step of the proof, we extend the pair of maps τ_l and δ_l to A_{l-1} for all $1 < l \leq n$. Then, by induction, we will show that each (τ_l, δ_l) extends to a skew derivation on S_{l-1} and that each A_l forms a ring S_l .

To start, let $f = \sum_i c_i Y_{l-1}^i$ be a power series in A_{l-1}. As in 4.1.6 (i), we can write

$$f = \sum_{k=0}^{\infty} F_k,$$

where each

$$F_k := \sum_{s(c_i,i)=k} c_i Y_{l-1}^i$$

is a finite sum (possibly equal to 0). Our goal now is to extend τ_l and δ_l to A_{l-1}. For $k = 0, 1, 2, \ldots$, if $\tau_l(F_k) \neq 0$, we can write

$$\tau_l(F_k) = \sum_{j \in T_k} t_j^{(k)} Y_{l-1}^j,$$

for some subset $T_k \subseteq \mathbb{N}^{l-1}$ and some $t_j^{(k)} \in C$. Next, let

$$G_m = \sum_{k=0}^{\infty} \sum_{j \in N_{m,k}} t_j^{(k)} Y_{l-1}^j,$$

where

$$N_{m,k} = \{j \in T_k \mid \text{the degree of } t_j^{(k)} Y_{l-1}^j \text{ is } m\}.$$

Then

$$\tau_l(F_0) + \tau_l(F_1) + \ldots + \tau_l(F_k) + \ldots \; = \; G_0 + G_1 + \ldots + G_m + \ldots \qquad (4.1)$$

It follows from 4.1.5 that any nonzero $\tau_l(F_k)$ is a finite sum and that each $t_j^{(k)} Y_{l-1}^j$ has degree $\geq k$. Hence each G_m is a finite sum by the construction. Recall from 4.1.6 (ii) that

$$G_0 + G_1 + \ldots + G_m + \ldots$$

is a formal power series in A_{l-1}. Therefore,

$$\sum_{k=0}^{\infty} \tau_l \left(F_k\right) \in A_{l-1}.$$

Using the same argument (replacing τ_l with δ_l), we also have

$$\sum_{k=0}^{\infty} \delta_l \left(F_k\right) \in A_{l-1}.$$

Then, for $1 \leq l \leq n$ and $f = \sum_i c_i Y_{l-1}^i \in A_{l-1}$, we extend t_l and δ_l by setting the following maps

$$\widehat{\tau}_l(f) = \sum_{k=0}^{\infty} \tau_l \left(\sum_{s(c_i,i)=k} c_i Y_{l-1}^i \right) \quad \text{and} \quad \widehat{\delta}_l(f) = \sum_{k=0}^{\infty} \delta_l \left(\sum_{s(c_i,i)=k} c_i Y_{l-1}^i \right). \tag{4.2}$$

It is clear that $\widehat{\tau}_l \mid_{R_{l-1}} = \tau_l$ and $\widehat{\delta}_l \mid_{R_{l-1}} = \delta_l$.

Now, let $n \geq 2$. Assume that the abelian group A_{n-1} is a well-defined power series ring, which we will denote as S_{n-1}, and also assume that S_{n-1} is a complete local ring with maximal ideal $\mathfrak{m}_{n-1} = \mathfrak{m} + \langle y_1, \ldots, y_{n-1} \rangle$. Next we show that $(\widehat{\tau}_n, \widehat{\delta}_n)$, from (4.2), is a skew derivation on S_{n-1}; that is, $\widehat{\tau}_n$ is an automorphism of S_{n-1} and $\widehat{\delta}_n$ is a left $\widehat{\tau}_n$-derivation.

Let t be a positive integer. Choose two arbitrary elements a and b in S_{n-1}. Write $a = a_t + a'_t$ and $b = b_t + b'_t$, where a_t (respectively b_t) is the sum of the monomials appearing in a (respectively b) with degree $\leq t$. Then it follows from (4.2) that

$$\widehat{\tau}_n(a) = \widehat{\tau}_n(a_t) + \widehat{\tau}_n(a'_t) \quad \text{and} \quad \widehat{\tau}_n(b) = \widehat{\tau}_n(b_t) + \widehat{\tau}_n(b'_t).$$

Therefore, we have

$$\widehat{\tau}_n(ab) = \tau_n\left(a_t \cdot b_t\right) + \widehat{\tau}_n(a'_t \cdot b_t + a_t \cdot b'_t + a'_t \cdot b'_t), \quad \text{and}$$

$$\widehat{\tau}_n(a) \cdot \widehat{\tau}_n(b) = \tau_n(a_t) \cdot \tau_n(b_t) + \widehat{\tau}_n(a'_t) \cdot \widehat{\tau}_n(b_t) + \widehat{\tau}_n(a_t) \cdot \widehat{\tau}_n(b'_t) + \widehat{\tau}_n(a'_t) \cdot \widehat{\tau}_n(b'_t).$$

Note that $\tau_n(a_t \cdot b_t) = \tau_n(a_t) \cdot \tau_n(b_t)$. It follows from 4.1.5 that

$$\widehat{\tau}_n(ab) - \widehat{\tau}_n(a) \cdot \widehat{\tau}_n(b) \in \mathfrak{m}_{n-1}^{t+1}.$$

Let $t \to \infty$, then it follows from the completeness of S_{n-1} that

$$\widehat{\tau}_n(ab) = \widehat{\tau}_n(a) \cdot \widehat{\tau}_n(b).$$

Using the same argument (replacing $\widehat{\tau}_n$ with $\widehat{\delta}_n$), we can get

$$\widehat{\delta}_n(ab) = \widehat{\delta}_n(a)b + \widehat{\tau}_n(a)\widehat{\delta}_n(b).$$

Therefore $(\widehat{\tau}_n, \widehat{\delta}_n)$ is a skew derivation on S_{n-1}.

In view of the assumptions in 4.1.1 and (4.2), we see that

$$\widehat{\tau}_n(\mathfrak{m}_{n-1}) \subseteq \mathfrak{m}_{n-1}, \quad \widehat{\delta}_n(S_{n-1}) \subseteq \mathfrak{m}_{n-1}, \quad \text{and} \quad \widehat{\delta}_n(\mathfrak{m}_{n-1}) \subseteq \mathfrak{m}_{n-1}^2.$$

Following 2.5.9, the skew power series ring $S_n = S_{n-1}[[y_n;\, \tau_n,\, \delta_n]]$ is well defined, and S_n is a complete local ring with maximal ideal $\mathfrak{m}_n = \mathfrak{m} + \langle y_1, \ldots, y_n \rangle$. This completes the inductive step. The theorem is proved by induction. □

The results in the following corollary are consequences of 2.5.10, 2.5.11 and 4.1.7.

Corollary 4.1.8. *The power series extension S_n in 4.1.7 is the completion of R_n with respect to the ideal $\mathfrak{m}_n = \mathfrak{m} + \langle y_1, \ldots, y_n \rangle$.*

(i) *Any power series in S_n is a unit (in S_n) if and only if its constant term is a unit in C.*

(ii) *The associated graded ring $\operatorname{gr} S_n$ is isomorphic to an iterated skew polynomial ring over $\operatorname{gr} C$ with endomorphisms $\bar{\tau}_1, \ldots, \bar{\tau}_n$ and with derivations all zero.*

(iii) *Assume further that $\bar{\tau}_1, \ldots, \bar{\tau}_n$ are automorphisms. If $\operatorname{gr} C$ is a domain, S_n is a domain. If $\operatorname{gr} C$ is right (respectively left) noetherian, so is S_n. If $\operatorname{gr} C$ is Auslander regular, then S_n is also Auslander regular.*

(iv) *Suppose that $\operatorname{gr} C$ is right noetherian and that $\bar{\tau}_1, \ldots, \bar{\tau}_n$ are automorphisms. Then it holds that $\operatorname{rKdim} S_n \leq \operatorname{rKdim} \operatorname{gr} C + n$ and $\operatorname{rgl} S_n \leq \operatorname{rgl} \operatorname{gr} C + n$.*

4.2 Applications to quantum coordinate rings

In this section, we will apply Theorem 4.1.7 to some well-known quantum coordinate rings, including quantum matrices, quantum symplectic spaces, and quantum Euclidean spaces to obtain power series completions of these rings. These power series rings are new examples of quantum algebras.

Example 4.2.1. Quantum Matrices

Let $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbf{k}))$ be the multiparameter quantum coordinate ring of $n \times n$ matrices over $\mathbf{k}$, as in 2.6.2. It is not hard to see that these skew derivations satisfy the assumptions in 4.1.1. Hence, by Theorem 4.1.7, the power series extension of $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbf{k}))$ is the iterated skew power series ring

$$\mathbf{k}[[y_{11}]]\,[[y_{12};\widehat{\tau}_{12}]]\ \cdots\ [[y_{lm};\ \widehat{\tau}_{lm},\widehat{\delta}_{lm}]]\ \cdots\ [[y_{nn};\widehat{\tau}_{nn},\widehat{\delta}_{nn}]],$$

where each extended skew derivation is defined as in (4.2). Also note that each τ_{lm} acts by nonzero scalar multiplication on the generators. It now follows from 4.1.8 that the preceding power series completion is a local, noetherian, Auslander regular domain.

Example 4.2.2. Quantized k-algebras K_n

Let $K_n = K^{P,Q}_{n,\Gamma}(\mathbf{k})$ be the class of Horton algebras, as in 2.6.3; that is, iterated skew polynomial rings with certain skew derivations. In view of the defining relations in 2.6.3, we see that the automorphisms and derivations satisfy the assumptions in 4.1.1, and so, by Theorem 4.1.7, K_n has the power series extension

$$\mathbf{k}[[x_1]][[y_1;\ \widehat{\tau}_1]][[x_2;\ \widehat{\sigma}_2]][[y_2;\ \widehat{\tau}_2,\ \widehat{\delta}_2]]\cdots[[x_l;\ \widehat{\sigma}_l]][[y_l;\ \widehat{\tau}_l,\ \widehat{\delta}_l]]\cdots[[x_n;\ \widehat{\sigma}_n]][[y_n;\ \widehat{\tau}_n,\ \widehat{\delta}_n]],$$

where the extended skew derivations are defined as in (4.2). Again, it follows from 4.1.8 that this completion is a local, noetherian, Auslander regular domain.

Moreover, comparing with 4.1.8 (iv), the dimensions of the power series completions in 4.2.2 can be more precisely determined as follows:

Remark. Let E be an algebra in the class K_n, and $\widehat{E}$ be the power series completion of E with respect to the ideal $\langle x_1, y_1, \ldots, x_n, y_n \rangle$. From 4.2.2, we see that, among the defining commutation relations, nonzero derivations only occur in the following cases:

$$y_i x_i = \tau_i(x_i) y_i + \delta_i(x_i), \quad \text{for } i = 2, \ldots, n.$$

Also note that $\delta_i(x_i) \in I_{i-1} = \langle x_1, y_1, \ldots, x_{i-1}, y_{i-1} \rangle$. Hence, the set of generators $\{x_1, y_1, \ldots, x_n, y_n\}$ forms a *regular normalizing set* (see 2.4.10). Since $J(\widehat{E}) = \langle x_1, y_1, \ldots, x_n, y_n \rangle$, it now follows from 2.4.11 that the Krull dimension, classical Krull dimension and global dimension of $\widehat{E}$ are all equal to $2n$.

Remark. For the quantum coordinate rings and quantum algebras in examples 4.2.1 and 4.2.2, it is well known that the derivations δ_{lm} and δ_l are locally nilpotent. In [8], using this fact (and other assumptions), Cauchon constructed the "Derivation-Elimination Algorithm". But, for power series completions of these examples, the extended derivations $\widehat{\delta}_{lm}$ and $\widehat{\delta}_l$ are not locally nilpotent.

REFERENCES

[1] V. I. Arnautov, S. T. Glavatsky and A.v. Mikhalev, *Introduction to the Theory of Topological Rings and Modules*, Pure and Applied Mathematics 197, Marcel Dekker, Inc. New York 1996.

[2] M. Artin, W. Schelter, and J. Tate, *Quantum deformations of GL_n*, communic. Pure Appl. Math. 44(1991), 879-895.

[3] M.F. Atiyah and I.G. MaCdonald *Introduction to Commutative Algebra*, Addison-Wesley publishing Company, 1969.

[4] A. D. Bell, Localization and ideal theory in noetherian strongly group-graded rings, *J. Algebra*, 105 (1987), 76–115.

[5] K. A. Brown and K. R. Goodearl, *Lectures on Algebraic Quantum Groups*, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser Verlag, Basel, 2002.

[6] K. A. Brown and K. R. Goodearl, Prime spectra of quantum semisimple groups, *Trans. Amer. Math. Soc.*, 348 (1996), 2465–2502.

[7] K. A. Brown and R. B. Warfield, Jr., The influence of ideal structure on representation theory, *J. Algebra*, 116 (1988), 294–315.

[8] G. Cauchon, Effacement des dérivations et spectres premiers des algèbres quantiques, *J. Alg.*, 260(2003), 476-518.

[9] P. M. Cohn, *Skew Fields: Theory of General Division Rings*, Encyclopedia of Mathematics and its Applications 57, Cambridge University Press, Cambridge, 1995.

[10] C. De Concini, V. Kac, and C. Procesi, Some remarkable degenerations of quantum groups, *Comm. Math. Phys.*, 157 (1993), 405–427.

[11] A. Goldie and G. Michler, Ore extensions and polycyclic group rings, *J. London Math. Soc. (2)*, 9 (1974/75), 337–345.

[12] K. R. Goodearl and T. H. Lenagan, Catenarity in quantum algebras, *J. Pure Appl. Algebra*, 111 (1996), 123–142.

[13] K. R. Goodearl and E. S. Letzter, Prime and primitive spectra of multiparameter quantum affine spaces, in *Trends in ring theory (Miskolc, 1996)*, CMS Conference Proceedings 22, American Mathematical Society, Providence, RI, 1998, 39–58.

[14] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings, Second Edition*, London Mathematical Society Student Texts 61, Cambridge University Press, Cambridge, 2004.

[15] K. L. Horton, The prime and primitive spectra of multiparameter quantum symplectic and Euclidean spaces, *Comm. Alg.*, (10) 31 (2003), 4713-4743.

[16] T. J. Hodges, Quantum tori and poisson tori, unpublished notes, 1994.

[17] T. J. Hodges, T. Levasseur, and M. Toro, Algebraic structure of multi-parameter quantum groups, *Advances in Math.*, 126 (1997), 52-92.

[18] A. V. Jategaonkar, *Localization in Noetherian Rings*, London Mathematical Society Lecture Notes 98, Cambridge University Press, Cambridge, 1986.

[19] A. Joseph, *Quantum Groups and Their Primitive Ideals*, Ergebnisse der Math. (3) 29, Springer-Verlag, Berlin, 1995.

[20] R. M. Kashaev, Heisenberg double and pentagon relation, *Algebra i Analiz*, 8 (1996), no. 4, 63-74.

[21] T. H. Koornwinder, Special functions and q-commuting variables, in *Special Functions, q-series and Related Topics (Toronto, Ontariao, 1995)*, Fields Institute Communications 14, Amer. Math. Soc., Providence, 1997, 131–166.

[22] E. S. Letzter, L. Wang, Prime ideals of q-commutative power series rings, submitted for publication.

[23] H. Li and F. Van Oystaeyen, *Zariskian filtrations*, Kluwer Academic Publishers, Dordrecht, 1996.

[24] J. C. McConnell, J. C. Robson, *Noncommutative Noetherian Rings*, Graduate studies in mathematics, V. 30, American Mathematics Society, Providence, Rhode Island, 2000.

[25] D. S. Passman, *The Algebraic Structure of Group Rings*, Wiley-Interscience, New York, 1977.

[26] P. Schneider and O. Venjakob, On the codimension of modules over skew power series rings with applications to Iwasawa algebras, *J. Pure Appl. Algebra*, 204 (2006), 349–367.

[27] A. A. Tuganbaev, The Jacobson radical of the Laurent series ring, *Fundam. Prikl. Mat.*, 12 (2006), 209–215.

[28] O. Venjakob, A non-commutative Weierstrass preparation theorem and applications to Iwasawa theory (with an appendix by Denis Vogel), *J. Reine Angew. Math.*, 559 (2003), 153–191.

[29] R. Walker, Local rings and normalizing sets of elements, *Pro. London Math. Soc.*, (3) 24 (1972), 27-45.

[30] L. Wang, Completions of quantum coordinate rings, *Proc. Amer. Math. Soc.*, to appear.

[31] O. Zariski, P. Samuel, *Commutative algebra*, Vol. I and II, D. Van Nostrand, 1958,1960, New printing, Springer, New York.