EXACT RELATIONS SATISFIED BY THE EFFECTIVE TENSORS OF TWO-DIMENSIONAL TWO-PHASE THERMOELECTRIC COMPOSITES.

A Thesis
Submitted to
the Temple University Graduate Board

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in Partial Fulfillment
of the Requirements for the Degree of
MASTER OF SCIENCE

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by
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August, 2020

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ABSTRACT

EXACT RELATIONS SATISFIED BY THE EFFECTIVE TENSORS OF TWO-DIMENSIONAL TWO-PHASE THERMOELECTRIC COMPOSITES.

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Thermoelectric materials have been used for cooling and heating systems for over a hundred years. Today practical applications of thermoelectric devices include cooling car seats, power generation, and refrigeration. Thermoelectric materials are special for their ability to convert temperature imbalances into electricity. Their applications can inform the discourse about the transition to renewable energy sources—something that our Earth most desperately needs. The goal of this dissertation is to describe how the effective tensors of two-dimensional thermoelectric composites made from two isotropic materials depend on thermoelectric parameters of the constituents. Using the theory of exact relations and links developed by Grabovsky and his collaborators, we describe all equations satisfied by the thermoelectric effective tensor of a composite without the explicit knowledge of its microstructure. In some special cases, the effective tensor can be determined completely. Even in the general case, four out of 10 components of the two-dimensional thermoelectric tensor can be expressed in terms of the remaining 6, regardless of the microstructure. We started with special cases and worked our way up to the more general ones.

ACKNOWLEDGEMENTS

I want to thank my advisor, Professor Yury Grabovsky, for his creative metaphors, and exceptional analogies that helped my understanding of this theoretical material. As well as his patience and determination to create a safe learning environment. I want to also thank all of the members of my committee that have taken the time to read my dissertation, and provide their guidance.

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CHAPTER 1

EQUATIONS OF THERMOELECTRICITY

1.1 Physics of thermoelectricity

Thermoelectric properties of a material are described by the relations between the gradient $\nabla\mu$ of an electrochemical potential, temperature gradient ∇T, current density $\boldsymbol{j}_E$ and entropy flux $\boldsymbol{j}_S$ [2]. The total energy $U = U(S, N)$ is a function of entropy and the number of charge carriers N. Therefore, the energy flux $\dot{U}$ is given by

$$\dot{U} = T\dot{S} + \mu\dot{N}, \quad T = \frac{\partial U}{\partial S}, \quad \mu = \frac{\partial U}{\partial N}.$$

where T is the absolute temperature and μ is the electrochemical potential. Thus, in a general heterogeneous medium we have

$$\boldsymbol{j}_U = T\boldsymbol{j}_S + \mu\boldsymbol{j}_E,$$

where $\boldsymbol{j}_U$ is the total energy flux, $\boldsymbol{j}_S$ is the entropy flux and $\boldsymbol{j}_E$ is the electric current (charge carrier flux). The conservation of charge and energy laws are expressed by the equations

$$\nabla \cdot \boldsymbol{j}_E = 0, \qquad \nabla \cdot \boldsymbol{j}_U = 0. \tag{1.1}$$

In addition to conservation laws we also postulate linear constitutive laws that relate the electric current and the entropy flux to the non-uniformity of electrochemical potential and temperature. In a thermoelectric material these two driving forces are coupled:

$$\begin{cases} \boldsymbol{j}_E = \boldsymbol{M}_{11}\nabla(-\mu) + \boldsymbol{M}_{12}\nabla(-T), \\ \boldsymbol{j}_S = \boldsymbol{M}_{12}^t\nabla(-\mu) + \boldsymbol{M}_{22}\nabla(-T), \end{cases} \qquad \boldsymbol{M}_{11}^t = \boldsymbol{M}_{11}, \quad \boldsymbol{M}_{22}^t = \boldsymbol{M}_{22}, \tag{1.2}$$

where the Onsager reciprocity relation are incorporated in the above constitutive laws in the form of the symmetry of the tensor relating the fluxes to the gradients of the thermodynamic driving forces. We will now examine the physical meaning of the constitutive tensors $\boldsymbol{M}_{ij}$, and change notation for them accordingly.

When the temperature is constant, we have $\boldsymbol{j}_E = \boldsymbol{M}_{11}\nabla(-\mu)$, which is an ordinary conductivity. Indeed, the electrochemical potential μ is a sum of the electrostatic potential and a chemical potential. The latter depends only on the temperature and is therefore constant. In this case $\boldsymbol{E} = \nabla(-\mu)$ is the electric field and the first equation in (1.2) should read $\boldsymbol{j}_E = \boldsymbol{\sigma}\boldsymbol{E}$, where $\boldsymbol{\sigma}$ is the isothermal conductivity tensor. Thus, $\boldsymbol{M}_{11} = \boldsymbol{\sigma}$, which is a symmetric, positive definite $d \times d$ matrix, where $d = 2$ or 3.

In the absence of the electrical current ($\boldsymbol{j}_E = \boldsymbol{0}$) the gradient of $-\mu$ has the meaning of the electromotive force generated by a temperature gradient. This is called the *Seebeck effect* [8]. Writing

$$\boldsymbol{e}_{\mathrm{emf}} = -\nabla\mu = \boldsymbol{S}\nabla T,$$

where the $d \times d$ matrix $\boldsymbol{S}$ is called the Seebeck coefficient (tensor). We see, from the first equation in (1.2), that $\boldsymbol{M}_{12} = \boldsymbol{\sigma}\boldsymbol{S}$.

The heat flux at zero electric current is characterized by the heat conductivity tensor $\boldsymbol{j}_U = -\boldsymbol{\kappa}\nabla T$, which gives a formula for $\boldsymbol{M}_{22}$:

$$\boldsymbol{M}_{22} = \frac{\boldsymbol{\kappa}}{T} + \boldsymbol{S}^t\boldsymbol{\sigma}\boldsymbol{S}.$$

We observe that the assumption of symmetry and positive definiteness of the electrical and heat conductivity tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\kappa}$, respectively, is equivalent to the symmetry and positive definiteness of the $2d \times 2d$ matrix

$$\mathsf{M} = \begin{bmatrix} \boldsymbol{\sigma} & \boldsymbol{\sigma}\boldsymbol{S} \\ \boldsymbol{S}^t\boldsymbol{\sigma} & \boldsymbol{\kappa}/T + \boldsymbol{S}^t\boldsymbol{\sigma}\boldsymbol{S} \end{bmatrix}, \tag{1.3}$$

that describes constitutive relation (1.2)

$$\begin{bmatrix} \boldsymbol{j}_E \\ \boldsymbol{j}_S \end{bmatrix} = \mathsf{M} \begin{bmatrix} \nabla(-\mu) \\ \nabla(-T) \end{bmatrix}. \tag{1.4}$$

Notwithstanding the assumption of linear dependence of fluxes on the gradients of the thermodynamic driving forces, the equations of thermoelectricity are inherently nonlinear, especially since all the "material" tensors $\boldsymbol{\sigma}$, $\boldsymbol{S}$ and $\boldsymbol{\kappa}$ depend on the temperature. Recalling that the flux $\boldsymbol{j}_U$ and not $\boldsymbol{j}_S$, which is conserved (i.e., divergence-free), we should rewrite the constitutive equations relating $(\boldsymbol{j}_E, \boldsymbol{j}_U)$ to gradients of the potentials. However, this will break the symmetry of the constitutive relation. To retain the symmetry we follow Callen [2] and define the new potentials

$$\psi_1 = \frac{\mu}{T}, \qquad \psi_2 = \frac{1}{T},$$

denoting

$$\boldsymbol{e}_1 = \nabla\psi_1, \quad \boldsymbol{e}_2 = \nabla\psi_2, \quad \boldsymbol{j}_1 = -\boldsymbol{j}_E, \quad \boldsymbol{j}_2 = \boldsymbol{j}_U,$$

we obtain

$$\begin{cases} \boldsymbol{j}_1 = \boldsymbol{L}_{11}\boldsymbol{e}_1 + \boldsymbol{L}_{12}\boldsymbol{e}_2, \\ \boldsymbol{j}_2 = \boldsymbol{L}_{12}^t\boldsymbol{e}_1 + \boldsymbol{L}_{22}\boldsymbol{e}_2, \end{cases} \tag{1.5}$$

where

$$\boldsymbol{L}_{11} = T\boldsymbol{\sigma}, \quad \boldsymbol{L}_{12} = -T(\mu\boldsymbol{\sigma} + T\boldsymbol{\sigma}\boldsymbol{S}), \quad \boldsymbol{L}_{22} = T[\mu^2\boldsymbol{\sigma} + T\boldsymbol{\gamma} + T\mu(\boldsymbol{\sigma}\boldsymbol{S} + \boldsymbol{S}^t\boldsymbol{\sigma})]. \tag{1.6}$$

Now the new "material tensor"

$$\mathsf{L} = T\begin{bmatrix} \boldsymbol{\sigma} & -\boldsymbol{\sigma}(\mu + T\boldsymbol{S}) \\ -(\mu + T\boldsymbol{S})^t\boldsymbol{\sigma} & T\boldsymbol{\kappa} + (\mu + T\boldsymbol{S})^t\boldsymbol{\sigma}(\mu + T\boldsymbol{S}) \end{bmatrix} \tag{1.7}$$

is symmetric and positive definite if and only if M, given by (1.3), is symmetric and positive definite, i.e. if and only if $\boldsymbol{\sigma}$ and $\boldsymbol{\kappa}$ are symmetric and positive definite $d \times d$ matrices, $d = 2$ or 3.

1.2 Linearized equations of thermoelectricity

In what follows we will be interested in the "small fluctuation regime"

$$\mu = \mu_0 + \epsilon\widetilde{\mu}, \qquad T = T_0 + \epsilon\widetilde{T}.$$

As $\epsilon \to 0$ the equations of thermoelectricity (1.1), (1.3) and (1.4) become linear with respect to $\widetilde{\mu}$ and $\widetilde{T}$, with temperature-dependent coefficients set to their values corresponding to $T = T_0$. We have already observed that the full thermoelectric system is invariant with respect to addition of a constant to the electrochemical potential μ. Therefore, we can set $\mu_0 = 0$, without loss of generality.

In what follows we use notation μ and T instead of $\widetilde{\mu}$ and $\widetilde{T}$. Thus, modifying the potential ψ_1

$$\psi_1 \mapsto \frac{\mu - \mu_0}{T},$$

Thus, for linearized problems we can write

$$\begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix} = \mathsf{L}\begin{bmatrix} \nabla\psi_1 \\ \nabla\psi_2 \end{bmatrix}, \quad \mathsf{L} = T_0^2\begin{bmatrix} \boldsymbol{\sigma}/T_0 & -\boldsymbol{\sigma}\boldsymbol{S} \\ -\boldsymbol{S}^t\boldsymbol{\sigma} & \boldsymbol{\kappa} + T_0\boldsymbol{S}^t\boldsymbol{\sigma}\boldsymbol{S} \end{bmatrix}, \tag{1.8}$$

where all physical property tensors $\boldsymbol{\sigma}$, $\boldsymbol{\kappa}$ and $\boldsymbol{S}$ are evaluated at $T = T_0$—the working temperature. From the block-components of L in (1.8) we can recover the physical parameters:

$$\boldsymbol{\sigma} = \beta_0 \boldsymbol{L}_{11}, \quad \boldsymbol{S} = -\beta_0 \boldsymbol{L}_{11}^{-1} \boldsymbol{L}_{12}, \quad \boldsymbol{\kappa} = \beta_0^2 (\boldsymbol{L}_{22} - \boldsymbol{L}_{12}^t \boldsymbol{L}_{11}^{-1} \boldsymbol{L}_{12}), \quad \beta_0 = \frac{1}{T_0}. \tag{1.9}$$

CHAPTER 2

ISOTROPIC MATERIALS

From this point on we will be dealing only with block-components $\boldsymbol{L}_{ij}$ of the thermoelectric tensor L. All our conclusions could then be translated into statements about physical parameters via (1.9), while the physical parameters themselves will no longer be used or referred to in the sequel.

A general thermoelectric tensor L is *anisotropic*, i.e. its N components will change when we rotate the material. However, of particular interest are *isotropic* materials, which look the same in all orientations, like a ball. Such materials have thermoelectric tensors of the form

$$\mathsf{L} = \begin{bmatrix} \varsigma_{11}\boldsymbol{I}_d & \varsigma_{12}\boldsymbol{I}_d \\ \varsigma_{12}\boldsymbol{I}_d & \varsigma_{22}\boldsymbol{I}_d \end{bmatrix} = \boldsymbol{\varsigma} \otimes \boldsymbol{I}_d, \qquad \boldsymbol{\varsigma} = \begin{bmatrix} \varsigma_{11} & \varsigma_{12} \\ \varsigma_{12} & \varsigma_{22} \end{bmatrix}, \tag{2.1}$$

when $d = 3$. When $d = 2$ the form is a little more complicated

$$\mathsf{L} = \boldsymbol{\sigma} \otimes \boldsymbol{I}_2 + r\boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp, \qquad \boldsymbol{R}_\perp = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.2}$$

Here $\boldsymbol{I}_d$ denotes a $d \times d$ identity matrix. The tensor (2.2) is positive definite if and only if $\boldsymbol{\sigma} > 0$ and $\det \boldsymbol{\sigma} > r^2$. In this study we will be interested in two-phase composites, i.e., composites made of two different thermoelectric materials L_A and L_B. If tensors L_A and L_B are anisotropic, it means that in a composite described above we have to use these materials in one fixed orientation. This is very often impractical, and we will restrict our attention to *polycrystals*, where we are permitted to use each anisotropic material in any orientation, so that at different points we may have different orientation of the same material. Because of this, tensors L_A and L_B have to be isotropic, if we wish to consider two-phase composites that are also polycrystalline.

CHAPTER 3

PERIODIC COMPOSITES

Let $Q = [0,1]^d$. It is a unit square when $d = 2$ and unit cube when $d = 3$. Let us suppose that Q is divided into two complementary subsets A and B. We place one thermoelectric material in A and another in B. If the corresponding tensors of material properties are denoted by L_A and L_B, then the function

$$\mathsf{L}(\boldsymbol{x}) = \mathsf{L}_A \chi_A(\boldsymbol{x}) + \mathsf{L}_B \chi_B(\boldsymbol{x})$$

describes this situation mathematically, since $\mathsf{L}(\boldsymbol{x}) = \mathsf{L}_A$, if and only if $\boldsymbol{x} \in A$ and $\mathsf{L}(\boldsymbol{x}) = \mathsf{L}_B$, if and only if $\boldsymbol{x} \in B$. Here $\chi_S(\boldsymbol{x})$ is the characteristic function of a subset S, taking value 1, when $\boldsymbol{x} \in S$ and value 0, otherwise.

Now we are going to tile the entire space $\mathbb{R}^d$ with copies of the "period cell" Q, generating a Q-periodic function $\mathsf{L}_{\mathrm{per}}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$. Specifically, in order to find the value of $\mathsf{L}_{\mathrm{per}}(\boldsymbol{x})$ at a specific point $\boldsymbol{x} \in \mathbb{R}^d$ we first find a vector $\boldsymbol{z}$ with integer components, such that $\boldsymbol{x} - \boldsymbol{z} \in Q$ and then define $\mathsf{L}_{\mathrm{per}}(\boldsymbol{x}) = \mathsf{L}(\boldsymbol{x} - \boldsymbol{z})$. In general $\mathsf{L}_{\mathrm{per}}(\boldsymbol{x}_1) = \mathsf{L}_{\mathrm{per}}(\boldsymbol{x}_2)$, whenever $\boldsymbol{x}_1 - \boldsymbol{x}_2$ has integer components.

A periodic composite material [9, 1] would have such a structure on a *microscopic level*. Mathematically, we choose $\epsilon > 0$, representing a microscopic length scale and define $\mathsf{L}_\epsilon(\boldsymbol{x}) = \mathsf{L}_{\mathrm{per}}(\boldsymbol{x}/\epsilon)$, restricting $\boldsymbol{x}$ to lie in a subset $\Omega \subset \mathbb{R}^d$ occupied by our composite. On a macroscopic level, such a composite will look as though it is a homogeneous thermoelectric material. Its thermoelectric tensor L^*, called the effective tensor of the composite, is a complicated function not only of the tensors L_A and L_B of its constituents, but also of the set A ($B = Q \setminus A$) [5]. Specifically, if we keep L_A and L_B fixed and change only the shape of A, then the effective tensor L^* will change as well. Understanding how L^* depends on the shape of A is an important (and difficult) problem that could help design thermoelectric composites with desired properties. Even though, there is a mathematical description of L^* as a function of A, it is complicated and we will not be needing or using this description.

CHAPTER 4

EXACT RELATIONS AND LINKS

Let us recall that the thermoelectric tensor L is a 2×2 block-matrix

$$\mathsf{L} = \begin{bmatrix} \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \boldsymbol{L}_{12}^t & \boldsymbol{L}_{22} \end{bmatrix}, \tag{4.1}$$

where $\boldsymbol{L}_{11}$ and $\boldsymbol{L}_{22}$ are symmetric $d \times d$ matrices. Therefore, we are going to think of each such tensor as a point in an N-dimensional vector space, where $N = 2d^2 + d$, the dimension of all symmetric $2d \times 2d$ matrices.

Now, let us imagine that we have fixed two such points, representing tensors L_A and L_B and we are making periodic composites with all possible subsets $A \subset Q$. For each choice of the set A we get a point L^* in our N-dimensional vector space. The topological closure of the set of all such points corresponding to all possible subsets $A \subset Q$ is called the G-closure of a two-point set $\{\mathsf{L}_A, \mathsf{L}_B\}$ [6, 11]. We will see that in the case of two-dimensional thermoelectric composites such G-closures will in general be compact subsets of 6-dimensional submanifolds of the 10-dimensional space of all two-dimensional thermoelectic tensors. In particular, making composites made of thermoelectic materials from such a submanifold will produce effective tensors lying on that same submanifold. Equations describing such a submanifold are called exact relations [7, 4]. In the language of composite materials, these relations will be satisfied by *all* composites, as long as they are made of materials that satisfy these equations.

In addition to exact relations we will also be interested in links between effective tensors of composites with the same microstructure but different constituents. The links of special interest to us relate thermoelectric properties of our composite to conducting properties of the composite with the same microstructure.

4.1 Exact relations

One example of exact relations is physically obvious. If all materials in a composite do not exhibit thermoelectric effect, then the composite will not be thermoelectric, no matter what microstructure is used to create the composite. In our geometric point of view such an exact relation is given by the 6-dimensional submanifold in the 10-dimensional space of symmetric 4×4 matrices, representing thermoelectric tensors.

$$\mathbb{M}_{16} = \left\{ \begin{bmatrix} \boldsymbol{L}_1 & 0 \\ 0 & \boldsymbol{L}_2 \end{bmatrix} : \boldsymbol{L}_1 > 0, \quad \boldsymbol{L}_2 > 0 \right\}. \tag{4.2}$$

In order to distinguish and refer to various exact relations we use the same numbering as in [3], where all exact relations, including all intersections have been enumerated and described. Here the numbers carry no other information or function than to distinguish different exact relations, which are fully described here.

A nontrivial example of an exact relation from [3] is

$$\mathbb{M}_{17} = \{\mathsf{L} > 0 : \mathsf{L}(\boldsymbol{J} \otimes \boldsymbol{R}_\perp)\mathsf{L} = \boldsymbol{J} \otimes \boldsymbol{R}_\perp\}, \quad \boldsymbol{J} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.3}$$

In block-components we can rewrite this as

$$\mathbb{M}_{17} = \left\{ \mathsf{L} > 0 : \boldsymbol{L}_{11} = \frac{\boldsymbol{L}_{12}\mathrm{cof}(\boldsymbol{L}_{22})\boldsymbol{L}_{12}^t}{\det \boldsymbol{L}_{12}},\ \det \boldsymbol{L}_{22} - \det \boldsymbol{L}_{12} = 1 \right\}. \tag{4.4}$$

Of course, there is symmetry between indices and we also have

$$\mathbb{M}_{17} = \left\{ \mathsf{L} > 0 : \boldsymbol{L}_{22} = \frac{\boldsymbol{L}_{12}^t\mathrm{cof}(\boldsymbol{L}_{11})\boldsymbol{L}_{12}}{\det \boldsymbol{L}_{12}},\ \det \boldsymbol{L}_{11} - \det \boldsymbol{L}_{12} = 1 \right\}.$$

4.2 Links

A physically obvious example of a link comes from the physically obvious exact relation (4.2). The effective tensor

$$\mathsf{L}^* = \begin{bmatrix} \boldsymbol{L}_1^* & 0 \\ 0 & \boldsymbol{L}_2^* \end{bmatrix}$$

of such a composite is expressed in terms of the effective conductivity of composites $\boldsymbol{L}_1(x)$, $\boldsymbol{L}_2(\boldsymbol{x})$.

Let us now give an important nontrivial example of a link, which we call a *global link*. Suppose that $\mathsf{L}(\boldsymbol{x})$ is a local thermoelectric tensor of a two-dimensional periodic composite, and L^* is its effective thermoelectric tensor. A global link is a function $\Psi(\mathsf{L})$ with values in the space of thermoelectric tensors that has the property that $\Psi(\mathsf{L}^*)$ is the effective thermoelectric tensor of the periodic composite with the local thermoelectric tensor $\Psi(\mathsf{L}(\boldsymbol{x}))$. In short,

$$\Psi(\mathsf{L}(\boldsymbol{x}))^* = \Psi(\mathsf{L}^*). \tag{4.5}$$

All such links have been computed in [3]. They are given by

$$\Psi_{\boldsymbol{A},\boldsymbol{B}}(\mathsf{L}) = (\boldsymbol{B} \otimes \boldsymbol{I}_2)(a_{11}\mathsf{L} + a_{12}\mathsf{T})(a_{12}\mathsf{L} + a_{22}\mathsf{T})^{-1}\mathsf{T}(\boldsymbol{B}^t \otimes \boldsymbol{I}_2), \quad \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{4.6}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2×2 invertible matrices, and $\mathsf{T} = \boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp$. We remark that

$$\Psi_{\boldsymbol{A},\boldsymbol{B}}(\mathsf{L}) = (\boldsymbol{B} \otimes \boldsymbol{I}_2)\mathsf{T}(a_{12}\mathsf{L} + a_{22}\mathsf{T})^{-1}(a_{11}\mathsf{L} + a_{12}\mathsf{T})(\boldsymbol{B}^t \otimes \boldsymbol{I}_2).$$

It is important to note that different pairs of matrices $\{\boldsymbol{A}, \boldsymbol{B}\} \subset GL(2, \mathbb{R})$ can define the same transformation $\Psi_{\boldsymbol{A},\boldsymbol{B}}$. Specifically,

$$\Psi_{\lambda\boldsymbol{A},\boldsymbol{B}} = \Psi_{\boldsymbol{A},\boldsymbol{B}}, \quad \Psi_{\boldsymbol{A},\lambda\boldsymbol{B}} = \Psi_{\boldsymbol{A}_\lambda,\boldsymbol{B}}, \qquad \boldsymbol{A}_\lambda = \begin{bmatrix} \lambda^2 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{A} \tag{4.7}$$

for any nonzero real number λ. Thus, without loss of generality, we may assume that $|\det \boldsymbol{A}| = |\det \boldsymbol{B}| = 1$. Even with this assumption we still have symmetries

$$\Psi_{-\boldsymbol{A},\boldsymbol{B}} = \Psi_{\boldsymbol{A},\boldsymbol{B}}, \qquad \Psi_{\boldsymbol{A},-\boldsymbol{B}} = \Psi_{\boldsymbol{A},\boldsymbol{B}}.$$

We note that transformations (4.7) form a Lie group with the composition law

$$\Psi_{\boldsymbol{A}_1,\boldsymbol{B}_1} \circ \Psi_{\boldsymbol{A}_2,\boldsymbol{B}_2} = \Psi_{\boldsymbol{A}_1^{\boldsymbol{B}_2}\boldsymbol{A}_2,\boldsymbol{B}_1\boldsymbol{B}_2}, \qquad \boldsymbol{A}^{\boldsymbol{B}} = \begin{bmatrix} \det \boldsymbol{B} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \boldsymbol{A} \begin{bmatrix} \det \boldsymbol{B} & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.8}$$

We note that if $\det \boldsymbol{B}_2 = 1$, then $\Psi_{\boldsymbol{A}_1,\boldsymbol{B}_1} \circ \Psi_{\boldsymbol{A}_2,\boldsymbol{B}_2} = \Psi_{\boldsymbol{A}_1\boldsymbol{A}_2,\boldsymbol{B}_1\boldsymbol{B}_2}$. For future reference we observe that the most general transformation $\Psi_{\boldsymbol{A},\boldsymbol{B}}$, such that $\Psi_{\boldsymbol{A},\boldsymbol{B}}(\mathsf{I}) = \mathsf{I}$ has

$$\boldsymbol{A} = \begin{bmatrix} a_0 & 1 \\ 1 & a_0 \end{bmatrix}, \quad \boldsymbol{B} \in O(2, \mathbb{R}). \tag{4.9}$$

We note that each of the global links (4.6) maps an exact relation into an exact relation. We therefore, have the option of deriving only the exact relations passing though $\mathsf{L}_0 = \mathsf{I}$. Any other exact relations can be obtained as images of the ones passing though L_0 by an application of the link (4.6).

CHAPTER 5

TWO PHASE COMPOSITES WITH ISOTROPIC PHASES

5.1 Problem formulation

Suppose, we are given two isotropic thermoelectric tensors

$$\mathsf{L}_1 = \varsigma_1 \otimes \boldsymbol{I}_2 + r_1\mathsf{T}, \qquad \mathsf{L}_2 = \varsigma_2 \otimes \boldsymbol{I}_2 + r_2\mathsf{T}.$$

Our goal is to say as much as we can about the effective thermoelectric tensor L^* of a two phase composite made with L_1 and L_2, provided we know about the microstructure at most the volume fractions with which they are used in the composite. The most natural approach is to use the global link (4.6) to eliminate the thermoelectric coupling coefficients r_j and $(\varsigma_j)_{12}$. For three-dimensional thermoelectric composites this has been done in [10] by means of the linear links (4.6) of the form $\Psi_{\boldsymbol{I}_2,\boldsymbol{B}}$. In two space dimensions the same method applies only when $r_1 = r_2 = 0$. We can eliminate the thermoelectric coupling from L_1 by the transformation

$$\Psi_1(\mathsf{L}) = (\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)(\mathsf{L} - r_1\mathsf{T})(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2) \tag{5.1}$$

maps L_1 to I, while

$$\mathsf{L}' = \Psi_1(\mathsf{L}_2) = \varsigma \otimes \boldsymbol{I}_2 + \rho \mathsf{T},$$

where

$$\varsigma = \varsigma_1^{-1/2} \varsigma_2 \varsigma_1^{-1/2}, \qquad \rho = \frac{r_2 - r_1}{\sqrt{\det \varsigma_1}}. \tag{5.2}$$

Next we apply transformation $\Psi_2 = \Psi_{\boldsymbol{A}, \boldsymbol{I}_2}$, where $\boldsymbol{A}$ is given by (4.9).

$$\Psi_2(\mathsf{L}) = (a_0 \mathsf{L} + \mathsf{T})(\mathsf{L} + a_0 \mathsf{T})^{-1} \mathsf{T} = \mathsf{T}(\mathsf{L} + a_0 \mathsf{T})^{-1}(a_0 \mathsf{L} + \mathsf{T}). \tag{5.3}$$

This family of transformations parametrized by $a_0 \in \mathbb{R} \setminus \{1, -1\}$ have the property that $\Psi_2(\mathsf{I}) = \mathsf{I}$. A direct calculation gives the formula

$$\Psi_2(\varsigma \otimes \boldsymbol{I}_2 + \rho \mathsf{T}) = \mu \varsigma \otimes \boldsymbol{I}_2 + \nu \mathsf{T},$$

where

$$\mu = \frac{1 - a_0^2}{\det \varsigma - (\rho + a_0)^2}, \qquad \nu = \frac{a_0 \det \varsigma - (\rho + a_0)(a_0 \rho + 1)}{\det \varsigma - (\rho + a_0)^2}. \tag{5.4}$$

Our goal is to choose the value of a_0 so that $\nu = 0$ in (5.4) and ς is diagonal. It will therefore be convenient to work in the frame in which ς is diagonal. The requirement $\nu = 0$ gives the quadratic equation for a_0:

$$(a_0^2 + 1)\rho = a_0(\det \varsigma - (\rho^2 + 1)). \tag{5.5}$$

This equation has distinct real roots if and only if

$$|r_1 - r_2| < \left| \sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2} \right|. \tag{5.6}$$

For this reason we call this a "weakly coupled regime". If we have equality in (5.6), then we will say that we are in a "borderline regime". Let us consider the weakly coupled regime first. If (5.6) holds and a_0 solves (5.5), then

$$\Psi_2(\varsigma \otimes \boldsymbol{I}_2 + \rho \boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp) = (a_0 \rho + 1) \frac{\varsigma}{\det \varsigma} \otimes \boldsymbol{I}_2. \tag{5.7}$$

This choice of a_0 (by design) eliminates the thermoelectric coupling coefficients, permitting us to express the effective thermoelectric tensor of the composite in terms of the effective tensors of 2D conducting composites.

When inequality in (5.6) is reversed we are in a "strongly coupled regime". In this case we need to find a transformation (4.6) that maps L_j into a different exact relation. The appropriate one is (4.3) A simple calculation shows that

$$\varsigma\otimes\boldsymbol{I}_2+r\mathsf{T}\in\mathbb{M}_{17}\quad\Longleftrightarrow\quad\sigma_{11}=\sigma_{22}\text{ and }\det\varsigma=r^2+1. \tag{5.8}$$

We therefore choose a different chain of global links. We first apply

$$\Psi_0(\mathsf{L})=(\varsigma_1^{-1/2}\otimes\boldsymbol{I}_2)\mathsf{L}(\varsigma_1^{-1/2}\otimes\boldsymbol{I}_2) \tag{5.9}$$

and compute

$$\Psi_0(\mathsf{L}_1)=\mathsf{I}+\rho_1\mathsf{T}=\mathsf{L}_1',\quad\Psi_0(\mathsf{L}_2)=\varsigma\otimes\boldsymbol{I}_2+\rho_2\mathsf{T}=\mathsf{L}_2',\quad\rho_j=\frac{r_j}{\sqrt{\det\varsigma_1}}.$$

In this case we will choose to work in the frame in which $\sigma_{11}=\sigma_{22}$, as suggested by (5.8). It turns out that affine transformations

$$\Psi_3(\mathsf{L})=a\mathsf{L}+b\mathsf{T},\qquad a>0,\ b\in\mathbb{R} \tag{5.10}$$

are sufficient for our goal to map L_j' into $\mathbb{M}_{17}$. For $\mathsf{L}_j''=\Psi_3(\mathsf{L}_j')$, $j=1,2$, we compute

$$\mathsf{L}_1''=a\mathsf{I}+(a\rho_1+b)\mathsf{T},\qquad\mathsf{L}_2''=a\varsigma\otimes\boldsymbol{I}_2+(a\rho_2+b)\mathsf{T}.$$

In our frame the requirement that L_j'' lie on $\mathbb{M}_{17}$ is equivalent to

$$a^2=(a\rho_1+b)^2+1,\qquad a^2\det\varsigma=(a\rho_2+b)^2+1.$$

The values of $a>0$ and $b\in\mathbb{R}$ with these properties are

$$b=a\frac{\det\varsigma-1+\rho_1^2-\rho_2^2}{2\rho},\quad a^2=\frac{4\rho^2}{((\rho+1)^2-\det\varsigma)(\det\varsigma-(\rho-1)^2)}. \tag{5.11}$$

Recalling that $|\rho_1|<1$ and $|\rho_2|<\sqrt{\det\varsigma}$, we obtain

$$|\rho|=|\rho_2-\rho_1|\le|\rho_2|+|\rho_1|<\sqrt{\det\varsigma}+1.$$

The positivity of the right-hand side in the expression for a^2 in (5.11) is then equivalent to

$$(\rho + 1 - \sqrt{\det \varsigma})(\sqrt{\det \varsigma} - 1 + \rho) > 0,$$

which is also equivalent to

$$\rho^2 > (\sqrt{\det \varsigma} - 1)^2 \quad \Longleftrightarrow \quad |r_1 - r_2| > |\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2}|.$$

We conclude that, the desired parameters $a > 0$ and $b \in \mathbb{R}$ exist if and only if we are in the strongly coupled regime. Hence, we can always use (5.10) with $a > 0$ and b, given by (5.11) to map L'_j onto $\mathbb{M}_{17}$ exact relation. Applying this exact relation permits us to express the 10 thermoelectric moduli of the composite in terms of only 6 microstructure-dependent parameters. The explicit structure of the effective tensor will be derived in the next section.

5.2 Summary of results

In the list below we will use several common notations.

- In those cases when ς_1 and ς_2 are not scalar multiples of one another we denote λ_1 and λ_2 the two distinct (and positive) generalized eigenvalues solving

$$\det(\varsigma_2 - \lambda\varsigma_1) = 0. \tag{5.12}$$

  The ordering of the eigenvalues (as well as of ς_1 and ς_2) is unimportant.

- The expressions for the effective tensor of the composite will frequently involve the matrices

$$\boldsymbol{S}_1 = \frac{\varsigma_2 - \lambda_1\varsigma_1}{\lambda_2 - \lambda_1}, \qquad \boldsymbol{S}_2 = \frac{\varsigma_2 - \lambda_2\varsigma_1}{\lambda_1 - \lambda_2}. \tag{5.13}$$

- The notation $\varsigma^*(h)$ refers to the effective conductivity of the 2D conducting composite where the conductivity of material 1 is 1 and the conductivity of material 2 is $h > 0$.

Within the context of weakly coupled, strongly coupled and borderline cases there are several special case where we can say a lot more about the effective thermoelectric tensor of a composite. All such special cases correspond to smaller exact relations lying inside the ones corresponding to the three general cases above. The knowledge of a complete list of exact relations permits us to identify all the special cases, which we presently describe.

Deterministic case

$\varsigma_1 = \theta_1\varsigma_0$, $\varsigma_2 = \theta_2\varsigma_0$, $|r_1 - r_2| = |\theta_1 - \theta_2|\sqrt{\det\varsigma_0}$. The formula for the effective thermoelectric tensor is $\mathsf{L}^* = \varsigma^* \otimes \boldsymbol{I}_2 + r^*\mathsf{T}$, where

$$\varsigma^* = \langle\varsigma(\boldsymbol{x})^{-1}\rangle^{-1}, \qquad r^* = \frac{\langle r(x)\theta(x)^{-1}\rangle}{\langle\theta(x)^{-1}\rangle} \tag{5.14}$$

In this case the effective thermoelectric tensor is determined regardless of the microstructure, which is why we call this the "deterministic case".

Special borderline case

$\boldsymbol{L}_1 = \varsigma_1 + ir_0\boldsymbol{R}_\perp$, $\boldsymbol{L}_2 = \varsigma_2 + ir_0\boldsymbol{R}_\perp$, moreover, $\det\varsigma_1 = \det\varsigma_2$. The formula for the effective thermoelectric tensor is

$$\mathsf{L}^* = r_0\mathsf{T} + \left(\frac{\boldsymbol{S}_1}{\det\varsigma^*} + \boldsymbol{S}_2\right) \otimes \varsigma^*, \qquad \varsigma^* = \varsigma^*(\lambda), \tag{5.15}$$

where $\lambda > 0$ is one of the two roots of (5.12). (The other is then $1/\lambda$.) The result is independent of the choice of the root in (5.12) and the choice of which material is has index 1 and which has index 2.

Generic borderline case

$|r_1 - r_2| = \left|\sqrt{\det\varsigma_1} - \sqrt{\det\varsigma_2}\right|$, $r_1 \neq r_2$. The formula for the effective thermoelectric tensor is

$$\mathsf{L}^* = \boldsymbol{S}_1 \otimes \varsigma^* \mathrm{cof}(\boldsymbol{L}^*)\varsigma^* + \boldsymbol{S}_2 \otimes \boldsymbol{L}^* + \frac{\alpha}{\det \varsigma_1}(\boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2 \otimes \varsigma^* \boldsymbol{R}_\perp \boldsymbol{L}^* + \boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1 \otimes \boldsymbol{L}^* \boldsymbol{R}_\perp \varsigma^*) + \\ + (r_1 - \alpha)\mathsf{T} \quad (5.16)$$

where

$$\alpha = \frac{\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2}}{r_1 - r_2}\sqrt{\det \varsigma_1}, \qquad \varsigma^* = \varsigma^*\left(\sqrt{\frac{\lambda_2}{\lambda_1}}\right).$$

Special weakly coupled case

$\varsigma_1 = \theta_1 \varsigma_0$, $\varsigma_2 = \theta_2 \varsigma_0$, $|r_1 - r_2| < |\theta_1 - \theta_2|\sqrt{\det \varsigma_0}$. The formula for the effective thermoelectric tensor is

$$\mathsf{L}^* = \frac{(1 - a_0^2)}{\det \varsigma^* - a_0^2}\varsigma_1 \otimes \varsigma^* + \left(r_1 + \frac{a_0(1 - \det \varsigma^*)\sqrt{\det \varsigma_1}}{\det \varsigma^* - a_0^2}\right)\mathsf{T}, \quad (5.17)$$

where a_0 solves (5.5) and

$$\varsigma^* = \varsigma^*\left(\frac{(r_2 - r_1)a_0 + \sqrt{\det \varsigma_1}}{\sqrt{\det \varsigma_2}}\right)$$

A uniform field relation

$|r_1 - r_2|^2 = \det(\varsigma_1 - \varsigma_2)$. This equation states that the tensor $\Delta\mathsf{L} = \mathsf{L}_1 - \mathsf{L}_2$ is singular. Since the tensors L_1, L_2 are isotropic the null-space of $\Delta\mathsf{L}$ is rotationally invariant, and therefore, two-dimensional, since $\Delta\mathsf{L} \neq 0$. For any vector $\boldsymbol{e}$ in the null-space of $\Delta\mathsf{L}$ we have $\mathsf{L}_1\boldsymbol{e} = \mathsf{L}_2\boldsymbol{e}$. This means that the two constituents of the composite respond identically to the applied uniform fields $\boldsymbol{e}$, and therefore $\mathsf{L}^*\boldsymbol{e} = \mathsf{L}_1\boldsymbol{e} = \mathsf{L}_2\boldsymbol{e}$. That's why these equations are called the uniform field relations. In addition to these formulas our general theory shows that L^* can be expressed in terms of the effective conductivity of a specific conducting composite. This fact is not captured by the uniform field relations. We therefore follow our general procedure to obtain an explicit representation of the effective tensor in terms of the material properties of the constituents and the effective conductivity of a composite with the same microstructure. The effective tensor of a composite is

$$\mathsf{L}^* = \alpha^* \left(\boldsymbol{S}_1 \otimes \boldsymbol{A}_1^* + \boldsymbol{S}_2 \otimes \boldsymbol{A}_2^* + \frac{a_0}{\sqrt{\det \varsigma_1}} (\boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2 \otimes \varsigma^* \boldsymbol{R}_\perp + \boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1 \otimes \boldsymbol{R}_\perp \varsigma^*) \right) + \beta^* \mathsf{T}, \quad (5.18)$$

where

$$\boldsymbol{A}_1^* = \varsigma^* - a_0^2 \boldsymbol{I}_2, \quad \boldsymbol{A}_2^* = \boldsymbol{I}_2 \det \varsigma^* - a_0^2 \varsigma^*, \quad \alpha_* = \frac{1 - a_0^2}{\det \boldsymbol{A}_1^*},$$
$$\beta^* = (r_1 - a_0 \sqrt{\det \varsigma_1} (1 + a_0^2 \alpha^*)), \quad a_0 = \frac{\lambda_1 - 1}{r_2 - r_1} \sqrt{\det \varsigma_1}, \quad \varsigma^* = \varsigma^* \left(\frac{\lambda_1}{\lambda_2} \right).$$

The case $a_0 = \infty$ corresponding to $r_1 = r_2 = r_0$ and $\det(\varsigma_1 - \varsigma_2) = 0$ is also included by taking a limit as $a_0 \to \infty$ in (5.18). In this case

$$\mathsf{L}^* = r_0 \mathsf{T} + \boldsymbol{S}_1 \otimes \boldsymbol{I}_2 + \boldsymbol{S}_2 \otimes \varsigma^*(\lambda_1), \quad \lambda_1 \neq 1 = \lambda_2.$$

Generic weakly coupled case

$|r_1 - r_2| < \left| \sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2} \right|$, assuming that $\varsigma_1 \neq \theta \varsigma_2$ and $|r_1 - r_2|^2 \neq \det(\varsigma_1 - \varsigma_2)$. The formula for the effective thermoelectric tensor is

$$\mathsf{L}^* = \alpha^* \left(\boldsymbol{S}_1 \otimes \boldsymbol{A}_1^* + \boldsymbol{S}_2 \otimes \boldsymbol{A}_2^* + \frac{a_0}{\sqrt{\det \varsigma_1}} (\boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2 \otimes \varsigma_1^* \boldsymbol{R}_\perp \varsigma_2^* + \boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1 \otimes \varsigma_2^* \boldsymbol{R}_\perp \varsigma_1^*) \right) + \gamma \mathsf{T} \quad (5.19)$$

where a_0 is one of the roots of (5.5), $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ are given in (5.13), and

$$\gamma := r_1 - a_0 \sqrt{\det \varsigma_1} (1 + a_0^2 \alpha^*)$$

$$\boldsymbol{A}_1^* = \varsigma_1^* \det \varsigma_2^* - a_0^2 \varsigma_2^*, \quad \boldsymbol{A}_2^* = \varsigma_2^* \det \varsigma_1^* - a_0^2 \varsigma_1^*, \quad \alpha^* = \frac{(1 - a_0^2) \det \varsigma_2^*}{\det \boldsymbol{A}_1^*} = \frac{(1 - a_0^2) \det \varsigma_1^*}{\det \boldsymbol{A}_2^*}$$

and

$$\varsigma_1^* = \varsigma^* \left(\frac{a_0 \rho + 1}{\lambda_2} \right), \qquad \varsigma_2^* = \varsigma^* \left(\frac{a_0 \rho + 1}{\lambda_1} \right).$$

Generic strongly coupled case

$|r_1-r_2| > \left|\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2}\right|$. The effective tensor L^* satisfies the following equation:

$$(\mathsf{L}^* + A\mathsf{T})\mathsf{T}(\boldsymbol{Z}_0 \otimes \boldsymbol{R}_\perp)\mathsf{T}(\mathsf{L}^* + A\mathsf{T}) + B\boldsymbol{Z}_0 \otimes \boldsymbol{R}_\perp = 0, \qquad \boldsymbol{Z}_0 = \boldsymbol{S}_2\boldsymbol{R}_\perp\boldsymbol{S}_1 - \boldsymbol{S}_1\boldsymbol{R}_\perp\boldsymbol{S}_2,$$

where

$$A = \frac{\det \varsigma_2 - \det \varsigma_1 + r_1^2 - r_2^2}{2\Delta r}, \qquad \Delta r = r_2 - r_1. \tag{5.20}$$

$$B = \frac{((\Delta r)^2 - (\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2})^2)((\sqrt{\det \varsigma_1} + \sqrt{\det \varsigma_2})^2 - (\Delta r)^2)}{4(\Delta r)^2}. \tag{5.21}$$

Special strongly coupled case

$\varsigma_1 = \theta_1\varsigma_0$, $\varsigma_2 = \theta_2\varsigma_0$, $|r_1 - r_2| > |\theta_1 - \theta_2|\sqrt{\det \varsigma_0}$. The formula for the effective tensor is

$$\mathsf{L}^* = \varsigma_1 \otimes \boldsymbol{L}^* + t^*\mathsf{T}, \qquad \det \varsigma_1 \det \boldsymbol{L}^* = (t^* + A)^2 + B,$$

where A and B are given by (5.20) and (5.21), respectively.

CHAPTER 6

COMPUTATIONS

6.1 Deterministic case

$\varsigma_1 = \theta_1\varsigma_0$, $\varsigma_2 = \theta_2\varsigma_0$, $|r_1 - r_2| = |\theta_1 - \theta_2|\sqrt{\det\varsigma_0}$. The relevant exact relation and link are

$$\mathbb{M}_2 = \left\{ \begin{bmatrix} \mu \boldsymbol{I}_2 & \pm(\mu-1)\boldsymbol{R}_\perp \\ \mp(\mu-1)\boldsymbol{R}_\perp & \mu\boldsymbol{I}_2 \end{bmatrix} : \mu > \frac{1}{2} \right\}, \qquad (\mu^*)^{-1} = \langle \mu^{-1} \rangle.$$

We apply Ψ_1 given by (5.1), mapping:

$$\Psi_1(\mathsf{L}_1) = \mathsf{I} \qquad \Psi_1(\mathsf{L}_2) =: \mathsf{P}$$

where P is a new isotropic tensor of the form $\varsigma \otimes \boldsymbol{I}_2 + \rho\mathsf{T}$ given by (5.2). We calculate P directly, substituting $\mathsf{L}_2 := \varsigma_2 \otimes \boldsymbol{I}_2 + r_2\mathsf{T}$:

$$\mathsf{P} = (\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)(\mathsf{L}_2 - r_1\mathsf{T})(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)$$

We simplify via the constraints above:

$$\mathsf{P} = \begin{bmatrix} \dfrac{\theta_2}{\theta_1}\boldsymbol{I}_2 & \dfrac{-(r_2 - r_1)}{\theta_1\sqrt{\det\varsigma_0}}\boldsymbol{R}_\perp \\ \dfrac{(r_2 - r_1)}{\theta_1\sqrt{\det\varsigma_0}}\boldsymbol{R}_\perp & \dfrac{\theta_2}{\theta_1}\boldsymbol{I}_2 \end{bmatrix} \qquad (6.1)$$

To solve for λ such that P will have the same form as $\mathbb{M}_2(\lambda)$, we only need to consider their first block components. Then, we can see that

$$\lambda = \frac{\theta_2}{\theta_1}, \qquad \lambda - 1 = \frac{\theta_2 - \theta_1}{\theta_1} \tag{6.2}$$

From the given parameters, we see that $\mathbb{M}_2$ is associated with the following property, respectively:

$$r_2 - r_1 = \mp(\theta_2 - \theta_1)\sqrt{\det \varsigma_0} \tag{6.3}$$

Now, we use the fractional linear relation given for all tensors on $\mathbb{M}_2$, where $\langle\rangle$ is the average value notation. By design,

$$\Psi(\mathsf{L}^*) = \mathbb{M}_2(\lambda^*) := \mathsf{P}^*$$

$$\mathsf{L}_1 \longmapsto \mathbb{M}_2(1) = \mathsf{I} \text{ with } f_1$$

$$\mathsf{L}_2 \longmapsto \mathbb{M}_2\left(\frac{\theta_2}{\theta_1}\right) = \mathsf{P} \text{ with } f_2$$

where $f_1 + f_2 = 1$. Note that $L^{-1} = \frac{1}{\lambda}\boldsymbol{I}_2$. Then, our linear fractional relation gives

$$\frac{1}{\lambda^*} = \left\langle \frac{1}{\lambda} \right\rangle = f_1 \cdot 1 + f_2 \cdot \frac{1}{\lambda} = \frac{\lambda f_1 + f_2}{\lambda} \tag{6.4}$$

which gives the following definition:

$$\lambda^* = \frac{\lambda}{\lambda f_1 + f_2} \tag{6.5}$$

Plugging in for λ and simplifying, we get

$$\lambda^* = \frac{\theta_2}{\theta_2 f_1 + \theta_1 f_2} \quad \text{and} \quad \lambda^* - 1 = \frac{(\theta_2 - \theta_1) f_2}{\theta_2 f_1 + \theta_1 f_2} \tag{6.6}$$

Undoing Ψ_1 via (6.12), we get

$$\mathsf{L}^* = \Psi_1^{-1}(\mathbb{M}_2(\lambda^*)) = (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) \begin{bmatrix} \lambda^* \boldsymbol{I}_2 & \pm(\lambda^* - 1)\boldsymbol{R}_\perp \\ \mp(\lambda^* - 1)\boldsymbol{R}_\perp & \lambda^* \boldsymbol{I}_2 \end{bmatrix} (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) + r_1 \mathsf{T}$$

In order to distribute and simplify this formula, we write the matrix as a sum of tensor products of two matrices:

$$\mathsf{L}^* = (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2)[(\lambda^* \boldsymbol{I}_2 \otimes \boldsymbol{I}_2) \mp (\lambda^* - 1)\boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp](\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) + r_1 \mathsf{T}$$

Then, we can distribute on either side to get

$$\mathsf{L}^* = \lambda^* \varsigma_1 \otimes \boldsymbol{I}_2 \mp (\lambda^* - 1)\varsigma_1^{1/2} \boldsymbol{R}_\perp \varsigma_1^{1/2} \otimes \boldsymbol{R}_\perp + r_1 \mathsf{T}$$

To simplify this, consider that ς_1 is 2x2 symmetric, then by properties of the characteristic equation of the determinant, we get :

$$\mathsf{L}^* = \lambda^* \varsigma_1 \otimes \boldsymbol{I}_2 \mp ((\lambda^* - 1)\sqrt{\det \varsigma_1})\mathsf{T} + r_1 \mathsf{T}$$

Plugging in our original constraints, we have

$$\mathsf{L}^* = \theta_1 \lambda^* \varsigma_0 \otimes \boldsymbol{I}_2 \mp ((\lambda^* - 1)\theta_1 \sqrt{\det \varsigma_0})\mathsf{T} + r_1 \mathsf{T} \tag{6.7}$$

Now, we can substitute λ^* given by (6.6):

$$\mathsf{L}^* = \left(\frac{\theta_1 \theta_2}{\theta_2 f_1 + \theta_1 f_2}\right) \varsigma_0 \otimes \boldsymbol{I}_2 \mp \left[\left(\frac{(\theta_2 - \theta_1) f_2}{\theta_2 f_1 + \theta_1 f_2}\right) \theta_1 \sqrt{\det(\varsigma_0)}\right] \mathsf{T} + r_1 \mathsf{T}$$

where we note the following simplification:

$$\left(\frac{\theta_1 \theta_2}{\theta_2 f_1 + \theta_1 f_2}\right) \varsigma_0 = \left(\frac{1}{f_1 \theta_1^{-1} + f_2 \theta_2^{-1}}\right) \varsigma_0$$

By definition of ς_1 and ς_2, we have $(\theta_1 \varsigma_0)^{-1} = \theta_1^{-1} \varsigma_0^{-1}$ and $(\theta_2 \varsigma_0)^{-1} = \theta_2^{-1} \varsigma_0^{-1}$. Then, we have

$$f_1 \varsigma_1^{-1} + f_2 \varsigma_2^{-1} = (f_1 \theta_1^{-1} + f_2 \theta_2^{-1}) \varsigma_0^{-1}$$

Let

$$\varsigma^* := (f_1 \varsigma_1^{-1} + f_2 \varsigma_2^{-1})^{-1} = \langle \varsigma_0(\boldsymbol{x})^{-1} \rangle^{-1} \tag{6.8}$$

where ς^* is defined to be the **harmonic mean** of ς_0. So, we have

$$\mathsf{L}^* = \varsigma^* \otimes \boldsymbol{I}_2 \mp \left[\left(\frac{(\theta_2 - \theta_1) f_2}{\theta_2 f_1 + \theta_1 f_2}\right) \theta_1 \sqrt{\det(\varsigma_0)}\right] \mathsf{T} + r_1 \mathsf{T}$$

By (6.3), we get:

$$\mathsf{L}^* = \varsigma^* \otimes \boldsymbol{I}_2 + \left[\left(\frac{(r_2 - r_1)f_2}{\theta_2 f_1 + \theta_1 f_2}\right)\theta_1\right]\mathsf{T} + r_1\mathsf{T}$$

Note that for any two phase composite, we define the average, $\langle q\rangle := q_1 f_1 + q_2 f_2$. Let

$$r^* := \frac{\langle r(x)\theta(x)^{-1}\rangle}{\langle \theta(x)^{-1}\rangle} \tag{6.9}$$

Then, our final form of our isotropic material composite tensor for the deterministic case is given by

$$\mathsf{L}^* = \varsigma^* \otimes \boldsymbol{I}_2 + r^*\mathsf{T} \tag{6.10}$$

such that ς^* and r^* are defined in (6.8) and (6.9).

6.2 Special borderline case

$\boldsymbol{L}_1 = \varsigma_1 + ir_0\boldsymbol{R}_\perp$, $\boldsymbol{L}_2 = \varsigma_2 + ir_0\boldsymbol{R}_\perp$, moreover, $\det\varsigma_1 = \det\varsigma_2$. The relevant exact relation is

$$\mathbb{M}_5 = \left\{ \begin{bmatrix} \boldsymbol{L} & 0 \\ 0 & \dfrac{\boldsymbol{L}}{\det \boldsymbol{L}} \end{bmatrix} : \boldsymbol{L} > 0 \right\}.$$

and the relevant link is that $\boldsymbol{L}^*$ is the effective conductivity tensor of the conducting composite with local conductivity $\boldsymbol{L}(\boldsymbol{x})$. As in the previous case, we apply Ψ_1 as defined in (5.1):

$$\Psi_1(\mathsf{L}_1) = (\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)(\varsigma_1 \otimes \boldsymbol{I}_2 + r_0\mathsf{T} - r_0\mathsf{T})(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)$$

$$= \varsigma_1^{-1/2}\varsigma_1\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2 = \boldsymbol{I}_2 \otimes \boldsymbol{I}_2$$

$$\Psi_1(\mathsf{L}_2) = \mathsf{P} := (\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)(\varsigma_2 \otimes \boldsymbol{I}_2 + r_0\mathsf{T} - r_0\mathsf{T})(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)$$

$$= \varsigma_1^{-1/2}\varsigma_2\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2 = \varsigma \otimes \boldsymbol{I}_2$$

by definition of ς given by (5.2). We can see that P lies in our exact relation as $\varsigma > 0$ is symmetric, with determinant:

$$\det(\varsigma) = \frac{1}{\sqrt{\det\varsigma_1}}\det\varsigma_2\frac{1}{\sqrt{\det\varsigma_1}} = \frac{\det\varsigma_2}{\det\varsigma_1} = 1$$

where we work in the frame where ς is diagonal such that its eigenvalues are λ and $\dfrac{1}{\lambda}$. Let us denote the effective conductivity of a composite made with conductors 1 and λ by $\varsigma^* := \Sigma^*(\lambda)$. Considering our chosen exact relation $\mathbb{M}_5$, we write our effective tensor as follows:

$$\mathsf{P}^* = \left\{ \begin{bmatrix} \varsigma^* & 0 \\ 0 & \frac{\varsigma^*}{\det \varsigma^*} \end{bmatrix} : \varsigma^* = (\varsigma^*)^t, \varsigma^* > 0 \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\det \varsigma^*} \end{bmatrix} \otimes \varsigma^* \right\} \tag{6.11}$$

Note that P^* depends on the microstructure of ς, and hence dependent upon λ, the eigenvalues of ς with an invariant coordinate system. During most computations, we need to invert the effective tensors via Ψ_1 given by (5.1). We calculate the inverse the map of Ψ_1 to be:

$$\Psi_1^{-1}(\mathsf{W}) = (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2)(\mathsf{W})(\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) + r_1 \mathsf{T} \tag{6.12}$$

Now, we undo Ψ to compute the effective tensor L^*:

$$\mathsf{L}^* = r_0 \mathsf{T} + (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) \left(\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\det \varsigma^*} \end{bmatrix} \otimes \varsigma^* \right) (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2)$$

$$\mathsf{L}^* = r_0 \mathsf{T} + \left(\varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\det \varsigma^*} \end{bmatrix} \varsigma_1^{1/2} \right) \otimes \varsigma^*$$

Since we are working in the frame where ς is diagonal, we can use a "2D-linear machine" that we describe below to get a more simplified form of L^*. We have

$$\varsigma_1^{1/2} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \varsigma_1^{1/2} = \varsigma_2 \quad , \quad \varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \varsigma_1^{1/2} = \varsigma_1 \tag{6.13}$$

by definition of ς given by (5.2). In this case, we have $\lambda_1 = \lambda$ and $\lambda_2 = \dfrac{1}{\lambda}$, so that we may solve the following system of linear equations:

$$x(1,1) + y\left(\lambda, \frac{1}{\lambda}\right) = \left(1, \frac{1}{\det \varsigma^*}\right)$$

Solving via maple, we get

$$x = \frac{\lambda^2 - \det \varsigma^*}{\det \varsigma^* (\lambda^2 - 1)} \qquad , \qquad y = \frac{\lambda(\det \varsigma^* - 1)}{\det \varsigma^* (\lambda^2 - 1)}$$

Then, by (5.13), we get the following simplification for the linear combination:

$$x\varsigma_1 + y\varsigma_2 = \frac{\boldsymbol{S}_1}{\det \varsigma^*} + \boldsymbol{S}_2$$

We have the following form of our effective tensor L^* for the special borderline case:

$$\mathsf{L}^* = r_0 \mathsf{T} + \left(\frac{\mathbf{S}_1}{\det \varsigma^*} + \mathbf{S}_2 \right) \otimes \varsigma^* \tag{6.14}$$

such that $\varsigma^* = \varsigma^*(\lambda)$.

6.3 Generic borderline case

$|r_1 - r_2| = \left| \sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2} \right|$, $r_1 \neq r_2$. The relevant exact relations are

$$\mathbb{M}_{20} = \left\{ \begin{bmatrix} \boldsymbol{L} & \pm(\boldsymbol{L}\boldsymbol{M} - \boldsymbol{R}_\perp) \\ \pm(\boldsymbol{M}^t\boldsymbol{L} + \boldsymbol{R}_\perp) & \boldsymbol{M}^t\boldsymbol{L}\boldsymbol{M} \end{bmatrix} : \mathrm{Tr}\, \boldsymbol{M} = 0,\ \boldsymbol{L} > 0,\ \boldsymbol{L}^{-1} + 2\boldsymbol{M}\boldsymbol{R}_\perp < 0 \right\} \tag{6.15}$$

The relevant link is as follows. $\boldsymbol{M}^* = \boldsymbol{R}_\perp \varsigma^*$, where ς^* is the 2D effective conductivity tensor of the composite with local conductivity $\boldsymbol{c}(\boldsymbol{x}) = -\boldsymbol{R}_\perp \boldsymbol{M}(\boldsymbol{x})$, which is symmetric and positive definite for $\boldsymbol{M}(\boldsymbol{x})$ satisfying the constraints in (6.15). For this case, we only need to use the automorphism Ψ_1 such that:

$$\Psi_1(\mathsf{L}_1) = \mathsf{I}, \qquad \Psi_1(\mathsf{L}_2) = \mathsf{P}$$

where P is a new isotropic tensor of the form $\varsigma \otimes \boldsymbol{I}_2 + \rho \mathsf{T}$. Then, substituting $\mathsf{L}_2 = \varsigma_2 \otimes \boldsymbol{I}_2 + r_2 \mathsf{T}$ into (5.1):

$$\mathsf{P} = (\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)(\varsigma_2 \otimes \boldsymbol{I}_2 + r_2 \mathsf{T} - r_1 \mathsf{T})(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)$$

$$\mathsf{P} = (\varsigma_1^{-1/2} \varsigma_2 \otimes \boldsymbol{I}_2 + (r_2 - r_1)\varsigma_1^{-1/2} \boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp)(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)$$

$$\mathsf{P} = \varsigma_1^{-1/2}\varsigma_2\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2 + (r_2 - r_1)\varsigma_1^{-1/2}\boldsymbol{R}_\perp\varsigma_1^{-1/2} \otimes \boldsymbol{R}_\perp$$

$$\mathsf{P} = \varsigma \otimes \boldsymbol{I}_2 + \frac{(r_2 - r_1)}{\sqrt{\det \varsigma_1}}\boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp$$

$$\mathsf{P} = \varsigma \otimes \boldsymbol{I}_2 + \rho\boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp$$

$$\mathsf{P} = \begin{bmatrix} \lambda_1\boldsymbol{I}_2 & 0 \\ 0 & \lambda_2\boldsymbol{I}_2 \end{bmatrix} + \begin{bmatrix} 0 & -\rho\boldsymbol{R}_\perp \\ \rho\boldsymbol{R}_\perp & 0 \end{bmatrix}$$

Finally, we get the form

$$\mathsf{P} = \begin{bmatrix} \lambda_1\boldsymbol{I}_2 & -\rho\boldsymbol{R}_\perp \\ \rho\boldsymbol{R}_\perp & \lambda_2\boldsymbol{I}_2 \end{bmatrix} \tag{6.16}$$

There are two equalities implied by the original constraint that can be written in terms of ς and ρ given by (5.2):

$$\pm (r_1 - r_2) = \sqrt{\det(\varsigma_1)} - \sqrt{\det(\varsigma_2)} \tag{6.17}$$

Later, we will verify that this in fact corresponds to $\mathbb{M}_{20}^{\mp}$. We can rewrite this property in the following way:

$$\mp\frac{(r_2 - r_1)}{\sqrt{\det(\varsigma_1)}} = 1 - \frac{\sqrt{\det(\varsigma_2)}}{\sqrt{\det(\varsigma_1)}}$$

Since we continue to work in the frame where ς is diagonal, we have

$$\det(\varsigma) = \frac{\det \varsigma_2}{\det \varsigma_1} = \lambda_1\lambda_2$$

Then, (6.17) gives the following relation between λ_1, λ_2 and ρ:

$$\sqrt{\lambda_1\lambda_2} = 1 \pm \rho \tag{6.18}$$

Note that we have not guaranteed that all signs align with cases given by form of $\mathbb{M}_{20}$. (In fact, we will show that it is the opposite).

We want to see if P, given by (6.16), lies either in $\mathbb{M}_{20}^{+}$ or $\mathbb{M}_{20}^{-}$. Considering the components of $\mathbb{M}_{20}$ and (6.16), we get the following equalities:

$$\boldsymbol{L} = \lambda_1\boldsymbol{I}_2 \tag{6.19}$$

$$\boldsymbol{M}^t \boldsymbol{L} \boldsymbol{M} = \lambda_2 \boldsymbol{I}_2 \tag{6.20}$$

$$\pm\left(\boldsymbol{L}\boldsymbol{M} - \boldsymbol{R}_\perp\right) = -\rho \boldsymbol{R}_\perp \tag{6.21}$$

After plugging in (6.19) into (6.21), we get :

$$\boldsymbol{M} = \frac{(1 \mp \rho)}{\lambda_1} \boldsymbol{R}_\perp \tag{6.22}$$

Then, plugging (6.19) and (6.22) into (6.20):

$$\boldsymbol{M}^t \boldsymbol{L} \boldsymbol{M} = \left(\frac{(1 \mp \rho)}{-\lambda_1} \boldsymbol{R}_\perp\right) (\lambda_1 \boldsymbol{I}_2) \left(\frac{(1 \mp \rho)}{\lambda_1} \boldsymbol{R}_\perp\right) = \frac{(1 \mp \rho)^2}{\lambda_1} \boldsymbol{I}_2 = \lambda_2 \boldsymbol{I}_2$$

Therefore, we can see that (6.16) and (6.17) implies

$$\mathsf{P} \subset \mathbb{M}_4^{\pm} \quad \Longrightarrow \quad \sqrt{\lambda_1 \lambda_2} = (1 \mp \rho) \tag{6.23}$$

Comparing these results with (6.18), we have the following sign correspondence:

$$\pm \text{ in (6.17) corresponds to } \mathbb{M}_{20}^{\mp} \tag{6.24}$$

From (6.23) and (6.22), we see that $\boldsymbol{M}$ can also be written as:

$$\boldsymbol{M} = \sqrt{\frac{\lambda_2}{\lambda_1}} \boldsymbol{R}_\perp \tag{6.25}$$

Let ς^* be the effective conductivity of the composite made with conductors 1 and $\sqrt{\frac{\lambda_2}{\lambda_1}}$ such that $\varsigma^* := \varsigma^*\left(\sqrt{\frac{\lambda_2}{\lambda_1}}\right)$, which by definition is symmetric. Then, by the relevant link our final form of our effective conductivity tensor lies $\mathbb{M}_{20}$ and can be given by:

$$\mathsf{P}^* := \begin{bmatrix} \boldsymbol{L}^* & \mp(\boldsymbol{L}^* \boldsymbol{M}^* - \boldsymbol{R}_\perp) \\ \mp((\boldsymbol{M}^*)^t \boldsymbol{L}^* + \boldsymbol{R}_\perp) & (\boldsymbol{M}^*)^t \boldsymbol{L}^* \boldsymbol{M}^* \end{bmatrix} \tag{6.26}$$

where $\boldsymbol{L}^*$ is a 2x2 symmetric, positive definite matrix describing the composite's thermal electricity. In general, $\boldsymbol{L}^*$ cannot be related nicely to 2-dimensional conductivity. We apply the relevant link, $\boldsymbol{M}^* = \boldsymbol{R}_\perp \varsigma^*$, to get:

$$\mathsf{P}^* = \begin{bmatrix} \boldsymbol{L}^* & \mp(\boldsymbol{L}^*\boldsymbol{R}_\perp\varsigma^* - \boldsymbol{R}_\perp) \\ \mp(-\varsigma^*\boldsymbol{R}_\perp\boldsymbol{L}^* + \boldsymbol{R}_\perp) & -\varsigma^*\boldsymbol{R}_\perp\boldsymbol{L}^*\boldsymbol{R}_\perp\varsigma^* \end{bmatrix}$$

$$\mathsf{P}^* = \begin{bmatrix} \boldsymbol{L}^* & \mp\boldsymbol{L}^*\boldsymbol{R}_\perp\varsigma^* \\ \pm\varsigma^*\boldsymbol{R}_\perp\boldsymbol{L}^* & \varsigma^*\mathrm{cof}(\boldsymbol{L}^*)\varsigma^* \end{bmatrix} \mp \mathsf{T}$$

Via Maple, we have verified that components (12) and (21) are transposes of one another. We write P^* as a sum of tensor products:

$$\mathsf{P}^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \boldsymbol{L}^* + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \boldsymbol{A} \mp \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \boldsymbol{B} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \boldsymbol{B}^t + \mathsf{T} \right)$$

where

$$\boldsymbol{A} := \varsigma^*\mathrm{cof}(\boldsymbol{L}^*)\varsigma^* \tag{6.27}$$

$$\boldsymbol{B} := \boldsymbol{L}^*\boldsymbol{R}_\perp\varsigma^* \tag{6.28}$$

Applying (6.12) to P^*, we get

$$\Psi_1^{-1}(\mathsf{P}^*) = \left(\left(\varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \varsigma_1^{1/2} \right) \otimes \boldsymbol{L}^* + \left(\varsigma_1^{1/2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \varsigma_1^{1/2} \right) \otimes \boldsymbol{A} \mp ... \right.$$

$$\left. ... \mp \left(\varsigma_1^{1/2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \varsigma_1^{1/2} \otimes \boldsymbol{B} + \varsigma_1^{1/2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \varsigma_1^{1/2} \otimes \boldsymbol{B}^t + \mathsf{T} \right) \right) (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) + r_1\mathsf{T}$$

To simplify, we apply the "2D linear machine", given by (6.13), which was introduced in the previous case. We solve the following systems of equations:

$$x(1,1) + y(\lambda_1, \lambda_2) = (1,0)$$

Via Maple, we get

$$x = \frac{-\lambda_2}{\lambda_1 - \lambda_2}, \qquad y = \frac{1}{\lambda_1 - \lambda_2}$$

Therefore,

$$\varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \varsigma_1^{1/2} = \frac{\varsigma_2 - \lambda_2\varsigma_1}{\lambda_1 - \lambda_2} = \boldsymbol{S}_2$$

which we defined by (5.13). Similarly,

$$x(1,1) + y(\lambda_1, \lambda_2) = (0,1)$$

Via Maple, we get

$$x = \frac{\lambda_1}{\lambda_1 - \lambda_2}, \qquad y = \frac{-1}{\lambda_1 - \lambda_2}$$

Therefore,

$$\varsigma_1^{1/2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \varsigma_1^{1/2} = \frac{\varsigma_2 - \lambda_1 \varsigma_1}{\lambda_2 - \lambda_1} = \boldsymbol{S}_1$$

which we defined by (5.13). In order to compute the other two systems seen in our formula for $\Psi_1^{-1}(\mathsf{P}^*)$, which are no longer diagonal, we must solve two other systems with our linear machine. First, we solve

$$x(1,1) + y(\lambda_1, \lambda_2) = (1,-1)$$

This linear system can be found by taking the difference of the last two systems solved:

$$\varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \varsigma_1^{1/2} = \varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \varsigma_1^{1/2} - \varsigma_1^{1/2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \varsigma_1^{1/2}$$
$$= \frac{(-\lambda_2 - \lambda_1)\varsigma_1 + 2 + 2\varsigma_2}{\lambda_1 - \lambda_2}$$

So that we have

$$\varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \varsigma_1^{1/2} = \boldsymbol{S}_2 - \boldsymbol{S}_1 \tag{6.29}$$

This next linear system can be found by multiplying previously calculated systems in the following way:

$$\varsigma_1^{1/2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \varsigma_1^{1/2} = \varsigma_1^{1/2} \left(\boldsymbol{R}_\perp \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \varsigma_1^{1/2}$$

$$= \sqrt{\det(\varsigma_1)} \boldsymbol{R}_\perp \varsigma_1^{-1} \varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \varsigma_1^{1/2}$$

$$= \sqrt{\det(\varsigma_1)}\boldsymbol{R}_\perp \varsigma_1^{-1}\left(\frac{(-\lambda_2-\lambda_1)\varsigma_1+2\varsigma_2}{\lambda_1-\lambda_2}\right)$$

So that we have

$$\varsigma_1^{1/2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\varsigma_1^{1/2} = \sqrt{\det(\varsigma_1)}\boldsymbol{R}_\perp\varsigma_1^{-1}\cdot(\boldsymbol{S}_2-\boldsymbol{S}_1) \tag{6.30}$$

Finally we can compute the following for our current case:

$$\begin{aligned}\varsigma_1^{1/2}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\varsigma_1^{1/2} &= \varsigma_1^{1/2}\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)\varsigma_1^{1/2}\\ &= \boldsymbol{S}_2\varsigma_1^{-1}\sqrt{\det\varsigma_1}\boldsymbol{R}_\perp\varsigma_1^{-1}(\boldsymbol{S}_2-\boldsymbol{S}_1)\\ &= \frac{\sqrt{\det\varsigma_1}}{\det\varsigma_1}\boldsymbol{S}_2\boldsymbol{R}_\perp(\boldsymbol{S}_2-\boldsymbol{S}_1)\\ &= \frac{1}{\sqrt{\det\varsigma_1}}\boldsymbol{S}_2\boldsymbol{R}_\perp\boldsymbol{S}_2 - \frac{1}{\sqrt{\det\varsigma_1}}\boldsymbol{S}_2\boldsymbol{R}_\perp\boldsymbol{S}_1\end{aligned}$$

Note that within this case, ς_1 and ς_2 are not scalar multiples of each other, and as such (5.12) implies that $\det(\boldsymbol{S}_2)=0$, which gives the following simplification:

$$\varsigma_1^{1/2}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\varsigma_1^{1/2} = -\frac{1}{\sqrt{\det\varsigma_1}}\boldsymbol{S}_2\boldsymbol{R}_\perp\boldsymbol{S}_1 \tag{6.31}$$

Similarly, we compute

$$\begin{aligned}\varsigma_1^{1/2}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\varsigma_1^{1/2} &= \varsigma_1^{1/2}\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)\varsigma_1^{1/2}\\ &= \boldsymbol{S}_1\varsigma_1^{-1}\sqrt{\det\varsigma_1}\boldsymbol{R}_\perp\varsigma_1^{-1}(\boldsymbol{S}_2-\boldsymbol{S}_1)\\ &= \frac{\sqrt{\det\varsigma_1}}{\det\varsigma_1}\boldsymbol{S}_1\boldsymbol{R}_\perp(\boldsymbol{S}_2-\boldsymbol{S}_1)\\ &= \frac{1}{\sqrt{\det\varsigma_1}}\boldsymbol{S}_1\boldsymbol{R}_\perp\boldsymbol{S}_2 - \frac{1}{\sqrt{\det\varsigma_1}}\boldsymbol{S}_1\boldsymbol{R}_\perp\boldsymbol{S}_1\end{aligned}$$

Once again, (5.12) implies that $\det(\boldsymbol{S}_1)=0$, which gives the following simplification:

$$\varsigma_1^{1/2}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\varsigma_1^{1/2} = \frac{1}{\sqrt{\det\varsigma_1}}\boldsymbol{S}_1\boldsymbol{R}_\perp\boldsymbol{S}_2 \tag{6.32}$$

Finally we substitute the values given by (5.13), (6.32), (6.31) into $\Psi_1^{-1}(\mathsf{P}^*)$ to get

$$= \boldsymbol{S}_2 \otimes \boldsymbol{L}^* + \boldsymbol{S}_1 \otimes \boldsymbol{A} \mp \left(\frac{-\boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1}{\sqrt{\det \varsigma_1}} \otimes \boldsymbol{B} + \frac{\boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2}{\sqrt{\det \varsigma_1}} \otimes \boldsymbol{B}^t + \sqrt{\det \varsigma_1} \mathsf{T} \right) + r_1 \mathsf{T}$$

$$= \boldsymbol{S}_2 \otimes \boldsymbol{L}^* + \boldsymbol{S}_1 \otimes \boldsymbol{A} \pm \left(\frac{\boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1}{\sqrt{\det \varsigma_1}} \otimes \boldsymbol{L}^* \boldsymbol{R}_\perp \varsigma^* + \frac{\boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2}{\sqrt{\det \varsigma_1}} \otimes \varsigma^* \boldsymbol{R}_\perp \boldsymbol{L}^* \right) \mp \sqrt{\det \varsigma_1} \mathsf{T} + r_1 \mathsf{T}$$

Note that our final form of L^* should not have any representation of cases - i.e. we need to remove our "±" by substituting the definition given by (6.17). We have the following form of our effective tensor L^* for the generic borderline case:

$$\mathsf{L}^* = \boldsymbol{S}_1 \otimes \varsigma^* \mathrm{cof}(\boldsymbol{L}^*) \varsigma^* + \boldsymbol{S}_2 \otimes \boldsymbol{L}^* + \frac{\alpha}{\det \varsigma_1} (\boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2 \otimes \varsigma^* \boldsymbol{R}_\perp \boldsymbol{L}^* + \boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1 \otimes \boldsymbol{L}^* \boldsymbol{R}_\perp \varsigma^*)$$
$$+ (r_1 - \alpha) \mathsf{T} \quad (6.33)$$

where $\boldsymbol{B}$, $\boldsymbol{A}$, $\boldsymbol{S}_1$, and $\boldsymbol{S}_2$ are given by (6.27), (6.28), and (5.13), and we define where

$$\alpha = \frac{\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2}}{r_1 - r_2} \sqrt{\det \varsigma_1}, \qquad \varsigma^* = \varsigma^* \left(\sqrt{\frac{\lambda_2}{\lambda_1}} \right).$$

6.4 Special weakly coupled case

$\varsigma_1 = \theta_1 \varsigma_0$, $\varsigma_2 = \theta_2 \varsigma_0$, $|r_1 - r_2| < |\theta_1 - \theta_2| \sqrt{\det \varsigma_0}$. The relevant exact relation is

$$\mathbb{M}_4 = \left\{ \begin{bmatrix} \boldsymbol{L} & 0 \\ 0 & \boldsymbol{L} \end{bmatrix} : \boldsymbol{L} > 0 \right\}.$$

and the relevant link is that $\boldsymbol{L}^*$ is the effective conductivity tensor of the conducting composite with local conductivity $\boldsymbol{L}(\boldsymbol{x})$.

Note that the given constraint ensures that the discriminant of the quadratic equation given by (5.5) is positive, which means that the roots, $a_0 \in \mathbb{R}$. We, also, redefine ς and ρ in terms of our new conditions:

$$\varsigma = (\theta_1 \varsigma_0)^{-1/2} (\theta_2 \varsigma_0) (\theta_1 \varsigma_0)^{-1/2} = \frac{\theta_2 \varsigma_0}{\theta_1 \varsigma_0} = \frac{\theta_2}{\theta_1} \boldsymbol{I}_2 \quad (6.34)$$

$$\rho = \frac{r_2 - r_1}{\sqrt{\det(\theta_1 \varsigma_0)}} = \frac{r_2 - r_1}{\theta_1 \sqrt{\det(\varsigma_0)}} \quad (6.35)$$

Note that since ($\mathbb{M}_4$) is mirror-symmetric (self-symmetric), we do not have $\pm$ cases - instead we have two mirror copies of the same case. The goal is to rewrite the given constraint in terms of ρ, θ_1, θ_2 such that we find the admissible values of the parameters. In this case, we let Ψ be a composition of global automorphisms Ψ_1 and Ψ_2 given by (5.1) and (5.3):

$$\Psi_1(\mathsf{L}_1) = \boldsymbol{I}_2 \otimes \boldsymbol{I}_2$$

$$\Psi_1(\mathsf{L}_2) := \mathsf{L}_2^{'} = \varsigma \otimes \boldsymbol{I}_2 + \rho \mathsf{T}$$

$$\Psi_2(\boldsymbol{I}_2 \otimes \boldsymbol{I}_2) = \boldsymbol{I}_2 \otimes \boldsymbol{I}_2$$

$$\Psi_2(\mathsf{L}_2^{'}) = \mathsf{P} \in \mathbb{M}_4$$

We can use (5.4) to find a formula for P:

$$\Psi_2\left(\frac{\theta_2}{\theta_1}\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 + \frac{r_2 - r_1}{\theta_1\sqrt{\det \varsigma_0}}\boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp\right) = \left(a_0\frac{r_2 - r_1}{\theta_1\sqrt{\det \varsigma_0}} + 1\right)\frac{\theta_1}{\theta_2}\boldsymbol{I}_2 \otimes \boldsymbol{I}_2$$

$$\mathsf{P} = \left(\frac{a_0 \Delta r}{\theta_1\sqrt{\det \varsigma_0}}\frac{\theta_1}{\theta_2} + \frac{\theta_1}{\theta_2}\right)\boldsymbol{I}_2 \otimes \boldsymbol{I}_2$$

where $\Delta r = r_2 - r_1$. Simplifying via common denominator, we have

$$\mathsf{P} = \left(\frac{a_0 \Delta r + \theta_1\sqrt{\det \varsigma_0}}{\theta_2\sqrt{\det \varsigma_0}}\right)\boldsymbol{I}_2 \otimes \boldsymbol{I}_2$$

By the new conditions for ς_1 and ς_2, we have:

$$\mathsf{P} = \left(\frac{a_0 \Delta r + \sqrt{\det \varsigma_1}}{\sqrt{\det \varsigma_2}}\right)\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 \tag{6.36}$$

which lies on the exact relation $\mathbb{M}_4$. Our effective tensor of the composite, P^*, mixes the following conductivity tensors from materials $\mathsf{L}_1\prime$ and $\mathsf{L}_2\prime$:

$$\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 \quad \text{and} \quad \boldsymbol{I}_2 \otimes \left(\frac{a_0 \Delta r + \sqrt{\det \varsigma_1}}{\sqrt{\det \varsigma_2}}\right)\boldsymbol{I}_2$$

Let ς^* be the effective conductivity defined by:

$$\varsigma^* = \varsigma^*\left(\frac{a_0 \Delta r + \sqrt{\det \varsigma_1}}{\sqrt{\det \varsigma_2}}\right) \tag{6.37}$$

We get the following form of P^*:

$$\mathsf{P}^* := \begin{bmatrix} \varsigma^* & 0 \\ 0 & \varsigma^* \end{bmatrix} = \boldsymbol{I}_2 \otimes \varsigma^* \tag{6.38}$$

In order to find L^*, we need to undo $\Psi_2(\Psi_1(L^*)) = \mathsf{P}^*$. We know that the map $A \otimes B \mapsto B \otimes A$ is an algebra automorphism. Therefore, to invert Ψ_2, we need to switch the order of tensor products in (5.4) and replace all occurrences of a_0 with $-a_0$. We also need $\rho = 0$ by the form of P^* given by (6.38).

$$\mathsf{Q}^* := \Psi_2^{-1}(\mathsf{P}^*) = \mu^* \boldsymbol{I}_2 \otimes \varsigma^* + \nu^* \boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp$$

where

$$\mu^* := \frac{1 - a_0^2}{\det \varsigma^* - a_0^2} \qquad \nu^* := \frac{a_0(1 - \det \varsigma^*)}{\det \varsigma^* - a_0^2} \tag{6.39}$$

Finally, we can compute L^* by applying Ψ_1^{-1} given by (6.12):

$$\Psi_1^{-1}(Q^*) = (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) Q^* (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) + r_1 \mathsf{T}$$

$$= (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2)(\mu^* \boldsymbol{I}_2 \otimes \varsigma^* + \nu^* \boldsymbol{R}_\perp \otimes \boldsymbol{R}_\perp)(\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) + r_1 \mathsf{T}$$

$$= \mu^* (\varsigma_1^{1/2} \boldsymbol{I}_2 \varsigma_1^{1/2} \otimes \varsigma^*) + \nu^* \varsigma_1^{1/2} \boldsymbol{R}_\perp \varsigma_1^{1/2} \otimes \boldsymbol{R}_\perp) + r_1 \mathsf{T}$$

By properties of $\boldsymbol{R}_\perp$, we simplify to get

$$= \mu^* \varsigma_1 \otimes \varsigma^* + \nu^* \sqrt{\det \varsigma_1} \mathsf{T} + r_1 \mathsf{T}$$

Therefore, we have the following form of the effective tensor L^* for the special weakly coupled case:

$$\mathsf{L}^* := \left(\frac{1 - a_0^2}{\det \varsigma^* - a_0^2} \right) \varsigma_1 \otimes \varsigma^* + \left(r_1 + \frac{a_0(1 - \det \varsigma^*) \sqrt{\det \varsigma_1}}{\det \varsigma^* - a_0^2} \right) \mathsf{T} \tag{6.40}$$

where a_0 is a root of (5.5) and ς^* is given by (6.37).

6.5 A uniform field relation

$|r_1 - r_2|^2 = \det(\varsigma_1 - \varsigma_2)$. The relevant exact relation is

$$\mathbb{M}_{11} = \left\{ \begin{bmatrix} \boldsymbol{L} & 0 \\ 0 & \boldsymbol{I}_2 \end{bmatrix} : \boldsymbol{L} > 0 \right\}.$$

and the relevant link is that $\boldsymbol{L}^*$ is the effective conductivity tensor of the conducting composite with local conductivity $\boldsymbol{L}(\boldsymbol{x})$. We can factor the given constraint into the following helpful form:

$$\det(\varsigma_1 - \varsigma_2) = \det(\varsigma_1)\det(\boldsymbol{I}_2 - \varsigma)$$

Then, dividing by $\det \varsigma_1$ and using formulas given by (5.2), we get:

$$\rho^2 = \det(\boldsymbol{I}_2 - \varsigma) \tag{6.41}$$

We can use this equation to find roots of the quadratic equation. Let λ_1, λ_2 be the eigenvalues of ς, which we have once again assumed to be diagonal. Then,

$$\rho^2 = (1 - \lambda_1)(1 - \lambda_2) \tag{6.42}$$

For simplification, we solve (6.42) for λ_2 to get:

$$\lambda_2 = \frac{\rho^2 + \lambda_1 - 1}{\lambda_1 - 1} \tag{6.43}$$

As in the previous case, we let Ψ be a composition of global automorphisms Ψ_1 and Ψ_2 given by (5.1) and (5.3):

$$\Psi_1(\mathsf{L}_1) = \mathsf{I}$$

$$\Psi_1(\mathsf{L}_2) := \mathsf{L}_2' = \varsigma \otimes \boldsymbol{I}_2 + \rho \otimes \mathsf{T}$$

$$\Psi_2(\boldsymbol{I} \otimes \boldsymbol{I}) = \mathsf{I}$$

$$\Psi_2(\mathsf{L}_2') = \mathsf{P} \in \mathbb{M}_{11}$$

Let a_0 be the root of (5.5) such that when we apply P^*, the antisymmetric part will disappear. To describe P, Maple allows us to substitute $\mathsf{L}_2^{'}$ into (5.3): :

$$\mathsf{P} = \begin{bmatrix} \dfrac{\lambda_1(a_0^2-1)}{(\rho+a_0)^2-\lambda_1\lambda_2} & \dfrac{-i(\rho^2 a_0+\rho a_0^2-a_0\lambda_1\lambda_2+\rho+a_0)}{(\rho+a_0)^2-\lambda_1\lambda_2} \\ \dfrac{i(\rho^2 a_0+\rho a_0^2-a_0\lambda_1\lambda_2+\rho+a_0)}{(\rho+a_0)^2-\lambda_1\lambda_2} & \dfrac{\lambda_1(a_0^2-1)}{(\rho+a_0)^2-\lambda_1\lambda_2} \end{bmatrix} \tag{6.44}$$

After factoring (6.44), and by definition of eigenvalues, $\lambda_1\lambda_2 = \det(\varsigma)$, we have

$$\mathsf{P} = \begin{bmatrix} \dfrac{\lambda_1(a_0^2-1)}{(\rho+a_0)^2-\det\varsigma} & \dfrac{-i((a_0+\rho)(a_0\rho+1)-a_0\det\varsigma)}{(\rho+a_0)^2-\det\varsigma} \\ \dfrac{i((a_0+\rho)(a_0\rho+1)-a_0\det\varsigma)}{(\rho+a_0)^2-\det\varsigma} & \dfrac{\lambda_1(a_0^2-1)}{(\rho+a_0)^2-\det\varsigma} \end{bmatrix} \tag{6.45}$$

In order to land on our chosen exact relation, we must solve for a_0 such that (6.45) is of the form $\mathbb{M}_{11}$. We divide by $-i$, substitute λ_2, given by (6.43), into the second block component of (6.45) and set equal to 0 to get:

$$0 = -\frac{(\rho a_o - \lambda_1 + 1)(-a_0\lambda_1 + \rho + a_0)}{\lambda_1 - 1} \tag{6.46}$$

Although there are two solutions for a_0, we choose the following definition of a_0 such that P is of the necessary form:

$$a_0 := \frac{\lambda_1 - 1}{\rho} \tag{6.47}$$

Then, substituting in for a_0, and λ_2 as in (6.43), we can simplify to get the following:

$$\mathsf{P} = \begin{bmatrix} \dfrac{\lambda_1}{\lambda_2}\boldsymbol{I}_2 & 0 \\ 0 & \boldsymbol{I}_2 \end{bmatrix} \tag{6.48}$$

Our effective tensor of the composite, P^*, mixes the following conductivity tensors from materials $\mathsf{L}_1\prime$ and $\mathsf{L}_2\prime$:

$$\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 \quad \text{and} \quad \begin{bmatrix} \dfrac{\lambda_1}{\lambda_2} & 0 \\ 0 & 1 \end{bmatrix} \otimes \boldsymbol{I}_2$$

Let $\varsigma^* = \Sigma^*(h)$ be the effective conductivity of P, where $h := \dfrac{\lambda_1}{\lambda_2}$. We can write

$$\mathsf{P}^* := \begin{bmatrix} \varsigma^* & 0 \\ 0 & \boldsymbol{I}_2 \end{bmatrix} \tag{6.49}$$

which lies on the exact relation $\mathbb{M}_{11}$. Note that P^* is no longer isotropic. To find L^*, we need to undo $\Psi_2(\Psi_1(\mathsf{L}^*)) = \mathsf{P}^*$. We compute the inverse map of Ψ_2 to be:

$$\Psi_2^{-1}(\mathsf{W}) = \mathbf{T}(\mathsf{W} - a_0\mathsf{T})^{-1}(\mathsf{T} - a_0\mathsf{W}) \tag{6.50}$$

In some cases, including this one, we need to directly compute the above (6.50) formula. To make these calculations easier, Professor Grabovsky has created the following method. Let F be a 2x2 block matrix defined by components $\boldsymbol{F}_{ij}$ which are 2x2 matrices. We can compute its inverse via the following formula:

$$\mathsf{F}^{-1} := \begin{bmatrix} \mathbf{S}_{11}^{-1} & -\mathbf{S}_{11}^{-1}\mathbf{F}_{12}\mathbf{F}_{22}^{-1} \\ -\mathbf{S}_{22}^{-1}\mathbf{F}_{21}\mathbf{F}_{11}^{-1} & \mathbf{S}_{22}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{-1} & -\mathbf{F}_{11}^{-1}\mathbf{F}_{12}\mathbf{S}_{22}^{-1} \\ -\mathbf{F}_{22}^{-1}\mathbf{F}_{21}\mathbf{S}_{11}^{-1} & \mathbf{S}_{22}^{-1} \end{bmatrix} \tag{6.51}$$

such that

$$\mathbf{S}_{11} := \mathbf{F}_{11} - \mathbf{F}_{12}\mathbf{F}_{22}^{-1}\mathbf{F}_{21} \tag{6.52}$$

and

$$\mathbf{S}_{22} := \mathbf{F}_{22} - \mathbf{F}_{21}\mathbf{F}_{11}^{-1}\mathbf{F}_{12} \tag{6.53}$$

Now, we apply Ψ_2^{-1} to (6.49), which needs to be computed piece by piece:

$$\mathsf{T} - a_0\mathsf{P}^* = \begin{bmatrix} 0 & -\boldsymbol{R}_\perp \\ \boldsymbol{R}_\perp & 0 \end{bmatrix} - \begin{bmatrix} a_0\varsigma^* & 0 \\ 0 & a_0\boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} -a_0\varsigma^* & -\boldsymbol{R}_\perp \\ \boldsymbol{R}_\perp & -a_0\boldsymbol{I}_2 \end{bmatrix} \tag{6.54}$$

$$\mathsf{P}^* - a_0\mathsf{T} = \begin{bmatrix} \varsigma^* & a_0\boldsymbol{R}_\perp \\ -a_0\boldsymbol{R}_\perp & \boldsymbol{I}_2 \end{bmatrix} \tag{6.55}$$

Then, by (6.51), we calculate the 2x2 block matrix's inverse:

$$(\mathsf{P}^* - a_0\mathsf{T})^{-1} := \begin{bmatrix} (\varsigma^* - a_0^2\boldsymbol{I}_2)^{-1} & -a_0(\varsigma^* - a_0^2\boldsymbol{I}_2)^{-1}\boldsymbol{R}_\perp \\ a_0\left(\boldsymbol{I}_2 - a_0^2\dfrac{\varsigma^*}{\det(\varsigma^*)}\right)^{-1}\boldsymbol{R}_\perp(\varsigma^*)^{-1} & \left(\boldsymbol{I}_2 - a_0^2\dfrac{\varsigma^*}{\det(\varsigma^*)}\right)^{-1} \end{bmatrix} \tag{6.56}$$

We combined our pieces by matrix multiplication and simplified via the characteristic equation of the determinant for $2x2$ matrices, (i.e. $\boldsymbol{R}_\perp A^{-1}\boldsymbol{R}_\perp^t = \operatorname{cof}(A^{-1}) = \frac{A}{\det(A)}$). After verifying with Maple that components (12) and (21) are transposes of each other, we have that $\Psi_2^{-1}(\mathsf{P}^*)$ is symmetric:

$$\Psi_2^{-1}(\mathsf{P}^*) = \begin{bmatrix} \frac{(1-a_0^2)}{\det(\varsigma^* - a_0^2\boldsymbol{I}_2)}\left(\det(\varsigma^*)\boldsymbol{I}_2 - a_0^2\varsigma^*\right) & \left(-a_0\boldsymbol{R}_\perp(\varsigma^* - a_0^2\boldsymbol{I}_2)^{-1}(\varsigma^* - \boldsymbol{I}_2)\right)^t \\ -a_0\boldsymbol{R}_\perp(\varsigma^* - a_0^2\boldsymbol{I}_2)^{-1}(\varsigma^* - \boldsymbol{I}_2) & \frac{(1-a_0^2)}{\det(\varsigma^* - a_0^2\boldsymbol{I}_2)}\left(\varsigma^* - a_0^2\boldsymbol{I}_2\right) \end{bmatrix} \tag{6.57}$$

Let $\Psi_2^{-1}(\mathsf{P}^*) := \mathsf{Q}$, and after more algebraic simplification, Q can be written as:

$$\mathsf{Q} = \alpha\mathbf{H} - a_0\mathsf{T} \tag{6.58}$$

where

$$\boldsymbol{A}_1^* := \varsigma^* - a_0^2\boldsymbol{I}_2 \tag{6.59}$$

$$\boldsymbol{A}_2^* := \det(\varsigma^*)\boldsymbol{I}_2 - a_0^2\varsigma^* \tag{6.60}$$

$$\alpha := \frac{(1-a_0^2)}{\det(\boldsymbol{A}_1^*)} \tag{6.61}$$

$$\mathbf{H} := \begin{bmatrix} \boldsymbol{A}_2^* & -a_0\boldsymbol{R}_\perp\left(\varsigma^* - a_0^2\boldsymbol{I}_2\right) \\ a_0\left(\varsigma^* - a_0^2\boldsymbol{I}_2\right)\boldsymbol{R}_\perp & \boldsymbol{A}_1^* \end{bmatrix} \tag{6.62}$$

To substitute Q given by (6.58) into Ψ_1^{-1} we must write $\mathbf{H}$ as a sum of three tensor products:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \boldsymbol{A}_2^* + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \boldsymbol{A}_1^* + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes (-a_0\boldsymbol{R}_\perp\varsigma^*) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes (a_0\varsigma^*\boldsymbol{R}_\perp) - a_0^3\mathsf{T}$$

Once again, we have verified that the last two tensor product components are transposes via Maple. Then, Q has the following form:

$$\mathsf{Q} := \alpha\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \boldsymbol{A}_2^* + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \boldsymbol{A}_1^* + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \boldsymbol{B} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \boldsymbol{B}^t - a_0^3\mathsf{T}\right) - a_0\mathsf{T} \tag{6.63}$$

where

$$\boldsymbol{B} := -a_0 \boldsymbol{R}_\perp \varsigma^* \tag{6.64}$$

Next, we apply (6.12) to (6.63) to get

$$\Psi_1^{-1}(\mathsf{Q}) = \alpha \left(\varsigma_1^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \varsigma_1^{1/2} \otimes \boldsymbol{A}_2^* + \varsigma_1^{1/2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \varsigma_1^{1/2} \otimes \boldsymbol{A}_1^* + \varsigma_1^{1/2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \varsigma_1^{1/2} \otimes \boldsymbol{B} \right)$$

$$+\alpha \left(\varsigma_1^{1/2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \varsigma_1^{1/2} \otimes \boldsymbol{B}^t - a_0^3 \varsigma_1^{1/2} \boldsymbol{R}_\perp \varsigma_1^{1/2} \otimes \boldsymbol{R}_\perp \right) - a_0 \varsigma_1^{1/2} \boldsymbol{R}_\perp \varsigma_1^{1/2} \otimes \boldsymbol{R}_\perp + r_1 \mathsf{T}$$

To simplify, we apply the "2D-linear machine" given by (6.13). We substitute the previously computed (5.13), (6.31), and (6.31) to get

$$\mathsf{L}^* = \alpha \left(\boldsymbol{S}_1 \otimes \boldsymbol{A}_1^* + \boldsymbol{S}_2 \otimes \boldsymbol{A}_2^* - \frac{\boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1}{\sqrt{\det \varsigma_1}} \otimes \boldsymbol{B} + \frac{\boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2}{\sqrt{\det \varsigma_1}} \otimes \boldsymbol{B}^t - a_0^3 \sqrt{\det \varsigma_1} \mathsf{T} \right) - ...$$

$$... - a_0 \sqrt{\det \varsigma_1} \mathsf{T} + r_1 \mathsf{T}$$

Substituting for $\boldsymbol{B}$ and $\boldsymbol{B}^t$, we have a final form of the effective tensor L^* for the uniform field relation:

$$\mathsf{L}^* = \alpha^* \left(\boldsymbol{S}_2 \otimes \boldsymbol{A} + \boldsymbol{S}_1 \otimes \boldsymbol{B} + \frac{a_0 \boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1}{\sqrt{\det(\varsigma_1)}} \otimes \boldsymbol{R}_\perp \varsigma^* + \frac{a_0 \boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2}{\sqrt{\det(\varsigma_1)}} \otimes \varsigma^* \boldsymbol{R}_\perp \right) + \beta^* \mathsf{T} \tag{6.65}$$

where

$$\beta^* : (r_1 - a_0 \sqrt{\det \varsigma_1} (1 + \alpha^* a_0^2)) \tag{6.66}$$

$$a_0 = \frac{\lambda_1 - 1}{r_2 - r_1} \sqrt{\det \varsigma_1} \tag{6.67}$$

where α^* is given by (6.61).

6.6 Generic weakly coupled case

$|r_1 - r_2| < \left|\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2}\right|$, assuming that $\varsigma_1 \neq \theta\varsigma_2$ and $|r_1 - r_2|^2 \neq \det(\varsigma_1 - \varsigma_2)$. The relevant exact relation is

$$\mathbb{M}_{16} = \left\{ \begin{bmatrix} \boldsymbol{L}_{11} & 0 \\ 0 & \boldsymbol{L}_{22} \end{bmatrix} : \boldsymbol{L}_{11} > 0,\ \boldsymbol{L}_{22} > 0 \right\}.$$

and the relevant link is that $\boldsymbol{L}_{11}^*$ and $\boldsymbol{L}_{11}^*$ are the effective conductivity tensors of the conducting composites with local conductivity $\boldsymbol{L}_{11}(\boldsymbol{x})$ and $\boldsymbol{L}_{22}(\boldsymbol{x})$, respectively. For this case, let Ψ be a composition of global automorphisms Ψ_1 and Ψ_2 given by (5.1) and (5.3).

$$\Psi_1(\mathsf{L}_1) = \mathsf{I}$$

$$\Psi_1(\mathsf{L}_2) := \mathsf{L}_2' = \varsigma \otimes \boldsymbol{I}_2 + \rho\mathsf{T}$$

$$\Psi_2(\boldsymbol{I}_2 \otimes \boldsymbol{I}_2) = \mathsf{I}$$

$$\Psi_2(\mathsf{L}_2') = \mathsf{P} \in \mathbb{M}_{16}$$

By (5.4), we have

$$\mathsf{P} = (a_0\rho + 1)\frac{\varsigma}{\det \varsigma} \otimes \boldsymbol{I}_2$$

We are, once again, working in the frame where ς is diagonal with eigenvalues λ_1 and λ_2. Then, we have

$$\mathsf{P} = (a_0\rho + 1)\begin{bmatrix} \frac{1}{\lambda_2} & 0 \\ 0 & \frac{1}{\lambda_1} \end{bmatrix} \otimes \boldsymbol{I}_2 \tag{6.68}$$

Our effective tensor of the composite, P^*, mixes the following:

$$\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 \quad \text{and} \quad \begin{bmatrix} \frac{(a_0\rho+1)}{\lambda_2}\boldsymbol{I}_2 & 0 \\ 0 & \frac{(a_0\rho+1)}{\lambda_1}\boldsymbol{I}_2 \end{bmatrix}$$

Let $\varsigma_1^* = \Sigma^*\left(\frac{(a_0\rho+1)}{\lambda_2}\right)$ and $\varsigma_2^* = \Sigma^*\left(\frac{(a_0\rho+1)}{\lambda_1}\right)$, then we write

$$\mathsf{P}^* := \begin{bmatrix} \varsigma_1^* & 0 \\ 0 & \varsigma_2^* \end{bmatrix} \tag{6.69}$$

To find L^*, we need to undo $\Psi_2(\Psi_1(L^*)) = \mathsf{P}^*$. First, we must substitute (6.69) into Ψ_2^{-1} given by (6.50), which will need to be computed piece by piece:

$$\mathsf{T} - a_0\mathsf{P}^* = \begin{bmatrix} 0 & -\boldsymbol{R}_\perp \\ \boldsymbol{R}_\perp & 0 \end{bmatrix} - a_0 \begin{bmatrix} \varsigma_1^* & 0 \\ 0 & \varsigma_2^* \end{bmatrix} = \begin{bmatrix} -a_0\varsigma_1^* & -\boldsymbol{R}_\perp \\ \boldsymbol{R}_\perp & -a_0\varsigma_2^* \end{bmatrix} \tag{6.70}$$

$$\mathsf{P}^* - a_0\mathsf{T} = \begin{bmatrix} \varsigma_1^* & a_0\boldsymbol{R}_\perp \\ -a_0\boldsymbol{R}_\perp & \varsigma_2^* \end{bmatrix} \tag{6.71}$$

Then, as in the last case, we calculate the 2x2 block matrix's inverse by (6.51):

$$(\mathsf{P}^* - a_0\mathsf{T})^{-1} = \begin{bmatrix} \left(\varsigma_1^* - a_0^2\dfrac{\varsigma_2^*}{\det\varsigma_2^*}\right)^{-1} & -a_0\left(\varsigma_1^* - a_0^2\dfrac{\varsigma_2^*}{\det\varsigma_2^*}\right)^{-1}\boldsymbol{R}_\perp\varsigma_2^{*-1} \\ a_0\left(\varsigma_2^* - a_0^2\dfrac{\varsigma_1^*}{\det\varsigma_1^*}\right)^{-1}\boldsymbol{R}_\perp\varsigma_1^{*-1} & \left(\varsigma_2^* - a_0^2\dfrac{\varsigma_1^*}{\det\varsigma_1^*}\right)^{-1} \end{bmatrix} \tag{6.72}$$

Then, after multiplying T by (6.72), we have

$$\begin{bmatrix} -a_0\boldsymbol{R}_\perp\left(\varsigma_2^* - a_0^2\dfrac{\varsigma_1^*}{\det\varsigma_1^*}\right)^{-1}\boldsymbol{R}_\perp\varsigma_1^{*-1} & -\boldsymbol{R}_\perp\left(\varsigma_2^* - a_0^2\dfrac{\varsigma_1^*}{\det\varsigma_1^*}\right)^{-1} \\ \boldsymbol{R}_\perp\left(\varsigma_1^* - a_0^2\dfrac{\varsigma_2^*}{\det\varsigma_2^*}\right)^{-1} & -a_0\boldsymbol{R}_\perp\left(\varsigma_1^* - a_0^2\dfrac{\varsigma_2^*}{\det\varsigma_2^*}\right)^{-1}\boldsymbol{R}_\perp\varsigma_2^{*-1} \end{bmatrix}$$

Next, we multiply the above matrix by our formula given by (6.70) to calculate $\Psi_2^{-1}(\mathsf{P}^*)$. We want to simplify such that we can factor out a shared fractional scalar from each component. Note that Maple verifies the following equality:

$$\det\left(\varsigma_1^* - a_0^2\frac{\varsigma_2^*}{\det\varsigma_2^*}\right)\det(\varsigma_2^*) = \det\left(\varsigma_2^* - a_0^2\frac{\varsigma_1^*}{\det\varsigma_1^*}\right)\det(\varsigma_1^*)$$

For simplification, let us define

$$\boldsymbol{A}_1^* := \left(\varsigma_1^*\det\varsigma_2^* - a_0^2\varsigma_2^*\right) \tag{6.73}$$

$$\boldsymbol{A}_2^* := \left(\varsigma_2^*\det\varsigma_1^* - a_0^2\varsigma_1^*\right) \tag{6.74}$$

$$\alpha^* := \frac{(1-a_0^2)\det\varsigma_2^*}{\det\boldsymbol{A}_1^*} = \frac{(1-a_0^2)\det\varsigma_1^*}{\det\boldsymbol{A}_2^*} \tag{6.75}$$

As we compute each component of $\Psi_2^{-1}(\mathsf{P}^*)$, we verify with Maple that component (12) and component (21) are transposes of one another. Let $\mathsf{Q}^* := \Psi_2^{-1}(\mathsf{P}^*)$ for simplicity. Then, after much algebraic simplification, we can write:

$$Q^* := \alpha^* \begin{bmatrix} \boldsymbol{A}_1^* & \boldsymbol{B} \\ \boldsymbol{B}^t & \boldsymbol{A}_2^* \end{bmatrix} - a_0(1+\alpha^* a_0^2)\mathsf{T} \tag{6.76}$$

where we define the following:

$$\boldsymbol{B} := -a_0 \varsigma_2^* \boldsymbol{R}_\perp \varsigma_1^* \tag{6.77}$$

Next, we need substitute (6.76) into Ψ_1^{-1} given by (6.12). We need split Q up into a sum of tensor products:

$$Q^* = \alpha^* \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \boldsymbol{A} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \boldsymbol{B} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \boldsymbol{C} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \boldsymbol{C}^t \right) - a_0(1+\alpha^* a_0^2)\mathsf{T} \tag{6.78}$$

After substituting (6.78) into (6.12), we apply the "2D-linear machine" given by (6.13). We substitute the previously computed (5.13), (6.31), and (6.31) to get the final form of L^* for the generic weakly coupled case:

$$\mathsf{L}^* = \alpha^* \left(\boldsymbol{S}_1 \otimes \boldsymbol{A}_1^* + \boldsymbol{S}_2 \otimes \boldsymbol{A}_2^* + \frac{a_0}{\sqrt{\det \varsigma_1}} (\boldsymbol{S}_1 \boldsymbol{R}_\perp \boldsymbol{S}_2 \otimes \varsigma_1^* \boldsymbol{R}_\perp \varsigma_2^* + \boldsymbol{S}_2 \boldsymbol{R}_\perp \boldsymbol{S}_1) \otimes \varsigma_2^* \boldsymbol{R}_\perp \varsigma_1^* \right)$$

$$.. + \gamma \mathsf{T} \tag{6.79}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are given by (5.13), α^* is given by (6.75), $\boldsymbol{A}_1^*$ by (6.74), and $\boldsymbol{A}_2^*$ by (6.73) and we define

$$\gamma := r_1 - a_0 \sqrt{\det \varsigma_1}(1 + a_0^2 \alpha^*) \tag{6.80}$$

where a_0 is one of the roots of (5.5).

6.7 Special strongly coupled case

$\varsigma_1 = \theta_1\varsigma_0$, $\varsigma_2 = \theta_2\varsigma_0$, $|r_1 - r_2| > |\theta_1 - \theta_2|\sqrt{\det\varsigma_0}$. The relevant exact relation is

$$\mathbb{M}_6 = \left\{ \begin{bmatrix} \boldsymbol{L} & -t\boldsymbol{R}_\perp \\ t\boldsymbol{R}_\perp & \boldsymbol{L} \end{bmatrix} : \boldsymbol{L} > 0,\ \det\boldsymbol{L} = 1 + t^2,\ t \in \mathbb{R} \right\}.$$

What is special for this case is that ς is a multiple of the identity matrix and therefore we have $\sigma_{11} = \sigma_{22}$ *in every frame.* It remains to compute $\mathsf{L}^* = \Psi_0^{-1} \circ \Psi_3^{-1}(\mathsf{L}_0^*)$, where

$$\mathsf{L}_0^* = \begin{bmatrix} \boldsymbol{L}^* & -t^*\boldsymbol{R}_\perp \\ t^*\boldsymbol{R}_\perp & \boldsymbol{L}^* \end{bmatrix},$$

satisfying $\det\boldsymbol{L}^* = 1 + (t^*)^2$. Applying the given conditions, we can redefine ς as in the special weakly coupled case:

$$\varsigma = \frac{\theta_2}{\theta_1}\boldsymbol{I}_2 \tag{6.81}$$

As we are in a general case, we will need to use a different composition of automorphisms than in the previously calculated cases. The transformation $\Psi_3 \circ \Psi_0$, given by (5.9) and (5.10) with a and b given by (5.11) takes L_j to the relevant exact relation. First, we apply Ψ_0 given by (5.9) to our materials to get

$$\Psi_0(\mathsf{L}_j) = \mathsf{L}_j'$$

$$\mathsf{L}_1' = \boldsymbol{I} + \rho_1\mathsf{T}, \qquad \mathsf{L}_2' = \frac{\theta_2}{\theta_1}\boldsymbol{I}_2 + \rho_2\mathsf{T} : \qquad \rho_j = \frac{r_j}{\sqrt{\det\varsigma_1}}$$

We then apply Ψ_3 given by (5.10) to map L_j' into $\mathbb{M}_6$:

$$\Psi_3(\mathsf{L}_1') = \mathsf{L}_1'' = \mathsf{L}_1'' = a\mathsf{I} + (a\rho_1 + b)\mathsf{T}$$

$$\Psi_3(\mathsf{L}_2') = \mathsf{L}_2'' = a\frac{\theta_2}{\theta_1}\mathsf{I} + (a\rho_2 + b)\mathsf{T}$$

Our effective tensor of the composite, P^*, mixes the following conductivity tensors of materials L_1'' and L_2'':

$$\begin{bmatrix} a\boldsymbol{I}_2 & -(a\rho_1+b)\boldsymbol{R}_\perp \\ (a\rho_1+b)\boldsymbol{R}_\perp & a\boldsymbol{I}_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a\frac{\theta_2}{\theta_1}\boldsymbol{I}_2 & -(a\rho_2+b)\boldsymbol{R}_\perp \\ (a\rho_2+b)\boldsymbol{R}_\perp & a\frac{\theta_2}{\theta_1}\boldsymbol{I}_2 \end{bmatrix}$$

To land on the exact relation $\mathbb{M}_6$, our effective tensor of the composite, P^*, is given by

$$\mathsf{P}^* := \begin{bmatrix} \boldsymbol{L}_0^* & -t_0^*\boldsymbol{R}_\perp \\ t_0^*\boldsymbol{R}_\perp & \boldsymbol{L}_0^* \end{bmatrix} \tag{6.82}$$

such that $\det \boldsymbol{L}_0^* = 1+(t_0^*)^2$. It remains to compute $\mathsf{L}^* = \Psi_0^{-1}\circ\Psi_3^{-1}(\mathsf{P}^*)$. Calculating the inverses of both transformations we have,

$$\Psi_0^{-1}(\mathsf{L}) = (\varsigma_1^{1/2}\otimes\boldsymbol{I}_2)\mathsf{L}(\varsigma_1^{1/2}\otimes\boldsymbol{I}_2) \tag{6.83}$$

$$\Psi_3^{-1}(\mathsf{L}) = a^{-1}(\mathsf{L}-b\mathsf{T}) \tag{6.84}$$

Applying (6.84) to (6.82), we get

$$\Psi_3^{-1}(\mathsf{P}^*) = a^{-1}(\mathsf{P}^*-b\mathsf{T}) = \frac{1}{a}\begin{bmatrix} \boldsymbol{L}_0^* & (b-t_0^*)\boldsymbol{R}_\perp \\ (t_0^*-b)\boldsymbol{R}_\perp & \boldsymbol{L}_0^* \end{bmatrix}$$

Before we apply Ψ_3^{-1}, we write $\Psi_3^{-1}(\mathsf{P}^*) := \mathsf{Q}^*$ as a sum of tensor products

$$\mathsf{Q}^* = \frac{1}{a}\left(\boldsymbol{I}_2\otimes\boldsymbol{L}_0^* + (t_0^*-b)\mathsf{T}\right) \tag{6.85}$$

Finally, we substitute Q^* into (6.84). By relation of ς and ς_1 and properties of $\boldsymbol{R}_\perp$, we get:

$$\Psi_0^{-1}(\mathsf{Q}^*) = \frac{1}{a}\left(\varsigma_1\otimes\boldsymbol{L}_0^* + (t_0^*-b)\sqrt{\det\varsigma_1}\mathsf{T}\right)$$

To create a better presentation of our results, let $\boldsymbol{L}_0^* = a\boldsymbol{L}^*$ and $t_0^* = \frac{a}{\sqrt{\det\varsigma_1}}t^*+b$. We compute

$$\det\boldsymbol{L}_0^* = t_0^{*2}+1 = \left(\frac{a}{\sqrt{\det\varsigma_1}}t^*+b\right)^2+1 \tag{6.86}$$

Since $\det(a\boldsymbol{L}^*) = \det \boldsymbol{L}_0^*$, we have

$$\det \boldsymbol{L}^* = \frac{1}{a^2} \det \boldsymbol{L}_0^* \tag{6.87}$$

Substituting (6.86) into (6.87) to eliminate $\boldsymbol{L}_0^*$, we have:

$$\det \boldsymbol{L}^* = \frac{1}{a^2} \left(\frac{a}{\sqrt{\det \varsigma_1}} t^* + b \right)^2 + \frac{1}{a^2}$$

which can be simplified to the form

$$\det \boldsymbol{L}^* = \left(\frac{t^*}{\sqrt{\det \varsigma_1}} + \frac{b}{a} \right)^2 + \frac{1}{a^2}$$

Multiplying both sides by $\det \varsigma_1$, we write

$$\det \varsigma_1 \det \boldsymbol{L}^* = \left(t^* + \frac{b\sqrt{\det \varsigma_1}}{a} \right)^2 + \frac{\det \varsigma_1}{a^2}$$

Let us define the following:

$$A = \frac{b\sqrt{\det \varsigma_1}}{a}, \qquad B = \frac{\det \varsigma_1}{a^2}$$

We can derive explicit formulas for A and B, by substituting in for a^2, and b given by (5.11). For compactness of notation, let us denote $\Delta r = r_2 - r_1$. Computing

$$A = \sqrt{\det \varsigma_1} \left(\frac{\dfrac{\det \varsigma_2}{\det \varsigma_1} - 1 + \dfrac{r_1^2}{\det \varsigma_1} - \dfrac{r_2^2}{\det \varsigma_1}}{2\dfrac{\Delta r}{\sqrt{\det \varsigma_1}}} \right)$$

$$A = \sqrt{\det \varsigma_1} \frac{\sqrt{\det \varsigma_1}}{\det \varsigma_1} \frac{\det \varsigma_2 - \det \varsigma_1 + r_1^2 - r_2^2}{2\Delta r}$$

We obtain the following

$$A = \frac{\det \varsigma_2 - \det \varsigma_1 + r_1^2 - r_2^2}{2\Delta r} \tag{6.88}$$

Similarly, calculating for B we get:

$$B = \frac{(\Delta r + \sqrt{\det \varsigma_1})^2 - \det \varsigma_2)^2 - \det \varsigma_2)(\det \varsigma_2 - (\Delta r - \sqrt{\det \varsigma_1})^2)}{4(\Delta r)^2}$$

Rewriting the numerator of B into a more symmetric form, we have

$$\mathrm{Num}(B) = (\Delta r + \sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2})(\Delta r + \sqrt{\det \varsigma_1} + \sqrt{\det \varsigma_2}) \times ...$$

$$... \times (\sqrt{\det \varsigma_2} + \Delta r - \sqrt{\det \varsigma_1})(\sqrt{\det \varsigma_2} - \Delta r + \sqrt{\det \varsigma_1})$$

We combine the first and third terms together, as well as the second and fourth terms together to get:

$$\text{Num}(B) = ((\Delta r)^2 - (\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2})^2)((\sqrt{\det \varsigma_1} + \sqrt{\det \varsigma_2})^2 - (\Delta r)^2)$$

Therefore, we found a more symmetric formula for B:

$$B := \frac{((\Delta r)^2 - (\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2})^2)((\sqrt{\det \varsigma_1} + \sqrt{\det \varsigma_2})^2 - (\Delta r)^2)}{4(\Delta r)^2} \tag{6.89}$$

Then, the final form of L^* for the special strongly coupled case is

$$L^* := \varsigma_1 \otimes \boldsymbol{L}^* + t^* \mathsf{T} \tag{6.90}$$

such that

$$\det \varsigma_1 \det \boldsymbol{L}^* := (t^* + A)^2 + B \tag{6.91}$$

where A and B are given by (6.88) and (6.89).

6.8 Generic strongly coupled case

$|r_1 - r_2| > \left|\sqrt{\det \varsigma_1} - \sqrt{\det \varsigma_2}\right|$. The relevant exact relation is $\mathbb{M}_{17}$

$$\mathbb{M}_{17} = \{\mathsf{L} > 0 : \mathsf{L}(\boldsymbol{J} \otimes \boldsymbol{R}_\perp)\mathsf{L} = \boldsymbol{J} \otimes \boldsymbol{R}_\perp\}, \quad \boldsymbol{J} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In this case it will be convenient to describe L^* by the equation that it solves in the same way the exact relation $\mathbb{M}_{17}$ is defined. However, while there is also a fairly compact equivalent description of $\mathbb{M}_{17}$ in terms of its block-components, such a description does not seem to be available for L^*. As in the previous case, we use the automorphism $\Psi = \Psi_3 \circ \Psi_0$. Applying Ψ_0 given by (5.9), we get:

$$\boldsymbol{L}_1' = \boldsymbol{I}_2 + i\rho_1 \boldsymbol{R}_\perp, \qquad \boldsymbol{L}_2' = \varsigma + i\rho_2 \boldsymbol{R}_\perp : \qquad \rho_j = \frac{r_j}{\sqrt{\det \varsigma_1}}$$

Note that we are, once again, working in the frame where $\varsigma_{11} = \varsigma_{22}$. Then, we apply Ψ_3 given by (5.10) to map L'_j into $\mathbb{M}_{17}$:

$$\Psi_3(\mathsf{L}'_1) = \mathsf{L}''_1 = a\mathsf{I} + (a\rho_1 + b)\mathsf{T}$$

$$\Psi_3(\mathsf{L}'_2) = \mathsf{L}''_2 = a\varsigma \otimes \boldsymbol{I}_2 + (a\rho_2 + b)\mathsf{T}$$

Our effective tensor of the composite, P^*, mixes the following conductivity tensors of materials L''_1 and L''_2:

$$\begin{bmatrix} a\boldsymbol{I}_2 & -(a\rho_1 + b)\boldsymbol{R}_\perp \\ (a\rho_1 + b)\boldsymbol{R}_\perp & a\boldsymbol{I}_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a\varsigma_{11}\boldsymbol{I}_2 & a\varsigma_{12} - (a\rho_2 + b)\boldsymbol{R}_\perp \\ a\varsigma_{12} + (a\rho_2 + b)\boldsymbol{R}_\perp & a\varsigma_{11}\boldsymbol{I}_2 \end{bmatrix}$$

Note that our values of a and b, given by (5.11), ensures that L''_1 and L''_2 lie on $\mathbb{M}_{17}$. Therefore, our effective tensor of the composite, P^*, is also guaranteed to lie on $\mathbb{M}_{17}$. We want to find $\mathsf{P}^* = \Psi_3(\Psi_0(\mathsf{L}^*))$. All we need to do is use the description of the exact relation as an equation. We have:

$$\mathsf{P}^* = \Psi_3(\Psi_0(\mathsf{L}^*)) = \Psi_3((\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)\mathsf{L}^*(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2))$$

$$= \left(a(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)\mathsf{L}^*(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2) + b\mathsf{T}\right)$$

Then, in order for P^* to lie on the exact relation $\mathbb{M}_{17}$, it must satisfy the following equation: $(\boldsymbol{J} \otimes \boldsymbol{R}_\perp) =$

$$= \left(a(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)\mathsf{L}^*(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2) + b\mathsf{T}\right)(\boldsymbol{J} \otimes \boldsymbol{R}_\perp)\left(a(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)L^*(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2) + b\mathsf{T}\right)$$

For compactness, let $\mathsf{S} = (\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)$, and $\mathsf{J} = \boldsymbol{J} \otimes \boldsymbol{R}_\perp$. We work with the following version of the equation for L^* written to highlight the structure.

$$(a\mathsf{S}\mathsf{L}^*\mathsf{S} + b\mathsf{T})\mathsf{J}(a\mathsf{S}\mathsf{L}^*\mathsf{S} + b\mathsf{T}) = \mathsf{J}$$

Factoring out S on both sides,

$$\mathsf{S}(aL^* + b\mathsf{S}^{-1}\mathsf{T}\mathsf{S}^{-1})\mathsf{S}\mathsf{J}\mathsf{S}(aL^* + b\mathsf{S}^{-1}\mathsf{T}\mathsf{S}^{-1})\mathsf{S} = \mathsf{J}$$

Multiplying by S^{-1} on both sides:

$$(a\mathsf{L}^* + b\mathsf{S}^{-1}\mathsf{T}\mathsf{S}^{-1})\mathsf{S}\mathsf{J}\mathsf{S}(a\mathsf{L}^* + b\mathsf{S}^{-1}\mathsf{T}\mathsf{S}^{-1}) = \mathsf{S}^{-1}\mathsf{J}\mathsf{S}^{-1}$$

Observe now that all tensors, aside from L^*, are nice tensor products. We multiply them separately:

$$\mathsf{S}^{-1}\mathsf{T}\mathsf{S}^{-1} = (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2)\mathsf{T}(\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) = \sqrt{\det \varsigma_1}\mathsf{T}$$

$$\mathsf{S}\mathsf{J}\mathsf{S} = (\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2)\mathsf{J}(\varsigma_1^{-1/2} \otimes \boldsymbol{I}_2) = \varsigma_1^{-1/2}\boldsymbol{J}\varsigma_1^{-1/2} \otimes \boldsymbol{R}_\perp$$

$$\mathsf{S}^{-1}\mathsf{J}\mathsf{S}^{-1} = (\varsigma_1^{1/2} \otimes \boldsymbol{I}_2)J(\varsigma_1^{1/2} \otimes \boldsymbol{I}_2) = \varsigma_1^{1/2}\boldsymbol{J}\varsigma_1^{1/2} \otimes \boldsymbol{R}_\perp$$

Note that $\boldsymbol{J}^{-1} = \boldsymbol{J}$, so that we may write:

$$\varsigma_1^{-1/2}\boldsymbol{J}\varsigma_1^{-1/2} = \frac{\operatorname{cof}(\varsigma_1^{1/2}\boldsymbol{J}\varsigma_1^{1/2})}{\det(\varsigma_1^{1/2}\boldsymbol{J}\varsigma_1^{1/2})} = -\frac{\operatorname{cof}(\varsigma_1^{1/2}\boldsymbol{J}\varsigma_1^{1/2})}{\det \varsigma_1}$$

It remains to simplify $\varsigma_1^{1/2}\boldsymbol{J}\varsigma_1^{1/2}$. Let us first divide our equation by a^2 given by (5.11), so that we may use the same constants A and B from the previous case given by (6.88) and (6.89). We have

$$(L^* + A\mathsf{T})SJS(L^* + A\mathsf{T}) = B\frac{S^{-1}JS^{-1}}{\det \varsigma_1}$$

Denoting $\boldsymbol{Z}_0 = \varsigma_1^{1/2}\mathbf{J}\varsigma_1^{1/2}$, we obtain

$$-(\mathsf{L}^* + A\mathsf{T})\mathsf{T}(\boldsymbol{Z}_0 \otimes \boldsymbol{R}_\perp)\mathsf{T}(\mathsf{L}^* + A\mathsf{T}) = B\boldsymbol{Z}_0 \otimes \boldsymbol{R}_\perp$$

We work in the frame where

$$\varsigma = \begin{bmatrix} \mu & \nu \\ \nu & \mu \end{bmatrix}$$

Let us denote the eigenvalues of ς as

$$\lambda_1 = \mu + \nu, \qquad \lambda_2 = \mu - \nu$$

Solving for μ and ν, we get

$$\mu = \frac{\lambda_1 + \lambda_2}{2}, \qquad \nu = \frac{\lambda_1 - \lambda_2}{2}$$

By definition of ς given by (5.2), we can write:

$$\varsigma_2 = \varsigma_1^{1/2} \begin{bmatrix} \mu & \nu \\ \nu & \mu \end{bmatrix} \varsigma_1^{1/2} = \mu\varsigma_1 + \nu\mathbf{X} \tag{6.92}$$

where $\boldsymbol{X} := \varsigma_1^{1/2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \varsigma_1^{1/2}$. Solving (6.92) for $\boldsymbol{X}$ we obtain

$$\boldsymbol{X} = \frac{\varsigma_2 - \mu\varsigma_1}{\nu} = \frac{2\varsigma_2 - (\lambda_1 + \lambda_2)\varsigma_1}{\lambda_1 - \lambda_2} = \boldsymbol{S}_2 - \boldsymbol{S}_1$$

Now, we use that $\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{R}_\perp$, to obtain:

$$\boldsymbol{Z}_0 = \varsigma_1^{1/2}\mathbf{J}\varsigma_1^{1/2} = \mathbf{X}\varsigma_1^{-1}\varsigma_1^{1/2}\boldsymbol{R}_\perp\varsigma_1^{1/2} = \mathbf{X}\frac{\boldsymbol{R}_\perp\varsigma_1\boldsymbol{R}_\perp^{\mathsf{T}}}{\det\varsigma_1}\sqrt{\det\varsigma_1}\boldsymbol{R}_\perp$$

We use our previously defined variables $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ given by (5.13) to obtain

$$\boldsymbol{Z}_0 = \frac{\mathbf{X}\boldsymbol{R}_\perp\varsigma_1}{\sqrt{\det\varsigma_1}} = \frac{(\boldsymbol{S}_2 - \boldsymbol{S}_1)\boldsymbol{R}_\perp(\boldsymbol{S}_2 + \boldsymbol{S}_1)}{\sqrt{\det\varsigma_1}} = \frac{\boldsymbol{S}_2\boldsymbol{R}_\perp\boldsymbol{S}_1 - \boldsymbol{S}_1\boldsymbol{R}_\perp\boldsymbol{S}_2}{\sqrt{\det\varsigma_1}}$$

Then, the final form of the effective tensor L^* for the generic strongly coupled case is given by

$$(\mathsf{L}^* + A\mathsf{T})\mathsf{T}(\boldsymbol{Z} \otimes \boldsymbol{R}_\perp)\mathsf{T}(\mathsf{L}^* + A\mathsf{T}) + B\boldsymbol{Z} \otimes \boldsymbol{R}_\perp = 0 \tag{6.93}$$

where

$$\boldsymbol{Z} = \boldsymbol{S}_2\boldsymbol{R}_\perp\boldsymbol{S}_1 - \boldsymbol{S}_1\boldsymbol{R}_\perp\boldsymbol{S}_2 \tag{6.94}$$

and where A, and B are given by (6.88), and (6.89).

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