The Refractor Problem with Loss of Energy and Monge-Ampère Type Equations

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ABSTRACT

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In this dissertation we study The Refractor Problem and its analytic formulation which leads to Monge-Ampère type equation. This problem can be described as follows: suppose that Ω, Ω^* are two domains of S^{n-1} and g, f are two positive functions integrable on Ω and Ω^* respectively. Consider two homogeneous, isotropic media; medium I and medium II, which have different optical densities and assume that from a point O inside medium I, light emanates with intensity $g(x)$, $x \in \Omega$. When an incident ray of light hits an interface between two media with different indices of refraction, it splits into two rays; a reflected ray that propagates back into medium I and a refracted ray that proceeds into medium II. Consequently, the incident ray loses some of its energy as it proceeds into medium II. By using Fresnel equations, which are consequences of Maxwell's Equations, one can determine precisely how much of the energy is lost due to internal reflection. The problem is to take into account this loss and construct a surface $\mathcal{R}$ such that all rays emitted from the point O with directions in Ω are refracted by $\mathcal{R}$ into media II with directions in Ω^* and the prescribed illumination intensity received in the direction $m \in \Omega^*$ is $f(m)$. We propose a model to this problem. We introduce weak solutions for the problem and prove their existence by using approximation by ellipsoids or hyperboloids depending on whether $n_1 < n_2$ or $n_1 > n_2$. We will also prove that a solution of the problem satisfies a Monge-Ampére type of PDE.

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EGZIABHER YEMESGEN!!

to

my mom, Aziza

my wife, Tigist

my daughters, Naomi and Abigail
with all my love

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CHAPTER 1

Introduction

The objective of this thesis is to investigate the refractor problem. This problem arises in geometric optics in relation to construction of refracting lenses. We shall propose a new model that takes into account the loss of energy due to internal reflection. The analytic formulation of the model leads to a differential equation of Monge-Ampère type on the unit sphere.

The Monge-Ampère type equation has a general form

$$\det[D^2u - A(x, u, Du)] = f(x, u, Du)$$

where $\det D^2u$ denotes the determinant of the Hessian matrix of a function u in the Euclidean space $\mathbf{R}^n$, A is a matrix and f is a given function. This is a fully nonlinear, second order partial differential equation. In recent years, this equation, has received a lot of attention for its role as a model for optimal transport problems, optimal antenna design and front formation in meteorology.

The equation draws its name from its initial formulation in two dimensions, by the French mathematicians, Monge and Ampère, about two hundred years ago, by considering the case when $A = 0$ and f is a function of x. It was subsequently studied by Minkowski, Bernstein, Schauder, Lewy and many others. In the last century the development of the Monge-Ampère type equation was closely related to geometric problems, such as the Minkowski problem of finding a convex body, whose boundary has its Gauss curvature prescribed as a function of its normal, the problem of local isometric embedding of Riemannian surfaces in $\mathbf{R}^3$ and the related Weyl problem. The first notable result is by Minkowski, who proved the existence of a weak solution to the Minkowski problem by approximation by convex polyhedra with given face areas. Using convex polyhedra with given generalized curvatures at the vertices, Aleksandrov [Al42a] also proved the existence of a weak solution to the Minkowski problem (in all dimensions), as well as the C^1 smoothness of solutions in two dimensions [Al42b]. Moreover, Minkowski, Aleksandrov, and Lewy [Le38], also proved the uniqueness of weak solutions. Both notions of weak solutions by Minkowski and Aleksandrov have continued to be frequently used in recent years. In fact, Aleksandrov's generalized solution corresponds to the curvature measure in the theory of convex bodies while the weak solution of Minkowski is related to the area measure [Sc93]. We refer the reader to [Gu01] for more discussion on the Monge-Ampère equations.

To explain the refractor problem consider two homogeneous, isotropic media, medium I and medium II with indices of refraction n_1 and n_2 respectively. Set $\kappa = n_2/n_1$ and let Γ be an interface between media I and media II. Suppose a point source of light is located at the origin in medium I. We identify each direction of a ray of light emanating from $\mathcal{O}$ with a point on S^{n-1}. According to *Snell's law* (law of refraction) if a ray of light in the direction x hits the interface Γ at a point P, it will be refracted by Γ in to a direction m given by

$$m = \frac{1}{\kappa}\,(x - \lambda\nu) \tag{1.1}$$

where ν is the normal to Γ at the point P, $\lambda = \Phi(x \cdot \nu)$ and

$$\Phi(t) = t - \kappa\sqrt{1 - \kappa^{-2}(1 - t^2)}. \tag{1.2}$$

Here $x \cdot \nu$ represents the usual inner product of x and ν in $\mathbf{R}^n$.

The refractor problem is described as follows: Let $\bar{\Omega}$, $\bar{\Omega}^*$ be two domains on S^{n-1} and $f \in L^1(\bar{\Omega}^*)$, $g \in L^1(\bar{\Omega})$ be nonnegative. Suppose that from a point O inside medium I, an incident light emanates with intensity $g(x)$ for $x \in \bar{\Omega}$. We want to construct a refracting surface $\mathcal{R}$ parameterized by a radial function $\rho \in C(\bar{\Omega})$ as $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$, separating media I and II such that all rays refracted by $\mathcal{R}$ into medium II have directions in Ω^* and the prescribed illumination intensity received in the direction $m \in \bar{\Omega}^*$ is $f(m)$.

When a ray of light hits an interface between the two media with different indices of refraction, it splits into two rays; a reflected ray that propagates back into medium I and a refracted (transmitted) ray that proceeds to medium II, and the energy (intensity) it carries also splits among these two rays.

In the case that

$$\int_{\bar{\Omega}} g(x)dx = \int_{\bar{\Omega}^*} f(x)dx \tag{1.3}$$

that is, when the loss of energy due to internal reflection is not considered, the answer to the existence and uniqueness of such a surface is given in [GH09]. This result which is proved via optimal mass transport methods is given as follows:

Theorem 1.1. *Let f, g, $\bar{\Omega}$, and $\bar{\Omega}^*$ be as in above and 1.3 holds. Then there exists a weak solution $\mathcal{R}$, unique up to dilations, of the refractor problem with emitting illumination intensity g and prescribed refracted illumination intensity f.*

Assuming 1.3 holds, progress has also been made regarding the regularity of the solution in [GH09] and [KA09] by using the regularity condition (A3 condition) introduced in [MTW05].

The objective of this thesis is to study the refractor problem with loss of energy, proposing a model that takes this loss into account. The notions of refractor and weak solution are defined in a way parallel to the corresponding notions in [GH09], except that the energy of the incident ray will be modified to account for the energy lost due to reflection. These modifying factors are given by *Fresnel's equations,* (Equations 2.14 and 2.15.) The presence of these factors which depend on the normal to the refractor present difficulties in extending the method used in [GH09] to tackle the problem. As a result we use approximation by ellipsoids (for $\kappa < 1$) and hyperboloids (for $\kappa > 1$) to prove existence of solution to the problem and, in particular, obtain a different proof of some of the results in [GH09].

The refractor problem has a resemblance with the Minkowski problem in which a closed convex surface has to be recovered from its Gauss curvature given as a function of the unit normal [[BU58], [PO78]]. In fact the approach used in chapters 4, 5 and 6 of the thesis to construct weak solutions is similar to the classical approach for solving the Minkowski problem [[BU58], [PO78]] and has been used in relation to the reflector problem [[CH09], [CO08]]. However, in contrast to the others, our approximations involve surfaces determined by finite number of ellipsoids of revolution or hyperboloids of revolution depending on whether $\kappa < 1$ or $\kappa > 1$.

In chapter 2 we study the notions of refractors and introduce the Fresnel's equations. We will also discuss surfaces with uniform refraction property. Weak solution to the refractor problem will be introduced in the third chapter. We will, in chapter 4, prove existence of solution in the case where no energy is lost. We will then prove the existence of solution to the refractor problem by accounting for the loss of energy in Chapter 5. In Chapter 6 we show that similar results can be proved for the case $\kappa > 1$. The fact that the analytic formulation of the problem leads to a Monge-Ampère type of differential equation will be discussed in chapter 7. We conclude the thesis with an appendix discussing the details of the derivation of Fresnel's equations.

CHAPTER 2

Preliminaries

2.1 Refraction

When a ray of light traveling through a certain medium encounters a boundary leading to another medium of different optical property, it usually changes its direction; this physical phenomenon is called *refraction.*

Suppose Γ is a surface in $\mathbf{R}^n$ that separates two homogeneous, isotropic and dielectric media I and II. By definition the index of refraction n of a medium is $n = c/v$ where c and v represent the speed of light in vacuum and the medium respectively. Let n_1 and n_2 be the indices of refraction of medium I and medium II respectively.

Light undergoes reflection and refraction simultaneously. That is, if a *monochromatic*[1] wave of light having direction of propagation $x \in S^{n-1}$ hits the boundary Γ between the two media I and II, at the point P, the wave is split between two rays; a refracted (transmitted) wave proceeding into medium II in a direction $m \in S^{n-1}$ and a reflected wave propagated back in to medium I in a direction $r \in S^{n-1}$.

In the next two subsections we shall briefly discuss the kinematic and dynamic properties of the incident, reflected and transmitted waves.

[1]Since the direction of refraction depends on frequency and a monochromatic wave has a single constant frequency, we assume our light wave to be monochromatic.

2.1.1 Kinematic properties

Let ν be the unit normal to Γ at P going towards medium II. The plane determined by x and ν is called the *incidence plane.* If θ_i is the angle between x and ν (*the angle of incidence*), θ_t is the angle between m and ν (*the angle of refraction*), θ_r is the angle between r and ν (*the angle of reflection*) then according to the law of reflection $\theta_i = \theta_r$ and according to the law of refraction (Snell's Law) $n_1 \sin\theta_i = n_2 \sin\theta_t$.

The law of refraction can alternatively be written in vector form as

$$n_1(x \times \nu) = n_2(m \times \nu). \tag{2.1}$$

If we set $\kappa = n_2/n_1$ we can also write 2.1 as

$$x - \kappa m = \lambda\nu \tag{2.2}$$

where $\lambda \in \mathbf{R}$ is given by

$$\begin{aligned} \lambda &= x \cdot \nu - m \cdot \nu \\ &= x \cdot \nu - \kappa\sqrt{1 - \kappa^{-2}(1 - (x \cdot \nu)^2)}. \end{aligned}$$

Notice here that $\lambda = \Phi(x \cdot \nu)$ where Φ is as given in 1.2.

When medium I is optically denser than medium II and consequently $v_1 < v_2$, or equivalently $\kappa < 1$, the refracted ray bends away from the normal. As a result, there is a critical value $\theta_i = \theta_c$, of the angle of incidence given by $\sin\theta_c = \kappa$ for which the refracted ray emerges in a direction tangent to the boundary; that is the angle of refraction $\theta_t = \pi/2$. If θ_i exceeds this critical value θ_c, no light enters medium II. In this case all the incident light is reflected back into the first medium and a phenomenon called *total internal reflection* happens. Thus there cannot be refraction unless $0 \leq \theta_i \leq \theta_c$. It can be verified that

$$\theta_t - \theta_i = \arcsin(\kappa^{-1}\sin\theta_i) - \sin\theta_i \tag{2.3}$$

is strictly increasing for $\theta_i \in [0, \theta_c]$, and therefore $0 \leq \theta_t - \theta_i \leq \dfrac{\pi}{2} - \theta_c$. By taking cos we arrive at the physical constraint $x \cdot m \geq \kappa$. Conversely, given

$x, m \in S^{n-1}$ with $x \cdot m \geq \kappa$ and $\kappa < 1$, it follows from 2.3 that there exists a hyperplane refracting any ray through medium I with direction x into a ray of direction m in medium II.

The discussion in the preceding paragraph can be summarized in the following lemma.

Lemma 2.1. *Suppose that the indices of refraction of media I and II are given by n_1 and n_2 respectively with $\kappa = n_2/n_1 < 1$. A light ray in medium I in the direction $x \in S^{n-1}$ is refracted by some surface into a light ray in medium II in the direction $m \in S^{n-1}$ if and only if $x \cdot m \geq \kappa$.*

On the other hand if medium II is optically denser than medium I, and consequently $v_1 > v_2$, or equivalently $\kappa > 1$, the refracted ray bends towards the normal. The maximum angle of refraction $\theta_t = \theta_c^*$ occurs when $\theta_i = \dfrac{\pi}{2}$ and it is given by $\theta_c^* = \arcsin(1/\kappa)$. As in 2.3 we can show that

$$\theta_i - \theta_t = \arcsin(\kappa \sin\theta_t) - \theta_t \tag{2.4}$$

is strictly increasing for $\theta_t \in [0, \theta_c^*]$ and $0 \leq \theta_i - \theta_i \leq \dfrac{\pi}{2} - \theta_c^*$. We thus obtain the physical constraint that $x \cdot m \geq \dfrac{1}{\kappa}$ for the case $\kappa > 1$. Also from 2.4 we conclude that if $x, m \in S^{n-1}$ with $x \cdot m \geq \dfrac{1}{\kappa}$ and $\kappa > 1$ then any ray in the direction x through medium I can be refracted into medium II to a ray with direction m by an appropriate hyperplane. The discussion in this paragraph can be summarized in the following lemma:

Lemma 2.2. *Suppose that the indices of refraction of media I and II are given by n_1 and n_2 respectively with $\kappa = n_2/n_1 > 1$. A light ray in medium I in the direction $x \in S^{n-1}$ is refracted by some surface into a light ray in medium II in the direction $m \in S^{n-1}$ if and only if $x \cdot m \geq \dfrac{1}{\kappa}$.*

2.1.2 Dynamic properties

When a light wave is incident on an interface separating two media of different optical properties, it gives rise to two waves; the reflected wave and the transmitted wave. Accordingly, the energy carried by the incident wave will be divided between the energies of the reflected and transmitted waves. By considering the electromagnetic field theory of propagation of light, it is possible to examine how the energy of the incident wave is divided between the secondary waves. This is briefly discussed in this subsection. For a more detail explanation we refer the reader to Appendix A, and for a more comprehensive discussion we refer the reader to [BW80], Chapter 1.

If $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ where $\mathbf{r} = (x, y, z)$ represents a point in 3-d space and t is the time, are the electric vector and magnetic fields of an electromagnetic wave, then in the absence of charge the system of Maxwell's equations will have the form:

$$\nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{2.5}$$

$$\nabla \times \mathbf{B} = \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{2.6}$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 0 \tag{2.7}$$

$$\nabla \cdot (\mu \mathbf{B}) = 0 \tag{2.8}$$

where c is the speed of light in vacuum, $\mu = \mu(x, y, z)$ is the magnetic permeability of the medium, and $\epsilon = \epsilon(x, y, z)$ is the electric permittivity of the medium. As usual ∇ stands for the gradient vector $(\partial_x, \partial_y, \partial_z)$.

By using these equations and assuming that the waves are plane waves [2] we can deduce the following important equations relating the electric and magnetic fields.

$$\mathbf{E} = -v(s \times \mathbf{B}) \tag{2.9}$$

$$\mathbf{B} = \frac{1}{v}(s \times \mathbf{E}). \tag{2.10}$$

Definition 2.1. *The flow of the energy (both with regard to its magnitude and direction) in an electromagnetic wave with electric field* $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ *and*

[2] having the same value at all points of any plane perpendicular to the direction of propagation

magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ *is given by the* Poynting vector *defined by*

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B},$$

where c *is the speed of light in free space.*

Thus the light intensity from (2.10) will be

$$\mathbf{S} = \frac{c}{4\pi v}\mathbf{E} \times (\mathbf{s} \times \mathbf{E}) = \frac{n}{4\pi}|\mathbf{E}|^{\mathbf{2}}\mathbf{s}. \tag{2.11}$$

If we denote quantities referring to the incident wave by suffix (i), to the reflected wave by (r), and to the refracted wave by (t), then the electric fields for the three waves are

$$\begin{aligned}
\mathbf{E}^{(\mathbf{i})}(\mathbf{r}, t) &= \mathrm{A}e^{-i(\omega t - \mathbf{s}^{(\mathbf{i})}\cdot\mathbf{r})} \\
\mathbf{E}^{(\mathbf{r})}(\mathbf{r}, t) &= \mathrm{R}e^{-i(\omega t - \mathbf{s}^{(\mathbf{r})}\cdot\mathbf{r})} \\
\mathbf{E}^{(\mathbf{t})}(\mathbf{r}, t) &= \mathrm{T}e^{-i(\omega t - \mathbf{s}^{(\mathbf{t})}\cdot\mathbf{r})}
\end{aligned}$$

where A, T and R are the amplitude vectors and $\mathbf{s}^{(\mathbf{i})}$, $\mathbf{s}^{(\mathbf{r})}$ and $\mathbf{s}^{(\mathbf{t})}$ are the directions of propagation of the corresponding fields.

Let's choose a system of coordinates such that the normal ν to the interface Γ at the point of incidence is on the z-axis and the x and y axes are on the plane perpendicular to ν and in such a way that the vector $\mathbf{s}^{i}$ lies on the xz-plane. This means that the tangent plane to Γ at P is the xy-plane and the incidence plane is the xz-plane.

Resolving each of the electric field vectors into components parallel (denoted by subscript $||$) and perpendicular (denoted by subscript $\perp$) to the incidence plane we obtain:

$$\begin{aligned}
\mathbf{E}^{(\mathbf{i})}(\mathbf{r}, t) &= \left(-\mathrm{A}_{\|}\cos\theta_i, \mathrm{A}_{\perp}, \mathrm{A}_{\|}\sin\theta_i\right) e^{-i(\omega t - \mathbf{s}^{(\mathbf{i})}\cdot\mathbf{r})} \\
&= \mathbf{E}_{\mathbf{0}}^{\mathbf{i}} e^{-i(\omega t - \mathbf{s}^{(\mathbf{i})}\cdot\mathbf{r})} \\
\mathbf{E}^{(\mathbf{r})}(\mathbf{r}, t) &= \left(-\mathrm{R}_{\|}\cos\theta_r, \mathrm{R}_{\perp}, \mathrm{R}_{\|}\sin\theta_r\right) e^{-i(\omega t - \mathbf{s}^{(\mathbf{r})}\cdot\mathbf{r})} \\
&= \mathbf{E}_{\mathbf{0}}^{\mathbf{r}} e^{-i(\omega t - \mathbf{s}^{(\mathbf{r})}\cdot\mathbf{r})} \\
\mathbf{E}^{(\mathbf{t})}(\mathbf{r}, t) &= \left(-\mathrm{T}_{\|}\cos\theta_t, \mathrm{T}_{\perp}, \mathrm{T}_{\|}\sin\theta_t\right) e^{-i(\omega t - \mathbf{s}^{(\mathbf{r})}\cdot\mathbf{r})} \\
&= \mathbf{E}_{\mathbf{0}}^{\mathbf{t}} e^{-i(\omega t - \mathbf{s}^{(\mathbf{t})}\cdot\mathbf{r})}
\end{aligned}$$

The components of the magnetic vector are then obtained by 2.10 to be:

$$\begin{aligned}
\mathbf{B}^{(\mathbf{i})}(\mathbf{r},t) &= \frac{1}{v_1}\left(-\mathrm{A}_\perp\cos\theta_i, -\mathrm{A}_\parallel, \mathrm{A}_\perp\sin\theta_i\right)e^{-i(\omega t-\mathbf{s}^{(\mathbf{i})}\cdot\mathbf{r})} \\
&= \mathbf{B}_\mathbf{0}^\mathbf{i} e^{-i(\omega t-\mathbf{s}^{(\mathbf{i})}\cdot\mathbf{r})} \\
\mathbf{B}^{(\mathbf{r})}(\mathbf{r},t) &= \frac{1}{v_1}\left(-\mathrm{R}_\perp\cos\theta_r, -\mathrm{R}_\parallel, \mathrm{R}_\perp\sin\theta_r\right)e^{-i(\omega t-\mathbf{s}^{(\mathbf{r})}\cdot\mathbf{r})} \\
&= \mathbf{B}_\mathbf{0}^\mathbf{r} e^{-i(\omega t-\mathbf{s}^{(\mathbf{r})}\cdot\mathbf{r})} \\
\mathbf{B}^{(\mathbf{t})}(\mathbf{r},t) &= \frac{1}{v_2}\left(-\mathrm{T}_\perp\cos\theta_t, -\mathrm{T}_\parallel, \mathrm{T}_\perp\sin\theta_t\right)e^{-i(\omega t-\mathbf{s}^{(\mathbf{r})}\cdot\mathbf{r})} \\
&= \mathbf{B}_\mathbf{0}^\mathbf{t} e^{-i(\omega t-\mathbf{s}^{(\mathbf{t})}\cdot\mathbf{r})}
\end{aligned}$$

The boundary conditions expressing the continuity of the tangential components [BW80][section 1.1, Equations (23) and (25)] of the electric and magnetic fields across the interface require that

$$\mathbf{k}\times\mathbf{E}_\mathbf{0}^\mathbf{i} + \mathbf{k}\times\mathbf{E}_\mathbf{0}^\mathbf{r} = \mathbf{k}\times\mathbf{E}_\mathbf{0}^\mathbf{t} \tag{2.12}$$

$$\mathbf{k}\times\mathbf{B}_\mathbf{0}^\mathbf{i} + \mathbf{k}\times\mathbf{B}_\mathbf{0}^\mathbf{r} = \mathbf{k}\times\mathbf{B}_\mathbf{0}^\mathbf{t} \tag{2.13}$$

From (2.12) we obtain the equations

$$\mathrm{A}_\perp + \mathrm{R}_\perp = \mathrm{T}_\perp, \qquad \cos\theta_i(\mathrm{A}_\parallel - \mathrm{R}_\parallel) = \cos\theta_t\mathrm{T}_\parallel;$$

and from (2.13) we obtain

$$\frac{\mathrm{A}_\parallel}{v_1} + \frac{\mathrm{R}_\parallel}{v_1} = \frac{T_\parallel}{v_2}, \qquad \cos\theta_i\left(\frac{\mathrm{A}_\perp}{v_1} - \frac{\mathrm{R}_\perp}{v_1}\right) = \cos\theta_t\frac{\mathrm{T}_\perp}{v_2}.$$

Using $n_1 = c/v_1$ and $n_2 = c/v_2$ and solving the last two sets of equations yields the *Fresnel formulae* given by:

$$\begin{aligned}
\mathrm{T}_\parallel &= \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}\mathrm{A}_\parallel \\
\mathrm{T}_\perp &= \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}\mathrm{A}_\perp \\
\mathrm{R}_\parallel &= \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}\mathrm{A}_\parallel \\
\mathrm{R}_\perp &= \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}\mathrm{A}_\perp.
\end{aligned}$$

We will replace $\mathbf{s^i}$ by x and $\mathbf{s^t}$ by m, and we also set $\kappa = n_2/n_1$. Recall ν is the normal to the interface. We have $\cos\theta_i = x\cdot\nu$ and $\cos\theta_t = m\cdot\nu$. In addition, from the Snell law $x - \kappa m = \lambda\nu$, so the Fresnel formulae have the form:

$$\begin{aligned}
\mathrm{T}_\parallel &= \frac{2\,x\cdot\nu}{\kappa\,x\cdot\nu + m\cdot\nu}\,\mathrm{A}_\parallel = \frac{2\,x\cdot\nu}{(\kappa\,x+m)\cdot\nu}\,\mathrm{A}_\parallel = \frac{2\,x\cdot(x-\kappa\,m)}{(\kappa\,x+m)\cdot(x-\kappa\,m)}\,\mathrm{A}_\parallel\\
\mathrm{T}_\perp &= \frac{2\,x\cdot\nu}{x\cdot\nu + \kappa\,m\cdot\nu}\,\mathrm{A}_\perp = \frac{2\,x\cdot\nu}{(x+\kappa\,m)\cdot\nu}\,\mathrm{A}_\perp = \frac{2\,x\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\,\mathrm{A}_\perp\\
\mathrm{R}_\parallel &= \frac{\kappa\,x\cdot\nu - m\cdot\nu}{\kappa\,x\cdot\nu + m\cdot\nu}\,\mathrm{A}_\parallel = \frac{(\kappa\,x-m)\cdot\nu}{(\kappa\,x+m)\cdot\nu}\,\mathrm{A}_\parallel = \frac{(\kappa\,x-m)\cdot(x-\kappa\,m)}{(\kappa\,x+m)\cdot(x-\kappa\,m)}\,\mathrm{A}_\parallel\\
\mathrm{R}_\perp &= \frac{x\cdot\nu - \kappa\,m\cdot\nu}{x\cdot\nu + \kappa\,m\cdot\nu}\,\mathrm{A}_\perp = \frac{(x-\kappa\,m)\cdot\nu}{(x+\kappa\,m)\cdot\nu}\,\mathrm{A}_\perp = \frac{(x-\kappa\,m)\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\,\mathrm{A}_\perp.
\end{aligned}$$

By 2.11, the amount of energy, J^i, flowing through a unit area of the boundary per second of the incident wave is then

$$J^{(i)} = |\mathbf{S^i}|\cos\theta_i = \frac{n_1}{4\pi}|\mathrm{A}|^2 x\cdot\nu.$$

Similarly, the amount of energies of the reflected and transmitted waves leaving a unit area of the boundary per second is given by

$$\begin{aligned}
J^{(r)} &= |\mathbf{S^r}|\cos\theta_i = \frac{n_1}{4\pi}|\mathrm{R}|^2 x\cdot\nu\\
J^{(t)} &= |\mathbf{S^t}|\cos\theta_t = \frac{n_2}{4\pi}|\mathrm{T}|^2 m\cdot\nu.
\end{aligned}$$

The reflection and transmission coefficients are defined by

$$r_\Gamma(x) = \frac{J^{(r)}}{J^{(i)}} = \left(\frac{|\mathrm{R}|}{|\mathrm{A}|}\right)^2, \text{ and } t_\Gamma(x) = \frac{J^{(t)}}{J^{(i)}} = \frac{n_2 m\cdot\nu}{n_1 x\cdot\nu}\left(\frac{|\mathrm{T}|}{|\mathrm{A}|}\right)^2.$$

From Fresnels formulae and the fact that

$$|\mathrm{R}|^2 = \mathrm{R}_\parallel^2 + \mathrm{R}_\perp^2$$

we obtain the ratio of the intensity of the incident wave that will be reflected to be

$$r_\Gamma(x) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{x\cdot m} - (1+\kappa^2)\right]^2 \frac{\mathrm{A}_\parallel^2}{\mathrm{A}_\parallel^2 + \mathrm{A}_\perp^2} + \left[1 - 2\kappa\,x\cdot m + \kappa^2\right]^2 \frac{\mathrm{A}_\perp^2}{\mathrm{A}_\parallel^2 + \mathrm{A}_\perp^2}\right) \tag{2.14}$$

and by conservation of energy the ratio of the intensity of the incident wave that will be transmitted is

$$t_\Gamma(x) = 1 - r_\Gamma(x). \tag{2.15}$$

Equations 2.14 and 2.15 are called *Fresnel's Equations* and the numbers $r_\Gamma(x)$ and $t_\Gamma(x)$ are called *Fresnel coefficients.*

Remark 2.1.

i. For brevity let us define

$$\phi(t) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2 \alpha + \left[1 - 2\kappa\, t + \kappa^2\right]^2 \beta\right), \tag{2.16}$$

where $\alpha = \dfrac{\mathrm{A}_\parallel^2}{\mathrm{A}_\parallel^2 + \mathrm{A}_\perp^2}$*, and* $\beta = \dfrac{\mathrm{A}_\perp^2}{\mathrm{A}_\parallel^2 + \mathrm{A}_\perp^2}$*. Then* $r_\Gamma(x) = \phi(x \cdot m)$*,* $t_\Gamma(x) = 1 - \phi(x \cdot m)$ *and* $\alpha + \beta = 1$.

ii. By using Snell's law we can write $r_\Gamma(x) = \phi(x) = F(x, \nu_x)$ *where* ν_x *is the normal to* Γ *at the point of incidence.*

iii. By definition $0 \leq t_\Gamma(x) \leq 1$.

2.2 Uniform Refraction Property

It is well known that any paraboloid of revolution reflects all rays of light emitted from its focus into light rays in the axial direction.

In Descartes Eighth Discourse on Optics [DE01], it is discussed that in the plane, if $\kappa < 1$ and the ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ with $a > b$, is filled with a material having refraction index n_1 and the outside of the ellipse is filled with a material having refraction index n_2, then all rays emanating from one focus are refracted by the half of the ellipse closer to the other focus into rays parallel to the x axis, if the eccentricity $e = \sqrt{1-(b/a)^2} = \kappa$. Similarly, if $\kappa > 1$ and the region containing (h, k) and bounded by the hyperbola $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, with $a > b$, is filled with a material having refraction index n_1, and the complement of this region is filled with a material having refraction index n_2, then all rays emanating from one focus are refracted by the branch of the hyperbola closer to the other focus into rays parallel to the x axis, if the eccentricity $e = \sqrt{1-(b/a)^2} = \kappa$.

In [GH09], a similar result is proved for higher dimensions. For convenience we shall reproduce this result. To that end fix $m \in S^{n-1}$ and suppose that a surface Γ parameterized by the polar representation $\rho(x)x$, $x \in S^{n-1}$ is an interface between media I and II with the property that all rays that emanate from the origin inside medium I, are refracted into rays parallel to m. If we consider a curve on Γ given by $r(t) = \rho(x(t))x(t)$ for $x(t) \in S^{n-1}$ by 1.1, the tangent vector $r'(t)$ satisfies $r'(t) \cdot (x(t) - \kappa m) = 0$. That is, $([\rho(x(t)]'x(t) + \rho(x(t)))x'(t)) \cdot (x(t) - \kappa m) = 0$, which yields $(\rho(x(t))(1 - \kappa m \cdot x(t)))' = 0$. Therefore there exists some $b \in \mathbf{R}$ such that

$$\rho(x) = \frac{b}{1 - \kappa m \cdot x} \tag{2.17}$$

for $x \in S^{n-1}$. If $\kappa < 1$, then for $b > 0$ the surface given by 2.17 is an ellipsoid of revolution about the axis of direction m. To see this suppose for simplicity that $m = e_n$, the $nth-$coordinate vector. If $y = (y', y_n) \in \mathbf{R}^n$ is a point on Γ, then $y = \rho(x)x$ with $x = y/|y|$. From 2.17, $|y| = b + \kappa y_n$, which yields $|y'|^2 + (1-\kappa^2)y_n^2 - 2\kappa b y_n - b^2 = 0$. This can then be written as

$$\frac{|y'|^2}{\left(\dfrac{b}{\sqrt{1-\kappa^2}}\right)^2} + \frac{\left(y_n - \dfrac{\kappa b}{1-\kappa^2}\right)^2}{\left(\dfrac{b}{1-\kappa^2}\right)^2} = 1 \tag{2.18}$$

which is an ellipsoid of revolution about the y_n axis with foci at O and $\dfrac{2\kappa b}{1-\kappa^2}m$. Since $|y| = \kappa y_n + b$ and the physical constraint for refraction requires $\dfrac{y}{|y|} \cdot e_n \geq \kappa$, we have $y_n \geq \dfrac{\kappa b}{1-\kappa^2}$. This means, for refraction to occur, y must be in the upper part of the ellipsoid given in 2.18. We denote this semi-ellipsoid by $E(e_n, b)$. Moreover the normal to $E(e_n, b)$ at y which is given by $\dfrac{y}{|y|} - \kappa e_n$ satisfies

$$\left(\frac{y}{|y|} - \kappa e_n\right) \cdot \frac{y}{|y|} \geq 1 - \kappa > 0$$

and

$$\left(\frac{y}{|y|} - \kappa e_n\right) \cdot e_n \geq 0$$

from which we conclude that $\left(\dfrac{y}{|y|} - \kappa e_n\right)$ is an outward normal to $E(e_n, b)$ at y. Rotating the coordinates, we notice that with $\kappa < 1$ and $b > 0$ 2.17 defines an ellipsoid of revolution about the axis of direction m with foci O and $\dfrac{2\kappa b}{1-\kappa^2} m$. Thus we obtain the following

Lemma 2.3. *Let n_1 and n_2 be the indices of refraction of two media I and II and let $n_2/n_1 = \kappa < 1$. Then $E(m, b)$, the semi-ellipsoid of revolution which has its foci at O and $\dfrac{2\kappa b}{1-\kappa^2} m$, axis of revolution parallel to m and which is given by*

$$E(m,b) = \{\rho(x)x : \rho(x) = \frac{b}{1 - \kappa m \cdot x},\ x \in S^{n-1},\ m \cdot x \geq \kappa\} \tag{2.19}$$

has a uniform refraction property; any ray emanating from the origin O is refracted in the direction m.

If $\kappa > 1$, then the physical constraint requires that $b < 0$. If for $b > 0$ we consider

$$H(m,b) = \{\rho(x)x : \rho(x) = \frac{b}{\kappa m \cdot x - 1},\ x \in S^{n-1},\ m \cdot x \geq 1/\kappa\}.$$

then $H(m, b)$ is the sheet with opening in direction m of a hyperboloid of revolution of two sheets about the axis of direction m which has its foci at O and $\dfrac{2\kappa b}{1-\kappa^2} m$. To see this assume $m = e_n$. If $y = (y', y_n) \in H(e_n, b)$ then $y = \rho(x)x$ with $x = y/|y|$. Then

$$|y'|^2 + y_n^2 = (\kappa y_n - b)^2$$

from which we obtain

$$|y'|^2 - (\kappa^2 - 1)\left[\left(y_n - \frac{\kappa b}{\kappa^2 - 1}\right)^2 - \left(\frac{\kappa b}{\kappa^2 - 1}\right)^2\right] = b^2.$$

Thus, any point y on $H(e_n, b)$ satisfies the equation

$$\frac{\left(y_n - \dfrac{\kappa b}{\kappa^2 - 1}\right)^2}{\left(\dfrac{b}{\kappa^2 - 1}\right)^2} - \frac{|y'|^2}{\left(\dfrac{b}{\sqrt{\kappa^2 - 1}}\right)^2} = 1 \tag{2.20}$$

which represents a hyperboloid of revolution of two sheets about the y_n axis with foci $(0, 0)$ and $(0, \dfrac{2\kappa b}{\kappa^2 - 1})$. Moreover the upper sheet of this hyperboloid is given by

$$y_n = \frac{\kappa b}{\kappa^2 - 1} + \frac{b}{\kappa^2 - 1} \sqrt{1 + \frac{|y'|^2}{\dfrac{b}{\sqrt{\kappa^2 - 1}}}}$$

and satisfies $ky_n - b > 0$, and hence has the polar equation $\rho(x) = \dfrac{b}{\kappa e_n \cdot x - 1}$. Similarly, the lower sheet satisfies $\kappa y_n - b < 0$ and has polar equation $\rho(x) = \dfrac{b}{\kappa e_n \cdot x + 1}$.

Once again by a rotation, we can see that $H(m, b)$ is the sheet with opening in direction m of a hyperboloid of revolution of two sheets about the axis of direction m with foci at O and $\dfrac{2\kappa b}{1 - \kappa^2} m$.

We notice that the focus $(0, 0)$ is outside the region enclosed by $H(m, b)$ and the focus $\dfrac{2\kappa b}{\kappa^2 - 1} m$ is inside that region. Moreover the vector $\kappa m - \dfrac{y}{|y|}$ satisfies $\left(\kappa m - \dfrac{y}{|y|}\right) \cdot m \geq \kappa - 1 > 0$ and $\left(\kappa m - \dfrac{y}{|y|}\right) \cdot \dfrac{y}{|y|} > 0$ and thus is an inward normal to $H(m, b)$ at y. We then obtain the following Lemma.

Lemma 2.4. *Let n_1 and n_2 be the indices of refraction of two media I and II and let $n_2/n_1 = \kappa > 1$. Then $H(m, b)$, the sheet with opening in direction m of a hyperboloid of revolution of two sheets about the axis of direction m with foci at O and $\dfrac{2\kappa b}{1 - \kappa^2} m$ and which is given by*

$$H(m, b) = \{\rho(x)x : \rho(x) = \frac{b}{\kappa m \cdot x - 1},\ x \in S^{n-1},\ m \cdot x \geq 1/\kappa\} \tag{2.21}$$

has a uniform refraction property; any ray emanating from the origin O is refracted in the direction m.

CHAPTER 3

Weak Solutions

Let n_1 and n_2 be the indices of refraction of two homogeneous and isotropic media, medium I and medium II, respectively. Suppose medium I is denser than medium II and therefore $\kappa = n_2/n_1 < 1$.

Suppose that Ω and Ω^* are two domains of the unit sphere S^{n-1} of $\mathbf{R}^n$ with the property that

$$\inf_{m\in\bar{\Omega}^*, x\in\bar{\Omega}} m \cdot x \geq \kappa, \tag{3.1}$$

where $m \cdot x$ is the usual inner product of m and x in $\mathbf{R}^n$. We further suppose that $g \in L^1(\bar{\Omega})$ and $\inf_\Omega g > 0$.

For $m \in \bar{\Omega}^*$ and $b > 0$, let $E(m, b)$ be the semi-ellipsoid given by 2.19.

Definition 3.1. *A surface $\mathcal{R}$ in $\mathbf{R}^n$ parameterized by $\rho(x)x$ with $\rho \in C(\bar{\Omega})$ is a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$ for the case $\kappa < 1$, if for any $x_o \in \bar{\Omega}$ there exists a semi-ellipsoid $E(m, b)$, $m \in \bar{\Omega}^*$ such that $\rho(x_o) = \dfrac{b}{1 - \kappa m \cdot x_o}$ and $\rho(x) \leq \dfrac{b}{1 - \kappa m \cdot x}$ for all $x \in \bar{\Omega}$. We call $E(m, b)$ a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_o)x_o$ or simply at x_o.*

Lemma 3.1. *Any refractor is globally Lipschitz on $\bar{\Omega}$ and hence the set of singular points has measure zero.*

Proof. Let $\mathcal{R}$ be a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$ parameterized by $\rho(x)x$, $x \in \bar{\Omega}$. Let $x \in \Omega$, and $E(m, b)$ be a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x)x$. Then

$\rho(x) = \dfrac{b}{1-\kappa m\cdot x}$ and for any $y \in \Omega$, $\rho(y) \leq \dfrac{b}{1-\kappa m \cdot y}$. Moreover since $\rho \in C(\bar{\Omega})$, we notice that there exists M such that $\rho(x) \leq M$ for all $x \in \bar{\Omega}$. Thus $b \leq M$. Hence

$$\begin{aligned} \rho(y) - \rho(x) &\leq \frac{b\kappa m \cdot (x-y)}{(1-\kappa m\cdot y)(1-\kappa m \cdot x)} \\ &\leq \frac{b\kappa \|x-y\|}{(1-\kappa)^2} \\ &\leq \frac{M\kappa\|x-y\|}{(1-\kappa)^2}. \end{aligned}$$

Interchanging the roles of x and y we obtain a similar inequality. Thus $|\rho(x) - \rho(y)| \leq L\|x-y\|$ for some L and ρ is globally Lipshcitz. Consequently by Rademacher's theorem [Ev98] the set of singular points of ρ has measure zero.

$\square$

Remark 3.1. *If a refractor $\mathcal{R}$ parameterized by ρ has two distinct supporting semi-ellipsoids, $E(m_1, b_1)$ and $E(m_2, b_2)$ at $\rho(x)x$, then $\rho(x)x$ is a singular point of $\mathcal{R}$. Otherwise, if Π is the tangent hyperplane at $\rho(x)x$, then Π must coincide both with the tangent hyperplane of $E(m_1, b_1)$ and $E(m_2, b_2)$. From 1.1, $m_1 = m_2$.*

Definition 3.2. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined by for $x_o \in \overline{\Omega}$*

$$\mathcal{N}_{\mathcal{R}}(x_o) = \{m \in \bar{\Omega}^* : E(m,b) \text{ supports } \mathcal{R} \text{ at} \rho(x_o)x_o \text{ for some } b > 0\}.$$

Given $m_o \in \bar{\Omega}^*$ the tracing mapping of $\mathcal{R}$ is defined by

$$\mathcal{T}_{\mathcal{R}}(m_o) = \{x \in \bar{\Omega} : m_o \in \mathcal{N}_{\mathcal{R}}(x)\}.$$

We now prove some basic properties about the refractor and tracing mapping.

Lemma 3.2. *If $m \in \bar{\Omega}^*$, then $\mathcal{T}_{\mathcal{R}}(m)$ is closed in $\bar{\Omega}$.*

Proof. Let $x_n \in \mathcal{T}_{\mathcal{R}}(m)$ for $n \geq 1$, with $x_n \to x_o$. There exists $b > 0$ such that $E(m, b)$ supports $\mathcal{R}$ at $\rho(x_n)x_n$ for all n. Since ρ is Lipschitz and $x_o \in \bar{\Omega}$ $\rho(x_n)x_n \to \rho(x_o)x_o$ and $\rho(x_o)x_o \in \mathcal{R}$. Moreover, $\rho(x_n) = \dfrac{b}{1 - \kappa m \cdot x_n}$ for all n, thus $\rho(x_o) = \dfrac{b}{1 - \kappa m \cdot x_o}$. Also for $x \in \bar{\Omega}$, $\rho(x) \leq \dfrac{b}{1 - \kappa m \cdot x_n}$, and hence $\rho(x) \leq \dfrac{b}{1 - \kappa m \cdot x_o}$. Thus $E(m, b)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$ and $x_o \in \mathcal{T}_{\mathcal{R}}(m)$. □

Lemma 3.3.

i. $[\mathcal{T}_{\mathcal{R}}(F)]^c \subset \mathcal{T}_{\mathcal{R}}(F^c)$ *for all* $F \subset \bar{\Omega}^*$.

ii. The set $\mathcal{C} = \{F \subset \bar{\Omega}^* : \mathcal{T}_{\mathcal{R}}(F)$ *is Lebesgue measurable*$\}$ *is a* σ*-algebra containing all Borel sets in* $\bar{\Omega}^*$

Proof. The proof of i, follows from

$$x \in [\mathcal{T}_{\mathcal{R}}(F)]^c \Rightarrow \mathcal{N}_{\mathcal{R}}(x) \cap F = \emptyset \Rightarrow \mathcal{N}_{\mathcal{R}}(x) \cap F^c \neq \emptyset \Rightarrow x \in \mathcal{T}_{\mathcal{R}}(F^c).$$

To prove ii, we first note that $\mathcal{T}_{\mathcal{R}}(\bar{\Omega}^*) = \bar{\Omega}$ and $\mathcal{T}_{\mathcal{R}}(\emptyset) = \emptyset$. For $F_i \in \mathcal{C}$, $\mathcal{T}_{\mathcal{R}}(\cup_{i=1}^{\infty} F_i) = \cup_{i=1}^{\infty} \mathcal{T}_{\mathcal{R}}(F_i)$, hence $\mathcal{C}$ is closed under countable union. Also if $F \in \mathcal{C}$, then

$$\begin{aligned}
\mathcal{T}_{\mathcal{R}}(F^c) &= (\mathcal{T}_{\mathcal{R}}(F^c) \cap [\mathcal{T}_{\mathcal{R}}(F)]^c) \cup (\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)) \\
&= [\mathcal{T}_{\mathcal{R}}(F)]^c \cup [\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)]
\end{aligned}$$

where the last equality is by *i*. $[\mathcal{T}_{\mathcal{R}}(F)]^c$ is measurable as a complement of a measurable set and $\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)$ is measurable since it is a set of measure zero. So $\mathcal{C}$ is closed under taking complements and we have proved that $\mathcal{C}$ is a σ-algebra.

To prove that $\mathcal{C}$ contains all Borel subsets of $\bar{\Omega}^*$ consider a closed subset K of $\bar{\Omega}^*$. Clearly K is compact. Let $x_i \in \mathcal{T}_{\mathcal{R}}(K)$, for $i \geq 1$. There exists $m_i \in \mathcal{N}_{\mathcal{R}}(x_i) \cap K$. Let $E(m_i, b_i)$ be a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_i)x_i$. We have

$$\rho(x)(1 - \kappa m_i \cdot x) \leq b_i \qquad \text{for } x \in \bar{\Omega}.$$

with equality at $x = x_i$. Assume that $a_1 \le \rho(x) \le a_2$ on $\bar{\Omega}$ for some constants $0 < a_1 \le a_2$. Then

$$a_1(1-\kappa) \le a_1(1-\kappa m_i \cdot x) \le \rho(x)(1-\kappa m_i \cdot x) \le b_i$$

and

$$b_i = \rho(x_i)(1-\kappa m_i \cdot x_i) \le a_2(1-\kappa m_i \cdot x_i) \le a_2(1-\kappa^2).$$

Thus

$$a_1(1-\kappa) \le b_i \le a_2(1-\kappa^2)$$

and the $b_i s$ are bounded. Assume through a subsequence that $x_i \to x_o \in \bar{\Omega}$, $m_i \to m_o \in K$ and $b_i \to b_o$ as $i \to \infty$. Then for $x \in \bar{\Omega}$, and for all i

$$\rho(x)(1-\kappa m_i \cdot x) \le b_i \text{ and } \rho(x_i)(1-\kappa m_i \cdot x_i) = b_i.$$

Taking the limit as $i \to \infty$

$$\rho(x)(1-\kappa m_o \cdot x) \le b_o \text{ and } \rho(x_o)(1-\kappa m_o \cdot x_o) = b_o.$$

So $E(m_o, b_o)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$ and $x_o \in \mathcal{T}_{\mathcal{R}}(m_o)$, proving that $\mathcal{T}_{\mathcal{R}}(K)$ is compact. This completes the proof that $\mathcal{C}$ is a σ-algebra. □

Lemma 3.4. *Let $E(m_l, b_l)$ be a sequence of semi-ellipsoids with $m_l \to m$ and $b_l \to b$, as $l \to \infty$. Let $z_l \in E(m_l, b_l)$ with $z_l \to z_0$ as $l \to \infty$. Then $z_0 \in E(m, b)$, and the normal $\nu_l(z_l)$ to the semi-ellipsoid $E(m_l, b_l)$ at z_l satisfies $\nu_l(z_l) \to \nu(z_0)$ the normal to the semi-ellipsoid $E(m, b)$ at the point z_0.*

Proof. The equation of $E(m_l, b_l)$ in rectangular coordinates is $|z| - \kappa m_l \cdot z = b_l$, then the normal vector at z is $\nu_l(z) = \dfrac{z}{|z|} - \kappa m_l$, and so

$$\nu_l(z_l) = \frac{z_l}{|z_l|} - \kappa m_l \to \frac{z_0}{|z_0|} - \kappa m$$

which is the normal to $E(m, b)$ at z_0. □

Proposition 3.1. *Let $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ be a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$. Let S be the singular set of ρ. Then the Fresnel coefficient $t_{\mathcal{R}}(x)$ is continuous relative to the set $\bar{\Omega} \setminus S$.*

Proof. We shall prove that $r_{\mathcal{R}}$ is continuous.

Suppose $C_1 \leq \rho(x) \leq C_2$, with C_i, $i = 1, 2$ positive constants. From Remark 2.1, $r_{\mathcal{R}}(x)$ is a function $\phi(x) = F(x, \nu(x))$ which is defined in $\bar{\Omega} \setminus S$ where the function $F(x, m)$ is continuous in $\bar{\Omega} \times \bar{\Omega}^*$.

To prove that $r_{\mathcal{R}}(x)$ is continuous we shall prove that it is both lower and upper semicontinuous relative to the set $\bar{\Omega} \setminus S$.

To prove that $r_{\mathcal{R}}(x)$ is lower semicontinuous relative to the set $\bar{\Omega} \setminus S$, we shall prove that the set

$$E_\alpha = \{x \in \bar{\Omega} \setminus S : \phi(x) \leq \alpha\}$$

is closed relative to $\bar{\Omega} \setminus S$, for all α, that is, if $x_j, x_0 \in \bar{\Omega} \setminus S$, with $x_j \to x_0$ and $x_j \in E_\alpha$, then $x_0 \in E_\alpha$.

First we claim that if $x_j \to x_0$ with $x_j, x_0 \in \bar{\Omega} \setminus S$, then there exists a subsequence x_{j_l} such that $\nu(x_{j_l}) \to \nu(x_0)$, as $l \to \infty$. Let $E(m_j, b_j)$ be a supporting ellipsoid to the refractor at $\rho(x_j)x_j$. So

$$\rho(x) \leq \frac{b_j}{1 - \kappa m_j \cdot x}$$

with equality at $x = x_j$. Hence

$$C_1(1 - \kappa) \leq C_1(1 - \kappa m_j \cdot x_j) \leq b_j \leq C_2(1 - \kappa m_j \cdot x_j) \leq C_2(1 - \kappa^2).$$

Therefore, there is a subsequence $b_{j_l} \to b_0 > 0$ and $m_{j_l} \to m_0$, as $l \to \infty$. Then by Lemma 3.4 we have the claim.

Now if $x_j \in E_\alpha$, then $\phi(x_j) = F(x_j, \nu(x_j)) \leq \alpha$, but from the claim, there exists a subsequence x_{j_l} such that $\nu(x_{j_l}) \to \nu(x_0)$ as $l \to \infty$. Then from the continuity of F we are done.

By a similar argument $r_{\mathcal{R}}(x)$ is upper semicontinuous and hence $r_{\mathcal{R}}(x)$ is continuous.

In particular, we obtain that the function $r_{\mathcal{R}}(x)$, which is defined a.e. in Ω, is measurable in Ω. □

Lemma 3.5. *Let $\mathcal{R}$ be a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$ and let $g \in L^1(\bar{\Omega})$ with $\inf_{\bar{\Omega}} g > 0$. Define a set function on Borel subsets of $\bar{\Omega}^*$, by*

$$G_{\mathcal{R}}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} g(x) t_{\mathcal{R}}(x) dx$$

where dx is the surface measure on S^{n-1}. Then $G_{\mathcal{R}}$ is a finite Borel measure defined on $\mathcal{C}$, where $\mathcal{C}$ is the σ-algebra defined in 3.3, ii. $G_{\mathcal{R}}$ is called the refractor measure associated with $\mathcal{R}$ and g.

Proof. Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{C}$. Let $H_1 = \mathcal{T}_{\mathcal{R}}(F_1)$ and $H_k = \mathcal{T}_{\mathcal{R}}(F_k) \setminus \cup_{i=1}^{k-1} \mathcal{T}_{\mathcal{R}}(F_i)$ for $k \geq 2$. Then the collection $\{H_k\}_{k=1}^{\infty}$ is a disjoint collection and $\cup_{k=1}^{\infty} H_k = \cup_{k=1}^{\infty} \mathcal{T}_{\mathcal{R}}(F_k)$. Thus

$$\begin{aligned} G_{\mathcal{R}}(\cup_{k=1}^{\infty} F_k) &= \int_{\mathcal{T}_{\mathcal{R}}(\cup_{k=1}^{\infty} F_k)} g(x) t_{\mathcal{R}}(x) dx \\ &= \int_{\cup_{k=1}^{\infty} \mathcal{T}_{\mathcal{R}}(F_k)} g(x) t_{\mathcal{R}}(x) dx \\ &= \int_{\cup_{k=1}^{\infty} H_k} g(x) t_{\mathcal{R}}(x) dx = \sum_{k=1}^{\infty} \int_{H_k} g(x) t_{\mathcal{R}}(x) dx \end{aligned}$$

Since $H_k \subset \mathcal{T}_{\mathcal{R}}(F_k)$, $\mathcal{T}_{\mathcal{R}}(F_k) = H_k \cup [\mathcal{T}_{\mathcal{R}}(F_k) \setminus H_k]$. But

$$\mathcal{T}_{\mathcal{R}}(F_k) \setminus H_k = \mathcal{T}_{\mathcal{R}}(F_k) \cap \left(\cup_{i=1}^{k-1} \mathcal{T}_{\mathcal{R}}(F_i) \right)$$

and therefore $\mathcal{T}_{\mathcal{R}}(F_k) \setminus H_k$ is a subset of the singular set of $\mathcal{R}$ and has measure zero for $k \geq 2$. So

$$\int_{\mathcal{T}_{\mathcal{R}}(F_k)} g(x) t_{\mathcal{R}}(x) dx = \int_{H_k} g(x) dx,$$

and

$$G_{\mathcal{R}}(\cup_{k=1}^{\infty} F_k) = \sum_{k=1}^{\infty} \int_{H_k} g(x) t_{\mathcal{R}}(x) dx = \sum_{k=1}^{\infty} \int_{\mathcal{T}_{\mathcal{R}}(F_k)} g(x) t_{\mathcal{R}}(x) dx = \sum_{k=1}^{\infty} G_{\mathcal{R}}(F_k).$$

This proves that $G_{\mathcal{R}}$ is additive and hence a measure. □

Lemma 3.6. *Let $\mathcal{R}_j = \{\rho_j(x) x : x \in \bar{\Omega}\}$, $j \geq 1$ be refractors from $\bar{\Omega}$ to $\bar{\Omega}^*$. Suppose that $0 < a_1 \leq \rho_j \leq a_2$ and $\rho_j \to \rho$ uniformly on $\bar{\Omega}$. Then:*

i. $\mathcal{R} := \{\rho(x)x : x \in \bar{\Omega}\}$ *is a refractor from* $\bar{\Omega}$ *to* $\bar{\Omega}^*$.

ii. For any compact set $K \subset \bar{\Omega}^*$

$$\limsup_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(K) \subset \mathcal{T}_{\mathcal{R}}(K).$$

iii. For any open set $G \subset \bar{\Omega}^*$,

$$\mathcal{T}_{\mathcal{R}}(G) \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S,$$

where S *is the singular set of* $\mathcal{R}$.

Proof. (i) Obviously $\rho \in C(\bar{\Omega})$ and $\rho > 0$. Fix $x_o \in \bar{\Omega}$. Then there exists $m_j \in \bar{\Omega}^*$ and $b_j > 0$ such that $E(m_j, b_j)$ supports $\mathcal{R}_j$ at $\rho(x_o)x_o$ and thus

$$\rho_j(x_o) = \frac{b_j}{1 - \kappa m_j \cdot x_o} \qquad \text{and} \qquad \rho_j(x) \le \frac{b_j}{1 - \kappa m_j \cdot x}$$

for all $x \in \bar{\Omega}$. Consequently

$$\frac{b_j}{1 - \kappa m_j \cdot x_o} \le a_2 \qquad \text{and} \qquad a_1 \le \frac{b_j}{1 - \kappa m_j \cdot x}$$

for all j and therefore

$$a_1(1 - \kappa) \le b_j \le a_2$$

for all j. If need be by passing to a subsequence we obtain m_o and b_o such that $m_j \to m_o \in \bar{\Omega}^*$ and $b_j \to b_o$. We claim $E(m_o, b_o)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$. Indeed

$$\rho(x_o) = \lim_j \rho_j(x_o) = \lim_j \frac{b_j}{1 - \kappa m_j \cdot x_o} = \frac{b_o}{1 - \kappa m_o \cdot x_o}$$

and

$$\rho(x) = \lim_j \rho_j(x) \le \lim_j \frac{b_j}{1 - \kappa m_j \cdot x} = \frac{b_o}{1 - \kappa m_o \cdot x}$$

for all $x \in \bar{\Omega}$. Thus $\mathcal{R}$ is a refractor.

(ii.) Let $x_o \in \limsup \mathcal{T}_{\mathcal{R}_j}(K)$. Without loss of generality assume that $x_o \in \mathcal{T}_{\mathcal{R}_j}(K)$ for all $j \ge 1$. Then there exist $m_j \in \mathcal{N}_{\mathcal{R}_j}(x_o) \cap K$ and $b_j > 0$ such that

$$\rho_j(x_o) = \frac{b_j}{1 - \kappa m_j \cdot x_o} \qquad \text{and} \qquad \rho_j(x) \le \frac{b_j}{1 - \kappa m_j \cdot x}$$

for all $x \in \bar{\Omega}$. As in the proof of (i) we may assume that $m_j \to m_o \in K$ and $b_j \to b_o$ and conclude that $E(m_o, b_o)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$, proving that $x_o \in \mathcal{T}_{\mathcal{R}}(m_o)$. Hence $x_o \in \mathcal{T}_{\mathcal{R}}(K)$.

(iii.) Let G be an open subset of $\bar{\Omega}^*$. By (ii), $\limsup_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G^c) \subset \mathcal{T}_{\mathcal{R}}(G^c)$ as G^c is compact. Also

$$\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c \subset \limsup_{j\to\infty}\{[\mathcal{T}_{\mathcal{R}_j}(G)]^c \cup [\mathcal{T}_{\mathcal{R}_j}(G) \cap \mathcal{T}_{\mathcal{R}_j}(G^c)]\} \tag{3.2}$$

and by Lemma 3.3 the right hand side of (3.2) is equal to $\limsup_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G^c)$. By (ii) we will then have

$$\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c \subset \mathcal{T}_{\mathcal{R}}(G^c) = \{[\mathcal{T}_{\mathcal{R}}(G)]^c \cup [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]\}.$$

Taking complements we obtain

$$\{\limsup_{j\to\infty}[\mathcal{T}_{\mathcal{R}_j}(G)]^c\}^c \supset [\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c.$$

Consequently

$$\liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \supset [\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c$$

and thus

$$[[\mathcal{T}_{\mathcal{R}}(G)] \cap [\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c)]^c] \cup S \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S.$$

But $\mathcal{T}_{\mathcal{R}}(G) \cap \mathcal{T}_{\mathcal{R}}(G^c) \subset S$. Thus

$$\mathcal{T}_{\mathcal{R}}(G) \subset \mathcal{T}_{\mathcal{R}}(G) \cup S \subset \liminf_{j\to\infty} \mathcal{T}_{\mathcal{R}_j}(G) \cup S$$

as required. □

We shall now define the notion of weak solution for the refractor problem.

The notion of weak solutions is defined through energy relation as is done in [GH09].

Let μ be a Radon measure defined on the Borel subsets of $\bar{\Omega}^*$.

Definition 3.3. *A refractor $\mathcal{R}$ is a weak solution of the refractor problem for the case $\kappa < 1$ with emitting illumination intensity g on $\bar{\Omega}$ and prescribed refracted illumination intensity μ on $\bar{\Omega}^*$ if*

$$G_{\mathcal{R}}(\omega) = \int_{\mathcal{T}_{\mathcal{R}}(\omega)} g(x) t_{\mathcal{R}}(x) dx = \mu(\omega)$$

for every Borel subset of ω of $\bar{\Omega}^*$.

Remark 3.2. *If $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ is a weak solution of the refractor problem and $c > 0$, the so is $c\mathcal{R} = \{c\rho(x)x : x \in \bar{\Omega}\}$.*

Proof. Indeed if $E(m, b)$ is a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x)x$, then $E(m, cb)$ is a supporting ellipsoid to $c\mathcal{R}$ at $c\rho(x)x$. Thus for any $\omega \subset \bar{\Omega}^*$, $\mathcal{T}_{\mathcal{R}}(\omega) = \mathcal{T}_{c\mathcal{R}}(\omega)$ and $t_{\mathcal{R}}(x) = t_{c\mathcal{R}}(x)$. □

CHAPTER 4

Existence of Weak Solution for the Case $\kappa < 1$ with No Loss of Energy

In this chapter we establish the existence of weak solution to the refractor problem assuming that the interface is an ideal refractor in which all the energy carried by the incident wave will be fully transmitted; that is we assume that $t_{\mathcal{R}}(x) \equiv 1$. Therefore, in this case, we define weak solution to be as follows:

Definition 4.1. *A refractor $\mathcal{R}$ is a weak solution of the refractor problem for the case $\kappa < 1$ with emitting illumination intensity g on $\bar{\Omega}$ and prescribed refracted illumination intensity μ on $\bar{\Omega}^*$ if*

$$G_{\mathcal{R}}(\omega) = \int_{\mathcal{T}_{\mathcal{R}}(\omega)} g(x)dx = \mu(\omega)$$

for every Borel subset of ω of $\bar{\Omega}^*$.

4.1 Existence of Weak Solution when μ equals sum of delta measures

In this section we establish the existence of weak solution to the refractor problem by assuming that all the light rays will be refracted into only finitely many directions.

For the remaining part of this section let $m_1, m_2, \ldots, m_k$, $k \geq 2$ be distinct points in $\bar{\Omega}^*$.

For $\mathbf{b} = (b_1, \ldots, b_k) \in \mathbf{R}^k$ with each $b_i > 0$, we denote by $\mathcal{R}(\mathbf{b})$ the refractor defined by a finite number of semi-ellipsoids and given by

$$\mathcal{R}(\mathbf{b}) = \left\{ \rho(x)x : x \in \bar{\Omega}, \rho(x) = \min_{1 \leq i \leq k} \frac{b_i}{1 - \kappa m_i \cdot x} \right\}. \tag{4.1}$$

Theorem 4.1. *Let $g \in L^1(\bar{\Omega})$ and $\inf_{\bar{\Omega}} g > 0$. Let $f_1, \ldots, f_k$ be positive numbers and μ be a measure defined on Borel subsets of $\bar{\Omega}^*$ by $\mu = \sum_{i=1}^k f_i \delta_i$ where δ_i is a unit dirac measure concentrated at m_i, for $1 \leq i \leq k$. Suppose also the energy conservation condition*

$$\int_{\bar{\Omega}} g(x)dx = \mu(\bar{\Omega}^*) \tag{4.2}$$

is satisfied. Then there exists a refractor $\mathcal{R}$ such that

i. $\bar{\Omega} = \cup_{i=1}^k \mathcal{T}_{\mathcal{R}}(m_i)$

ii. $G_{\mathcal{R}}(m_i) := \int_{\mathcal{T}_{\mathcal{R}}(m_i)} g(x)dx = f_i$ *for* $1 \leq i \leq k$.

Remark 4.1. *If g is as in Theorem 4.1 and $\mu = f_1 \delta_{m_1}$ with $f_1 > 0$ then a solution $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ of the refractor problem with emitting illumination intensity g and refracted illumination intensity μ equals $E(m_1, b_1)$ for some $b_1 > 0$ all over $\bar{\Omega}$.*

Proof. Let g be as in Theorem 4.1 and $\mu = f_1 \delta_{m_1}$ with $f_1 > 0$. Let $\mathcal{R} = \{\rho(x)x : x \in \Omega\}$ be a solution of the refractor problem with emitting illumination intensity g on $\bar{\Omega}$ and refracted illumination intensity μ on $\bar{\Omega}^*$. We claim that

$$\mathcal{N}_{\mathcal{R}}(\bar{\Omega}) = \{m_1\}. \tag{4.3}$$

Let $x_o \in \bar{\Omega}$ and $E(m_o, b_o)$, $b_o > 0$, be a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_o)x_o$ and S be the singular set of $\mathcal{R}$.

If $x_o \notin S$ then $m_o = m_1$. To prove this suppose the contrary that $m_o \neq m_1$. Then we notice that there exists a neighborhood $\mathcal{O} \subset \bar{\Omega}$ of x_o such that $m_1 \notin \mathcal{N}_{\mathcal{R}}(\mathcal{O})$. Indeed if not, for each n and for the ball $B_{1/n}(x_o)$ there exists a $b > 0$ and $x_n \in B_{1/n}(x_o)$ such that $E(m_1, b)$ is supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_n)x_n$ for all n. Therefore for all n

$$\rho(x_n) = \frac{b}{1 - \kappa x_n \cdot m_1}$$

and by continuity of ρ

$$\rho(x_o) = \frac{b}{1 - \kappa x_o \cdot m_1}.$$

Thus $E(m_1, b)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$. Since $x_o \in \bar{\Omega} \setminus S$ this is a contradiction. Therefore there exists a neighborhood $\mathcal{O}$ of x_o such that $m_1 \notin \mathcal{N}_{\mathcal{R}}(\mathcal{O})$.

Consequently,

$$\int_{\mathcal{O}} g(x)dx \leq \int_{\mathcal{T}_{\mathcal{R}}(\Omega^* \setminus \{m_1\})} g(x)dx = \mu(\Omega^* \setminus \{m_1\}) = 0$$

which is a contradiction to the fact that $\inf_{\bar{\Omega}} g > 0$.

Thus

$$\mathcal{N}_{\mathcal{R}}(\Omega \setminus S) = \{m_1\}. \tag{4.4}$$

However, this implies that $\mathcal{R}$ equals $E(m_1, b_0)$ for some $b_0 > 0$ all over $\bar{\Omega}$. Indeed, if $x, y \in \bar{\Omega} \setminus S$, then there are two supporting ellipsoids $E(m_1, b_x)$, $E(m_1, b_y)$ and from the inequalities of the defining functions we get that $b_x = b_y$. So for each $x \in \bar{\Omega} \setminus S$, there is a unique supporting ellipsoid $E(m_1, b_0)$. This means that $\mathcal{R}$ equals $E(m_1, b_0)$ in $\bar{\Omega} \setminus S$. Since $|S| = 0$ the set $\bar{\Omega} \setminus S$ is dense in $\bar{\Omega}$ and since both $\mathcal{R}$ and the ellipsoid are continuous functions they must be equal in all $\bar{\Omega}$. □

To prove Theorem 4.1, we first prove the following Lemmas.

Lemma 4.1. *Let $W \subset \mathbf{R}^k$ be given by $W = \{\boldsymbol{b} = (1, b_2, \ldots, b_k) : b_i > 0\}$ and for all $\boldsymbol{b} \in W$, $\mathcal{R}(\boldsymbol{b})$ satisfies*

$$G_{\mathcal{R}(\boldsymbol{b})}(m_i) = \int_{\mathcal{T}_{\mathcal{R}(\boldsymbol{b})}(m_i)} g(x)dx \leq f_i \qquad \text{for } i = 2, \ldots, k.$$

where $f_i's$, g, and, $m_i's$ are as in Theorem(4.1). Then:

i. $W \neq \emptyset$.

ii. If $\boldsymbol{b} = (1, b_2, \ldots, b_k) \in W$, *then* $\dfrac{1}{1+\kappa} \leq b_i$, *for* $i = 2, \ldots, k$.

iii. For $i = 1, \ldots, k$, $G_{\mathcal{R}(\boldsymbol{b})}(m_i)$ *is continuous on W with respect to the Euclidean norm.*

Proof. (i.) If for some $i \neq 1$, $E(m_i, b_i)$, is a supporting semi-ellipsoid to $\mathcal{R}(\mathbf{b})$ at $\rho(x)x$ then

$$\frac{b_i}{1-\kappa^2} \leq \frac{b_i}{1-\kappa m_i \cdot x} = \rho(x) \leq \frac{1}{1-\kappa m_1 \cdot x} \leq \frac{1}{1-\kappa}$$

and therefore $b_i \leq 1 + \kappa$.

We claim that if for $i \neq 1$, $b_i > 1 + \kappa$, then $\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i) \subset S$, where S is the singular set of $\mathcal{R}(\mathbf{b})$. Indeed if $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)$ there exist $b > 0$ such that $E(m_i, b)$ supports $\mathcal{R}$ at $\rho(x_o)x_o$. Thus

$$\rho(x) = \min_{1 \leq i \leq k} \frac{b_i}{1-\kappa m_i \cdot x} \quad \text{and} \quad \rho(x) \leq \frac{b}{1-\kappa m_i \cdot x}$$

with

$$\rho(x_o) = \frac{b}{1-\kappa m_i \cdot x_o}.$$

Then

$$\frac{b}{1-\kappa m_i \cdot x_o} = \rho(x_o) \leq \frac{b_i}{1-\kappa m_i \cdot x_o}$$

and hence $b \leq b_i$. If $b = b_i$, then $E(m_i, b_i)$ is a supporting semi-ellipsoid to $\mathcal{R}$, and this is a contradiction. Thus $b < b_i$. We then have

$$\rho(x) \leq \frac{b}{1-\kappa m_i \cdot x} < \frac{b_i}{1-\kappa m_i \cdot x} \quad \text{for all} \quad x \in \bar{\Omega}$$

and hence

$$\rho(x) = \min_{j \neq i} \frac{b_j}{1 - \kappa m_j \cdot x}.$$

In particular there exists $l \neq i$ such that at x_o,

$$\rho(x_o) = \frac{b_l}{1 - \kappa m_l \cdot x_o} \quad \text{and} \quad \rho(x) \leq \frac{b_l}{1 - \kappa m_l \cdot x} \quad \text{for all} \quad x \in \bar{\Omega}.$$

Thus $x_o \in S$ and the claim is proved. Consequently $G_{\mathcal{R}(\mathbf{b})}(m_i) \leq \int_S g(x)dx < f_i$.

Now choose $\mathbf{b} = (1, b_2, \ldots, b_k)$ satisfying $b_i > 1 + \kappa$ for $2 \leq i \leq k$. Then $\mathbf{b} \in W$ and we conclude that $W \neq \emptyset$.

(ii.) Let $\mathbf{b} = (1, b_2, \ldots, b_k) \in W$. First notice that $G_{\mathcal{R}(\mathbf{b})}(m_1) \geq f_1$. Indeed, since

$$\begin{aligned} \sum_{i=1}^{k} G_{\mathcal{R}(\mathbf{b})}(m_i) &= \sum_{i=1}^{k} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} g(x)dx \\ &= \int_{\bigcup_{i=1}^{k} \mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} g(x)dx = \int_{\bar{\Omega}} g(x)dx = \sum_{i=1}^{k} f_i, \end{aligned}$$

we have

$$[f_1 - G_{\mathcal{R}(\mathbf{b})}(m_1)] + \sum_{i=2}^{k} [f_i - G_{\mathcal{R}(\mathbf{b})}(m_i)] = 0.$$

But $f_i - G_{\mathcal{R}(\mathbf{b})}(m_i) \geq 0$ for $i = 2, \ldots, k$. Therefore $G_{\mathcal{R}(\mathbf{b})}(m_1) \geq f_1$.

We claim that there is a point x_o in $\bar{\Omega}$ such that $\rho(x_o)x_o \in \mathcal{R}(\mathbf{b}) \cap E(m_1, b_1)$ and $\rho(x_o)x_o \notin \mathcal{R}(\mathbf{b}) \cap E(m_i, b_i)$ for $i \neq 1$. Assume the contrary. Then $\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_1) \subset S$ where S is the singular set of $\mathcal{R}(\mathbf{b})$. But then,

$$G_{\mathcal{R}(\mathbf{b})}(m_1) = \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_1)} g(x)dx \leq \int_S g(x)dx = 0$$

and consequently $f_1 \leq 0$, which is a contradiction.

We then obtain

$$\rho(x_o) = \frac{1}{1 - \kappa m_1 \cdot x_o} < \frac{b_i}{1 - \kappa m_i \cdot x_o} \qquad \text{for all } i \neq 2,$$

and hence

$$\frac{1}{1 + \kappa} \leq \frac{1 - \kappa m_i \cdot x_o}{1 - \kappa m_1 \cdot x_o} < b_i \tag{4.5}$$

as required.

(iii.) Let $\mathbf{b}^j = (1, b_2^j, \ldots, b_k^j), \quad j \geq 1$ be a sequence in W converging to $\mathbf{b} = (1, b_2, \ldots, b_k)$ in W. Suppose also $\mathcal{R}(\mathbf{b}^j) = \{\rho_j(x)x : x \in \bar{\Omega}\}$ and $\mathcal{R}(\mathbf{b}) = \{\rho(x)x : x \in \bar{\Omega}\}$. If $x_o \in \bar{\Omega}$,

$$\begin{aligned}\rho_j(x_o) - \rho(x_o) &= \rho_j(x_o) - \frac{b_l}{1 - \kappa m_l \cdot x_o} \quad \text{for some } l. \\ &\leq \frac{b_l^j}{1 - \kappa m_l \cdot x_o} - \frac{b_l}{1 - \kappa m_l \cdot x_o} \leq \frac{\|\mathbf{b}^j - \mathbf{b}\|}{1 - \kappa},\end{aligned}$$

thus $\rho_j \to \rho$ uniformly on $\bar{\Omega}$.

Moreover if $x \in \bar{\Omega}$ and $j \geq 1$, there exists i_o such that by 4.5,

$$\frac{1}{1+\kappa} \leq \frac{b_{i_o}^j}{1 - \kappa m_{i_o} \cdot x} = \rho_j(x) = \min_{1 \leq i \leq k} \frac{b_i^j}{1 - \kappa m_i \cdot x} \leq \frac{M}{1 - \kappa}$$

for some M such that $\|\mathbf{b}^j\| \leq M$. Then by Lemma (3.6)

$$\int_{\overline{lim}_{j \to \infty} \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x)dx \leq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} g(x)dx = G_{\mathcal{R}(\mathbf{b})}(m_i).$$

Thus by reverse Fatou's Lemma,

$$\limsup_{j \to \infty} G_{\mathcal{R}(\mathbf{b}_j)}(m_i) \leq G_{\mathcal{R}(\mathbf{b})}(m_i). \tag{4.6}$$

Also by Lemma (3.6) for any open set $G \subset \bar{\Omega}^*$, we have

$$\liminf_{j \to \infty} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^j)}(G)} g(x)dx \geq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(G)} g(x)dx.$$

In particular if we choose G to be a neighborhood of m_i such that $m_l \notin G$ for $l \neq i$, then $\mathcal{T}_{\mathcal{R}(\mathbf{b}^j)}(G \setminus \{m_i\}) \subset S(\mathbf{b}^j)$ where $S(\mathbf{b}^j)$ is the singular set of $\mathcal{R}(\mathbf{b}^j)$, and consequently,

$$\liminf_{j \to \infty} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^j)}(m_i)} g(x)dx \geq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} g(x)dx. \tag{4.7}$$

Combining (4.6) and (4.7) we obtain

$$G_{\mathcal{R}(\mathbf{b})}(m_i) = \lim_{j \to \infty} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^j)}(m_i)} g(x)dx$$

and thus conclude that $G_{\mathcal{R}(b)}(m_i)$ is continuous on W. □

Lemma 4.2. *Let W be as in the previous Lemma and let $\boldsymbol{b}_o = (1, b_2^o, \ldots, b_k^o) \in W$. The set*

$$W_o = \{\boldsymbol{b} = (1, b_2, \ldots, b_k) \in W : b_i \leq b_i^o \quad \text{for } i = 2, \ldots, k\}$$

is compact.

Proof. By definition and Lemma (4.1) it is clear that W_o is bounded. Let $\{\mathbf{b}^j\}_{j=1}^{\infty}$ be a sequence in W_o converging to $\mathbf{b}' = (b_1', b_2', \ldots, b_k')$. Since

$$G_{\mathcal{R}(\mathbf{b}^j)}(m_i) \leq f_i,$$

for $i = 2, \ldots, k$, and $G_{\mathcal{R}(\mathbf{b}^j)}(m_i)$ is continuous with respect to $\mathbf{b}$, by Lemma (4.1, iii) we have $G_{\mathcal{R}(\mathbf{b}')}(m_i) \leq f_i$. Thus $\mathbf{b}' \in W$. Clearly $b_1' = 1$ and $b_i' \leq b_i^o$ for $i = 2, \ldots, k$, and $\mathbf{b}' \in W_o$. □

We shall now prove Theorem (4.1).

Proof. Proof of Theorem (4.1)
Define

$$d : W_o \to \mathbf{R}$$

by

$$\mathbf{b} = (1, b_2, \ldots, b_k) \mapsto \sum_{i=1}^{k} b_i.$$

Let

$$\mathbf{b}^* = (1, b_2^*, \ldots, b_k^*) = arg \min_{\mathbf{b} \in W_o} d(\mathbf{b}).$$

Claim: $\mathcal{R}(\mathbf{b}^*)$ is a solution declared in the theorem.

Suppose the contrary. Without loss of generality assume that $G_{\mathcal{R}(\mathbf{b}^*)}(m_2) < f_2$. Take λ, $0 < \lambda < 1$ and $\mathbf{b}_\lambda^* = (1, \lambda b_2^*, \ldots, b_k^*)$. Let $\mathcal{R}(\mathbf{b}^*) = \{\rho^*(x)x : x \in \bar{\Omega}\}$ and $\mathcal{R}(\mathbf{b}_\lambda^*) = \{\rho_\lambda^*(x)x : x \in \bar{\Omega}\}$, and S be the singular set of $\mathcal{R}(\mathbf{b}_\lambda^*)$. If $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}_\lambda^*)}(m_i)$ and $x_o \notin S$ then for $i \neq 2$,

$$\rho_\lambda^*(x_o) = \frac{b_i^*}{1 - \kappa m_i \cdot x_o} \quad \text{and} \quad \rho_\lambda^*(x) \leq \frac{b_i^*}{1 - \kappa m_i \cdot x} \quad \forall x \in \bar{\Omega}.$$

and consequently

$$\rho^*(x_o) = \frac{b_i^*}{1 - \kappa m_i \cdot x_o} \quad \text{and} \quad \rho^*(x) \leq \frac{b_i^*}{1 - \kappa m_i \cdot x} \qquad \forall x \in \bar{\Omega}$$

and therefore $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)$.

Thus $\mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i) \setminus S \subset \mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)$ for $i \neq 2$ and hence,

$$G_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i) \leq G_{\mathcal{R}(\mathbf{b}^*)}(m_i) \leq f_i \quad \text{for} \quad i \neq 2.$$

Also by Lemma (4.1, iii), we can choose λ sufficiently close to 1 so that $G_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_2) < f_2$. Thus we found $\mathbf{b}^*_\lambda \in W_o \subset W$ with $d(\mathbf{b}^*_\lambda) < d(\mathbf{b}^*)$ which is a contradiction.

This completes the proof of the theorem.

Corollary 4.1. *Let g and μ be as given in Theorem 4.1 satisfying the relation 4.2. Then there exists a weak solution; having the form 4.1, of the refractor problem for the case $\kappa < 1$, with emitting illumination intensity g and prescribed refracted illumination intensity μ.*

□

4.2 Existence of weak solution when μ is a finite Radon measure

In this section we will prove the existence of weak solution for the refractor problem by lifting the assumption that the number of directions in which light ray is refracted is finite. First we prove the following:

Lemma 4.3. *Let $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ be a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$ such that* $\inf_{x\in\bar{\Omega}} \rho(x) = 1$. *Then there is a constant C such that*

$$\sup_{x\in\bar{\Omega}} \rho(x) \leq C.$$

Proof. Suppose $\inf_{x\in\bar{\Omega}} \rho(x)$ is attained at $x_o \in \bar{\Omega}$, and let $E(m_o, b_o)$ be a supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_o)x_o$. Then

$$1 = \rho(x_o) = \frac{b_o}{1 - \kappa m_o \cdot x_o} \quad \text{and} \quad \rho(x) \leq \frac{b_o}{1 - \kappa m_o \cdot x_o} \quad \forall x \in \bar{\Omega}.$$

From the first equation we get that $b_o = 1 - \kappa m_o \cdot x_o \leq 1 + \kappa$ and using this in the inequality we obtain

$$\rho(x) \leq \frac{1+\kappa}{1-\kappa^2} \quad \text{for all } x \in \bar{\Omega}$$

and this proves the lemma. □

Theorem 4.2. *Let $g \in L^1(\bar{\Omega})$ with $\inf_{\bar{\Omega}} g > 0$, and let μ be a Radon measure on $\bar{\Omega}^*$, such that*

$$\int_{\bar{\Omega}} g(x)dx = \mu(\bar{\Omega}^*).$$

Then, there exists a weak solution $\mathcal{R}$ of the refractor problem for the case $\kappa < 1$, with emitting illumination intensity g and prescribed refracted illumination intensity μ.

Proof. To begin the proof fix $l \in \mathbf{N}, l \geq 2$. Partition $\bar{\Omega}^*$ in to a finite number of disjoint Borel subsets $\omega_1^l, \ldots, \omega_{k_l}^l$ such that $diam(\omega_i^l) \leq \frac{1}{l}$. Choose points $m_i^l \in \omega_i^l$ and define a measure on $\bar{\Omega}^*$

$$\mu_l = \sum_{i=1}^{k_l} \mu(\omega_i^l)\delta_{m_i^l}.$$

Then

$$\mu_l(\bar{\Omega}^*) = \sum_{i=1}^{k_l} \mu(\omega_i^l) = \mu(\bar{\Omega}^*) = \int_{\bar{\Omega}} g(x)dx.$$

If $h \in C(\bar{\Omega}^*)$, then

$$\begin{aligned}
\int_{\bar{\Omega}^*} h d\mu_l - \int_{\bar{\Omega}^*} h d\mu &= \sum_{i=1}^{k_l} \int_{\bar{\Omega}^*} h\mu(\omega_i^l) d\delta_{m_i^l} - \int_{\bar{\Omega}^*} h(x)d\mu \\
&= \sum_{i=1}^{k_l} \int_{\omega_i^l} h(m_i^l)d\mu - \sum_{i=1}^{k_l} \int_{\omega_i^l} h(x)d\mu \\
&= \sum_{i=1}^{k_l} \int_{\omega_i^l} (h(m_i^l) - h(x))d\mu.
\end{aligned}$$

Since $h \in C(\bar{\Omega}^*)$ and $diam(\omega_i^l) < \dfrac{1}{l}$, we obtain that

$$\int_{\bar{\Omega}^*} h d\mu_l \to \int_{\bar{\Omega}^*} h d\mu \quad \text{as } l \to \infty$$

and hence μ_l *converges weakly* to μ.

Moreover by Theorem 4.1, corresponding to μ_l, there exists a refractor $\mathcal{R}_l = \{\rho_l(x)x : x \in \bar{\Omega}\}$, satisfying

$$G_{\mathcal{R}_l}(\omega) = \mu_l(\omega)$$

for every Borel subset ω of $\bar{\Omega}^*$. Normalize ρ_l so that $\inf_{\bar{\Omega}} \rho_l(x) = 1$. Then by Lemma (4.3) there exists a uniform bound C such that

$$\sup_{x \in \bar{\Omega}} \rho_l(x) \le C \qquad \text{for all } l \ge 1.$$

Also if $x_o, x_1 \in \bar{\Omega}$ and $E(m_o, b_o)$ is a supporting semi ellipsoid to $\mathcal{R}_l$ at $\rho_l(x_o)x_o$ then

$$\begin{aligned} \rho_l(x_1) - \rho_l(x_o) &\le \frac{b_o}{1 - \kappa m_o \cdot x_1} - \frac{b_o}{1 - \kappa m_o \cdot x_o} \\ &= \frac{\kappa m_o}{1 - \kappa m_o \cdot x_1} \frac{b_o}{1 - \kappa m_o \cdot x_o} \|x_1 - x_o\| \\ &\le \frac{C}{1 - \kappa} \|x_1 - x_o\|. \end{aligned}$$

By changing the roles of x_o and x_1 we conclude that

$$|\rho_l(x_1) - \rho_l(x_o)| \le \frac{C}{1 - \kappa} \|x_1 - x_o\| \text{ for all } \quad l \ge 1.$$

Thus $\{\rho_l : l \ge 1\}$ is an equicontinuous family which is bounded uniformly. By *Arzelà - Ascoli* Theorem, if need be by taking subsequence, we conclude that $\rho_l \to \rho$ uniformly on $\bar{\Omega}$. By Lemma 3.6, i, $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ is a refractor.

We claim that $G_{\mathcal{R}_l} \rightharpoonup G_{\mathcal{R}}$. Indeed if F is any closed subset of $\bar{\Omega}^*$ then by Lemma (3.6, ii) and reverse Fatou's Lemma we have

$$\limsup_{l \to \infty} G_{\mathcal{R}_l}(F) \le \int_{\limsup_{l \to \infty} \mathcal{T}_{\mathcal{R}_l}(F)} g(x) dx \le \int_{\mathcal{T}_{\mathcal{R}}(F)} g(x) dx = G_{\mathcal{R}}(F).$$

But then by Portmanteau's Theorem [Bi99] $G_{\mathcal{R}_l}$ converges weakly to $G_{\mathcal{R}}$. Since $G_{\mathcal{R}_l} = \mu_l$ and the weak limit is unique $G_{\mathcal{R}}(\omega) = \mu(\omega)$ and this completes the proof. □

CHAPTER 5

Existence of Weak Solution for the Case $\kappa < 1$ with Loss of Energy

In this chapter we will consider the more general refractor problem. That is, we shall take into account the physical fact that the energy carried by the incident wave will be divided between the energies of the reflected and transmitted waves and therefore $t_{\mathcal{R}}(x)$ is not identity. Throughout this chapter we assume that there exists $\epsilon > 0$ such that $\bar{\Omega}$ and $\bar{\Omega}^*$ satisfy the geometric constraint

$$\inf_{m \in \bar{\Omega}^*, x \in \bar{\Omega}} m \cdot x \geq \kappa + \epsilon. \tag{5.1}$$

First we prove some properties of Fresnel coefficients.

5.1 Properties of Fresnel coefficients

Proposition 5.1. *If $\bar{\Omega}$ and $\bar{\Omega}^*$ satisfy the geometric condition 5.1 and $\mathcal{R}$ is a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$, then there exists $\epsilon > 0$ and $C(\kappa, \epsilon) > 0$ such that $C(\kappa, \epsilon) < t_{\mathcal{R}}(x) < 1$ for all $x \in \bar{\Omega}$.*

Proof. Recall that $t_{\mathcal{R}}(x) = 1 - r_{\mathcal{R}}(x)$ and $r_{\mathcal{R}}(x) = \phi(x \cdot m)$ where ϕ is given by

$$\phi(t) = \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2 \alpha + \left[1 - 2\kappa\, t + \kappa^2\right]^2 \beta\right),$$

where $\alpha = \dfrac{\mathrm{A}_\parallel^2}{\mathrm{A}_\parallel^2 + \mathrm{A}_\perp^2}$, and $\beta = \dfrac{\mathrm{A}_\perp^2}{\mathrm{A}_\parallel^2 + \mathrm{A}_\perp^2}$.

To prove the proposition, it suffices to show that there exists a constant C_ϵ, $0 < C_\epsilon < 1$ such that

$$r_{\mathcal{R}}(x) \leq C_\epsilon, \qquad \text{for all } \kappa + \epsilon \leq x \cdot m \leq 1. \tag{5.2}$$

This is equivalent to proving that

$$\phi(t) \leq C_\epsilon, \qquad \text{for all } \kappa + \epsilon \leq t \leq 1.$$

Set

$$g(t) = \left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2, \qquad h(t) = \left[1 - 2\kappa\, t + \kappa^2\right]^2.$$

We have $g'(t) = -4\kappa\left[\dfrac{2\kappa}{t} - (1+\kappa^2)\right]\dfrac{1}{t^2}$, so $g'(t) > 0$ for $t > \dfrac{2\kappa}{1+\kappa^2}$, and $g'(t) < 0$ for $t < \dfrac{2\kappa}{1+\kappa^2}$. Since $\kappa < 1$, we have $\kappa + \epsilon < \dfrac{2\kappa}{1+\kappa^2} < 1$ for $\epsilon > 0$ small. Therefore, g decreases in the interval $[\kappa+\epsilon, \dfrac{2\kappa}{1+\kappa^2}]$, and g increases in the interval $[\dfrac{2\kappa}{1+\kappa^2}, 1]$. Hence

$$\max_{[\kappa+\epsilon,1]} g(t) = \max\{g(\kappa+\epsilon), g(1)\}.$$

We have that $g(1) = (1-\kappa)^4$, and $g(\kappa+\epsilon) > g(1)$ for ϵ small, so

$$\max_{[\kappa+\epsilon,1]} g(t) = g(\kappa+\epsilon).$$

On the other hand, $h'(t) = -4\kappa\left[1 - 2\kappa\, t + \kappa^2\right]$, and so $h'(t) > 0$ for $t > \dfrac{1+\kappa^2}{2\kappa}$ and $h'(t) < 0$ for $t < \dfrac{1+\kappa^2}{2\kappa}$. Since $\dfrac{1+\kappa^2}{2\kappa} > 1$, the function h is decreasing in the interval $[\kappa+\epsilon, 1]$ and so

$$\max_{[\kappa+\epsilon,1]} h(t) = h(\kappa+\epsilon).$$

Therefore we obtain that

$$\max_{[\kappa+\epsilon,1]} \phi(t) \leq \frac{1}{(1-\kappa^2)^2} \left(\alpha\, g(\kappa+\epsilon) + \beta\, h(\kappa+\epsilon)\right).$$

It is easy to see that

$$g(\kappa+\epsilon) < (1-\kappa^2)^2, \quad \text{and} \quad h(\kappa+\epsilon) < (1-\kappa^2)^2$$

and so we obtain the bound

$$\max_{[\kappa+\epsilon,1]} \phi(t) \leq C_\epsilon < 1,$$

with

$$C_\epsilon = \max\left\{\frac{g(\kappa+\epsilon)}{(1-\kappa^2)^2}, \frac{h(\kappa+\epsilon)}{(1-\kappa^2)^2}\right\}$$

independent of α and β. □

Lemma 5.1. *Suppose $\mathcal{R}_j$ and $\mathcal{R}$ are refractors with defining functions $\rho_j(x)$ and $\rho(x)$ and corresponding Fresnel coefficients t_j and t, respectively. Suppose $\rho_j \to \rho$ pointwise in $\bar{\Omega}$ and there exist positive constants C_1 and C_2 such that $C_1 \leq \rho_j(x) \leq C_2$ in $\bar{\Omega}$. Let S be the union of all singular points of all the refractors $\mathcal{R}_j$ and $\mathcal{R}$. Then for each $y \notin S$ there is subsequence $t_{j_l}(y) \to t(y)$ as $l \to \infty$.*

Proof. Given $y \notin S$ and j, there exist $b_j > 0$ and $m_j \in \bar{\Omega}^*$ such that $\rho_j(z) \leq \dfrac{b_j}{1-\kappa m_j \cdot z}$ for all $z \in \bar{\Omega}$ with equality at $z = y$. We then have that

$$C_1 \leq \frac{b_j}{1-\kappa m_j \cdot y} \leq C_2$$

and so

$$C_1(1-\kappa) \leq C_1(1-\kappa m_j \cdot y) \leq b_j \leq C_2(1-\kappa m_j \cdot y) \leq C_2(1-\kappa^2),$$

that is, b_j is bounded away from zero and infinity. Therefore there exist subsequences $b_{j_l} \to b > 0$, $m_{j_l} \to m \in \bar{\Omega}^*$, and so the semi-ellipsoid $E(m,b)$ supports $\mathcal{R}$ at y, so $y \in \mathcal{T}_{\mathcal{R}}(m)$. If $y \notin S$, then the normal $\nu_{j_l}(y)$ to the ellipsoid

$E(m_{j_l}, b_{j_l})$ equals the normal to the refractor $\mathcal{R}_{j_l}$ at y, and the normal $\nu(y)$ to the ellipsoid $E(m, b)$ equals the normal to the refractor $\mathcal{R}$ at y. Since $E(m_{j_l}, b_{j_l})$ tends to $E(m, b)$ as $l \to \infty$, it follows that $\nu_{j_l}(y) \to \nu(y)$ for $y \notin S$ as $l \to \infty$ from Lemma 3.4. Consequently $t_{j_l}(y) \to t(y)$ for $y \notin S$. □

Theorem 5.1. *Assume the hypotheses and notation of Lemma 5.1. Let* $F \subset \bar{\Omega}^*$ *compact,* $F_j = \mathcal{T}_{\mathcal{R}_j}(F)$. *Then*

$$\limsup_{j\to\infty} \chi_{F_j}(y) t_j(y) = t(y) \limsup_{j\to\infty} \chi_{F_j}(y), \tag{5.3}$$

$$\liminf_{j\to\infty} \chi_{F_j}(y) t_j(y) = t(y) \liminf_{j\to\infty} \chi_{F_j}(y), \tag{5.4}$$

for all $y \notin S$.

Proof. Let $F^* = \limsup_{j\to\infty} F_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} F_j$. If $y \notin F^*$, then $y \in \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} F_j^c$, that is, $y \notin F_j$ for all $j \geq k$ and hence both sides of (5.3) are zero.

Fix $y \notin S$. If $y \in F^*$, then the right hand side of (5.3) equals $t(y)$. If $\limsup_{j\to\infty} \chi_{F_j}(y) t_j(y) = B(y) > 0$, then there is a subsequence F_{j_l} such that $\lim_{l\to\infty} \chi_{F_{j_l}}(y) t_{j_l}(y) = B(y) > 0$. Hence $y \in F_{j_l}$ for all l sufficiently large. This means $y \in \mathcal{T}_{\mathcal{R}_{j_l}}(F)$ and so $y \in \mathcal{T}_{\mathcal{R}_{j_l}}(m_{j_l})$ for some $m_{j_l} \in F$ for all l sufficiently large. By compactness there is a subsequence of m_{j_l} converging to some $m \in F$, and therefore $y \in \mathcal{T}_{\mathcal{R}}(F)$. Now from Lemma 5.1, there is a subsequence (depending on y) such that $t_{j_{l_k}}(y) \to t(y)$ as $k \to \infty$. Consequently, $B(y) = \lim_{l\to\infty} \chi_{F_{j_{l_k}}}(y) t_{j_{l_k}}(y) = t(y)$, so we obtain (5.3) for $y \in F^*$.

On the other hand, if $y \in F^*$ and $\limsup_{j\to\infty} \chi_{F_j}(y) t_j(y) = 0$, then $\lim_{l\to\infty} \chi_{F_{j_l}}(y) t_{j_l}(y) = 0$ for a subsequence, and $y \in F_{j_l}$ for all l. Therefore $\lim_{l\to\infty} t_{j_l}(y) = 0$. Once again from Lemma 5.1 $\lim_{k\to\infty} t_{j_{l_k}}(y) = t(y) = 0$ for a subsequence. This completes the proof of (5.3).

It remains to prove (5.4). Let $F_* = \liminf_{j\to\infty} F_j = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} F_j$. If $y \notin F_*$, then the right hand side of (5.4) is zero; and $y \in \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} F_j^c$, so there exists a subsequence F_{j_l} with $\chi_{F_{j_l}}(y) = 0$ for all l. Therefore $\lim_{l\to\infty} \chi_{F_{j_l}}(y) t_{j_l}(y) = 0$ and consequently $\liminf_{j\to\infty} \chi_{F_j}(y) t_j(y) = 0$. So both sides of (5.4) are zero. Suppose now that $y \in F_*$. There are two possibilities $\liminf_{j\to\infty} \chi_{F_j}(y) t_j(y) =$

0 or $\liminf_{j\to\infty} \chi_{F_j}(y)t_j(y) > 0$. In the first case, there is a subsequence such that $\lim_{l\to\infty} \chi_{F_{j_l}}(y)t_{j_l}(y) = 0$, and since $y \in F_*$, $\chi_{F_j}(y) = 1$ for j large and so $\lim_{l\to\infty} t_{j_l}(y) = 0$. From Lemma 5.1, $t_{j_{l_k}}(y) \to t(y)$ as $k \to \infty$ and so $t(y) = 0$. In case $\liminf_{j\to\infty} \chi_{F_j}(y)t_j(y) = B(y) > 0$, we have $\liminf_{j\to\infty} t_j(y) = B(y) > 0$ and so $\lim_{l\to\infty} t_{j_l}(y) = B(y)$ and once again from Lemma 5.1, $t_{j_{l_k}}(y) \to t(y) = B(y)$ as $k \to \infty$ and the proof is complete.

□

5.2 Existence of Weak Solution when μ equals sum of delta measures

Let $m_1, m_2, \ldots, m_k$, $k \geq 2$ be distinct points in $\bar{\Omega}^*$.

For $\mathbf{b} = (b_1, \ldots, b_k) \in \mathbf{R}^k$ with each $b_i > 0$, we denote by $\mathcal{R}(\mathbf{b})$ the refractor defined by

$$\mathcal{R}(\mathbf{b}) = \left\{ \rho(x)x : x \in \bar{\Omega}, \rho(x) = \min_{1\leq i\leq k} \frac{b_i}{1 - \kappa m_i \cdot x} \right\}.$$

The following theorem states existence of weak solution for the refractor problem when μ is a linear combination of dirac measures concentrated at $m_1, \ldots, m_k$.

Theorem 5.2. *Let $g \in L^1(\bar{\Omega})$ with $\inf_{\bar{\Omega}} g > 0$,and let $m_1, \ldots, m_k$, $k \geq 2$ be distinct points in $\bar{\Omega}^*$. Let $f_1, \ldots, f_k > 0$ and suppose that μ is a Borel measure defined on $\bar{\Omega}^*$ by $\mu(\omega) = \sum_{i=1}^k f_i \delta_{m_i}(\omega)$ for every Borel subset ω of $\bar{\Omega}^*$. Suppose also that*

$$\int_{\bar{\Omega}} g(x)\, dx \geq \frac{1}{1 - C_\epsilon}\, \mu(\bar{\Omega}^*),$$

where C_ϵ is as defined in proposition 5.1, then there exists a $\boldsymbol{b}_o$ in $\mathbf{R}^k$ and a refractor $\mathcal{R}(\boldsymbol{b}_o)$ such that

$$\int_{\mathcal{T}_{\mathcal{R}(\boldsymbol{b}_o)}(m_i)} g(x) t_{\mathcal{R}(\boldsymbol{b}_o)}(x) dx = f_i$$

for $i = 1, \ldots, k$.

Remark 5.1. *If g is as in Theorem 5.2 and $\mu = f_1\delta_{m_1}$ with $f_1 > 0$ then there is no solution to the refractor problem with emitting illumination intensity g and refracted illumination intensity μ unless it is assumed* a priori *that*

$$\int_{\bar{\Omega}} g(x)t_{\mathcal{R}}(x)dx = f_1.$$

Indeed if $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ is a solution of the refractor problem with emitting illumination intensity g and refracted illumination intensity μ, we can use a similar argument as in Remark 4.1 to show that $\mathcal{R}$ equals $E(m_1, b_1)$ for some $b_1 > 0$ all over $\bar{\Omega}$. Clearly this predetermines the value of $t_{\mathcal{R}}(x)$ and consequently predetermines the value of

$$\int_{\bar{\Omega}} g(x)t_{\mathcal{R}}(x)dx.$$

To prove Theorem (5.2) we first prove the following

Lemma 5.2. *Let $W \subset \mathbf{R}^k$ be the set given as $W = \{\boldsymbol{b} = (1, b_2, \ldots, b_k) : b_i > 0\}$ with the property that for all $b \in W$, $\mathcal{R}(\boldsymbol{b})$ satisfies*

$$G_{\mathcal{R}(\boldsymbol{b})}(m_i) = \int_{\mathcal{T}_{\mathcal{R}(\boldsymbol{b})}(m_i)} g(x)t_{\mathcal{R}(\boldsymbol{b})}(x)dx \leq f_i \quad \text{for all } i = 2, \ldots, k.$$

where f_is and g are as in Theorem (5.2,) then

i. $W \neq \emptyset$.

ii. *If $\boldsymbol{b} = (1, b_2, \ldots, \boldsymbol{b}_k) \in W$ then*

$$\frac{1}{1+\kappa} \leq b_i \tag{5.5}$$

for all $i = 2, \ldots, k$.

Proof. (i.) Similar to the proof of (i) of Lemma 4.1.

(ii.) By using Remark 3.1 we first prove that if $\mathbf{b} \in W$, $f_1 \leq G_{\mathcal{R}(\mathbf{b})}(m_1)$. Indeed,

$$\begin{aligned}\sum_{i=1}^{k} G_{\mathcal{R}(\mathbf{b})}(m_i) &= \sum_{i=1}^{k} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b})}(x) dx \\ &= \int_{\cup_{i=1}^{k} \mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b})}(x) dx \\ &= \int_{\bar{\Omega}} g(x) t_{\mathcal{R}(\mathbf{b})}(x) dx \\ &\geq (1 - C_\epsilon) \int_{\bar{\Omega}} g(x) dx \\ &\geq \mu(\bar{\Omega}^*) = \sum_{i=1}^{k} f_i,\end{aligned}$$

and so we have that

$$[f_1 - G_{\mathcal{R}(\mathbf{b})}(m_1)] + \left[\sum_{i=2}^{k} f_i - G_{\mathcal{R}(\mathbf{b})}(m_i)\right] \leq 0.$$

If $\mathbf{b} \in W$,

$$\sum_{i=2}^{k} f_i - G_{\mathcal{R}(\mathbf{b})}(m_i) \geq 0.$$

Thus $f_1 \leq G_{\mathcal{R}(\mathbf{b})}(m_1)$.

Suppose that $\mathcal{R}(\mathbf{b}) = \{\rho(x)x : x \in \bar{\Omega}\}$. We shall prove that there exist a point $\rho(x_o)x_o$ such that $\rho(x_o)x_o \in \mathcal{R}(b) \cap E(m_1, 1)$ and $\rho(x_o)x_o \notin E(m_i, b_i)$ for all $i \geq 2$. Otherwise $\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_1) \subset S$ where S is the singular set of ρ. But then,

$$G_{\mathcal{R}(\mathbf{b})}(m_1) = \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(m_1)} g(x) t_{\mathcal{R}(\mathbf{b})}(x) dx \leq \int_S g(x) t_{\mathcal{R}(\mathbf{b})}(x) dx = 0$$

contradicting the fact that $f_1 > 0$. Then

$$\rho(x_o) = \frac{1}{1 - \kappa m_1 \cdot x_o} < \frac{b_i}{1 - \kappa m_i \cdot x_o}$$

for all $i = 2, \ldots, k$, from which we conclude that

$$\frac{1}{1 + \kappa} < b_i$$

for all $i = 2, \ldots, k$. $\square$

Lemma 5.3. *Let* $\boldsymbol{b}_j = (b^j_1, \ldots, b^j_k)$ *and* $\boldsymbol{b}_o = (b^o_1, \ldots, b^o_k)$ *with* $\boldsymbol{b}_j \to \boldsymbol{b}_o$ *in* $\mathbf{R}^\mathbf{k}$ *as* $j \to \infty$. *Suppose* $\mathcal{R}_j = \mathcal{R}(\boldsymbol{b}_j) = \{\rho_j(x)x : x \in \bar{\Omega}\}$ *and* $\mathcal{R}_o = \mathcal{R}(\boldsymbol{b}_o) = \{\rho(x)x : x \in \bar{\Omega}\}$. *Then,* $\rho_j \to \rho$ *uniformly on* $\bar{\Omega}$.

Proof. If $x_o \in \bar{\Omega}$,

$$\begin{aligned} \rho_j(x_o) - \rho(x_o) &= \rho_j(x_o) - \frac{b_l}{1-\kappa m_l \cdot x_o} \quad \text{for some } l, \\ &\leq \frac{b^j_l}{1-\kappa m_l \cdot x_o} - \frac{b_l}{1-\kappa m_l \cdot x_o} \leq \frac{\|\mathbf{b}^j - \mathbf{b}\|}{1-\kappa}, \end{aligned}$$

thus $\rho_j \to \rho$ uniformly on $\bar{\Omega}$. □

Theorem 5.3. *Let* $\epsilon > 0$. *Then* $G_{\mathcal{R}(\boldsymbol{b})}(m_i)$ *are continuous on the region* $R_\epsilon = \{(1, b_2, \ldots, b_k) : b_i \geq \epsilon,\ i = 2, \cdots, k\}$, *for all* $1 \leq i \leq k$.

Proof. Let $\mathbf{b}_j,\ j \geq 1$ be a sequence in R_ϵ converging to $\mathbf{b}_o \in R_\epsilon$. Suppose also that $\mathcal{R}(\mathbf{b}_j) = \{\rho_j(x)x : x \in \bar{\Omega}\}$ and $\mathcal{R}(\mathbf{b}_o) = \{\rho_o(x)x : x \in \bar{\Omega}\}$. By Lemma (5.3), $\rho_j \to \rho$ uniformly on $\bar{\Omega}$. Moreover for any $x \in \bar{\Omega}$ and $j \geq 1$, by (5.5),

$$\epsilon \leq \frac{b^j_l}{1-\kappa m_l \cdot x} = \rho_j(x)$$

for some l in $\{1, 2, \ldots, k\}$. Also

$$\rho_j(x) = \min_{1 \leq i \leq k} \frac{b^j_i}{1-\kappa m_i \cdot x} \leq \frac{1}{1-\kappa m_1 \cdot x} \leq \frac{1}{1-\kappa}.$$

We thus obtain $a_1, a_2 > 0$ such that $0 < a_1 \leq \rho_j(x) \leq a_2$. Let $G \subset \bar{\Omega}^*$ be a neighborhood of m_i such that $m_l \notin G$ for $l \neq i$. If $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(G)$ and $x_o \notin S$, there exists a unique $m \in G$ and $b > 0$ such that

$$\rho_j(x_o) = \frac{b}{1-\kappa m \cdot x_o} \quad \text{and} \quad \rho_j(x) \leq \frac{b}{1-\kappa m \cdot x} \quad \text{for all} \quad x \in \bar{\Omega}.$$

But by definition of $\mathcal{R}(\mathbf{b}_j)$, $m = m_l$ for some $l = 1, \ldots, k$. Thus $m = m_i$. From this we conclude that $\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(G) \subset \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i) \cup S$. Combining this with Lemma

(3.6) and the fact that S has measure zero, we obtain

$$\begin{aligned}
&\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(G)} g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \\
\leq\ & \int_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)\cup S} g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \\
\leq\ & \int_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx + \int_S g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \\
=\ & \int_{\bar{\Omega}} \chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \qquad (5.6)
\end{aligned}$$

By applying equation 5.4, to equation (5.6) we get,

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(G)} g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \leq \int_{\bar{\Omega}} \chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)g(x)dx. \qquad (5.7)$$

It is also true that

$$\chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) = \liminf \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x).$$

Using this in (5.7), we obtain

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx \leq \int_{\bar{\Omega}} \liminf \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)g(x)dx$$

from which we deduce by Fatou's lemma that

$$\begin{aligned}
\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx &\leq \liminf \int_{\bar{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)g(x)dx \qquad (5.8) \\
&= \liminf \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x)g(x)dx \qquad (5.9)
\end{aligned}$$

To complete the proof we shall prove that

$$\limsup \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x)g(x)dx \leq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x)t_{\mathcal{R}(\mathbf{b}_o)}(x)dx. \qquad (5.10)$$

First notice that

$$\begin{aligned}
&\limsup \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x)g(x)dx \\
=\ & \limsup \int_{\bar{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)g(x)dx \\
\leq\ & \int_{\bar{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)t_{\mathcal{R}(\mathbf{b}_j)}(x)g(x)dx \qquad (5.11)
\end{aligned}$$

where the last inequality is due to Fatou's Lemma.
By 5.3 and the fact that

$$\limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) = \chi_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)$$

we have

$$\begin{aligned}
& \int_{\bar{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx \\
= & \int_{\bar{\Omega}} \chi_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \\
= & \int_{\bar{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \\
= & \int_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx.
\end{aligned}$$

But then by Lemma(3.6),

$$\int_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \leq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx$$

from which we conclude equation (5.10) and therefore concluding the proof of the theorem. □

We shall now prove Theorem 5.2.

Proof of Theorem 5.2. Fix $\tilde{\mathbf{b}} = (1, \tilde{b}_2, \ldots, \tilde{b}_k)$ in W and consider $\widetilde{W} = \{\mathbf{b} = (1, b_2, \ldots, b_k) \in W : b_i \leq \tilde{b}_i \text{ for } i = 2, \ldots, k\}$. Then $\widetilde{W}$ is compact. Consider the map

$$d : \widetilde{W} \to \mathbf{R}$$

given by

$$d(\mathbf{b}) = \sum_{i=1}^{k} b_i$$

where $\mathbf{b} = (1, b_2, \ldots, b_k)$. Let $\mathbf{b}^* = (1, b_2^*, \ldots, b_k^*)$ be such that

$$\mathbf{b}^* = arg \min_{\mathbf{b} \in \widetilde{W}} d(\mathbf{b}).$$

Note that such a $\mathbf{b}^*$ exists as $\widetilde{W}$ is compact and d is continuous.

Take $\mathbf{b}_o = \mathbf{b}^*$.

Suppose the contrary and without loss of generality that

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(m_2)} g(x) t_{\mathcal{R}(b_o)}(x) < f_2.$$

Take $0 < \lambda < 1$ and consider $\mathbf{b}^*_\lambda = (1, \lambda b^*_2, \ldots, b^*_k)$. Let S^*_λ be the singular set of $\mathcal{R}(\mathbf{b}^*_\lambda)$. If $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i) \setminus S^*_\lambda,\ i \neq 2$, then

$$\rho(x_o) = \frac{b^*_i}{1 - \kappa m_i x_o}, \text{ and } \rho(x) \leq \frac{b^*_i}{1 - \kappa m_i x}$$

for all $x \in \bar{\Omega}$ and therefore $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)$. Hence $\mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i) \setminus S^*_\lambda \subset \mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)$ and consequently

$$\begin{aligned} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}^*_\lambda)}(x) dx &= \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}^*)}(x) dx \\ &\leq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}^*)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}^*)}(x) dx. \end{aligned}$$

Also by using Theorem 5.3, choose λ sufficiently close 1 so that $G_{\mathcal{R}(\mathbf{b}^*_\lambda)}(m_2) < f_2$.

From this we conclude that $\mathbf{b}^*_\lambda \in \widetilde{W}$. But this is a contradiction as $d(\mathbf{b}^*_\lambda) < d(\mathbf{b}^*)$. □

Corollary 5.1. *Let $g \in L^1(\bar{\Omega})$ and let $m_1, \ldots, m_k,\ k \geq 2$ be as given above. Let $f_1, \ldots, f_k > 0$ and suppose that μ is a Borel measure defined on $\bar{\Omega}^*$ by $\mu(\omega) = \sum_{i=1}^k f_i \delta_{m_i}(\omega)$ for every Borel subset ω of $\bar{\Omega}^*$. Suppose also that*

$$\int_{\bar{\Omega}} g(x)\, dx \geq \frac{1}{1 - C_\epsilon}\, \mu(\bar{\Omega}^*),$$

where C_ϵ is as defined in proposition 5.1. Then there exists a weak solution for the refractor problem.

5.3 Existence of weak solution when μ is a finite Radon measure

Theorem 5.4. *Let g be integrable on $\bar{\Omega}$ with $\inf_{\bar{\Omega}} g > 0$. Let μ be a Radon measure on $\bar{\Omega}^*$. Suppose also that*

$$\int_{\bar{\Omega}} g(x)\, dx \geq \frac{1}{1 - C_\epsilon}\, \mu(\bar{\Omega}^*), \tag{5.12}$$

where C_ϵ is as defined in proposition 5.1. Then there exists a refractor $\mathcal{R}$ such that for every Borel set $\omega \subset \bar{\Omega}^*$,

$$\mu(\omega) = \int_{\mathcal{T}_{\mathcal{R}}(\omega)} g(x) t_{\mathcal{R}}(x) dx.$$

i.e. there is a weak solution for the refractor problem for $\kappa < 1$ with emitting illumination intensity g and prescribed refracted intensity μ.

Proof. Let l be an integer, $l \geq 2$. Partition $\bar{\Omega}^*$ into a finite number of disjoint Borel subsets $\omega_1^l, \ldots, \omega_{k_l}^l$ so that *diam* $(\omega_i^l) \leq \dfrac{1}{l}$ for $i = 1, 2, \ldots, k_l$. Choose $m_i^l \in \omega_i^l$, and consider the measure on $\bar{\Omega}^*$ defined by

$$\mu_l = \sum_{i=1}^{k_l} \mu(\omega_i^l) \delta_{m_i^l}.$$

We note that μ_l converges weakly to μ as $l \to \infty$. Moreover by 5.12,

$$\mu_l(\bar{\Omega}^*) = \mu(\bar{\Omega}^*) \leq (1 - C_\epsilon) \int_{\bar{\Omega}} g(x) t_{\mathcal{R}}(x) dx. \tag{5.13}$$

Thus by Theorem 5.2, there exists a refractor

$$\mathcal{R}_l = \left\{ \rho_l(x) x : \rho_l(x) = \min_{1 \leq i \leq k_l} \frac{b_i}{1 - \kappa m_i^l \cdot x} \right\}$$

satisfying

$$\mu_l(\omega) = \int_{\mathcal{T}_{\mathcal{R}_l}(\omega)} g(x) t_{\mathcal{R}_l}(x) dx. \tag{5.14}$$

Normalize $\mathcal{R}_l$ so that $\inf_{\bar{\Omega}} \rho_l(x) = 1$.

Then by Lemma (4.3) there exists C such that

$$\sup_{x \in \bar{\Omega}} \rho_l(x) \leq C \tag{5.15}$$

for all $l \geq 1$.

Also if $x_o, x_1 \in \bar{\Omega}$ and $E(m_o, b_o)$ is a supporting semi ellipsoid to $\mathcal{R}_l$ at $\rho_l(x_o)x_o$ then for $x_1 \in \bar{\Omega}$ we have

$$\begin{aligned}
\rho_l(x_1) - \rho_l(x_o) &\leq \frac{b_o}{1-\kappa m_o x_1} - \frac{b_o}{1-\kappa m_o x_o} \\
&= \frac{\kappa m_o}{1-\kappa m_o x_1}\frac{b_o}{1-\kappa m_o x_o}\|x_1 - x_o\| \\
&\leq \frac{C}{1-\kappa}\|x_1 - x_o\|.
\end{aligned}$$

By changing the roles of x_o and x_1 we conclude that

$$|\rho_l(x_1) - \rho_l(x_o)| \leq \frac{C}{1-\kappa}\|x_1 - x_o\| \text{ for all } \; l \geq 1.$$

Thus $\{\rho_l : l \geq 1\}$ is an equicontinuous family which is bounded uniformly. Then by $Arzel\grave{a} - Ascoli$ Theorem, if need be by taking subsequence, we have that $\rho_l \to \rho$ uniformly on $\bar{\Omega}$. By Lemma (3.6, i) $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ is a refractor.

To prove that μ_l converges weakly also to $\int_{\mathcal{T}_{\mathcal{R}}(\omega)} g(x)t_{\mathcal{R}}(x)dx$ we would like to prove that for any closed subset F of $\bar{\Omega}^*$

$$\limsup \mu_l(F) \leq \int_{\mathcal{T}_{\mathcal{R}}(F)} g(x)t_{\mathcal{R}}(x)dx \tag{5.16}$$

To complete the proof of Theorem 5.4, it now remains to prove 5.16. To that end we combine reverse Fatou's Lemma with equation 5.3 and obtain,

$$\begin{aligned}
\limsup \mu_l(F) &= \limsup \int_{\mathcal{T}_{\mathcal{R}_l}(F)} g(x)t_{\mathcal{R}_l}(x)dx \\
&\leq \int_{\bar{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}_l}(F)}(x)g(x)t_{\mathcal{R}_l}(x)dx \\
&\leq \int_{\mathcal{T}_{\mathcal{R}}(F)} g(x)t_{\mathcal{R}}(x)dx
\end{aligned}$$

completing the proof of the Theorem. □

CHAPTER 6

The problem for the case $\kappa > 1$

In this chapter we shall discuss the existence of weak solution for the refractor problem in the case $n_2/n_1 = \kappa > 1$. That is we assume that n_1 and n_2 are the indices of refraction of two homogeneous and isotropic media, medium I and medium II respectively, where medium II is denser than medium I.

Suppose that Ω and Ω^* are two domains of the unit sphere S^{n-1} of $\mathbf{R}^n$ with the physical property that

$$\inf_{m\in\Omega^*, x\in\Omega} m \cdot x \geq 1/\kappa + \epsilon \tag{6.1}$$

for some $\epsilon > 0$.

The refractor problem in the case $\kappa > 1$ can be solved in a similar manner to the case $\kappa < 1$. The main difference is to use semi-hyperboloids of two sheets $H(m, b)$, defined by 2.21 in place of the semi-ellipsoids $E(m, b)$.

Definition 6.1. *A surface $\mathcal{R}$ in $\mathbf{R}^n$ parameterized by $\rho(x)x$ with $\rho \in C(\bar{\Omega})$ is a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$ for the case $\kappa > 1$ (often simply called a refractor in this chapter) if for any $x_o \in \bar{\Omega}$ there exists a semi-hyperboloid $H(m, b)$, $m \in \Omega^*$ such that $\rho(x_o) = \dfrac{b}{\kappa m \cdot x_o - 1}$ and $\rho(x) \geq \dfrac{b}{\kappa m \cdot x - 1}$ for all $x \in \bar{\Omega}$. We call such $H(m, b)$ a supporting semi-hyperboloid to $\mathcal{R}$ $\rho(x_o)x_o$ or simply at x_o.*

Using a similar argument as in the proof of Lemma 3.1, we can show that $\mathcal{R}$ is Lipschitz in $\overline{\Omega}$. The refractor mapping of $\mathcal{R}$ and the tracing mapping of $\mathcal{R}$ are also defined similarly.

Definition 6.2. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined by for $x_o \in \bar{\Omega}$*

$$\mathcal{N}_{\mathcal{R}}(x_o) = \{m \in \bar{\Omega}^* : H(m,b) \text{ supports } \mathcal{R} \text{ at } \rho(x_o)x_o \text{ for some } b > 0\}.$$

Given $m_o \in \bar{\Omega}^*$ the tracing mapping of $\mathcal{R}$ is defined by

$$\mathcal{T}_{\mathcal{R}}(m_o) = \{x \in \bar{\Omega} : m_o \in \mathcal{N}_{\mathcal{R}}(x)\}.$$

The proofs of the following lemmas are analogous to those of Lemmas 3.2 and 3.3 respectively.

Lemma 6.1. *Let $\mathcal{R}$ be a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$. If $m \in \bar{\Omega}^*$, then $\mathcal{T}_{\mathcal{R}}(m)$ is closed in $\bar{\Omega}$.*

Lemma 6.2.

i. $[\mathcal{T}_{\mathcal{R}}(F)]^c \subset \mathcal{T}_{\mathcal{R}}(F^c)$ *for all* $F \subset \bar{\Omega}^*$.

ii. The set $\mathcal{C} = \{F \subset \bar{\Omega}^* : \mathcal{T}_{\mathcal{R}}(F)$ is Lebesgue measurable$\}$ is a $\sigma-$ algebra containing all Borel sets in $\bar{\Omega}^*$

The following proposition regarding the Fresnel coefficient $t_{\mathcal{R}}(x)$ is proved in a similar way as in proposition 3.1.

Proposition 6.1. *Let $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ be a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$ for the case $\kappa > 1$. Let E be the singular set of ρ. Then the Fresnel coefficient $t_{\mathcal{R}}(x)$ is continuous relative to the set $\bar{\Omega} \setminus E$.*

Proposition 6.2. *If $\bar{\Omega}$ and $\bar{\Omega}^*$ satisfy the geometric condition 6.1 and $\mathcal{R}$ is a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$ for the case $\kappa > 1$, then there exists $\epsilon > 0$ and $C(\kappa, \epsilon) > 0$ such that $C(\kappa, \epsilon) < t_{\mathcal{R}}(x) < 1$ for all $x \in \bar{\Omega}$.*

Proof. Recall that

$$t_{\mathcal{R}}(x) = 1 - r_{\mathcal{R}}(x) \tag{6.2}$$

and $r_{\mathcal{R}}(x) = \phi(x \cdot m)$ where ϕ is given by

$$\phi(t) = \frac{1}{(1-\kappa^2)^2} \left(\left[\frac{2\kappa}{t} - (1+\kappa^2) \right]^2 \alpha + \left[1 - 2\kappa\, t + \kappa^2 \right]^2 \beta \right),$$

where $\alpha = \dfrac{\mathrm{A}_{\parallel}^2}{\mathrm{A}_{\parallel}^2 + \mathrm{A}_{\perp}^2}$, and $\beta = \dfrac{\mathrm{A}_{\perp}^2}{\mathrm{A}_{\parallel}^2 + \mathrm{A}_{\perp}^2}$.

As in proposition 5.1, set

$$g(t) = \left[\frac{2\kappa}{t} - (1+\kappa^2)\right]^2, \qquad h(t) = \left[1 - 2\kappa\, t + \kappa^2\right]^2.$$

For ϵ small we have $\dfrac{1}{\kappa} + \epsilon < \dfrac{2\kappa}{1+\kappa^2} < 1$, so as in the proof of proposition 5.1, g decreases in the interval $[\dfrac{1}{\kappa} + \epsilon, \dfrac{2\kappa}{1+\kappa^2}]$, and g increases in the interval $[\dfrac{2\kappa}{1+\kappa^2}, 1]$. Hence

$$\max_{[(1/\kappa)+\epsilon,1]} g(t) = \max\left\{ g\left(\frac{1}{\kappa} + \epsilon\right), g(1) \right\}.$$

Since now $\kappa > 1$ we have that $g(1) > g\left(\dfrac{1}{\kappa} + \epsilon\right)$, for ϵ small, and so

$$\max_{[(1/\kappa)+\epsilon,1]} g(t) = g(1).$$

Since we always have $\dfrac{1+\kappa^2}{2\kappa} > 1$, the function h is decreasing in the interval $[(1/\kappa)+\epsilon, 1]$ and so

$$\max_{[(1/\kappa)+\epsilon,1]} h(t) = h((1/\kappa)+\epsilon).$$

Therefore we obtain that

$$\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \leq \frac{1}{(1-\kappa^2)^2} \left(\alpha\, g(1) + \beta\, h((1/\kappa)+\epsilon)\right).$$

It is clear that $g(1) = (1-\kappa)^2 < (1-\kappa^2)^2$, and also $h((1/\kappa)+\epsilon) < (1-\kappa^2)^2$ when $0 < \epsilon < 1 - (1/\kappa)$. So we obtain the bound

$$\max_{[(1/\kappa)+\epsilon,1]} \phi(t) \leq C_\epsilon < 1,$$

with

$$C_\epsilon = \max\left\{ \left(\frac{1-\kappa}{1+\kappa}\right)^2, \frac{h((1/\kappa)+\epsilon)}{(1-\kappa^2)^2} \right\}$$

independent of α and β. Choose $C(\kappa,\epsilon) = 1 - C_\epsilon$ and this completes the proof. □

We also have the following Lemma whose proof is similar to that of Lemma 3.4.

Lemma 6.3. *Let $H(m_l, b_l)$ be a sequence of semi-hyperboloids with $m_l \to m$ and $b_l \to b$, as $l \to \infty$. Let $z_l \in H(m_l, b_l)$ with $z_l \to z_0$ as $l \to \infty$. Then $z_0 \in H(m, b)$, and the normal $\nu_l(z_l)$ to the semi-hyperboloid $H(m_l, b_l)$ at z_l satisfies $\nu_l(z_l) \to \nu(z_0)$ the normal to the semi-hyperboloid $H(m, b)$ at the point z_0.*

Proof. In rectangular coordinates the equation of $H(m_l, b_l)$ is $\kappa m_l \cdot z - |z| = b_l$. Then the normal vector at z is $\nu_l(z) = \kappa m_l - \dfrac{z}{|z|}$, and so

$$\nu_l(z_l) = \kappa m_l - \frac{z_l}{|z_l|} \to \kappa m - \frac{z_0}{|z_0|}$$

which is the normal to $H(m, b)$ at z_0. The normal at z written in polar coordinates, i.e., $z = \rho_l(x)x$ has the form $\nu_l(x) = \kappa m_l - x$ so $\nu_l(x) \to \kappa m - x = \nu(x)$, the normal to the hyperboloid $H(m, b)$. □

The proof of the following Lemma goes verbatim with that of 5.1.

Lemma 6.4. *Suppose $\mathcal{R}_j$ and $\mathcal{R}$ are refractors with defining functions $\rho_j(x)$ and $\rho(x)$ and corresponding transmission coefficients t_j and t, respectively. Suppose $\rho_j \to \rho$ pointwise $\bar{\Omega}$ with $C_1 \le \rho_j(x) \le C_2$ in $\bar{\Omega}$ for some positive constants C_1 and C_2. Let S be the union of all singular points of all the refractors $\mathcal{R}_j$ and $\mathcal{R}$. Then for each $y \notin S$ there is subsequence $t_{j_l}(y) \to t(y)$ as $l \to \infty$.*

We can also prove the following Lemma about the refractor measure.

Lemma 6.5. *Let $\mathcal{R}$ be a refractor from $\bar{\Omega}$ to $\bar{\Omega}^*$. Let $g \in L^1(\bar{\Omega})$ with $\inf_{\bar{\Omega}} g > 0$. Define a set function on Borel subsets of $\bar{\Omega}^*$, by*

$$G_{\mathcal{R}}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} g(x) t_{\mathcal{R}}(x) dx$$

where dx is the surface measure on S^{n-1} and $t_{\mathcal{R}}(x)$ is as given by 6.2. Then $G_{\mathcal{R}}$ is a finite Borel measure defined on $\mathcal{C}$. $G_{\mathcal{R}}$ is called the refractor measure associated with $\mathcal{R}$ and g.

Weak solutions of the refractor problem are defined in a similar way.

Definition 6.3. *Let $g \in L^1(\bar{\Omega})$ and μ be a Radon measure on $\bar{\Omega}^*$. A refractor $\mathcal{R}$ is a weak solution of the refractor problem for the case $\kappa > 1$ with emitting illumination intensity $g(x)$ and prescribed refracted illumination intensity μ if for any Borel set $\omega \subset \bar{\Omega}^*$*

$$G_{\mathcal{R}}(\omega) = \int_{\mathcal{T}_{\mathcal{R}}(\omega)} g(x) t_{\mathcal{R}}(x) dx = \mu(\omega).$$

We now discuss the existence of solution for the case $\kappa > 1$.

6.1 Existence of Weak Solution when μ equals sum of delta measures

Let $m_1, m_2, \ldots, m_k$, $k \geq 2$ be distinct points in $\bar{\Omega}^*$. For $\mathbf{b} = (b_1, \ldots, b_k) \in \mathbf{R}^k$ with each $b_i > 0$, we denote by $\mathcal{R}(\mathbf{b})$ the refractor defined by

$$\mathcal{R}(\mathbf{b}) = \left\{ \rho(x)x : x \in \bar{\Omega}, \rho(x) = \max_{1 \leq i \leq k} \frac{b_i}{\kappa m_i \cdot x - 1} \right\}.$$

The proofs of the following Lemmas are similar to that of Lemmas 5.2 and 5.3 respectively.

Lemma 6.6. *Let $W \subset \mathbf{R}^k$ be the set given as $W = \{\boldsymbol{b} = (1, b_2, \ldots, b_k) : b_i > 0\}$ with the property that for all $\boldsymbol{b} \in W$, $\mathcal{R}(\boldsymbol{b})$ satisfies*

$$G_{\mathcal{R}(\boldsymbol{b})}(m_i) = \int_{\mathcal{T}_{\mathcal{R}(\boldsymbol{b})}(m_i)} g(x) t_{\mathcal{R}(\boldsymbol{b})}(x) dx \leq f_i \quad \text{for all } i = 2, \ldots, k.$$

where f_is and g are as in Theorem (6.2), then

i. $W \neq \emptyset$.

ii. If $\boldsymbol{b} = (1, b_2, \ldots, b_k) \in W$ then

$$b_i \leq \frac{\kappa - 1}{\epsilon \kappa} \tag{6.3}$$

for all $i = 2, \ldots, k$.

Lemma 6.7. *Let* $\boldsymbol{b}_j = (b_1^j, \ldots, b_k^j)$ *and* $\boldsymbol{b}_o = (b_1^o, \ldots, b_k^o)$ *with* $\boldsymbol{b}_j \to \boldsymbol{b}_o$ *in* $\mathbf{R}^\mathbf{k}$ *as* $j \to \infty$. *Suppose* $\mathcal{R}_j = \mathcal{R}(\boldsymbol{b}_j) = \{\rho_j(x)x : x \in \bar{\Omega}\}$ *and* $\mathcal{R}_o = \mathcal{R}(\boldsymbol{b}_o) = \{\rho(x)x : x \in \bar{\Omega}\}$. *Then,* $\rho_j \to \rho$ *uniformly on* $\bar{\Omega}$.

We can also prove the following property about the set W in Lemma 6.6.

Theorem 6.1. *Let* W *be as in Lemma (6.6). For* $1 \leq i \leq k$, $G_{\mathcal{R}(\boldsymbol{b})}(m_i)$ *is continuous on* W.

Proof. Let $\mathbf{b}_j,\ j \geq 1$ be a sequence in W converging to $\mathbf{b}_o \in W$. Suppose also that $\mathcal{R}(\mathbf{b}_j) = \{\rho_j(x)x : x \in \bar{\Omega}\}$ and $\mathcal{R}(\mathbf{b}_o) = \{\rho(x)x : x \in \bar{\Omega}\}$. By Lemma (6.7), $\rho_j \to \rho$ uniformly on $\bar{\Omega}$. Moreover for any $x \in \bar{\Omega}$ and $j \geq 1$, by (6.3),

$$\rho_j(x) = \frac{b_l^j}{\kappa m_l \cdot x - 1} \leq \max\left\{\frac{\kappa - 1}{(\epsilon\kappa)^2}, \frac{1}{(\epsilon\kappa)^2}\right\}$$

for some l in $\{1, 2, \ldots, k\}$ and also

$$\frac{1}{\kappa - 1} \leq \frac{1}{\kappa m_1 \cdot x - 1} \leq \rho_j(x) = \max_{1 \leq i \leq k} \frac{b_i^j}{\kappa m_i \cdot x - 1}.$$

We thus obtain $a_1, a_2 > 0$ such that $0 < a_1 \leq \rho_j(x) \leq a_2$. Let $G \subset \bar{\Omega}^*$ be a neighborhood of m_i such that $m_l \notin G$ for $l \neq i$. If $x_o \in \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(G)$ and $x_o \notin S$, there exists a unique $m \in G$ and $b > 0$ such that

$$\rho_j(x_o) = \frac{b}{1 - \kappa m \cdot x_o} \quad \text{and} \quad \rho_j(x) \geq \frac{b}{1 - \kappa m \cdot x} \quad \text{fro all} \quad x \in \bar{\Omega}.$$

But by definition of $\mathcal{R}(\mathbf{b}_j)$, $m = m_l$ for some $l = 1, \ldots, k$. Thus $m = m_i$. From this we conclude that $\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(G) \subset \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i) \cup S$. Combining this with Lemma (3.6) and the fact that S has measure zero, we obtain

$$\begin{aligned} & \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \\ \leq & \int_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i) \cup S} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \\ \leq & \int_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx + \int_S g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \\ = & \int_{\bar{\Omega}} \chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \end{aligned} \tag{6.4}$$

By applying equation 5.4, to equation (6.4) we get,

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b})}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \leq \int_{\bar{\Omega}} \chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx. \tag{6.5}$$

It is also true that

$$\chi_{\liminf \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) = \liminf \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x).$$

Using this in (6.5), we obtain

$$\int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \leq \int_{\bar{\Omega}} \liminf \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx$$

from which we deduce by Fatou's lemma that

$$\begin{aligned} \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx &\leq \liminf \int_{\bar{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx && (6.6) \\ &= \liminf \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx && (6.7) \end{aligned}$$

To complete the proof we shall prove that

$$\limsup \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx \leq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx. \tag{6.8}$$

First notice that

$$\begin{aligned} & \limsup \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx \\ = & \limsup \int_{\bar{\Omega}} \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx \\ \leq & \int_{\bar{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx && (6.9) \end{aligned}$$

where the last inequality is due to Fatou's Lemma.

By 5.3 and the fact that

$$\limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) = \chi_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x)$$

we have

$$\begin{aligned}
& \int_{\bar{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) t_{\mathcal{R}(\mathbf{b}_j)}(x) g(x) dx \\
= & \int_{\bar{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \\
= & \int_{\bar{\Omega}} \chi_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)}(x) g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \\
= & \int_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx.
\end{aligned}$$

But then by Lemma(3.6),

$$\int_{\limsup \mathcal{T}_{\mathcal{R}(\mathbf{b}_j)}(m_i)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx \leq \int_{\mathcal{T}_{\mathcal{R}(\mathbf{b}_o)}(G)} g(x) t_{\mathcal{R}(\mathbf{b}_o)}(x) dx$$

from which we conclude equation (6.8) and therefore completing the proof of the theorem. □

We now state the existence of weak solution for the case $\kappa > 1$ when refraction happens only in finitely many directions.

Theorem 6.2. *Let $g \in L^1(\bar{\Omega})$ with $\inf_{\bar{\Omega}} g > 0$ and let $m_1, \ldots, m_k$, $k \geq 2$ be distinct points in $\bar{\Omega}^*$. Let $f_1, \ldots, f_k > 0$, and μ be a Borel measure defined on $\bar{\Omega}^*$ by $\mu(\omega) = \sum_{i=1}^k f_i \delta_{m_i}(\omega)$ for every Borel subset ω of $\overline{\Omega^*}$. Suppose also that*

$$\int_{\bar{\Omega}} g(x) dx \geq \frac{1}{1 - C_\epsilon} \mu(\bar{\Omega}^*) \tag{6.10}$$

where C_ϵ is as in proposition 6.2. Then there exists a $\boldsymbol{b}_o \in \mathbf{R}^k$ and a refractor $\mathcal{R}(\boldsymbol{b}_o)$ such that

$$\int_{\mathcal{T}_{\mathcal{R}(\boldsymbol{b}_o)}(m_i)} g(x) t_{\mathcal{R}(\boldsymbol{b}_o)}(x) dx = f_i$$

for $i = 1, \ldots, k$.

The proof is analogous to the proof of theorem 5.2.

Corollary 6.1. *Let $g \in L^1(\bar{\Omega})$ with $\inf_{\bar{\Omega}} g > 0$ and let $m_1, \ldots, m_k$ $k \geq 2$ be distinct points in $\bar{\Omega}^*$. Let $f_1, \ldots, f_k > 0$ and suppose that μ is a Borel measure* *defined on* $\bar{\Omega}^*$ *and given by* $\mu(\omega) = \sum_{i=1}^{k} f_i \delta_{m_i}(\omega)$ *for every Borel subset* ω *of* $\bar{\Omega}^*$. *Suppose also that*

$$\int_{\bar{\Omega}} g(x)\, dx \geq \frac{1}{1 - C_\epsilon}\, \mu(\bar{\Omega}^*),$$

where C_ϵ *is as defined in proposition 6.2. Then there exists a weak solution for the refractor problem.*

6.2 Existence of weak solution when μ is a finite Radon measure

To prove the existence theorem for the case when μ is a finite nonnegative Radon measure we first state the following lemma which is the counterpart of Lemma 4.3.

Lemma 6.8. *Let* $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ *be any refractor from* $\bar{\Omega}$ *to* $\bar{\Omega}^*$ *such that* $\inf_{x \in \bar{\Omega}} \rho(x) = 1$. *Then there is a constant* C *such that*

$$\sup_{x \in \bar{\Omega}} \rho(x) \leq C.$$

Proof. Let $H(m_o, b_o)$ be the supporting hyperboloid of $\mathcal{R}$ at $\rho(x_o)x_o$ where x_o is given by

$$\rho(x_o) = \sup_{x \in \bar{\Omega}} \rho(x).$$

Then

$$\rho(x_o) = \frac{b_o}{\kappa m_o \cdot x_o - 1} \quad \text{and} \quad \rho(x) \geq \frac{b_o}{\kappa m_o \cdot x - 1} \quad \forall x \in \bar{\Omega}.$$

Since

$$\frac{b_o}{\kappa - 1} \leq \frac{b_o}{\kappa m_o \cdot x - 1}$$

for all $x \in \bar{\Omega}$,

$$\frac{b_o}{\kappa - 1} \leq \inf_{x \in \bar{\Omega}} \frac{b_o}{\kappa m_o x - 1} \leq \inf_{x \in \bar{\Omega}} \rho(x) = 1.$$

Thus $b_o < \kappa - 1$ and

$$\rho(x_o) < \frac{\kappa - 1}{\epsilon \kappa}$$

as required. □

We now state and prove the main existence theorem.

Theorem 6.3. *Let g be integrable on Ω with $\inf_{\bar{\Omega}} g > 0$ and μ be a Radon measure on $\bar{\Omega}^*$. Suppose also that*

$$\int_{\bar{\Omega}} g(x)dx \geq \frac{1}{1-C_\epsilon}\mu(\bar{\Omega}^*) \tag{6.11}$$

where C_ϵ is as in proposition 6.2. Then there exists a weak solution for the refractor problem for $\kappa > 1$ with emitting illumination intensity g and prescribed refracted intensity μ. i.e. there exists a refractor $\mathcal{R}$ such that for every Borel set $\omega \subset \bar{\Omega}^*$,

$$\mu(\omega) = \int_{\mathcal{T}_{\mathcal{R}}(\omega)} g(x)t_{\mathcal{R}}(x)dx.$$

Proof. Let l be an integer, $l \geq 2$. Partition $\bar{\Omega}^*$ into a finite number of disjoint Borel subsets $\omega_1^l, \ldots, \omega_{k_l}^l$ so that *diam* $(\omega_i^l) \leq \dfrac{1}{l}$ for $i = 1, 2, \ldots, k_l$. Choose $m_i^l \in \omega_i^l$, and consider the measure on $\bar{\Omega}^*$ defined by

$$\mu_l = \sum_{i=1}^{k_l} \mu(\omega_i^l)\delta_{m_i^l}.$$

We note that μ_l converges weakly to μ as $l \to \infty$. By 6.11, we see that μ_l satisfies 6.10. Thus by Theorem 6.2, there exist a refractor

$$\mathcal{R}_l = \left\{\rho_l(x)x : \rho_l(x) = \max_{1\leq i\leq k_l} \frac{b_i}{\kappa m_i^l \cdot x - 1}\right\}$$

satisfying

$$\mu_l(\omega) = \int_{\mathcal{T}_{\mathcal{R}_l}(\omega)} g(x)t_{\mathcal{R}_l}(x)dx. \tag{6.12}$$

Normalize $\mathcal{R}_l$ so that $\inf_{\bar{\Omega}} \rho_l(x) = 1$.
Then by Lemma (6.8) there exists C such that

$$\sup_{x\in\bar{\Omega}} \rho_l(x) \leq C \tag{6.13}$$

for all $l \geq 1$.

Also if $x_o, x_1 \in \bar{\Omega}$ and $H(m_o, b_o)$ is a supporting hyperboloid to $\mathcal{R}_l$ at $\rho_l(x_o)x_o$ then we have

$$\begin{aligned} \rho_l(x_o) - \rho_l(x_1) &\leq \frac{b_o}{\kappa m_o \cdot x_o - 1} - \frac{b_o}{\kappa m_o \cdot x_1 - 1} \\ &= \frac{\kappa}{\kappa m_o \cdot x_o - 1} \frac{b_o}{\kappa m_o \cdot x_1 - 1} \|x_o - x_1\| \\ &\leq \frac{C}{\epsilon \kappa} \|x_o - x_1\|. \end{aligned}$$

By changing the roles of x_o and x_1 we conclude that

$$|\rho_l(x_1) - \rho_l(x_o)| \leq \frac{C}{\epsilon \kappa} \|x_1 - x_o\| \text{ for all } l \geq 1.$$

Thus $\{\rho_l : l \geq 1\}$ is an equicontinuous family which is bounded uniformly. Then by $Arzelà - Ascoli$ Theorem, if need be by taking subsequence, we have that $\rho_l \to \rho$ uniformly on $\bar{\Omega}$. By Lemma (3.6, i) $\mathcal{R} = \{\rho(x)x : x \in \bar{\Omega}\}$ is a refractor.

Moreover by equation 5.3 we can conclude

$$\limsup_{l \to \infty} \chi_{\mathcal{T}_{\mathcal{R}_l}(F)}(x) t_{\mathcal{R}_l}(x) = t_{\mathcal{R}}(x) \limsup_{l \to \infty} \chi_{\mathcal{T}_{\mathcal{R}_l}(F)}(x). \tag{6.14}$$

Let now F be a closed subset of Ω^*. Using Fatou's Lemma with 6.14 we obtain,

$$\begin{aligned} \limsup \mu_l(F) &= \limsup \int_{\mathcal{T}_{\mathcal{R}_l}(F)} g(x) t_{\mathcal{R}_l}(x) dx \\ &\leq \int_{\bar{\Omega}} \limsup \chi_{\mathcal{T}_{\mathcal{R}_l}(F)}(x) g(x) t_{\mathcal{R}_l}(x) dx \\ &\leq \int_{\mathcal{T}_{\mathcal{R}}(F)} g(x) t_{\mathcal{R}}(x) dx \end{aligned}$$

Thus by Portmanteau's Theorem [Bi99] μ_l converges weakly to $\int g(x) t_{\mathcal{R}}(x) dx$. By uniqueness of weak limits we conclude

$$\mu(\omega) = \int_{\mathcal{T}_{\mathcal{R}}(\omega)} g(x) t_{\mathcal{R}}(x) dx$$

as required. $\square$

CHAPTER 7

Derivation of the Differential Equation

7.1 The Jacobian Equation

Let X denote a point in the sphere S^{n-1}, and set $X = (x, x_n)$ with $x = (x_1, \ldots, x_{n-1})$. Let $\mathcal{R} = \{\rho(X)X : X \in \bar{\Omega}\}$ be a solution of the refractor problem from $\bar{\Omega}$ to $\bar{\Omega}^*$ with emitting illumination intensity g and prescribed refracted illumination intensity f. Let $\mathcal{U} = \{x = (x_1, \ldots, x_{n-1}) : (x, \sqrt{1-|x|^2}) \in \bar{\Omega}\}$ be the orthogonal projection of $\bar{\Omega}$. If we assume $\bar{\Omega}$ is a subset of upper unit sphere $S^{n-1}_+ = S^{n-1} \cap \{x_n > 0\}$ then we can identify $\bar{\Omega}$ with $\mathcal{U}$. Moreover we can consider ρ as a function of x, with $x \in \mathcal{U}$. For the purpose of deriving the partial differential equation, we assume throughout this chapter that ρ is C^2 smooth.

Let $Y \in S^{n-1}$ be the refracted direction of the ray X by the surface $\rho(X)X$. Recall, from Snell's law 1.1 that

$$Y = \frac{1}{\kappa}\left(X - \Phi(X \cdot \nu)\nu\right), \tag{7.1}$$

where

$$\Phi(t) = t - \kappa\sqrt{1 - \kappa^{-2}(1 - t^2)}, \tag{7.2}$$

and ν is the outward unit normal to the refractor at the point $\rho(X)X$. We denote by T the map $X \mapsto Y$ and we considered it defined in $\mathcal{U}$; that is, $T : U \to \bar{\Omega}^*$.

Since $Y = (y_1, \cdots, y_n) \in S^{n-1}$, we have $Y \cdot \partial_k Y = 0$ for $1 \le k \le n-1$. Therefore the vectors $\partial_k Y$, $1 \le k \le n-1$ are in the tangent plane to the sphere at the point Y. Let $u_0 \in \mathcal{U}$. The tangent plane to the sphere at $Y(u_0)$ is the collection of points

$$P(u) = Y(u_0) + (\partial_1 Y(u_0), \cdots, \partial_{n-1} Y(u_0))(u - u_0)$$

$u \in \mathbf{R}^{n-1}$. Let R be the $(n-1)$-dimensional box given by $R = [s_1, t_1] \times \cdots \times [s_{n-1}, t_{n-1}]$. Let $P_0 = P(s_1, \cdots, s_{n-1})$, $P_1 = P(t_1, s_2, \cdots, s_{n-1})$, $P_2 = P(s_1, t_2, s_3, \ldots, s_{n-1})$, $P_j = P(s_1, \ldots, s_{j-1}, t_j, s_{j+1}, \ldots, s_{n-1})$, and $P_{n-1} = P(s_1, \ldots, s_{n-2}, t_{n-1})$. We have that the vectors $\overrightarrow{P_0 P_j}$ satisfy $\overrightarrow{P_0 P_j} = (t_j - s_j)\, \partial_j Y(u_0)$ for $1 \le j \le n-1$. Notice that the $(n-1)$-dimensional volume of $T(R)$ is approximately the $(n-1)$-dimensional volume of the box B on the tangent plane. Recall that the volume of the box generated by n vectors in $\mathbf{R}^n$ is given by the determinant of the matrix whose columns are the given vectors. Since $Y(u_0)$ is perpendicular to the tangent plane at this point and $|Y| = 1$, the $(n-1)$-dimensional volume of the box B is equal to the n-dimensional volume of the box B' generated by B and Y. We then obtain that the $(n-1)$ dimensional volume of B is given by

$$|B| = (t_1 - s_1) \cdots (t_{n-1} - s_{n-1})\, |\det J| \tag{7.3}$$

where J is the matrix (which we call Jacobian matrix of T) given by

$$J = \begin{bmatrix} \partial_1 y_1 & \cdots & \partial_{n-1} y_1 & y_1 \\ \partial_1 y_2 & \cdots & \partial_{n-1} y_2 & y_2 \\ & \cdots & & \\ \partial_1 y_{n-1} & \cdots & \partial_{n-1} y_{n-1} & y_{n-1} \\ \partial_1 y_n & \cdots & \partial_{n-1} y_n & y_n \end{bmatrix}.$$

Since $\partial_k y_n = -\dfrac{1}{y_n}(y_1 \partial_k y_1 + \cdots + y_{n-1} \partial_k y_{n-1})$ for $1 \le k \le n-1$, replacing these values in the last row of J and using the fact that the determinant is multilinear in the rows yields

$$\det J = \frac{1}{y_n} \det \begin{bmatrix} \partial_1 y_1 & \cdots & \partial_{n-1} y_1 \\ \partial_1 y_2 & \cdots & \partial_{n-1} y_2 \\ & \cdots & \\ \partial_1 y_{n-1} & \cdots & \partial_{n-1} y_{n-1} \end{bmatrix}. \tag{7.4}$$

If we denote the matrix in 7.4 by Dy then $Dy = (\partial_j y_i),\ 1 \le i, j \le n-1$ and

$$\det J = \frac{1}{y_n} \det Dy.$$

If $dS_{\bar{\Omega}^*}$ and $dS_{\mathcal{U}}$ denote the area elements corresponding to $\bar{\Omega}^*$ and the volume element corresponding to $\mathcal{U}$ respectively then

$$|\det J| = \frac{dS_{\bar{\Omega}^*}}{dS_{\mathcal{U}}} \tag{7.5}$$

Also if $dS_{\bar{\Omega}}$ is the area element corresponding to $\bar{\Omega}$ then $\dfrac{dS_{\bar{\Omega}}}{dS_{\mathcal{U}}} = \dfrac{1}{\sqrt{1-|x|^2}}$.

Let $x_o \notin S$ where S is the singular set of $\mathcal{R}$ and $m_o = \mathcal{T}_{\mathcal{R}}(x_o) = T(x_o)$ where T is viewed as a map on $\bar{\Omega}$. Let $r > 0$ and $B_r(m_o)$ be the ball centered at m_o with radius r and contained in Ω^*. Then from the energy condition

$$\int_{\mathcal{T}_{\mathcal{R}}(\omega)} g(x) t_{\mathcal{R}}(x) dx = \int_{\omega} f(m) dm$$

we have

$$\left| \frac{\mathcal{T}_{\mathcal{R}}(B_r(m_o))}{B_r(m_o)} \right| \frac{1}{|\mathcal{T}_{\mathcal{R}}(B_r(m_o))|} \int_{\mathcal{T}_{\mathcal{R}}(B_r(m_o))} g(x) t_{\mathcal{R}}(x) dx = \frac{1}{|B_r(m_o)|} \int_{B_r(m_o)} f(m) dm.$$

If as $r \to 0$, both $|B_r(m_o)|$ and $|\mathcal{T}_{\mathcal{R}}(B_r(m_o))|$ tend to 0, then by Lebesgue Differentiation Theorem, we deduce that $g(x) t_{\mathcal{R}}(x) dS_{\bar{\Omega}} = f(T(x)) dS_{\bar{\Omega}^*}$.

Combining this result with 7.5 , we obtain

$$|\det J| = \frac{dS_{\bar{\Omega}^*}}{dS_{\mathcal{U}}} = \frac{g(x) t_{\mathcal{R}}(x)}{\sqrt{1-|x|^2} f(T(x))} \tag{7.6}$$

We now find the $|\det Dy|$ explicitly to show that ρ satisfies a differential equation of Monge-Ampère type.

7.2 The Monge-Ampère equation for ρ

In this section we shall derive the Monge-Ampère equation satisfied by ρ. First, we prove that the normal ν to $\mathcal{R}$ has the following expression.

Lemma 7.1. *The unit outer normal ν to the surface $\mathcal{R}$ at the point $\rho(x)X$, $X \in \bar{\Omega}$, is given by*

$$\nu = \frac{-\hat{D}\rho(x) + X\left(\rho(x) + D\rho(x)\cdot x\right)}{\sqrt{\rho^2 + |D\rho|^2 - (D\rho\cdot x)^2}}, \tag{7.7}$$

where $X = (x, \sqrt{1-|x|^2})$ and $\hat{D}\rho(x) = (\partial_1\rho(x), \cdots, \partial_{n-1}\rho(x), 0) = (D\rho(x), 0)$.

In addition, we have

$$X\cdot\nu = \frac{\rho}{\sqrt{\rho^2 - (x\cdot D\rho)^2 + |D\rho|^2}}. \tag{7.8}$$

Proof. First notice that the vectors $\partial_{x_k}\left((x, x_n)\rho(x)\right)$ are tangential to the graph of the refractor for $k = 1, \cdots, n-1$. Therefore we have

$$\partial_{x_k}\left((x, x_n)\rho(x)\right)\cdot\nu = 0$$

for $k = 1, \cdots, n-1$. Write $\nu = (\nu', \nu_n)$. Calculating explicitly the derivatives we get,

$$\rho\sum_{i=1}^{n-1}\delta_{ik}\nu_i + \partial_{x_k}\rho\sum_{i=1}^{n-1}x_i\nu_i = \left(\rho\frac{x_k}{\sqrt{1-|x|^2}} - \sqrt{1-|x|^2}\,\partial_{x_k}\rho\right)\nu_n, \tag{7.9}$$

for $k = 1, \cdots, n-1$.

If η, ξ are row vectors in $\mathbf{R}^n$, the tensor product is the $n\times n$ matrix defined by

$$\xi\otimes\eta = \xi^t\eta,$$

with the multiplication of matrices. Moreover if I is the $n\times n$ identity matrix and C is any constant, then the Shermann-Morrison formula says

$$(I + C\xi\otimes\eta)^{-1} = I - \frac{C\xi\otimes\eta}{1 + C(\xi\cdot\eta)}. \tag{7.10}$$

In matrix form, 7.9 will then become

$$\left(\rho I + D\rho\otimes x\right)(\nu')^t = \left(\rho\frac{x^t}{\sqrt{1-|x|^2}} - \sqrt{1-|x|^2}(D\rho)^t\right)\nu_n.$$

From 7.10 we have

$$\begin{aligned}
(\rho I + D\rho \otimes x)^{-1} &= \rho^{-1}\left(I + \frac{D\rho}{\rho} \otimes x\right)^{-1} \\
&= \rho^{-1}\left(I - \frac{\dfrac{D\rho}{\rho} \otimes x}{1 + \dfrac{D\rho}{\rho} \cdot x}\right) \\
&= \rho^{-1}\left(I - \frac{D\rho \otimes x}{\rho + D\rho \cdot x}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
(\nu')^t &= \rho^{-1}\left(I - \frac{D\rho \otimes x}{\rho + D\rho \cdot x}\right)\left(\rho\frac{x^t}{\sqrt{1-|x|^2}} - \sqrt{1-|x|^2}(D\rho)^t\right)\nu_n \\
&= \rho^{-1}\left(\frac{\rho}{\sqrt{1-|x|^2}}x^t - \sqrt{1-|x|^2}(D\rho)^t - \frac{\rho}{\sqrt{1-|x|^2}(\rho + D\rho\cdot x)}(D\rho\otimes x)x^t \right. \\
&\qquad \left. + \frac{\sqrt{1-|x|^2}}{\rho + D\rho\cdot x}(D\rho\otimes x)(D\rho)^t\right)\nu_n.
\end{aligned}$$

Now observe that for any row vectors ξ, η, γ we have $(\xi \otimes \eta)\gamma^t = (\eta \cdot \gamma)\xi^t$. So $(D\rho \otimes x)x^t = |x|^2(D\rho)^t$ and $(D\rho \otimes x)(D\rho)^t = (x \cdot D\rho)(D\rho)^t$. Therefore

$$\begin{aligned}
(\nu')^t &= \rho^{-1}\left(\frac{\rho}{\sqrt{1-|x|^2}}x^t - \left(\sqrt{1-|x|^2} + \frac{|x|^2\rho}{\sqrt{1-|x|^2}(\rho + D\rho\cdot x)} \right.\right. \\
&\qquad \left.\left. - \frac{\sqrt{1-|x|^2}}{\rho + D\rho\cdot x}(x\cdot D\rho)\right)(D\rho)^t\right)\nu_n \\
&= \rho^{-1}\left(\frac{\rho}{\sqrt{1-|x|^2}}x^t - \frac{\rho}{\sqrt{1-|x|^2}(\rho + D\rho\cdot x)}(D\rho)^t\right)\nu_n \\
&= \frac{1}{\sqrt{1-|x|^2}}\left(x^t - \frac{1}{\rho + D\rho\cdot x}(D\rho)^t\right)\nu_n.
\end{aligned}$$

So writing the normal as a row vector we get

$$\nu = (\nu', \nu_n) = \left(\frac{1}{\sqrt{1-|x|^2}}\left(x - \frac{1}{\rho + D\rho\cdot x}D\rho\right), 1\right)\nu_n. \tag{7.11}$$

Using this formula we get that

$$X \cdot \nu = \frac{1}{\sqrt{1-|x|^2}}\left(\frac{\rho}{\rho + D\rho\cdot x}\right)\nu_n.$$

Since ν is the outer normal to the refractor at X, we must have $X \cdot \nu \geq 0$. Therefore ν_n and $\rho + D\rho \cdot x$ must have the same sign. Also since $|\nu'|^2 + \nu_n^2 = 1$, we obtain from 7.11 that

$$\left(\frac{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}{(1-|x|^2)(\rho + D\rho \cdot x)^2}\right) \nu_n^2 = 1.$$

Notice here that $\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2 > 0$ since $|X| = 1$. So

$$\nu_n = \pm\sqrt{\frac{(1-|x|^2)(\rho + D\rho \cdot x)^2}{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}} = \pm|\rho + D\rho \cdot x|\sqrt{\frac{1-|x|^2}{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}}. \tag{7.12}$$

Hence from 7.11

$$\begin{aligned}
\nu &= (\nu', \nu_n) \\
&= \pm|\rho + D\rho \cdot x|\sqrt{\frac{(1-|x|^2)}{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}} \left(\frac{1}{\sqrt{1-|x|^2}}\left(x - \frac{1}{\rho + D\rho \cdot x} D\rho\right), 1\right) \\
&= \pm\frac{|\rho + D\rho \cdot x|}{\rho + D\rho \cdot x}\left(\frac{-D\rho + (\rho + D\rho \cdot x)}{\sqrt{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}} x, \frac{\sqrt{1-|x|^2}(\rho + D\rho \cdot x)}{\sqrt{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}}\right)
\end{aligned}$$

and 7.7 follows. Moreover,

$$X \cdot \nu = \pm\frac{|\rho + D\rho \cdot x|}{\rho + D\rho \cdot x} \frac{\rho}{\sqrt{\rho^2 - (x \cdot D\rho)^2 + |D\rho|^2}}.$$

If $\rho + D\rho \cdot x > 0$, then ν_n is positive and so in (7.12) we need to choose the plus sign. If $\rho + D\rho \cdot x < 0$, then $\nu_n < 0$ and so in (7.12) we need to choose the minus sign. Therefore in any case we obtain (7.8). □

For brevity lets first introduce the functions;

$$h(x, z, p) = \frac{\Phi\left(\frac{z}{\sqrt{z^2 + |p|^2 - (p \cdot x)^2)}}\right)}{\sqrt{z^2 + |p|^2 - (p \cdot x)^2}}, \tag{7.13}$$

where Φ is defined in (7.2), and

$$\begin{aligned}
w(x, z, p) &= 1 - \Phi\left(\frac{z}{\sqrt{z^2 + |p|^2 - (p \cdot x)^2}}\right) \frac{z + p \cdot x}{\sqrt{z^2 + |p|^2 - (p \cdot x)^2}} \qquad (7.14) \\
&= 1 - h(x, z, p)(z + p \cdot x).
\end{aligned}$$

We now prove the main result of this section.

Theorem 7.1. *If ρ is the function defining a refractor $\mathcal{R}$, solution to the refractor problem with intensity $g \in L^1(\bar{\Omega})$ on $\bar{\Omega}$ and $f \in L^1(\bar{\Omega}^*)$ on $\bar{\Omega}^*$, then*

$$|\det(D^2\rho + C^{-1}B)| = \frac{g(x)t_{\mathcal{R}}(x)\kappa^{n-1}\, w}{f(T(x))\, h^{n-1}\left(1 - h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right)\cdot D_p h\right)}, \tag{7.15}$$

where C^{-1} is given in (7.22)*, B given by* (7.20)*, h is defined in* (7.13)*, and w defined in* (7.14)*.*

Proof. From (7.1), (7.7), (7.8), and the definitions of h and w we get that the components of y in (7.1) can be written as

$$y_i = \frac{1}{\kappa}\left(w(x,\rho(x),D\rho(x))x_i + h(x,\rho(x),D\rho(x))\,\rho_{x_i}\right), \qquad 1 \le i \le n-1,$$

and

$$y_n = \frac{1}{\kappa} w\sqrt{1-|x|^2}. \tag{7.16}$$

Differentiating y_i with respect to x_j, with $1 \le i, j \le n-1$, we get

$$\partial_j y_i = \frac{1}{\kappa}\left(w\,\delta_{ij} + x_i\left(w_{x_j} + w_z\,\rho_{x_j} + \sum_{k=1}^{n-1} w_{p_k}\,\rho_{x_k x_j}\right) + h(x,\rho,D\rho)\rho_{x_i x_j}\right.$$
$$\left. + \rho_{x_i}\left(h_{x_j} + h_z\,\rho_{x_j} + \sum_{k=1}^{n-1} h_{p_k}\,\rho_{x_k x_j}\right)\right).$$

Recall that all the vectors $x, D\rho, D_x w, D_p w, D_z w, D_x h, D_p h$ are regarded as row vectors. The matrix $Dy = (\partial_j y_i)$, $1 \le i, j \le n-1$, can then be written as

$$Dy = \frac{1}{\kappa}\Big(w\,I + x\otimes D_x w + w_z x\otimes D\rho + x\otimes\left((D^2\rho)(D_p w)^t\right)^t$$
$$+ h(x,\rho,D\rho)\,D^2\rho + D\rho\otimes D_x h + h_z\,D\rho\otimes D\rho + D\rho\otimes\left((D^2\rho)(D_p h)^t\right)^t\Big).$$

Note that if u, v are both row vectors and A is an $n\times n$ symmetric matrix, then $u\otimes(Av^t)^t = (u\otimes v)A$, and we obtain the formula

$$Dy = \frac{1}{\kappa}\big(w\,I + x\otimes D_x w + w_z x\otimes D\rho + D\rho\otimes D_x h + h_z\,D\rho\otimes D\rho$$
$$+ (x\otimes D_p w)\,D^2\rho + h(x,\rho,D\rho)\,D^2\rho + (D\rho\otimes D_p h)\,D^2\rho\big).$$

Let

$$C(x) = (x \otimes D_p w) + h(x, \rho, D\rho)\, I + (D\rho \otimes D_p h) \tag{7.17}$$

$$B(x) = w\, I + x \otimes D_x w + w_z x \otimes D\rho + D\rho \otimes D_x h + h_z\, D\rho \otimes D\rho. \tag{7.18}$$

So

$$Dy = \frac{1}{\kappa}\left[B(x) + C(x)\, D^2\rho\right]. \tag{7.19}$$

We have $D_x w = -D_x h\,(z + p\cdot x) - h\,p$, $w_z = -h_z\,(z + p\cdot x) - h$, and $D_p w = -D_p h\,(z + p\cdot x) - h\,x$. So we can write

$$\begin{aligned}
C(x) &= -(\rho + D\rho\cdot x)\,(x \otimes D_p h) - h(x\otimes x) + h(x,\rho,D\rho)\, I + (D\rho \otimes D_p h)\\
&= ((-(\rho + D\rho\cdot x)x + D\rho) \otimes D_p h) - h(x\otimes x) + h(x,\rho,D\rho)\, I\\
&= ((-(\rho + D\rho\cdot x)x + D\rho) \otimes D_p h) - h\,((x\otimes x) - I)\\
&= h\left(\left(h^{-1}\,(-(\rho + D\rho\cdot x)x + D\rho) \otimes D_p h\right) + (((-x)\otimes x) + I)\right)\\
&= h\,(\mathcal{M}_1 + \mathcal{M}_2)\\
&= h\mathcal{M}_2\left(I + \mathcal{M}_2^{-1}\mathcal{M}_1\right),
\end{aligned}$$

with

$$\mathcal{M}_1 = -h^{-1}\,((\rho + D\rho\cdot x)x - D\rho) \otimes D_p h, \qquad \mathcal{M}_2 = ((-x)\otimes x) + I;$$

and

$$\begin{aligned}
B(x) &= wI - x \otimes (D_x h(\rho + D\rho\cdot x) + hD\rho) - (h_z(\rho + D\rho\cdot x) + h)\, x\otimes D\rho\\
&\qquad + D\rho \otimes D_x h + h_z D\rho \otimes D\rho.\\
&= (1 - (\rho + D\rho\cdot x)h)\, I - ((\rho + D\rho\cdot x)x - D\rho) \otimes D_x h\\
&\qquad - x \otimes ((2h + h_z(\rho + D\rho\cdot x))\, D\rho) + h_z\, D\rho \otimes D\rho.
\end{aligned} \tag{7.20}$$

From the Shermann-Morrison formula we have that if $\mathcal{M} = I + \xi^t\eta$ where ξ and η are any vectors, then

$$\det\,\mathcal{M} = 1 + \xi\cdot\eta, \qquad \mathcal{M}^{-1} = I - \frac{\xi^t\eta}{1 + \xi\cdot\eta}. \tag{7.21}$$

We can calculate explicitly the inverse of $C(x)$ noticing it is the product of two matrices having the form of $\mathcal{M}$. We have

$$C^{-1} = \frac{1}{h}\left(I + \mathcal{M}_2^{-1}\mathcal{M}_1\right)^{-1}\mathcal{M}_2^{-1}, \text{ and } \mathcal{M}_2^{-1} = I + \frac{x \otimes x}{1-|x|^2}.$$

If we set $v = h^{-1}\left((\rho + D\rho \cdot x)x - D\rho\right)$, then $I + \mathcal{M}_2^{-1}\mathcal{M}_1 = I - \mathcal{M}_2^{-1}(v \otimes D_p h) = I + \left(-\mathcal{M}_2^{-1}v^t\right) D_p h$. Therefore

$$\left(I + \mathcal{M}_2^{-1}\mathcal{M}_1\right)^{-1} = I + \frac{\left(\mathcal{M}_2^{-1}v^t\right) D_p h}{1 - \left(\mathcal{M}_2^{-1}v^t\right)^t \cdot D_p h} := \mathcal{N},$$

and so

$$C^{-1} = \frac{1}{h}\mathcal{N}\left(I + \frac{x \otimes x}{1-|x|^2}\right). \tag{7.22}$$

Let us calculate this matrix more explicitly. We have

$$\begin{aligned}
\mathcal{M}_2^{-1}v^t &= v^t + \frac{1}{1-|x|^2}x^t x v^t \\
&= h^{-1}\left[(\rho + D\rho \cdot x)x^t - (D\rho)^t + \frac{1}{1-|x|^2}x^t x\left((\rho + D\rho \cdot x)x^t - (D\rho)^t\right)\right] \\
&= h^{-1}\left[(\rho + D\rho \cdot x)x^t - (D\rho)^t + \frac{1}{1-|x|^2}\left(|x|^2(\rho + D\rho \cdot x)x^t - (D\rho \cdot x)x^t\right)\right] \\
&= h^{-1}\left[\frac{\rho}{1-|x|^2}x^t - (D\rho)^t\right].
\end{aligned}$$

So

$$\mathcal{N} = I + \frac{h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right) \otimes D_p h}{1 - h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right) \cdot D_p h}, \tag{7.23}$$

and

$$\det \mathcal{N} = \frac{1}{1 - h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right) \cdot D_p h}.$$

Therefore from (7.22)

$$\det C = h^{n-1}(1-|x|^2)\left(1 - h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right) \cdot D_p h\right). \tag{7.24}$$

Notice that for row vectors α, β, ξ, η, we have $(\alpha \otimes \beta)(\xi \otimes \eta) = (\beta \cdot \xi)(\alpha \otimes \eta)$.

Then

$$
\begin{aligned}
C^{-1} &= \frac{1}{h}\left[I + \frac{x \otimes x}{1-|x|^2} + \frac{\left(\mathcal{M}_2^{-1} v^t\right) D_p h}{1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h} + \left(\frac{\left(\mathcal{M}_2^{-1} v^t\right) D_p h}{1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h}\right)\left(\frac{x \otimes x}{1-|x|^2}\right)\right] \\
&= \frac{1}{h}\left[I + \frac{x \otimes x}{1-|x|^2} + \frac{\left(\mathcal{M}_2^{-1} v^t\right) D_p h}{1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h}\right. \\
&\qquad \left. + \frac{1}{(1-|x|^2)\left(1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h\right)}\left(\left(\mathcal{M}_2^{-1} v^t\right) D_p h\, x^t\, x\right)\right] \\
&= \frac{1}{h}\left[I + \frac{x \otimes x}{1-|x|^2} + \frac{\left(\mathcal{M}_2^{-1} v^t\right) D_p h}{1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h}\right. \\
&\qquad \left. + \frac{x \cdot D_p h}{(1-|x|^2)\left(1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h\right)}\left(\left(\mathcal{M}_2^{-1} v^t\right)^t \otimes x\right)\right] \\
&= \frac{1}{h}\left[I + \frac{x \otimes x}{1-|x|^2}\right. \\
&\qquad \left. + \frac{1}{1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h}\left(\mathcal{M}_2^{-1} v^t\right)\left(D_p h + \frac{x \cdot D_p h}{(1-|x|^2)} x\right)\right] \\
&= \frac{1}{h}\left[I + \frac{x \otimes x}{1-|x|^2}\right. \\
&\qquad \left. + \frac{1}{1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h}\left(h^{-1}\left[\frac{\rho}{1-|x|^2} x^t - (D\rho)^t\right]\right)\left(D_p h + \frac{x \cdot D_p h}{(1-|x|^2)} x\right)\right].
\end{aligned}
$$

Let

$$
\mathcal{A} = \frac{1}{1-\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h}\left(h^{-1}\left[\frac{\rho}{1-|x|^2} x^t - (D\rho)^t\right]\right)\left(D_p h + \frac{x \cdot D_p h}{(1-|x|^2)} x\right).
$$

We have

$$
\left(\mathcal{M}_2^{-1} v^t\right)^t \cdot D_p h = h^{-1}\left[\frac{\rho}{1-|x|^2}(x \cdot D_p h) - D\rho \cdot D_p h\right]
$$

So

$$
\mathcal{A} = \frac{1}{h - \left[\frac{\rho}{1-|x|^2}(x \cdot D_p h) - D\rho \cdot D_p h\right]}\left(\frac{\rho}{1-|x|^2} x - D\rho\right) \otimes \left(D_p h + \frac{x \cdot D_p h}{(1-|x|^2)} x\right),
$$

and

$$C^{-1} = \frac{1}{h}\left[I + \frac{x \otimes x}{1-|x|^2} + \mathcal{A}\right].$$

From (7.19)

$$Dy = \frac{1}{\kappa}\,C\,\left(C^{-1}B(x) + D^2\rho\right),$$

and so

$$\det Dy = \frac{1}{\kappa^{n-1}}\det C\,\det\left(C^{-1}B + D^2\rho\right). \tag{7.25}$$

Combining (7.4), (7.6), (7.16), and (7.25), we obtain

$$\left|\frac{1}{\kappa^{n-1}}\det C\,\det\left(C^{-1}B + D^2\rho\right)\right| = \frac{g(x)t_{\mathcal{R}}(x)w}{\kappa f(T(x))}.$$

Finally from (7.24) we get

$$|\det(D^2\rho + C^{-1}B)| = \frac{g(x)t_{\mathcal{R}}(x)\kappa^{n-2}\,w}{f(T(x))\,h^{n-1}\left(1 - h^{-1}\left(\frac{\rho}{1-|x|^2}x - D\rho\right)\cdot D_p h\right)}.$$

□

APPENDIX A

Maxwell's equations and Fresnel Equations

A.1 Maxwell's equations

The *electromagnetic field* (EM) is a region of space in which electric and magnetic forces are acting. It is established in space by the presence of electric charges. It extends indefinitely throughout space and is described by the electric field vector **E** and a magnetic field vector **B**.

These are three-dimensional vector fields that have a value defined at every point of space and time: $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$, where $\mathbf{r}$ represents a point in 3-d space $\mathbf{r} = (x, y, z)$. The electric field is produced by stationary charges, and the magnetic field by moving charges (currents); these two are often described as the sources of the field. The way in which **E** and **B** interact is described by Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \text{Gauss's law} \tag{A.1}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \text{Gauss's law for magnetism} \tag{A.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{Faraday's law} \tag{A.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \text{Ampère-Maxwell's law.} \tag{A.4}$$

Here

$\nabla = (\partial_x, \partial_y, \partial_z)$	the gradient
$\rho = \rho(\mathbf{r}, t)$	charge density
ϵ_0	permittivity of free space
$\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$	current density vector
μ_0	permeability of free space

We have $c = 1/\sqrt{\epsilon_0 \mu_0}$, the speed of light in vacuum.

A.2 General case

In several situations is necessary to consider a medium where the magnetic permeability $\mu = \mu(x, y, z)$[1] and the electric permittivity $\epsilon = \epsilon(x, y, z)$[2] are not constants. This is the case when the physical properties of the medium change from point to point, in particular, this happens in geometric optics when materials of different refractive indices are considered. In such case the Maxwell's equations have the form:

$$\nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{A.5}$$

$$\nabla \times \mathbf{B} = \frac{2\pi}{c}\sigma \mathbf{E} + \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{A.6}$$

$$\nabla \cdot (\epsilon \mathbf{E}) = 4\pi\rho \tag{A.7}$$

$$\nabla \cdot (\mu \mathbf{B}) = 0, \tag{A.8}$$

c being the speed of light in vacuum. Recall that substances for which $\sigma \neq 0$ are conductors and if σ is negligibly small, the substances are called insulators or dielectrics, see [BW80][Section 1.1.2]. Under certain assumptions on the field and the physical set up we have that $\mathbf{J} = \sigma \mathbf{E}$, see [BW80][Section 1.1.2, formula (9)].

It is important to notice that these equations are written in Gaussian units, and the Maxwell's equations in the first section written in SI units.

[1]For values of μ for different substances see `http://en.wikipedia.org/wiki/Permeability_(electromagnetism)\#Values_for_some_common_materials`.

[2]For relative permittivity of some substances see `http://en.wikipedia.org/wiki/Relative_permittivity`.

A.3 Maxwell's equations in integral form

Points in $\mathbf{R}^4$ are denoted by (x, y, z, t), and suppose $D \subset \mathbf{R}^4$ is a domain for which the divergence theorem holds, for example, the boundary is piecewise smooth, that is, a finite union of C^1 surfaces. For a point $P = (x, y, z, t)$ on the boundary ∂D, the unit outer normal at P is denoted by $\nu = (\nu_x, \nu_y, \nu_z, \nu_t)$. From equation (A.6)

$$\nabla \times \mathbf{B} - \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{2\pi}{c}\sigma \mathbf{E}. \tag{A.9}$$

Recall we assume $\epsilon = \epsilon(x, y, z)$, and we want to derive an integral form of the last equation that does not require differentiability of the fields. In order to do that, we initially assume the fields are smooth and applying the divergence theorem we will obtain formulas independent of the derivatives of the fields. Set $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$. We have

$$\begin{aligned}
&\int_D \nabla \times \mathbf{B}\, dxdydzdt \\
&= \mathbf{i}\int_D (\partial_y \mathbf{B}_3 - \partial_z \mathbf{B}_2) - \mathbf{j}\int_D (\partial_x \mathbf{B}_3 - \partial_z \mathbf{B}_1) + \mathbf{k}\int_D (\partial_x \mathbf{B}_2 - \partial_y \mathbf{B}_1) \\
&= \mathbf{i}\int_D \text{div}\,(0, \mathbf{B}_3, -\mathbf{B}_2, 0) - \mathbf{j}\int_D \text{div}\,(\mathbf{B}_3, 0, -\mathbf{B}_1, 0) + \mathbf{k}\int_D \text{div}\,(\mathbf{B}_2, -\mathbf{B}_1, 0, 0) \\
&= \mathbf{i}\int_{\partial D} (0, \mathbf{B}_3, -\mathbf{B}_2, 0)\cdot(\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma \\
&\qquad - \mathbf{j}\int_{\partial D} (\mathbf{B}_3, 0, -\mathbf{B}_1, 0)\cdot(\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma + \mathbf{k}\int_{\partial D} (\mathbf{B}_2, -\mathbf{B}_1, 0, 0)\cdot(\nu_x, \nu_y, \nu_z, \nu_t)\, d\sigma \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)\, d\sigma.
\end{aligned}$$

So integrating (A.9) over D yields

$$\begin{aligned}
&\int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)\, d\sigma - \int_D \frac{\epsilon}{c}\mathbf{E}_t\, dxdydzdt \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)\, d\sigma - \int_D \left(\frac{\epsilon}{c}\mathbf{E}\right)_t dxdydzdt \\
&= \int_{\partial D} (\nu_x, \nu_y, \nu_z) \times (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)\, d\sigma - \int_{\partial D} \frac{\epsilon}{c}\mathbf{E}\,\nu_t\, d\sigma \\
&= \int_{\partial D} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3) - \frac{\epsilon}{c}\mathbf{E}\,\nu_t\right) d\sigma = \int_D \frac{2\pi}{c}\sigma \mathbf{E}\, dxdydzdt.
\end{aligned}$$

Therefore the surface integral

$$\int_{\partial D} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3) - \frac{\epsilon}{c} \mathbf{E}\, \nu_t \right) d\sigma = \int_D \frac{2\pi}{c} \sigma \mathbf{E}\, dxdydzdt, \quad \text{(A.10)}$$

for each closed hypersurface ∂D in $\mathbf{R}^4$. Proceeding in the same way with equation (A.5) we obtain that the surface integral

$$\int_{\partial D} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3) + \frac{\mu}{c} \mathbf{B}\, \nu_t \right) d\sigma = 0, \quad \text{(A.11)}$$

for each closed hypersurface ∂D in $\mathbf{R}^4$.

Concerning equations (A.7) and (A.8), proceeding in the same way as before we obtain that

$$\int_{\partial D} \epsilon \mathbf{E} \cdot \nu \, d\sigma = 4\pi \int_D \rho \, dxdydzdt \quad \text{(A.12)}$$

$$\int_{\partial D} \mu \mathbf{B} \cdot \nu \, d\sigma = 0, \quad \text{(A.13)}$$

for each domain $D \subset \mathbf{r^4}$ for which the divergence theorem holds. Equations (A.10), (A.11), (A.12), and (A.13) are Maxwell's equations in integral form.

A.4 Boundary conditions at a surface of discontinuity

Let us consider a point $P_0 = (x_0, y_0, z_0, t_0)$, a hypersurface Γ_0 passing through P_0 and suppose that the fields $\mathbf{B}$ and $\mathbf{E}$, solutions to the Maxwell's equations in integral form, as well as the functions ϵ and μ, are discontinuous on Γ_0. Suppose that all these quantities are defined locally around P_0 say in the 4-dimensional ball $B_R(P_0)$. This situation is typical when we have two media with different indices of refraction and the surface Γ_0 is the one separating the two media. The surface Γ_0 divides the open ball $B_R(P_0)$ into two open pieces: B_R^+ and B_R^-. In order to make sense of the integrals we assume the surface Γ_0 is C^1, the fields $\mathbf{E}$ and $\mathbf{B}$, and ϵ and μ are bounded in $B_R(P_0)$, and all continuous on $B_R(P_0) \setminus \Gamma_0$. We assume also that for each $Q \in \Gamma_0 \cap B_R(P_0)$ the following limits exist and are finite:

$$\lim_{P \to Q, P \in B_R^+} \mathbf{E}(P) = \mathbf{E}^+(Q), \qquad \lim_{P \to Q, P \in B_R^+} \mathbf{B}(P) = \mathbf{B}^+(Q)$$
$$\lim_{P \to Q, P \in B_R^-} \mathbf{E}(P) = \mathbf{E}^-(Q), \qquad \lim_{P \to Q, P \in B_R^-} \mathbf{B}(P) = \mathbf{B}^-(Q),$$

and similar quantities for ϵ and μ. We consider the parallel surfaces to Γ_0. That is, given $Q \in \Gamma_0$, denote by $\nu(Q)$ the unit normal at Q in the same direction for each Q, and given $\delta > 0$ small, consider the parallel surfaces

$$\Gamma_\delta^+ = \{Q + \delta\nu(Q) : Q \in \Gamma_0\}, \qquad \Gamma_\delta^- = \{Q - \delta\nu(Q) : Q \in \Gamma_0\}.$$

If $F(Q)$ is a bounded function in a neighborhood of Γ_0, then the surface integral

$$\int_{\Gamma_\delta^\pm \cap B_R(P_0)} F(P)\, d\sigma(P) = \int_{\Gamma_0 \cap B_R(P_0)} F(Q \pm \delta\nu(Q))\, d\sigma(Q) + \text{error}_\delta, \quad \text{(A.14)}$$

where error_δ depending on the surface integral on the difference between $\Gamma_\delta^\pm \cap B_R(P_0)$ and its projection on $\Gamma_0 \cap B_R(P_0)$, and tends to zero when $\delta \to 0$. Since F is bounded, by Lebesgue bounded convergence theorem we have that

$$\lim_{\delta \to 0} \int_{\Gamma_\delta^\pm \cap B_R(P_0)} F(P)\, d\sigma(P) = \int_{\Gamma_0 \cap B_R(P_0)} F^\pm(Q)\, d\sigma(Q), \quad \text{(A.15)}$$

where $F^\pm(Q) = \lim_{P \to Q, P \in B_R^\pm} F(P)$ if these limits exist for each $Q \in \Gamma_0$.

Let us consider the region D_δ bounded by the surfaces $\Gamma_\delta^\pm$ and the boundary of the ball $B_R(P_0)$. The boundary of D_δ has three pieces: $\Gamma_\delta^\pm \cap B_R(P_0)$ and the lateral side S_δ of the ball $B_R(P_0)$ sandwiched by $\Gamma_\delta^\pm$. Applying (A.10) in D_δ yields

$$\int_{\Gamma_\delta^+ \cap B_R(P_0)} \left((\nu_x, \nu_y, \nu_z) \times \mathbf{B} - \frac{\epsilon}{c} \mathbf{E}\, \nu_t \right) d\sigma + \int_{\Gamma_\delta^- \cap B_R(P_0)} \left((\nu_x, \nu_y, \nu_z) \times \mathbf{B} - \frac{\epsilon}{c} \mathbf{E}\, \nu_t \right) d\sigma$$
$$+ \int_{S_\delta} \left((\nu_x, \nu_y, \nu_z) \times \mathbf{B} - \frac{\epsilon}{c} \mathbf{E}\, \nu_t \right) d\sigma = \int_{D_\delta} \frac{2\pi}{c} \sigma \mathbf{E}\, dxdydzdt.$$

If $P = Q \pm \delta\nu(Q) \in \Gamma_\delta^\pm$, then $\nu(P) = \pm\nu(Q)$, and so from (A.14) we get

$$\int_{\Gamma_\delta^\pm \cap B_R(P_0)} \left((\nu_x, \nu_y, \nu_z) \times \mathbf{B} - \frac{\epsilon}{c} \mathbf{E}\, \nu_t \right) d\sigma = \int_{\Gamma_0 \cap B_R(P_0)} \left(\pm(\nu_x, \nu_y, \nu_z) \times \mathbf{B} \mp \frac{\epsilon}{c} \mathbf{E}\, \nu_t \right) d\sigma$$
$$+ \text{error}_\delta.$$

Since both the area of S_δ and the volume of D_δ tend to zero as $\delta \to 0$, the integrals over S_δ and D_δ also tend to zero. So letting $\delta \to 0$, from (A.15) we obtain

$$\int_{\Gamma_0 \cap B_R(P_0)} \left((\nu_x, \nu_y, \nu_z) \times (\mathbf{B}^+ - \mathbf{B}^-) - \frac{1}{c}(\epsilon^+ \mathbf{E}^+ - \epsilon^- \mathbf{E}^-)\, \nu_t \right) d\sigma = 0.$$

Now letting $R \to 0$ we obtain the following equation valid at P_0

$$(\nu_x, \nu_y, \nu_z) \times (\mathbf{B}^+ - \mathbf{B}^-) - \frac{1}{c}(\epsilon^+ \mathbf{E}^+ - \epsilon^- \mathbf{E}^-)\, \nu_t = 0, \qquad \text{(A.16)}$$

where ν is the normal to the interface Γ_0 at the point P_0.

Suppose the interface is independent of time and is given by a function $\phi(x, y, z) = 0$, then the normal at a point is $\nu = (\phi_x, \phi_y, \phi_z, 0)$, therefore equation (A.16) becomes

$$\nabla\phi \times (\mathbf{B}^+ - \mathbf{B}^-) = 0.$$

We can write $\mathbf{B}^\pm = \mathbf{B}^\pm_{\text{tan}} + \mathbf{B}^\pm_{\text{perp}}$, where $\mathbf{B}^\pm_{\text{perp}}$ is the component in the direction of the normal $\nabla\phi$ and $\mathbf{B}^\pm_{\text{tan}}$ is the component perpendicular to the normal. We have $\nabla\phi \times \mathbf{B}^\pm = \nabla\phi \times \mathbf{B}^\pm_{\text{tan}} + \nabla\phi \times \mathbf{B}^\pm_{\text{perp}} = \nabla\phi \times \mathbf{B}^\pm_{\text{tan}}$. So

$$0 = \nabla\phi \times (\mathbf{B}^+ - \mathbf{B}^-) = \nabla\phi \times (\mathbf{B}^+_{\text{tan}} - \mathbf{B}^-_{\text{tan}}) = |\nabla\phi|\, |\mathbf{B}^+_{\text{tan}} - \mathbf{B}^-_{\text{tan}}|,$$

since the vectors are perpendicular. So if $\nabla\phi \neq 0$, the obtain the important relation that

$$\mathbf{B}^+_{\text{tan}} - \mathbf{B}^-_{\text{tan}} = 0,$$

that is, *the tangential components of the magnetic field are continuous across the boundary*. Similarly, applying (A.11) on D_δ and proceeding in the same manner we obtain the equation

$$(\nu_x, \nu_y, \nu_z) \times (\mathbf{E}^+ - \mathbf{E}^-) + \frac{1}{c}(\mu^+ \mathbf{B}^+ - \mu^- \mathbf{B}^-)\, \nu_t = 0, \qquad \text{(A.17)}$$

where ν is the normal to the interface Γ_0 at the point P_0. If the interface is independent of t, proceeding exactly as before, we obtain

$$\nabla\phi \times (\mathbf{E}^+ - \mathbf{E}^-) = 0,$$

and

$$\mathbf{E}^{+}_{\text{tan}} - \mathbf{E}^{-}_{\text{tan}} = 0,$$

that is, also *the tangential components of the electric field are continuous across the boundary.*

In regard to equations (A.12) and (A.13), we obtain similarly that

$$(\epsilon^{+}\mathbf{E}^{+} - \epsilon^{-}\mathbf{E}^{-}) \cdot \nu = 0, \text{ and } (\mu^{+}\mathbf{B}^{+} - \mu^{-}\mathbf{B}^{-}) \cdot \nu = 0.$$

Since $\mathbf{B}^{\pm} \cdot \nu = \mathbf{B}^{\pm}_{\text{perp}} \cdot \nu$, and similarly for $\mathbf{E}$, assuming $\phi = \phi(x, y, z)$ yields

$$0 = (\epsilon^{+}\mathbf{E}^{+}_{\text{perp}} - \epsilon^{-}\mathbf{E}^{-}_{\text{perp}}) \cdot \nabla\phi = |\epsilon^{+}\mathbf{E}^{+}_{\text{perp}} - \epsilon^{-}\mathbf{E}^{-}_{\text{perp}}|\,|\nabla\phi|,$$

and

$$0 = (\mu^{+}\mathbf{B}^{+}_{\text{perp}} - \mu^{-}\mathbf{B}^{-}_{\text{perp}}) \cdot \nabla\phi = |\mu^{+}\mathbf{B}^{+}_{\text{perp}} - \mu^{-}\mathbf{B}^{-}_{\text{perp}}|\,|\nabla\phi|,$$

and therefore

$$|\epsilon^{+}\mathbf{E}^{+}_{\text{perp}} - \epsilon^{-}\mathbf{E}^{-}_{\text{perp}}| = |\mu^{+}\mathbf{B}^{+}_{\text{perp}} - \mu^{-}\mathbf{B}^{-}_{\text{perp}}| = 0.$$

Therefore *the perpendicular components of the fields* $\epsilon\mathbf{E}$ *and* $\mu\mathbf{B}$ *are continuous across the interface.*

A.5 Maxwell's equations in the absence of charges

This is the case when $\rho = 0$ and $\mathbf{J} = 0$. So the equations become

$$\nabla \cdot \mathbf{E} = 0, \tag{A.18}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{A.19}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{A.20}$$

$$\nabla \times \mathbf{B} = \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}, \tag{A.21}$$

A.6 The wave equation

Recall the following formula from vector analysis for a vector $\mathbf{A} = \mathbf{A}(x, y, z) = (\mathbf{A}_x(x, y, z), \mathbf{A}_y(x, y, z), \mathbf{A}_z(x, y, z))$:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A}. \tag{A.22}$$

Denote $\nabla \cdot \nabla = \nabla^2$, the Laplacian, and so

$$\nabla^2 \mathbf{A} = \left(\frac{\partial^2 \mathbf{A}_x}{\partial x^2} + \frac{\partial^2 \mathbf{A}_x}{\partial y^2} + \frac{\partial^2 \mathbf{A}_x}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 \mathbf{A}_y}{\partial x^2} + \frac{\partial^2 \mathbf{A}_y}{\partial y^2} + \frac{\partial^2 \mathbf{A}_y}{\partial z^2}\right)\mathbf{j} + \left(\frac{\partial^2 \mathbf{A}_z}{\partial x^2} + \frac{\partial^2 \mathbf{A}_z}{\partial y^2} + \frac{\partial^2 \mathbf{A}_z}{\partial z^2}\right)\mathbf{k}.$$

From Faraday's law and Ampère's law

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial(\nabla \times \mathbf{B})}{\partial t} = -\epsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

and so from formula (A.22) we obtain that $\mathbf{E}$ satisfies the wave equation

$$\epsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}.$$

Proceeding in the same manner for $\mathbf{B}$ we obtain

$$\epsilon_0\mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla^2 \mathbf{B}.$$

That is, both the electric and magnetic fields satisfy the wave equation. We have from physics that

$$v = c = \frac{1}{\sqrt{\epsilon_0\mu_0}},$$

c being the speed of propagation of light in free space, which in this case is the velocity v of propagation. If free space is changed by a material with other values of μ_0 and ϵ_0, the velocity v represent the speed of propagation of waves in this material.

A.7 Plane waves

Let $\mathbf{s}$ be a unit vector. Any solution to the wave equation

$$\frac{1}{v^2}\partial_t^2 V = \nabla^2 V,$$

of the form $V(\mathbf{r}, t) = F(\mathbf{r} \cdot \mathbf{s}, t)$ is called a *plane wave*, since at each time t, V is constant on each plane of the form $\mathbf{r} \cdot \mathbf{s}$=constant. That is, for each t the vector $V(\mathbf{r}, t)$ is the same on each plane $\mathbf{r} \cdot \mathbf{s}$=constant. The plane wave propagates in the direction $\mathbf{s}$. It can be proved that any solution to the wave equation of this form can be written as

$$V(\mathbf{r}, t) = V_1(\mathbf{r} \cdot \mathbf{s} - vt) + V_2(\mathbf{r} \cdot \mathbf{s} + vt)$$

where V_1, V_2 are arbitrary functions, see [BW80][Section 1.3.1]. Since the fields $\mathbf{E}$ and $\mathbf{B}$ both satisfy the wave equation, it is then natural to consider the case when

$$\mathbf{E} = \mathbf{E}(\mathbf{r} \cdot \mathbf{s} - \mathbf{vt}), \quad \mathbf{B} = \mathbf{B}(\mathbf{r} \cdot \mathbf{s} - \mathbf{vt}),$$

that is, $\mathbf{E}$ and $\mathbf{B}$ are functions of the scalar variable $\mathbf{r} \cdot \mathbf{s} - \mathbf{vt}$. We have

$$\frac{\partial \mathbf{E}}{\partial t} = -v\mathbf{E}', \text{ and } \nabla \times \mathbf{E} = s \times \mathbf{E}';$$

and similarly for $\mathbf{B}$ under the assumption that $\mathbf{J} = 0$. Thus, from the Faraday and Ampère laws, and since $v^2 = \dfrac{1}{\epsilon_0 \mu_0}$, we obtain the equations

$$\begin{aligned} s \times \mathbf{E}' &= v\mathbf{B}' \\ s \times \mathbf{B}' &= -\frac{1}{v}\mathbf{E}'. \end{aligned}$$

Since s is a constant vector $s \times \mathbf{E}' = (s \times \mathbf{E})'$, and so the equations are

$$\begin{aligned} (s \times \mathbf{E})' &= v\mathbf{B}' \\ (s \times \mathbf{B})' &= -\frac{1}{v}\mathbf{E}'. \end{aligned}$$

Integrating these equations and taking constants of integration zero (which amounts to neglect constant fields), we obtain the very important equations relating the electric and magnetic fields

$$\mathbf{E} = -v(s \times \mathbf{B}) \tag{A.23}$$

$$\mathbf{B} = \frac{1}{v}(s \times \mathbf{E}). \tag{A.24}$$

This shows that $\mathbf{s} \cdot \mathbf{E} = \mathbf{s} \cdot \mathbf{B} = \mathbf{0}$, that means, the electric and magnetic field are always *perpendicular* to the direction of propagation $\mathbf{s}$. In addition, $\mathbf{E} \cdot \mathbf{B} = v(s \times \mathbf{B}) \cdot \mathbf{B} = 0$, that is, $\mathbf{E}$ and $\mathbf{B}$ are always perpendicular. We also obtain taking absolute values that

$$|\mathbf{E}| = v|\mathbf{B}|.$$

A.8 Fresnel formulas

We consider plane waves whose components have the form

$$a \cos \left(\omega \left(t - \frac{\mathbf{r} \cdot \mathbf{s}}{v} \right) + \delta \right) = a \cos \left(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta \right),$$

that is, $\mathbf{k} = \frac{\omega}{v}\mathbf{s}$ and a, δ are real numbers. The quantity $\omega t - \mathbf{k} \cdot \mathbf{r} + \delta$ is called the *phase*, and a is called the *amplitude*.

Let $\mathbf{s}^i$ be the direction (unit) of an incident plane wave traveling for a while in media I with velocity of propagation v_1 that hits, at a point P, a boundary Γ between I and another media II where the velocity of propagation is v_2 (I and II are also called dielectrics as they are materials with zero conductivity, that is $\sigma = 0$ and so the current density vector $\mathbf{J} = 0$, see Subsection A.2). Then the wave splits into two waves: a *transmitted wave* propagating in media II and a *reflected wave* propagated back into media I. We shall assume that these two waves are also plane. The plane determined by ν and $\mathbf{s}^i$ is called the *incidence plane*. We choose a system of coordinates such that the normal ν is on the z-axis and the x and y axes are on the plane perpendicular to ν and in such a way that the vector $\mathbf{s}^i$ lies on the xz-plane. This means that the tangent plane to Γ at P is the xy-plane. So we assume that

$$\mathbf{s}^i = \sin \theta_i \mathbf{i} + \cos \theta_i \mathbf{k}$$

that is, $\mathbf{s}^i$ lives on the xz-plane and so the direction of propagation is perpendicular to the y-axis and θ_i is the angle between the normal vector ν to the boundary at P (the z-axis) and the incident direction $\mathbf{s}^i$.

The electric field corresponding to this incident field is

$$\mathbf{E}^i(\mathbf{r},\mathbf{t}) = \left(-I_\parallel \cos\theta_i, I_\perp, I_\parallel \sin\theta_i\right)\,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^i}}{v_1}\right)\right)$$

$$= \mathbf{E}_0^i\,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^i}}{v_1}\right)\right).$$

Notice that $\mathbf{E}$ has this form because, as we proved before, $\mathbf{E}$ is always perpendicular to the direction of propagation $\mathbf{s}^i$. Notice also that the field $\mathbf{E}^i$ has a component that is perpendicular to the plane of incidence and a component that is parallel to this plane, indeed,

$$\mathbf{E}_\perp^i = I_\perp \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^i}}{v_1}\right)\right)\mathbf{j},$$

and

$$\mathbf{E}_\parallel^i = \left(-I_\parallel \cos\theta_i\,\mathbf{i} + I_\parallel \sin\theta_i\,\mathbf{k}\right)\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^i}}{v_1}\right)\right).$$

Also notice that

$$|\mathbf{E}^i|^2 = \left(I_\parallel^2 + I_\perp^2\right)\cos^2\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^i}}{v_1}\right)\right)$$

From (A.24), the magnetic field is then

$$\mathbf{B}^i(\mathbf{r},\mathbf{t}) = \frac{1}{v_1}\left(-I_\perp \cos\theta_i, -I_\parallel, I_\perp \sin\theta_i\right)\,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^i}}{v_1}\right)\right)$$

$$= \mathbf{B}_0^i\,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^i}}{v_1}\right)\right).$$

Let us now introduce $\mathbf{s}^t$, the direction of propagation of the transmitted wave, and θ_t the angle between the normal ν and $\mathbf{s}^t$, and similarly, $\mathbf{s}^r$ is the direction of propagation of the reflected wave and θ_r is the angle between the normal ν and $\mathbf{s}^r$. We have that $\mathbf{s}^r = \sin\theta_r\mathbf{i} + \cos\theta_r\mathbf{k} = \sin\theta_i\mathbf{i} - \cos\theta_i\mathbf{k}$. Then the corresponding electric and magnetic fields corresponding to transmission are

$$\mathbf{E}^t(\mathbf{r},\mathbf{t}) = \left(-T_\parallel \cos\theta_t, T_\perp, T_\parallel \sin\theta_t\right)\,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^t}}{v_2}\right)\right)$$

$$= \mathbf{E}_0^t\,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^t}}{v_2}\right)\right)$$

$$\mathbf{B}^t(\mathbf{r},\mathbf{t}) = \frac{1}{v_2}\left(-T_\perp \cos\theta_t, -T_\parallel, T_\perp \sin\theta_t\right)\,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^t}}{v_2}\right)\right)$$

$$= \mathbf{B}_0^t\,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^t}}{v_2}\right)\right);$$

and similarly the fields corresponding to reflection are

$$\begin{aligned}
\mathbf{E}^r(\mathbf{r},\mathbf{t}) &= \left(-R_\parallel \cos\theta_r, R_\perp, R_\parallel \sin\theta_r\right)\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^r}}{v_1}\right)\right)\\
&= \mathbf{E}_0^r \,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^r}}{v_1}\right)\right)\\
\mathbf{B}^r(\mathbf{r},\mathbf{t}) &= \frac{1}{v_1}\left(-R_\perp \cos\theta_r, -R_\parallel, R_\perp \sin\theta_r\right)\, \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^r}}{v_1}\right)\right)\\
&= \mathbf{B}_0^r \,\cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^r}}{v_1}\right)\right).
\end{aligned}$$

Recall that from Snell's law all vectors $\mathbf{s}^i, \mathbf{s}^t, \mathbf{s}^r$ and ν all live on the same plane, that is, the xz-plane. Each of the electric and magnetic fields can be decomposed uniquely as a sum of a component in the direction of the normal (normal component) or on the z-axis plus another component perpendicular to the normal (tangential component) or on the xy-plane. From the integral form of Maxwell's equations, the tangential components of $\mathbf{E}$ (and also of $\mathbf{B}$ if $\mathbf{J} = 0$) at the interface are continuous (see [BW80][Section 1.1.3, formula (23)]) and since the electric field on media 1 near Γ equals $\mathbf{E}^i + \mathbf{E}^r$, we get $\mathbf{E}^i_{\text{tan}} + \mathbf{E}^r_{\text{tan}} = \mathbf{E}^t_{\text{tan}}$ on Γ, since $\mathbf{E}^t_{\text{tan}}$ is the transmitted electric field in media 2 near Γ . From the configuration we have, we can write $\mathbf{E}^i = E^i_{\text{normal}}\,\mathbf{k} + E^i_{\text{tan}}$, and so $\mathbf{k}\times\mathbf{E}^i = \mathbf{k}\times E^i_{\text{tan}}$. Similarly, $\mathbf{k}\times\mathbf{E}^r = \mathbf{k}\times E^r_{\text{tan}}$ and $\mathbf{k}\times\mathbf{E}^t = \mathbf{k}\times E^t_{\text{tan}}$. So $\mathbf{k}\times\mathbf{E}^i + \mathbf{k}\times\mathbf{E}^r = \mathbf{k}\times\mathbf{E}^t$. Then

$$\begin{aligned}
&\mathbf{k}\times\mathbf{E}_0^i \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^i}}{v_1}\right)\right) + \mathbf{k}\times\mathbf{E}_0^r \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^r}}{v_1}\right)\right)\\
&= \mathbf{k}\times\mathbf{E}_0^t \cos\left(\omega\left(t - \frac{\mathbf{r}\cdot\mathbf{s^t}}{v_2}\right)\right),
\end{aligned}$$

for all $\mathbf{r}\in\mathbf{\Gamma}$ and all t. The interface point P is $\mathbf{r} = (\mathbf{0},\mathbf{0},\mathbf{0})$, so we obtain

$$\mathbf{k}\times\mathbf{E}_0^i \cos(\omega t) + \mathbf{k}\times\mathbf{E}_0^r \cos(\omega t) = \mathbf{k}\times\mathbf{E}_0^t \cos(\omega t)$$

for all t. Eliminating the cosines we get

$$\mathbf{k}\times\mathbf{E}_0^i + \mathbf{k}\times\mathbf{E}_0^r = \mathbf{k}\times\mathbf{E}_0^t. \tag{A.25}$$

Since we are assuming the current density vector $\mathbf{J} = 0$, it follows from [?][Section 1.1.3, formula (25)] that the tangential component of the magnetic field is also continuous across the interface. So as before with the electric field, we have $\mathbf{B}^i_{\tan} + \mathbf{B}^r_{\tan} = \mathbf{B}^t_{\tan}$ on Γ, and so

$$\mathbf{k} \times \mathbf{B}^i_0 + \mathbf{k} \times \mathbf{B}^r_0 = \mathbf{k} \times \mathbf{B}^t_0. \tag{A.26}$$

From (A.25) we obtain the equations

$$I_\perp + R_\perp = T_\perp, \qquad \cos\theta_i (I_\parallel - R_\parallel) = \cos\theta_t T_\parallel;$$

and from (A.26) we obtain

$$\frac{I_\parallel}{v_1} + \frac{R_\parallel}{v_1} = \frac{T_\parallel}{v_2}, \qquad \cos\theta_i \left(\frac{I_\perp}{v_1} - \frac{R_\perp}{v_1} \right) = \cos\theta_t \frac{T_\perp}{v_2}.$$

We have $n_1 = c/v_1$ and $n_2 = c/v_2$ so solving the last two sets of equations yields

$$\begin{aligned} T_\parallel &= \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} I_\parallel \\ T_\perp &= \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t} I_\perp \\ R_\parallel &= \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} I_\parallel \\ R_\perp &= \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} I_\perp. \end{aligned}$$

These are the *Fresnel equations* expressing the amplitudes of the reflected and transmitted waves in terms of the amplitude of the incident wave.

A.9 Rewriting the Fresnel Equations

We will replace $\mathbf{s^i}$ by x and $\mathbf{s^t}$ by m, and we also set $\kappa = n_2/n_1$. Recall ν is the normal to the interface. We have $\cos\theta_i = x \cdot \nu$ and $\cos\theta_t = m \cdot \nu$. In addition, from the Snell law $x - \kappa m = \lambda\nu$, so the Fresnel equations the have the form

$$\begin{aligned}
T_\parallel &= \frac{2\,x\cdot\nu}{\kappa\,x\cdot\nu+m\cdot\nu}\,I_\parallel = \frac{2\,x\cdot\nu}{(\kappa\,x+m)\cdot\nu}\,I_\parallel = \frac{2\,x\cdot(x-\kappa\,m)}{(\kappa\,x+m)\cdot(x-\kappa\,m)}\,I_\parallel\\
T_\perp &= \frac{2\,x\cdot\nu}{x\cdot\nu+\kappa\,m\cdot\nu}\,I_\perp = \frac{2\,x\cdot\nu}{(x+\kappa\,m)\cdot\nu}\,I_\perp = \frac{2\,x\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\,I_\perp\\
R_\parallel &= \frac{\kappa\,x\cdot\nu-m\cdot\nu}{\kappa\,x\cdot\nu+m\cdot\nu}\,I_\parallel = \frac{(\kappa\,x-m)\cdot\nu}{(\kappa\,x+m)\cdot\nu}\,I_\parallel = \frac{(\kappa\,x-m)\cdot(x-\kappa\,m)}{(\kappa\,x+m)\cdot(x-\kappa\,m)}\,I_\parallel\\
R_\perp &= \frac{x\cdot\nu-\kappa\,m\cdot\nu}{x\cdot\nu+\kappa\,m\cdot\nu}\,I_\perp = \frac{(x-\kappa\,m)\cdot\nu}{(x+\kappa\,m)\cdot\nu}\,I_\perp = \frac{(x-\kappa\,m)\cdot(x-\kappa\,m)}{(x+\kappa\,m)\cdot(x-\kappa\,m)}\,I_\perp.
\end{aligned}$$

Notice that the denominators of the perpendicular components are the same and likewise for the parallel components.

A.10 The Poynting vector

It is defined by

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B},$$

where c is the speed of light in free space. The vector $\mathbf{S}$ represents the flux of energy through a surface. Suppose dA is the area of a surface element at a point P and let ν be the normal at P. Then the energy flux of energy through dA is given by

$$dF = \mathbf{S}\cdot\nu\,dA.$$

From (A.24) we get that

$$\mathbf{S} = \frac{c}{4\pi v}\mathbf{E}\times(\mathbf{s}\times\mathbf{E}) = \frac{n}{4\pi}|\mathbf{E}|^2\mathbf{s}.$$

Using the form of the incident wave from the previous section the amount of energy, flowing through a unit area of the boundary per second, J^i of the incident wave $\mathbf{E}^i$ is then

$$J^i = |\mathbf{S}^i|\cos\theta_i = \frac{n_1}{4\pi}|\mathbf{E}_0^i|^2\cos\theta_i.$$

Similarly, the amount of energies of the reflected and transmitted waves (also given in the previous section) leaving a unit area of the boundary per second is given by

$$J^r = |\mathbf{S}^r| \cos\theta_i = \frac{n_1}{4\pi} |\mathbf{E}_0^r|^2 \cos\theta_i$$
$$J^t = |\mathbf{S}^t| \cos\theta_t = \frac{n_2}{4\pi} |\mathbf{E}_0^t|^2 \cos\theta_t.$$

The reflection and transmission coefficients are defined by

$$\mathcal{R} = \frac{J^r}{J^i} = \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2, \text{ and } \mathcal{T} = \frac{J^t}{J^i} = \frac{n_2 \cos\theta_t}{n_1 \cos\theta_i} \left(\frac{|\mathbf{E}_0^t|}{|\mathbf{E}_0^i|}\right)^2.$$

By conservation of energy or by direct verification $\mathcal{R} + \mathcal{T} = 1$.

A.11 Polarization

Polarization is a property of the field that describes the orientation of their oscillations. Since the electric vector is assumed a plane wave and as we showed it is perpendicular to the direction of propagation $\mathbf{s}$, then for each $\mathbf{r}$ in the plane $\mathbf{r} \cdot \mathbf{s} = \mathbf{c}$ and t fixed, the vector $\mathbf{E}(\mathbf{r}, t)$ is constant. We visualize $\mathbf{E}(\mathbf{r}, t)$ as a vector with origin at the intersection of the direction $\mathbf{s}$ with the plane $\mathbf{r} \cdot \mathbf{s} = \mathbf{c}$. That is, as a vector with origin at the point $(\mathbf{r} \cdot \mathbf{s})\mathbf{s}$ and terminal point $(\mathbf{r} \cdot \mathbf{s})\mathbf{s} + \mathbf{E}(\mathbf{r}, \mathbf{t})$. Then t is fixed and $\mathbf{r}$ runs over all space, the end point of this vector describes a curve in 3-d. If we now move t, this curve is shifted (and keeps the same shape) by changing the phase because of the presence of ωt in the cos function. So when $\mathbf{r} \cdot \mathbf{s} = \mathbf{c}$ and t moves the vector $\mathbf{E}(\mathbf{r}, \mathbf{t})$ describes a curve in the plane $\mathbf{r} \cdot \mathbf{s} = \mathbf{c}$. If this curve is an ellipse we say that the wave is *elliptically polarized* and when the ellipse is a circle we say the wave is *circulary polarized*, and if the the ellipse degenerates to a segment we say the wave is *linearly polarized*. If the wave describing the incident field has components that have different phases, then this changes the sense of circulation and inclination of the ellipse (for elliptically polarized light), see [BW80][Section 1.4.2]. See `http://en.wikipedia.org/wiki/Polarization` for pictures.

Suppose for example that the wave is linearly polarized perpendicularly to the plane of incidence. That is, $I_{\|} = 0$. Then from Fresnel equations

$T_\parallel = R_\parallel = 0$ and

$$\mathcal{R} = \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2 = \left(\frac{R_\perp}{I_\perp}\right)^2 = \left(\frac{|x - \kappa\, m|^2}{1 - \kappa^2}\right)^2,$$

and

$$\begin{aligned}\mathcal{T} &= \kappa\frac{m\cdot\nu}{x\cdot\nu}\left(\frac{T_\perp}{I_\perp}\right)^2 = \kappa\frac{m\cdot(x-\kappa\, m)}{x\cdot(x-\kappa\, m)}\left(\frac{2\,x\cdot(x-\kappa\, m)}{(x+\kappa\, m)\cdot(x-\kappa\, m)}\right)^2\\ &= \frac{4\kappa}{(1-\kappa^2)^2}\left(m\cdot(x-\kappa\, m)\right)\left(x\cdot(x-\kappa\, m)\right).\end{aligned}$$

For the case when no polarization is assumed, that is, radiation has no particular preference for the direction in which it vibrates, we have from Fresnel's equations that

$$|\mathbf{E}_0^r|^2 = R_\parallel^2 + R_\perp^2 = \left[\frac{(\kappa\, x - m)\cdot(x-\kappa\, m)}{(\kappa\, x + m)\cdot(x-\kappa\, m)}\right]^2 I_\parallel^2 + \left[\frac{(x-\kappa\, m)\cdot(x-\kappa\, m)}{(x+\kappa\, m)\cdot(x-\kappa\, m)}\right]^2 I_\perp^2,$$

and so

$$\begin{aligned}\mathcal{R} &= \left(\frac{|\mathbf{E}_0^r|}{|\mathbf{E}_0^i|}\right)^2 = \frac{R_\parallel^2 + R_\perp^2}{I_\parallel^2 + I_\perp^2}\\ &= \left[\frac{(\kappa\, x - m)\cdot(x-\kappa\, m)}{(\kappa\, x + m)\cdot(x-\kappa\, m)}\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[\frac{(x-\kappa\, m)\cdot(x-\kappa\, m)}{(x+\kappa\, m)\cdot(x-\kappa\, m)}\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\\ &= \frac{1}{(1-\kappa^2)^2}\left(\left[\frac{2\kappa}{x\cdot m} - (1+\kappa^2)\right]^2 \frac{I_\parallel^2}{I_\parallel^2 + I_\perp^2} + \left[1 - 2\kappa\, x\cdot m + \kappa^2\right]^2 \frac{I_\perp^2}{I_\parallel^2 + I_\perp^2}\right)\end{aligned}$$

which is a function only of $x \cdot m$. In principle the coefficients $I_\parallel$ and $I_\perp$ might depend on the direction x, in other words, for each direction x we would have a wave that changes its amplitude with the direction of propagation. The energy of the incident wave would be $f(x) = |\mathbf{E}_0^i|^2 = I_\parallel(x)^2 + I_\perp(x)^2$. Notice that if the incidence is normal, that is, $x = m$, then $\mathcal{R} = \left(\frac{1-\kappa}{1+\kappa}\right)^2$ which shows that even for radiation normal to the interface we lose energy by reflection. For example, if we go from air to glass, $n_1 = 1$ and $n_2 = 1.5$, we have $\kappa = 1.5$ so $\mathcal{R} = .04$, which means that 4% of the energy is lost in reflection.

REFERENCES

[Al42a] Aleksandrov, A.D., *Existence of a uniqueness of a convex surface with a given integral curvature,* Dokl. Akad. Nauk. SSSR 35(1942), 131-134.

[Al42b] Aleksandrov, A.D., *Smoothness of a convex surface of bounded Gaussian curvature,* Dokl. Akad. Nauk. SSSR 36(1942), 195-199.

[Al50] Aleksandrov, A.D., *Convex Polyhedra,* Gosudarstv, Moscow-Leningrad, 1950.

[Bi99] Billingsley, P. *Convergence of Probability Measures*, 2nd ed, 1999, Wiley Int.

[BW80] Born, M., and Wolf, E., *Principles of Optics,* Sixth ed., Pergamon Press, 1980.

[BU58] Busemann, H. *Convex Surfaces*, Wiley, New York, 1958.

[CH09] Caffarelli, L. A., and Huang Q., *Reflector problem in* $\mathbf{R}^n$ *Endowed with Non-Euclidean Norm*, Arch. Rational Mech. Anal. 193(2009)

[CO08] Caffarelli, L. A., and Oliker, V. A. *Weak Solutions of One Inverse Problem in Geometric Optics*, Journal of Mathematical Sciences, Vol. 154, No. 1, 2008.

[DE01] Descartes, R. *Discourse on Method, optics, Geometry, and Meteorology*, Hackett Publishing Co., 2001.

[Ev98] Evans, L.C. *Partial differential equations,* AMS, Vol 19, 1998.

[Gu01] Gutiérrez, C.E., *The Monge-Ampère Equation,* Birkhauser, 2001.

[GH09] Gutiérrez, C.E. and Huang, Q. , *The refractor problem in reshaping light beams,* Arch. Rational Mech. Anal, 193(2009).

[KA09] Karakhanyan, A. *On the regularity of weak solutions to refractor problem*, Armenian Journal of Mathematics Volume 2, Number 1, 2009, 28-37

[KW08] Karakhanyan,A. and Wang, X. *On the reflector shape design,* to appear on Journal of Differential Geometry.

[KK65] Kline, M. and Kay, I. *Electromagnetic theory and Geometrical Optics*, 1965, J Wiley and Sons, Inc.

[Le38] Lewy, H., *On differential geometry in the large I, Minkowski's problem,* Trans. Amer. Math. Soc. 43(1938), 258-270.

[MTW05] Ma, X.-N., Trudinger, N., Wang, X.-J.,*Regularity of potential functions of the optimal transportation problem*, Arch. Rational Mech. Anal. 177(2), 151183, 2005.

[PO78] Pogorelov, A. V. *The Minkowski Multidimensional Problem*, Wiley, New York, 1978.

[Sc93] Schneider, R., *Convex bodies: The Brunn-Minkowski theory,* Cambridge University Press, 1993.