A PERRON-FROBENIUS TYPE OF THEOREM FOR QUANTUM OPERATIONS

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Matthew Lagro
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Thesis approvals:

Wei-Shih Yang, Thesis Advisor, Mathematics
Daniel Szyld, Mathematics
Brian Rider, Mathematics
Chiu Tan, Computer and Information Sciences

ABSTRACT

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Matthew Lagro
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Professor Wei-Shih Yang, Chair

Quantum random walks are a generalization of classical Markovian random walks to a quantum mechanical or quantum computing setting. Quantum walks have promising applications but are complicated by quantum decoherence.

We prove that the long-time limiting behavior of the class of quantum operations which are the convex combination of norm one operators is governed by the eigenvectors with norm one eigenvalues which are shared by the operators. This class includes all operations formed by a coherent operation with positive probability of orthogonal measurement at each step. We also prove that any operation that has range contained in a low enough dimension subspace of the space of density operators has limiting behavior isomorphic to an associated Markov chain. A particular class of such operations are coherent operations followed by an orthogonal measurement.

Applications of the convergence theorems to quantum walks are given.

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To Pauline, my partner.

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CHAPTER 1

INTRODUCTION

Markov chains have been studied for just over a century. The first publication was in 1907 when Andrei Markov constructed a chain out of the first 20,000 letters of *Eugeny Onegin* as an example of a process that was dependent but still exhibited a law of large numbers type of convergence. Since then, Markov chains have become pervasive in mathematics, science, computing, finance, industry, and other fields. A small sampling of applications include models of information sources and channels in information theory [Sha48], protein folding in biochemistry [Voe15] queuing theory [Erlang1917], and hidden Markov models in speech recognition. Markov chains have proven invaluable in science and technology because they provide a model with discrete transitions that are unpredictable beyond conditioning on the current state of the system.

One of the most omnipresent examples, Google's PageRank algorithm [BPMW99] is a good example of the themes of this work. The algorithm conceptually starts with a random walk process, whereby a "surfer" at some web page follows any one of the outgoing links at random (or if there are none, any web page). Since the pages that are linked to more often are more likely to be traveled to, the average fraction of visits a given page receives captures a notion of how central a page is to the web. This process is not guaranteed to converge to a stationary distribution so the actual random walk used is a convex combination of the process above and the simpler (but ergodic) walk which just picks a succession of pages on the web at random.

Quantum computation is still in its long infancy, starting with Richard Feynman's reflections on the simulation of quantum mechanics on the computer [Fey82]. Simulation

of a quantum system is difficult because the number of variables needed even for a discrete system is exponential in the number of independent components. However, it is clear that a quantum system only consists of the (linear) number of components that it consists of. David Deutsch [Deu85] proposed the converse of the difficulty: a computing machine that is discrete and finite like a classical computer but exploits the quantum mechanical interactions between its parts to efficiently simulate systems that are too large for classical computers. Since then significant progress in the theory of quantum computation has been made, including Grover's search algorithm [Gro96] which is quadratically faster than its classical equivalent, and Shor's factoring algorithm [Sho97] which is nearly exponentially faster than its best classical competitor.

Connecting these two domains is the notion of a quantum random walk, introduced by Aharonov, Davidovich, and Zagury in 1992 [ADZ93]. A classical random walk evolves a probability density in discrete steps by a stochastic operator, a quantum walk instead evolves a vector of probability amplitudes by a unitary operator. Much like Markov chains are well suited to computational simulation, being discrete in time and space, quantum walks are well suited to representation by qubits and stepwise transformation by quantum computers.

In this work we focus our attention on discrete time quantum random walks that take their states form large but finite spaces (for example in the space given by a collection of qubits). The fundamental challenge to quantum computation is decoherence: the collapse of a coherent vector of entangled systems to a statistical ensemble of states with less interaction. To account for decoherence we model quantum systems as statistical ensembles of states represented in the density operator formulation. Density operators are transformed in time by quantum operations.

Our central results are convergence results to stationary or periodic limiting distributions for a generalized family of quantum operations. We generalize the Perron-Frobenius theorem to a special family of quantum operations we dub *Markovian*. These operations arise naturally when considering the effect of measurement or decoherence on a quantum process. The spectral properties of Markovian operations give elegant convergence of any initial density operator in terms of the eigenvectors of the operation. We additionally show that the convergence of any quantum operation which is the convex combination of a Markovian operation and any other operation of norm less than or equal to one has

convergence to a limit governed by the eigenvectors of the Markovian operation. The representation of an operation as the convex combination of a fully decoherent operation and a coherent one is a common model of decoherence for quantum systems.

Our results show that the limiting behavior of a large family of quantum systems is governed by the behavior of a related classical process.

CHAPTER 2

BACKGROUND

2.1 Introduction

In order to best present the formulation of quantum mechanics that we use in this work we present the definitions in an interative fashion. The greatest difficulties come from the connected issues of measurement and indeterminacy. We first discuss the case of an isolated, undisturbed system evolving without measurement. Next we present the simplest cases of measurement of a system in an orthonormal basis, then generalize to arbitrary measurements. To describe statistical ensembles of states we introduce the density operator formulation. Finally, we present quantum operators which transform ensembles (as represented by density operators) in time.

This work focuses on quantum walks which are processes on a discrete space. These discrete spaces have particularly natural representations in terms of finite or countable bases. Because we perform so many manipulations of vectors and their duals, we will use the convenient bra-ket notation borrowed from physics.

2.2 Coherent evolution of quantum systems

Axiom 1. *The state space of a quantum system can be represented as a Hilbert space over the complex numbers. That is, a space $\mathcal{H}$ which satisfies:*

1. *$\mathcal{H}$ is a vector space with scalars in $\mathbb{C}$.*

2. *$\mathcal{H}$ is an inner product space, with inner product $\langle|\rangle : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ where for any $u, v, w \in \mathcal{H}$ and $\alpha \in \mathbb{C}$*

   - *The inner product is linear in the second component, $\langle u|\alpha v + w\rangle = \alpha \langle u|v\rangle + \langle u|w\rangle$*
   - *The inner product is skew-symmetric, $\langle v|u\rangle = \overline{\langle u|v\rangle}$*
   - *The inner product is positive definite, $\langle u|u\rangle$ is real and non-negative, and equals 0 if and only if $u = 0$.*

Coherent evolution of a quantum system is evolution without disturbance or observation from the outside environment. We formalize this as evolution by unitary linear maps.

Axiom 2. *An isolated quantum system that evolves any initial state $\phi_0 \in \mathcal{H}$ at time 0 to a state $\phi_t \in \mathcal{H}$ at time t may be written as a unitary map $U : \mathcal{H} \to \mathcal{H}$ that is, for a given process there exists a unitary U such that for any $\phi_0 \in \mathcal{H}$, $\phi_t = U\phi_0$.*

Definition 2.1. *Recall that a* unitary map *is a linear map the inverse of which is the Hermitian conjugate, that is $U^{-1} = U^*$ or $UU^* = U^*U = \mathrm{Id}_{\mathcal{H}}$. In the case of finite dimensional Hilbert spaces, this is equivalent to U^{-1} is the complex-conjugate transpose of U, that the columns of U considered as vectors in $\mathcal{H}$ are orthogonal and norm 1 in the inner product of $\mathcal{H}$, or that the rows of U are orthonormal vectors in $\mathcal{H}$.*

Several facts are implicit in the above axiom. The first is the *principle of superposition*: the system is linear in the evolution of its states, that is if u_0, v_0 are states in $\mathcal{H}$, and $u_t = Uu_0$, $v_t = Uv_0$, then $u_t + v_t = U(u_0 + v_0)$. Since $\mathcal{H}$ is a *complex* Hilbert space, for any state can be multiplied by any complex number of norm 1 and retain the same norm and lie in the same one-dimensional subspace. These two properties are analogous to the behavior of linear physical systems of waves; that is the sum of two waves is a wave, many physical processes are linear and show superposition, and two waves of identical amplitude and frequency may differ in phase. By analogy with waves we say that u and αu differ in phase.

Axiom 3. *If a quantum system is composed of two sub-systems with state spaces $\mathcal{I}, \mathcal{J}$ respectively then the state space of the combined system is*

$$\mathcal{H} = \mathcal{I} \otimes \mathcal{J}.$$

2.3 Measurement of quantum systems

Definition 2.2. *Let $A_i \in L(\mathcal{H})$. A family of linear operators $\{A_1, \ldots, A_m\}$ is called a measurement if*

$$\sum_{i=1}^{m} A_i^* A_i = \mathrm{Id}_{\mathcal{H}}$$

here $\mathrm{Id}_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$.

Definition 2.3. *We say that a measurement is unital if $\sum_i A_i A_i^* = 1_{\mathcal{H}}$. A unital measurement forms a quantum operation which preserves the identity matrix (corresponding to the uniform distribution over all vectors in $\mathcal{H}$).*

2.4 Ensembles, density operators, and quantum operations

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Let $L(\mathcal{H})$ be the space of linear operators from $\mathcal{H}$ to $\mathcal{H}$, with inner product $\langle A|B\rangle = \mathrm{Tr}\,(A^*B)$.

A linear operator $\rho \in L(\mathcal{H})$ is called a density operator (or state) if $\rho \geq 0$ and $\mathrm{Tr}\,(\rho) = 1$, here $\rho \geq 0$ means $\langle \phi, \rho\phi\rangle \geq 0$, for all $\phi \in \mathcal{H}$. We call the set of all density operators $\mathcal{E} \subset L(\mathcal{H})$.

It is well-known that if $\rho \geq 0$, then ρ is self-adjoint. It follows by the Spectral theorem that $\rho = \sum_{i=1}^{n} p_i |\phi_j\rangle \langle\phi_j|$, for some orthonormal basis $\{\phi_j\}$ of $\mathcal{H}$, with $p_i \geq 0$, and $\sum_{i=1}^{n} p_i = 1$ since $\mathrm{Tr}\,(\rho) = 1$.

Definition 2.4. *A linear operator $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$ is called a quantum operation if there is a measurement $\{A_1, \ldots, A_m\}$ such that*

$$\Phi(B) = \sum_{i=1}^{m} A_i B A_i^*, \tag{4.1}$$

for all $B \in L(\mathcal{H})$. We shall call Φ the quantum operation associated with $\{A_1, \ldots, A_m\}$.

It can be easily checked that $\Phi(B)$ is a density operator if B is a density operator. By the postulates of quantum statistical mechanics, a density operator ρ known to an observer becomes $\Phi(\rho) = \sum_{i=1}^{m} A_i \rho A_i^*$ after a measurement $\{A_1, \ldots, A_m\}$ has been performed somewhere and the outcomes of the measurement have not been known to the

observer. Therefore, the operation Φ that gives the transition of ρ from before to after a measurement is a quantum operation.

Axiom 4. *The effect of measurement on a quantum system described as a density matrix is*

$$\rho' \Phi(\rho) = \sum_i A_i \rho {A^*}_i$$

and the probability of observing any given measurement outcome is

$$\mathrm{Tr}\left({A_i}^* A_i \rho\right)$$

2.5 Classical Markov chains

Markov chains are stochastic processes on countable state spaces that have the Markov property. Intuitively they are processes with transitions between discrete states with transition probabilities that depend only on the current state of the system, not its history.

Definition 2.5. *A* stochastic process *is a collection of random vectors X_t indexed by $t \in I$ a totally ordered set. That is, $X : I \times \Omega \to V$ where $X_t(\cdot)$ is a measurable function from probability space Ω to V for every $t \in I$.*

"Random vector" should be understood here to be an ordered tuple of values, or an indexed set of values, and V to be the collection of possible random vectors. V needs to be a measure space with a Borel measure but it does not strictly need to be a vector space. In practice V is generally $\mathbb{R}^n$ or $\mathbb{C}^n$.

Intuitively, a stochastic process is a process in time where the values at some time t are random in some way.

Definition 2.6. *A stochastic process $X : I \times \Omega \to V$ is a* Markov chain *if $I \subseteq \mathbb{Z}$, the range $X(I, \Omega) =: B \subseteq V$ is a countable set, and X has the Markov property that the conditional probability of transition is independent of the history of X, that is for any finite collection of times $s > t > t_1, \ldots, t_k$ and $x, y, y_1, \ldots y_k \in V$*

$$\Pr\{\, X_s = x | X_t = y \,\} = \Pr\{\, X_s = x | X_t = y, X_{t_1} = y_1, \ldots, X_{t_k} = y_k \,\}.$$

That is, a Markov chain is a stochastic process on a discrete set of times (or alternatively which is constant on the intervals $[t, t+1)$) and for which the conditional probability of being at some state at time s given the state at some previous time t is independent of all the states of the chain at times before t.

A Markov chain can have some structural conditions about which states can transition to which states. We define the three most important below.

Definition 2.7. *A Markov chain is* irreducible *if for any two states $x, y \in V$ and time t there exists a time $s > t$ such that*

$$\Pr[X_s = y | X_t = x] > 0.$$

That is, there is a time such that the probability of transitioning from any given state to any other given state is non-zero.

Definition 2.8. *A state $x \in V$ is* d-periodic *if for any time t and any time s such that if $\Pr[X_s = x | X_t = x] > 0$ then $t \equiv s \mod d$.*

A Markov chain is d-periodic if every state $x \in V$ is d-periodic.

Definition 2.9. *A state $x \in V$ is* positive recurrent *if for any time $t \in I$*

$$\Pr[X = x \text{ i.o.} | X_t = x] = 1.$$

That is, if any state reappears infinitely often in its future orbit. A Markov chain is positive recurrent if every state in $x \in B$ is positive recurrent.

Finally as a specific example of a Markov chain, consider a random walk on a graph. The random walk describes a stochastic process where transitions from one state to another occur at discrete time steps and the next state is chosen randomly from the states adjacent on the graph to the current state. This satisfies the definition of a Markov chain since the set of states (vertices) is countable and the transition probabilities are dependent only on the current state independent of previous states.

A basic question to answer is the probability of being at any given state at a long time in the future, or similarly the fraction of the time spent at a given state if the walk is allowed to proceed forever. This question has practical implications, for example Google's PageRank algorithm [BPMW99] ranks a web page according to the limit of the

probability that a random walk on the graph formed by links between web pages is at the ranked page as the number of time steps goes to infinity. To make this more precise we will consider Markov chains as linear operators on the space of distributions over the states of the chain.

A Markov chain was defined as a stochastic process, but the definition was given in terms of conditional probabilities. We can therefore view a Markov chain as a transformation of probability distributions in the following manner. Suppose $x \in B$ and $s > t \in I$. Then

$$\Pr(X_s = x) = \sum_{y \in B} \Pr(X_s = X | X_t = y) \Pr(X_t = y),$$

but this sum is of the same form as a matrix-vector product. If we take the space of distributions on B, $C(B)$ with independent set of vectors $\{ \delta_x : x \in B \}$ then we have the linear transformation $A : \delta_x \mapsto \sum_{x \in B} \Pr(X_s = x | X_t = y) \delta_y$. In the finite dimensional case the delta functionals δ_x span the space of distributions on B, hence we can represent a transition by the Markov chain of a probability distribution over states of the chain at time t to a probability distribution at time s by a stochastic matrix. Because the Markov property applies to Markov chains, we have for $s > t > t_1$ that $\Pr(X_s = x | X_t = y, X_{t_1} = y_1) = \Pr(X_s = x | X_t = y)$ therefore

$$\begin{aligned}
\Pr(X_s = x) &= \sum_{y \in B} \Pr(X_s = x | X_t = y, X_{t_1} = y_1) \Pr(X_t = y) \\
&= \sum_{y \in B} \Pr(X_s = X | X_t = y) \Pr(X_t = y) \\
&= \sum_{y \in B} \Pr(X_s = X | X_t = y) \sum_{y_1 \in B} \Pr(X_t = y | X_{t_1} = y_1) \Pr(X_{t_1} = y_1) \\
&= \sum_{y \in B} \sum_{y_1 \in B} \Pr(X_s = X | X_t = y) \Pr(X_t = y | X_{t_1} = y_1) \Pr(X_{t_1} = y_1).
\end{aligned}$$

Hence the transition matrix for times $s > t_1$ is the product of the transition matrix for times $s > t$ and $t > t_1$.

2.6 The Perron-Frobenius theorem and Markov chains

The Perron-Frobenius gives strong conditions on the nature of the largest eigenvalues of an irreducible non-negative square matrix and their associated eigenspaces. Since the stochastic matrix for a Markov chain is necessarily non-negative and square we only need irreducibility to satisfy the Perron-Frobenius theorem. Large powers of a matrix are more easily understood in their eigenbasis, so we can control the behavior of the walk as time goes to infinity.

Theorem 2.10. *If A is an irreducible non-negative square matrix with period d and has spectral radius r (i.e. the modulus of the largest eigenvalue is r) then r is a simple eigenvalue of A and has strictly positive left and right eigenvectors, no other eigenvalue has strictly positive eigenvectors, each one of the d many d-roots of unity ω^k $r\omega^k$ is a simple eigenvalue of A, and all other eigenvalues are strictly smaller than r in modulus.*

In lemma 4.6 we will show that as $n \to \infty$, $v_n = A^n v$ becomes arbitrarily close $A^n w$, where w is a sum of the eigenvectors matching the modulus r eigenvalues. We give a more specific version of this convergence behavior in corollary 4.7 and a more detailed version of the Perron-Frobenius theorem tailored to Markov chains in lemma 5.4.

2.7 A comment about vector notation

Our results are an application of results from linear algebra and stochastic processes to the physical setting of quantum computing. Unfortunately, there are differences of notation between the fields. Most notable is that quantum computing prefers the bra-ket notation with operators applied on the left to kets on the right. Most references on linear algebra focus on operators applied on the left to column vectors on the right. The standard convention in probability is to represent distributions over a finite set of states as a row vector multiplied on the left against a row stochastic matrix on the right. We have chosen to follow these conventions for results which are in one particular subject.

Therefore our linear algebra lemmata are written with the conventions that a vector v is a column vector with components $(v)_i$ and that the product $(Av)_i = \sum_{j=1}^{n} a_{ij} v_j$.

Our notation for Markov chains is the transpose of this, with $(vA)_j = \sum_{i=1}^{n} a_{ij} v_i$. The two conventions only appear together in proposition 3.4 where a quantum operation is related to a special Markov chain.

It has also proved useful to make use of notation from quantum mechanics. In particular, we adopt the braket notation where the ket $|x\rangle$ is a vector in a Hilbert space $\mathcal{H}$ and x is understood as a purely formal label identifying the vector. In certain cases the labels x have some other mathematical structure (for example the members of a group for a walk on the group's Cayley graph). In this case we have tried to arrange the definitions so that relationships between the kets in $\mathcal{H}$ correspond to the natural relationship between the objects. However, it is important to notice that the Hilbert space is distinct from the object from which we draw the labels, for example $\mathbb{Z}$ is a cyclic group with a single generator but the corresponding position space $\mathcal{P}$ is a Hilbert space isomorphic to ℓ^2. Although $1 + 2 = 3$, $|1\rangle + |2\rangle \neq |3\rangle$, in fact they are in orthogonal subspaces, $|1\rangle + |2\rangle \perp |3\rangle$. However the shift operator is written to use the addition structure on $\mathbb{Z}$ to simplify the notation.

We also have adopted the convention from physics that the inner product $\langle v|w\rangle$ is conjugate symmetric $\langle w|v\rangle = \overline{\langle v|w\rangle}$ and linear in the *second* variable w, that is $\langle v|\alpha w\rangle = \alpha \langle v|w\rangle$ whereas $\langle \beta v|w\rangle = \overline{\beta} \langle v|w\rangle$. This choice of convention simplifies the notation for probability amplitudes and expected values of observables such as $\langle \phi|M^* M|\phi\rangle$. The dual of a ket is the bra with the same label $(|x\rangle)^* = \langle x|$. The inner product $\langle v, w\rangle$ is then just the product of the dual of a vector and a vector $\langle v|w\rangle$, or in typical linear algebra notation $v^* w$.

CHAPTER 3

DEFINITIONS AND BASIC PROPOSITIONS

3.1 Diagonal spaces and Markovian operations

A quantum operation is the quantum analogue of the transition matrix of a Markov chain in classical dynamics. For some cases it is exactly classical Markov transition matrix and for some other cases, it is completely quantum and, more generally, it appears to be a mixture of both classical and quantum dynamics as we shall see below.

Definition 3.1. *Let $E = \{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathcal{H}$.*

$L(\mathcal{H})$ is the linear space composed of linear operators $\mathcal{H} \to \mathcal{H}$ which has basis $\{|e_i\rangle \langle e_j| : e_i, e_j \in E\}$

The set of all density operators in $L(\mathcal{H})$ is denoted $\mathcal{E}$.

We define $\mathcal{C}$ as the linear space spanned by $\mathcal{E}$. We define $\mathcal{C}_d$ as the linear space spanned by the diagonal projectors, $\mathrm{Span}\,\{|e_i\rangle \langle e_i|\}_{i=1}^n$, *$\mathcal{C}_d \subset L(\mathcal{H})$.*

The classical domain $\mathcal{E}_d$ (with respect to $\{e_i\}_{i=1}^n$) is the set of all density matrices in the linear space spanned by the projectors, $\{|e_i\rangle \langle e_i|\}_{i=1}^n$. Equivalently, this is $\mathcal{E}_d = \mathcal{E} \cap \mathcal{C}_d$.

Note that $\{|e_i\rangle \langle e_i|\}_{i=1}^n$ is a measurement and the classical domain with respect to $\{e_i\}_{i=1}^n$ is the range of the quantum operation associated with $\{|e_i\rangle \langle e_i|\}_{i=1}^n$.

Definition 3.2. *A quantum operation $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$ is called Markovian with respect to an orthonormal basis $\{e_i\}_{i=1}^n$ if $\Phi : \mathcal{E}_d \to \mathcal{E}_d$ and for any $B \in L(\mathcal{H})$, there exists t such*

that $\Phi^t(B) \in \mathcal{E}_d$, where $\mathcal{E}_d$ is the classical domain with respect to $\{e_i\}_{i=1}^n$. In this case, we shall call $\mathcal{E}_d$ the classical domain of Φ.

3.2 Markovian parts and matrices for quantum operations

Definition 3.3. *Let $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$ be Markovian with respect to $\{e_i\}_{i=1}^n$ and $\mathcal{E}_d$ its classical domain. The Markovian part Φ^M of Φ is defined by the restriction of Φ on $\mathcal{E}_d$.*

Proposition 3.4. *Suppose Φ is a Markovian quantum operator, then associated Markov chain A^Φ defined by*

$$\left(A^\Phi\right)_{jk} = \mathrm{Tr}\left(|e_k\rangle\langle e_k| \, \Phi(|e_j\rangle\langle e_j|)\right)$$

is a row stochastic matrix and $\left(A^\Phi\right)^n = A^{(\Phi^n)}$.

Furthermore, there is an linear isomorphism T between $\mathcal{C}_d$ and the space of functions $C(B)$ which commutes with Φ and A^Φ, that is $T \circ A^\Phi = T \circ \Phi$.

Proof. The operation Φ is trace-preserving, hence

$$\sum_j \left(A^\Phi\right)_{jk} = \sum_j \mathrm{Tr}\left(|e_k\rangle\langle e_k| \, \Phi\left(|e_j\rangle\langle e_j|\right)\right) = \mathrm{Tr}\left(\Phi\left(|e_j\rangle\langle e_j|\right)\right) = 1$$

Further, $\left(A^\Phi\right)_{jk} \geq 0$ because $\Phi\left(|e_j\rangle\langle e_j|\right)$ is a density matrix, and therefore non-negative.

Now consider $\Phi^n\left(|e_j\rangle\langle e_j|\right)$. By induction assume that $\left(A^\Phi\right)^n = A^{(\Phi^n)}$.

We define a bijection T between $\mathcal{C}_d$ and the space $C(B)$ of functions on the state space B of A^Φ via $T : |e_j\rangle\langle e_j| \to \delta_{e_j}(x)$. Then we note that the j-th component of $T\left(|e_i\rangle\langle e_i|\right) A^\Phi$ is

$$\sum_k \left(A^\Phi\right)_{jk} = \sum_k \mathrm{Tr}\left(|e_k\rangle\langle e_k| \, \Phi\left(|e_i\rangle\langle e_i|\right)\right),$$

but this is the same as the j-th component of $T\left(\Phi\left(|e_i\rangle\langle e_i|\right)\right)$ So we have that $T \circ \Phi = A^\Phi \circ T$ that is, Φ and A^Φ are homomorphic. Finally, $T^{-1}A^\Phi T = \Phi$, so Φ and A^Φ are actually isomorphic. □

Definition 3.5. *We call the matrix A^Φ in proposition (3.4) the* Markov matrix *in $\mathcal{E}_d$ associated with Φ.*

By Proposition 3.4, there is a Markov chain with state space $\{i = 1, 2, ..., n\}$ such that the transition probability from k to j is $(A^{\Phi})_{kj}$. We shall call A^{Φ} the Markov matrix associated with the Markovian quantum operation Φ. The matrix representation of Φ^M with respect to the basis $\{e_i\}_{i=1}^n$ is $(A^{\Phi})^*$.

3.3 The Perron-Frobenius properties

Definition 3.6. *We say that an operator* Φ *has the* Perron-Frobenius properties *if* 1 *is a simple eigenvalue of* Φ *with an eigenvector* $\rho \in \mathcal{E}$ *and all other eigenvalues* λ *are strictly smaller in modulus,* $|\lambda| < 1$.

Notice that if an operator Φ has the Perron-Frobenius properties on the whole vector space the definition above is equivalent to one result of the Perron-Frobenius theorem, namely that the operator has a simple dominant eigenvalue larger in absolute value than all other eigenvalues. The restriction to the set of density operators $\mathcal{E}$ is analogous to the existence of a non-negative dominant eigenvector.

The Perron-Frobenius properties are fundamental for the limiting behavior of the dynamics generated by the operator. For a Markov transition matrix, they are essential for the ergodicity of the Markov chain.

Our goal in this dissertation is to give a sufficient condition for a quantum operation to satisfy the Perron-Frobenius properties on $\mathcal{E}$. Our theorems are generalizations of the Perron-Frobenius Theorem.

CHAPTER 4

CONVERGENCE OF CONVEX COMBINATION OPERATORS

4.1 Eigenvalues of convex combinations

For a linear operator A we denote the spectrum $\sigma(A) = \{\lambda; \lambda \text{ is an eigenvalue of } A\ \}$. For $\lambda \in \sigma(A)$, we let $dim_A(\lambda)$ be the geometric dimension of λ and $mult_A(\lambda)$ the algebraic dimension of λ. It is known that $1 \leq dim_A(\lambda) \leq mult_A(\lambda)$.

Quantum operators map the set of states $\mathcal{E}$ to itself. Since we are interested in the behavior of a quantum operator as an iterated linear map, but only need its action on $\mathcal{E}$, we will generalize concepts with regard to a restricted subset of vectors.

Definition 4.1. *We say that $S \subseteq \mathcal{H}$ is* closed under operator A if the image of S is contained in S, that is $A(S) \subseteq S$.

Since quantum operators map density operators to density operators, the set $\mathcal{E}$ is closed under any quantum operator.

Definition 4.2. *Let V, W be normed vector spaces. We define the induced operator norm of an operator $A : V \to W$ by*

$$\|A\| := \sup_{x \in V} \frac{\|Ax\|_W}{\|x\|_V}.$$

Definition 4.3. *Let $L(V,W)$ be the space of linear operators from normed spaces V to W. A norm on the space $L(V,W)$ is* consistent *if for every $x \in V$,*

$$\|Ax\| \leq \|A\|\|x\|.$$

By definition every induced norm is consistent.

Lemma 4.4. *Let $A : V \to V$ be a linear operator. Suppose for some consistent norm $\|A\| \leq 1$. If λ is an eigenvalue of A with $|\lambda| = 1$ then there are no properly generalized eigenvectors for eigenvalue λ in $\mathcal{H}$.*

Proof. Suppose to the contrary that ρ_i is a generalized eigenvector of order $i \geq 2$ for λ with $|\lambda| = 1$; i.e. $(A-\lambda)^{i-1}\rho_i \neq 0$ and $(A-I)^i\rho_i = 0$. Then

$$\begin{aligned} A^m\rho_i = (A - \lambda I + \lambda I)^m\rho_i &= \sum_{k=0}^{m} \binom{m}{k} \lambda^{m-k}(A-\lambda I)^k\rho_i \\ &= \sum_{k=0}^{i-1} \binom{m}{k} \lambda^{m-k}(A-\lambda I)^k\rho_i. \end{aligned} \tag{1.1}$$

Since $\|A^m\rho_i\| \leq \|A^m\|\|\rho_i\|$ we have $\left\|\sum_{k=0}^{i-1} \binom{m}{k} \lambda^{m-k}(A-\lambda I)^k\rho_i\right\| \leq \|\rho_i\|$.

We have the lower bound on binomial coefficients $\frac{m^k}{k^k} \leq \binom{m}{k}$ and since $k \leq i-1$ in the summation, we can divide by m^{i-1} to get $\left(k^k m^{k-i+1}\right)^{-1} \leq m^{1-i}\binom{m}{k}$.

We can therefore divide (1.1) by m^{i-1} and take norms to get

$$\begin{aligned} m^{1-i}\|\rho_i\| &\geq m^{1-i}\left\|\sum_{k=0}^{i-1} \binom{m}{k} \lambda^{m-k}(A-\lambda I)^k\rho_i\right\| \\ &\geq \left\|\sum_{k=0}^{i-1} \lambda^{m-k}\frac{1}{k^k m^{k-i+1}}(A-\lambda I)^k\rho_i\right\| \geq 0. \end{aligned}$$

However, $\lim_{m\to\infty} \frac{1}{k^k m^{k-i+1}} = 0$ except when $k-i+1 = 0$, hence taking limits of the finite sum we get

$$\begin{aligned} \lim_{m\to\infty} m^{1-i}\|\rho_i\| = 0 &\geq \left\|\sum_{k=0}^{i-1} \lim_{m\to\infty} \lambda^{m-k}\frac{1}{k^k m^{k-i+1}}(A-\lambda I)^k\rho_i\right\| \\ &= \left\|\lambda^{m-i+1}\frac{1}{(i-1)^{i-1}}(A-\lambda I)^{i-1}\rho_i\right\| \geq 0. \end{aligned}$$

Therefore, we get that $\left\| \frac{1}{(i-1)^{i-1}} (A - \lambda I)^{i-1} \rho_i \right\| = 0$ which contradicts the claim that ρ_i is a generalized eigenvalue of order i. □

The next lemma follows a similar argument about the convergence of convex combinations of operators in [LP11b] and [LP11a], as well as in [XY13]:

Lemma 4.5. *Let $\mathcal{H}$ be a strictly convex space, such as a Hilbert space, and $A, B \in L(\mathcal{H})$. Let $L := pA + (1-p)B$ with $0 < p < 1$. Suppose $\|A\|, \|B\| \leq 1$, λ is an eigenvalue of L where $|\lambda| = 1$, with matching eigenvector ρ. Then (λ, ρ) is an eigenpair of both A and B.*

Proof.

$$\begin{aligned} \|\rho\| &= \|\lambda\rho\| && (1.2)\\ &= \|pA\rho + (1-p)B\rho\| && (1.3)\\ &\leq \|pA\rho\| + \|(1-p)B\rho\| && (1.4)\\ &\leq p\|A\|\|\rho\| + (1-p)\|B\|\|\rho\| && (1.5)\\ &\leq \|\rho\|, && (1.6) \end{aligned}$$

thus all the terms are equal. In a strictly convex space, $\|x + y\| = \|x\| + \|y\|$ if and only if $y = ax$ where $a > 0$. Hence by (1.4) we have $A\rho$ and $B\rho$ positive scalar multiples of each other. Moreover, the sum of the two is $\lambda\rho$, therefore $A\rho = a\lambda\rho$ and $B\rho = b\lambda\rho$. Further by (1.5) we have $\|A\rho\| = \|\rho\|$ and $\|B\rho\| = \|\rho\|$. Therefore, $a = b = 1$ and (λ, ρ) is an eigenpair of both A and B. □

4.2 Convergence in terms of generalized eigendecompositions

Lemma 4.6. *Suppose V is a normed vector space and $A : V \to V$ is a linear operator.*

If $x \in V$ can be written with generalized eigenvectors of A as

$$x = y + \sum_{j=1}^{N} \beta_j w_j,$$

where N is either finite or $+\infty$, the coefficient sum is absolutely convergent, i.e. $\sum_{j=1}^{N} |\beta_j| < \infty$, the vectors w_k are generalized eigenvectors of A with ranks bounded by ℓ for bounded eigenvalues $|\mu_j| \leq s < r$, then

$$r^{-n}A^n x = r^{-n}A^n y + O\big(n^\ell s^n r^{-n}\big).$$

Proof. Since A is linear we may consider each vector in the sum separately.

Take a non-dominant generalized eigenpair (μ_j, w_j) with rank ℓ.

$$\begin{aligned} A^n w_j &= (A - \mu_j I + \mu_j I)^n w_j = \sum_{i=0}^{n} \binom{n}{i} (A - \mu_j I)^i {\mu_j}^{n-i} w_j \\ &= \sum_{i=0}^{\ell-1} \binom{n}{i} (A - \mu_j I)^i {\mu_j}^{n-i} w_j. \end{aligned}$$

Notice that since $i < \ell$ we have $\binom{n}{i} < Cn^{\ell-1}$ for some constant C while $|{\nu_j}^{n-i}| < |{\nu_j}^{-i}| s^n$. We have assumed that $|\mu_j| \leq s$. Therefore

$$\begin{aligned} \big\|r^{-n}A^n w_j\big\| &= \left\| r^{-n} \sum_{i=0}^{\ell-1} \binom{n}{i} (A - \mu_j I)^i {\mu_j}^{n-i} w_j \right\| \\ &\leq Cn^{\ell-1}\Big(\frac{s}{r}\Big)^n \left\| \sum_{i=0}^{\ell-1} (A - \mu_j I)^i {\mu_j}^{-i} w_j \right\|. \end{aligned}$$

Since $\sum_{j=1}^{N} |\beta_j|$ is absolutely convergent, we can write $\|r^{-n}A^n w_j\| \leq C' n^{\ell-1}\big(\frac{s}{r}\big)^n$ and so

$$\left\| r^{-n}A^n \left(\sum_{j=1}^{N} \beta_j w_j \right) \right\| \leq C' n^{\ell-1}\Big(\frac{s}{r}\Big)^n \sum_{j=1}^{N} |\beta_j|.$$

□

Corollary 4.7. *If A is a linear operator on a normed space V with dimension $N < \infty$ and A has spectral radius 1, then for any $x \in V$ we can write*

$$x = \sum_{k=1}^{m} \alpha_k v_k + \sum_{k=m+1}^{N} \beta_k w_k$$

where the v_k are generalized eigenvectors of A with eigenvalues $|\lambda_k| = 1$ and the w_k are generalized eigenvectors with eigenvalues $|\lambda_k| \leq s < 1$ and then

$$A^n x = \sum_{k=1}^{m} \alpha_k A^n v_k + O(s^n).$$

Proof. By the Jordan normal form there is a generalized eigenbasis of A for V. Since we assumed spectral radius 1 we can divide the eigenvalues λ_k into classes $|\lambda_k| = 1$ and $|\lambda_k| < 1$. Since the generalized eigenbasis is finite the coefficient sums are absolutely convergent and the ranks of the generalized eigenvectors are all bounded by $N < \infty$. Since there are only finitely many vectors in the second class it has a maximum norm, $|\lambda_k| \leq s < 1$.

Therefore, we meet all the conditions of lemma (4.6) which gives us the conclusion. □

Lemma 4.8. *Let $A \in L(\mathcal{H})$ with $\|A\| \leq 1$, where $\mathcal{H}$ is a finite dimensional Hilbert space. Then A satisfies the Perron-Frobenius properties if and only if there exists $\phi_\infty \neq 0$ such that for any $\phi \in \mathcal{H}$, $\lim_{m\to\infty} A^m(\phi) \to \alpha(\phi)\phi_\infty$ and $\lim_{m\to\infty} A^m \neq 0$.*

Proof. Suppose A satisfies the Perron-Frobenius properties. We note that since A satisfies the Perron-Frobenius properties by lemma (4.4) there is only one eigenvalue with $|\lambda| = 1$ which is $\lambda = 1$, it has an eigenvector $\phi_1 \in \mathcal{E}$, and there are no linearly independent generalized eigenvectors for the eigenvalue $\lambda = 1$ in $\mathcal{E}$.

Write ϕ in the generalized eigenbasis of A, $\phi = \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_n\phi_n$. By corollary (4.7), $A^n = \alpha_1\phi_1 + O(s^n)$. Therefore $A^m\phi \to \alpha_1\phi_1$ as $m \to \infty$.

To show the converse, again write ϕ in the generalized eigenbasis of A,

$$\phi = \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_n\phi_n.$$

Then by Lemma (4.6),

$$r^{-n}A^n\phi = r^{-n}A^n \sum_{k=1}^{m} \alpha_k v_k + O\big(n^\ell s^n r^{-n}\big) \tag{2.7}$$

where the maximal rank of a generalized eigenvector $\ell < N$, r is the modulus of the largest eigenvalue, and s is the largest modulus of an eigenvalue which is smaller than r.

We proceed to eliminate the possibilities that $r \neq 1$, that there are other eigenvalues $|\lambda| = 1$, or that 1 is not a simple eigenvalue.

First suppose $r = 0$, that is all eigenvalues are 0. By taking the Jordan normal form to the N-th power we get $A^N = 0$ but this contradicts the assumption that $\lim_{n\to\infty} A^n \neq 0$.

Second, $1 \leq r$. We have a nonzero ϕ_∞ such that $\lim_{n\to\infty} A^n\phi = \alpha\phi_\infty$. Since all linear operators on finite dimensional spaces are continuous this implies

$$A(\alpha(\phi_\infty)\phi_\infty) = \lim_{n\to\infty} AA^n(\phi_\infty) = \lim_{n\to\infty} A^{n+1}(\phi_\infty) = \alpha(\phi_\infty)\phi_\infty.$$

So 1 is an eigenvalue of A with eigenvector ϕ_∞.

Third, consider the case $r > 0$. Then there is an eigenvector ϕ_r for $|\lambda_r| = r$. Thus $A^n\phi_r = {\lambda_r}^n\phi_r$ but $\lim_{n\to\infty} |{\lambda_r}^n| = \infty$ which contradicts the assumption that there is a constant $\alpha(\phi_r)$ such that $\lim_{n\to\infty} A^n\phi_r = \alpha(\phi_r)\phi_\infty$.

Fourth for a contradiction assume that there is another eigenvalue $\lambda \neq 1$. Again by the Jordan form, there is at least one eigenvector ϕ_λ for this eigenvalue. Then

$$A^n\phi_\lambda = \lambda^n\phi_\lambda$$

which converges to a limit if and only if $\lim_{n\to\infty} \lambda^n$ exists, but since $|\lambda| = 1$ the limit exists only if $\lambda = 1$.

Fifth if there is more than one independent eigenvector for $\lambda = 1$, say ϕ_∞ and ϕ_1 then $\lim_{n\to\infty} \phi_1 = \phi_1 \neq \phi_\infty$ contradicting the assumption that there is a unique limiting vector.

Finally, suppose eigenvalue 1 has a generalized eigenvector ρ. From the Jordan normal form we can assume without loss of generality that there is a ρ of rank 2. Then $A^n\rho = \sum_{k=0}^{n} \binom{n}{k}(A-I)^k I^{n-k}\rho = \rho + n(A\rho - \rho)$ which does not converge to a limit unless $A\rho - \rho = 0$ which contradicts the assumption that ρ is a generalized eigenvector.

Therefore we can conclude that 1 is a simple eigenvalue of A and all other eigenvalues are smaller in modulus than 1, that is A has the Perron-Frobenius properties.

□

CHAPTER 5

PERRON FROBENIUS TYPE THEOREMS FOR QUANTUM OPERATORS

5.1 Aperiodic case

Theorem 5.1. *Suppose* Φ_1 *and* Φ_0 *quantum operators with* $\|\Phi_1\|, \|\Phi_0\| \leq 1$. *Then for* $\Phi_p := (1-p)\Phi_0 + p\Phi_1$ *the following are equivalent:*

1. Φ_p *has the Perron-Frobenius property for all* $0 < p \leq 1$,
2. Φ_1 *has the Perron-Frobenius property,*
3. *There exists* $\rho_\infty \in \mathcal{E}$ *such that for any* $\rho \in \mathcal{E}$, $\Phi_1^n(\rho) \to \alpha\rho_\infty$ *as* $n \to \infty$,
4. *There exists* $\rho_\infty \in \mathcal{E}$ *such that for any* $\rho \in \mathcal{E}$, $\Phi_p^n(\rho) \to \alpha\rho_\infty$ *as* $n \to \infty$.

Proof. Condition 1 directly implies Condition 2.

Using Lemma 4.5 we can see that condition 2 implies 1. Since Φ_0, Φ_1 are quantum operators they satisfy all three requirements of the lemma on $\mathcal{E}$. Therefore any eigenvector ρ eigenvalue $|\lambda| = 1$ for Φ_p is also an eigenvector of eigenvalue λ for Φ_1.

It follows from Lemma 4.8 that Conditions 1 and 4 are equivalent and Conditions 2 and 3 are equivalent. □

We have the following theorem for quantum operators which are Markovian on a classical part.

Theorem 5.2. *If Φ_1 is Markovian and A^{Φ_1} is irreducible, aperiodic, and positive recurrent, then Φ_p, $0 < p \leq 1$, has the Perron-Frobenius property.*

Proof. By the Ergodic Theorem of Markov chain on finite state space, A^{Φ_1} is ergodic, i.e., there exists μ such that

$$\lim_{m\to\infty}[f(A^{\Phi_1})^m](j) = \alpha_f \mu(j), \tag{1.1}$$

for all f, where $\alpha_f = \sum_i f(i)$. For $f \in \mathcal{E}_d$, we have

$$\lim_{m\to\infty}(\Phi_1^M)^m f = \lim_{m\to\infty} f^*(\Phi_1^{M^*})^m = \lim_{m\to\infty} f^*(A^{\Phi_1})^m = \alpha_{f^*}\mu. \tag{1.2}$$

Now let $\rho \in L(\mathcal{H})$. Since there exists m_0 such that $(\Phi_1)^m(\rho) \in \mathcal{E}_d$ for all $m \geq m_0$, we have

$$\lim_{m\to\infty}(\Phi_1)^m\rho = \lim_{m\to\infty}(\Phi_1^M)^m f = \alpha_{f^*}\mu, \tag{1.3}$$

where $f = (\Phi_1)^{m_0}(\rho)$.

By lemma 4.8, Φ_1 has the Perron–Frobenius property. By Theorem 5.1, Φ_p has the Perron–Frobenius property, for all $0 < p \leq 1$. □

5.2 Periodic cases

We recall results about irreducible periodic Markov chains, following the discussion in Durrett [Dur14].

Lemma 5.3. *Suppose a Markov chain P is irreducible, positive recurrent, and d-periodic. Then we have the following facts about P and its state space B:*

1. *For any two states $x, y \in B$ we can define the set $K_{x,y} := \{n \geq 1 : P^n(x,y) > 0\}$, and all transition times $n \in K_{x,y}$ are congruent modulo d.*

2. *The relation $x \sim y$ defined by having transition times $n \in K_{x,y}$ is an equivalence relation, and there is an ordering of equivalence classes $B_0, \ldots, B_{d-1}$ such that P strictly maps B_k into $B_{k+1(\mod d)}$.*

3. P^d is irreducible, closed, and aperiodic on each of the B_k.

Proof. To prove (1) Fix $x, y \in B$. Let $n \in K_{x,y}$ and $m \in K_{y,x}$. Then $P^{n+m}(x,x) > 0$ and so $n+m \in K_{x,x}$. However, since P is d-periodic, $n+m \equiv 0 \mod d$, hence $n \equiv -m \mod d$. However, this is true for all n, independent of the choice of m. Hence all $n \in K_{x,y}$ are congruent. Define $0 \leq r_{x,y} < d$ as the integer $n \equiv r_{x,y} \mod d$.

To prove (2), we define $x \sim y$ by $K_{x,y}$ being in congruence class 0. This relation is reflexive because P is d-periodic and recurrent, so $r_{x,x} = 0$. Further, $x \sim y$ is symmetric, since P is irreducible, there is some $m \in K_{y,x}$ such that $P^m(y,x) > 0$, and some $n \in K_{x,y}$, but then $P^{m+n}(x,x) \geq P^n(x,y)P^m(y,x)$. Since $x \sim y$ by assumption, $n \equiv 0 \mod d$, but $n+m \equiv 0 \mod d$ since P is d-periodic, thus $m \equiv 0 \mod d$, that is $y \sim x$. Finally $x \sim y$ is transitive. Take $x, y, z \in B$ such that $x \sim y$ and $y \sim z$. Then for $n \in K_{x,y}$ and $m \in K_{y,z}$, we have $P^{n+m}(x,z) \geq P^n(x,y)P^m(y,z) > 0$, thus $n+m \in K_{x,z}$, but $n+m \equiv 0 \mod d$.

Since $x \sim y$ is an equivalence relation, the state space B is divided into disjoint equivalence classes. Choose a distinguished B_0, say the one where $0 \in B_0$. We show that the image of any equivalence class B_k under one step of P is contained in a single equivalence class. Take any $x, x' \in B_k$ and any $y, y' \in B$ such that $P(x,y) > 0$ and $P(x',y') > 0$. By the irreducibility of P, there are n, m, s such that $P^n(y,y') > 0$, $P^m(y',x') > 0$ Then we have that $P(x,y)P^n(y,y')P^m(y',x') > 0$ and so $1+n+m \in K_{x,x'}$, i.e. $1+n+m \equiv 0 \mod d$. However, we also have that $P(x',y')P^m(y',x') > 0$, thus $1+m \in K_{x',x'}$, so $1+m \equiv 0 \mod d$. Therefore $n \equiv 0 \mod d$, that is $y \sim y'$.

Let us begin numbering the equivalence classes such that P maps B_0 into B_1, B_1 into B_2, etc. Note that since P is d-periodic, B_0 and B_d must be the same class. Further, since d is the gcd of the recurrence times for any $x \in B$, the $B_0, B_1, \ldots, B_{d-1}$ must be distinct since otherwise there would be an element with recurrence time not congruent to $0 \mod d$. Finally, since P is irreducible, every element in B must be contained in one of the B_k.

Lastly, we prove (3). P^d is closed on B_k in the sense that $\left(P^d\right)^m(x,y) > 0$ only if x, y are in the same class B_k. Since $\left(P^d\right)^m(x,y) = P^{dm}(x,y)$ and P is d-periodic, $P^{dm}(x,y) > 0$ only if x, y are in the same equivalence class.

Further, $P^d\big|_{C(B_k)}$ is irreducible. Since P is irreducible, for any $x, y \in B_k$ there is n such that $P^n(x,y) > 0$. Since P is d-periodic, $n = md$. Thus $\left(P^d\right)^m(x,y) = P^{md}(x,y) >$

0.

Finally, P^d is aperiodic. The set of recurrence times $K_{x,x}$ for $x \in B_k$ has gcd d because P is periodic. Therefore every $n \in K_{x,x}$ is a product $n = md$. Moreover, d is the greatest common divisor, so the set of factors m have gcd 1. However, since $(P^m)^d(x,x) = P^{md}(x,x) > 0$ these are the recurrence times of P^d for $x \in B_k$. Thus every $x \in B_k$ is period 1 under P^d, and so P^d is aperiodic. □

We say that a Markovian operator Φ is *periodic* if A^Φ is a periodic Markov chain. The operator Φ has the same period as A^Φ.

Lemma 5.4. *If A is stochastic, irreducible, and period d then*

- *Every d-root of unity ω^k is a simple eigenvalue of A with eigenvectors v_k. Here $\omega = e^{2\pi i/d}$.*
- *Of the d-roots of unity only $\omega^0 = 1$ has an eigenvector $v_0 \in \mathcal{E}$ and the eigenspaces are pairwise orthogonal.*
- *All other eigenvalues λ of A are smaller than one, $|\lambda| < 1$.*
- *There is a cyclic decomposition $B = B_0 \cup B_1 \cup \cdots \cup B_{d-1}$ such that $A(C(B_k)) \subseteq C(B_{k+1 \mod d})$.*
- *$(A)^d\Big|_{C(B_k)}$ is closed, irreducible, and aperiodic on each of the subspaces spanned by the components of the cyclic decomposition.*
- *A has unique periodic limits in the sense that if $b \in C(B_k)$ then there is a unique nonnegative vector $\pi_{k+\ell} \in C(B_{k+\ell})$ such that*

$$\lim_{n\to\infty} bA^{nd+\ell} = \alpha_b \pi_{k+\ell},$$

and if b is a nonzero, nonnegative vector, then $\alpha_b \geq 0$.

Proof. We assume that Φ is Markovian and A is irreducible period d. Since A is a Markov chain we get a cyclic decomposition $B = B_0 \cup B_1 \cup \cdots \cup B_{d-1}$ such that $A : C(B_k) \to C(B_{k+1 \mod d})$ and $(A)^d\Big|_{B_k}$ is irreducible, aperiodic, and recurrent; equivalently it satisfies the Perron-Frobenius theorem when restricted to each subspace $C(B_k)$.

Therefore $1 \in \operatorname{Spec}\left((A)^d \Big|_{B_k} \right)$ and any other eigenvalues of $(A)^d \Big|_{B_k}$ are strictly smaller in norm than 1. Moreover these eigenvalues exhaust the modulus 1 eigenvalues of $(A)^d$

Now we note that if $\lambda \in \operatorname{Spec} A$ then $\lambda^d \in \operatorname{Spec} (A)^d$ since states in $C(B_k)$ are isomorphic to states in ${\mathcal{C}_d}^k$. Thus if $|\lambda| = 1$ then $\lambda^d = 1$ so $\lambda = \omega^k$ where $\omega = e^{2\pi i/d}$ is a primitive d-root of unity. Hence all the dominant eigenvalues of A are d-roots of unity.

Next, we wish to show that every d-root of unity is a simple eigenvalue of A. First, we show that every d-root of unity $e^{2\pi ik/d}$ is actually an eigenvalue of A and will subsequently show that each one of these eigenvalues is simple via a counting argument on eigenvectors.

Let π_k be the left Perron-Frobenius eigenvector for $A|_{B_k}$ or equivalently the limiting distribution of A restricted to $C(B_k)$. We also have that $\pi_k A = \pi_{k+1 \mod d}$. To see this, we know $\pi_k \in C(B_k)$ and so $\pi_k A \in C(B_{k+1 \mod d})$. Therefore $\pi_k A^{nd+1} \in C(B_{k+1})$ but π_k and π_{k+1} are the unique limits of $A = A$ on $C(B_k)$ and $C(B_{k+1})$. Thus

$$\pi_{k+1} = \lim_{n\to\infty} \pi_k A^{nd+1} = \lim_{n\to\infty} \pi_k A^1 = \pi_k A.$$

We use a Vandermonde matrix to make a Fourier-type transform of the matrix of Perron-Frobenius vectors of A to find eigenvectors of A for d-roots of unity eigenvalues. Writing the eigenvectors π_k as rows in a matrix we take the product

$$\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \dots & \omega^{-(d-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \dots & \omega^{-2(d-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(d-1)} & \omega^{-2(d-1)} & \dots & \omega \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_d \end{pmatrix}.$$

Here the Vandermonde matrix V has entries $V_{jk} = \omega^{-j(k-1)}$. Then the rows v_k are eigenvectors of A. Applying A and recalling that $\omega^d = 1$ we get

$$\begin{aligned} v_k A &= \left(\pi_0 + \omega^{-k}\pi_1 + \omega^{-2k}\pi_2 + \cdots + \omega^{-(d-1)k}\pi_{d-1}\right) A \\ &= \pi_1 + \omega^{-k}\pi_2 + \cdots + \omega^{-(d-1)k}\pi_{d \mod d} \\ &= \omega^k v_k. \end{aligned}$$

We have already mentioned that all eigenvalues of A such that $|\lambda| = 1$ must be d-roots of unity. The above construction shows that every d-root of unity has a matching eigenvector for A. Now we wish to show that the ω^k are simple eigenvalues. Suppose

for a contradiction that ω^k is not simple. Then there exists a generalized eigenvector u_k of rank r (possibly $r = 1$) which is linearly independent of v_k. Since the $(A)^d$ always converges to a limit, we have that there is a limit $u_{k,\infty}$ such that

$$\lim_{n\to\infty} u_k A^{dn} = u_{k,\infty}.$$

However, since u_k is a generalized eigenvalue of rank r, we also have that $\lim_{n\to\infty} u_k A^{dnr} = u_k$. Therefore u_k is a proper eigenvector of $(A)^d$ for eigenvalue $\omega^{kd} = 1$. Moreover, since the generalized eigenvectors of A are linearly independent, u_k is independent of the $\pi_0, \ldots, \pi_{d-1}$. However, this contradicts the fact that the π_k are the complete set of eigenvectors for $\lambda = 1$ of $(A)^d$.

Now concluding that for any $b \in C(B)$ the limit $\lim_{n\to\infty} b(A)^{nd+\ell}$ exists is straightforward. We can write $C(B)$ with a basis of generalized eigenvectors of A. Let

$$b = \sum_{k=0}^{d-1} \alpha_k v_k + \sum_{k=d}^{N} \alpha_k w_k$$

where $\left(\omega^k, v_k\right)$ are the simple dominant eigenpairs and (λ_k, w_k) are the other generalized eigenpairs. Since $|\lambda_k| < 1$ for the (λ_k, w_k) eigenpairs the limit

$$\lim_{n\to\infty} \left(\sum_{k=d}^{N} \alpha_k w_k \right) A^n = 0$$

so we get

$$\lim_{n\to\infty} b(A)^{nd+\ell} = \lim_{n\to\infty} \sum_{k=0}^{d-1} \alpha_k \omega_k^{nd+\ell} v_k = \sum_{k=0}^{d-1} \alpha_k \omega_k^{\ell} v_k.$$

□

Theorem 5.5. *If Φ is a Markovian quantum operator and A^Φ is irreducible and period δ then*

- *Every d-root of unity is a simple eigenvalue of Φ*
- *Of the d-roots of unity only $\lambda = 1$ has an eigenvector in $\mathcal{E}_d$ and the eigenspaces are pairwise orthogonal.*
- *All other eigenvalues λ of Φ are smaller than one, $|\lambda| < 1$.*

- *There is a cyclic decomposition* $\mathcal{C}_d = \mathcal{C}_d^0 \cup \mathcal{C}_d^1 \cup \cdots \cup \mathcal{C}_d^{d-1}$ *such that* $\Phi\big(\mathcal{C}_d^k\big) \subseteq \mathcal{C}_d^{k+1 \mod \delta}$.

- Φ *has unique periodic limits in the sense that if* $\rho \in \mathcal{E}_d^k$ *then there is a unique state* $\xi_{k+\ell} \in \mathcal{E}_d^{k+\ell \mod \delta}$ *such that*

$$\lim_{n\to\infty} \Phi^{n\delta+\ell}(\rho) = \alpha_b \xi_{k+\ell}.$$

Proof. Recall from definition (3.2), that Φ is Markovian if $\Phi^t(B) \in \mathcal{C}_d$ for any $B \in \mathcal{C}$ and t big enough. From this we can conclude that any eigenpair (λ, ρ) of Φ where $|\lambda| = 1$ must have $\rho \in \mathcal{C}_d$. Suppose not, then $\Phi^t(\rho) = \omega^{kt}\rho \notin \mathcal{C}_d$ which is a contradiction. Conversely, since T commutes with Φ and A^Φ if $\rho \in \mathcal{C}_d$ is an eigenvector of Φ for eigenvalue λ then

$$T(\rho)A^\Phi = \big(A^\Phi \circ T\big)(\rho) = (T \circ \Phi)(\rho) = T(\lambda\rho) = \lambda T(\rho),$$

and so $(\lambda, T(\rho))$ is an eigenpair of A^ϕ.

Since Φ is Markovian, by proposition (3.4) A^Φ is stochastic. Since A^Φ is stochastic, irreducible, and period δ we can apply lemma 5.4 to get the eigenpairs $\big(\omega^k, v_k\big)$ of A^Φ. By the isomorphism T in proposition (3.4) we get corresponding eigenvalues ω^k and eigenvectors $\zeta^k = T^{-1}(v_k)$. Because the eigenvalues are simple for A^Φ, they are simple for Φ by the isomorphism T.

In lemma (5.4) we get the cyclic decomposition $B = B_0 \cup B_1 \cup \cdots \cup B_{d-1}$. B has one matching element i corresponding to each $e_i \in \mathcal{H}$ by the isomorphism $i = T(|e_i\rangle\langle e_i|)$. Therefore, we may define the subspaces

$$\mathcal{C}^k = \text{Span}\{\ |e_i\rangle\langle e_j|\ : i, j \in B_k\}$$

and the sets ${\mathcal{E}_d}^k = \mathcal{E}_d \cap \mathcal{C}^k$. The subspaces $\mathcal{C}^k$ are pairwise orthogonal, which we can check by comparing basis elements. For $|e_i\rangle\langle e_i| \in \mathcal{C}^k$ and $|e_j\rangle\langle e_j| \in \mathcal{C}^\ell$, the Hilbert-Schmidt inner product is

$$\langle |e_i\rangle\langle e_i| \,|\, |e_j\rangle\langle e_j| \rangle_{HS} = \text{Tr}\left(|e_i\rangle\langle e_i|\, |e_j\rangle\langle e_j|\right) = \delta ij.$$

Furthermore, since $L(\mathcal{H}) = \mathcal{C} = \text{Span}\{\ |e_i\rangle\langle e_i|\ : e_i \in \mathcal{H}\}$ we have

$$L(\mathcal{H}) = \mathcal{C} = \mathcal{C}^0 \oplus \ldots \oplus \mathcal{C}^{\delta-1}.$$

Now we wish to check that the eigenvectors ζ^k are orthogonal. By the definition of the eigenvectors v_k in lemma (5.4)

$$\zeta^k = \sum_{j=0}^{\delta-1} \omega^{-jk} \xi^j$$

that is, one of the rows of the image of the matrix of row vectors $\xi^j := T^{-1}(\pi_j)$ under the Vandermonde matrix in lemma (5.4). Since $\pi_j \in C(B_j)$, we know $T^{-1}(\pi_j) \in \mathcal{C}^j$. Therefore the π_j are pairwise orthogonal, since they reside in orthogonal subspaces. However since the Vandermonde matrix is unitary, it preserves orthogonality, and so the ζ^k are orthogonal.

Finally, to see that only the eigenvalue $\omega^0 = 1$ has an eigenvector $\zeta^0 \in \mathcal{E}_d$, consider the images $T\big(\zeta^k\big)$. If $\zeta^k = \sum_{i \in B_k} a_i \left|e_i\right\rangle \left\langle e_i\right|$ then $v_k = T\big(\zeta^k\big) = \sum_{i \in B_k} a_i \delta_i(x)$. However, this is a nonnegative vector in $C(B_k)$ if and only if the $a_i > 0$. If the a_i are all nonnegative, then $\zeta^k \in \mathcal{C}^k$ is a positive semidefinite matrix. If v_k nonnegative and not the zero vector then we may rescale so that $\sum_i (v_k)_i = 1$, in which case $\operatorname{Tr} \zeta^k = 1$. The set of density matrices $\mathcal{E}_d$ is the set of matrices in $L(\mathcal{H})$ which are Hermitian, positive semidefinite, and have trace 1. Therefore $\zeta^k \in \mathcal{E}_d$ if and only if v_k is nonnegative and the entries sum to 1. However, there is only one nonnegative eigenvector v_0 up to scaling for eigenvalue $\omega^0 = 1$ amongst the dominant eigenpairs of A^Φ. Therefore only ζ^0 can be scaled as a positive semidefinite matrix with trace 1 amongst the dominant eigenvectors of Φ. □

Definition 5.6. *We say that a quantum operation Φ has the periodic Perron-Frobenius properties if*

- *1 is a simple eigenvalue of Φ with an eigenvector $\zeta_0 \in \mathcal{E}$,*
- *all eigenvalues λ of Φ are either $|\lambda| < 1$ or λ is a δ-root of unity,*
- *if λ is a δ-root of unity that is an eigenvalue of Φ then it is simple and it does* not *have an eigenvector in $\mathcal{E}$.*

Note that the periodic Perron-Frobenius properties are a subset of the conclusions about Markovian quantum operations in theorem (5.5).

Lemma 5.7. *For a sub-collection of density operators $\mathcal{E}^k$ defined as*

$$\mathcal{E}^k := \mathcal{C}^k \cap \mathcal{E}$$

where $\mathcal{C} := \operatorname{Span}\{\ |e_i\rangle\langle e_j|\ : i,j \in B_k\}$ *is a linear subspace as in theorem (5.5) and* $\mathcal{E}$ *is the set of all density operators in* $L(\mathcal{H})$ *then*

$$\operatorname{Span}\mathcal{E}^k = \mathcal{C}^k.$$

Proof. Each one of the pure outer products for $i,j \in B_k$ is in the subspace

$$|e_i\rangle\langle e_j| \in \mathcal{C}^k.$$

Therefore if we designate $\mathcal{H}^k = \operatorname{Span}\{\ |e_i\rangle : i \in B_k\}$ then alternatively $\mathcal{C}^k = L\big(\mathcal{H}^k\big)$. Thus for any $|x\rangle \in \mathcal{H}^k$ we have $|x\rangle\langle x| \in \mathcal{C}^k$. However this matrix is Hermitian, positive definite, and has trace 1, so $|x\rangle\langle x| \in \mathcal{E}^k$. We notice that for $x, y \in \mathcal{H}^k$,

$$|x+y\rangle\langle x+y| = |x\rangle\langle x| + |x\rangle\langle y| + |y\rangle\langle x| + |y\rangle\langle y|$$
$$|x+iy\rangle\langle x+iy| = |x\rangle\langle x| - i|x\rangle\langle y| + i|y\rangle\langle x| + |y\rangle\langle y|$$

Therefore

$$\begin{aligned}&(|x+y\rangle\langle x+y| - |x\rangle\langle x| - |y\rangle\langle y|) + i(|x+iy\rangle\langle x+iy| - |x\rangle\langle x| - |y\rangle\langle y|)\\ &= |x\rangle\langle y| + |y\rangle\langle x| + i|x\rangle\langle y| - i|y\rangle\langle x|\\ &= 2|x\rangle\langle y|\end{aligned}$$

and so every mixed outer product $|x\rangle\langle y|$ is in the span of the pure outer products all of which are contained in $\mathcal{E}^k$.

Thus $\mathcal{C}^k \subseteq \operatorname{Span}\mathcal{E}^k \subseteq \mathcal{C}^k$. □

Theorem 5.8. *Suppose* Φ_1 *and* Φ_0 *are quantum operations on finite dimensional spaces with* $\|\Phi_1\|, \|\Phi_0\| \leq 1$. *Then* $\Phi_p := (1-p)\Phi_0 + p\Phi_1$ *has the periodic Perron-Frobenius properties for every* $0 < p \leq 1$ *if and only if at least one of* Φ_0 *or* Φ_1 *have the periodic Perron-Frobenius properties. If* Φ_p *has the periodic Perron-Frobenius properties then for any* $\rho \in L(\mathcal{H})$

$$\lim_{n\to\infty} {\Phi_p}^{\delta n+\ell}(\rho)$$

exists.

If we have the additional assumptions that Φ_1 is Markovian; A^{Φ_1} is stochastic, irreducible, positive recurrent, and has period δ; and Φ_0, Φ_1 both satisfy the compatibility condition with A^{Φ_1} that

$$\Phi_i\big(\mathcal{E}^k\big) \subseteq \mathcal{E}^{k+1 \mod \delta}$$

then Φ_p has the periodic Perron-Frobenius properties if and only if there are vectors $\xi_0, \ldots \xi_{\delta-1}$ with $\xi_k \in \mathcal{E}^k$ and for any $\rho_k \in \mathcal{C}^k$,

$$\lim_{n\to\infty} {\Phi_p}^{n\delta+\ell}(\rho_k) = \alpha_{\rho_k}\xi_{k+\ell \mod \delta}$$

Proof. Trivially Φ_1 has the periodic Perron-Frobenius properties if Φ_p does for every $0 < p \leq 1$. To get the reverse implication, we use lemma (4.5) to see that if (λ, ρ) is an eigenpair of Φ_p with $|\lambda| = 1$ then it is an eigenpair of Φ_1 and therefore λ is a δ-root of unity and a simple eigenvalue.

If Φ_p has the periodic Perron-Frobenius properties then $\lim_{n\to\infty} {\Phi_p}^{n\delta+\ell}(\rho)$ exists. By lemma (4.6) ${\Phi_p}^{n\delta+\ell}(\rho) = {\Phi_p}^{n\delta+\ell}(\zeta) + O\big(n^\ell s^n\big)$ where ℓ is the maximum rank of a generalized eigenvector of Φ_p, s is the maximal modulus of an eigenvalue which has modulus smaller than 1, and ζ is a linear combination of generalized eigenvectors with eigenvalues $|\lambda_k| = 1$.

Since all the eigenvalues $|\lambda_k| = 1$ of Φ_p are δ-roots of unity without generalized eigenvectors of rank greater than or equal to 2, if $\zeta = \sum_{k=1}^{M} \alpha_k \zeta_k$ where ζ_k is an eigenvector for ω^k then

$${\Phi_p}^{n\delta+\ell}(\rho) = {\Phi_p}^{n\delta+\ell}(\zeta) = \sum_{k=1}^{M} \alpha_k \omega^{k(n\delta+\ell)} \zeta_k + O\big(n^\ell s^n\big) \xrightarrow[n\to\infty]{} \sum_{k=1}^{M} \alpha_k \omega^{k\ell} \zeta_k$$

Now, we may consider the case with the additional assumption that Φ_1 is Markovian; A^{Φ_1} is stochastic, irreducible, positive recurrent, and period δ; and that both Φ_1 and Φ_0 map $\mathcal{E}^k$ to $\mathcal{E}^{k+1 \mod \delta}$. Then by lemma (5.7) both ${\Phi_1}^\delta$ and ${\Phi_0}^\delta$ map each subspace $\mathcal{C}^k$ to itself. Therefore ${\Phi_p}^\delta$ is reducible into δ components, each a quantum operation on $\mathcal{C}^k$. We will denote the restriction of an operator to $\mathcal{C}^k$ by $\big({\Phi_p}^\delta\big)_k := {\Phi_p}^\delta\big|_{\mathcal{C}^k}$

Our next goal is to view ${\Phi_p}^\delta$ as the convex combination of a Markovian quantum operation and a quantum operation with norm bounded by 1. We consider the δ power of Φ_p, which is a non-commutative binomial expansion

$${\Phi_p}^\delta = \sum_{\alpha} p^{|\alpha|}(1-p)^{k-|\alpha|} \prod_{i=1}^{\delta} \Phi_{\alpha_i}$$

where $\alpha = (\alpha_1, \ldots, \alpha_\delta)$ is a vector where α_i are either 0 or 1, $|\alpha| = \sum_{i=1} \delta|\alpha_i|$, and Φ_{α_i} is Φ_0 or Φ_1 depending on α_i. This is simply the straightforward expansion of the product, however we notice first that each of the products $\prod_{i=1}^{\delta} \Phi_{\alpha_i}$ are products of operators with norm less than or equal to 1, and so has norm bounded by 1, and second that we can write this as

$$\Phi_p{}^\delta = p^\delta \Phi_1{}^\delta + \sum_{\alpha \neq (1,\ldots,1)} p^{|\alpha|}(1-p)^{k-|\alpha|} \prod_{i=1}^{\delta} \Phi_{\alpha_i}.$$

Since each of the operators remaining in the sum have norm less than or equal to 1 and since $(p+1-p)^\delta = \sum_{k=0}^{\delta} p^k (1-p)^{\delta-k}$ we conclude by the triangle inequality that

$$\left\| \sum_{\alpha \neq (1,\ldots,1)} p^{|\alpha|}(1-p)^{k-|\alpha|} \prod_{i=1}^{\delta} \Phi_{\alpha_i} \right\| \leq 1-p.$$

Thus we define

$$A := \frac{1}{1-p} \sum_{\alpha \neq (1,\ldots,1)} p^{|\alpha|}(1-p)^{k-|\alpha|} \prod_{i=1}^{\delta} \Phi_{\alpha_i}$$

which has $\|A\| \leq 1$. Therefore we can view $\Phi_p{}^\delta$ as the convex combination of two norm 1 operations

$$\Phi_p{}^\delta = p^\delta \Phi_1{}^\delta + (1-p)A.$$

We can apply theorem (5.5) to Φ_1 to conclude that it has a set of vectors ξ^k which are orthogonal and $\Phi_1(\xi^k) = \xi^{k+1 \mod \delta}$ which are also the unique vectors (up to scalar multiples) which have the property that if $\phi_k \in \mathcal{E}_d{}^k$ then $\lim_{n\to\infty} \Phi_1{}^{n\delta+\ell}(\rho_k) = \alpha_{\rho_k}\xi_{k+\ell}$ Therefore we can apply theorem (5.1) to each component $\left(\Phi_p{}^\delta\right)_k = p^\delta \left(\Phi_1{}^\delta\right)_k + (1-p)(A)_k$ to conclude that each component has a unique limiting vector ξ_k in $\mathcal{C}^k$. That is for any $\rho_k \in \mathcal{C}^k$,

$$\lim_{n\to\infty} (\Phi_p)_k{}^{n\delta}(\rho_k) = \alpha(\rho_k)\xi_k$$

and if $\rho_k \in \mathcal{E}^k$ then since Φ_p is trace preserving $\alpha(\rho_k) = 1$. Each ξ_k is the only dominant eigenvector of $\left(\Phi_p{}^\delta\right)_k$. This limiting vector can also be viewed as an eigenvector of $\Phi_p{}^\delta$ for eigenvalue 1.

We now consider the collection of dominant eigenvectors $\{\xi_0, \ldots, \xi_{\delta-1}\}$. In particular by the uniqueness of the limits and the continuity of Φ_p for every ξ_k,

$$\xi_{k+\ell \mod \delta} = \lim_{n\to\infty} \Phi_p{}^{n\delta+\ell}(\xi_k) = \Phi_p{}^\ell \left(\lim_{n\to\infty} \Phi_p{}^{n\delta}(\xi_k) \right) = \Phi_p{}^\ell(\xi_k).$$

Therefore we may apply the same Fourier transform to the ξ_k as in theorem 5.5 to get the same set of eigenpairs $\left(\omega^k, \zeta_k\right)$ as eigenpairs of Φ_p. $\square$

CHAPTER 6

APPLICATIONS AND EXAMPLES

6.1 Classical random walks

Random walks are an important class of classical Markov chains. Here the Markov chain is defined over states $x \in \mathbb{R}^d$ we have a sequence of independent identically distributed variables $\xi_i \in \mathbb{R}^d$ and the transition is defined as

$$x_{t+n} - x_t = \sum_{i=1}^{n} \xi_i.$$

See [Dur14, p. 236] for examples. This walk has Lebesgue measure as a stationary measure [Dur14, p. 252] which is *not* a probability measure. However, we can apply the central limit theorem [Dur14, p152] and we have that if the ξ_i are symmetric in $\mathbb{R}^d$ with variance 1 and $x_0 = 0$ then

$$\lim_{n\to\infty} \frac{x_{t+n} - x_t}{\sqrt{n}} \sim N(0; 1).$$

For a concrete example, if we take $x \in \mathbb{R}$ and define ξ_i as the variable with $\Pr(\xi_i = +1) = \Pr(\xi_i = -1) = \frac{1}{2}$ then the walk starting at $x = 0$ has standard deviation of position $\sqrt{t}$ at time t. In the limit $\lim_{t\to\infty} \frac{x_t}{\sqrt{t}} \sim N(0; 1)$. We also notice that if the variables take discrete values ± 1 then any change in position is constrained to be in $\mathbb{Z}^d$ therefore this walk may as well be defined on $\mathbb{Z}^d$ as $\mathbb{R}^d$.

6.2 Coherent quantum walks

6.2.1 Definition

We wish to define a coherent quantum process that is reminiscent of a classical random walk. The simplest and first way to do this, introduced in [ADZ93] is to introduce an additional degree of freedom we call the "coin space". We have a position space P which is spanned by $\{\ |x\rangle : x \in \mathbb{Z}\}$ where the $|x\rangle$ are a formal notation for a countable collection of orthogonal unit length vectors, but should be thought of as the state that corresponds to being located at the position $x \in \mathbb{Z}$. The position space is then the collection of all sums $\sum_{x\in\mathbb{Z}} \alpha_x |x\rangle$ where the coefficients are square summable, $\sum_{x\in\mathbb{Z}} |\alpha_x|^2 < \infty$. We define the coin space as any two-dimensional Hilbert space $C \cong \mathbb{C}^2$. For convenience, we will choose two orthonormal basis vectors for C and call them respectively $|+1\rangle, |-1\rangle$. We then define our full state space as $\mathcal{H} = P \otimes C$.

The evolution operator $U : \mathcal{H} \to \mathcal{H}$ is defined as the composition of a unitary shift operator and a unitary coin flip operator. The shift operator S is defined on the orthonormal basis of $\mathcal{H}$ formed by the simple tensors of basis vectors of P and C, and can be written as

$$S|x\rangle \otimes |+1\rangle = |x+1\rangle \otimes |+1\rangle$$
$$S|x\rangle \otimes |-1\rangle = |x-1\rangle \otimes |-1\rangle$$

and extended by linearity to a map on $\mathcal{H}$. As a convenient abuse of notation, we can write the above as $S|x\rangle|l\rangle = |x+l\rangle|l\rangle$. S is unitary because it takes distinct orthogonal vectors to distinct orthogonal vectors.

Our coin flip operator can be any unitary transformation on C, as a concrete example take the Hadamard matrix

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

This choice of flip operator is attractive because it is similar in behavior to independent classical coin tosses. If the coin is in state $|+1\rangle \in C$ before the flip then after the flip $H|+1\rangle = \frac{1}{\sqrt{2}}|+1\rangle + \frac{1}{\sqrt{2}}|-1\rangle$. Likewise $H|-1\rangle = \frac{1}{\sqrt{2}}|+1\rangle - \frac{1}{\sqrt{2}}|-1\rangle$. If the coin is measured by the projectors $|+1\rangle\langle+1|, |-1\rangle\langle-1|$ immediately before after after the flip all four possible behaviors ($+1$ to $+1$, $+1$ to -1, -1 to $+1$, and -1 to -1) are equally

likely. However, it is important to note that this operation is *not* a classical coin toss. Since it is unitary it is reversible in time, in fact $H^2 = \mathrm{Id}$. Furthermore, even when the coin's state is in a superposition of states with equal squared-amplitudes, the coin is in a single well-defined quantum state, not a statistical mix of states.

We finally define the transition operator

$$U = S(\mathrm{Id}_P \otimes H).$$

The walk is iterated for some number of steps t to transform an initial state $|\Psi_0\rangle$ to $|\Psi_t\rangle = U^t |\Psi_0\rangle$ and measure the position observable, that is measure the position of the walk as p_t with probability

$$\Pr(p_t = x) = |(\langle x| \otimes \langle +1|)|\Psi_t\rangle|^2 + |(\langle x| \otimes \langle -1|)|\Psi_t\rangle|^2.$$

The distribution of positions does not converge to a normal distribution as time $t \to \infty$. The variance of the position grows as t^2 as opposed to t in the classical case. After scaling by t the position distribution has Cesàro mean equal to a uniform distribution on the interval $\left[-t/\sqrt{2}, t/\sqrt{2}\right]$. For an excellent survey of results see [Kem08]

Generalizations of the discrete time quantum walk are simple for d-regular graphs, see [Kem08], [Ken06], [ADSS07]. Instead of a coin space $C \cong \mathbb{C}^2$, we choose a d-dimensional space isomorphic to $\mathbb{C}^d$, fix an orthonormal basis of C, and d-color the edges of the graph. The position space is spanned by one vector $|v\rangle$ for each node of the graph v. The shift operator then matches each one of the basis vectors $|d\rangle$ of C to one of the colors and takes the basis vector $|v\rangle \otimes |d\rangle \mapsto |v'\rangle \otimes |d\rangle$ where v and v' are connected by the edge of color d.

This construction is especially elegant when the graph is the Cayley graph of a group, for example $\mathbb{Z}^d$ generated by ± 1, $C_n = \mathbb{Z}/n\mathbb{Z}$ generated by ± 1, or the hypercube ${\mathbb{Z}_2}^d$ generated by $e_1, \ldots, d_2$ where $e_k = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ which flips the value of the k-th coordinate from 1 to 0 or vice versa.

6.2.2 One dimensional example

As an illustration, take the example of a walk on a one dimensional lattice, for example $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. The graph is the Cayley graph generated by the elements $+1, -1$, and the

vertices x of G are labeled by the integers $x \in \mathbb{Z}$ or $x \in \mathbb{Z}/n\mathbb{Z}$ respectively. The position space P is spanned by the orthonormal vectors $|x\rangle$ and the coin space C is spanned by $|+1\rangle, |-1\rangle$. The complete state space $\mathcal{H} = P \otimes C$.

The shift operator S is defined on the basis elements

$$S : |x\rangle \otimes |+1\rangle \mapsto |x+1\rangle \otimes |+1\rangle$$
$$S : |x\rangle \otimes |-1\rangle \mapsto |x-1\rangle \otimes |-1\rangle$$

and extends by linearity. S is unitary since the adjoint operator S^* is S^{-1}. As a coin flip operator we can use any unitary operator on $\mathbb{C}^2$ but to be concrete we will use $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then we can trace the image of the basis vectors above as

$$\mathrm{Id}_P \otimes H : |x\rangle \otimes |+1\rangle \mapsto \frac{1}{\sqrt{2}} |x\rangle \otimes |+1\rangle + \frac{1}{\sqrt{2}} |x\rangle \otimes |-1\rangle$$
$$\mathrm{Id}_P \otimes H : |x\rangle \otimes |-1\rangle \mapsto \frac{1}{\sqrt{2}} |x\rangle \otimes |+1\rangle - \frac{1}{\sqrt{2}} |x\rangle \otimes |-1\rangle .$$

Therefore $U = S(\mathrm{Id}_P \otimes H)$ acts on the basis correspondingly

$$U : |x\rangle \otimes |+1\rangle \mapsto \frac{1}{\sqrt{2}} |x+1\rangle \otimes |+1\rangle + \frac{1}{\sqrt{2}} |x-1\rangle \otimes |-1\rangle$$
$$U : |x\rangle \otimes |-1\rangle \mapsto \frac{1}{\sqrt{2}} |x+1\rangle \otimes |+1\rangle - \frac{1}{\sqrt{2}} |x-1\rangle \otimes |-1\rangle .$$

Since this takes pure states to pure states, the density matrix formulation is not essential. However, it is not difficult to compute. A convenient basis for the space of linear operators on $\mathcal{H}$ are the simple tensors $|\phi\rangle \langle\psi|$ where $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. Here the pure states which correspond to the basis states $|x\rangle \otimes |\pm 1\rangle \in \mathcal{H}$ are

$$(|x\rangle \otimes |\pm 1\rangle) \otimes (|x\rangle \otimes |\pm 1\rangle)^* = |x\rangle \langle x| \otimes |\pm 1\rangle \langle \pm 1|$$

which we may abbreviate as $|x\rangle |\pm 1\rangle \langle \pm 1| \langle x|$.

In this basis we can take the outer product of the pure states to get a matrix repre-

sentation

$$U|x\rangle|+1\rangle\langle+1|\langle x|U^* = \frac{1}{2}|x+1\rangle|+1\rangle\langle+1|\langle x+1| + \frac{1}{2}|x+1\rangle|+1\rangle\langle-1|\langle x-1| + \frac{1}{2}|x-1\rangle|-1\rangle\langle+1|\langle x+1| + \frac{1}{2}|x-1\rangle|-1\rangle\langle-1|\langle x-1|, \quad (2.1)$$

$$U|x\rangle|+1\rangle\langle+1|\langle x|U^* = \frac{1}{2}|x+1\rangle|+1\rangle\langle+1|\langle x+1| - \frac{1}{2}|x+1\rangle|+1\rangle\langle-1|\langle x-1| - \frac{1}{2}|x-1\rangle|-1\rangle\langle+1|\langle x+1| + \frac{1}{2}|x-1\rangle|-1\rangle\langle-1|\langle x-1|. \quad (2.2)$$

Notice that this representation is fairly awkward for the coherent problem because it represents a rank one orthogonal projector in terms of four projectors. However, it is convenient for the position measurement and will subsequently be convenient for decoherence related to position measurement.

For example, the position observable A that gives classical position x for states $|x\rangle|+1\rangle$ and $|x\rangle|-1\rangle$ corresponds to the measurement which is a collection of orthogonal rank two projectors:

$$\{\, A_x = \ |\,x\rangle\ |+1\rangle\langle+1\,|\ \langle x\,|\ +\ |\,x\rangle\ |-1\rangle\langle-1\,|\ \langle x\,|\ :\ |\,x\rangle \in E_P \,\},$$

where E is our distinguished basis of P. The observable in matrix form is then

$$A = \sum_{|x\rangle\in E_P} x|x\rangle|+1\rangle\langle+1|\langle x| + x|x\rangle|-1\rangle\langle-1|\langle x|$$

and the expected value of the position observable for density operator ρ is $\langle A\rangle = \operatorname{Tr}\rho A$. We note that the measurement operators each have range which are diagonal, that is which are contained in $\mathcal{E}_d$. Therefore after measurement

$$\sum_{|x\rangle\in E} A_x \rho {A^*}_x = \rho' \in \mathcal{E}_d$$

and the density matrix ρ' is diagonal in the basis we used above.

6.2.3 Hypercube example

As another example we work out the coherent quantum walk on the d-dimensional hypercube. Our graph G is the Cayley graph of $(\mathbb{Z}/2\mathbb{Z})^d$ generated by the elements

$e_k = 0 \otimes \cdots \otimes 0 \otimes 1 \otimes 0 \otimes \cdots \otimes 0$, that is, the image of the 1 element in the kth place, and the identity 0 elsewhere. The graph is a 2 by 2 by 2 ... by 2 hypercube. The vertices x can be labeled by members of $(\mathbb{Z}/2\mathbb{Z})^d$ by envisioning the elements as coordinates in $\mathbb{Z}^d$ with one vertex at the origin and the edges parallel to the axes.

The position space P is spanned by the vectors $\{\ |x\rangle : x \text{ is a vertex of G }\}$. The coin space C is spanned by $\{\ |k\rangle : e_k = 0 \otimes \cdots \otimes 1 \otimes \cdots \otimes 0\}$ the generators of the Cayley graph.

The shift operator is defined on this basis via

$$S\,|x\rangle \otimes |k\rangle = |x + e_k\rangle \otimes |k\rangle$$

and the coin operator is some unitary operator on C. As an example, we can choose

$$F = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{d-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{d-1} & \omega^{2(d-1)} & \cdots & \omega^{(d-1)(d-1)} \end{pmatrix}$$

which is unitary and takes a coin in any one of the basis states to a vector with projection of equal modulus on to each of the basis states.

Thus we can calculate the action of U on the basis vectors

$$\begin{aligned} & U\,|x\rangle\,|e_k\rangle\,\langle e_k|\,\langle x|\,U^* \\ & \quad \mapsto \left(\frac{1}{\sqrt{d}} \sum_{j \in E_C} \omega^{(k-1)(j-1)}\,|x + e_j\rangle \otimes |e_j\rangle\right)\left(\frac{1}{\sqrt{d}} \sum_{j' \in E_C} \omega^{(k-1)(j'-1)}\,|x + e_{j'}\rangle \otimes |e_{j'}\rangle\right)^* \\ & \quad = \sum_{j \in E_C} \sum_{j' \in E_C} \frac{\omega^{(k-1)(j-j')}}{d}\,|x + e_j\rangle\,|e_j\rangle\,\langle e_{j'}|\,\langle x + e_{j'}|\,. \end{aligned} \tag{2.3}$$

In particular notice that the diagonal entries all have coefficient $\frac{1}{d}$, hence the trace is preserved.

6.3 Fully decoherent quantum walks

We can easily situate the unitary operation U in the previous section in terms of density operators by letting $\rho_0 = |\Psi_0\rangle \langle\Psi_0|$ and defining the operation

$$\Phi(\rho) = U\rho U^*$$

which is a quantum operation because $U^*U = \mathrm{Id}$.

The measurement we applied in the section on coherent operators 6.2 can be understood as a projective measurement $\{ A_{x,l} \}$ where $|x\rangle$ are an orthonormal basis of the position space P, $|l\rangle$ are an orthonormal basis of the coin space C, and

$$A_{x,l} = (|x\rangle \otimes |l\rangle)(|x\rangle \otimes |l\rangle) = |x\rangle \langle x| \otimes |l\rangle \langle l|$$

We can notice first that $A_{x,l}\rho {A_{x,l}}^* = \alpha_{x,l;x,l} |x\rangle \langle x| \otimes |l\rangle \langle l|$ is rank 1. Additionally, the sum of these are diagonal in the basis given by $\{ \ |\ x\rangle \langle x\ | \otimes |\ l\rangle \langle l\ |\ \}$. Recalling our definition of a Markovian operation 3.2 we can see that

$$\Phi_1(\rho) = \sum_{x,l} A_{x,l} U \rho U^* {A^*}_{x,l}$$

is Markovian for the classical domain

$$\mathcal{E}_d = \mathrm{Span} \{ \ |\ x\rangle \langle x\ | \otimes |\ l\rangle \langle l\ |\ \}.$$

6.3.1 Decoherent one dimensional example

For example, if we use the walk on $\mathbb{Z}/n\mathbb{Z}$ with the Hadamard coin flip operator, $U = S(\mathrm{Id}_P \otimes H)$ from 6.2.2 then for the fully decoherent walk

$$\Phi(\rho) = \sum_{x \in \mathbb{Z}, l = \pm 1} A_{x,l} U \rho U^* {A^*}_{x,l}$$

the image of basis vectors under measurement $A_{x,l} |x'\rangle \otimes |l'\rangle$ are nonzero only if $x = x', l = l'$ so the computation on the basis vectors simplifies dramatically. Referring back

to 2.1 and 2.2

$$\Phi(|x\rangle |+1\rangle \langle -1| \langle x|) = \sum_{|x\rangle \in E_P, l=\pm 1} A_{x,l} U |x\rangle |+1\rangle \langle +1| \langle x| U^* A^*{}_{x,l}$$
$$= \frac{1}{2} |x+1\rangle |+1\rangle \langle +1| \langle x+1| + \frac{1}{2} |x-1\rangle |-1\rangle \langle -1| \langle x-1|, \quad (3.4)$$
$$\Phi(|x\rangle |+1\rangle \langle -1| \langle x|) = \sum_{|x\rangle \in E_P, l=\pm 1} A_{x,l} U |x\rangle |-1\rangle \langle -1| \langle x| U^* A^*{}_{x,l}$$
$$= \frac{1}{2} |x+1\rangle |+1\rangle \langle +1| \langle x+1| + \frac{1}{2} |x-1\rangle |-1\rangle \langle -1| \langle x-1|. \quad (3.5)$$

Notice that this behavior is exactly the same as that of a classical symmetric one dimensional nearest neighbor random walk on $\mathbb{Z}$. That is, Φ is Markovian since for any $\rho \in \mathcal{E}$ $\Phi(\rho) \in \mathcal{E}_d$ for our distinguished basis $E_P \otimes E_C$ and the walk takes the states in one position and coin basis state to an equal probability mix of the position states to the left and right with coin in $-1, +1$ respectively. If we take the Markovian matrix associated with the measured Φ (see definition 3.5), then we have

$$\left(A^\Phi\right)_{x+1,+1,x,\pm 1} = \frac{1}{2}$$
$$\left(A^\Phi\right)_{x-1,-1,x,\pm 1} = \frac{1}{2}$$

and all other entries of A^Φ are 0. This matrix structure is precisely $W \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where W is the Toeplitz operator with 0 diagonal, $\frac{1}{2}$ on the super and sub diagonal, $\frac{1}{2}$ in the upper right and lower left corners if finite dimensional, k and 0 elsewhere. W is the same stochastic matrix as for the symmetric one dimensional random walk.

6.3.2 Decoherent hypercube example

Referring back to Section 6.2.3 we can take the complete measurement $\{ A_{x,k} \}$ where x is a vertex of the graph and k is one of the distinguished basis elements of the coin space C

$$A_{x,k} = (|x\rangle \otimes |k\rangle)(|x\rangle \otimes |k\rangle) = |x\rangle \langle x| \otimes |k\rangle \langle k|.$$

When we apply this to the image of a pure state in Equation 2.3 the components where $j \neq j'$ are eliminated by the measurement, that is,

$$\sum_{x,k} A_{x,k} U |x\rangle |e_k\rangle \langle e_k| \langle x| U^* A^*{}_{x,k} = \sum_j \frac{1}{d} |x + e_j\rangle |e_j\rangle \langle e_j| \langle x + e_j|.$$

The fully measured walk is therefore a Markovian operation, and in this case the walk is isomorphic to the classical nearest neighbor walk on the hypercube graph. Let us consider what we can conclude about this walk. The nearest neighbor walk on the hypercube is period 2 since the graph is two colorable. By Lemma 5.4 we can conclude that $+1, -1$ are simple eigenvalues of the transition matrix A^{Φ}. By the symmetry of the graph we can see that the eigenvector v_1 for eigenvalue $+1$ must be the uniform distribution on the vertices of the graph. The eigenvector v_{-1} for -1 is not positive (and therefore not a probability distribution), but by the construction of the dominant eigenvectors in lemma 5.4 we can see that it has a eigenvector that is the difference $\pi_0 - \pi_1$ where $\pi_0\big(A^{\Phi}\big)^2 = \pi_0$ and π_0 is the uniform distribution over the "even" colored vertices of the graph, likewise π_1 is the uniform distribution over the "odd" colored vertices. We then have that for any initial probability distribution $x \in C(B)$

$$\lim_{n\to\infty} x\big(A^{\Phi}\big)^{2n} = \alpha v_1 + \beta v_{-1},$$
$$\lim_{n\to\infty} x\big(A^{\Phi}\big)^{2n+1} = \alpha v_1 - \beta v_{-1},$$

with the constraints that $\alpha + \beta \geq 0$ and $\alpha - \beta \geq 0$ and the sum of the components $\sum_{i=1}^{N} \alpha v_{+1} + \beta v_{-1} = 1$ so that both limits are probability distributions. However, since v_{-1} is uniform in magnitude with an equal number of positive and negative components, $\sum_{i=1}^{N} \beta v_{-1} = 0$ hence $\|\alpha v_{+1}\| = 1$. Without loss of generality we can assume that $\alpha = 1$ and $1 > \beta > -1$. Our possible solutions can therefore be seen as a one parameter family with a phase β.

By the isomorphism in Proposition 3.4 we correspondingly get a pair of operators ζ_{+1}, ζ_{-1} where ζ_{+1} is the uniform density matrix and ζ_{-1} is the difference of the uniform density matrix supported on the even vertices and the uniform distribution on the odd vertices. The possible limiting densities are $\rho_{2n} = \zeta_{+1} + \beta\zeta_{-1}$ and $rho_{2n+1} = \zeta{+}1 - \beta\zeta_{-1}$.

By Theorem 5.8, we can conclude that if we have a quantum operation which is the convex combination of the fully measured walk and any norm one operator that the limits are taken from the same space.

CHAPTER 7

CONCLUSION

We have proven a convergence result that gives sufficent conditions for the convergence of an aperiodic quantum operation to a unique limiting density operator, or the convergence of a periodic quantum operator to a unique family of density operators which have periodic behavior and respect a cyclic decomposition of the state space. The convergence theorem applies to any quantum operation which is the convex combination of any norm one operation and a special kind of operation we term a Markovian operation.

We defined a diagonal subset of density operators as a formalization of classical (decoherent) states, more precisely, the set of density operators which are contained in a subspace of bounded linear operators from the state Hilbert space to itself which are diagonalizable by a prefered orthonormal basis. Markovian operations with respect to that basis are the quantum operations which map the set of diagonal density operators to itself, and under iteration eventually map all density operators into the diagonal subspace.

In addition, we have given a technique for calculating the limiting density operators by examining a classical Markov chain that can be derived from a Markovian operation with respect to the prefered basis.

Our results provide a convenient and efficient way to deal with a large class of the examples of decoherent discrete time quantum walks that have been studied in the literature. A prevalent model of decoherent quantum walks is the composition of a unitary quantum walk with an operation which is equivalent to the process of applying either a one or more projective measurement or the identity operation to the density operator

independently at each time step. Both the unitary operation and the measurements are norm one operations, and the complete process can be viewed as a convex combination of the operations. In most cases, the operation in the convex combination which corresponds to a complete projective measurement in some basis applied after one step of the unitary walk is easily checked to be Markovian. Since a quantum walk respects a graph structure, the classical Markov chain associated with the Markovian operation is often a particularly tractable chain.

The convergence results for this class of operations provides some insight into the emergence of classical behavior from quantum systems. Classical Markov chains exhaust the limiting behavior of quantum operations which can be modeled as unitary operations experiencing a non-zero rate of projective measurement.

Future directions are in three major directions. The first is to generalize the conditions for the convergence theorem. Although norm one operations encompass many common examples, not all operations are norm one. Since the conclusion is about the spectral radius of the convex combination of operations which is bounded above by the operator norm, it is likely that the condition can be expanded to a more general case. Second, the convergence theorem applies to the limiting behavior of the convex combination of quantum operations, we wish to also study shorter time phenomena, such as hitting and mixing times of combined operations. Finally, our results cover a large class of the studied examples, so the calculation of examples that do not meet the conditions and do not show the same limiting behaviors would be valuable guidance to future research.

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