BLOCK DESIGNS UNDER AUTOCORRELATED ERRORS

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To the memory of my parents

ABSTRACT

This research work is focused on the balanced and partially balanced incomplete block designs when observations within blocks are correlated. The topic for this dissertation was motivated by a problem in pharmaceutical research, when several treatments are allocated to individuals, and repeated measurements are taken on each individual. In that case, there is correlation among the observations taken on the same individual. Typically, it is reasonable to assume that the observations within individual close to each other are highly correlated than observations that are far away from each other. It is also reasonable to assume that the correlation between any two observations within each individual is same.

We have characterized balanced and partially balanced incomplete block designs when observations within blocks are autocorrelated. In Chapter 3, we have provided an explicit expression for the average variance of estimated elementary treatment contrasts for designs obtained by Type I and II series of orthogonal arrays, under autocorrelated errors, and compared them with the corresponding balanced incomplete block designs with uncorrelated errors. The relative efficiency of balanced incomplete block design compared to the corresponding balanced incomplete block design obtained by Types I and II series of orthogonal array under autocorrelated errors does not depend on the number of treatments (v) and is an increasing function of the block size (k). When orthogonal arrays of Type I or Type II do not exist for a given number of treatments, we provided alternative partially balanced designs with autocorrelated errors.

In Chapter 4, we rearranged the treatments in each block of symmetric balanced incomplete block designs and used them with autocorrelated error structure of the plots in a block. The C-matrix of estimated treatment effects under autocorrelation was given and the relative efficiency of symmetric balanced incomplete block designs with independent errors compared to the autocorrelated designs is given.

In Chapter 5, we discussed the compound symmetry correlation structure within blocks. An explicit expression of the average variance of designs obtained by Type I and II series of orthogonal arrays and symmetric balanced incomplete block designs under compound symmetric errors has been provided and compared them with the corresponding balanced incomplete block designs with uncorrelated errors. Finally, the relative efficiencies of these designs with autocorrelated errors vs. compound symmetric error structure are given.

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CHAPTER 1

INTRODUCTION

Experimental designs are important in scientific research and industrial applications. Commonly used experimental design is completely randomized design, which can be used in many areas, such as agriculture, social science, and especially medicine. This design is used when all the experimental units are homogenous. When all units are not homogenous, the experimental units are arranged into groups (blocks) having homogeneous units. The completely randomized block design is the widely used block design in research. In this design each treatment is used once in each block. The completely randomized block design is not applied in all circumstances although it has many advantages over other designs. Incomplete block designs are used when experimenters may not be able to run all the treatments in each block due to shortage of experimental units in a block, the physical size and nature of the blocks, or cost is a consideration. Balanced incomplete block (BIB) designs are the incomplete block designs in which all treatment contrasts are confounded with blocks to the same extent. These designs estimate all elementary contrasts of treatment effects with equal variance and are shown to be universal optimal by Kiefer (1975). Limited number of BIB designs is available and partially balanced incomplete block (PBIB) designs may be chosen if BIB design is not available and the experimenter does not require each pair of treatments to occur together the same number of times, but each pair of treatments occur together in λ_i blocks depending on a specified association scheme. These designs do not give equal variance for all estimated elementary contrasts of treatment effects. The details on the existence and non existence of these block designs and their constructions are very well discussed in Raghavarao (1971), and Raghavarao and Padgett (2005). Most of the block designs consider the responses in each block to be uncorrelated. However, it is more reasonable to assume in some experiments, especially including humans and animals in repeated trials, that observations within a block are correlated. Further, it is likely that observations within a block close to each other are highly correlated than observations that are far from each other. In some cases every pair of observations within a block may be equally correlated. In these circumstances, the standard optimal block designs may not be efficient. Orthogonal arrays of Type I and Type II (OA I and OA II) introduced by Rao (1961) have been recently shown to be optimal block designs by Majumdar and Martin (2004). They showed that such designs are variance balanced and universal optimal. They did not provide actual expressions of variances of estimated contrasts of treatment effects. We will extend their work and discuss other block designs with autocorrelated error structures and compound symmetry error structures in this dissertation.

Continuing the work of Majumdar and Martin, we find the average variance of designs obtained by OA I and OA II under autocorrelation errors and compound symmetric errors and compare them with the corresponding balanced incomplete block designs with uncorrelated errors. This comparison will enable us to make some useful recommendations in planning and analyzing the experiment. In some problems we may not be able to construct OA I or OA II for the given number of treatments and required number of rows (block size). Thus we cannot construct optimal autocorrelated designs always. However, in such cases we may be able to construct equi-block sized partially balanced incomplete block (PBIB) designs for the given number of treatments v in block size K, where K is a prime or prime power. We can construct an OA I or OA II in k rows with K symbols such that we can have an autocorreclated errors design in v treatments with block size k. We call such designs partially balanced incomplete block designs with autocorrelated errors (PBIBAC designs).

1.1 Motivation to the Problem

Block designs sum up one of Fisher's basic concepts of the statistical design of experiments: the importance of setting off experimental runs into small blocks that are highly homogenous, in order to increase the precision of the experiment. There is an enormous amount of mathematical and statistical research about block designs. Block designs are used in many fields, such as agriculture, social science, and especially in pharmaceutical research and development. The ordinary work for block designs assumes that there is no correlation between units or observations within each block; but, it is possible that units, (plots or observations) within a block are correlated, when repeated observations are taken on the same individual. These situations arise in many experiments in different areas including environmental studies, marketing, medicine, and so on. For example, in pharmaceutical research and studies, when several treatments are allocated to individuals, and repeated measurements are taken on each individual, there is correlation among the observations taken on the same individual. This correlation may be constant for all pairs of observations or autocorrelated giving higher correlation for closer observations and smaller correlation for distant pairs. Under autocorrelation of errors between units in a single block of a factorial experiment, Jenkins and Chanmugam

(1962), Cheng and Steinberg (1991), Saunders et al. (1995), Martin et al. (1998), Elliot et al. (1998) found optimal designs. Sethuraman and Raghavarao (2009) provided balanced 2^n factorial designs using multiple replications. Their work consists of using complete block designs. Often it becomes necessary not to use all treatments on each individual and we need to use incomplete block designs. Balanced incomplete block designs for v treatments in b blocks of k units each are the optimal incomplete block designs. However, balanced incomplete block designs cannot be constructed for all combinations of v, b and k. In such cases partially balanced incomplete designs for the given number of treatments v in block size K may be constructed.

Classes of block designs based on orthogonal arrays of Type I and Type II that possess useful properties, including balance, high efficiency and optimality, have been researched recently. Majumdar and Martin (2004) proved that the orthogonal arrays of Type I and Type II considered as block designs are variance balanced with autocorrelation of errors between units in each block and are universal optimal. However, they did not give explicit expression of the average variance of estimated elementary contrasts of treatment effects. They showed that the information matrix of treatment effects is completely symmetric for those designs and they are universal optimal. There is a need to provide variances of elementary contrasts of estimated treatment effects for OA I and OA II designs and extend their work to other useful block designs to increase the scope of block designs with autocorrelated errors and compound symmetry of errors.

1.2 Scope of this Dissertation

This dissertation focuses on the development of statistical methodology in experimental design with applications to block designs under autocorrelation errors and compound symmetric errors. It has five chapters.

After the introduction in Chapter 1, the literature on block designs, OA I and OA II, partially balanced incomplete block design (PBIB), symmetric balanced incomplete block (SBIB) and designs with autocorrelation and compound symmetric error structures are reviewed in Chapter 2.

In the third chapter, we use difference method to construct BIBAC of type I and type II, Partially Balanced incomplete Block Design with autocorrelation errors (PBIBAC) of type I and type II. The statistical analysis of these designs is provided and the relative efficiency (RE) of BIB designs with uncorrelated errors compared to BIBAC designs is given. We will also discuss PBIBAC analysis. These results are published in Shu and Raghavarao (2010).

In the fourth chapter, we rearrange the treatments in each block of a symmetric balanced incomplete block design as a Youden Square design and use it under autocorrelation error structure of the plots in a block. The C-matrix of estimated treatment effects under autocorrelation is given and relative efficiency of the symmetric balanced incomplete block designs with independent errors compared with auto correlated error structure is provided. These results are published in Raghavarao and Shu (2011) when $\lambda=1$ for the symmetric BIB design.

The fifth chapter is devoted to the statistical analysis of OA I and OA II orthogonal array designs and SBIB designs when observations within blocks have compound symmetry or intra-class correlation structure. We have provided RE of these designs with uncorrelated errors compared to BIB designs and SBIB designs with compound symmetric error structure, respectively. This helps us to see the robustness of autocorrelation error structure compared to compound symmetry error structure.

In Chapter 6, we summarize our findings and propose future work based on this dissertation.

CHAPTER 2

Literature Review

Since Sir Ronald A. Fisher laid the foundation of experimental design for agricultural trials in 1920, experimental designs have received much attention and the applications of statistical designs in the areas of biological sciences, pharmaceuticals, social sciences, marketing etc., are considerable. A significant work which impacted on the expanding use of designs for industrial experiments was done by Taguchi (1987). We consider block designs in this work.

An experiment consists of n experimental units or observations arranged in b blocks or plots of sizes $k_1, k_2, \cdots, k_b$ such that $\sum_i k_i = n$. Assume that the statistical model relating the response and the explanatory variables can be written as,

$$E(\mathbf{Y_i}) = X_i \boldsymbol{\theta} \ , \ i = 1, 2, \cdots, b, \tag{2.1}$$

where $E(.)$ is the expected value of the random vector in parentheses, $\mathbf{Y_i}$ denotes the vector of observations within block i, X_i is the design matrix for block i and $\boldsymbol{\theta}' = (\mu, \boldsymbol{\beta}', \boldsymbol{\tau}')$, where $\boldsymbol{\beta}' = (\beta_1, \beta_2, \cdots, \beta_b)$ and $\boldsymbol{\tau}' = (\tau_1, \tau_2, \cdots, \tau_v)$, are vectors of block effects and treatment effects, respectively, and μ is the overall mean.

Uncorrelated error models assume that the observations are uncorrelated, i.e., the variance-covariance matrix of the observational vector, $\mathbf{Y}' = (\mathbf{Y}_1', \mathbf{Y}_2', \cdots, \mathbf{Y}_b')$, is given by Var($\mathbf{Y}$) = $\sigma^2 I_n$ where I_n is an identity matrix. This assumption is appropriate when all the treatments of the experiment have been completely randomized within each block. The theory of complete or incomplete block design usually assumed uncorrelated errors. In a complete block design, every treatment is allocated to every block, i.e., every combination of treatments and blocks are tested. By contrast, in an incomplete block design, the number of units in each block (i.e., block size) is less than the number of treatments (Yates 1936a). The commonly used incomplete block designs are balanced incomplete block (BIB) designs and partially incomplete block (PBIB) designs. Yates (1936a) introduced the concept of BIB designs and proposed other incomplete block designs which he referred as quasi-factorial or lattice designs (Yates 1936b, 1940). Later, Bose (1939, 1942) and Fisher (1940) explored the structure and construction of BIB designs. Bose and Nair (1939) generalized the notion of BIB designs to that of PBIB designs. Now we define BIB and PBIB designs.

Definition 2.1 . A BIB design is an arrangement of v symbols in b sets each of size $k\,(<v)$, such that

1. every symbol occurs at most once in a set
2. every symbol occurs in r sets
3. every pair of distinct symbols occurs together in λ sets.

We need the concept of association scheme to introduce PBIB designs and association scheme on v symbols is defined below:

Definition 2.2 . Given v symbols $1, 2, \ldots, v$, a relation satisfying the following conditions is said to be an *association scheme with* m *classes*:

1. Any two symbols α and β are either first, second, …, or mth associates and this relationship is symmetrical. We denote $(\alpha, \beta) = i$, when α and β are ith associates.

2. Each symbol α has n_i, ith associates, the number n_i being independent of α.

3. If $(\alpha, \beta) = i$, the number of symbols γ that satisfy simultaneously $(\alpha, \gamma) = j$, $(\beta, \gamma) = j'$, is $p^i_{jj'}$ and this number is independent of α and β. Further, $p^i_{jj'} = p^i_{j'j}$.

The numbers v, n_i, $p^i_{jj'}$ are called the parameters of the association scheme. The parameters $p^i_{jj'}$ can be written in m matrices of order $m \times m$ as follows:

$$P_i = (p^i_{jj'}),\ i = 1, 2, \ldots, m;\ j, j' = 1, 2, \ldots, m. \tag{2.2}$$

The commonly used association scheme is Group Divisible (GD) association scheme defined as below:

Definition 2.3 . In a group divisible association scheme there are $v = mn$ symbols arranged in m groups of n symbols. Two symbols in the same group are first associates and two symbols in different groups are second associates.

Clearly

$$n_1 = n - 1,\ n_2 = n(m-1),$$

$$P_1 = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix},\ P_2 = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{pmatrix}. \quad (2.3)$$

Given an m - class association scheme on v symbols, a PBIB design with m associate classes is defined in Definition 2.4.

Definition 2.4 . A PBIB design with m associate classes is an arrangement of v symbols in b sets of size $k\ (< v)$ such that

1. Every symbol occurs at most once in a set.
2. Every symbol occurs in r sets.
3. Two symbols α and β occur in λ_i sets, if $(\alpha, \beta) = i$ and λ_i is independent of the symbols α and β.

Youden (1951) used symmetrical BIB designs to eliminating heterogeneity in two directions, generalizing the concept of the Latin square design, and such design are known as Youden Square designs.

However the observations or units are taken sequentially in time or in space and it may be realistic to assume some sort of dependence for many experimental settings in biological, pharmaceutical, and other experiments. In the circumstance where the experimental observations or units are correlated within a block or plot, the information matrix for the model (2.1) will depend on the correlation structure.

The useful correlation structures in clinical trials are the first-order autocorrelation (or AR(1)), and intra-class, i.e., compound symmetry, between the observations taken on the same subject. The variance-covariance matrix of **Y** is a block diagonal matrix given by

$$Var(\mathbf{Y}) = \sigma^2 \begin{bmatrix} \Sigma_1 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \Sigma_b \end{bmatrix}. \qquad (2.4)$$

Autocorrelation, AR(1):

Let the correlation between observations w time periods apart in a block be $\rho^{|w|}$, i.e., the observations lying close to each other in time or space are highly correlated than observations that lie far from each other. Thus, the Var(Y) is given by (2.4) where

$$\Sigma_i = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{k_i-1} \\ \rho & 1 & \rho & \cdots & \rho^{k_i-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{k_i-1} & \rho^{k_i-2} & \rho^{k_i-3} & \cdots & 1 \end{bmatrix}.$$

The value of ρ is assumed to be known prior to the design of the experiment. The inverse of Σ_i is

$$\Sigma_i^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}$$

Compound Symmetry:

Let ρ be the correlation between observations on the i^{th} individual or block, then the Var(**Y**) is given by (2.4) where $\Sigma_i = (1-\rho)I_{k_i} + \rho J_{k_i}$, k_i is the number of observations in block i, $(i = 1, 2, \cdots, b)$, J_{k_i} is a $k_i \times k_i$ matrix of all ones and I_{k_i} is the identity matrix of order k_i

The inverse of Σ_i for compound symmetry is

$$\Sigma_i^{-1} = \frac{1}{1-\rho}\left[I_{k_i} - \frac{\rho}{(1+\rho(k_i - 1))} J_{k_i} \right]$$

Under autocorrelation of errors between units in a single block of a factorial experiment, Jenkins and Chanmugam (1962), Cheng and Steinberg (1991), Saunders et al. (1995), Martin et al. (1998), Elliott et al. (1998) found optimal designs. Sethuraman and Raghavarao (2009) provided balanced 2^n factorial designs using multiple replications.

Orthogonal arrays were introduced by Rao (1947) and two variants of the orthogonal array called Orthogonal Array of Type I (OA I) and Orthogonal Array of Type II (OA II) were also introduced by Rao (1961). The definitions of orthogonal arrays of Type I and Type II are:

Definition 2.5 . In orthogonal arrays of Type I (OA I) designs, s symbols are arranged in a $k \times N$ array such that in any two rows each ordered pair of distinct symbols occurs once. Here $N = s(s-1)$.

Definition 2.6 . In orthogonal arrays of Type II (OA II) designs, s symbols are arranged in a $k \times N$ array such that in any two rows each unordered pair of distinct symbols occurs once. Here $N = s(s-1)/2$.

Bose, Shrikhande and Parker (1960) and Parker (1958) used the OA I earlier in their work on disproving Euler's Conjecture. Morgan and Chakravarti (1988) researched on the construction of efficient statistical designs based on OA II, and Stufken (1991) researched on OA I designs. Majumdar and Martin (2004) showed that BIB designs based on OA I and OA II for experiments using units ordered over time or space are universal optimal under atuocorrelated errors within each block.

CHAPTER 3

Balanced and Partially Balanced Incomplete Block Designs with Autocorrelated Errors

In this chapter, balanced and partially balanced incomplete block designs when observations within blocks are autocorrelated are considered. We construct the OA I and OA II by the method of differences in Section 3.1. In some problems we may not be able to construct OA I or OA II for the given number of treatments and required number of rows (block size). Thus we cannot construct optimal autocorrelated designs always. However, in such cases we may be able to construct partially balanced incomplete designs for the given number of treatments N in block size K, where K is a prime or prime power. Form the treatments each block, we can construct BIBAC designs of type I as given in Section 3.1. In Sections 3.2 and 3.3, we give the analysis of BIBAC designs and the relative efficiency of BIB design compared to BIBAC design, respectively. We also prove that the relative efficiency does not depend on the number of treatments (v), and is an increasing function of k. Finally we discuss PBIBAC designs analysis in Section 3.4.

3.1 BIBAC and PBIBAC Designs with Autocorrelated Errors

Consider an experiment with v treatments where v is a prime or prime power. When v is odd, we can construct a block design in $v(v-1)/2$ (or $v(v-1)$) blocks of size k such that every unordered (or ordered) pair of treatments occur together in $k-1$

blocks in successive units, $k-2$ blocks in units differing by 1, $k-3$ blocks in units differing by 2, etc. By writing the blocks of the design as columns in a $k \times v(v-1)/2$ (or $k \times v(v-1)$), we get OA II (or OA I), and we will call such block designs as Balanced Incomplete Block Autocorrelation (BIBAC) designs of type II (or type I).

The construction of OA I (or OA II) designs are given by several authors (See Rao (1961), Mukhopadhyay (1972)). We sketch their construction by the method of differences for completeness.

Let v be a prime or prime power with x as a primitive root, and let $\alpha_0 = 0, \alpha_1 = x, \alpha_2 = x^2, \cdots, \alpha_{v-1} = x^{v-1} = 1$ be the elements of the Galois Field of v elements GF(v). When v is even prime number, the column vectors

$$(\alpha_0, \alpha_i x, \alpha_i x^2, \cdots, \alpha_i x^{k-1})', \quad i = 1, 2, \cdots, (v-1),$$

developed mod v (See Raghavarao, 1971) gives BIBAC design of type I. When v is odd prime number, the column vectors

$$(\alpha_0, \alpha_i x, \alpha_i x^2, \cdots, \alpha_i x^{k-1})', \quad i = 1, 2, \cdots, (v-1)/2,$$

developed mod v form a BIBAC design of type II. The above BIBAC type I designs are clearly balanced incomplete block designs (See Raghavarao, 1971) with parameters

$$v, \mathrm{b} = v(v-1), \mathrm{k}, \mathrm{r} = \mathrm{k}(v-1), \lambda = k(k-1),$$

while the BIBAC type II designs are balanced incomplete block designs with parameters

$$v, \mathrm{b} = v(v-1)/2, \mathrm{k}, \mathrm{r} = \mathrm{k}(v-1)/2, \lambda = k(k-1)/2.$$

Here developing a column $(\alpha_0, \alpha_i x, \alpha_i x^2, \cdots, \alpha_i x^{k-1})'$, we get the j[th] column

$$(\alpha_0 + \alpha_{\mathrm{j}}, \alpha_i x + \alpha_{\mathrm{j}}, \alpha_i x^2 + \alpha_{\mathrm{j}}, \cdots, \alpha_i x^{k-1} + \alpha_{\mathrm{j}})',$$

where all the elements belong to $GF(v)$. In this construction, when v is a prime and not a prime power, we represent the element α_i by i. For example, when $v = 5$, and $k = 4$, we get the BIBAC design of type II by developing the following 2 columns:

$$(0,1,2,3)' \;;\; (0,2,4,1)';$$

and the BIBAC design is

0	1	2	3	4	0	1	2	3	4
1	2	3	4	0	2	3	4	0	1
2	3	4	0	1	4	0	1	2	3
3	4	0	1	2	1	2	3	4	0

When $v = 6$, we cannot construct OA I or OA II designs in 4 rows. However, we can construct a partially balanced incomplete block design with autocorrelated errors.

Consider a partially balanced incomplete block design with m associate classes in blocks of size K, where K is a prime or prime power. Using the treatments from each of the blocks of that design, we can construct BIBAC design of type I or type II. The resulting design under autocorrelation of errors will have different variances for the estimated elementary contrasts as shown in Section 3.4, and we will call them PBIBAC designs of type I or type II, depending on the type of AC design used for the treatments in each of the blocks of the partially balanced incomplete block design.

As an example consider a partially balanced incomplete block (PBIB) design with group divisible association scheme with $v = 6$ treatments in 3 groups of 2 treatments in each group as follows:

$$\{0,1\}, \{2,3\}, \{4,5\}.$$

We have the group divisible design

$$\{0,1,2,3\},\ \{0,1,4,5\},\ \{2,3,4,5\},$$

and forming OA I from each block we get the following array in 4 rows and 36 columns.

$$\begin{array}{cccccccccccc|cccccccccccc|cccccccccccc}
0 & 2 & 3 & 1 & 0 & 2 & 3 & 1 & 0 & 2 & 3 & 1 & 0 & 4 & 5 & 1 & 0 & 4 & 5 & 1 & 0 & 4 & 5 & 1 & 2 & 4 & 5 & 3 & 2 & 4 & 5 & 3 & 2 & 4 & 5 & 3 \\
1 & 3 & 2 & 0 & 2 & 0 & 1 & 3 & 3 & 1 & 0 & 2 & 1 & 5 & 4 & 0 & 4 & 0 & 1 & 5 & 5 & 1 & 0 & 4 & 3 & 5 & 4 & 2 & 4 & 2 & 3 & 5 & 5 & 3 & 2 & 4 \\
2 & 0 & 1 & 3 & 3 & 1 & 0 & 2 & 1 & 3 & 2 & 0 & 4 & 0 & 1 & 5 & 5 & 1 & 0 & 4 & 1 & 5 & 4 & 0 & 4 & 2 & 3 & 5 & 5 & 3 & 2 & 4 & 3 & 5 & 4 & 2 \\
3 & 1 & 0 & 2 & 1 & 3 & 2 & 0 & 2 & 0 & 1 & 3 & 5 & 1 & 0 & 4 & 1 & 5 & 4 & 0 & 4 & 0 & 1 & 5 & 5 & 3 & 2 & 4 & 3 & 5 & 4 & 2 & 4 & 2 & 3 & 5
\end{array}$$

It may not be possible to construct an OA II design with $v = 6$ in 3 rows. However by using the PBIB design with group divisible association scheme in 4 blocks:

$$\{0,2,4\}, \{0,3,5\}, \{1,2,5\}, \{1,3,4\},$$

we get the PBIBAC design of type II as follows:

$$\begin{array}{cccccccccccc}
0 & 2 & 4 & 0 & 3 & 5 & 1 & 2 & 5 & 1 & 3 & 4 \\
2 & 4 & 0 & 3 & 5 & 0 & 2 & 5 & 1 & 3 & 4 & 1 \\
4 & 0 & 2 & 5 & 0 & 3 & 5 & 1 & 2 & 4 & 1 & 3
\end{array}$$

3.2 Analysis of BIBAC Designs of Type I

We will give detailed results for BIBAC design of type I and sketch the results for BIBAC type II designs and PBIBAC designs. In the blocks of BIBAC design, we refer the row label of the $k \times v(v-1)$ array as the position in the block. Let Y_{ij} be the response in the j^th^ position of the i^th^ block $\left(i = 1, 2, \cdots, v(v-1); j = 1, \cdots, k\right)$. We assume the model:

$$Y_{ij} = \mu + \beta_i + \tau_{d(i,j)} + e_{ij} \ , \tag{3.1}$$

where $d(i, j)$ is the treatment in the j^th^ position of the i^th^ block, μ is the general mean, β_i is the i^th^ block effect, τ_l is the l^{th} treatment effect and e_{ij} are random errors. Let us denote:

$\mathbf{Y}_i' = \left(Y_{i1}, Y_{i2}, \cdots, Y_{ik}\right)$, $\mathbf{e}_i' = \left(e_{i1}, e_{i2}, \cdots, e_{ik}\right)$, $\boldsymbol{\beta}' = \left(\beta_1, \beta_2, \cdots, \beta_{v(v-1)}\right)$, $\boldsymbol{\tau}' = \left(\tau_1, \tau_2, \cdots, \tau_v\right)$.

We assume $\mathrm{E}(\mathbf{e}_i) = \mathbf{0}$ and $\mathrm{var}(\mathbf{e}_i) = \sigma^2 \Sigma$, where $\mathbf{0}$ is a zero vector and

$$\Sigma = (\rho^{|i - j|}) \, .$$

We further assume that $\rho > 0$ is known; otherwise, we will estimate it from the data. In matrix form we write the model

$$\mathbf{Y}_i = X_i \boldsymbol{\theta} + \mathbf{e}_i, \quad i = 1, 2, \cdots, v(v-1), \tag{3.2}$$

where $\boldsymbol{\theta}' = \left(\mu, \boldsymbol{\beta}', \boldsymbol{\tau}'\right)$ and X_i is a $k \times \left(v^2 + 1\right)$ design matrix. Put $\mathbf{Y}' = \left(\mathbf{Y}_1', \mathbf{Y}_2', \cdots, \mathbf{Y}_{v(v-1)}'\right)$, $X' = \left(X_1', X_2', \cdots, X_{v(v-1)}'\right)$, and $\mathbf{e}' = \left(\mathbf{e}_1', \mathbf{e}_2', \cdots, \mathbf{e}_{v(v-1)}'\right)$. We then have

$$\mathbf{Y} = X \, \boldsymbol{\theta} + \mathbf{e} \, . \tag{3.3}$$

The information matrix of $\boldsymbol{\theta}$ is then given by

$$X'\left[I_{v(v-1)} \otimes \Sigma^{-1}\right]X = \frac{1}{1-\rho^2}\begin{bmatrix} v(v-1)t & t\,\mathbf{1}'_b & t\,(v-1)\mathbf{1}'_v \\ t\,\mathbf{1}_b & t\,\mathrm{I}_b & M \\ t(v-1)\mathbf{1}_v & M' & S \end{bmatrix}, \tag{3.4}$$

where $b = v(v-1)$, and I_b is the identity matrix of order b, '$\otimes$' is the kronecker product sign, $t = 2(1-\rho) + (k-2)(1-\rho)^2$, $M = (m_{ij})$ is a $b \times v$ order matrix, where

$m_{ij} = 1-\rho$, if jth treatment occurs in 1st or kth position in the ith block;

$= (1-\rho)^2$, if jth treatment occurs in 2nd, 3rd, …, or $(k-1)^{th}$ position in the ith block;

$= 0$, if j^{th} treatment doesn't occur in the i^{th} block;

and $S = (s_{ij})$ is a $v \times v$ order matrix, where

$s_{ii} = 2(v-1) + (1+\rho^2)(k-2)(v-1)$;

$s_{ij} = -2(k-1)\rho$, $i \neq j$.

The complete symmetric information matrix of $\boldsymbol{\tau}$ is then given by

$$C_{\tau|\beta} = (c_{ij}) = \frac{1}{1-\rho^2}\left(S - \frac{1}{t}M'M\right), \tag{3.5}$$

where

$$c_{ii} = \frac{1}{1-\rho^2}\left[2(v-1) + (1+\rho^2)(k-2)(v-1) - \frac{1}{t}\{2(1-\rho)^2(v-1) + (1-\rho)^4(k-2)(v-1)\}\right]$$

$i = 1, \ldots, v$, and

$$c_{ij} = \frac{1}{1-\rho^2}\left[-2(k-1)\rho - \frac{1}{t}\{4(1-\rho)^3(k-2) + (1-\rho)^4(k-2)(k-3) + 2(1-\rho)^2\}\right],$$

for $i \neq j; i, j = 1, \ldots, v$.

The average variance, $\overline{V}$, of all estimated elementary contrasts of treatment effects is

$$\overline{V} = \frac{2\sigma^2}{c_{ii} - c_{ij}}. \tag{3.6}$$

3.3 Relative Efficiency

Since BIBAC type I design is a BIB design, the average variance of estimated elementary treatment effect contrasts for the corresponding BIB design is

$$\overline{V}_1 = \frac{2\sigma^2}{(k-1)v}. \tag{3.7}$$

Obviously, when $\rho = 0$, $\overline{V}$ simplifies to $\overline{V}_1$. Assuming σ^2 is same for BIBAC type I design and the corresponding BIB design, the relative efficiency (RE) of BIB design compared to BIBAC type I design is given by

$$RE_{\text{BIB:BIBACI}} = \frac{\overline{V}}{\overline{V}_1} = \frac{(k-1)v}{c_{ii} - c_{ij}}. \tag{3.8}$$

Now $c_{ii} - c_{ij} = A\,v + B$, where

$$A = \frac{1}{1-\rho^2}\left[t + 2\rho(k-1) - (1-\rho)^2 + \frac{2\rho(1-\rho)^2}{t}\right],$$

an expression not containing v, and

$$\begin{aligned} B &= \frac{1}{1-\rho^2}\Big[-2-(1+\rho^2)(k-2)+2(k-1)\rho-k+\frac{1}{t}\{2(1-\rho)^2+(1-\rho)^4(k-2)+ \\ &\qquad 4(1-\rho)^3(k-2)+(1-\rho)^4(k-2)(k-3)+2(1-\rho)^2\}\Big] \\ &= \frac{1}{1-\rho^2}\left[-2-(1+\rho^2)(k-2)+2(k-1)\rho-k+t\right] \\ &= 0 \end{aligned}.$$

Hence

$$RE_{\text{BIB:BIBACI}} = \frac{k-1}{A},$$

and does not depend on v. RE is a function of k and ρ.

Further

$$\begin{aligned} \frac{d}{dk}\left(\frac{A}{k-1}\right) &= \frac{-2\rho(1-\rho)t^2 - 2\rho(1-\rho)^2\{t+(1-\rho)^2(k-1)\}}{t^2(k-1)^2} \\ &< 0 \end{aligned}.$$

Hence $\frac{A}{k-1}$ is a decreasing function of k, implying that $RE_{\text{BIB:BIBACI}}$ is an increasing function of k. Since ρ is inherent in the experimental material, for any v, we should use minimum k in planning the BIBAC1 experiment to get maximum benefit of the autocorrelation compared to using standard BIB design analysis. For illustration purpose, for selected values of v, k and $\rho > 0$, we tabulate the relative efficiency in Table 3.1 and plot in Figure 3.1. From Table 3.1 and Figure 3.1, we note that the relative efficiency decreases with increasing ρ. For any given ρ, there is not much difference in the relative efficiencies for different values of v and k. Further, it may be noted that the RE approaches zero in the limit as ρ approaches to 1.

Table 3.1: Relative Efficiencies of BIB Designs Compared to BIBAC Type I Designs

v	k	Relative Efficiency (RE)				
		$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
5	3	1.000	0.840	0.642	0.427	0.208
	4	1.000	0.859	0.661	0.438	0.211
7	3	1.000	0.840	0.642	0.427	0.208
	4	1.000	0.859	0.661	0.438	0.211
	5	1.000	0.871	0.673	0.444	0.213
	6	1.000	0.879	0.681	0.448	0.214
11	3	1.000	0.840	0.642	0.427	0.208
	4	1.000	0.859	0.661	0.438	0.211

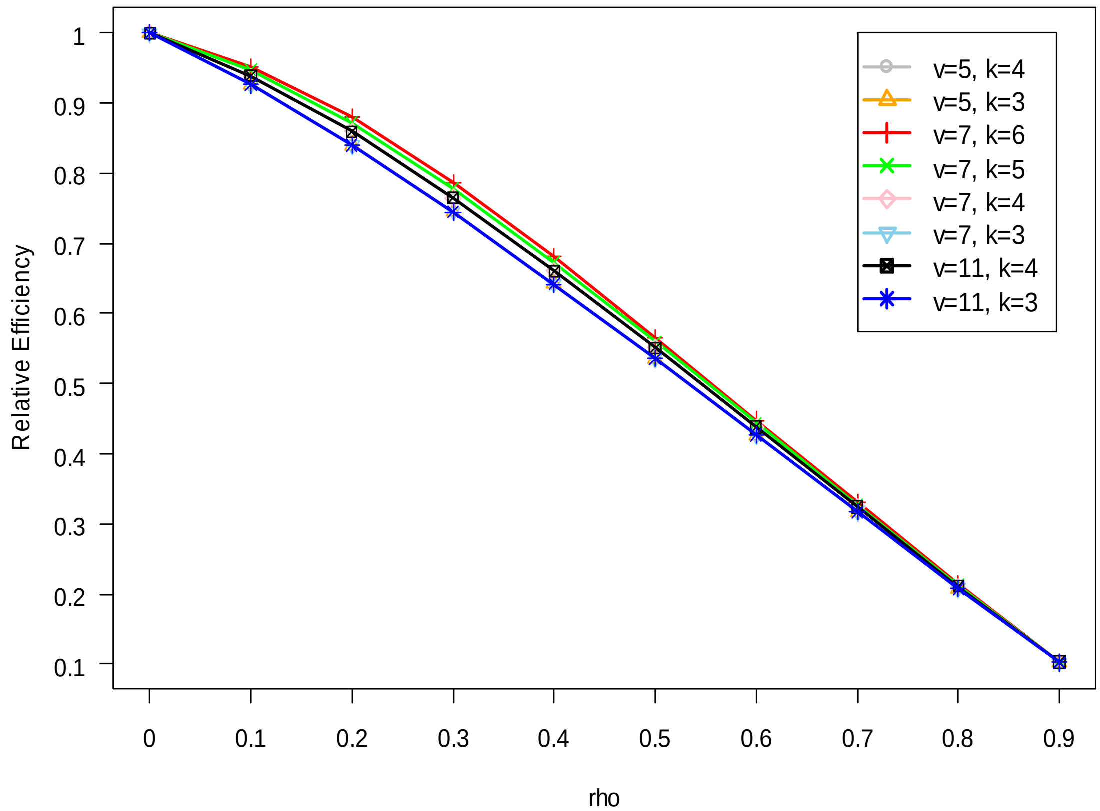

Figure 3.1: Relative Efficiencies of BIB Designs Compared to BIBAC Type I Designs.

In BIBAC type II designs, all occurrences of treatments in blocks are half of BIBAC type I design. Hence the information matrix of treatment effects eliminating block effects is

$$C^*_{\tau|\beta} = (c^*_{ii} - c^*_{ij})I_v + c^*_{ij}J_{vv}, \tag{3.9}$$

where $c^*_{ii} = c_{ii}/2$ and $c^*_{ij} = c_{ij}/2$. In this case, the relative efficiency compared to the corresponding BIB design is same as the RE given in formula (3.8).

3.4 PBIBAC Designs

Consider a PBIB design with m associate classes in v treatments of block size K where every treatment occurs in r blocks and every pair of treatments of i^{th} class occur together in λ_i blocks. Let $B_l = (b_{lij}), l = 1, 2, \cdots, m; i, j = 1, 2, \cdots, v$ be the l^{th} association matrix where $b_{lij} = 1\,(or\; 0)$ according as the i and j treatments are l^{th} associates (or not)

Let c_{ii}^{**} and c_{ij}^{**} be the expressions c_{ii} and c_{ij} given in Section 3.3 replacing v by K. Since we are forming BIBAC design from each block of PBIB design, in the PBIBAC design c_{ii}^{**} occur r times for each treatment, and every pair of l^{th} associate class treatments provide $\lambda_l c_{ij}^{**}$ expression in the information matrix. Hence the information matrix of treatment effects assuming autocorrelation of errors is

$$C_{\tau|\beta}^{**} = rc_{ii}^{**} I_v + \sum_{l=1}^{m} \lambda_l c_{ij}^{**} B_l \,. \tag{3.10}$$

Let us consider the PBIBAC design given in Section 3.1, with $\rho = 0.6$, $v = 6, \mathrm{K} = 4$. In the expressions c_{ii} and c_{ij} by plugging $K = v = 4$ and $k = 4$, we get $c_{ii}^{**} = 20.5714$, and $c_{ij}^{**} = -6.8571$.

The PBIB design has group divisible association scheme with

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

and $r = 2$, $K = 4$, $\lambda_1 = 2$, $\lambda_2 = 1$. Hence the information matrix for treatment effects eliminating block effects is

$$C^{**}_{\tau|\beta} = 41.1428 I_6 - 13.7142 B_1 - 6.8571 B_2, \tag{3.11}$$

A g-inverse of $C^{**}_{\tau|\beta}$ is

$$C^{-}_{\tau|\beta} = I_3 \otimes \begin{bmatrix} 0.0212 & 0.003 \\ 0.003 & 0.0212 \end{bmatrix}. \tag{3.12}$$

The average variance for testing elementary contrasts of treatment effects is $0.0418\sigma^2$.

In this design we are using 36 subjects. Hence the information per subject (ignoring $\frac{1}{\sigma^2}$) is 0.674. A BIBAC1 or 2 does not exist for $v = 6, k = 4$. However, the information per subject can be calculated for BIBAC1 with $v = 6, k = 4$ and is 0.6803. The information per subject for PBIBAC type I is very close to the information per subject we could have obtained from BIBAC1, had it existed.

CHAPTER 4

A Class of Symmetric Balanced Incomplete Block Designs with Autocorrelated Errors

In experiments when several treatments are used in a sequence on an experimental unit, the errors in the responses will be autocorrelated. Designs under autocorrelation structure using orthogonal arrays of Type I and Type II are studied by Majumdar and Martin (2004), and Shu and Raghavarao (2010) and some aspects of these designs are given in Chapter 3. In this Chapter we use symmetric balanced incomplete block designs under autocorrelated error structure. We first discuss the case $\lambda = 1$, and then generalize it to $\lambda > 1$.

The method of differences originally introduced by Bose (1939) is a powerful tool of constructing balanced incomplete block (BIB) designs. By using this method we construct Symmetric BIB designs with autocorrelated errors.

We find the average variance of symmetric BIB designs under autocorrelated errors. The relative efficiency of symmetric BIB design compared to the corresponding symmetric balanced incomplete block designs with autocorrelated errors is then shown to be a decreasing function of the autocorrelation ρ.

4.1 Symmetric BIB Designs with $\lambda = 1$ and Autocorrelated Errors

By using project geometries PG(2, s) (See Raghavarao, 1971) or the method of differences, we can construct a symmetric BIB design with parameters

$$v = b = s^2 + s + 1, \; r = k = s + 1, \; \lambda = 1, \tag{4.1}$$

where s is a prime or prime power. In the method of differences, there exists one initial set $S = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ in which $\alpha_j - \alpha_i, \; j \neq i, \; i, j = 1,2,\cdots k$ gives the members of the set $\{1, 2, \cdots, (v-1)\}$ exactly once. Then we form v blocks

$$(\alpha_1 + i, \alpha_2 + i, \cdots, \alpha_k + i), \;\; i = 0, 1, 2, \cdots, v-1,$$

developed mod v (See Raghavarao, 1971) to get symmetric BIB designs which can be used with autocorrleated errors. For example, we get the symmetric BIB design with parameters $v = 21$and $k = 5$ with autocorrelated errors by developing the column

$(0, 1, 6, 8, 18)'$,

The required design with autocorrelated errors with columns as blocks is:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0
6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5
8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7
18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

4.2. Analysis of Symmetric BIB Designs with $\lambda = 1$ and Autocorrelated Errors

Continuing the analysis of BIBAC designs given in Section 3.2, we use 5 matrices to express the estimated treatment effects. Let $B_l^* = (b_{lij}^*), l = 1, 2, \cdots, 5; i, j = 1, 2, \cdots, v$, where the 5 matrices are defined as below:

$$B_1^* = (b_{1ij}^*) = \begin{cases} \text{1, if i and j occur in 1st and kth positions in the same block} \\ \text{0, otherwise} \end{cases}, \tag{4.3}$$

$$B_2^* = (b_{2ij}^*) = \begin{cases} \text{1, if i or j occurs in 1st or kth position, the other occurs} \\ \quad \text{in the adjacent position in the same block} \\ \text{0, otherwise} \end{cases}, \tag{4.4}$$

$$B_3^* = (b_{3ij}^*) = \begin{cases} \text{1, if i or j occurs in 1st or kth position, the other occurs in} \\ \quad \text{the non - adjacent position in the same blcok} \\ \text{0, otherwise} \end{cases}, \tag{4.5}$$

$$B_4^* = (b_{4ij}^*) = \begin{cases} \text{1, if i and j occur in 2nd, 3rd,} \cdots \text{, (k - 1)th positions and} \\ \quad \text{they are in adjacent positions in the same block} \\ \text{0, otherwise} \end{cases}, \tag{4.6}$$

$$B_5^* = (b_{5ij}^*) = \begin{cases} \text{1, if i and j occur in 2nd, 3rd,} \cdots \text{, (k - 1)th positions and} \\ \quad \text{they are in non - adjacent positions in the smae block} \\ \text{0, otherwise} \end{cases}. \tag{4.7}$$

The information matrix of θ is then given by

$$X'\left[\mathrm{I}_{v(v-1)} \otimes \Sigma^{-1}\right]X = \frac{1}{1-\rho^2}\begin{bmatrix} vt & t1'_b & t1'_v \\ t1_b & t\mathrm{I}_b & M \\ t1_v & M' & S \end{bmatrix}, \tag{4.8}$$

where $b = v$, I_b is the identity matrix of order b, '$\otimes$' is the kronecker product sign, and $t = 2(1-\rho) + (k-2)(1-\rho)^2$.

The information matrix of τ is then given by

$$C_{\tau|\beta} = (c_{ij}) = \frac{1}{1-\rho^2}\left(S - \frac{1}{t}M'M\right). \tag{4.9}$$

In terms of t, B_1^*, B_2^*, B_3^*, B_4^*, and B_5^*, we have:

$$S = [2 + (1+\rho^2)(k-2)]I_v - \rho B_2^* - \rho B_4^*;$$

$$\begin{aligned} M'M &= [2(1-\rho)^2 + (k-2)(1-\rho)^4]I_v + (1-\rho)^2 B_1^* + (1-\rho)^3 B_2^* + (1-\rho)^3 B_3^* \\ &\quad + (1-\rho)^4 B_4^* + (1-\rho)^4 B_5^* \end{aligned},$$

and the information matrix of τ is

$$\begin{aligned} C_{\tau|\beta} = \frac{1}{1-\rho^2}\Bigg[&\{2 + (1+\rho^2)(k-2) - \frac{2(1-\rho)^2 + (k-2)(1-\rho)^4}{t}\}I_v - \frac{(1-\rho)^2}{t}B_1^* \\ &- \{\rho + \frac{(1-\rho)^3}{t}\}B_2^* - \frac{(1-\rho)^3}{t}B_3^* - \{\rho + \frac{(1-\rho)^4}{t}\}B_4^* - \frac{(1-\rho)^4}{t}B_5^* \Bigg] \end{aligned}. \tag{4.10}$$

Let us consider the symmetric BIB design with autocorrelated errors with $\rho = 0.6$, $v = b = 21$, $k = 5$, $\lambda = 1$. The 5 B_l^* matrices for the symmetric BIB design given in Section 4.1 can be shown in one matrix as follows:

$$
B_l^* = \begin{bmatrix}
0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla \\
\nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * \\
* & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial \\
\partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond \\
\Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * \\
* & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond \\
\Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# \\
\# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond \\
\Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond \\
\Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla \\
\nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla \\
\nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond \\
\Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond \\
\Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond & \# \\
\# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * & \Diamond \\
\Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond & * \\
* & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial & \Diamond \\
\Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * & \partial \\
\partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla & * \\
* & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0 & \nabla \\
\nabla & * & \partial & \Diamond & * & \Diamond & \# & \Diamond & \Diamond & \nabla & \nabla & \Diamond & \Diamond & \# & \Diamond & * & \Diamond & \partial & * & \nabla & 0
\end{bmatrix}
$$

where

$$
B_l^* = \begin{cases}
B_1^*, \text{if } \partial = 1 \text{ and the other equal } 0 \\
B_2^*, \text{if } \nabla = 1 \text{ and the other equal } 0 \\
B_3^*, \text{if } \Diamond = 1 \text{ and the other equal } 0 \\
B_4^*, \text{if } * = 1 \text{ and the other equal } 0 \\
B_5^*, \text{if } \# = 1 \text{ and the other equal } 0
\end{cases} .
$$

Here the information matrix for treatment effects eliminating block effects with autocorrelated errors is

$$C_{\tau|\beta} = 9.016I_{21} - 0.195B_1^* - 1.016B_2^* - 0.078B_3^* - 0.969B_4^* - 0.031B_5^*. \quad (4.11)$$

Now we can find $C_{\tau|\beta}^-$, a g-inverse of $C_{\tau|\beta}$, and calculate average variance, $\overline{V}$, of all estimated elementary contrasts of treatment effects as

$$\overline{V} = \frac{2\sigma^2}{v(v-1)}\left[v\, trC_{\tau|\beta}^- - 1'C_{\tau|\beta}^- 1\right]. \quad (4.12)$$

4.3. Relative Efficiency of Symmetric BIB Design with $\lambda = 1$ and Autocorrelated Errors

The average variance of estimated elementary treatment contrasts for the corresponding symmetric BIB design with independent errors is

$$\overline{V}_1 = \frac{2k\sigma^2}{v}. \quad (4.13)$$

Assuming σ^2 is same for symmetric BIB design with autocorrelated errors and the corresponding symmetric BIB design, the relative efficiency (RE) of a symmetric BIB design compared to symmetric BIB design with autocorrelated errors is given by

$$RE_{\text{SBIB:SBIBAC}} = \frac{\overline{V}}{\overline{V}_1} = \frac{v\, trC_{\tau|\beta}^- - 1'C_{\tau|\beta}^- 1}{k(v-1)}. \quad (4.14)$$

For selected values of v, k and $\rho > 0$, we tabulate the relative efficiencies in Table 4.1

From Table 4.1, we note that the relative efficiency decreases with increasing ρ. For any given ρ, there is not much difference in the relative efficiencies for different values of v. Further, it may be noted that the RE approaches zero in the limit as ρ approaches to 1.

Table 4.1: Relative Efficiencies of Symmetric BIB Designs Compared to Symmetric BIB Designs with $\lambda = 1$ and Autocorrelated Errors

v	k	Relative Efficiency (RE)				
		$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
7	3	1.000	0.844	0.653	0.441	0.219
13	4	1.000	0.867	0.679	0.459	0.227
21	5	1.000	0.879	0.692	0.466	0.228
31	6	1.000	0.887	0.704	0.479	0.238

4.4. Symmetric BIB Designs with $\lambda > 1$ and Autocorrelated Errors

By using the method of differences (Raghavarao, 1971), we can construct a symmetric BIB design with parameters

$$v = b, r = k, \lambda . \tag{4.15}$$

The details on the existence and non existence of these BIB designs and other configurations using the Hasse-Minkowski invariant and rational congruence of matrices are very well described in Raghavarao (1971), Raghavarao and Padgett (2005). A symmetric BIB design with parameter $v = b = 11, r = k = 6, \lambda = 3$ with autocorrelated errors is obtained by developing the following initial column

$$(0, 2, 3, 5, 6, 7)',$$

and is given below:

0	1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6

4.5. Analysis of Symmetric BIB Designs with $\lambda > 1$ and Autocorrelated Errors

We need the B_l^* matrices of Section 4.2 to give the analysis of these designs. While the entries of B_l^* are 0 or 1 for designs given in Section 4.2, in the present case they can be any positive integer in the closed interval $[0, \lambda]$. They satisfy

$$\sum_{i=1}^{5} B_i^* = \lambda(J_v - I_v), \text{ if } \lambda > 1 \ .$$

For example, let us consider the symmetric BIB design with parameters $v = b = 11$, $r = k = 6$, $\lambda = 3$ given in Section 4.4. The 5 B_l^* matrices for that design are

$$B_1^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_2^* = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$B_3^* = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 3 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 \\ 3 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 3 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 3 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$B_4^* = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

$$B_5^* = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \end{bmatrix}$$

In this case the coefficients of B_l^* are same as Section 4.2 in the $C_{\tau|\beta}$ matrix. This information matrix for treatment effects eliminating block effects for the above example with autocorrelation $\rho = 0.6$ is

$$C_{\tau|\beta} = 11.167 I_{11} - 0.174 B_1^* - 1.007 B_2^* - 0.069 B_3^* - 0.965 B_4^* - 0.028 B_5^* . \qquad (4.16)$$

Now we can find $C_{\tau|\beta}^-$, a g-inverse of $C_{\tau|\beta}$, and calculate the average variance, $\overline{V}$, of all estimated elementary contrasts of treatments as formulation (4.12).

4.6. Relative Efficiency of Symmetric BIB Design with $\lambda > 1$ and Autocorrelated Errors

Since a symmetric BIB design with autocorrelated errors is a symmetric BIB design, the average variance of estimated elementary treatment contrasts for the corresponding symmetric BIB design with independent errors is

$$\overline{V}_1 = \frac{2k\sigma^2}{\lambda v}. \tag{4.17}$$

Assuming σ^2 is same for symmetric BIB design with autocorrelated errors and the corresponding symmetric BIB design, the relative efficiency (RE) of a symmetric BIB design compared to symmetric BIB design with autocorrelated errors is given by

$$RE_{\text{SBIB:SBIBAC}} = \frac{\overline{V}}{\overline{V}_1} = \frac{\lambda(v\, trC^{-}_{\tau|\beta} - 1'C^{-}_{\tau|\beta}1)}{k(v-1)}. \tag{4.18}$$

For selected values of v, k, λ and $\rho > 0$, we tabulate the relative efficiencies in Table 4.2

From Table 4.2, we note that the relative efficiency decreases with increasing ρ. For any given ρ, there is not much difference in the relative efficiencies for different values of v.

Table 4.2: Relative Efficiencies of Symmetric BIB Designs Compared to Symmetric BIB Designs with $\lambda > 1$ and Autocorrelated Errors

v	k	λ	Relative Efficiency (RE)				
			$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
7	4	2	1.000	0.860	0.662	0.438	0.212
11	5	2	1.000	0.875	0.681	0.452	0.218
11	6	3	1.000	0.897	0.736	0.526	0.275
13	9	6	1.000	0.922	0.790	0.601	0.337

For these designs, the RE will also be depending on the structure of the symmetric BIB designs in addition to the parameters. Further, it may be noted that the RE approaches zero in the limit as ρ approaches to 1.

CHAPTER 5

OA Type and Symmetric Balanced Incomplete Block Designs with Compound Symmetric Error Structure

In the previous two chapters, OA Type balanced incomplete block designs and symmetric balanced incomplete block designs with autocorrelated errors were considered. However, it is also reasonable to assume that the correlation between any two observations within each individual is same. We also want to see whether autocorrelation error structure is robust against compound symmetric error structure with respect to relative efficiency. In this chapter, we consider designs of the last two chapters when the observations within each block have compound symmetry correlation structure. In Sections 5.1 and 5.2, we give the analysis of OA Type balanced incomplete block designs with compound symmetric (BIBCS) error structure and give the relative efficiency of BIB design compared to BIBCS design. We also show that the relative efficiency does not depend on the number of treatments and the block sizes, and is a decreasing function of the correlation, ρ. In Sections 5.3-5.6, we provide the analysis of symmetric balanced incomplete block designs with compound symmetric (SBIBCS) error structure and give the relative efficiency of SBIB design compared to SBIBCS design. Finally, in Section 5.7, the relative efficiencies of BIBAC designs compared to the corresponding BIBCS, and SBIBAC designs compared to the corresponding SBIBCS designs are given.

5.1. Analysis of BIBCS Designs of OA Type I

Continuing the analysis of BIBAC designs given in Section 3.2, we have

$$\mathbf{Y} = X\,\boldsymbol{\theta} + \mathbf{e}\,, \tag{5.1}$$

where $\mathbf{e}$ has variance-covariance matrix $I_{v(v-1)} \otimes \Sigma$, where $\Sigma = (1-\rho)I_k + \rho J_k$.

The information matrix of $\boldsymbol{\theta}$ is then given by

$$X'\left[\mathrm{I}_{v(v-1)} \otimes \Sigma^{-1}\right]X = \frac{1}{1-\rho}\begin{bmatrix} v(v-1)kt & kt\,\mathbf{1}'_b & kt\left(v-1\right)\mathbf{1}'_v \\ kt\,\mathbf{1}_b & kt\,I_b & M \\ kt\left(v-1\right)\mathbf{1}_v & M' & S \end{bmatrix}, \tag{5.2}$$

where $b = v\left(v-1\right)$, I_b is the identity matrix of order b, '$\otimes$' is the kronecker product sign,

$t = \dfrac{1-\rho}{1+\rho(k-1)}$, $M = \left(m_{ij}\right)$ is a $b \times v$ order matrix, where

$m_{ij} = \dfrac{1-\rho}{1+\rho(k-1)}$, if j^{th} treatment occurs in the i^{th} block;

$= 0$, if j^{th} treatment does not occur in the i^{th} block

and $S = \left(s_{ij}\right)$ is a $v \times v$ order matrix, where

$s_{ii} = \dfrac{k(v-1)[1+\rho(k-2)]}{1+\rho(k-1)}$;

$s_{ij} = -\dfrac{\rho k(k-1)}{1+\rho(k-1)}$, $i \neq j$.

The complete symmetric information matrix of $\boldsymbol{\tau}$ is then given by

$$C_{\tau|\beta} = \left(c_{ij}\right) = \frac{1}{1-\rho}\left(S - \frac{1}{kt}M'M\right), \tag{5.2}$$

where

$$c_{ii} = \frac{1}{1-\rho}\left[\frac{k(v-1)\{1+\rho(k-2)\}}{1+\rho(k-1)} - (v-1)t\right]$$

$i = 1, \ldots, v$, and

$$c_{ij} = \frac{1}{1-\rho}\left[-\frac{\rho k(k-1)}{1+\rho(k-1)} - (k-1)t\right],$$

for $i \neq j; i, j = 1, \ldots, v$.

The average variance, $\overline{V}$, of all estimated elementary contrasts of treatment effects is

$$\overline{V} = \frac{2\sigma^2}{c_{ii} - c_{ij}} = \frac{2(1-\rho)\sigma^2}{(k-1)v}.$$

5.2. Relative Efficiency of BIB Designs vs. BIBCS Designs of OA Type I

Since BIBCS Type I design is a BIB of Type I design, the average variance of estimated elementary treatment effect contrasts for the corresponding BIB design is

$$\overline{V_1} = \frac{2\sigma^2}{(k-1)v}. \tag{5.3}$$

Obviously, when $\rho = 0$, $\overline{V}$ simplifies to $\overline{V_1}$. Assuming σ^2 is same for BIBCS Type I design and the corresponding BIB design, the relative efficiency (RE) of BIB design compared to BIBCS Type I design is given by

$$
\begin{aligned}
RE_{\text{BIB : BIBCSI}} &= \frac{\overline{V}}{\overline{V}_1} = \frac{(k-1)v}{c_{ii}-c_{ij}} \\
&= \frac{(1-\rho)(k-1)v}{(k-1)v} . \qquad (5.4) \\
&= 1-\rho
\end{aligned}
$$

From equation (5.4), we note that the relative efficiency does not depend on v and k. It is a decreasing function ρ.

In BIBCS Type II designs, all occurrences of treatment and blocks are half of BIBCS Type I design. Hence the information matrix of treatment effects eliminating block effects is

$$C^*_{\tau|\beta} = (c^*_{ii} - c^*_{ij}) I_v + c^*_{ij} J_{vv} , \qquad (5.5)$$

where $c^*_{ii} = c_{ii}/2$ and $c^*_{ij} = c_{ij}/2$. In this case, the relative efficiency compared to the corresponding BIB design is same as the RE given in formula (5.4).

5.3. Analysis of SBIB Designs with $\lambda = 1$ and Compound Symmetry Error Structure

Continuing the analysis of SBIBAC designs given in Section 4.2, we use the 5 B^*_l matrices to express the estimated treatment effects. For the definition of B^*_l matrices, we refer to Section 4.2.

The information matrix of $\boldsymbol{\theta}$ is given by

$$X'\left[I_v \otimes \Sigma^{-1}\right]X = \frac{1}{1-\rho}\begin{bmatrix} kvt & kt\mathbf{1}'_b & kt\mathbf{1}'_v \\ kt\mathbf{1}_b & ktI_{\mathbf{b}} & M \\ kt\mathbf{1}_v & M' & S \end{bmatrix},$$

where $b = v$, and I_b is the identity matrix of order b, '$\otimes$' is the kronecker product sign,

$t = \dfrac{1-\rho}{1+\rho(k-1)}$, S is a $v \times v$ matrix as given below:

$$S = \left[k\{1 - \frac{\rho}{1+\rho(k-1)}\}\right]I_v - \frac{\rho}{1+\rho(k-1)}\sum_{l=1}^{5} B_l^* .$$

The matrix M is a $b \times v$ matrix satisfying

$$M'M = kt^2 I_v + t^2 \sum_{l=1}^{5} B_l^*$$

The information matrix of τ is then given by

$$\begin{aligned}
C_{\tau|\beta} &= (c_{ij}) = \frac{1}{1-\rho}\left(S - \frac{1}{kt}M'M\right) \\
&= \frac{1}{1-\rho}\left[\{k(1 - \frac{\rho}{1+\rho(k-1)}) - t\}I_v - \{\frac{\rho}{1+\rho(k-1)} + \frac{t}{k}\}\sum_{l=1}^{5} B_l^*\right] \\
&= \frac{1}{1-\rho}\left[\{k(1 - \frac{\rho}{1+\rho(k-1)}) - t\}I_v - \{\frac{\rho}{1+\rho(k-1)} + \frac{t}{k}\}(J_v - I_v)\right] \\
&= \frac{1}{1-\rho}\left[\{k(1 - \frac{\rho}{1+\rho(k-1)}) - t\}I_v - \{\frac{\rho}{1+\rho(k-1)} + \frac{t}{k}\}(J_v - I_v)\right] \\
&= \frac{1}{1-\rho}\left[\{k - \frac{k\rho}{1+\rho(k-1)} - \frac{1-\rho}{1+\rho(k-1)} + \frac{\rho}{1+\rho(k-1)} + \frac{1-\rho}{k(1+\rho(k-1))}\}I_v - \right. \\
&\qquad \left. \{\frac{\rho}{1+\rho(k-1)} + \frac{1-\rho}{k(1+\rho(k-1))}\}\right]J_v \\
&= \frac{1}{1-\rho}\left[\frac{k^2\{1+\rho(k-1)\} - k(1-2\rho+k\rho) + 1 - \rho}{k\{1+\rho(k-1)\}}\right]I_v - \frac{1}{1-\rho}\left[\frac{k\rho + 1 - \rho}{k(1+\rho(k-1)}\right]J_v \quad .(5.6)
\end{aligned}$$

Now we can find $C_{\tau|\beta}^{-}$, a g-inverse of $C_{\tau|\beta}$, as given Equation (5.7).

$$C_{\tau|\beta}^{-} = \frac{(1-\rho)k\{1+\rho(k-1)\}}{k^2\{1+\rho(k-1)\} - k(1-2\rho+k\rho)+1-\rho} I_v + wJ_v, \tag{5.7}$$

where w is not of interest, and calculate the average variance, $\overline{V}$, of all estimated elementary contrasts of treatments as

$$\overline{V} = 2\sigma^2 \left[\frac{(1-\rho)k\{1+\rho(k-1)\}}{k^2\{1+\rho(k-1)\} - k(1-2\rho+k\rho)+1-\rho} \right]. \tag{5.8}$$

5.4. Relative Efficiency of SBIB Designs with $\lambda = 1$ with Compound Symmetry Error Structure

Since a symmetric BIB design with compound symmetric errors is a symmetric BIB design, the average variance of estimated elementary treatment contrasts for the corresponding SBIB design with independent errors is

$$\overline{V}_1 = \frac{2k\sigma^2}{v}. \tag{5.9}$$

Assuming σ^2 is same for SBIB design with compound symmetry error structure and the corresponding symmetric BIB design, the relative efficiency (RE) of a SBIB design compared to symmetric BIB design with compound symmetry error structure is given by

$$
\begin{aligned}
RE_{SBIB:S\,BIBCS} &= \frac{\overline{V}}{\overline{V}_1} = \frac{v(1-\rho)[1+\rho(k-1)}{k^2[1+\rho(k-1)]-k^2\rho-k+2k\rho+1-\rho} \\
&= \frac{\{k(k-1)+1\}(1-\rho)\{1+\rho(k-1)\}}{\{k(k-1)+1\}\{1+\rho(k-1)\}} \\
&= 1-\rho.
\end{aligned} \tag{5.10}
$$

From equation (5.10), we note that the relative efficiency does not depend on v and k. It is a decreasing function ρ.

5.5. Analysis of SBIB Designs with $\lambda > 1$ and Compound Symmetric Error Structure

Continuing the analysis of Section 5.3, the information matrix of τ eliminating block effects is

$$
\begin{aligned}
C_{\tau|\beta} &= \frac{1}{1-\rho}\left[\frac{k^2\{1+\rho(k-1)\}-k(k\rho+1-\rho-\lambda\rho)+\lambda(1-\rho)}{k\{1+\rho(k-1)\}}\right]I_v \\
&\quad -\frac{1}{1-\rho}\left[\frac{k\lambda\rho+\lambda(1-\rho)}{k\{1+\rho(k-1)\}}\right]J_v
\end{aligned}, \tag{5.11}
$$

And a g-inverse of $C_{\tau|\beta}$, is

$$
C^{-}_{\tau|\beta} = \frac{(1-\rho)k\{1+\rho(k-1)\}}{k^2\{1+\rho(k-1)\}-k(k\rho+1-\rho-\lambda\rho)+\lambda(1-\rho)}I_v + wJ_v, \tag{5.12}
$$

where w is not of interest. The average variance, $\overline{V}$, of all estimated elementary contrasts of treatments as

$$
\overline{V} = 2\sigma^2\left[\frac{(1-\rho)k\{1+\rho(k-1)\}}{k^2\{1+\rho(k-1)\}-k(k\rho+1-\rho-\lambda\rho)+\lambda(1-\rho)}\right]. \tag{5.13}
$$

5.6. Relative Efficiency of SBIB Designs with $\lambda > 1$ and Compound symmetric error structure

Since a SBIB design with compound symmetric errors is a SBIB design, the average variance of estimated elementary treatment contrasts for the corresponding SBIB design with independent errors is

$$\overline{V}_1 = \frac{2k\sigma^2}{\lambda v}. \tag{5.14}$$

The relative efficiency assuming σ^2 same for the designs is

$$\begin{aligned} RE_{\text{SBIB:SBIBCS}} &= \frac{\overline{V}}{\overline{V}_1} = \frac{(1-\rho)k\{1+\rho(k-1)\}}{k^2\{1+\rho(k-1)\} - k(k\rho+1-\rho-\lambda\rho) + \lambda(1-\rho)} \times \frac{\lambda v}{k} \\ &= \frac{\{k(k-1)+\lambda\}(1-\rho)\{1+\rho(k-1)\}}{\{k(k-1)+\lambda\}\{1+\rho(k-1)\}} \\ &= 1 - \rho \end{aligned} \tag{5.15}$$

From equation (5.15), we note that the relative efficiency does not depend on v and k as before. It is a decreasing function ρ.

5.7. Relative Efficiencies of BIB Designs and SBIB Designs with Autocorrelated Errors vs. with Compound Symmetry Error Structure

It is easy to verify that

$$RE_{\boldsymbol{BIBACI:BIBCSI}} = \frac{\overline{V}_{BIBCSI}}{\overline{V}_{BIBACI}} = \frac{\overline{V}_{BIBCSI} / \overline{V}_{BIBI}}{\overline{V}_{BIBACI} / \overline{V}_{BIBI}} = \frac{RE_{BIBI:BIBCSI}}{RE_{BIBI:BIBACI}}, \tag{5.16}$$

And for selected values of v, k and ρ >0, and this relative efficiency is tabulated in Table 5.1. From Table 5.1, we note that the relative efficiency is a decreasing function of k. For any given k, there is not much difference in the relative efficiencies for different values of ρ and v. In the limit as ρ approaches 1, this RE also approaches 1.

Table 5.1: Relative Efficiencies of BIB Designs with Autocorrelated Errors vs. with Compound symmetric error structure

V	k	Relative Efficiency (RE)				
		$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
5	3	1.000	0.952	0.934	0.937	0.960
	4	1.000	0.931	0.907	0.914	0.946
7	3	1.000	0.952	0.934	0.937	0.960
	4	1.000	0.918	0.907	0.914	0.946
	5	1.000	0.910	0.892	0.902	0.938
	6	1.000	0.879	0.882	0.893	0.934
11	3	1.000	0.952	0.934	0.937	0.960
	4	1.000	0.931	0.907	0.914	0.946

Similarly we give the RE of SBIBAC verse SBIBCS in Tables 5.2 and 5.3

Table 5.2: Relative Efficiencies of SBIB Designs with Autocorrelated Errors vs. Compound symmetric error structure when $\lambda = 1$

v	k	Relative Efficiency (RE)				
		$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
7	3	1.000	0.948	0.919	0.907	0.913
13	4	1.000	0.923	0.884	0.871	0.881
21	5	1.000	0.910	0.867	0.858	0.877
31	6	1.000	0.902	0.852	0.835	0.840

Table 5.3: Relative Efficiencies of SBIB Designs with Autocorrelated Errors vs. Compound symmetric error structure when $\lambda > 1$

v	k	λ	Relative Efficiency (RE)				
			$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$
7	4	2	1.000	0.930	0.906	0.913	0.943
11	5	2	1.000	0.914	0.881	0.885	0.917
11	6	3	1.000	0.892	0.815	0.760	0.727
13	9	6	1.000	0.868	0.759	0.666	0.593

It is worth noting that the RE's in Tables 5.2 and 5.3 are linearly decreasing or convex functions in ρ for a given set of parameters. This may be due to the structures of the used symmetric BIB designs.

CHAPTER 6

Summary and Future Research

This research focused on the block designs when observations within blocks are autocorrelated.

In Chapter 3, we studied balanced and partially balanced incomplete block designs with autocorrelated errors constructed from orthogonal arrays of Type I and II. We gave explicit expressions for the average variance of estimated elementary contrasts of treatment effects of designs obtained by OA I and OA II under autocorrelation of errors and the relative efficiency of BIB design compared to the corresponding BIB design with autocorrelated errors.

In Chapter 4, we studied a class of symmetric balanced incomplete block designs with autocorrelated errors. The average variance of symmetric BIB designs under autocorrelated errors was given and the relative efficiency of these designs compared with the corresponding symmetric balanced incomplete block designs with uncorrelated errors was obtained.

In Chapter 5, we studied balanced incomplete block designs constructed from OA Types I and II designs and symmetric balanced incomplete block designs with compound symmetric error structure. We found explicit expressions of the average variance of estimated elementary contrasts of treatment effects of designs obtained by OA I and OA II under compound symmetric error structure and determined the relative efficiency of these designs with the corresponding balanced incomplete block designs with uncorrelated errors. We also provided the average variance of SBIB designs under compound symmetric error structure and found the relative efficiency of these designs compared with the corresponding symmetric balanced incomplete block designs with compound symmetric error structure.

Topics for future research based on this dissertation are:

1. Study of asymmetric balanced incomplete block designs under autocorrelated errors, and.
2. Examine the possibility to extend the above work to other types of block designs.

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