THE USE OF TEMPORALLY AGGREGATED DATA ON DETECTING A STRUCTURAL CHANGE OF A TIME SERIES PROCESS

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To My Family:

Mother, *Namsoon Kim*,

Wife, *Yunjeong Choi*,

and

Daughter, *Diane J. Lee*

ABSTRACT

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A time series process can be influenced by an interruptive event which starts at a certain time point and so a structural break in either mean or variance may occur before and after the event time.

However, the traditional statistical tests of two independent samples, such as the t-test for a mean difference and the F-test for a variance difference, cannot be directly used for detecting the structural breaks because it is almost certainly impossible that two random samples exist in a time series. As alternative methods, the likelihood ratio (LR) test for a mean change and the cumulative sum (CUSUM) of squares test for a variance change have been widely employed in literature.

Another point of interest is temporal aggregation in a time series. Most published time series data are temporally aggregated from the original observations of a small time unit to the cumulative records of a large time unit. However, it is known that temporal aggregation has substantial effects on process properties because it transforms a high frequency nonaggregate process into a low frequency aggregate process.

In this research, we investigate the effects of temporal aggregation on the LR test and the CUSUM test, through the ARIMA model transformation. First, we derive

the proper transformation of ARIMA model orders and parameters when a time series is temporally aggregated.

For the LR test for a mean change, its test statistic is associated with model parameters and errors. The parameters and errors in the statistic should be changed when an AR(p) process transforms upon the mth order temporal aggregation to an ARMA(P, Q) process. Using the property, we propose a modified LR test when a time series is aggregated. Through Monte Carlo simulations and empirical examples, we show that the aggregation leads the null distribution of the modified LR test statistic being shifted to the left. Hence, the test power increases as the order of aggregation increases.

For the CUSUM test for a variance change, we show that two aggregation terms will appear in the test statistic and have negative effects on test results when an ARIMA(p, d, q) process transforms upon the mth order temporal aggregation to an ARIMA(P, d, Q) process. Then, we propose a modified CUSUM test to control the terms which are interpreted as the aggregation effects. Through Monte Carlo simulations and empirical examples, the modified CUSUM test shows better performance and higher test powers to detect a variance change in an aggregated time series than the original CUSUM test.

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INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

It is sometimes found that a time series is influenced by an interruptive event starting at a certain time point, and so a structural change in the series occurs before and after the event time. The discordance can be interpreted as either a mean change or a variance change of the time series.

However, when we are interested to detect the changes, the classical statistical methods of the t-test for a mean difference and the F-test for a variance difference between two independent samples cannot be directly employed on the given problem because time series observations are almost certainly dependent and no possibility for randomization exists (Box and Tiao, 1965). Also, since the event time is often unknown and needs to be estimated, it makes another issue how to distinguish between the pre-event and the post-event and compare their structural difference.

As an alternative approach, we may test for random errors which are generally assumed to be a Gaussian white noise in a linear time series model. A difference between two groups of the sequential errors indicates a structural change in the given time series. Also, we can find a threshold time point to separate the two error groups when computing the maximum value of test statistics. In literature, a likelihood ratio (LR) test (see Hinkley, 1970; Chang et al., 1988; Tsay, 1988; Chen and Tiao, 1990; Balke, 1993; Chen and Liu, 1993; Tsay et al., 2000; Sánchez and Peña, 2003; Galeano et al., 2006) and a cumulative sum (CUSUM) test (see Hsu, 1977; Tsay, 1988; Inclán and Tiao, 1994; Lee and Park, 2001; Jin and Zhang, 2011) with the random errors have been widely used in the mean change problem and the variance change problem, respectively.

Another point of interest is temporal aggregation. Most published time series data are temporally aggregated from the original observations of a small time unit to the cumulative records of a large time unit. However, it is known that temporal aggregation has substantial effects on process properties because it transforms a high frequency nonaggregate series into a low frequency aggregate series. Amemiya and Wu (1972), Brewer (1973), Abraham (1982), Weiss (1984), Stram and Wei (1986), and Silvestrini and Veredas (2008) study the changes of autoregressive integrated moving average (ARIMA) model structures and parameters which result from the aggregation. Tiao (1972) and Wei (1978a) show that the aggregate model converges to an integrated moving average (IMA) limiting model as the aggregation order goes to infinity. Tiao and Wei (1976) and Wei (1978b) discuss the information loss in parameter estimation. Lütkepohl (1984, 1986) investigates the aggregation effects on vector autoregressive moving average (VARMA) model structures and the efficiency of the multivariate forecasts. It is also known that the temporal aggregation strengthens the linearity (Granger and Lee, 1999; Teles and Wei, 2000), induce the normality (Teles and Wei, 2002), and reduce the unit-root characteristic (Teles et al., 2008).

The purpose of this research is to analyze the effects of temporal aggregation on the LR test for a mean change and the CUSUM test for a variance change in a time series. To this end, we will investigate the changes in the ARIMA models and in those test statistics when a time series is temporally aggregated. This study will provide some guideline on the effectiveness of the tests under the aggregation and propose modified tests to correct the aggregation effects.

We start by presenting the mean change and variance change problems and reviews the relevant literature of the LR test and the CUSUM test in Chapter 1. Chapter 2 presents the proper model transformation when an ARIMA(p, d, q) time series process is temporally aggregated. In Chapter 3, we investigate how the LR testing procedure for a mean change is affected by temporal aggregation and proposed an alternative form to correct the aggregation effects on the LR test. In Chapter 4, we analyze the effects of temporal aggregation on the CUSUM testing procedure for a variance change and propose a modified CUSUM test to control the aggregation effects on the CUSUM test.

1.2 Testing for a Mean Change

The problem of interest is to identify a mean change in a time series process $\{x_t; t = 1, \ldots, n\}$. It can be reworded as testing the null hypothesis of a constant mean μ, i.e.,

$$H_0 : \mu_1 = \cdots = \mu_n \equiv \mu$$

against the alternative of a mean change starting at time k, i.e.,

$$H_1 : \mu_1 = \cdots = \mu_{k-1} \neq \mu_k = \cdots = \mu_n$$

for an integer k and $1 < k \leq n$, where μ_t is the expected value of x_t at time t.

We note that an autoregressive (AR) model has been widely used to describe a time series process in practice because of its easy interpretation. Let us consider two time series processes:

1. A stationary process $\{x_t^{(0)}; t = 1, \ldots, n\}$, which follows an AR($p$) model,

$$\phi_p(B)x_t^{(0)} = a_t, \tag{1.1}$$

where a_t is a Gaussian white noise of mean zero and variance σ_a^2, and

$$\phi_p(B) = 1 - \sum_{i=1}^{p} \phi_i B^i \tag{1.2}$$

is a polynomial of backshift operator B. Here all the roots of $\phi_p(B)$ are assumed to be outside of a unit circle.

2. A discordant process $\{x_t; t = 1, \ldots, n\}$ with a mean change starting at a time point k, which can be modeled as

$$
\begin{aligned}
x_t &= x_t^{(0)} + w_k(1 + B + B^2 + \cdots)I_t(k) \\
&= x_t^{(0)} + \frac{w_k I_t(k)}{(1 - B)}, \qquad (1.3)
\end{aligned}
$$

or, equivalently,

$$
\phi_p(B)x_t = \phi_p(B)\left[x_t^{(0)} + \frac{w_k I_t(k)}{(1 - B)}\right], \qquad (1.4)
$$

where w_k is the magnitude of the mean change and

$$
I_t(k) = \begin{cases} 1, & \text{for } t = k, \\ 0, & \text{for } t \neq k. \end{cases}
$$

We define a contaminated error $e_t = \phi_p(B)x_t$ for $t = 1, \ldots, n$. From Eq. (1.4), we have

$$
\begin{aligned}
e_t &= \phi_p(B)x_t \\
&= a_t + w_k I_t(k)\left[\frac{\phi_p(B)}{(1 - B)}\right]. \qquad (1.5)
\end{aligned}
$$

Then, Eq. (1.5) can be rewritten as a linear form,

$$
e_t = w_k y_t + a_t, \qquad (1.6)
$$

where

$$
\begin{aligned}
y_t &= \left[\frac{\phi_p(B)}{(1-B)}\right] I_t(k) \\
&= \left[(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p)(1+B+B^2+\cdots)\right] I_t(k) \\
&= \left[1+(1-\phi_1)B+(1-\phi_1-\phi_2)B^2+\cdots\right] I_t(k) \\
&= \begin{cases} 0, & \text{for } t<k, \\ 1, & \text{for } t=k, \\ 1-\sum_{j=1}^{t-k}\phi_j, & \text{for } t>k. \end{cases}
\end{aligned}
\tag{1.7}
$$

The change-magnitude w_k is estimated by the OLS estimator,

$$
\begin{aligned}
\hat{w}_k &= \frac{\sum_{t=k}^{n} e_t y_t}{\sum_{t=k}^{n} y_t^2} \\
&= \frac{e_k+\sum_{t=k+1}^{n} e_t\left(1-\sum_{j=1}^{t-k}\phi_j\right)}{1+\sum_{t=k+1}^{n}\left(1-\sum_{j=1}^{t-k}\phi_j\right)^2}
\end{aligned}
\tag{1.8}
$$

and the variance of the OLS estimator $\hat{w}_k$ is

$$
\begin{aligned}
\sigma^2_{\hat{w}_k} &= \frac{\sigma_a^2}{\sum_{t=k}^{n} y_t^2} \\
&= \frac{\sigma_a^2}{1+\sum_{t=k+1}^{n}\left(1-\sum_{i=1}^{t-k}\phi_i\right)^2}.
\end{aligned}
\tag{1.9}
$$

To test for the mean change at a known time point k, Chang et al. (1988) and Tsay (1988) propose a likelihood ratio (LR) test and its test statistic is given by

$$
\begin{aligned}
\lambda_k = \frac{\hat{w}_k}{\sigma_{\hat{w}_k}} &= \frac{\sum_{t=k}^{n} e_t y_t}{\sigma_a\sqrt{\sum_{t=k}^{n} y_t^2}} \\
&= \frac{e_k+\sum_{t=k+1}^{n} e_t\left(1-\sum_{i=1}^{t-k}\phi_i\right)}{\sigma_a\sqrt{1+\sum_{t=k+1}^{n}\left(1-\sum_{i=1}^{t-k}\phi_i\right)^2}},
\end{aligned}
\tag{1.10}
$$

assuming

$$\phi_{p+1} = \phi_{p+2} = \cdots = \phi_{n-k} = 0 \tag{1.11}$$

for $n - k > p$. We remark that the LR statistic λ_k in Eq. (1.10) is dependent on the model parameter ϕ's and θ's.

As a special case, if the series $x_t^{(0)}$ follows an AR(1) model, the LR test statistic in Eq. (1.10) becomes

$$\lambda_k = \frac{e_k + \sum_{t=k+1}^{n} e_t(1-\phi)}{\sigma_a\sqrt{1+(n-k)(1-\phi)^2}}. \tag{1.12}$$

Since the time point k is unknown and needs to be estimated in general, we use the supremum of the absolute LR statistic, i.e.,

$$\sup_{k=2,\ldots,n} |\lambda_k|. \tag{1.13}$$

as a test statistic instead of λ_k. If the supremum exceeds a predetermined critical value $L > 0$, then we reject the null hypothesis and it is believed that there exists a mean change starting at the time point

$$\hat{k} = \arg \sup_{k=2,\ldots,n} |\lambda_k| \tag{1.14}$$

with the change-magnitude $\hat{w}_{\hat{k}}$. Otherwise, we are confident that no significant mean change can be found in the series x_t (For more discussions, see Chang et al., 1988; Tsay, 1988; Chen and Liu, 1993; Tsay et al., 2000; Galeano et al., 2006).

Under the null hypothesis of no mean change, the LR statistic λ_k is asymptotically distributed as the standard normal distribution for a given k (See Tsay, 1988; Balke, 1993; Chen and Liu, 1993). Various recommended choices of the critical value L have been given in literature. According to Tsay (1988), one value of 3.0, 3.5, and 4.0 can be used. Chen and Tiao (1990) suggest $L = 2.8$ for high sensitivity mean shift detection, $L = 3.0$ for moderate sensitivity, and $L = 3.3$ for low sensitivity. The choices of Chen and Liu (1993) range from $L = 2.3$ to $L = 3.4$ at significance level $\alpha = 0.05$. Galeano et al. (2006) list their choices between 2.9 and 4.0 at $\alpha = 0.05$. Through the Monte Carlo simulations in Section 3.2, we will more extensively look into the critical value L and create a table for proper L values to replace the subjective choices used in the literature.

In practice, the model parameters ϕ's and θ's and the error variance σ_a^2 are also estimated.

Estimating the Error Variance for the LR Test

Since the error e_t $(t = 1, \ldots, n)$ in Eq. (1.5) is contaminated by a mean change, its sample variance is likely to overestimate σ_a^2. Hence, Chen and Liu (1993) suggest alternative methods to obtain the estimate $\hat{\sigma}_a^2$.

1. The median absolute deviation (MAD) method:

   The MAD estimate of the error variance is defined by

   $$\hat{\sigma}_a^2 = \left(1.483 \times \text{median}\left|\hat{e}_t - \tilde{e}\right|\right)^2, \qquad (1.15)$$

   where $\tilde{e}$ is the median of the estimated errors $\hat{e}_t$ (For more details, see Andrews et al., 1972, p.239).

2. The α% trimmed method:

   The estimate is the sample variance of the contaminated errors excluding the upper α% of their absolute values.

Estimating the Model Parameter for the LR Test

From Chen and Liu (1993), to estimate model parameters and test for a mean change when the series model is unknown, we will proceed as follows.

1. Model the given series x_t by assuming that there exists no mean change and estimate its model parameters, using the Bayesian information criterion (BIC) and the maximum likelihood estimates (MLE).

2. From the estimated model, compute the estimated errors

$$\hat{e}_t = \hat{\phi}(B)x_t \tag{1.16}$$

and the estimated y-values

$$\begin{aligned}\hat{y}_t &= \left[\hat{\phi}(B)(1 + B + B^2 + \cdots)\right] I_t(k) \\ &= \left[\frac{\hat{\phi}(B)}{(1 - B)}\right] I_t(k),\end{aligned} \tag{1.17}$$

where

$$\hat{\phi}_p(B) = 1 - \sum_{i=1}^{p} \hat{\phi}_i B^i \tag{1.18}$$

3. Calculate the estimated change-magnitude

$$\tilde{w}_k = \frac{\sum_{t=k}^{n} \hat{e}_t \hat{y}_t}{\sum_{t=k}^{n} \hat{y}_t^2} \tag{1.19}$$

and the estimated variance of $\tilde{w}_k$, i.e.,

$$\hat{\sigma}^2_{\tilde{w}_k} = \frac{\hat{\sigma}_a^2}{\sum_{t=k}^{n} \hat{y}_t^2}, \tag{1.20}$$

and the estimate of the likelihood ratio statistic

$$\hat{\lambda}_k = \frac{\tilde{w}_k}{\hat{\sigma}_{\tilde{w}_k}} = \frac{\sum_{t=k}^{n} \hat{e}_t \hat{y}_t}{\hat{\sigma}_a \sqrt{\sum_{t=k}^{n} \hat{y}_t^2}} \quad \text{for } k = 2, \ldots, n, \tag{1.21}$$

where $\hat{\sigma}_a^2$ is either the MAD estimate or the $\alpha\%$ trimmed estimate.

4. Compare the supremum $\sup_k |\hat{\lambda}_k|$ with the predetermined critical value L.
   (a) If $\sup_k |\hat{\lambda}_k| > L$ is satisfied, then there exists a possible mean change starting at the time point $\hat{k} = \arg\sup_{k=2,\ldots,n} |\hat{\lambda}_k|$ with the estimated shift-magnitude $\tilde{w}_{\hat{k}}$.
   (b) Otherwise, it is believed that the series has no significant mean change.
5. Eliminate the change effect from the given series x_t and obtain the adjusted series $x_t^{(\text{adj})}$,
$$x_t^{(\text{adj})} = x_t - \frac{\tilde{w}_{\hat{k}} I_t(\hat{k})}{(1-B)}, \tag{1.22}$$
which is assumed to be a stationary process shown in Eq. (1.1).
6. Refine the parameter estimation of the model including the estimation of error variance using the adjusted series $x_t^{(\text{adj})}$.

1.3 Testing for a Variance Change

The problem of interest is to identify a variance change in a time series process $\{x_t; t = 1, \ldots, n\}$. Since the given problem is equivalent to testing for a variance change of the model error, it can be reworded as testing the null hypothesis of a constant error variance σ_a^2, i.e.,

$$H_0 : \sigma_{a_1}^2 = \cdots = \sigma_{a_n}^2 \equiv \sigma_a^2$$

against the alternative of a variance change starting at time k, i.e.,

$$H_1 : \sigma_{a_1}^2 = \cdots = \sigma_{a_{k-1}}^2 \neq \sigma_{a_k}^2 = \cdots = \sigma_{a_n}^2$$

for an integer k and $1 < k \leq n$, where $\sigma_{a_t}^2$ is the error variance of x_t at time t.

We consider two time series processes:

1. A base process $\{x_t^{(0)}; t = 1, \ldots, n\}$ without any variance change, which follows an ARIMA(p, d, q) model,

$$\phi_p(B)(1-B)^d x_t^{(0)} = \theta_q(B) a_t, \tag{1.23}$$

where a_t is a Gaussian white noise of mean zero and variance σ_a^2, and

$$\phi_p(B) = 1 - \sum_{i=1}^{p} \phi_i B^i \tag{1.24}$$

and

$$\theta_q(B) = 1 - \sum_{j=1}^{q} \theta_j B^j \tag{1.25}$$

are polynomials of backshift operator B. Here all the roots of $\phi_p(B)$ and $\theta_q(B)$ are assumed to be outside of a unit circle.

2. A discordant process $\{x_t; t = 1, \ldots, n\}$ with a variance change starting at a time point k, which can be modeled as

$$x_t = x_t^{(0)} + v_k J_t(k) \left[\frac{\theta_q(B)}{\phi_p(B)(1-B)^d} \right] a_t \tag{1.26}$$

or, equivalently,

$$\begin{aligned} &\phi_p(B)(1-B)^d x_t \\ &= \phi_p(B)(1-B)^d x_t^{(0)} + v_k J_t(k) \theta_q(B) a_t \\ &= [1 + v_k J_t(k)]\, \theta_q(B) a_t, \end{aligned} \tag{1.27}$$

where v_k is the magnitude of the variance change and

$$J_t(k) = \begin{cases} 1, & \text{for } t \geq k, \\ 0, & \text{for } t < k. \end{cases}$$

We define a contaminated error

$$e_t = \pi(B)(1-B)^d x_t \tag{1.28}$$

for $t = 1, \ldots, n$, where

$$\pi(B) = (1 - \pi_1 B - \pi_2 B^2 - \cdots) = \frac{\phi_p(B)}{\theta_q(B)}. \tag{1.29}$$

From Eq. (1.27), we have

$$\begin{aligned} e_t &= \pi(B)(1-B)^d x_t \\ &= \begin{cases} a_t, & \text{for } t < k, \\ (1+v_k)a_t, & \text{for } t \geq k, \end{cases} \end{aligned} \tag{1.30}$$

which implies that the error variance changes from σ_a^2 to $(1+v_k)^2\sigma_a^2$ at k (Tsay, 1988).

We consider the cumulative sum (CUSUM) of the squared errors, i.e, $\sum_{t=1}^{l} e_t^2$ for $l = 1, \ldots, n$. To test for a variance change at a fixed k, Tsay (1988) and Inclán and Tiao (1994) propose a CUSUM test statistic,

$$d_k = \frac{\sum_{t=1}^{k} e_t^2}{\sum_{t=1}^{n} e_t^2} - \frac{k}{n}, \tag{1.31}$$

which has the null distribution

$$\frac{k(n-1)}{n^2}\left(\frac{1-F_{n-k,k}}{\frac{k}{n}+\frac{n-k}{n}F_{n-k,k}}\right),$$

where $F_{n-k,k}$ is an F-distribution with degrees of freedom $n-k$ and k.

However, the change point k is unknown and needs to be estimated in general. Hence, we use the supremum of the absolute CUSUM statistic, i.e.,

$$\sup_{k=2,\ldots,n} |d_k| \tag{1.32}$$

as a test statistic instead of d_k. Also, it is known that

$$\sup_{k=2,\ldots,n} |d_k| \xrightarrow{d} \sup_{0<r\leq 1}\left|\sqrt{\frac{2}{n}}B(r)\right|, \tag{1.33}$$

under the null hypothesis of no variance change, where

$$B(r) = W(r) - rW(1) \tag{1.34}$$

is a Brownian bridge and $W(r)$ is a Wiener process. Let $\lfloor x\rfloor$ denote the largest integer not greater than a real number x. Then, the value r is chosen from the condition of $\lfloor nr\rfloor = k$ (Inclán and Tiao, 1994).

CHAPTER 2

TEMPORAL AGGREGATION EFFECTS ON ARIMA MODELS

In this chapter, we first review the effects of temporal aggregation on the model form of an ARIMA process. Then, for the purpose of our study, we will derive the specific parameter relations between aggregate and nonaggregate models.

2.1 Aggregation Effects on the ARIMA Model Form

The discordant series x_t is transformed into the mth order temporal aggregate X_T, defined by

$$X_T = \sum_{t=m(T-1)+1}^{mT} x_t = \sum_{j=0}^{m-1} B^j x_{mT}, \tag{2.1}$$

where the aggregation order m is a positive integer for $m < n$ and the aggregate time unit $T = 1, \ldots, N$ for $N = n/m$ (Tiao, 1972; Wei, 2006, p.508). Similarly, the base series $x_t^{(0)}$ is aggregated into

$$X_T^{(0)} = \sum_{t=m(T-1)+1}^{mT} x_t^{(0)} = \sum_{j=0}^{m-1} B^j x_{mT}^{(0)}. \tag{2.2}$$

It has been known that if $x_t^{(0)}$ follows an ARIMA(p, d, q) model, then its mth order aggregate series $X_T^{(0)}$ will follow an ARIMA(P, d, Q) model,

$$\Phi_P(\mathcal{B})(1 - \mathcal{B})^d X_T^{(0)} = \Theta_Q(\mathcal{B}) A_T, \tag{2.3}$$

where A_T is a Gaussian white noise of mean zero and variance σ_A^2, and

$$\Phi_P(\mathcal{B}) = 1 - \sum_{i=1}^{P} \Phi_i \mathcal{B}^i \tag{2.4}$$

and

$$\Theta_Q(\mathcal{B}) = 1 - \sum_{j=1}^{Q} \Theta_j \mathcal{B}^j \tag{2.5}$$

are polynomials of the aggregate backshift operator $\mathcal{B} = B^m$.

The orders P and Q are determined by the roots of the polynomials $\phi_p(B)$ and $\theta_q(B)$, respectively. When the equation $\phi_p(B) = 0$ has distinct roots $\delta_{i^*}^{-1}$ for $i^* = 1, \ldots, p^*$, each with multiplicity s_{i^*} of $\sum_{i^*=1}^{p^*} s_{i^*} = p$, and the numbers s_{i^*} are partitioned into c distinct sets U_l for $l = 1, \ldots, c$, such that $s_{i_1^*}$ and $s_{i_2^*} \in U_l$ if and only if $\delta_{i_1^*}^m = \delta_{i_2^*}^m$ for a given m and $i_1^* \neq i_2^*$, the orders of P and Q are given by

$$P = \sum_{l=1}^{c} \max U_l \leq \sum_{i^*=1}^{p^*} s_{i^*} = p, \tag{2.6}$$

and

$$Q = \left\lfloor P + d + 1 - \frac{(p + d + 1) - q}{m} \right\rfloor, \tag{2.7}$$

where c is the number of distinct values $\delta_{i^*}^m$ and $\max U_l$ denotes the largest element in the lth set U_l. For the details and proofs, we refers readers to Stram and Wei (1986) and Wei (2006, pp.513–515).

2.2 The Parameter Relationship between Aggregate and Nonaggregate Models

Amemiya and Wu (1972), Brewer (1973), Weiss (1984), and Silvestrini and Veredas (2008) have derived the parameter relationships between aggregate and nonaggregate ARIMA models when $P = p$. We now extend to the case $P \leq p$ in (2.6) and show its aggregate model structure and parameters which are associated with the nonaggregate model.

Theorem 2.1. *When a series $x_t^{(0)}$ follows an ARIMA(p,d,q) model of*

$$\left(1-\sum_{i=1}^{p}\phi_i B^i\right)(1-B)^d x_t^{(0)} = \left(1-\sum_{j=1}^{q}\theta_j B^j\right)a_t,$$

its mth order aggregate series $X_T^{(0)}$ is known to follow an ARIMA(P,d,Q) model of

$$\left(1-\sum_{i=1}^{P}\Phi_i \mathcal{B}^i\right)(1-\mathcal{B})^d X_T^{(0)} = \left(1-\sum_{j=1}^{Q}\Theta_j \mathcal{B}^j\right)A_T$$

for $m>1$ and $m \in \mathbb{Z}$, where a_t and A_T are Gaussian white noises of mean zero and variance σ_a^2 and σ_A^2, respectively. Then, the parameters of the aggregate model are

$$\Phi_i = (-1)^{i+1}\sum(\delta_{i_1^*}\delta_{i_2^*}\cdots\delta_{i_i^*})^m, \quad i=1,\ldots,P, \tag{2.8}$$

and Θ_j, $j=1\ldots,Q$, a real solution of quadratic equations

$$\left(1+\sum_{j=1}^{Q}\Theta_j^2\right)+\gamma_S^{-1}\left(\Theta_S-\sum_{j=1}^{Q-S}\Theta_j\Theta_{S+j}\right)=0, \quad S=1,\ldots,Q, \tag{2.9}$$

where $\sum(\delta_{i_1^*}\delta_{i_2^*}\cdots\delta_{i_i^*})^m$ is the sum of all possible $\binom{P}{i}$ products of i different δ^m's and

$$\gamma_S = \frac{(\psi_0\psi_{mS}+\psi_1\psi_{mS+1}+\psi_2\psi_{mS+2}+\cdots)}{(\psi_0^2+\psi_1^2+\psi_2^2+\cdots)}. \tag{2.10}$$

The values of finite ψ's are determined by the equation

$$\begin{aligned}&\psi_0+\psi_1 B+\psi_2 B^2+\cdots\\ &=\left(1-\sum_{i=1}^{q}\theta_i B^i\right)\left[\frac{(1-B^m)^d\prod_{j=1}^{P}(1-\delta_j^m B^m)}{(1-B)^d\prod_{j=1}^{p}(1-\delta_j B)}\right]\left(1+\sum_{k=1}^{m-1}B^k\right).\end{aligned} \tag{2.11}$$

Also, the variance σ_A^2 can be written as $\sigma_A^2 = (\rho\sigma_a)^2$ with

$$\rho^2 = \left(\frac{\psi_0^2 + \psi_1^2 + \psi_2^2 + \cdots}{1 + \sum_{j=1}^{Q} \Theta_j^2} \right) \tag{2.12a}$$

or, equivalently,

$$\rho^2 = \left(\frac{\psi_0\psi_{mS} + \psi_1\psi_{mS+1} + \psi_2\psi_{mS+2} + \cdots}{-\Theta_S + \sum_{j=1}^{Q-S} \Theta_j \Theta_{S+j}} \right). \tag{2.12b}$$

Proof.

1. $m = 1$ (nonaggregation): $X_T^{(0)} = x_t^{(0)}$.

2. $m \geq 2$ and $m \in \mathbb{Z}$ (aggregation):

   We consider a link polynomial

$$\left[\frac{(1-B^m)^d \prod_{j=1}^{P}(1-\delta_j^m B^m)}{(1-B)^d \prod_{j=1}^{p}(1-\delta_j B)} \right] \left(1 + \sum_{k=1}^{m-1} B^k \right)$$

   for $P \leq p$, where δ's are all the inverted roots of $(1 - \sum_{i=1}^{p} \phi_i B^i)$, i.e.,

$$\prod_{j=1}^{p}(1-\delta_j B) = (1 - \sum_{i=1}^{p} \phi_i B^i) \tag{2.13}$$

   and for the first P roots, $\delta_{j_1}^m \neq \delta_{j_2}^m$ $(j_1 \neq j_2)$. When multiplying both sides of $(1 - \sum_{i=1}^{p} \phi_i B^i)(1-B)^d x_t^{(0)} = (1 - \sum_{j=1}^{q} \theta_j B^j) a_t$ by the link polynomial, we have

$$\begin{aligned} &\left[(1-B^m)^d \prod_{j=1}^{P}(1-\delta_j^m B^m) \right] \left(1 + \sum_{k=1}^{m-1} B^k \right) x_t^{(0)} \\ &= \left(1 - \sum_{i=1}^{q} \theta_i B^i \right) \left[\frac{(1-B^m)^d \prod_{j=1}^{P}(1-\delta_j^m B^m)}{(1-B)^d \prod_{j=1}^{p}(1-\delta_j B)} \right] \left(1 + \sum_{k=1}^{m-1} B^k \right) a_t. \end{aligned} \tag{2.14}$$

Let Z_t denote the model in Eq. (2.14). At $t = mT$, the AR part becomes

$$\begin{aligned} Z_{mT} &= \left[(1-B^m)^d \prod_{j=1}^{P}(1-\delta_j^m B^m)\right]\left(1+\sum_{k=1}^{m-1} B^k\right) x_{mT}^{(0)} \\ &= \left[\prod_{j=1}^{P}(1-\delta_j^m B^m)\right](1-B^m)^d X_T^{(0)} \end{aligned} \tag{2.15}$$

and the MA part

$$\begin{aligned} Z_{mT} &= \left(1-\sum_{i=1}^{q}\theta_i B^i\right)\left[\frac{(1-B^m)^d \prod_{j=1}^{P}(1-\delta_j^m B^m)}{(1-B)^d \prod_{j=1}^{p}(1-\delta_j B)}\right]\left(1+\sum_{k=1}^{m-1} B^k\right) a_{mT} \\ &\overset{say}{=} (\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots) a_{mT}, \end{aligned} \tag{2.16}$$

where the number of ψ's is finite. From Eq. (2.15) and (2.16), we obtain

$$\begin{aligned} &\left[\prod_{j=1}^{P}(1-\delta_j^m \mathcal{B})\right](1-\mathcal{B})^d X_T^{(0)} \\ &= \left(\sum_{i=0}^{\infty}\psi_i B^i\right)(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots) a_{mT}, \end{aligned} \tag{2.17}$$

which is equivalent to the given aggregate model

$$\left(1-\sum_{i=1}^{P}\Phi_i \mathcal{B}^i\right)(1-\mathcal{B})^d X_T^{(0)} = \left(1-\sum_{j=1}^{Q}\Theta_j \mathcal{B}^j\right) A_T, \tag{2.18}$$

where $\mathcal{B} = B^m$. Then, δ^{-m}'s are the distinct roots of $(1-\sum_{i=1}^{P}\Phi_i \mathcal{B}^i)$ and the ith parameter Φ_i can be expressed as

$$\Phi_i = (-1)^{i+1}\sum(\delta_{j_1}\delta_{j_2}\cdots\delta_{j_i})^m,$$

where $\sum(\delta_{j_1}\delta_{j_2}\cdots\delta_{j_i})^m$ is the sum of all possible $\binom{P}{i}$ products of i different

δ^m's. From Eq. (2.16), the variance of Z_{mT} is

$$\mathrm{Var}(Z_{mT}) = (\psi_0^2 + \psi_1^2 + \psi_2^2 + \cdots)\sigma_a^2 \tag{2.19}$$

and the covariance between Z_{mT} and $Z_{m(T+S)}$ is

$$\mathrm{Cov}(Z_{mT}, Z_{m(T+S)}) = (\psi_0\psi_{mS} + \psi_1\psi_{mS+1} + \psi_2\psi_{mS+2} + \cdots)\sigma_a^2. \tag{2.20}$$

We note that $(\psi_0^2 + \psi_1^2 + \psi_2^2 + \cdots)$ and $(\psi_0\psi_{mS} + \psi_1\psi_{mS+1} + \psi_2\psi_{mS+2} + \cdots)$ are convergent due to the stationarity of Z_{mT}. From the aggregate model (2.3), the variance of Z_{mT} is

$$\mathrm{Var}(Z_{mT}) = \left(1 + \sum_{j=1}^{Q} \Theta_j^2\right) \sigma_A^2 \tag{2.21}$$

and the covariance between Z_{mT} and $Z_{m(T+S)}$ is

$$\mathrm{Cov}(Z_{mT}, Z_{m(T+S)}) = \left(-\Theta_S + \sum_{j=1}^{Q-S} \Theta_j\Theta_{S+j}\right) \sigma_A^2. \tag{2.22}$$

Using Eq. (2.19), (2.20), (2.21), and (2.22), we obtain the auto-correlation $\gamma_S = \mathrm{Cov}(Z_{mT}, Z_{m(T+S)})/\mathrm{Var}(Z_{mT})$, i.e.,

$$\begin{aligned}\gamma_S &= \frac{(\psi_0\psi_{mS} + \psi_1\psi_{mS+1} + \psi_2\psi_{mS+2} + \cdots)\sigma_a^2}{(\psi_0^2 + \psi_1^2 + \psi_2^2 + \cdots)\sigma_a^2} \\ &= \frac{\left(-\Theta_S + \sum_{j=1}^{Q-S} \Theta_j\Theta_{S+j}\right)\sigma_A^2}{\left(1 + \sum_{j=1}^{Q} \Theta_j^2\right)\sigma_A^2},\end{aligned} \tag{2.23}$$

and so quadratic equations

$$\left(1 + \sum_{j=1}^{Q} \Theta_j^2\right) + \gamma_S^{-1}\left(\Theta_S - \sum_{j=1}^{Q-S} \Theta_j\Theta_{S+j}\right) = 0, \qquad S = 1, \ldots, Q$$

with respect to Θ's. From the equations above, we can find a real solution for the jth parameter Θ_j. Also, the variance σ_A^2 is

$$\sigma_A^2 = \left(\frac{\psi_0^2 + \psi_1^2 + \psi_2^2 + \cdots}{1 + \sum_{j=1}^{Q} \Theta_j^2} \right) \sigma_a^2,$$

or, equivalently,

$$\sigma_A^2 = \left(\frac{\psi_0 \psi_{mS} + \psi_1 \psi_{mS+1} + \psi_2 \psi_{mS+2} + \cdots}{-\Theta_S + \sum_{j=1}^{Q-S} \Theta_j \Theta_{S+j}} \right) \sigma_a^2.$$

□

As illustrations, let us derive the exact parameter expressions for the aggregate model when the nonaggregate model is either ARIMA$(1, 0, 0)$ (or equivalently, AR(1)) or ARIMA$(1, 0, 1)$ (or equivalently, ARMA$(1, 1)$). The results are summarized in Lemmas 2.2. and 2.3, respectively.

Lemma 2.2. *If a series $x_t^{(0)}$ follows an AR(1) model of*

$$(1 - \phi B) x_t^{(0)} = a_t$$

for nonzero ϕ, then its mth order aggregate series $X_T^{(0)}$ follows an ARMA$(1, 1)$ model of

$$(1 - \Phi \mathcal{B}) X_T^{(0)} = (1 - \Theta \mathcal{B}) A_T$$

for $m > 1$ and $m \in \mathbb{Z}$, where a_t and A_T are Gaussian white noises of mean zero and variance σ_a^2 and σ_A^2, respectively. The parameters of the aggregate ARMA$(1, 1)$ model are

$$\Phi = \phi^m \quad \textit{and} \quad \Theta = -\frac{1}{2\gamma} \pm \sqrt{\left(\frac{1}{2\gamma} + 1 \right) \left(\frac{1}{2\gamma} - 1 \right)}, \tag{2.24}$$

where

$$\gamma = \frac{\sum_{j=0}^{m-2}\left(\sum_{i=1}^{j+1}\phi^{i-1}\right)\left(\sum_{i=j+2}^{m}\phi^{i-1}\right)}{\sum_{j=0}^{m-1}\left(\sum_{i=1}^{j+1}\phi^{i-1}\right)^2 + \sum_{j=0}^{m-2}\left(\sum_{i=j+2}^{m}\phi^{i-1}\right)^2}. \tag{2.25}$$

Also, the variance σ_A^2 *is written as* $\sigma_A^2 = (\rho\sigma_a)^2$ *where*

$$\rho^2 = \frac{1}{1+\Theta^2}\left[\sum_{j=1}^{m}\left(\sum_{i=1}^{j}\phi^{i-1}\right)^2 + \sum_{j=1}^{m-1}\left(\sum_{i=j+1}^{m}\phi^{i-1}\right)^2\right], \tag{2.26a}$$

or, equivalently,

$$\rho^2 = -\frac{1}{\Theta}\sum_{j=1}^{m-1}\left(\sum_{i=1}^{j}\phi^{i-1}\right)\left(\sum_{i=j+1}^{m}\phi^{i-1}\right). \tag{2.26b}$$

Proof. We note that $p = P = 1$ from Eq. (2.6) and $Q = \lfloor 2 - \frac{2}{m} \rfloor = 1$ from Eq. (2.7), for $m > 1$ and $m \in \mathbb{Z}$. Then the AR expression in Eq. (2.15) becomes

$$Z_{mT} = (1 - \phi^m \mathcal{B})X_T^{(0)}, \tag{2.27}$$

which indicates that the AR parameter in Eq. (2.18) is now $\Phi = \phi^m$. From the MA expression in Eq. (2.16), we have

$$\begin{aligned} Z_{mT} &= \left(1 + \sum_{i=1}^{m-1}\phi^i B^i\right)\left(1 + \sum_{k=1}^{m-1} B^k\right) a_{mT} \\ &= \left(\sum_{j=0}^{2m-2}\psi_j B^j\right) a_{mT}, \end{aligned} \tag{2.28}$$

where

$$\psi_j = \begin{cases} \sum_{i=1}^{j+1}\phi^{i-1}, & \text{for } j = 0,\ldots,m-1, \\ \sum_{i=j-m+2}^{m}\phi^{i-1}, & \text{for } j = m,\ldots,2m-2, \end{cases} \tag{2.29}$$

assuming $\sum_{i=j_1}^{j_2}\phi^i = 0$ if $j_1 > j_2$. So, Eq. (2.9) becomes a quadratic equation

$$\Theta^2 + \gamma^{-1}\Theta + 1 = 0, \tag{2.30}$$

where

$$\gamma = \frac{\sum_{j=0}^{m-2} \psi_j \psi_{m+j}}{\sum_{j=0}^{2m-2} \psi_j^2}$$
$$= \frac{\sum_{j=0}^{m-2} \psi_j \psi_{m+j}}{\sum_{j=0}^{m-1} \psi_j^2 + \sum_{j=0}^{2m-2} \psi_{m+j}^2}$$
$$= \frac{\sum_{j=0}^{m-2} \left(\sum_{i=1}^{j+1} \phi^{i-1}\right) \left(\sum_{i=j+2}^{m} \phi^{i-1}\right)}{\sum_{j=0}^{m-1} \left(\sum_{i=1}^{j+1} \phi^{i-1}\right)^2 + \sum_{j=0}^{m-2} \left(\sum_{i=j+2}^{m} \phi^{i-1}\right)^2}.$$

Here the real solutions of Θ are

$$\Theta = -\frac{1}{2\gamma} \pm \sqrt{\left(\frac{1}{2\gamma} + 1\right)\left(\frac{1}{2\gamma} - 1\right)}.$$

Furthermore, from Eq. (2.12a), we have the variance of white noise A_T as

$$\sigma_A^2 = \frac{1}{1+\Theta^2}\left[\sum_{j=1}^{m}\left(\sum_{i=1}^{j} \phi^{i-1}\right)^2 + \sum_{j=1}^{m-1}\left(\sum_{i=j+1}^{m} \phi^{i-1}\right)^2\right]\sigma_a^2,$$

or, equivalently, from Eq. (2.12b),

$$\sigma_A^2 = -\frac{1}{\Theta}\sum_{j=1}^{m-1}\left(\sum_{i=1}^{j} \phi^{i-1}\right)\left(\sum_{i=j+1}^{m} \phi^{i-1}\right)\sigma_a^2.$$

□

Table 2.1 displays the numerical computations for aggregate ARMA$(1,1)$ model parameters Φ and Θ, and variance σ_A^2, when its nonaggregate model is an AR(1), $(1-\phi B)x_t^{(0)} = a_t$, for every choice of parameter $\phi \in \{-0.5, 0.3, 0.5, 0.8, 0.95\}$ and aggregation order $m = 1, 3, 6, 12$, assuming $\sigma_a = 1$.

We remark that Theorem 2.1 of Teles et al. (2008) is a special case of our Lemma 2.2 for $\phi = 1$.

Table 2.1: Model parameters and error variance of an aggregate ARMA(1, 1) process for given nonaggregate model's parameter ϕ and aggregation order m

ϕ	m	Φ	Θ	σ_A^2	σ_A
−0.50	1	−0.50000	–	1.00000	1.00000
	3	−0.12500	0.06479	1.92940	1.38903
	6	0.01562	0.10479	3.22436	1.79565
	12	0.00024	0.05035	5.91094	2.43124
0.30	1	0.30000	–	1.00000	1.00000
	3	0.02700	−0.10724	4.72793	2.17438
	6	0.00073	−0.06114	10.85873	3.29526
	12	0.00000	−0.02909	23.12463	4.80881
0.50	1	0.50000	–	1.00000	1.00000
	3	0.12500	−0.16667	6.75000	2.59808
	6	0.01562	−0.12459	18.38839	4.28817
	12	0.00024	−0.06247	42.50082	6.51927
0.80	1	0.80000	–	1.00000	1.00000
	3	0.51200	−0.21382	12.12259	3.48175
	6	0.26214	−0.23003	53.97619	7.34685
	12	0.06872	−0.18815	184.30572	13.57592
0.95	1	0.95000	–	1.00000	1.00000
	3	0.85737	−0.22041	16.38931	4.04837
	6	0.73509	−0.25388	107.12548	10.35014
	12	0.54036	−0.25899	639.77737	25.29382

Lemma 2.3. *If a series $x_t^{(0)}$ follows an ARMA$(1,1)$ model of*

$$(1-\phi B)x_t^{(0)} = (1-\theta B)a_t$$

for nonzero ϕ and θ ($\phi \neq \theta$), then its mth order aggregate series $X_T^{(0)}$ follows an ARMA$(1,1)$ model of

$$(1-\Phi\mathcal{B})X_T^{(0)} = (1-\Theta\mathcal{B})A_T$$

for $m > 1$ and $m \in \mathbb{Z}$, where a_t and A_T are Gaussian white noises of mean zero and variance σ_a^2 and σ_A^2, respectively. The parameters of the aggregate ARMA$(1,1)$ model are

$$\Phi = \phi^m \quad \textit{and} \quad \Theta = -\frac{1}{2\gamma} \pm \sqrt{\left(\frac{1}{2\gamma}+1\right)\left(\frac{1}{2\gamma}-1\right)}, \tag{2.31}$$

where

$$\gamma = \frac{\sum_{j=0}^{m-1}\left(\frac{1}{\phi-\theta}+\sum_{i=1}^{j}\phi^{i-1}\right)\left(\sum_{i=j+1}^{m}\phi^{i-1}-\frac{\phi^m}{\phi-\theta}\right)}{\sum_{j=0}^{m-1}\left[\left(\frac{1}{\phi-\theta}+\sum_{i=1}^{j}\phi^{i-1}\right)^2+\left(\sum_{i=j+1}^{m}\phi^{i-1}-\frac{\phi^m}{\phi-\theta}\right)^2\right]}. \tag{2.32}$$

Also, the variance σ_A^2 is written as $\sigma_A^2 = (\rho\sigma_a)^2$ where

$$\rho^2 = \frac{(\phi-\theta)^2}{1+\Theta^2}\sum_{j=0}^{m-1}\left[\left(\frac{1}{\phi-\theta}+\sum_{i=1}^{j}\phi^{i-1}\right)^2+\left(\sum_{i=j+1}^{m}\phi^{i-1}-\frac{\phi^m}{\phi-\theta}\right)^2\right], \tag{2.33a}$$

or, equivalently,

$$\rho^2 = -\frac{(\phi-\theta)^2}{\Theta}\sum_{j=0}^{m-1}\left(\frac{1}{\phi-\theta}+\sum_{i=1}^{j}\phi^{i-1}\right)\left(\sum_{i=j+1}^{m}\phi^{i-1}-\frac{\phi^m}{\phi-\theta}\right). \tag{2.33b}$$

Proof. We note that $p = P = 1$ from Eq. (2.6) and $Q = \lfloor 2 - \frac{1}{m} \rfloor = 1$ from Eq. (2.7), for $m > 1$ and $m \in \mathbb{Z}$. Then the AR expression in Eq. (2.15) becomes

$$Z_{mT} = (1-\phi^m\mathcal{B})X_T^{(0)}, \tag{2.34}$$

which indicates that the AR parameter in Eq. (2.18) is now $\Phi = \phi^m$. From the MA expression in Eq. (2.16), we have

$$\begin{aligned} Z_{mT} &= (1-\theta B)\left(1+\sum_{i=1}^{m-1}\phi^i B^i\right)\left(1+\sum_{k=1}^{m-1}B^k\right)a_{mT} \\ &= \left(\sum_{j=0}^{2m-1}\psi_j B^j\right)a_{mT}, \end{aligned} \tag{2.35}$$

where

$$\psi_j = \begin{cases} 1+(\phi-\theta)\left(\sum_{i=1}^{j}\phi^{i-1}\right), & \text{for } j=0,\ldots,m-1, \\ (\phi-\theta)\left(\sum_{i=j-m+1}^{m}\phi^{i-1}\right)-\phi^m, & \text{for } j=m,\ldots,2m-1, \end{cases} \tag{2.36}$$

assuming $\sum_{i=j_1}^{j_2}\phi^i = 0$ if $j_1 > j_2$. So, Eq. (2.9) becomes a quadratic equation

$$\Theta^2 + \gamma^{-1}\Theta + 1 = 0, \tag{2.37}$$

where

$$\begin{aligned} \gamma &= \frac{\sum_{j=0}^{m-1}\psi_j\psi_{m+j}}{\sum_{j=0}^{2m-1}\psi_j^2} \\ &= \frac{\sum_{j=0}^{m-1}\psi_j\psi_{m+j}}{\sum_{j=0}^{m-1}\left(\psi_j^2+\psi_{m+j}^2\right)} \\ &= \frac{(\phi-\theta)^2\sum_{j=0}^{m-1}\left(\frac{1}{\phi-\theta}+\sum_{i=1}^{j}\phi^{i-1}\right)\left(\sum_{i=j+1}^{m}\phi^{i-1}-\frac{\phi^m}{\phi-\theta}\right)}{(\phi-\theta)^2\sum_{j=0}^{m-1}\left[\left(\frac{1}{\phi-\theta}+\sum_{i=1}^{j}\phi^{i-1}\right)^2+\left(\sum_{i=j+1}^{m}\phi^{i-1}-\frac{\phi^m}{\phi-\theta}\right)^2\right]} \end{aligned}$$

and $\phi \neq \theta$. Here the real solutions of Θ are

$$\Theta = -\frac{1}{2\gamma} \pm \sqrt{\left(\frac{1}{2\gamma}+1\right)\left(\frac{1}{2\gamma}-1\right)}.$$

Furthermore, from Eq. (2.12a), we have the variance of white noise A_T as

$$\sigma_A^2 = \frac{(\phi-\theta)^2}{1+\Theta^2}\sum_{j=0}^{m-1}\left[\left(\frac{1}{\phi-\theta}+\sum_{i=1}^{j}\phi^{i-1}\right)^2+\left(\sum_{i=j+1}^{m}\phi^{i-1}-\frac{\phi^m}{\phi-\theta}\right)^2\right]\sigma_a^2,$$

or, equivalently, from Eq. (2.12b),

$$\sigma_A^2 = -\frac{(\phi-\theta)^2}{\Theta}\sum_{j=0}^{m-1}\left(\frac{1}{\phi-\theta}+\sum_{i=1}^{j}\phi^{i-1}\right)\left(\sum_{i=j+1}^{m}\phi^{i-1}-\frac{\phi^m}{\phi-\theta}\right)\sigma_a^2.$$

□

Table 2.2 displays the numerical computations for aggregate ARMA(1, 1) model parameters Φ and Θ, and variance σ_A^2, when its nonaggregate model is an ARMA(1, 1), $(1-\phi B)x_t^{(0)} = (1-\theta B)a_t$, for every choice of parameters ϕ, $\theta \in \{-0.5, -0.3, 0.5, 0.8\}$ $(\phi \neq \theta)$ and aggregation order $m = 1, 3, 6, 12$, assuming $\sigma_a^2 = 1$.

In Lemmas 2.2 and 2.3, the aggregate model of $X_T^{(0)}$ is stationary and invertible if the nonaggregate model of $x_t^{(0)}$ is stationary. This follows because $0 < |\Phi| = |\phi^m| < 1$ for $m > 1$ and $m \in \mathbb{Z}$, and the MA parameter Θ is chosen to be

$$\Theta = \begin{cases} -\frac{1}{2\gamma} + \sqrt{\left(\frac{1}{2\gamma}+1\right)\left(\frac{1}{2\gamma}-1\right)} & \text{for } 0 < \gamma < \frac{1}{2}, \\ -\frac{1}{2\gamma} - \sqrt{\left(\frac{1}{2\gamma}+1\right)\left(\frac{1}{2\gamma}-1\right)} & \text{for } -\frac{1}{2} < \gamma < 0, \end{cases} \tag{2.38}$$

and so $|\Theta| < 1$.

Table 2.2: Model parameters and error variance of an aggregate ARMA$(1,1)$ process for given nonaggregate model's parameters ϕ and θ, and aggregation order m

ϕ	θ	m	Φ	Θ	σ_{A^2}	ϕ	θ	m	Φ	Θ	σ_{A^2}
−0.5	−0.3	1	−0.50000	−0.30000	1.00000	0.5	−0.5	1	0.50000	−0.50000	1.00000
		3	−0.12500	−0.07378	2.47341			3	0.12500	−0.23529	13.54688
		6	0.01562	0.03639	4.70297			6	0.01562	−0.14614	39.83235
		12	0.00024	0.01117	9.21367			12	0.00024	−0.07049	94.19858
−0.5	0.5	1	−0.50000	0.50000	1.00000	0.5	−0.3	1	0.50000	−0.30000	1.00000
		3	−0.12500	0.46553	1.47683			3	0.12500	−0.22000	10.43198
		6	0.01562	0.40776	1.84173			6	0.01562	−0.14166	30.15381
		12	0.00024	0.28460	2.60391			12	0.00024	−0.06886	70.97017
−0.5	0.8	1	−0.50000	0.80000	1.00000	0.5	0.8	1	0.50000	0.80000	1.00000
		3	−0.12500	0.77703	1.35773			3	0.12500	0.55109	1.82365
		6	0.01562	0.73481	1.46967			6	0.01562	0.38996	2.49964
		12	0.00024	0.64336	1.67646			12	0.00024	0.26810	3.58250
−0.3	−0.5	1	−0.30000	−0.50000	1.00000	0.8	−0.5	1	0.80000	−0.50000	1.00000
		3	−0.02700	−0.05801	3.76341			3	0.51200	−0.27920	24.55879
		6	0.00073	−0.01348	7.76568			6	0.26214	−0.24761	118.04818
		12	0.00000	−0.00702	15.75448			12	0.06872	−0.19308	411.27214
−0.3	0.5	1	−0.30000	0.50000	1.00000	0.8	−0.3	1	0.80000	−0.30000	1.00000
		3	−0.02700	0.42041	1.39340			3	0.51200	−0.26470	18.87176
		6	0.00073	0.31607	1.89473			6	0.26214	−0.24398	89.18561
		12	0.00000	0.21021	2.84583			12	0.06872	−0.19208	309.42848
−0.3	0.8	1	−0.30000	0.80000	1.00000	0.8	0.5	1	0.80000	0.50000	1.00000
		3	−0.02700	0.74827	1.18187			3	0.51200	0.07258	5.19145
		6	0.00073	0.67219	1.31961			6	0.26214	−0.10909	16.59745
		12	0.00000	0.57201	1.55054			12	0.06872	−0.14752	49.39727

CHAPTER 3

TEMPORAL AGGREGATION EFFECTS ON TESTING FOR A MEAN CHANGE

In this chapter, we show the effects of temporal aggregation on the LR test for a mean change and propose a modified LR test when an AR(p) series is temporally aggregated.

3.1 Aggregation Effects on the LR Test

From Eq. (2.6) and (2.7), if a series $x_t^{(0)}$ follows an AR(p) model,

$$\left(1 - \sum_{i=1}^{p} \phi_i B^i\right) x_t^{(0)} = a_t, \tag{3.1}$$

its mth order aggregate series $X_T^{(0)}$ follows an ARMA(P, Q) model,

$$\left(1 - \sum_{i=1}^{P} \Phi_i \mathcal{B}^i\right) X_T^{(0)} = \left(1 - \sum_{j=1}^{Q} \Theta_j \mathcal{B}^j\right) A_T, \tag{3.2}$$

where $a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$ and $A_T \overset{i.i.d.}{\sim} N(0, \sigma_A^2)$. The model parameter Φ's and Θ's are decided by the rules in Theorem 2.1.

Let K be the change point of the aggregate discordant series X_T in Eq. (2.1), for $1 < K \leq N$ and $K \in \mathbb{Z}$. Then, similarly to Eq. (1.3), X_T is written as

$$X_t = X_T^{(0)} + \frac{\mathcal{W}_K I_T(K)}{(1 - \mathcal{B})}, \tag{3.3}$$

where $\mathcal{W}_K$ is a change-magnitude and $I_T(K) = 1$ for $T = K$ or 0 for $T \neq K$. We note that, as discussed in Chapter 2, the stationary series $X_T^{(0)}$ satisfies invertibility.

Similarly to Eq. (1.5) and (1.6), we consider a contaminated error

$$E_T = \Pi(\mathcal{B})X_T = \mathcal{W}_K\mathcal{Y}_T + A_T, \tag{3.4}$$

where

$$\Pi(\mathcal{B}) = (1 - \Pi_1\mathcal{B} - \Pi_2\mathcal{B}^2 - \cdots) = \frac{1 - \sum_{i=1}^{P}\Phi_i\mathcal{B}^i}{1 - \sum_{j=1}^{Q}\Theta_j\mathcal{B}^j} \tag{3.5}$$

and

$$\mathcal{Y}_T = \left[\frac{\Pi(\mathcal{B})}{(1-\mathcal{B})}\right] I_T(K) = \begin{cases} 0, & \text{for } t < k, \\ 1, & \text{for } t = k, \\ 1 - \sum_{j=1}^{t-k}\Pi_j, & \text{for } t > k. \end{cases} \tag{3.6}$$

The OLS estimator of $\mathcal{W}_K$ is

$$\widehat{\mathcal{W}}_K = \frac{\sum_{T=K}^{N} E_T\mathcal{Y}_T}{\sum_{T=K}^{N}\mathcal{Y}_T^2} \tag{3.7}$$

and its variance

$$\sigma^2_{\widehat{\mathcal{W}}_K} = \frac{\sigma_A^2}{\sum_{T=K}^{N}\mathcal{Y}_T^2}. \tag{3.8}$$

In the same manner as Eq. (1.10) and (1.13), the LR statistic to test for a mean change starting at an unknown K is

$$\sup_{K=2,\ldots,N} |\Lambda_K|, \tag{3.9}$$

with

$$\Lambda_K = \frac{\widehat{\mathcal{W}}_K}{\sigma_{\widehat{\mathcal{W}}_K}} = \frac{\sum_{T=K}^{N} E_T \mathcal{Y}_T}{\sigma_A \sqrt{\sum_{T=K}^{N} \mathcal{Y}_T^2}}$$

$$= \frac{E_K + \sum_{T=K+1}^{N} E_T \left(1 - \sum_{i=1}^{T-K} \Pi_i\right)}{\sigma_A \sqrt{1 + \sum_{T=K+1}^{N} \left(1 - \sum_{i=1}^{T-K} \Pi_i\right)^2}}. \tag{3.10}$$

We remark that Λ_K in (3.10) is associated with AR parameter Φ's and MA parameter Θ's.

In Theorem 3.1, we clarify the association and propose the modified LR test statistic when the aggregate series is used.

Theorem 3.1. *Assume that a stationary series $x_t^{(0)}$ follows an AR(p) model of*

$$\left(1 - \sum_{i=1}^{p} \phi_i B^i\right) x_t^{(0)} = a_t$$

and so its mth order aggregate series $X_T^{(0)}$ follows an ARMA(P, Q) model of

$$\left(1 - \sum_{i=1}^{P} \Phi_i \mathcal{B}^i\right) X_T^{(0)} = \left(1 - \sum_{j=1}^{Q} \Theta_j \mathcal{B}^j\right) A_T,$$

where $a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$ and $A_T \overset{i.i.d.}{\sim} N(0, \sigma_A^2)$. Then, the LR test statistic to test for a mean change in the aggregate discordant series X_T is given by $\sup_{K=2,\ldots,N} |\Lambda_K|$ with

$$\Lambda_K = \frac{E_K + \sum_{T=K+1}^{N} E_T \left(1 - \sum_{i=1}^{T-K} \Phi_i\right) + G(\Omega, \boldsymbol{E})}{\sigma_A \sqrt{1 + \sum_{T=K+1}^{N} \left(1 - \sum_{i=1}^{T-K} \Phi_i\right)^2 + F(\Omega)}}, \tag{3.11}$$

where $F(\Omega)$ is a function of $\Omega = (\Phi_1, \ldots, \Phi_P, \Theta_1, \ldots, \Theta_Q)$, and $G(\Omega, \boldsymbol{E})$ is a function of Ω and $\boldsymbol{E} = (E_{K+1}, \ldots, E_N)$.

Proof. When multiplying both sides of Eq. (3.5) by the polynomial $(1 - \sum_{j=1}^{Q} \Theta_j \mathcal{B}^j)$

and collecting like terms, the parameter Π's are sequentially associated with Φ's and Θ's as follows,

$$\Pi_i = \begin{cases} \Phi_i - \Theta_i + \sum_{j=1}^{i-1} \Pi_j \Theta_{i-j} & \text{if } 1 \leq i \leq Q, \\ \Phi_i + \sum_{j=1}^{Q} \Pi_{i-Q+j-1} \Theta_{Q-j+1} & \text{if } Q+1 \leq i \leq P, \\ \sum_{j=1}^{Q} \Pi_{i-Q+j-1} \Theta_{Q-j+1} & \text{if } i \geq P+1, \end{cases} \tag{3.12}$$

where

$$Q = \left\lfloor P + 1 - \frac{(p+1)}{m} \right\rfloor \leq P. \tag{3.13}$$

Then

$$\begin{aligned} \sum_{T=K}^{N} \mathcal{Y}_T^2 &= 1 + \sum_{T=K+1}^{N} \left(1 - \sum_{i=1}^{T-K} \Pi_i\right)^2 \\ &= 1 + \sum_{T=K+1}^{N} \left(1 - \sum_{i=1}^{T-K} \Phi_i\right)^2 + F(\Omega), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \sum_{T=K}^{N} E_T \mathcal{Y}_T &= E_K + \sum_{T=K+1}^{N} E_T \left(1 - \sum_{i=1}^{T-K} \Pi_i\right) \\ &= E_K + \sum_{T=K+1}^{N} E_T \left(1 - \sum_{i=1}^{T-K} \Phi_i\right) + G(\Omega, \boldsymbol{E}), \end{aligned} \tag{3.15}$$

where $F(\Omega)$ is a function of $\Omega = (\Phi_1, \ldots, \Phi_P, \Theta_1, \ldots, \Theta_Q)$, and $G(\Omega, \boldsymbol{E})$ is a function of Ω and $\boldsymbol{E} = (E_{K+1}, \ldots, E_N)$. We note that $\Phi_i = 0$ for $P < i \leq N - K$. When plugging Eq. (3.14) and (3.15) into Eq. (3.10), we obtain the expression

$$\Lambda_K = \frac{E_K + \sum_{T=K+1}^{N} E_T \left(1 - \sum_{i=1}^{T-K} \Phi_i\right) + G(\Omega, \boldsymbol{E})}{\sigma_A \sqrt{1 + \sum_{T=K+1}^{N} \left(1 - \sum_{i=1}^{T-K} \Phi_i\right)^2 + F(\Omega)}}.$$

□

We note that Λ_K in (3.11) is a function of the nonaggregate parameters $\phi_1, \cdots, \phi_p$ and σ_a^2 because of their functional relationship with Φ's, Θ's, and σ_A^2 as shown in Theorem 2.1.

Comparing the two expressions of λ_k in Eq. (1.10) and Λ_K in Eq. (3.11), Λ_K includes three additional parameters—$F(\Omega)$, $G(\Omega, \boldsymbol{E})$, and ρ, where $\rho = \sigma_A/\sigma_a$ given in either Eq. (2.12a) or (2.12b). Therefore, we cannot expect that the null distribution of $\sup_{K=2,\ldots,N} |\Lambda_K|$ is identical to the null distribution of $\sup_{k=1,\ldots,n} |\lambda_k|$ when $m > 1$. However, Λ_K reduces to λ_k when $m = 1$ with $F(\Omega) = G(\Omega, \boldsymbol{E}) = 0$ and $\rho = 1$. We demonstrate the location and scale changes of the null distribution through the Monte Carlo studies in Section 3.2.

In Lemma 3.2, for the later illustration and analysis, we derive the LR test statistic for the aggregate model when the nonaggregate series follows an AR(1) model.

Lemma 3.2. *Assume that a stationary series $x_t^{(0)}$ follows an AR(1) model of*

$$(1 - \phi B)x_t^{(0)} = a_t$$

and so it mth order aggregate series $X_T^{(0)}$ follows an ARMA(1,1) model of

$$(1 - \Phi\mathcal{B})X_T^{(0)} = (1 - \Theta\mathcal{B})A_T,$$

where $a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$ and $A_t \overset{i.i.d.}{\sim} N(0, \sigma_A^2)$. Then, the LR test statistic to test for a mean change in the aggregate discordant series X_T is given by $\sup_{K=2,\ldots,N} |\Lambda_K|$ with

$$\Lambda_K = \frac{E_K + (1 - \Phi)\sum_{T=K+1}^{N} E_T + G(\Phi, \Theta, \boldsymbol{E})}{\sigma_A\sqrt{1 + (N - K)(1 - \Phi)^2 + F(\Phi, \Theta)}}, \tag{3.16}$$

where $F(\Phi, \Theta)$ is a function of Φ and Θ, and $G(\Phi, \Theta, \boldsymbol{E})$ is a function of Φ, Θ, and

$\boldsymbol{E} = (E_{K+1}, \ldots, E_N)$, *i.e.*,

$$F(\Phi, \Theta) = 2(1-\Phi) \sum_{T=K+1}^{N} \left[(1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K} \right] + \sum_{T=K+1}^{N} \left[(1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K} \right]^2 \tag{3.17}$$

and

$$G(\Phi, \Theta, \boldsymbol{E}) = E_{K+1}\Theta + \sum_{T=K+2}^{N} E_T \left[(1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K} \right]. \tag{3.18}$$

We note that Φ *and* Θ *are given in Lemma 2.2.*

Proof. When $P = Q = 1$ in Eq. (3.5), the parameter Π's are expressed as

$$\Pi_i = \Theta^{i-1}(\Phi - \Theta), \quad i = 1, 2, \ldots, \tag{3.19}$$

where Φ and Θ are given in (2.24). Now we have

$$\begin{aligned} 1 - \sum_{i=1}^{T-K} \Pi_i &= 1 - \sum_{j=1}^{T-K} \Theta^{j-1}(\Phi - \Theta) \\ &= (1-\Phi) + (1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K}. \end{aligned} \tag{3.20}$$

Then, Eq. (3.14) and (3.15) become

$$\begin{aligned} \sum_{T=K}^{N} \mathcal{Y}_T^2 &= 1 + \sum_{T=K+1}^{N} \left(1 - \sum_{i=1}^{T-K} \Pi_i \right)^2 \\ &= 1 + \sum_{T=K+1}^{N} \left[(1-\Phi) + (1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K} \right]^2 \\ &= 1 + (N-K)(1-\Phi)^2 + F(\Phi, \Theta) \end{aligned} \tag{3.21}$$

and

$$
\begin{aligned}
\sum_{T=K}^{N} E_T \mathcal{Y}_T &= E_K + \sum_{T=K+1}^{N} E_T \left(1 - \sum_{i=1}^{T-K} \Pi_i\right) \\
&= E_K + \sum_{T=K+1}^{N} E_T \left[(1-\Phi) + (1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K}\right] \\
&= E_K + (1-\Phi) \sum_{T=K+1}^{N} E_T + G(\Phi, \Theta, \boldsymbol{E}),
\end{aligned} \tag{3.22}
$$

where

$$
\begin{aligned}
F(\Phi, \Theta) = 2(1-\Phi) &\sum_{T=K+1}^{N} \left[(1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K}\right] \\
&+ \sum_{T=K+1}^{N} \left[(1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K}\right]^2
\end{aligned}
$$

and

$$
G(\Phi, \Theta, \boldsymbol{E}) = E_{K+1}\Theta + \sum_{T=K+2}^{N} E_T \left[(1-\Phi) \sum_{j=1}^{T-K-1} \Theta^j + \Theta^{T-K}\right].
$$

Therefore, Λ_K of Eq. (3.11) is rewritten as

$$
\Lambda_K = \frac{E_K + (1-\Phi) \sum_{T=K+1}^{N} E_T + G(\Phi, \Theta, \boldsymbol{E})}{\sigma_A \sqrt{1 + (N-K)(1-\Phi)^2 + F(\Phi, \Theta)}}.
$$

□

3.2 Simulation Studies of the Aggregation Effects

In this section, we obtain percentiles of the empirical null distributions of the LR test through the Monte Carlo simulations and investigate the power of the LR test for nonaggregate series given in Section 1.2 and that for aggregate series given in Section 3.1.

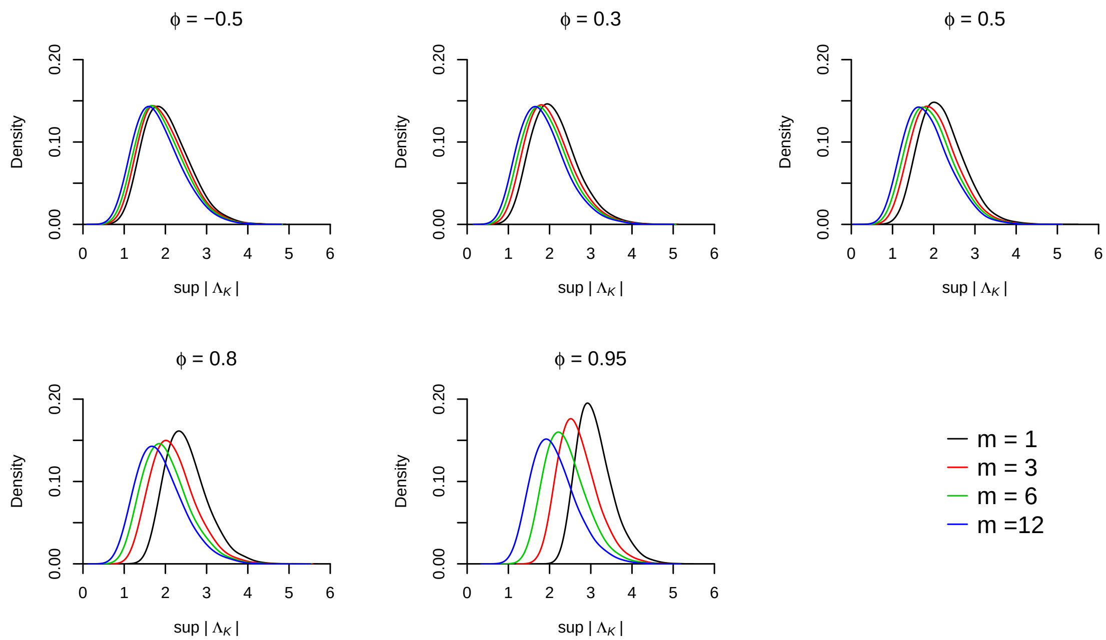

Figure 3.1: Empirical null distributions for the LR test

3.2.1 Empirical Null Distributions

To demonstrate the empirical properties, we consider the cases in which the nonaggregate stationary series $x_t^{(0)}$ follows an AR(1) model, $(1-\phi B)x_t^{(0)} = a_t$ with $\phi = -0.5$, 0.3, 0.5, 0.8, and 0.95, assuming $a_t \overset{iid}{\sim} N(0, 1)$. So, the aggregate stationary series $X_T^{(0)}$ becomes an $\text{ARMA}(1, 1)$ model as shown in Lemma 2.2. Under the null hypothesis of no mean change, we generate 10,000 different series of length $n = 1200$ for every ϕ. Also, we consider the mth order temporal aggregation of the simulated series for $m = 3$, 6, and 12.

For the LR test, all the model parameters and the error variance are assumed to be known. We compute the original test statistic $\sup_{k=2,\ldots,n} |\lambda_k|$ using λ_k in Eq. (1.12) for the nonaggregate and the modified test statistic $\sup_{K=2,\ldots,N} |\Lambda_K|$ using Λ_K in Eq. (3.16) for the aggregate series, where $N = n/m$. Through searching the supremum in every series, we obtain 10,000 suprema for the given ϕ and m. Then, we construct the distribution of the 10,000 values as the empirical null distribution.

Table 3.1: Percentiles of the empirical null distribution for the LR test

ϕ	m	N	25%	50%	75%	90%	95%	99%
−0.50	1	1200	1.634	1.980	2.389	2.790	3.045	3.569
	3	400	1.550	1.893	2.304	2.712	2.973	3.490
	6	200	1.491	1.825	2.246	2.661	2.920	3.463
	12	100	1.413	1.750	2.171	2.608	2.861	3.385
0.30	1	1200	1.730	2.066	2.453	2.866	3.128	3.683
	3	400	1.601	1.938	2.340	2.755	3.025	3.601
	6	200	1.521	1.860	2.264	2.687	2.977	3.544
	12	100	1.433	1.774	2.180	2.617	2.904	3.483
0.50	1	1200	1.813	2.148	2.524	2.913	3.151	3.695
	3	400	1.632	1.975	2.367	2.770	3.016	3.548
	6	200	1.541	1.884	2.282	2.690	2.934	3.457
	12	100	1.444	1.787	2.190	2.619	2.863	3.402
0.80	1	1200	2.144	2.446	2.812	3.200	3.440	3.947
	3	400	1.809	2.132	2.516	2.929	3.174	3.734
	6	200	1.616	1.952	2.347	2.776	3.049	3.597
	12	100	1.470	1.807	2.223	2.657	2.926	3.510
0.95	1	1200	2.786	3.036	3.342	3.672	3.892	4.334
	3	400	2.353	2.625	2.968	3.325	3.544	4.013
	6	200	2.027	2.332	2.699	3.076	3.323	3.826
	12	100	1.725	2.049	2.435	2.846	3.109	3.615

The results are listed in Table 3.1 for all the combinations of ϕ and m. We note that the corresponding values to higher percentiles, for example, 90%, 95%, or 99%, are now the critical values L at significance level α of 0.10, 0.05, or 0.01, respectively. Here we suggest that these critical values are employed in the LR test in replacing the subjective choices used in the literature.

The null distributions, which are computed through the kernel density estimation using Gaussian kernels, are also drawn in Figure 3.1. Through the plots, we notice the null distribution move its location and change its scale, depending on its choice of the aggregation order m and the model parameter ϕ. In general, the null distribution moves to left as m increases and this leftward location shift gets intense as ϕ increases. Also, for given ϕ, the maximum density is lower and the width is larger as m increases.

To investigate this distribution shift, we consider the effects on

$$|\Lambda_K| = \left|\widehat{\mathcal{W}}_K\right| / \sigma_{\widehat{\mathcal{W}}_K} \tag{3.23}$$

in terms of the estimated change-magnitude,

$$\begin{aligned}\left|\widehat{\mathcal{W}}_K\right| &= \left|\frac{\sum_{T=K}^{N} E_T \mathcal{Y}_T}{\sum_{T=K}^{N} \mathcal{Y}_T^2}\right| \\ &= \left|\frac{E_K + (1-\Phi)\sum_{T=K+1}^{N} E_T + G(\Phi, \Theta, \boldsymbol{E})}{1 + (N-K)(1-\Phi)^2 + F(\Phi, \Theta)}\right|\end{aligned} \tag{3.24}$$

and its standard deviation,

$$\begin{aligned}\sigma_{\widehat{\mathcal{W}}_K} &= \frac{\sigma_A}{\sqrt{\sum_{T=K}^{N} \mathcal{Y}_T^2}} \\ &= \frac{\rho\sigma_a}{\sqrt{1 + (N-K)(1-\Phi)^2 + F(\Phi, \Theta)}},\end{aligned} \tag{3.25}$$

expressed in Eq. (3.11) and (3.16).

We note that $\left|\widehat{\mathcal{W}}_K\right|$ in Eq. (3.24) and $\sigma_{\widehat{\mathcal{W}}_K}$ in Eq. (3.25) are functions of K, for

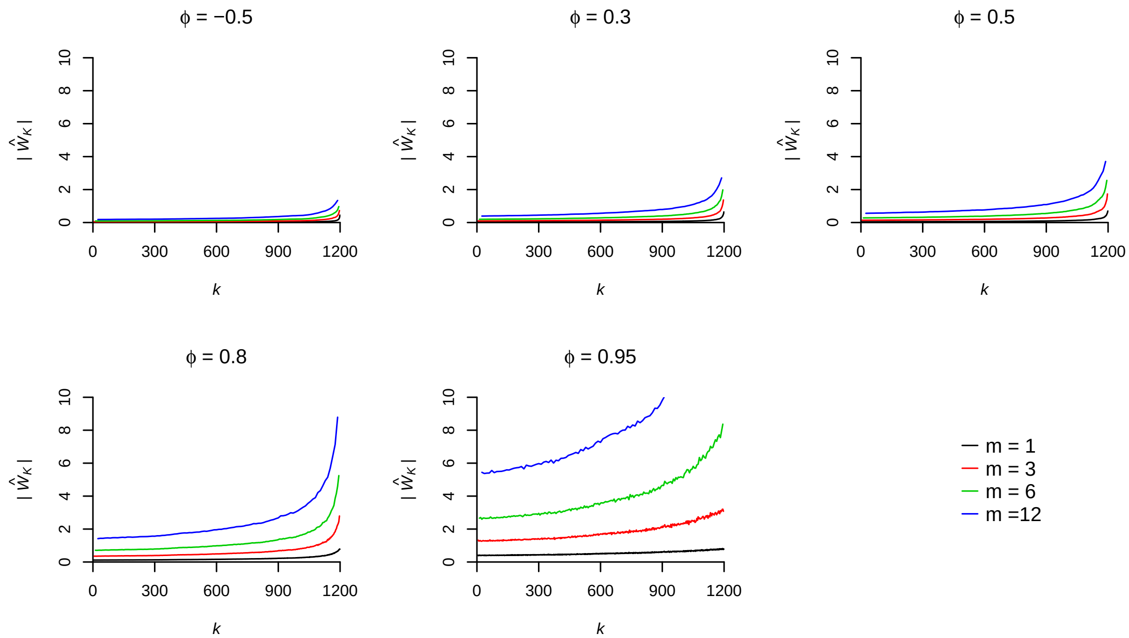

Figure 3.2: Values of $\left|\widehat{\mathcal{W}}_K\right|$ at the corresponding $k = K \times m$

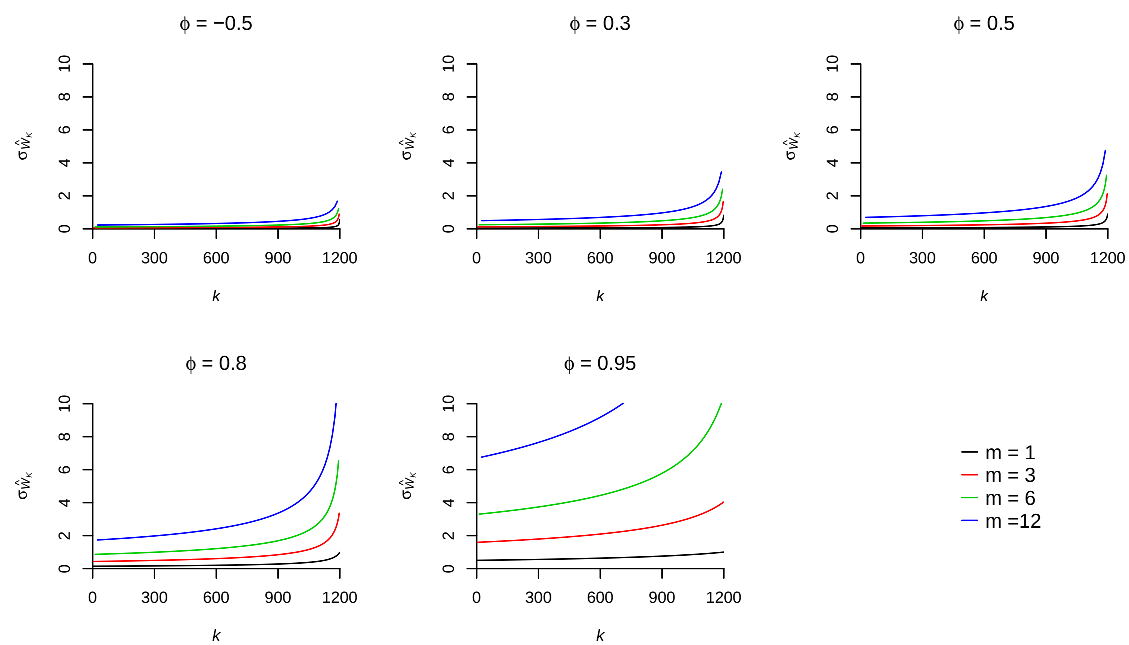

Figure 3.3: Values of $\sigma_{\widehat{\mathcal{W}}_K}$ at the corresponding $k = K \times m$

$1 < K \leq N$ and $K \in \mathbb{Z}$, when Φ, Θ, and $\sigma_A = \rho\sigma_a$ are given by ϕ and m in Lemmas 2.2 and 3.2. So, we calculate their numerical values for K in given ϕ and m, using the simulated series. We also plot the computed values of Eq. (3.24) and (3.25) on Figures 3.2 and 3.3, respectively. We remark that K is transformed into $k = K \times m$ for convenience of display.

Through all the plots of Figures 3.2 and 3.3, $\left|\widehat{\mathcal{W}}_K\right|$ and $\sigma_{\widehat{\mathcal{W}}_K}$ are increasing functions with a low growth rate for small K and a high growth rate for large K. Also, for given ϕ, the curves of $\left|\widehat{\mathcal{W}}_K\right|$ and $\sigma_{\widehat{\mathcal{W}}_K}$ for large m are higher than the curves for small m. It implies that the parameter changes caused by the temporal aggregation make the estimated change-magnitude and its standard deviation increase.

3.2.2 Test Powers

To examine the powers of the LR test, we generate other series. We consider the cases in which the nonaggregate $x_t^{(0)}$ follows an AR(1) process with model parameters $\phi = -0.5$, 0.3, 0.5, 0.8, and 0.95 and so the aggregate $X_T^{(0)}$ corresponds to an ARMA$(1,1)$ process, assuming $a_t \overset{iid}{\sim} N(0,1)$. For the alternative distribution, we assume that every series consists of 1200 observations and its mean shift starts at time point $k = 601$ with magnitude $\omega = 0.1$, 0.3, 0.5, 0.7, 1.0, 1.5, 2.0, 3.0, or 5.0. Then, we generate 10,000 different series under the change condition for selected ϕ and ω and aggregate each series for $m = 3$, 6, and 12.

Here we present the empirical powers of the LR test in Table 3.2, obtained from the simulations for 5% significance levels. In general, the test power increases as the change magnitude ω increases and so they become 100% in large ω, which are expected. Also, the power increases as the aggregation order m increases, for given ϕ. These results are consistent with what we discussed in Section 3.2.1. For the LR test, the leftward location shift of the null distribution indicates that the null hypothesis will be rejected more often when tested using aggregate series as compared to using

Table 3.2: Empirical powers of the LR test

ϕ	m	N	$\omega = 0.1$	0.3	0.5	0.7	1.0	1.5	2.0	3.0	5.0
−0.50	1	1200	0.857	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	3	400	0.871	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	6	200	0.881	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	12	100	0.898	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.30	1	1200	0.234	0.988	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	3	400	0.265	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	6	200	0.284	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	12	100	0.317	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.50	1	1200	0.145	0.856	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	3	400	0.182	0.888	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	6	200	0.215	0.907	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	12	100	0.243	0.917	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1	1200	0.063	0.185	0.445	0.783	0.983	1.000	1.000	1.000	1.000
	3	400	0.126	0.290	0.565	0.868	0.991	1.000	1.000	1.000	1.000
	6	200	0.167	0.359	0.625	0.894	0.992	1.000	1.000	1.000	1.000
	12	100	0.227	0.431	0.681	0.913	0.995	1.000	1.000	1.000	1.000
0.95	1	1200	0.036	0.047	0.076	0.075	0.163	0.334	0.510	0.926	1.000
	3	400	0.132	0.150	0.191	0.212	0.324	0.512	0.705	0.976	1.000
	6	200	0.243	0.263	0.335	0.352	0.463	0.645	0.810	0.987	1.000
	12	100	0.408	0.445	0.495	0.525	0.607	0.753	0.906	0.998	1.000

nonaggregate series.

Thus, from Table 3.1 of the null distributions or Table 3.2 of the test powers, we find that the information of the LR test for a mean change is strengthened through the mth order temporal aggregation ($m = 3$, 6, and 12).

3.3 Illustrative Examples

3.3.1 A Simulated Series

For better understanding of the results discussed earlier, let us consider a simulated example shown on Figure 3.4. The figure illustrates temporal aggregation $m = 1$, 3, 6, and 12 of an discordant AR(1) series with $n = 1200$, $\phi = 0.5$, $\omega = 0.3$, and $k = 601$. The horizontal blue line is the original mean level of series and the red line denotes the shift-magnitude starting at $K = \lceil k/m \rceil$. Through the pictures, we see that the temporal aggregation strengthens the shift magnitude as well as increases the error variance. This graphical result in Figure 3.4 is similar to the leftward shift phenomenon of the LR null distribution as the aggregation order m increases shown in Figure 3.1.

To examine more carefully, let us first consider the LR test of the nonaggregate series. Note that the AR(1) model parameter ϕ is assumed to be unknown. We estimate ϕ as shown in Section 1.2 and compute the test statistic $\sup_{k=2,\ldots,n} |\lambda_k|$ using λ_k in Eq. (1.12). Then, we obtain $\hat{\phi} = 0.5153$ (standard error: 0.0350) and the LR test statistics of 2.670 at $k = 603$ and 2.973 at $k = 603$ when the median absolute deviation (MAD) estimator and the 5% trimmed (TRM) estimator are employed as the error variance, respectively. Here we use the critical value $L = 3.15$ for the hypothesis testing at the significance level $\alpha = 0.05$. This is the critical value for the 95% percentile at $\phi = 0.5$ and $m = 1$ from Table 3.1 as well as within the reference range suggested by various researchers as discussed in Section 1.2. The conclusion

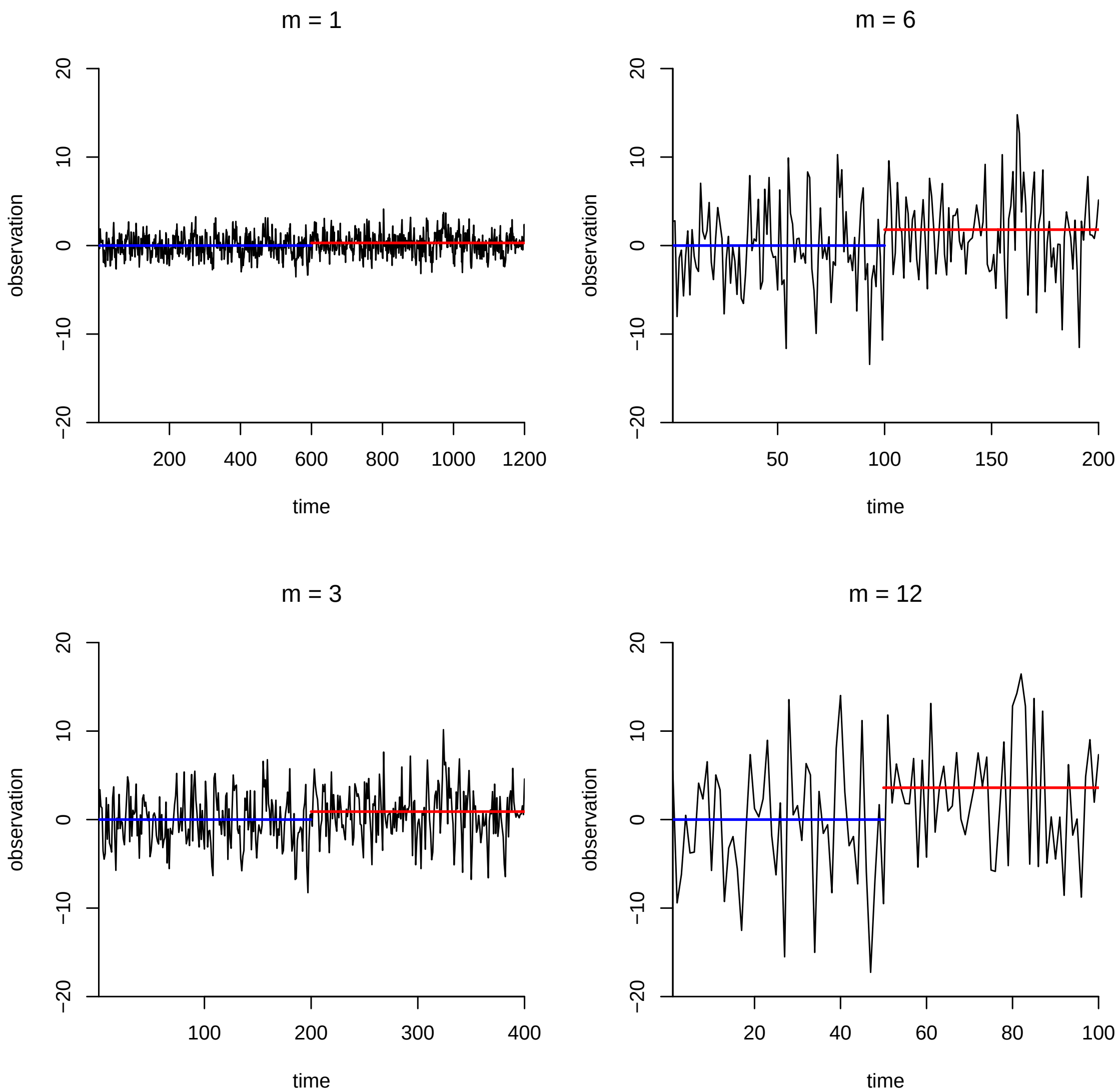

Figure 3.4: Temporal aggregation of an AR(1) series with $\phi = 0.5$ and $\omega = 0.3$

is that we do not reject the null hypothesis of no mean change and the test fails at $\alpha = 0.05$.

When keeping the AR(1) model and using the statistic λ_k in Eq. (1.12) and the critical value $L = 3.15$ at $\alpha = 0.05$, we conduct the LR tests of the mth order aggregate series. Then, the LR test statistics with the MAD method are 0.752 at $k = 198$, 1.156 at $k = 100$, and 1.798 at $k = 51$ for $m = 3$, 6, and 12, respectively. When the TRM method used, the LR test statistics are 0.7882 at $k = 198$, 1.354 at $k = 100$, and 2.106 at $k = 51$ for $m = 3$, 6, and 12, respectively. So, we cannot detect any significant mean shift either from the given aggregate series at $\alpha = 0.05$. The results contradict to the almost 100% test power shown in Table 3.2. That is, the tests fail because we do not consider the model transformation into an ARMA$(1, 1)$ process in spite of temporal aggregation.

Now for the aggregate series, let us consider the modified test statistic $\sup_{K=2,\ldots,N} |\Lambda_K|$ using Λ_K in (3.16). Then, we obtain the LR test statistics of 2.789 at $K = 202$, 3.142 at $K = 100$, and 2.644 at $K = 51$ for $m = 3$, 6, and 12, respectively, when the MAD method used. Also, the LR test statistics with the TRM method are 3.124 at $K = 202$, 3.191 at $K = 100$, and 3.097 at $K = 51$ for $m = 3$, 6, and 12, respectively. We note that these values are much higher than the ones when unmodified statistics in Eq. (1.12) were used.

For the hypothesis testing, we choose the critical values $L_{m=3} = 3.016$, $L_{m=6} = 2.934$, and $L_{m=12} = 2.863$ at $\alpha = 0.05$, which are the 95% percentiles of $\phi = 0.5$ for $m = 3$, 6, and 12 from Table 3.1. The tests indicate that the null hypothesis is rejected in most cases. This is especially true when the TRM method is used in estimating the error variance. These results are consistent with what were shown in the graphics of Figures 3.1 and 3.4.

3.3.2 Fish Recruitment

For a further illustration, we investigate a monthly time series of fish recruitment (number of fish) in the central Pacific Ocean between January 1950 and December 1986, taken from Shumway and Stoffer (2011). Figure 3.5 displays the monthly series ($n = 444, m = 1$) and its quarterly ($N = 148, m = 3$), semi-annually ($N = 74, m = 6$) and annually ($N = 37, m = 12$) aggregated series. Fish recruitment, defined as the number of new fishes in a population, is an important concept to explain population dynamics of fisheries.

The nonaggregate series is known to be fitted into an AR(2) model,

$$(1 - \phi_1 B - \phi_2 B^2)(x_t - \mu) = a_t \tag{3.26}$$

(Shumway and Stoffer, 2011, pp.110–111). From the estimation method presented in Section 1.2, we obtain the maximum-likelihood estimates $\hat{\phi}_1 = 1.3400$ (0.0422), $\hat{\phi}_2 = -0.4502$ (0.0423), and $\hat{\mu} = 62.5352$ (4.0488), where values in parentheses are standard errors of the estimates. Using Theorem 2.1, we can find an ARMA(2, 2) model for its mth temporal aggregation,

$$(1 - \Phi_1 \mathcal{B} - \Phi_2 \mathcal{B}^2)(X_T - \mathcal{U}) = (1 - \Theta_1 \mathcal{B} - \Theta_2 \mathcal{B}^2) A_T \tag{3.27}$$

where the parameter estimates $\hat{\Phi}_1 = 0.5963$, $\hat{\Phi}_2 = -0.0912$, $\hat{\Theta}_1 = -0.2814$, $\hat{\Theta}_2 = -0.0041$, and $\hat{\mathcal{U}} = 187.6056$ for $m = 3$, $\hat{\Phi}_1 = 0.1732$, $\hat{\Phi}_2 = -0.0083$, $\hat{\Theta}_1 = -0.7150$, $\hat{\Theta}_2 = -0.2901$, and $\hat{\mathcal{U}} = 375.2112$ for $m = 6$, and $\hat{\Phi}_1 = 0.0133$, $\hat{\Phi}_2 = -0.0001$, $\hat{\Theta}_1 = 0.0762$, $\hat{\Theta}_2 = 0.0861$, and $\hat{\mathcal{U}} = 750.4224$ for $m = 12$.

Similarly to Section 3.2.1, we generate 10,000 different series of size $n = 444$ for the nonaggregate AR(2) model under the null hypothesis of no mean change, and we compute the mth order temporal aggregation of the simulated series for $m = 3$, 6,

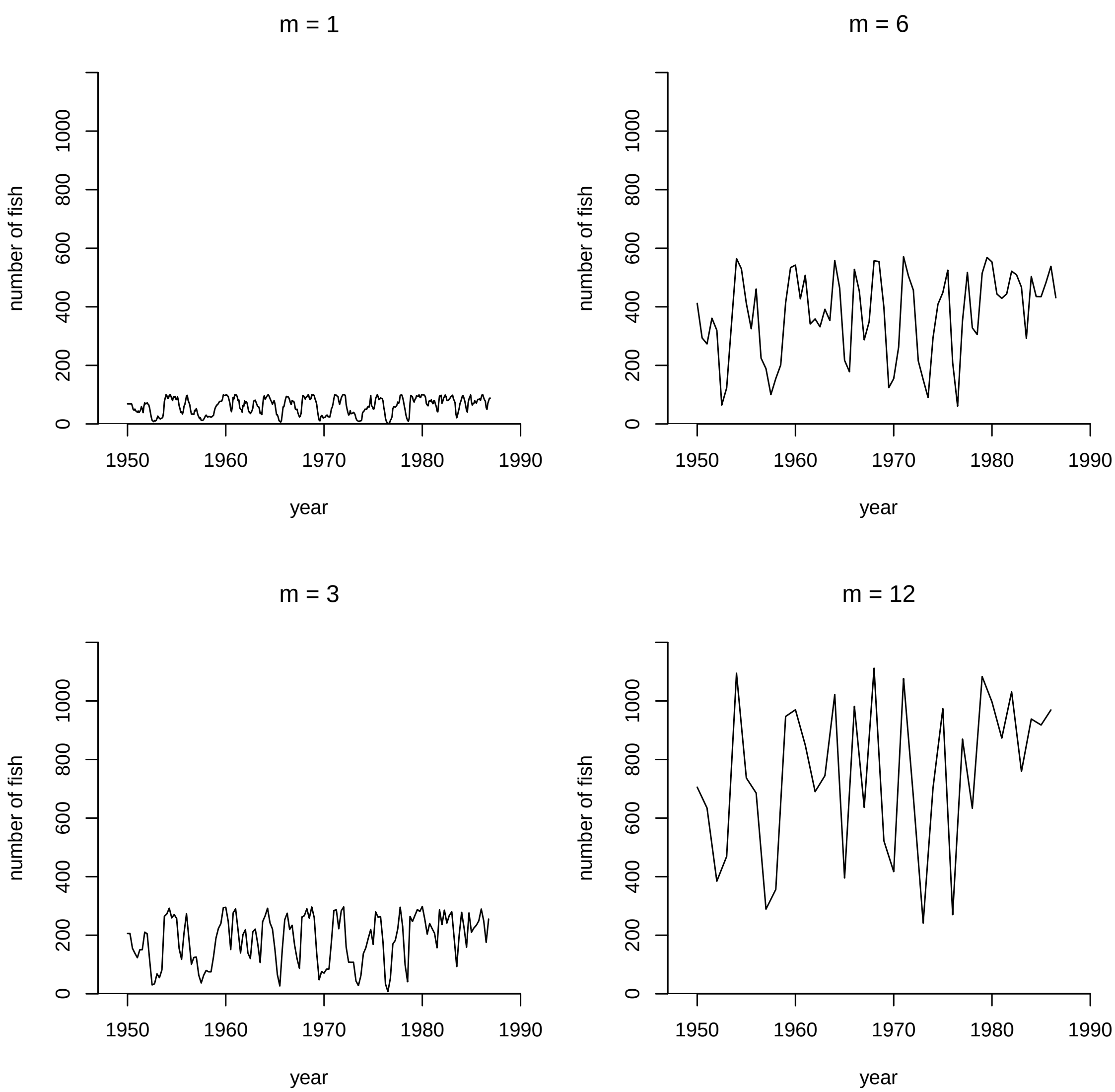

Figure 3.5: Fish recruitment in the central Pacific Ocean between 1950 and 1986

Table 3.3: Critical values when the nonaggregate model follows an AR(2) process

ϕ_1	ϕ_2	m	N	$\alpha = 0.10$	0.05	0.01
1.340	−0.450	1	444	2.643	2.888	3.321
		3	148	2.533	2.769	3.190
		6	74	2.430	2.654	3.063
		12	37	2.268	2.463	2.843

and 12. Then, we construct their empirical null distributions and find the 90%, 95%, and 99% percentiles which are now the critical values at $\alpha = 0.10$, 0.05, and 0.01, respectively. The critical values are given in Table 3.3.

When we retain the AR(2) model and apply the LR test shown in Eq. (1.10) and (1.13), the LR test statistics with the MAD method are 4.787 at $k = 346$, 2.182 at $k = 116$, 1.560 at $k = 69$, and 0.600 at $k = 28$ for $m = 1$, 3, 6, and 12, respectively. Through the TRM method, we obtain the LR test statistics of 4.830 at $k = 346$, 2.400 at $k = 116$, 1.608 at $k = 69$, and 0.566 at $k = 28$ for $m = 1$, 3, 6, and 12, respectively. For the critical value $L = 2.888$ at $\alpha = 0.05$, we reject the null hypothesis of no mean change in the monthly series and detect a significant mean change with the estimated change-magnitude $\hat{\omega}_{k=346} = 25.397$, starting at October, 1978. However, the unmodified tests for the aggregate series all fail to reject the null hypothesis of no mean change.

Now we consider the model transformation into ARMA$(2, 2)$ and employ the modified test shown in Eq. (3.9) and (3.10). Then, we obtain the LR test statistics of 2.903 at $K = 116$, 2.021 at $K = 69$, and 1.928 at $K = 30$ for $m = 3$, 6, and 12, respectively, through the MAD method. Also, the LR test statistics with the TRM method are 3.518 at $K = 116$, 2.041 at $K = 69$, and 2.441 at $K = 30$ for $m = 3$, 6, and 12, respectively. For the critical values $L_{m=3} = 2.769$, $L_{m=6} = 2.654$, and $L_{m=12} = 2.463$ at $\alpha = 0.05$, we reject the null hypothesis of no mean change in the quarterly series through either method and detect a mean change with the estimated shift-magnitude $\widehat{\mathcal{W}}_{K=116} = 72.554$, starting at the fourth quarter, 1978. Even though the tests for other cases fail, the differences between the test statistics and the critical values in the modified test are less than the differences in the unmodified test.

3.4 Concluding Remarks

In this Chapter, we analyze the temporal aggregation effects on a mean change of a time series. We have derived a modified LR test statistic when aggregate data are used. We show that the temporal aggregation leads the null distribution of the LR test statistic shifted to the left. In accordance with the distribution change, the test power increases as the aggregation order m increases. While aggregation is generally believed to cause information loss, our results show that the mth order temporal aggregation actually strengthens the LR test for a mean change, in terms of its empirical power.

Our underlying assumption is that the innovation distribution is Gaussian. Chen and Liu (1993) also investigate the performance of the test procedure under a certain non-Gaussian innovation, such as the exponential distribution. They find that the test procedure is effective in determining an extreme value in a time series, but it cannot distinguish such an extreme value as a mean shift or a regular observation associated with the inherent nature of the distribution. Therefore, we should consider other methods, for example, the quasi-likelihood method, when the innovation distribution is not Gaussian or unknown. Also, we can use a nonparametric test, for example, the CUSUM (cumulative sum) test of which idea is from the Kolmogorov-Smirnov test.

CHAPTER 4

TEMPORAL AGGREGATION EFFECTS ON TESTING FOR A VARIANCE CHANGE

In this chapter, we demonstrate the effects of temporal aggregation on the CUSUM test for a variance change and propose a modified CUSUM test to control the aggregation effects.

4.1 Aggregation Effects on the CUSUM Test

Let K be the change point of the aggregate discordant series X_T in Eq. (2.1), for $1 < K \leq N$ and $K \in \mathbb{Z}$. Then, similarly to Eq. (1.30), we define an error E_T,

$$E_T = \Pi(\mathcal{B})(1-\mathcal{B})^d X_T = \begin{cases} A_T, & \text{if } T < K, \\ (1+\mathcal{V}_K)A_T, & \text{if } T \geq K, \end{cases} \tag{4.1}$$

where

$$\Pi(\mathcal{B}) = (1 - \Pi_1\mathcal{B} - \Pi_2\mathcal{B}^2 - \cdots) = \Phi_P(\mathcal{B})/\Theta_Q(\mathcal{B}) \tag{4.2}$$

and $\mathcal{V}_K$ is a change-magnitude. Eq. (4.1) implies that the variance of A_T changes from σ_A^2 to $(1+\mathcal{V}_K)^2\sigma_A^2$ at time point K.

In the same manner as Eq. (1.31) and (1.32), the CUSUM statistic to test for a variance change at an unknown K is

$$\sup_{K=2,\ldots,N} |D_K| \tag{4.3}$$

with

$$D_K = \frac{\sum_{T=1}^{K} E_T^2}{\sum_{T=1}^{N} E_T^2} - \frac{K}{N}, \tag{4.4}$$

and the test statistic (4.3) converges into $\sup_{0<r\leq 1}\left|\sqrt{\frac{2}{N}}B(r)\right|$ in distribution under the null hypothesis of no variance change.

The error E_T in Eq. (4.1) can be rewritten as

$$\begin{aligned} E_T &= \Pi(\mathcal{B})\left(1+\sum_{j=1}^{m-1} B^j\right)\sum_{t=m(T-1)+1}^{mT}(1-B)^d x_t \\ &= \frac{\Pi(\mathcal{B})\theta_q(B)}{\phi_p(B)}\left(1+\sum_{j=1}^{m-1} B^j\right)\sum_{t=m(T-1)+1}^{mT} e_t \\ &= \eta(B)\sum_{t=m(T-1)+1}^{mT} e_t, \end{aligned} \tag{4.5}$$

where e_t is given in Eq. (1.30) and

$$\eta(B) = (1+\eta_1 B+\eta_2 B^2+\cdots) = \frac{\Pi(\mathcal{B})\theta_q(B)}{\phi_p(B)}\left(1+\sum_{j=1}^{m-1} B^j\right). \tag{4.6}$$

Then, we have the squared error

$$\begin{aligned} E_T^2 &= (1+\eta_1 B+\eta_2 B^2+\cdots)^2 \\ &\quad\times\left[\left(\sum_{t=m(T-1)+1}^{mT} e_t^2\right)+2\left(\sum_{t=m(T-1)+1}^{mT-1}\sum_{s=t+1}^{mT} e_t e_s\right)\right] \\ &= \sum_{t=m(T-1)+1}^{mT} e_t^2 + \alpha_{mT}, \end{aligned} \tag{4.7}$$

where

$$
\begin{aligned}
\alpha_{mT} = (1 + \eta_1 B + \eta_2 B^2 + \cdots)^2 \\
\times \left[\left(\sum_{t=m(T-1)+1}^{mT} e_t^2 \right) + 2 \left(\sum_{t=m(T-1)+1}^{mT-1} \sum_{s=t+1}^{mT} e_t e_s \right) \right] \\
- \sum_{t=m(T-1)+1}^{mT} e_t^2. \qquad (4.8)
\end{aligned}
$$

Therefore, D_K in Eq. (4.4) can be expressed as

$$
D_K = \frac{\sum_{t=1}^{mK} e_t^2 + \sum_{T=1}^{K} \alpha_{mT}}{\sum_{t=1}^{mN} e_t^2 + \sum_{T=1}^{N} \alpha_{mT}} - \frac{K}{N} \qquad (4.9)
$$

which implies that the statistic D_K is affected by the aggregation through the two terms $\sum_{T=1}^{K} \alpha_{mT}$ and $\sum_{T=1}^{N} \alpha_{mT}$. To eliminate the aggregation effects from the statistic, we propose a modified CUSUM test in Definition 4.1.

Definition 4.1. A modified CUSUM test statistic for a variance change of the aggregate series X_T is defined to be

$$
\sup_{K=2,\ldots,N} |M_K| \qquad (4.10)
$$

with

$$
M_K = \frac{\sum_{T=1}^{K} \mathcal{E}_T^2}{\sum_{T=1}^{N} \mathcal{E}_T^2} - \frac{K}{N}, \qquad (4.11)
$$

where $\mathcal{E}_T^2 = E_T^2 - \alpha_{mT}$.

In Theorem 4.2, we show the null distribution of the modified CUSUM test statistic (4.10).

Theorem 4.2. *Assume $K = \lfloor k/m \rfloor$ and $N = n/m$. For an unknown K, the CUSUM* *test statistic (4.10) converges in distribution into* $\sup_{0<r\leq 1}\left|\sqrt{\frac{2}{n}}B(r)\right|$, *i.e.,*

$$\sup_{K=2,\ldots,N}|M_K| \xrightarrow{d} \sup_{0<r\leq 1}\left|\sqrt{\frac{2}{n}}B(r)\right|. \tag{4.12}$$

Proof. We define a random variable $V_T = \mathcal{E}_T^2 - m\sigma_a^2$ for $T = 1, \ldots, N$. Under the null hypothesis of no variance change, we have the mean of V_T,

$$\mathrm{E}(V_T) = \mathrm{E}(\mathcal{E}_T^2) - m\sigma_a^2 = 0 \tag{4.13}$$

and the variance of V_T,

$$\mathrm{Var}(V_T) = \mathrm{Var}(\mathcal{E}_T^2) = 2m\sigma_a^4 \tag{4.14}$$

because

$$\mathcal{E}_T^2 = E_T^2 - \alpha_{mT} = \left(\sum_{t=m(T-1)+1}^{mT} e_t^2\right) \overset{iid}{\sim} \sigma_a^2\chi_m^2. \tag{4.15}$$

Let

$$Y_N(r) = \frac{1}{\sigma_V\sqrt{N}}\left[\sum_{j=1}^{\lfloor Nr\rfloor} V_j + (Nr - \lfloor Nr\rfloor)V_{\lfloor Nr\rfloor+1}\right] \tag{4.16}$$

for $\lfloor Nr\rfloor = K$ and $0 < r < 1$, where $\sigma_V^2 = \mathrm{Var}(V_T)$. We consider

$$\begin{aligned}
&Y_N(r) - rY_N(1)\\
&= \frac{1}{\sigma_a^2\sqrt{2mN}}\left[\sum_{j=1}^{\lfloor Nr\rfloor} V_j - r\sum_{j=1}^{N} V_j + (Nr - \lfloor Nr\rfloor)V_{\lfloor Nr\rfloor+1}\right]\\
&= \frac{1}{\sigma_a^2\sqrt{2mN}}\left(\sum_{j=1}^{\lfloor Nr\rfloor} V_j - r\sum_{j=1}^{N} V_j\right) + \frac{(Nr - \lfloor Nr\rfloor)V_{\lfloor Nr\rfloor+1}}{\sigma_a^2\sqrt{2mN}}\\
&= \frac{1}{\sigma_a^2\sqrt{2mN}}\left[\sum_{j=1}^{\lfloor Nr\rfloor}(\mathcal{E}_j^2 - m\sigma_a^2) - r\sum_{j=1}^{N}(\mathcal{E}_j^2 - m\sigma_a^2)\right]\\
&\quad + \frac{(Nr - \lfloor Nr\rfloor)V_{\lfloor Nr\rfloor+1}}{\sigma_a^2\sqrt{2mN}}.
\end{aligned} \tag{4.17}$$

For large N,

$$\begin{aligned}
&\sum_{j=1}^{\lfloor Nr \rfloor}(\mathcal{E}_j^2 - m\sigma_a^2) - r\sum_{j=1}^{N}(\mathcal{E}_j^2 - m\sigma_a^2) \\
&\approx \sum_{j=1}^{K}(\mathcal{E}_j^2 - m\sigma_a^2) - \frac{K}{N}\sum_{j=1}^{N}(\mathcal{E}_j^2 - m\sigma_a^2) \\
&= \sum_{j=1}^{K}\mathcal{E}_j^2 - \frac{K}{N}\sum_{j=1}^{N}\mathcal{E}_j^2 \\
&= \left(\frac{\sum_{j=1}^{K}\mathcal{E}_j^2}{\sum_{j=1}^{N}\mathcal{E}_j^2} - \frac{K}{N}\right)\sum_{j=1}^{N}\mathcal{E}_j^2 \\
&= M_K \sum_{j=1}^{N}\mathcal{E}_j^2 \qquad (4.18)
\end{aligned}$$

and so

$$\begin{aligned}
&M_K\left(\frac{\sum_{j=1}^{N}\mathcal{E}_j^2/(mN)}{\sigma_a^2}\right) \\
&\approx \sqrt{\frac{2}{mN}}\left(Y_N(r) - rY_N(1) - \frac{(Nr - \lfloor Nr \rfloor)V_{\lfloor Nr \rfloor + 1}}{\sigma_a^2\sqrt{2mN}}\right). \qquad (4.19)
\end{aligned}$$

We remark that

$$\frac{1}{mN}\sum_{j=1}^{N}\mathcal{E}_j^2 = \frac{1}{n}\sum_{t=1}^{n}e_t^2 \longrightarrow \sigma_a^2, \qquad (4.20)$$

$$Y_N(r) \xrightarrow{d} W(r), \qquad (4.21)$$

and

$$Y_N(r) - rY_N(1) \xrightarrow{d} B(r) \qquad (4.22)$$

where $W(r)$ is a Wiener process and $B(r)$ is a Brownian bridge (see Billingsley, 1999, p.90), and

$$\sup_{0 \le r < 1}\left|\frac{(Nr - \lfloor Nr \rfloor)V_{\lfloor Nr \rfloor + 1}}{\sigma_a^2\sqrt{2mN}}\right| \xrightarrow{p} 0 \qquad (4.23)$$

as $N \to \infty$. Therefore, we obtain

$$M_K \xrightarrow{d} \sqrt{\frac{2}{mN}} B(r) = \sqrt{\frac{2}{n}} B(r) \tag{4.24}$$

and

$$\sup_{K=2,\ldots,N} |M_K| \xrightarrow{d} \sup_{0<r\leq 1} \left| \sqrt{\frac{2}{n}} B(r) \right| \tag{4.25}$$

under the null hypothesis of no variance change. □

4.2 Simulation Studies of the Aggregation Effects

In this section, we compare the empirical null distributions of the unmodified CUSUM test statistic (4.3) and the modified CUSUM test statistic (4.10) through Monte Carlo simulations. We also investigate their statistical powers.

4.2.1 Empirical Null Distributions

Under the null hypothesis of no variance change, we generate 2500 different nonaggregate stationary series (each length $n = 1800$) which follow ARMA$(1, 1)$ models,

$$(1 - \phi B)x_t = (1 - \theta)a_t, \tag{4.26}$$

for every choice of parameters

$$\phi \text{ and } \theta \in \{-0.95, -0.8, -0.5, -0.3, -0.1, 0.1, 0.3, 0.5, 0.8, 0.95\},$$

assuming $\phi \neq \theta$ and $a_t \overset{iid}{\sim} N(0, 1)$.

Since there are 90 combinations of the two parameters, we have total 225,000 series. Then, we transform the simulated series into their mth order temporal aggregates X_T, as defined in (2.1), for $m =$ 3, 6, 12, 18, 24, 36, respectively. We note that

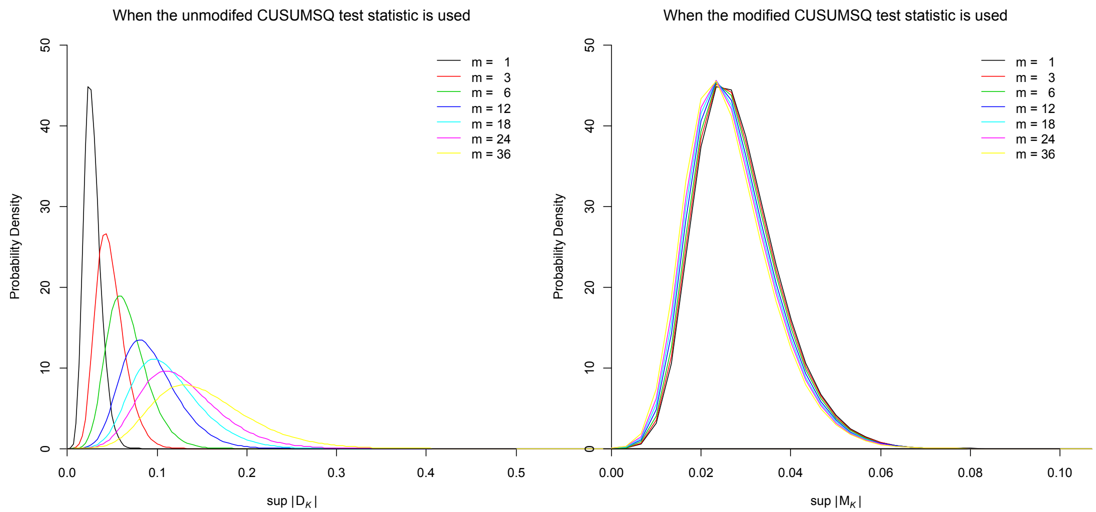

Figure 4.1: Simulated null distributions of the CUSUM test statistics

Table 4.1: Percentiles of the CUSUM test statistics

	$\sup_{K=2,\ldots,N} \lvert D_K \rvert$						$\sup_{K=2,\ldots,N} \lvert M_K \rvert$					
m	25th	50th	75th	90th	95th	99th	25th	50th	75th	90th	95th	99th
1	0.022	0.027	0.033	0.040	0.045	0.054	0.022	0.027	0.033	0.040	0.045	0.054
3	0.037	0.046	0.057	0.069	0.077	0.092	0.022	0.027	0.033	0.040	0.044	0.053
6	0.052	0.064	0.080	0.096	0.107	0.129	0.021	0.026	0.033	0.040	0.044	0.053
12	0.071	0.088	0.111	0.134	0.149	0.179	0.021	0.026	0.032	0.039	0.044	0.053
18	0.085	0.106	0.133	0.161	0.180	0.216	0.021	0.026	0.032	0.039	0.043	0.052
24	0.097	0.121	0.152	0.185	0.206	0.248	0.020	0.025	0.032	0.039	0.043	0.052
36	0.114	0.144	0.182	0.222	0.248	0.299	0.020	0.025	0.031	0.038	0.043	0.052

each aggregate series X_T follows a stationary ARIMA$(1,1)$ process,

$$(1-\Phi\mathcal{B})X_T = (1-\Theta\mathcal{B})A_T \tag{4.27}$$

and its length becomes N = 600, 300, 150, 100, 75, 50, respectively. The model parameters Φ and Θ are decided by Lemma 2.3 and the invertibility condition (2.38).

Under the null hypothesis, we compute every probability density of the unmodified statistic (4.4) and the modified statistic (4.11) for every parameter choice and aggregation order, respectively. The mean distribution of the all the density curves using (4.4) for each m is drawn on the left panel and the mean distribution using (4.11) on the right panel of Figure 4.1. We also present their 25th, 50th, 75th, 90th, 95th, and 99th percentiles in Table 4.1.

We notice that the null distribution of (4.4) changes its location rightward and shape downward as the aggregation order m increases, but the null distribution of (4.11) almost keeps its location and shape. For example, the median of the unmodified statistics increases approximately 433.33% from 0.027 to 0.144 when m changes from 1 to 36, but the median of the modified statistics decreases only approximately 7.40% from 0.027 to 0.025. When comparing their 50th percentiles, the average rightward-shift rate of the unmodified statistics is approximately +12.04% per one aggregation increment and the average rate of the modified statistics is approximately −0.21%.

4.2.2 Test Powers

To examine the statistical powers, we simulate alternative distributions. First, we generate 2500 different nonaggregate processes (each length $n = 1800$) which follow discordant ARIMA$(1,1)$ models

$$x_t = x_t^{(0)} + v_k \left[\frac{(1-\theta B)}{(1-\phi B)} a_t \right] I_t(k), \tag{4.28}$$

where

$$(1 - \phi B)x_t^{(0)} = (1 - \theta)a_t$$

for

$$\phi \text{ and } \theta \in \{-0.95, -0.8, -0.5, -0.3, -0.1, 0.1, 0.3, 0.5, 0.8, 0.95\},$$

assuming $\phi \neq \theta$, $a_t \overset{iid}{\sim} N(0, 1)$, and $v_k = 10$ at $k = 901$.

Since there are 90 combinations of the two parameters, we have total 225,000 nonaggregate discordant series and every series has a variances change of magnitude 10 starting at time point 901. Next, we transform the discordant series into their mth order temporal aggregates X_T for m = 3, 6, 12, 18, 24, 36, respectively. Then, we compute the two different CUSUM test statistics—the unmodified statistic $\sup_{K=1,\ldots,N} |D_K|$ and the modified statistic $\sup_{K=1,\ldots,N} |M_K|$ for all the nonaggregate and aggregate series and find their distributions.

In Figure 4.2 and Table 4.2, we present the test powers obtained from the simulations at significance level $\alpha = 0.05$. We use the 95th percentiles in Table 4.1 as our critical values. In Figure 4.2, the vertical blue dots indicate the test powers of the 90 discordant ARMA$(1, 1)$ processes at each m. In general, the test power becomes large as both $|\phi|$ and $|\theta|$ approach 1 at each m. When comparing their means or medians, the test power using the unmodified (4.4) decreases as m increases, but the test power using the modified (4.11) increases. For example, the mean test power for (4.4) decreases approximately 49.82% from 0.10391 to 0.05214 as m changes from 1 to 36, but the mean test power for (4.11) increases approximately 0.15% from 0.10391 to 0.10407. At $m = 36$, the mean test power for (4.11) is approximately 99.60% higher than the mean test power for (4.4).

Here the power decrements of the unmodified test can be interpreted as an information loss due to the aggregation. In other words, temporal aggregation has negative effects on the variance change test in terms of test power. Through the simulations,

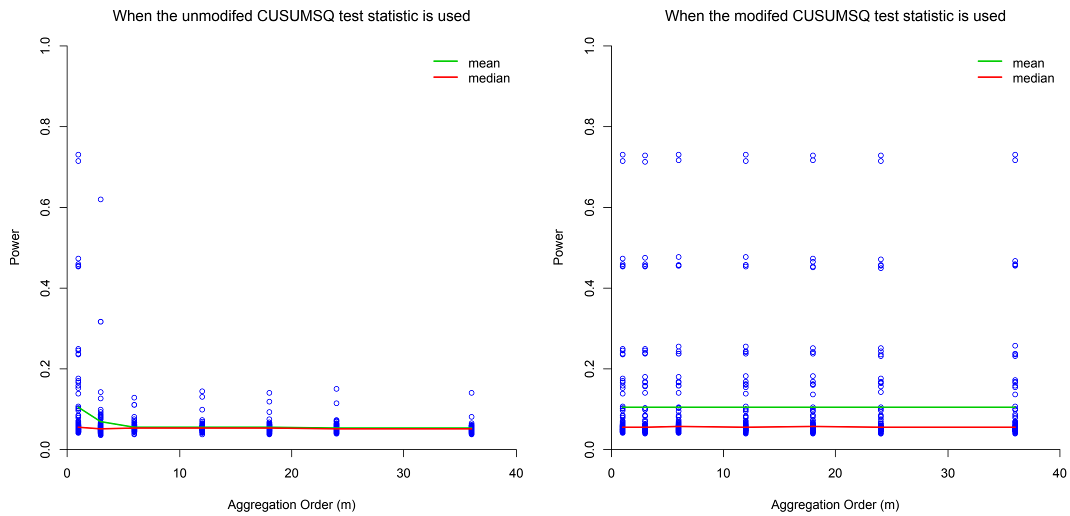

Figure 4.2: Test powers of the two CUSUM tests

Table 4.2: Test powers of the two CUSUM tests

	Test powers for every choice of model parameters when the unmodified (4.4) is used		Test powers for every choice of model parameters when the modified (4.11) is used	
m	mean	median	mean	median
1	0.10391	0.0552	0.10391	0.0552
3	0.06821	0.0514	0.10401	0.0546
6	0.05480	0.0520	0.10427	0.0562
12	0.05464	0.0524	0.10396	0.0548
18	0.05427	0.0518	0.10422	0.0558
24	0.05331	0.0516	0.10410	0.0550
36	0.05214	0.0516	0.10407	0.0552

we observe that the modified CUSUM test performs well without any decrease in test power even though the aggregation order m increases.

4.3 Illustrative Examples

The method of temporal aggregation is widely used in many areas of application like economics, business, and environmental science. In this section, we demonstrate the effects of temporal aggregation on the CUSUM test for a variance change, through the two real time series—the monthly U.S. international trade in goods and services between January 1992 and December 2013 and the monthly fish recruitment in the central Pacific Ocean between January 1950 and December 1986.

4.3.1 U.S. International Trade in Goods and Services

We investigate a monthly time series of U.S. international trade balance in goods and services, published through the U.S. Census Bureau (www.census.gov). The observations are seasonally adjusted balances (billions of dollars) of trade payments between January 1992 and December 2013 (total 264 months). A positive value indicates a trade surplus. A negative value indicates a trade deficit. The monthly series ($n = 264, m = 1$) and its temporal aggregations for $m = 3, 6, 12$ are displayed in Figure 4.3. Each aggregation represents a quarterly ($N = 88, m = 3$), a semi-annual ($N = 44, m = 6$), and an annual ($N = 22, m = 12$) balance, respectively.

Using the Bayesian information criterion (BIC), we choose an ARIMA model for the nonaggregate monthly series. The monthly series is now fitted into an IMA$(1, 3)$ model,

$$(1 - B)^d x_t = (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3) a_t \tag{4.29}$$

with maximum-likelihood estimates $\hat{\theta}_1 = 0.22765$ (0.05869), $\hat{\theta}_2 = 0.01112$ (0.05940) and $\hat{\theta}_3 = -0.32451$ (0.06108), and error variance estimate $\hat{\sigma}_a^2 = 6.50544$, where values

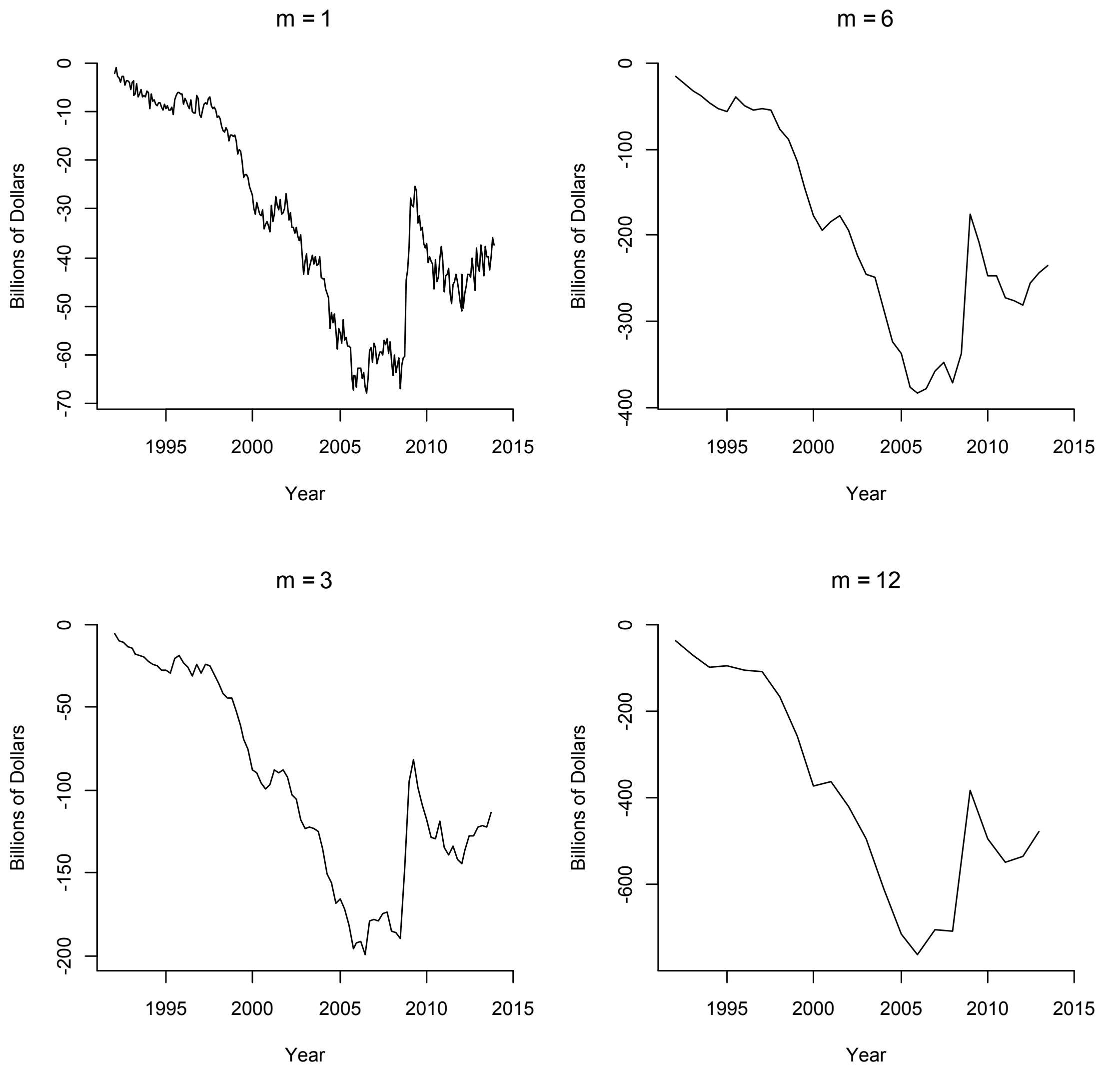

Figure 4.3: U.S. international trade balances between 1992 and 2013

Table 4.3: Model parameter estimates $\hat{\Theta}_1$ and $\hat{\Theta}_2$ when $\hat{\theta}_1 = 0.2277$, $\hat{\theta}_2 = 0.0111$, and $\hat{\theta}_3 = -0.3245$

m	$\hat{\Theta}_1$	$\hat{\Theta}_2$
3	−0.49679	−0.00195
6	−0.22033	0.17435
12	−0.00514	−0.07190

in parentheses are standard errors of the estimates. We note that the nonaggregate model transforms into an aggregate IMA(1, 2) model,

$$(1-\mathcal{B})^d X_T = (1-\Theta_1\mathcal{B}-\Theta_2\mathcal{B}^2)A_T \tag{4.30}$$

for $m = 3, 6, 12$. Using the rationale of Theorem 2.1, we can easily find MA parameter estimates $\hat{\Theta}_1$ and $\hat{\Theta}_2$ for every aggregation. The parameter estimates are shown in Table 4.3.

We calculate the estimated errors

$$\hat{e}_t = \left(\frac{1}{1-\hat{\theta}_1 B-\hat{\theta}_2 B^2-\hat{\theta}_3 B^3}\right) x_t \tag{4.31}$$

and

$$\hat{E}_T = \left(\frac{1}{1-\hat{\Theta}_1\mathcal{B}-\hat{\Theta}_2\mathcal{B}^2}\right) X_T, \tag{4.32}$$

and the aggregation effect $\sum_{T=1}^{K}\alpha_{mT}$ discussed in Sections 4.1 and 4.2. Figure 4.4 displays the computed values of $\sum_{T=1}^{K}\alpha_{mT}$ for every m. Then, we obtain test statistics from the unmodified form (4.4) and the modified form (4.11) for every aggregation, and estimate their change points

$$\hat{K}^{(1)} = \arg\max_{K=2,\ldots,N}|D_K| \tag{4.33}$$

and

$$\hat{K}^{(2)} = \arg\max_{K=2,\ldots,N}|M_K|, \tag{4.34}$$

respectively. We also find the estimated variances

$$\hat{\sigma}_A^2 = \frac{1}{\hat{K}^{(i)}-1}\sum_{T=1}^{\hat{K}^{(i)}-1}\hat{E}_T^2 \tag{4.35}$$

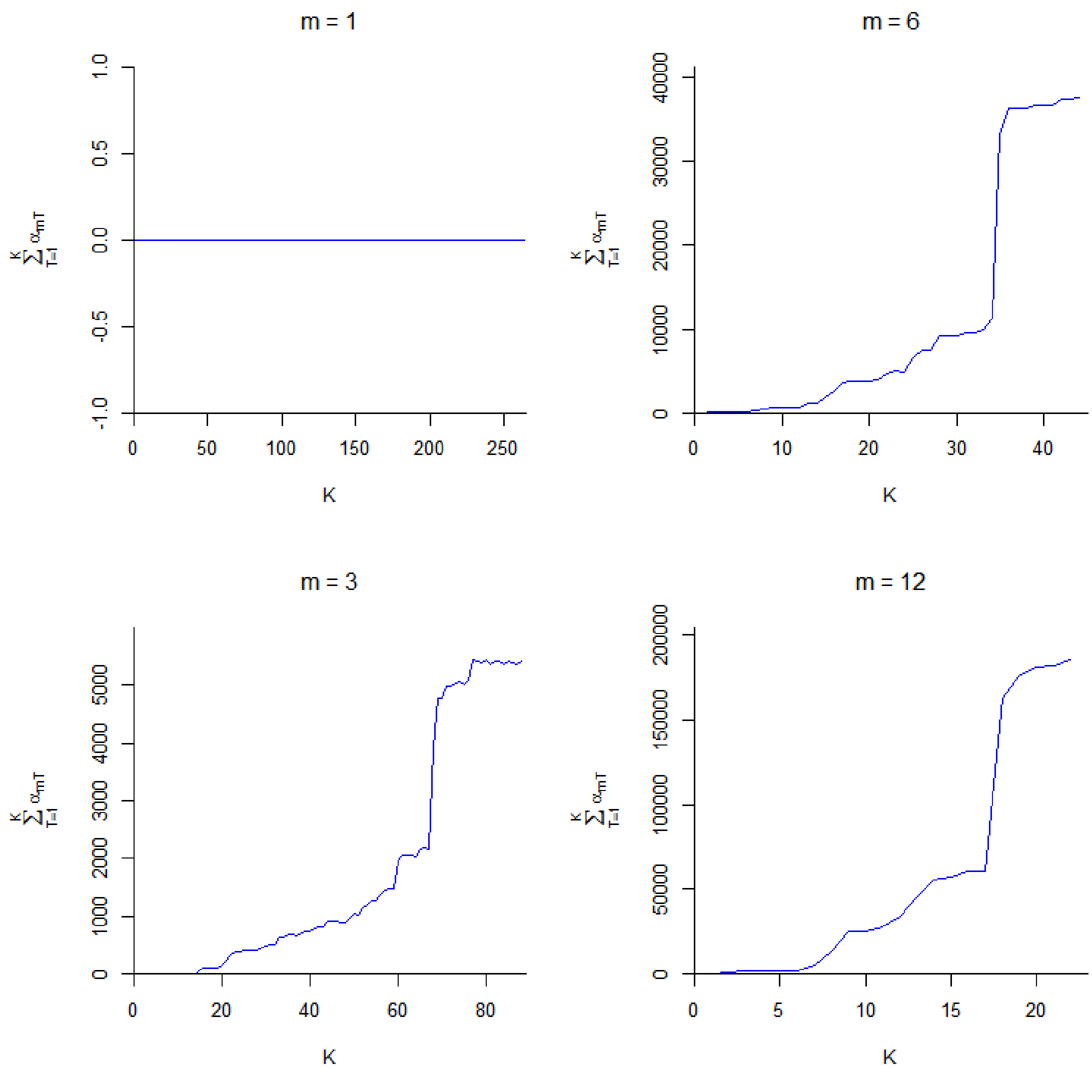

Figure 4.4: The aggregation effect $\sum_{T=1}^{K} \alpha_{mT}$ through K

Table 4.4: CUSUM test results

m	N	$\sup_{K=2,\ldots,N} \lvert D_K \rvert$	$\hat{K}^{(1)}$	$\hat{\sigma}_A^2$	$\hat{\mathcal{V}}_K$	$\sup_{K=2,\ldots,N} \lvert M_K \rvert$	$\hat{K}^{(2)}$	$\hat{\sigma}_A^2$	$\hat{\mathcal{V}}_K$
1	264	0.34728***	149	2.441	1.184	0.34728***	149	2.441	1.184
3	88	0.37858***	59	31.094	1.351	0.34049***	49	23.912	1.508
6	44	0.47663***	33	228.046	2.337	0.33107***	24	147.661	2.391
12	22	0.44280**	17	2539.795	2.047	0.33107***	12	1685.140	2.071

Note: * significant at $\alpha = 0.10$; ** significant at $\alpha = 0.05$; *** significant at $\alpha = 0.01$

and the estimated change magnitudes

$$\hat{\mathcal{V}}_K = \left[\frac{1}{(N - \hat{K}^{(i)} + 1)\hat{\sigma}_A^2} \sum_{T=\hat{K}^{(i)}}^{N} \hat{E}_T^2 \right]^{1/2} - 1, \tag{4.36}$$

for $i = 1, 2$.

The test results are shown in Table 4.4. All the test statistics from either Eq. (4.4) or (4.11) are significant at $\alpha = 0.05$ regardless of temporal aggregation. It implies that there exists a strong variance change on the given example. Here we compare the estimated change points $\hat{K}^{(1)}$ and $\hat{K}^{(2)}$ and find their differences—$\hat{K}_m^{(1)} \neq \left\lfloor \hat{K}_1^{(1)}/m \right\rfloor$ but $\hat{K}_m^{(2)} = \left\lfloor \hat{K}_1^{(2)}/m \right\rfloor$, where the subscript denotes the aggregation order. The modified method estimates the change points which are consistent with the temporal aggregation but the unmodified method does not. Through this example, we clearly see that the modified method needs to be used in order to accurately estimate the variance change points.

4.3.2 Fish Recruitment

We analyze another monthly time series taken from Shumway and Stoffer (2011). Figure 4.5 displays a monthly fish recruitment (number of fish) in the central Pacific Ocean between January 1950 and December 1986 ($n = 444, m = 1$) and its quarterly ($n = 148, m = 3$), semi-annual ($N = 74, m = 6$) and annual ($N = 37, m = 12$) aggregations. Fish recruitment is defined as the number of new fishes that enter a population in a given period. It is one of important concepts to explain population dynamics of fisheries.

The nonaggregate series is known to be fitted into an AR(2) model,

$$(1 - \phi_1 B - \phi_2 B^2)(x_t - \mu) = a_t \tag{4.37}$$

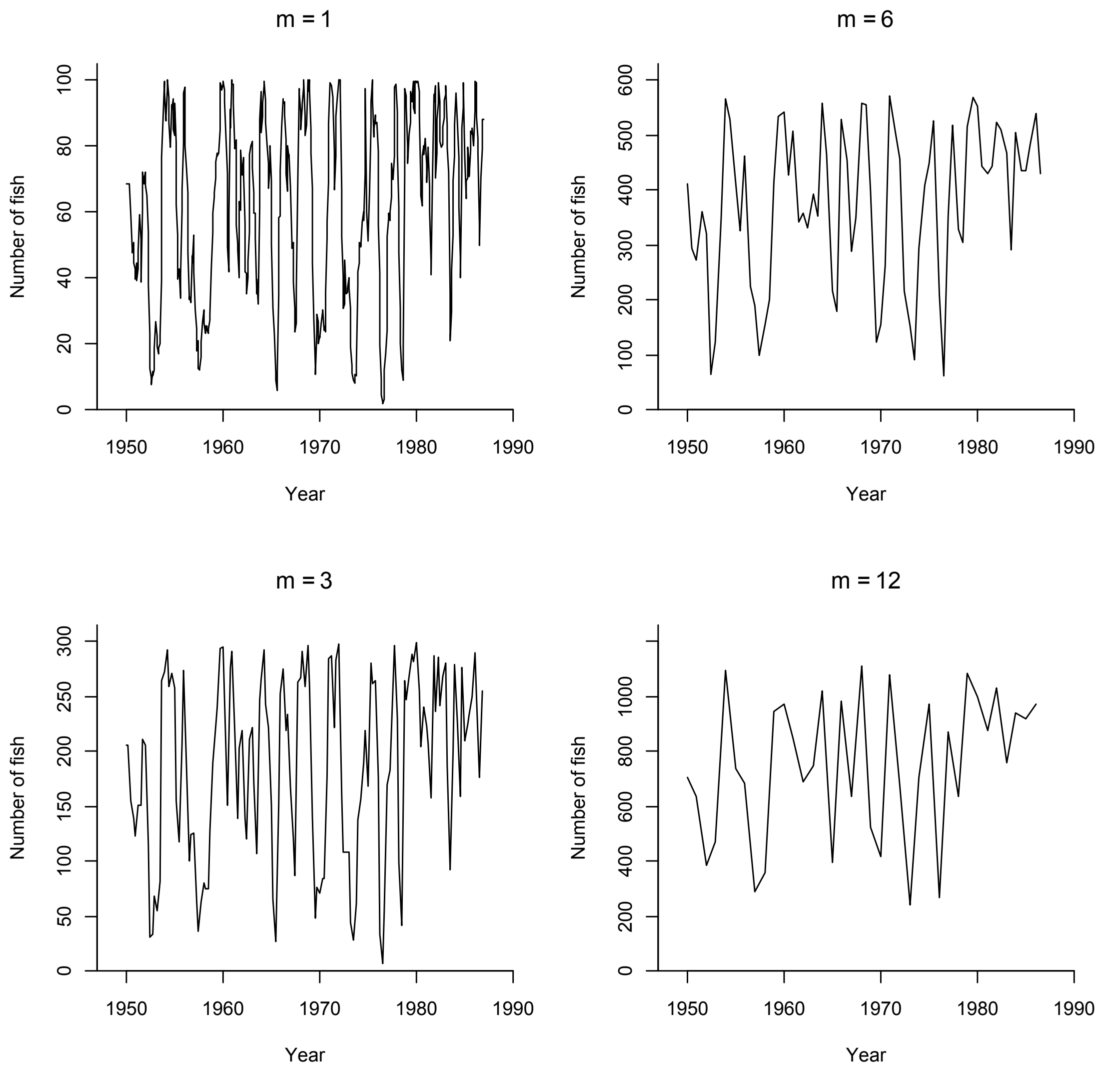

Figure 4.5: Fish recruitment in the central Pacific Ocean between 1950 and 1986

Table 4.5: Model parameter estimates $\hat{\Phi}_1$, $\hat{\Phi}_2$, $\hat{\Theta}_1$, $\hat{\Theta}_2$, and $\hat{\mathcal{U}}$ when $\hat{\phi}_1 = 1.34007$, $\hat{\phi}_2 = -0.45027$, and $\hat{\mu} = 62.27957$

m	$\hat{\Phi}_1$	$\hat{\Phi}_2$	$\hat{\Theta}_1$	$\hat{\Theta}_2$	$\hat{\mathcal{U}} = m\hat{\mu}$
3	0.59630	−0.09129	−0.28142	−0.00418	186.83871
6	0.17299	−0.00833	−0.71537	−0.29050	373.67742
12	0.01326	−0.00007	0.07608	0.08613	747.35484

(Shumway and Stoffer, 2011, pp.110–111). We now find the maximum-likelihood estimates $\hat{\phi}_1 = 1.34007$ (0.04223) and $\hat{\phi}_2 = -0.45027$ (0.04226), and error variance estimate $\hat{\sigma}_a^2 = 89.94383$, where values in parentheses are standard errors of the estimates. Also, the estimate of process mean μ is $\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} x_t = 62.27957$. From Theorem 2.1, we can find an ARMA$(2, 2)$ model for its mth temporal aggregation,

$$(1 - \Phi_1\mathcal{B} - \Phi_2\mathcal{B}^2)(X_T - \mathcal{U}) = (1 - \Theta_1\mathcal{B} - \Theta_2\mathcal{B}^2)A_T, \tag{4.38}$$

and the parameter estimates given in Table 4.5.

Now, we estimate the errors, i.e.,

$$\hat{e}_t = \left(1 - \hat{\phi}_1 B - \hat{\phi}_2 B^2\right) x_t \tag{4.39}$$

and

$$\hat{E}_T = \left(\frac{1 - \Phi_1\mathcal{B} - \Phi_2\mathcal{B}^2}{1 - \hat{\Theta}_1\mathcal{B} - \hat{\Theta}_2\mathcal{B}^2}\right) X_T, \tag{4.40}$$

and find the aggregation effect $\sum_{T=1}^{K} \alpha_{mT}$. The value of $\sum_{T=1}^{K} \alpha_{mT}$ for every m is drawn in Figure 4.6. From the unmodified form (4.4) and the modified form (4.11), we obtain their test statistics for every m. We also compute change point estimates $\hat{K}$, variance estimates $\hat{\sigma}_A^2$, and change magnitude estimates $\hat{\mathcal{V}}_K$ using the formulas given in the previous example.

We display the test results in Table 4.6. We note that the test statistics derived from Eq. (4.11) keep their significance as m increases, but the test statistics derived from Eq. (4.4) lose their significance once aggregated data are used. These results are consistent with the simulation studies in Section 4.2. Also, their change point estimates are $\hat{K}_m^{(1)} \neq \left\lfloor \hat{K}_1^{(1)}/m \right\rfloor$ but $\hat{K}_m^{(2)} = \left\lfloor \hat{K}_1^{(2)}/m \right\rfloor$. This example shows that the modified test statistic is needed not only for the accurate estimation of the variance change points but also about the higher test power.

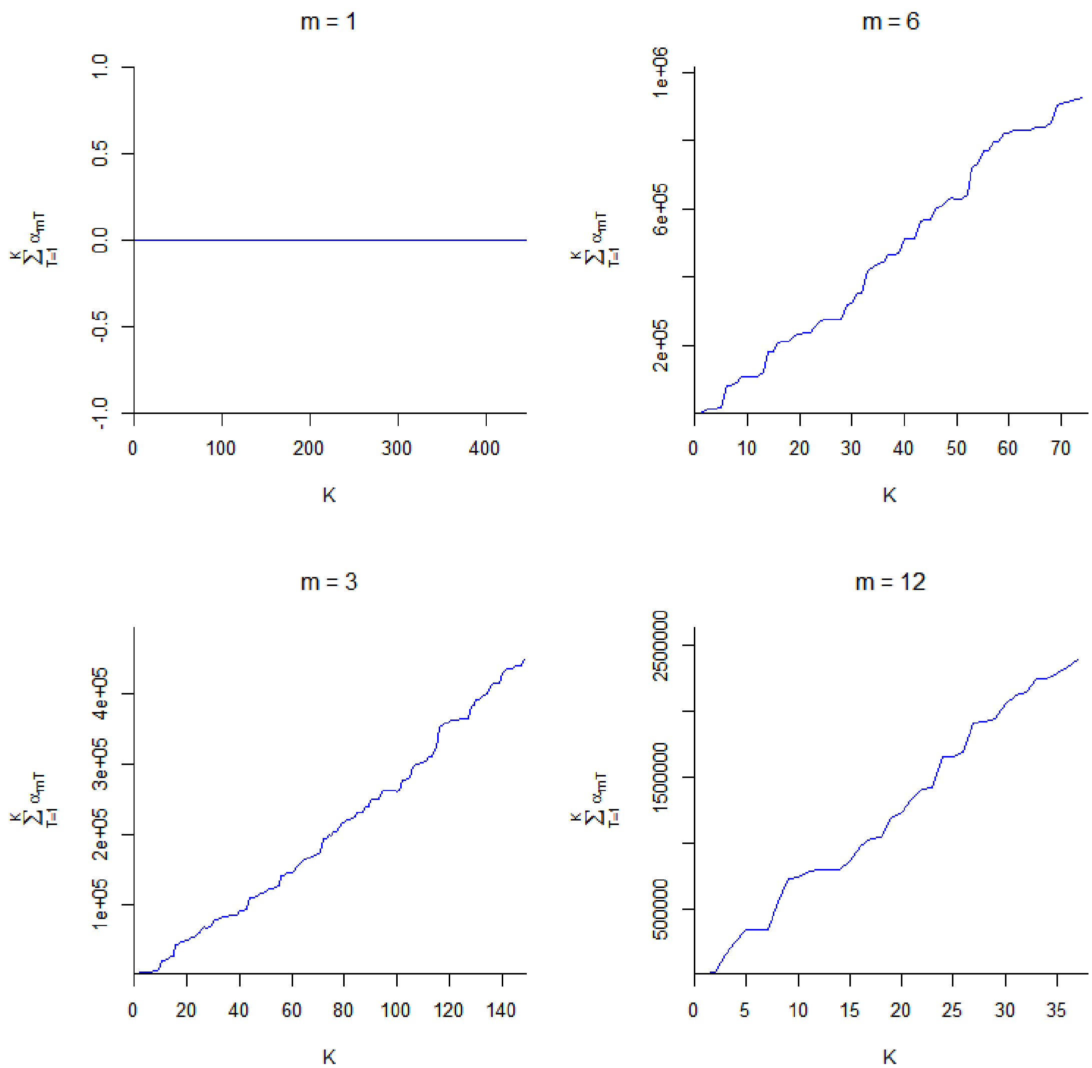

Figure 4.6: The aggregation effect $\sum_{T=1}^{K} \alpha_{mT}$ through K

Table 4.6: CUSUM test results

m	N	$\sup_{K=2,\dots,N} \lvert D_K \rvert$	$\hat{K}^{(1)}$	$\hat{\sigma}_A^2$	$\hat{\mathcal{V}}_K$	$\sup_{K=2,\dots,N} \lvert M_K \rvert$	$\hat{K}^{(2)}$	$\hat{\sigma}_A^2$	$\hat{\mathcal{V}}_K$
1	444	0.09718**	126	59.241	0.312	0.09718**	126	59.241	0.312
3	148	0.09348	101	2865.546	0.209	0.09717**	41	2369.769	0.245
6	74	0.08834	59	14366.738	−0.281	0.09717**	20	12203.364	0.056
12	37	0.06719	27	65745.939	−0.069	0.08644*	9	60858.112	−0.005

Note: * significant at $\alpha = 0.10$; ** significant $\alpha = 0.05$; *** significant at $\alpha = 0.01$

4.4 Concluding Remarks

In this Chapter, we analyze the effects of temporal aggregation on the CUSUM test for a variance change in a time series. We show that the CUSUM test statistic is affected by temporal aggregation. To remove the aggregation effects, we propose the modified CUSUM test statistic. Through the simulation study and the empirical examples, we demonstrate that the modified CUSUM test has a much better estimation for a variance change point and higher test power. Therefore, we conclude that the modified CUSUM test should be used for a proper variance test when temporally aggregated data are used in the analysis.

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