NON-PARASITIC WARLORDS AND GEOGRAPHICAL DISTANCE

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Jerry Hionis, Jr.
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Examining Committee Members:

Dr. Dimitrios Diamantaras, Advisory Chair, Department of Economics
Dr. Dai Zusai, Department of Economics
Dr. Moritz Ritter, Department of Economics
Dr. Edward Rosenthal, External Member, Temple University

ABSTRACT

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Chair: Dimitrios Diamantaras

This dissertation presents an extension of the warlord competition models found in Skaperdas (2002) and Konrad and Skaperdas (2012). I consider two non-parasitic warlords located on a line. Each warlord allocates resources for the extraction of natural resources, the production of goods and services, and conflict with the opposing warlord. Within the symmetric rates of seizure model, I use three different forms of the contest success function, a primary tool in the conflict theory literature, in my analysis. I show that the warlord closer to the point of conflict will invest less into the hiring of warriors and more into the production of goods and services, yet wins a larger proportion of total goods and services produced within the economy. Under certain conditions, the placement of the point of conflict at the midpoint between the two warlords maximizes the total resources toward war and minimizes total production. Under the asymmetric rates of seizure model, I find that the warlord closer to the point of conflict invests more in warfare and less in production; that is, results that counter what is found in the symmetric model.

For my beloved wife Dorothy and my daughter Effie. May God bless and bring peace, pleasure and prosperity to you both.

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ
ٱلْحَمْدُ لِلَّهِ رَبِّ ٱلْعَٰلَمِينَ
ٱلرَّحْمَٰنِ ٱلرَّحِيمِ
مَٰلِكِ يَوْمِ ٱلدِّينِ
إِيَّاكَ نَعْبُدُ وَ إِيَّاكَ نَسْتَعِينُ
آ هْدِنَا ٱلصِّرَٰطَ ٱلْمُسْتَقِيمَ
صِرَٰطَ ٱلَّذِينَ أَنْعَمْتَ عَلَيْهِمْ
غَيْرِ ٱلْمَغْضُوبِ عَلَيْهِمْ وَ لَا ٱلضَّا لِّينَ

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CHAPTER 1

Introduction and Statement of the Problem

1.1 Introduction and Statement of Problem

One of the goals within the conflict theory literature is to understand the factors that influence the decisions of people, groups, nations and so on to participate in wars of appropriation. Of course, not all conflicts are identical in terms of scope, structure and method. There exist within the literature a number of distinct conflict types:

State-to-State Two or more distinct states engaging in acts of war.

Insurgency Within a single state, two or more groups battling against an established ruler.

Civil War Within a single state, two or more groups battling against each other where no established ruler exists.

While state-to-state and insurgency style conflicts have been studied heavily within the literature, the role and effects of civil wars has begun to be empirically studied on a much larger scale to go beyond "anecdotal" results and relationships (See Sambanis (2002) for a review).

In two seminal studies, Fearon and Laitin (2003) and Collier and Hoeffer (2004) show that, against the conventional wisdom, areas with high risk of civil conflict are not ethnically/religiously diverse or suffer from “political shocks”, but conflicts occur when certain factors that favor insurgency and war are found; factors such as poverty, political instability, terrain and geography, high population densities and abundance of natural resources. In a similar study, Buhaug and Rød (2006) find two sets of results in relationship to geography. First, conflict is more likely to erupt in rural areas and along national borders. Second, there exists a positive relationship between distance to the “capital”, or the area with the highest population density, and the likelihood of conflict. These two results emphasize the difference between civil wars waged as an insurgency of a group against an established government presence and territorial conflicts amongst warlords. Buhaug and Gates (2002) and Buhaug et al (2009) study the determinants of conflict points within a geographic area and highlight the importance of ethnic identity, geography and ideology. These studies focus on the determinants of the location of conflicts and not the effects that the locations of conflicts have on the wars themselves.

While empirical studies show the important relationship between geography and civil war, less has been done within a theoretical framework. Early models of insurgency have focused on territorial expansion (Findlay (1996) and Wittman (2000)) and the effect that conflict has on geographic location of insurgency (Brito and Intriligator (1990,1992)), but not on the effects that geographic locations of conflict have on wartime decision making. To the author's knowledge, the only serious study on the geographical effects on decision making within a conflict is Gates (2002). Gates creates a principal-agent model which studies how an insurgent leader and an established ruler each construct a system of rewards and punishments for possible supports. The model is built upon a geographical framework in that both the insurgent leader and the established ruler are located at distinct areas with varying distances from possible supporters. In general, the model shows that the more distant supporters are rewarded more than those closer to the associated leader.

As opposed to Gates (2002), this dissertation specifically aims to explicitly explore the theoretical implications that geographical distance has on the decisions made by two warlords partaking in civil war. The models I present, and their results, show that the geographical location of conflict with respect to the location of the two warlords affects both the expenditure on conflict and the production of goods and services as well as the level of success each warlord gains from conflict. These models and results can, in the future, be tested with empirical data.

1.2 Model Overview

I present a guns-and-butter economy based upon the structure found in Skaperdas (2002) and Konrad and Skaperdas (2012). The economy modeled is one where no central government authority exists to enforce law and offer security for its populace; such as Yemen, Somalia, Afghanistan, Libya and, to an extent, Iraq, Nigeria, Mali and Pakistan. In lieu of such a government, two warlords, A and B, exist that offer protection and leadership for those within the economy. The models presented by Skaperdas and Konrad view warlords as kleptocratic, or parasitic, leaders who only offer protection to those willing to pay tribute and, hence, do not directly influence production within an economy. The model developed here allows warlords to offer protection as well as invest within the production of goods and services within the economy. In addition, each warlord is in control of a specific territory, known as a stronghold. The strongholds of warlord A and warlord B are separated by a linear distance.

Within his own territory, each warlord is endowed with certain exogenous resources. First, each warlord rules over a population group which dedicates its time to the economic activities dictated by the warlord. Second, each warlord has a cache of un-extracted natural resources that can be extracted and sold to some external purchaser for a fixed and exogenous price. To extract these natural resources, a warlord must both dedicate members of his population for the extraction as well as pay each extractor a fixed and exogenous wage. It is assumed within the model that the extraction wage per unit is less than the price paid per unit of a natural resource. Third, a pre-game budget exists for each warlord that can be interpreted as spoils earned in previous conflicts, foreign aid and so on. From his own set of resources, each warlord has two decisions: production of goods and services and the appropriation of goods and services through conflict.

Production of goods and services within the economy is done through capital investment by both warlord A and warlord B. Each warlord purchases units of capital from an external seller, for a fixed and exogenous price, using either the sales of extracted natural resources or the pre-game budget. Each warlord's production process is then a function of capital investment and the relative technology/productivity level of the warlord's production process.

Conflict within the economy is modeled through a Contest Success Function (CSF); that is, a function that shows the effect a warlord's effort has on the proportion of goods and services he is awarded through conflict. Each warlord's effort toward conflict in the model is the number of warriors he chooses to hire. A warlord hires warriors from his own population and also pays a fixed and exogenous wage to each. Hypothetically, most models within the conflict theory literature focus on battlefield conflicts, where players can implicitly be viewed as standing across from each other on different sides of a line. As a result, the CSF predominately used is the ratio form developed by Tullock (1980). In many modern conflicts in the developing world, there exist external factors that negatively affect the effort put forth toward conflict; such as climate, topography, geography and so on. The difference form of the Contest Success Function, developed by Hirshleifer (1988, 1989), addresses such factors.

The impact of a warlord's effort toward winning the conflict can be constructed in various ways. Employing the difference form of the CSF, I create three models of conflict to investigate the different effects that geographical distance can have on decision making: (1) the Base model, (2) the Gates-logit model and (3) the Ratio model. The first is a quasilinear formulation of a warlord's impact on the conflict which is based upon the CSF modeled in Buhaug et al (2009) to interpret distance having a subtractive effect on the number of warriors hired for conflict. The Gates-logit model uses the formula found in Gates (2002) in which distance has a subtractive effect on the log-linearized number of warriors hired for conflict. Finally, the Ratio model, views the relationship between warriors hired and distance from the point of conflict as a ratio.

The results found in Chapter 3 show that there are many similarities and contrasts in the results of the three above mentioned models. One constant theme found within the three models resembles what is known as the "Paradox of Power" (Hirshleifer (2001)); that is, the richer of two contestants within a conflict will invest more in the production of goods and services, while the poorer invests more in appropriation and less in production. Specifically, any increase in either warlord's population size or pre-game budget causes both warlords to increase their hiring of warriors. In regards to investment in capital, any increase in warlord A's population size or pre-game budget causes warlord A to increase his investment in capital and warlord B to decrease investment in capital. Any increase in warlord B's population size or pre-game budget causes warlord A to decrease his investment in capital and warlord B to increase investment in capital. In addition, as the point of conflict moves closer to warlord A's stronghold, warlord A invests more in capital and warlord B invests less. Interestingly, the Gates-logit model differs from the other two by showing that an increase in either warlord's population size and/or pre-game budget will result in an increase in the total amount of goods and services produced, on aggregate, within the economy. Alternatively, the Base and Ratio models show that the total amount of goods and services produced is unaffected by increases in population sizes and pre-games budgets.

All three models show that when both warlords face the same prices and the point of conflict is at the midpoint of the two strongholds, warlords hire the same number of warriors. As the point of conflict moves closer to a warlord's stronghold, the warlord hires fewer warriors and the opposing warlord hires more. The Base and Gates-logit models agree that the total number of warriors hired in the economy is unaffected by the point of conflict. The Ratio model differs in that when the point of conflict is at the midpoint between the two strongholds, total expenditure in conflict is at its highest and as the point of conflict moves away from the midpoint toward either warlords' stronghold, the total number of warriors hired decreases.

In terms of capital investment and the production of goods and services, all three models again agree: as the point of conflict moves closer to a warlord's stronghold, the warlord will invest in less capital while the opposing warlord invests more. The Base and Gates-logit model both show that the increase in one warlord's production is offset by the opposing warlord's decrease in production such that the total production of goods and services is unaffected by the point of conflict. The Ratio model's results, again, illustrate a different conclusion. Specifically, when the point of conflict is at the midpoint between the two strongholds, the total expenditure toward production of goods and services is at its lowest and as the point of conflict moves away from the midpoint toward either warlords' stronghold, the total production of goods and services increases.

While the Base and Gates-logit model have many similar results, the effect that the point of conflict has on the CSF in equilibrium under the Base model differs from both the Gates-logit and Ratio models. The Base model finds that the CSF in equilibrium depends upon the wages paid to the warriors hired and the wage paid to extractors of natural resources. The Gates-logit and Ratio model show that the

CSF in equilibrium is not impervious to other exogenous variables. Instead, the CSF depends on both the point of conflict as well as the wages paid to warriors and extractors. Assuming the wages for warriors and extractors are identical, warlord A wins a greater proportion of produced goods and services when the point of conflict moves closer to warlord A's stronghold while warlord B wins less. As the point of conflict moves closer to warlord B's stronghold, warlord B wins a larger proportion of goods and services and warlord A wins less.

The existence of an interior Nash equilibrium, where a positive number of warriors are hired and capital invested, differs between the three models. The existence of an interior Nash equilibrium within the Base model depends largely on the existence of pre-game budgets, a result that is in line with Collier and Hoeffler (2004); that is, conflicts and civil wars are more likely to take place when there is funding from international sources. When both warlords have a pre-game budget of zero, the only warrior equilibrium solution found is when no warriors are hired unless the wages paid to warriors are negative, which is a violation of a model assumption. As pre-game budgets get larger, an interior Nash equilibrium becomes possible as long as the profits made on the extraction and selling of natural resources outweigh the cost to hire each warrior. The dependence of the equilibrium's existence on the value of pre-game budgets seems to be caused by the requirement that warlords extract natural resources. In other words, each warrior that is hired costs a warlord a unit of population plus whatever units of population are needed to extract and sell the natural resources to pay the warrior. The Gates-logit and Ratio models establish that the existence of an interior Nash equilibrium is not as sensitive to the values of the pre-game budgets. Indeed, it is shown that an equilibrium solution with warriors being hired exists where both warlords' pre-game budgets are equal to zero.

The results stated above are found assuming a symmetric relationship between the amount of each warlord's production of goods and services available for appropriation.

One interpretation of such a situation is that both warlords are investing in a good and/or service that has a public value element, such as investing in the infrastructure of a country that both warlords share and benefit from. While this type of scenario does exist in modern civil conflict, it is also common to see an asymmetric relationship. Using the Base model formulation of the CSF, Chapter 4 extends the model developed in Chapter 3 by not assuming symmetry and allowing only a portion of a warlord's goods and services produced to be available for appropriation. The amount of a warlord's goods and services at risk is assumed to depend upon the location of the point of conflict; that is, more of a warlord's production of goods and services are subject to appropriation when the point of conflict moves closer to his own territory and less as it moves closer to the opposing warlord's territory.

In Chapter 4, the total expenditure on warfare is at its peak and the total investment on capital is at its lowest when the point of conflict is equidistant between the two stronghold. As the point of conflict moves closer to either warlord, total warrior hiring decreases and total capital investment increase. More specifically, when the point of conflict is closer to warlord A's stronghold, warlord A spends more on the hiring of warriors and less on capital investment than warlord B — the opposite holds true when the point of conflict is closer to warlord B. These contradictions can be explained by the importance that the point of conflict has on determining the rate of seizure. In the symmetric rate of seizure model, the closer the point of conflict is to a warlord's stronghold means fewer warriors need to be hired because the negative effect of travel and distance is small. With an asymmetric rate of seizure model, the closer the point of conflict is to a warlord's stronghold means that the warlord has more to gain from conflict because most of his production is being fought over. The opposing warlord has less to gain from engaging in conflict because little of his production is being fought over and the cost of conflict for the opposing warlord, due to the large distance, is high and the returns are low.

CHAPTER 2

Literature Review

This chapter is structured as follows. Section 2.1 introduces the concept and axiomatic treatment of Contest Success Functions, the essential "heart" of contest/conflict theory. Section 2.2 reviews the theoretic literature pertaining to both conventional and unconventional conflict types; such as, nation-to-nation, predator-prey, insurgency, guerilla warfare and so on. Section 2.3 introduces the concept of "warlord competition". Finally, Section 2.4 shows the contribution that this dissertation has on the literature. Readers who are familiar with the literature are encouraged to forgo Sections 2.1 through 2.3, while those unfamiliar to the study may find these sections helpful in gaining insight into the literature.

2.1 Contest Success Functions

The study of conflict as a theoretical economic problem appears to have originated as its own field through the developing and modeling of a contest. A contest can be defined as a competitive scenario where all players, simultaneously or sequentially, put forth effort to win a prize. A contest can be perfectly discriminating and imperfectly discriminating. A perfectly discriminating contest is such that that the player who contributes the most to the conflict wins the prize outright, such as a standard auction (Moldovanu and Sela (2001)). An imperfectly discriminating contest presumes that the player who contributes the highest toward the contest has the greatest probability of winning the prize. Imperfectly discriminating contest models are based upon a *Contest Success Function (CSF)*, or what Hirshleifer (1988) calls the "technology of conflict". While the functional form of a CSF can vary, all are constructed to relate a player's effort within the contest to his success in said contest. Effort levels enter in the CSF through an *Impact Function* which shows the per unit effect of a player's effort on the CSF.

Certain relationships and axioms must be satisfied to properly define a CSF within a contest: probability, marginal effects, anonymity, consistency and independence of irrelevant alternatives (Rai and Sarin (2009) for detailed descriptions). The probability axiom states that the CSF satisfies the properties of a probability distribution; that is, it is assumed the summation of all winning probabilities across all players is equal to one (additive to unity) [1]. The marginal effects axiom states that only a player's individual effort levels will increase his probability to win and, hence, one player's effort levels will not help another player's chance of winning (Münster (2009) for an extension into complementary effects and group contests). Anonymity requires that each player will have the same probability as an opposing player with the same effort and exogenous factors. In other words, the probability of winning a contest, given a specific effort level and set of exogenous traits, is not dependent upon the actual player himself. Consistency and the independence of irrelevant alternatives both deal specifically with a sub-contest between players. The consistency axiom implies that contests consisting of a smaller number of players will be qualitatively similar to the global contest with a large set players. Finally, the independence of irrelevant alternatives axiom states that only players active within the contest affect the CSF; that is, the contest should/can not depend on external players not participating within the contest itself.

[1] Blavatskyy (2004) presents an axiomatic model of a CSF which includes the possibility that no player wins and so there is a draw.

Skaperdas (1996) proves that there exists a specific class of CSFs that satisfy the above five axioms. Let a contest take place with $i \in N$ players. Let there exist L types of investments that a player can put efforts toward, where some are allowed to be fixed. Each player is willing to put forth effort levels $\mathbf{x}_i \geq 0 \quad \forall i \in N$, where $\mathbf{x}_i \in \mathbf{R}_+^L$ is the effort vector of player i for all L investments. Let $\pi_i : \mathbf{R}_+^{LN} \rightarrow \mathbf{R}_+$ be a probability of success where $\mathbf{x} \in \mathbf{R}_+^{LN}$ is the effort matrix of all N players for all L investments. The above axioms are then satisfied if and only if the CSF has the following form,

$$\pi_i(\mathbf{x}) = \frac{f_i(\mathbf{x}_i)}{\sum_{j \in N} f_j(\mathbf{x}_j)} \quad \forall \mathbf{x} \in \mathbf{R}_+^{LN}, \forall i \in N, \tag{2.1}$$

where $f_i(\cdot) : \mathbf{R}_+^{LN} \rightarrow \mathbf{R}_+$ is an *impact function* that is increasing in its arguments (Skaperdas(1996), Clark and Riis (1998) and Rai and Sarin (2009) for proofs). While the above CSF form is required for the five stated axioms to be satisfied, the impact function itself often varies depending on the model and its assumptions.

The two most common explicit impact functions used within the literature are the power and difference models. Originating in the rent-seeking literature, the power model, first presented by Tullock (1980), equates the impact function as $f(\mathbf{x}_i) = \alpha_i \cdot x_i^\delta$ where $L = 1$, $x_i \in \mathbf{R}_+$ is player i effort put toward the contest, $\alpha_i > 0$ is a positive scalar representing exogenous factors and $\delta > 0$ is commonly known as a *mass effect* factor; that is,

$$\pi_i(\mathbf{x}) = \begin{cases} \frac{\alpha_i \cdot x_i^\delta}{\sum_{j \in N} \alpha_j \cdot x_j^\delta} \quad \forall i \in N & \text{if} \quad \sum_{j \in N} \alpha_j \cdot x_j^\delta > 0; \\ \\ \frac{1}{N} & \text{Otherwise.} \end{cases}$$

The above CSF model is often the workhorse of the conflict theory literature that is also present in other economics fields that incorporate competition in their models such as rent-seeking models, auctions, advertising and sports (Konrad (2007) for a survey). The presence of the parameter α_i is meant to represent some exogenous factor that influences a player's probability of winning the contest that does not stem from effort (Clark and Riis (1998)). Such factors are a player's charisma, favoritism or some biased advantage, pre-game standings and so on. The mass effect variable δ is frequently interpreted as an element that captures the marginal increase in a player's probability of winning from an increase in effort. This variable can also be interpreted as a description of the type of conflict taking place. As δ approaches zero, the influence that effort has on the probability of winning the contest becomes less and the contest converges toward a random lottery. As δ approaches infinity, the contest converges toward a perfectly discriminating contest such as an all-pay auction (Jai et al (2011)).

The other common, albeit less popular, impact function form is known as the difference (Hirshleifer (1988,1989)) form:

$$\pi_i(\mathbf{x}) = \frac{1}{1 + \sum_{j \in N \setminus \{i\}} e^{\delta(\alpha_j \cdot x_j - \alpha_i \cdot x_i)}} \quad \forall i \in N,$$

where variables α and δ are positive scalars. There are a few notable advantages of the difference CSF form that are generally agreed upon. First, from an econometric standpoint, one can easily introduce an additive constant to the impact functions that would normally cause problems for the power form; that is, if $f(\mathbf{x}_i) = x_i$ and the impact function for each $i \in N$ is $f_i(x_i, c) = x_i + c$, where $c > 0$, then the difference CSF form is

$$\pi_i = \frac{1}{1 + \sum_{j \in N} e^{\delta(x_j + c - x_i - c)}} = \frac{1}{1 + \sum_{j \in N} e^{\delta(x_j - x_i)}} \quad \forall i \in N,$$

while the power CSF form is

$$\pi_i = \frac{1}{1 + \frac{(J-1)\cdot c + \sum_{j \in N \setminus \{i\}} x_j}{x_i + c}} \quad \forall i \in N.$$

Second, the functional form of the difference CSF is helpful in creating a decaying effect that territory, environment, climate factors and other causes of fatigue have on a player's effort level. Given the possibilities of imperfect conditions and imperfect information (hidden resources, numerous strongholds and battlefields, clandestine operations and so on), a player may lose a contest yet not lose everything he owns; that is, a player may hide some of his resources so that an opposing player may not win them during a conflict. In other words, the difference CSF form allows a player to not put forth any effort toward the contest and still be able to survive the contest with some nonnegative value. As an example, assume that a contest exists between N players for some exogenous prize and the impact function is again $f(\mathbf{x}_i) = x_i$ for all $i \in N$. If the power CSF is used and player i puts forth zero effort, then

$$\pi_i = \frac{x_i}{x_i + \sum_{j \in N \setminus \{i\}} x_j} = \frac{0}{\sum_{j \in N \setminus \{i\}} x_j} = 0 \quad \forall i \in N.$$

The difference CSF illustrates a different outcome where, again, if player i expends no effort toward the contest, then

$$\pi_i = \frac{1}{1 + \sum_{j \in N \setminus \{i\}} e^{x_j - x_i}} = \frac{1}{1 + \sum_{j \in N \setminus \{i\}} e^{x_j}} \quad \forall i \in N.$$

The two equations show that the difference CSF allows the opportunity for player i to survive the contest with some portion of the prize when he puts no effort into the contest while the power CSF form does not. Therefore, the difference form is said to represent a CSF which equates the proportion of the prize one wins and not the probability of winning the entire prize (Hirshleifer (1988), Skaperdas (1996)).

Likewise, there are two key problems with the difference CSF. The first and probably most obvious problem that arises with the use of the difference CSF is computation. Given the exponential form of the impact function, models using the difference CSF may not produce a tractable set of results. Secondly, a Nash equilibrium is not always guaranteed, when we seek a Nash Equilibrium in the interior of the strategy space (Garfinkel and Skaperdas (2007) have examples).

2.2 Economics of Conflict

As asserted above, many fields of economics use CSFs in their models, such as the development of alliances (Sandler (1999)), coalition games (Jordan (2006) and Piccione and Rubinstein (2007)), defense treaty organizations (Sandler and Hartley (2001)), sports economics (Szymanski (2003)), civil wars (Sambanis (2002) and Skaperdas (2008)), and organized crime (Skaperdas (2001)). Another field that uses CSFs within their models is the economic analysis of conflict and appropriation through warfare. The field itself is divided into many categories from the more unconventional conflicts to the more conventional conflicts.

2.2.1 Unconventional Conflicts

One of the unconventional types of conflict common within the literature is revolutionary insurgency; that is, a conflict between an established economic leader or regime and a group of players set upon usurping power. Early results show that in equilibrium, established regimes will always hire soldiers to defend their current wealth and insurrection will always take place (Grossman (1991,1999)). When a warring side must contend with a non-warring social class, the incentives and relationships between the two groups are such that a regime's level of tyranny and wealth distribution levels are directly linked to the usurping coalition's size and promises of future wealth distribution (Roemer (1985), Grossman (1995)). Anticipation and, hence, expectation of such revolutions can be rationalized as well. Specifically, the occurrence of a revolution depends upon the collective sentiment of a ruled population and the difference between those who are identified as non-activists and activists, known as a "threshold" function. As the threshold function, based upon self-interested preferences, becomes smaller, those identified as non-activists need fewer and fewer incentives to participate within a revolution (Kuran (1989)). There are also more non-conventional forms of insurrection, such as dynamic guerilla warfare models (Brito and Intriligator(1992)) and terrorism (Ferrero (2005)).

Such early results lead to the persistent question of the existence and/or rationale of Hobbesian stability within certain political states, such as anarchy, dynasties and despotism; that is, does the constant threat of war and appropriation lead populations to submit to a despotic government or ruler? The overwhelming answer is that anarchy itself is not stable, yet the reasoning varies depending on the model. Reasons include population growth versus income per capita (Usher (1989)), property rights (Skaperdas (1992), Grossman (1995), Grossman and Kim (1995)), morality (Grossman and Kim (2000)), the technology of predation (Grossman (2002)), technology of warfare and the effectiveness of conflict (Hirshleifer (1995)), the expected time in which warfare begins (Powell (1993)), religion (Ferrero (2008)), spatial distance and coalitions (Jung (2009)) and a competitive security market (Konrad and Skaperdas (2012)).

2.2.2 Conventional Conflicts

The models of unconventional conflicts, such as revolution and insurrection, usually assume that at least the usurper, if not the incumbent ruler as well, acts in a purely parasitic manner; that is, using an economy's resources to win the contest's prize and never investing within the economy itself. A rich set of models exists for conventional conflict scenarios, such as state-to-state conflicts and modern wars of secession, where the decision process involves both the expending of resources toward production of goods and services and/or the appropriation of goods and services, what is known as a *Guns and Butter* economy. One key distinction between contests within a guns and butter economy and other possible contests is the notion of how the prize — the object over which players are contesting — is developed. One class of models involves *exogenous prize* contests in which the players' actions do not affect the prize. The other class deals with *endogenous prize* contests where players' actions directly or indirectly affect the actual value of the contest's prize. A guns and butter economy can be placed in the later class of models.

The earliest attempt, to the author's knowledge, to present an analytic model of such an economy is by the econometrician Trygve Haavelmo in his treatise *A Study in the Theory of Economic Evolution*. Within a series of what he believes are oversights by the economics profession, Haavelmo presents a primordial model of a guns and butter economy influenced by the writings of Vilfredo Pareto. Even though a framework is present, Haavelmo does not produce any startling conclusions except for the intuitive result that the existence of conflict decreases the amount of resources toward goods and services produced (Haavelmo (1954)).

Excluding the work done by Haavelmo, guns and butter conflict models are generally believed to have originated as a distinct field through a series of papers by Jack Hirshleifer (See Skaperdas (1992), Garfinkel and Skaperdas (2006) and Sandler and Hartley (1995)). The guns and butter economy is modeled and solved using an array of CSFs with a focus on whether or not peace is possible. The result that many of these models have in common is known as the *Paradox of Power*: within the interior of the strategy space, poorer players benefit more from conflict than their richer counterparts (Hirshleifer(1988, 1991, 1995)). This conclusion is the product of their derived symmetric levels of effort toward conflict within the Nash equilibrium. The paradox of power holds for both the difference and the power CSF forms when the aim is for a symmetric solution. The existence of asymmetric solutions has been known to be more problematic. Specifically, some models show that all difference CSFs lead to an interior asymmetric Nash equilibrium, yet this notion has been discredited (Dixit (1987) and Garfinkel and Skaperdas (2007)). In terms of the possibility of a peaceful solution, the standard guns and butter economy can have a peaceful outcome when the logit CSF is used and the decisiveness parameter is low. It is demonstrated that a peaceful solution can not exist when the power CSF is applied (Hirshleifer (1991,1995)).

Many of the early models and results are limited by a few key assumptions. First, contests are two party interactions with no consideration of group formation (Noh (2002), Sandler (1999), Skaperdas (1998), and Sandler and Hartley (1995)). Second, it is assumed that full/complete information is held by both players (Fearon(1995), Sánchez-Pagés (2004) and Bester and Wärneryd (2006)). Third, the CSFs used are simplistic in nature in that they ignore the distinction between defensive and offensive measures (Grossman and Kim (1995)). Fourth, issues of timing and repeated interactions are ignored (McBride and Skaperdas (2005) and Bester and Konrad (2004)). Finally, geographical factors — such as distance of conflict and resources, environmental concerns, etc, — are left out (Findlay (1996), Gates (2002), Olsson (2007)).

2.3 Warlord Economies

The literature pertaining to guns and butter economies primarily focuses on battlefield conflicts, where players can implicitly be viewed as standing across from each other on different sides of a line. These types of conflicts are not as prevalent as they once were, especially within and between developed economics (Collier and Sambanis (2005a,b)). Incidentally, most conflicts and civil wars take place within developing economies such as Angola, Somalia, Sierra Leone, Afghanistan and the Republic of Congo and, hence, the standard methods will not effectively model said economies

(Collier and Sambanis (2005a,b) and Ali and Matthews (1999)). A key facet of these economies is that a state government is either too weak to enforce law or is completely non-existent. Individuals within such an economy need protection from the predator-prey system that exists which gives rise to warlords who offer protection to individuals within the economy from other competing warlords. Some studies have also shown that many warlord economies form and function in an almost identical way that many organized crime syndicates do (Reno (1998) and Skaperdas (2001)).

Skaperdas (2002) first developed a model of a "warlord economy" by defining warlords as warmongers who compete against each other over rents, such as oil, diamonds and other natural resources, instead of the production of marketable goods and services. To illustrate the model, let $L \geq 2$ be the number of warlords, $T \in \mathbf{R}_+$ be the total amount of resources or rents, $P \geq 0$ the total number of producers and $\alpha \geq 0$ the tax or "tribute rate" that each producer pays to their associated warlord for protection. Initially, each warlord is given an equal portion of resources; that is, $\frac{T+\alpha \cdot P}{L}$. Warlords can then compete with other warlords for a larger share of rents and producers by employing warriors, denoted by W. The warriors hired and producers by the warlords come from a single population set N such that $N = P + W$.

Under a localized competition model, warlords are spaced equally around a circle where each warlord $\ell \in L$ can only engage his two neighbors, warlord $\ell - 1$ and warlord $\ell + 1$. Let $\beta \in \mathbf{R}_+$ be the wage paid to each warrior, $w_{\ell-1}^{\ell}$ be the number of warriors that warlord ℓ hires to combat the warriors hired by warlord $\ell - 1$ and $w_{\ell+1}^{\ell}$ be the number of warriors that warlord ℓ hires to combat the warriors hired by warlord $\ell + 1$. Warlord ℓ's payoff function is defined as follows:

$$V_{\ell}^{loc} = \left(\frac{w_{\ell-1}^{\ell}}{w_{\ell-1}^{\ell} - w_{\ell}^{\ell-1}} + \frac{w_{\ell+1}^{\ell}}{w_{\ell+1}^{\ell} - w_{\ell}^{\ell+1}} \right) \cdot \left(\frac{T + \alpha \cdot P}{L} \right) - \beta \cdot \left(w_{\ell-1}^{\ell} + w_{\ell+1}^{\ell} \right) \quad \forall \ell \in L. \tag{2.2}$$

Under a globalized competition model, warlords are allowed to engage all other opposing warlords. The payoff function for warlord ℓ is then

$$V_{\ell}^{glob} = \left(\frac{w^{\ell}}{\sum_{j=1}^{L} w^{j}} \right) \cdot \left(\frac{T + \alpha \cdot P}{L} \right) - \beta \cdot \left(w^{\ell} \right) \quad \forall \ell \in L \tag{2.3}$$

Under the localized competition model, a symmetric Nash equilibrium is found where as the number of warlords increases, each warlord hires fewer warriors and receives a smaller payoff while the sum of all warlord payoffs is constant. As the amount of total rents within the economy increases, the number of producers decreases while the number of warriors hired increases; that is, as the total amount of rents increases, warlords prefer the appropriation of these rents over the taxation of producers. Within the globalized competition model, a Nash equilibrium exists that illustrates a conflict which is much more intense than its localized counterpart. Specifically, as the number of warlords increases, not only are more warriors hired, but both individual warlord payoffs and the sum of warlord payoffs decrease. In addition, for a large enough L, it is possible for there to be zero producers and $N = W$.

This model is constructed under seven important assumptions. First is that warlords, and hence warlord competition, are those who compete over natural resources and the taxation of a group of protectorates. Second, warlords are completely parasitic in appropriating the surplus of resources within the economy without investing in its future. Third, the aggregate of natural resources and the collection of producers within the economy are equally distributed before conflict begins. Fourth, warlords hire warriors and protect producers from a single population set. Fifth, warriors are implicitly being paid for their services after conflict from the warlord's post-contest spoils. Sixth, rents are assumed to be fully extracted and do not impose any cost on the warlords. Finally, the model assumes a quasi-linear utility function and, hence, sets the Lagrangian multiplier of the budget constraint equal to one.

Konrad and Skaperdas (2012) also construct a model of warlord competition but direct their attention more on the development of the protection market than the appropriation of an economy's rents. The model evaluates four types of political orders that have existed: anarchy, collective protection and self-governance, competing warlords and a Leviathan-like state. Like Skaperdas (2002), there exists a population of producers that need protection. Unlike Skaperdas (2002), the protection is needed against bandits, who prey upon the producers, and not from other warlords. Producers can choose to defend themselves against bandits or create a union of self-governance to defend themselves as a whole. As an alternative, warlords may exist that offer protection for a fee or "tribute rate". Konrad and Skaperdas show that self-governance produces lower payoffs for producers than under anarchy or warlord competition. In addition, the existence of multiple warlords competing over producers to protect does not increase the quality of protection against bandits, but increases the resources spent on the competition between warlords. Therefore, a single and unified autocratic state is preferred over a competitive market for protection.

2.4 Contribution of the Present Dissertation

The aim of my research is to extend the warlord competition model in a few important directions. First, the model found in Skaperdas (2002) focuses on purely parasitic warlords that do not invest in the economy but only extract an economy's resources through taxing protected producers. This is not the only type of warlord competition present within developing economies. There exist many situations where civil war has erupted between warlords who use an economy's resources for both conflict and investment in the future of the economy as a whole. Such cases can be found, in varying degrees, in the conflicts taken place in Kenya, Democratic Republic of Congo, Somalia, and Northern Ireland (Collier and Sambanis (2005a, 2005b) and Ali and Matthews (1999) for case studies), as well as in the history of organized crime syndicates such as the yakuza and the mafia (Reno (1998) and Skaperdas (2001)).

The model I present here studies an economy where warlords may be non-parasitic in that they can both produce goods and services and forcefully take opposing warlords' profits.

Skaperdas (2002) also assumes that the size of the population and the set of rents or natural resources are shared by all warlords. Again, there are many cases in practice, such as Sierra Leone and Liberia, where warlords control areas endowed with natural resources and a population set that is rarely, if ever, under explicit conflict by opposing warlords. Instead, the points of conflict within these economies are distant from the territories/strongholds of the warlords. Likewise, the budget constraints found in Skaperdas (2002) and Konrad and Skaperdas (2012) do not reflect the true costs of many warlords. Natural resources, such as diamonds, oil, timber and so on, need to be physically extracted in order to be sold and, hence, explicitly and implicitly affect the warlord's resources. Loyal subjects within a warlord's population set may also seek explicit compensation in the short-term, as opposed to a fraction of the warlord's spoils. My model includes both the necessity of natural resource extraction and wage compensation given by warlords to motivate their population.

In addition to this point, the model does not consider the role of geographical distance between the warlords. To the author's knowledge, there has been very little theoretical research on the relationship between geography and conflict[2]. Gates (2002) presents a model including both conflict and geographical distance between an established government and an insurrectionist movement. Using a principal-agent

[2]Findlay (1996) analyzes the role of territorial expansion by illustrating an economy where an individual leader seeks to increase his territory through force, but does not include the possibility of production of goods and services nor direct conflict against another player. Through a series of papers, Brito and Intrillgator (1988,1989,1992) develop a model of a dynamic guerilla warfare which includes movement toward optimal territories for both conflict advantages and the accumulation of membership. The works by Brito and Intrillgator deal strictly within an insurrection scenario against an established government and, again, do not include the possible production of goods and services. More so, their focus is placed on how conflict affects the location of the insurrectionist players and not the effect that the location of conflict has on decision making. Jung (2009) develops an extension of a pillage game with three players to include spatial movement. The model is concerned with the question of alliances and coalitions between the players and not the effects that geography and points of conflict have on the level of conflict.

framework, the focus of the analysis is on the set of rewards and penalties that both the establish government and insurrectionist leader create to attract and maintain supporters that are located along some geographical spectrum. Using Gates (2002) as a theoretical base, Buhaug and Scott (2002) and Buhaug et al (2009) empirically study the effects that the distance between warlords has on the timing and length of civil war. The above models assume that combatants are parasitic and do not include any form of investment into the production of goods and services. The model I present expands on these papers by narrowing in on how the location of conflict affects the expenditures on both the war effort and production of goods and services. In a further extension, I analyze the effect that asymmetric rates of seizure, based on the point of conflict, has on conflict and the production of goods and services by each warlord.

Third, Skaperdas (2002) and Konrad and Skaperdas (2012) use the standard power CSF. Gates (2002) uses the difference CSF to include the effects that geographical distance may have on an insurgent's war effort. Since the model developed below is giving special focus to the effects of geography on conflict and production, the appropriate CSF to use is the difference form presented earlier by Hirshleifer.

CHAPTER 3

Two Warlord Models Including Geographical Distance

3.1 Model Construct

Consider an economy where two *warlords*, A and B, are each in control of a distinct and separate territory. Within each territory of warlord A and B, there is a population of loyal subjects, $N_A \in \mathbf{R}_+$ and $N_B \in \mathbf{R}_+$, and a cache of unextracted natural resources, $R_A \in \mathbf{R}_+$ and $R_B \in \mathbf{R}_+$. Both warlord A's population size N_A and warlord B's population size N_B are taken to be continuous. Each member of the populations N_A and N_B is endowed with a single unit of resource that can be used toward one and only one economic activity. Let V_A and V_B measure the payoffs of warlord A and warlord B.

Each warlord's strategy set, denoted by $\mathcal{S}_A$ and $\mathcal{S}_B$, includes two economic activities: producing goods and services and appropriating goods and services produced by the opposing warlord through force. All decisions by the two warlords are made simultaneously during a one-stage game. In addition, it is assumed that both warlords have complete information in that each warlord knows with full certainty both the game structure as well as his own and the opposing warlord's payoff structure and abundance of resources.

3.1.1 Production of Goods and Services

The production of goods and services by warlord A, denoted by Q_A, is a function of the level of capital invested into production by the warlord A, denoted by $K_A \in \mathbf{R}_+$. This dissertation interprets capital as a basic input into the production of goods and services and does not consider intertemporal issues.

To help facilitate production within his territory, warlord A can invest in the capital stock at a price of $c_k > 0$ per unit of capital K_A. Assuming linearity, Q_A is defined by,

$$Q_A = \theta_A \cdot K_A \qquad \theta_A > 0, \tag{3.1}$$

where θ_A represents the effectiveness that each unit of capital invested by warlord A has on the total quantity produced. Warlord B's production function for goods and services is similarly defined:

$$Q_B = \theta_B \cdot K_B \qquad \theta_B > 0. \tag{3.2}$$

Both θ_A and θ_B are assumed to be equal to 1; that is, $Q_A = K_A$ and $Q_B = K_B$. It is assumed that the goods and services produced by both warlord A and warlord B are sold to an external *purchaser*, who pays a fixed exogenous price of $0 < m < 1$ per each unit of Q_A and Q_B.

Remark It should be emphasized here that the model does not include local consumption of the goods and services being produced on the part of either the two warlords and their respective population sizes. Therefore, all goods and services being produced within the economy are being sold to the external purchaser, regardless of any conflict that may occur.

3.1.2 Appropriation through Force

As opposed to financing his own territory's production, warlord A also has the ability to take revenues earned by the opposing warlord through force. The contest between the two warlords is determined by a *Contest Success Function (CSF).* A CSF is defined by the difference between the impact of each warlord's effort put toward the conflict. The effect or impact of each warlord's effort on the outcome of the conflict is explicitly defined as an *Impact Function.* The impact function of warlord A's effort toward the contest against warlord B is denoted by I_A. The impact function of warlord B's effort toward the contest against warlord A is denoted by I_B.

Definition 1. *Let $\pi_A : \mathbf{R}_+^2 \to \mathbf{R}_+$ be the CSF for the conflict between warlord A and B. From Hirshleifer (1989), the explicit form of π_A is defined as*

$$\pi_A = \frac{e^{\alpha \cdot I_A}}{e^{\alpha \cdot I_A} + e^{\alpha \cdot I_B}} = \frac{1}{1 + e^{\alpha \cdot (I_B - I_A)}}, \tag{3.3}$$

where $0 < \alpha < 1$ is an exogenous mass effect *variable and $\pi_A = 1 - \pi_B$.*

The CSF π_A should not be interpreted as the probability that warlord A will defeat B. Instead, π_A reflects the share of the total prize that warlord A is able to acquire while B acquires the remaining $1 - \pi_A$. From equation (3.3), if warlord A exerts no effort toward the conflict such that $I_A = 0$ while $I_B > 0$, the contest will not necessarily end in his receiving no income; that is, $I_A = 0$ does not imply $\pi_A = 0$.

3.1.3 Resources

Warlord A and B are each able to increase their chances within the conflict by employing *Warriors*, denoted by $W_A \in \mathbf{R}_+$ and $W_B \in \mathbf{R}_+$ respectively. Warriors hired by warlord A are hired from his population N_A and paid a compensating wage of c_w^A. Similarly, warriors hired by warlord B are hired from his population N_B and paid a compensating wage of c_w^B. The compensating wages, set exogenously, of c_w^A

and c_w^B are taken from the gains made by selling of a warlord's extracted natural resource stock $R_A \in \mathbf{R}_{++}$ and $R_B \in \mathbf{R}_{++}$, respectively, and some pre-existing cache of monetary resources $Y_A \in \mathbf{R}_+$ and $Y_B \in \mathbf{R}_+$, respectively. Extracting a unit of natural resources occupies a population unit, denoted by $E_A \in \mathbf{R}_+$ and $E_B \in \mathbf{R}_+$. Warlord A pays each extractor an exogenously set wage of $c_E^A \in \mathbf{R}_+$ and warlord B pays each of his extractors an exogenously set wage of $c_E^B \in \mathbf{R}_{++}$. Let $\hat{R}_A \in \mathbf{R}_+$ and $\hat{R}_B \in \mathbf{R}_+$ be the amount of natural resources that warlord A and warlord B, respectively, chooses to extract.

Assumption 1. *Both warlord A and warlord B are incapable of extracting all of the natural resources such that,*

$$\hat{R}_A < R_A; \tag{3.4}$$

$$\hat{R}_B < R_B. \tag{3.5}$$

Equations (3.4) and (3.5) state that each warlord's level of extracted natural resources is not constrained by the total amount that is endowed within his given territory.

Assumption 2. *Each unit of natural resources extracted is equal to a population unit of extractors. That is, $\hat{R}_A = E_A$ and $\hat{R}_B = E_B$.*

The goal of each warlord is to maximize his own income subject to two constraints. The first is labeled the *Population Constraint* and it is

$$N_A = W_A + E_A. \tag{3.6}$$

Equation (3.6) states that warlord A's decision on how to allocate his population amongst the two economic activities is restricted by the total population within his stronghold. Similarly, equation (3.6) states that everyone within warlord A's populace will be economically active. The second constraint is a *Resource Constraint*:

$$m_R \cdot \hat{R}_A + Y_A = c_k \cdot K_A + c_w^A \cdot W_A + c_E^A \cdot E_A, \tag{3.7}$$

where $m_R \in \mathbf{R}_+$ is the exogenously set price paid, by some external buyer, to each warlord for a single unit of natural resource extracted and sold. The population and resource constraints are similarly defined for warlord B.

Assumption 3. *The price for a unit of natural resource is greater than the cost to extract; that is,* $m_R > c_E^A$ *and* $m_R > c_E^B$.

Given the relationship between the variables $\hat{R}_A$ and E_A from Assumption 2, the population and resource constraints are simplified into a single equation. Let $\sigma_A = m_R - c_E^A$ and $\sigma_B = m_R - c_E^B$. In addition, let $\aleph_A = N_A + \frac{Y_A}{\sigma_A}$ and $\aleph_B = N_B + \frac{Y_B}{\sigma_B}$. Substituting the budget constraint from equation (3.7) into the population constraint from equation (3.6) and using Assumption 2, warlord A's income maximization decision is constrained by the total population at his disposal, N_A, such that

$$\aleph_A = \left(\frac{c_w^A}{\sigma_A} + 1 \right) \cdot W_A + \frac{c_k}{\sigma_A} \cdot K_A. \tag{3.8}$$

Likewise, warlord B's income maximization decision is constrained by the total population at his disposal, N_B, such that

$$\aleph_B = \left(\frac{c_w}{\sigma_B} + 1 \right) \cdot W_B + \frac{c_k}{\sigma_B} \cdot K_B. \tag{3.9}$$

3.1.4 Geography of the Economy

Within each territory, the prevailing warlord has an established stronghold where all economic operations take place. Let the location of warlord A's stronghold be denoted by ℓ_A and the location of warlord B's stronghold be denoted by ℓ_B.

Definition 2. *The* Geography *of the economy is defined as a line of fixed length on an interval* $[0, 1]$ *where* $\ell_A = 0$ *and* $\ell_B = 1$. *Figure 2 illustrates the basics of the economy's geography.*

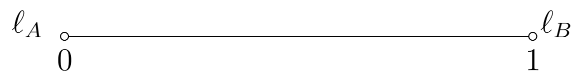


Figure 3.1: Geography of the Economy

Let ℓ_c denote the location that the conflict actually takes place between warlord A and B. The location of conflict ℓ_c is exogenously determined.

3.1.5 Income Gained from Conflict

The income gained by each warlord is the total amount of production profits he is able to defend and the amount he is able to take from the opposing warlord.

Definition 3. *Let* V_A *and* V_B *be the income gained by warlord* A *and warlord* B*, respectively, from both production and warfare. In addition, let the price paid for the sale of warlord production be non-negative; that is,* $m > 0$. *Warlord* A*'s income equation is*

$$V_A = \pi_A \cdot m \cdot (K_A + K_B), \tag{3.10}$$

where warlord B*'s income equation is similarly defined as*

$$V_B = \pi_B \cdot m \cdot (K_A + K_B). \tag{3.11}$$

Given W_B and K_B, warlord A seeks to solve his share of income made on the total production within the economy subject to his population size such that

$$\begin{aligned} \max_{W_A,K_A} V_A &= \pi_A \cdot m \cdot (K_A + K_B) \\ \text{s.t.} \qquad \aleph_A &= \left(\frac{c_w^A}{\sigma_A} + 1 \right) \cdot W_A + \frac{c_k}{\sigma_A} \cdot K_A, \end{aligned} \tag{3.12}$$

warlord B seeks to solve his share of income made on the total production within the economy subject to his population size such that

$$\begin{aligned} \max_{W_B,K_B} V_B &= \pi_B \cdot m \cdot (K_A + K_B) \\ \text{s.t.} \qquad \aleph_B &= \left(\frac{c_w^B}{\sigma_B} + 1 \right) \cdot W_B + \frac{c_k}{\sigma_B} \cdot K_B. \end{aligned} \tag{3.13}$$

Table 3.1 presents a quick list and explanation of all the model's variables.

Remark It is important to point out aspects of the model that may limit the robustness of the results. First, the conflict within the model is solely over the total production of goods and services within the economy and not over the natural resources, an assumption that is not made in the warlord competition model from Skaperdas (2002) in which a common set of natural resources are contested over. One alternative would be to model the competition solely over natural resources, as in Skaperdas (2002), while another would be to construct a dual conflict model in which each warlord can distribute resources toward the conflict over productive goods and services and the conflict over natural resources. The later has been modeled and attempted a number of different ways, but I could not find any tractable solutions. The former alternative is compelling and will possibly be pursued in future research.

Second, the model assumes that each warlord has the same rate of seizure and that it is set exogenously; that is, all of a warlord's production is subject to appropriation. While this assumption is later dropped in the next chapter, there still is relevancy to this symmetric assumption. In the current model, one can envision that the point of

Table 3.1: Summary of Variables within the Model

Endogenous	
Variable	Explanation
W_A, W_B	Number of warriors hired by warlord A and warlord B
K_A, K_B	Units of capital that warlord A and warlord B choose to invest in
E_A, E_B	Number of natural resource extractors hired by warlord A and warlord B
K	Total production of goods and services within the economy
Exogenous	
Variable	Explanation
N_A, N_B	Population sizes for warlord A and warlord B
Y_A, Y_B	Pre-game budget allocated to warlord A and warlord B
R_A, R_B	Set of un-extracted natural resources for warlord A and warlord B
$\hat{R}_A$, $\hat{R}_B$	Amount each warlord chooses to extract from R_A and R_B where $\hat{R}_A < R_A$ and $\hat{R}_B < R_B$
c_w^A, c_w^B	Wage paid to each warrior hired by warlord A and warlord B
c_k	Price per unit of capital
m	Price paid for each good and service produced and sold
m_R	Price paid on each unit of natural resource that is extracted and sold
c_E^A, c_E^B	Wage paid to each extractor hired by warlord A and warlord B
ℓ_A, ℓ_B	Location of warlord A's stronghold and warlord B's stronghold
ℓ_c	Location of conflict
ϕ	Scalar that represents the importance of distance on conflict
α	Mass-effect variable
Include Both Endogenous and Exogenous Variables	
Variable	Explanation
π_A	CSF function for warlord A against warlord B
π_B	CSF function for warlord B against warlord A
V_A, V_B	The income made by warlord A and warlord B from the production and appropriation of goods and services

conflict ℓ_c is the location where both warlord A and warlord B produce their goods and services and, hence, have an equal rate of seizure. For example, two separate individuals or groups may invest their own time and unique level or resources toward the completion of a single project yet also work against each other on securing a larger share of the project's benefit — such as the credit of the project's success. In the context of a civil war or an scenario involving organized crime syndicates, one can conceive two warlords choosing to invest their time, money and resources into some profitable good or services, such as the economy of a local area, while also battling against each other over who gains a larger segment of these profits.

3.1.6 Equilibrium

Before the three models are presented and solved, the appropriate equilibrium concept must be defined. Given that the model is constructed on the assumption of complete information and that both warlords are making their respective decisions simultaneously within a single stage game, a Pure Strategy Nash Equilibrium would be the most appropriate solution concept for the game.

Definition 4. *Let $\mathcal{S_A} \in \mathbf{R}^2_+ \times \mathbf{R}^2_+$ and $\mathcal{S_B} \in \mathbf{R}^2_+ \times \mathbf{R}^2_+$ denote the strategy set for warlords A and B, respectively. Let $s_A = \{W_A, K_A\} \in \mathcal{S_A}$ and $s_B = \{W_B, K_B\} \in \mathcal{S_B}$. In addition, let $\Gamma = \langle \mathcal{S}_A, \mathcal{S}_B, V_A, V_B \rangle$ be the two-player strategic-form game as constructed above. A* Pure Strategy Nash Equilibrium *for the game Γ specifies a set of strategy profiles (s_A^*, s_B^*) which solve maximization problems* (3.18) *and* (3.19)*.*

3.2 Base Model

Following Buhaug et al (2009), the impact function I_A and I_B are therefore explicitly defined as

$$I_A(\ell_c, \ell_A = 0; W_A) = W_A - \phi \cdot (\ell_c - \ell_A)^2 = W_A - \phi \cdot \ell_c^2; \tag{3.14}$$

$$I_B(\ell_c, \ell_B = 1; W_B) = W_B - \phi \cdot (\ell_c - \ell_B)^2 = W_B - \phi \cdot (\ell_c - 1)^2, \tag{3.15}$$

where ϕ is an exogenous scalar such that $1 > \phi > 0$. Equations (3.14) and (3.15) state that the impact function of warlord A's effort toward the contest against warlord B is dependent upon the amount of warriors hired and the distance between warlord A's stronghold and the area of conflict.

From Definition 1 and using equations (3.14) and (3.15), the CSF π_A for the conflict between warlord A and B equals

$$\pi_A = \frac{1}{1 + e^{\alpha \cdot (W_B - W_A + \phi \cdot (2 \cdot \ell_c - 1))}} \tag{3.16}$$

and the CSF π_{BA} for the conflict between warlord B and A equals

$$\pi_B = 1 - \pi_A = \frac{1}{1 + e^{\alpha \cdot (W_A - W_B - \phi \cdot (2 \cdot \ell_c - 1))}}. \tag{3.17}$$

As intuition would predict, π_A is a function increasing in W_A and decreasing in W_B. In addition, as the distance between warlord A's stronghold and the conflict zone location ℓ_c increases, π_A decreases. As the distance between warlord B's stronghold and the conflict zone location ℓ_c increases, π_A increases.

Warlord A then must decide on the optimal levels of W_A and K_A to maximize his income subject to the single constraint from equation (3.8). Similarly, warlord B simultaneously decides on the optimal levels of W_B and K_B to maximize his income subject to to the single constraint from equation (3.9). Using equations (3.10), (3.11),(3.8) and (3.9), each warlord's maximization problem is fully constructed.

Given W_B and K_B, warlord A seeks to solve his share of income made on the total production within the economy subject to his population size such that

$$\begin{aligned} \max_{W_A, K_A} V_A &= \left(\frac{1}{1 + e^{\alpha \cdot (W_B - W_A + \phi \cdot (2 \cdot \ell_c - 1))}} \right) \cdot m \cdot (K_A + K_B) \\ \text{s.t.} \quad \aleph_A &= \left(\frac{c_w^A}{\sigma_A} + 1 \right) \cdot W_A + \frac{c_k}{\sigma_A} \cdot K_A. \end{aligned} \tag{3.18}$$

Warlord B seeks to solve his share of income made on the total production within the economy subject to his population size such that

$$\max_{W_B, K_B} V_B = \left(\frac{1}{1 + e^{\alpha \cdot (W_A - W_B - \phi \cdot (2 \cdot \ell_c - 1))}}\right) \cdot m \cdot (K_A + K_B)$$

$$\text{s.t.} \quad \aleph_B = \left(\frac{c_w^B}{\sigma_B} + 1\right) \cdot W_B + \frac{c_k}{\sigma_B} \cdot K_B. \tag{3.19}$$

Solving maximization problems (3.18) and (3.19) for the choice variables and rearranging the appropriate variables gives rise to the following equilibrium result.

Theorem 1. *Let* $\hat{\ell} = \phi \cdot (2 \cdot \ell_c - 1)$. *Allowing Assumptions* 1, 2 *and* 3 *to hold, in the game* Γ *defined above, given that the following two conditions are satisfied,*

$$\left.\begin{array}{l} \dfrac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)}{c_w^B + \sigma_B} > \Omega_A > \dfrac{(\sigma_B)(\aleph_B)}{c_w^B + \sigma_B} - \dfrac{(\sigma_A)(\aleph_A)}{c_w^A + \sigma_A} \\ \\ \dfrac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)}{c_w^A + \sigma_A} > \Omega_B > \dfrac{(\sigma_A)(\aleph_A)}{c_w^A + \sigma_A} - \dfrac{(\sigma_B)(\aleph_B)}{c_w^B + \sigma_B} \end{array}\right\} \tag{3.20}$$

where

$$\Omega_A = \frac{1}{\alpha}\left(\ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right) + \left(\frac{c_w^A + c_w^B + \sigma_A + \sigma_B}{c_w^B + \sigma_B}\right)\right) - \phi \cdot (2\ell_c - 1);$$

$$\Omega_B = \frac{1}{\alpha}\left(\ln\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right) + \left(\frac{c_w^A + c_w^B + \sigma_A + \sigma_B}{c_w^A + \sigma_A}\right)\right) + \phi \cdot (2\ell_c - 1),$$

an interior pure strategy Nash equilibrium exists where,

1. *Warlord A and warlord B hire warrior numbers of*

$$W_A^* = \left(\frac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B) - \left(c_w^B + \sigma_B\right) \cdot \Omega_B}{c_w^A + c_w^B + \sigma_A + \sigma_B}\right) - \frac{1}{\alpha}; \tag{3.21}$$

$$W_B^* = \left(\frac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B) - \left(c_w^A + \sigma_A\right) \cdot \Omega_B}{c_w^A + c_w^B + \sigma_A + \sigma_B}\right) - \frac{1}{\alpha}. \tag{3.22}$$

2. Warlord A and B will invest in capital levels of

$$K_A^* = \left(\frac{\left(c_w^B+\sigma_B\right)(\sigma_A \cdot \aleph_A)+\left(c_w^A+\sigma_A\right)\cdot\left(c_w^B+\sigma_B\right)\cdot\Omega_A-\left(c_w^A+\sigma_A\right)(\sigma_B \cdot \aleph_B)}{c_k\cdot\left(c_w^A+c_w^B+\sigma_A+\sigma_B\right)}\right); \tag{3.23}$$

$$K_B^* = \left(\frac{\left(c_w^A+\sigma_A\right)(\sigma_B \cdot \aleph_B)+\left(c_w^A+\sigma_A\right)\cdot\left(c_w^B+\sigma_B\right)\cdot\Omega_B-\left(c_w^B+\sigma_B\right)(\sigma_A \cdot \aleph_A)}{c_k\cdot\left(c_w^A+c_w^B+\sigma_A+\sigma_B\right)}\right). \tag{3.24}$$

3. the total production of goods and services within the economy is

$$K^* = K_A^* + K_B^* = \frac{c_w^A + c_w^B + \sigma_A + \sigma_B}{\alpha \cdot c_k} \tag{3.25}$$

Proof. See page 108 in the Appendix. □

The two conditions in equation (3.20) spotlight the importance of the relationship between resources in the economy and the price paid to warriors. As the total population, pre-game budget and/or the profits made on the extraction and selling of a natural resource increase, the higher the wage paid to warriors can possibly be before it causes one of the decision variables to become negative. This problem derives from the fact that hiring a warrior costs the warlord in more ways that just the wage; that is, an increase in hired warriors decreases the warlord's population size available for extraction and yet also requires the extraction of more natural resources in order to pay the warriors' wages.

To illustrate one possible Nash equilibrium, let both population sizes be normalized to one, such that $N_A = N_B = 1$, and let each warlord have a pre-game budget of one, $Y_A = Y_B = 1$. Let the two warlords' warrior and extraction wages be identical, $c_w^A = c_w^B = c_w$ and $c_E^A = c_E^B = c_E$. Finally, let both the cost of hiring a warrior and the cost per unit of capital equal 1/10 of the profit made from the extraction and selling of a natural resource; that is, $c_w = c_k = (1/10) \cdot \sigma$. Finally, let the importance of location be $\phi = 1/4$, the mass-effect variable be $\alpha = 1$ and the point of conflict be located at the midpoint of the two warlord strongholds. From Theorem 1, both warlord A and B hire the same amount of warriors:

$$W_A^* = W_B^* = \frac{1}{11} \cdot \left(\frac{10}{\sigma} - 1 \right),$$

and capital investments of

$$K_A^* = K_B^* = 11.$$

Theorem 1 also states that equilibrium is found in the above example as long as the profit made from the extraction and selling of a natural resource is positive and no greater than ten; that is, $10 \geq \sigma \geq 0$.

The more interesting results of Theorem 1 are the effects that a warlord's population size has on not only his own production and appropriation levels, but on the opposing warlord's, as well. Equations (3.21) and (3.22) illustrate that an increase in N_A will increase the number of warriors hired by both warlord A and warlord B. Therefore, any increase in a warlord's population will cause the total war effort W^* to increase due to both W_A^* and W_B^* increasing.

An increase in population does not have such a positive effect on a warlord's production. Using equations (3.23) and (3.24), an increase in N_A will cause warlord A to increase the amount of capital invested, while warlord B responds by choosing to decrease his investments in capital. The interpretation within this scenario is that an increase in warlord A's population size will have his invest more in both warfare and the production of goods and services. To keep up with the escalated conflict effort by warlord A, warlord B puts more resources into hiring warriors and dedicates less toward the production of goods and services. As a result, the total level of production K^* is not affected by either warlords' resources.

By substituting equation (A.10) into the contest success function defined in equation (3.16), warlord A and B each receive a proportion of K^* dependent upon the wages per warrior hired and the profits made on the extraction and selling of a unit of natural resources.

Theorem 2. *In the game* Γ *defined above, given assumptions* 1 *through* 3 *hold and the two conditions in equation* (3.20) *are satisfied, the proportions of* K^* *that each warlord receive in equilibrium are dependent upon wages paid per warrior hired and the profits made on the extracting and selling of a unit of natural resources; that is,*

$$\pi_A^* = \frac{1}{1 + \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}} \tag{3.26}$$

$$\pi_B^* = 1 - \pi_A^* = \frac{1}{1 + \frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}} \tag{3.27}$$

Proof. See page 121 in the Appendix. □

3.2.1 Comparative Statics

3.2.1.1 Varying Population Sizes and Pre-Game Budgets

Let the point of conflict be equidistant from each warlord's stronghold such that $\ell_c = 1/2$; that is, $(\ell_c - \ell_A)^2 = (\ell_c - \ell_B)^2$ and $\hat{\ell} = 0$. In addition, $c_w^A = c_w^B = c_w$, $c_E^A = c_E^B = c_E$, $Y_A = Y_B$ and $\alpha = 1$. Suppose that both warlord A and warlord B have identical population sizes; that is, $N_A = N_B$. The equilibrium levels of W and K for each warlord are:

$$W_A^* = W_B^* = \frac{N_A}{\frac{c_w}{\sigma} + 1} + \frac{Y_A}{c_w + \sigma} - 1;$$

$$K_A^* = K_B^* = \left(\frac{c_w + \sigma}{c_k}\right).$$

The results above show that both warlords will invest symmetrically. Additionally, an increase in the cost per warrior will decrease the level of conflict and increase the role of production. Because each warlord will mirror the other's number of warriors hired, the proportion of goods and services produced, K^*, will be equally split [1].

Suppose, now, that warlord B's population size is double that of warlord A such that $N_B = 2 \cdot N_A$. The equilibrium levels of W and K are now:

$$W_A^* = \frac{3 \cdot N_A}{\frac{c_w}{\sigma} + 1} + \frac{Y_A}{c_w + \sigma} - 1 = W_B^*;$$

$$K_A^* = \frac{c_w + \sigma}{c_k} - \frac{N_A}{2 \cdot \frac{c_k}{\sigma}} < \frac{c_w + \sigma}{c_k} + \frac{N_A}{2 \cdot \frac{c_k}{\sigma}} = K_B^*.$$

This augmented example shows that when warlord B has double the population size of warlord A, warlord B will invest in more capital than warlord B. Unfortunately for warlord B, each warlord still maintains half of the total production K^* because both warlords are investing equal amounts of resources toward conflict.

To better emphasize this result, it can be assumed instead that warlord B has a population size that is 10 times as great as warlord A's population size; that is, $N_B = 10 \cdot N_A$. The equilibrium levels of W and K are then,

$$W_A^* = \frac{11 \cdot N_A}{\frac{c_w}{\sigma} + 1} + \frac{Y_A}{c_w + \sigma} - 1 = W_B^*;$$

$$K_A^* = \frac{c_w + \sigma}{c_k} - \frac{9 \cdot N_A}{2 \cdot \frac{c_k}{\sigma}} < \frac{c_w + \sigma}{c_k} + \frac{9 \cdot N_A}{2 \cdot \frac{c_k}{\sigma}} = K_B^*.$$

As in the previous example, warlord B's greater population size causes his to invest more into the production of goods and services while warlord A decreases his production levels equivalently. Warlord A instead chooses to spend his resources on

[1] This example, and those to follow, are presented as illustration for the reader. Standard partial derivative analysis is used and can be found in the accompanying proofs in the Appendix chapter.

the hiring of warriors so that both warlords again win half of K^*. The above three examples lead to an important characteristic of the model's equilibrium.

Corollary 1. *In the game* Γ *defined above, given assumptions* 1 *through* 3 *hold and the two conditions in equation* (3.20) *are satisfied, an increase in either a warlord's population size or pre-game budget in relation to the opposing warlord's population size or pre-game budget will cause*

1. *the warlord to spend more on the production of goods and services while the opposing warlord will invest in less;*
2. *no change in the total production of goods and services;*
3. *both warlords will increase their hiring of warriors.*

Proof. See page 122 in the Appendix. □

3.2.1.2 Varying Distance from the Point of Conflict

Let $\alpha = 1$, $Y_A = Y_B$ and $N_B = \sigma \cdot N_A$ where $\sigma > 0$ still holds. In addition, let $\phi = 1$, $c_w^A = c_w^B = c_w$ and $c_{E_A} = c_{E_B} = c_E$. Now allowing ℓ_c to vary from 1/2, suppose that the point of conflict is exogenously set at warlord A's stronghold such that $\ell_c = \ell_A = 0$ and $\hat{\ell} = -1$. The equilibrium levels of W^* and K^* are

$$W_A^* = \frac{(1+\sigma)\cdot N_A}{\frac{c_w}{\sigma}+1} + \frac{Y_A}{c_w+\sigma} - \frac{\phi+2}{2} \quad < \quad \frac{(1+\sigma)\cdot N_A}{\frac{c_w}{\sigma}+1} + \frac{Y_A}{c_w+\sigma} - \frac{2-\phi}{2} = W_B^*;$$

$$K_A^* = \frac{c_w+\sigma}{c_k}\left(\frac{(1-\sigma)\cdot N_A}{2\cdot(\frac{c_w}{\sigma}+1)} + \frac{\phi+2}{2}\right) \quad > \quad \frac{c_w+\sigma}{c_k}\left(\frac{(1-\sigma)\cdot N_A}{2\cdot(\frac{c_w}{\sigma}+1)} + \frac{2-\phi}{2}\right) = K_B^*.$$

The results given by the above equations show that when the point of conflict is set at warlord A's stronghold, warlord A will hire fewer warriors while warlord B will need to hire more warriors because of the distance that needs to be travelled. By hiring a smaller number of warriors, warlord A is given the opportunity to invest in more capital. Likewise, warlord B is unable to invest in greater quantities of goods and services due to the high number of warriors needed to be hired; that is, $K_A^* > K_B^*$.

The opposing extreme of $\ell_c = 0$ is to locate the point of conflict at warlord B's stronghold such that $\ell_c = 1$. The equilibrium levels of W^* and K^* now become

$$W_A^* = \frac{(1+\sigma)\cdot N_A}{\frac{c_W}{\sigma}+1} + \frac{Y_A}{c_W+\sigma} - \frac{2-\phi}{2} \quad > \quad \frac{(1+\sigma)\cdot N_A}{\frac{c_W}{\sigma}+1} + \frac{Y_A}{c_W+\sigma} - \frac{\phi+2}{2} = W_B^*;$$

$$K_A^* = \frac{c_W+\sigma}{c_k}\left(\frac{(1-\sigma)\cdot N_A}{2\cdot(\frac{c_W}{\sigma}+1)} + \frac{2-\phi}{2}\right) \quad < \quad \frac{c_W+\sigma}{c_k}\left(\frac{(1-\sigma)\cdot N_A}{2\cdot(\frac{c_W}{\sigma}+1)} + \frac{\phi+2}{2}\right) = K_B^*.$$

By changing the location of ℓ_c from 0 to 1, the equilibrium results are reversed. Warlord B now chooses to spend less on the hiring of warriors while investing more in capital. Warlord A, having to travel long distances to fight warlord B, must spend more on the hiring of warriors and forgo large investments into the production of goods and services; that is, $K_A^* < K_B^*$.

The illustrative examples above lead to another important characteristic of the model's equilibrium.

Corollary 2. *In the game* Γ *defined above, given assumptions* 1 *through* 3 *hold and the two conditions in equation* (3.20) *are satisfied, as* ℓ_c *increases away from* $\ell_A = 0$ *toward* $\ell_B = 1$*,*

1. *Warlord* B *spends more on the production of goods and services while warlord* A *invests in less;*

2. *the total production of goods and services stays the same;*

3. *Warlord* A *will increase his hiring of warriors while warlord* B *decreases hers;*

4. *the proportion of goods and services that each warlord takes is unaffected.*

Proof. See page 123 in the Appendix. □

3.2.1.3 Exogenous Prices

To begin, the price paid on the selling of a productive good or service m is absent from the equilibrium decision variables of both warlord and the equilibrium CSF; that is,

$$\frac{\partial W_A^*}{\partial m} = \frac{\partial W_B^*}{\partial m} = \frac{\partial K_A^*}{\partial m} = \frac{\partial K_B^*}{\partial m} = \frac{\partial K^*}{\partial m} = 0$$

and

$$\frac{\partial \pi_A}{\partial m} = \frac{\partial \pi_B}{\partial m} = 0.$$

This result is not surprising since the price m affects each warlord's post-conflict value identically.

The per unit price of capital c_k has the expected inverse relationship on both the individual and total production of goods and services. From equation (3.25), an increase in the per unit cost in capital causes the total production of goods and services to decrease: $\partial K^*/\partial c_k = -2\cdot(c_w^A + c_w^B + \sigma_A + \sigma_B)/c_k^2 < 0$. Similarly for warlord A, using equation (3.23)

$$\frac{\partial K_A^*}{\partial c_k} = -\frac{1}{c_k} \cdot K_A^*. \tag{3.28}$$

From definition, K_A^* cannot be negative and, therefore, $\partial K_A^*/\partial c_k < 0$. From equation (3.24),

$$\frac{\partial K_B^*}{\partial c_k} = -\frac{1}{c_k} \cdot K_B^*. \tag{3.29}$$

Again from definition, K_B^* cannot be negative and, therefore, $\partial K_B^*/\partial c_k < 0$.

Theorem 2 states that the proportion of goods and services each warlord will attain within equilibrium is independent of the per unit cost of capital but dependent upon the wages per warrior hired and the costs per extractor hired. Equations (3.26) and (3.27) state that, all other variables remaining constant, an increase in a warlord's wage per warrior results in a decrease in the amount of K^* won while an increase in a warlord's wage per extractor results in an increase in the proportion of K^* gained.

Corollary 3. *In the game* Γ *defined above, given assumptions* 1 *through* 3 *hold and the two conditions in equation* (3.20)*,*

1. *an increase in a warlord's wage paid per warrior hired will decrease the proportion of total goods and services won after conflict;*

2. *the proportion of goods and services that a warlord wins after conflict increases when the opposing warlord's wage paid per warrior hired increases;*

3. *an increase in a warlord's wage paid per extractor hired will increase the proportion of total goods and services won after conflict;*

4. *the proportion of goods and services that a warlord wins after conflict decrease when the opposing warlord's wage paid per extractor hired increases.*

Proof. See page 124 in the Appendix. □

Finding the effects that changes in c_w^A, c_w^B, c_E^A and c_E^B have on each warlord's warrior hiring and capital investment decisions are problematic and are based on variable relationships that have no clear interpretation. The effects of these variables can more easily be seen when there is a unilateral increase in both c_w^A and c_w^B as well c_E^A and c_E^B. From equations (3.21) and (3.22):

$$\frac{\partial W_A^*}{\partial c_w} = \frac{\partial W_B^*}{\partial c_w} = \left(\frac{1}{2}\right)\left((-1)\,\frac{\sigma}{(c_w+\sigma)^2}\right)(\aleph_A + \aleph_B) > 0.$$

The above equation affirms that a unilateral increase in warrior wages will result in both warlords decreasing the number of warriors they hire. While an increase in c_w has an identical effect on both warlords' warrior hiring decision, its effect on each warlord's production of goods and services differs. For warlord A, the effect that an increase in c_w has on K_A^* is

$$\frac{\partial K_A^*}{\partial c_w} = \left(\frac{1}{2 \cdot c_k}\right) \cdot \left(-\phi\left(2 \cdot \ell_c - 1\right)\right) + \frac{1}{\alpha \cdot c_k}$$

where,

$$\frac{\partial K_A^*}{\partial c_w} = \begin{cases} > 0 & \text{if} \quad \frac{1}{\phi \cdot \alpha} + \frac{1}{2} > \ell_c; \\ < 0 & \text{if} \quad \frac{1}{\phi \cdot \alpha} + \frac{1}{2} < \ell_c. \end{cases} \tag{3.30}$$

In similar fashion for warlord B

$$\frac{\partial K_B^*}{\partial c_w} = \left(\frac{1}{2 \cdot c_k}\right) \cdot \left(\phi\left(2 \cdot \ell_c - 1\right)\right) + \frac{1}{\alpha \cdot c_k}$$

where,

$$\frac{\partial K_B^*}{\partial c_w} = \begin{cases} > 0 & \text{if} \quad \frac{1}{2} - \frac{1}{\phi \cdot \alpha} < \ell_c; \\ < 0 & \text{if} \quad \frac{1}{2} - \frac{1}{\phi \cdot \alpha} > \ell_c. \end{cases} \tag{3.31}$$

Equation (3.30) states that the effect that wages paid to warriors has on the equilibrium depends upon the location of conflict, the importance of location variable ϕ and the mass-effect variable α. When the point of conflict passes $(1/2) + (1/\alpha \cdot \phi)$ and moves closer to $\ell_A = 0$, an increase in warrior wages will cause warlord A to spend less on capital and when the point of conflict moves toward $\ell_B = 1$ by passing $(1/2) + (1/\alpha \cdot \phi)$, an increase in the wage paid to warriors will result in a greater investment in capital. Likewise for warlord B, equation (3.31) asserts that when the point of conflict passes $(1/2) - (1/\alpha \cdot \phi)$ and moves closer to $\ell_B = 1$, an increase in warrior wages will cause warlord B to spend more on capital and when the point of conflict moves toward $\ell_A = 0$ by passing $(1/2) - (1/\alpha \cdot \phi)$, an increase in the wage paid to warriors will result in a less investment in capital. Therefore, when the location of conflict lies on the interval $\left(\frac{1}{2} - \frac{1}{\alpha \cdot \phi}, \frac{1}{2} + \frac{1}{\alpha \cdot \phi}\right)$, both warlords will increase their capital investments as the wage paid to warriors increases. These results are mildly intuitive since it was shown earlier that a warlord invests in fewer warriors and in more capital as the location of conflict moves toward his own stronghold.

Regarding a unilateral increase in extracted natural resource profits σ_A and σ_B, equations (3.21) and (3.22) expound the following:

$$\frac{\partial W_A^*}{\partial \sigma_A} = \frac{(N_A + N_B) \cdot \frac{c_w}{\sigma_A^2}}{2 \cdot \left(\frac{c_w}{\sigma_A} + 1\right)^2} - \frac{(Y_A + Y_B)}{2 \cdot (c_w + \sigma_A)^2},$$

where,

$$\frac{\partial W_A^*}{\partial \sigma_A} = \begin{cases} > 0 & \text{if} \quad c_w > \frac{Y_A + Y_B}{N_A + N_B}; \\ < 0 & \text{if} \quad c_w < \frac{Y_A + Y_B}{N_A + N_B}, \end{cases} \tag{3.32}$$

and

$$\frac{\partial W_B^*}{\partial \sigma_B} = \frac{(N_A + N_B) \cdot \frac{c_w}{\sigma_B^2}}{2 \cdot \left(\frac{c_w}{\sigma_B} + 1\right)^2} - \frac{(Y_A + Y_B)}{2 \cdot (c_w + \sigma_B)^2},$$

where

$$\frac{\partial W_B^*}{\partial \sigma_B} = \begin{cases} > 0 & \text{if} \quad c_w > \frac{Y_A + Y_B}{N_A + N_B}; \\ < 0 & \text{if} \quad c_w < \frac{Y_A + Y_B}{N_A + N_B}. \end{cases} \tag{3.33}$$

Equation (**??**) states that the effect σ has on each warlord's number of warriors hired is dependent upon the wages paid to warriors and the ratio of pre-game resources. For an increase in extracted natural resource profit to positively affect the number of warriors hired, the wage paid to each warrior must be more expensive than the ratio of pre-game resources. If the value of pre-game budgets outweighs the size of the populations, the wage must be very high for σ to have a positive effect on W_A^* and W_B^*. If the value of pre-game budgets is outweighed by the size of the populations, the wage does not need to be very high for σ to have a positive effect. In other words, if pre-game budgets are relatively more abundant than the population sizes, there is no real need to extract and sell natural resources and more warriors can be hired.

On the other hand, if pre-game budgets are low, warlords need to extract and sell resources in order to hire more warriors.

In general, the effect that extracted natural resource profits have on the production of goods and services is positive; that is, from equation (3.25),

$$\frac{\partial K^*}{\partial \sigma_A} = \frac{\partial K^*}{\partial \sigma_B} = \frac{1}{\alpha \cdot c_k} \cdot 2 > 0.$$

Therefore, production of goods and services increases within the economy when more is paid for extracted natural resources.

3.3 Examples

3.3.1 Simplified Example with No Distance Effects

Let both warlord A's and warlord B's population sizes be generalized to 1; that is, $N_A = N_B = 1$. Let $c_w^A = c_w^B = c_w$ and $c_E^A = c_E^B = c_E$. Finally, let the mass-effect variable be equal to one, $\alpha = 1$, and the point of conflict be equidistant from each warlord's stronghold; that is, $\ell_c = \frac{1}{2}$ or $\phi = 0$. From equations (3.21) and (3.22), the number of warriors hired by each warlord is

$$W_A^* = \frac{\sigma}{c_w + \sigma} + \frac{Y_A + Y_B}{2 \cdot (c_w + \sigma)} - 1;$$

$$W_B^* = \frac{\sigma}{c_w + \sigma} + \frac{Y_A + Y_B}{2 \cdot (c_w + \sigma)} - 1.$$

Given the above equations, the contest success functions π_{AB} and π_{BA} are equal at $\frac{1}{2}$. From equations (3.25), the total amount of goods and services produced within the economy is

$$K^* = 2 \cdot \frac{c_w + \sigma}{c_k},$$

where, from equations (3.23) and (3.24),

$$K_A^* = \frac{Y_A - Y_B}{2 \cdot c_k} + \frac{c_w + \sigma}{c_k};$$

$$K_B^* = \frac{Y_B - Y_A}{2 \cdot c_w} + \frac{c_k + \sigma}{c_k}.$$

3.3.1.1 No Pre-Game Budgets

Suppose, initially, that each warlord has a pre-game budget of zero such that $Y_A = Y_B = 0$. The number of warriors hired under the Nash Equilibrium are

$$W_A^* = W_B^* = \frac{\sigma}{c_w + \sigma} - 1,$$

and the amount of capital investment is

$$K_A^* = K_B^* = \frac{c_w + \sigma}{c_k}.$$

Theorem 1 states that for the Nash equilibrium to exist, $W_A^*, W_B^* \geq 0$. For this condition to be satisfied, the wage offered to warriors by both warlords must be equal to 0 and neither warlord will hire a single warrior. In other words, for $W_A^*, W_B^* > 0$, then $c_w < 0$; for $W_A^*, W_B^* = 0$, then $c_w = 0$. As a result of hiring no warriors, each warlord invests in capital levels of

$$K_A^* = K_B^* = \frac{\sigma}{c_k},$$

which is greater than zero by definition. The total production within the economy is then $K^* = 2 \cdot \frac{\sigma}{c_k}$.

The above example illustrates that when both warlords are not endowed with a pre-game budget are left with only their current set of resources, neither warlord is able to afford putting any resources toward conflict unless the wage paid to warriors is negative. Therefore, each warlord will dedicate his resources toward the extraction of natural resources and the production of goods and services.

3.3.1.2 An Established Warlord versus an Upstart

Now suppose that warlord A is an *established* warlord that has a pre-game budget of $Y_A = 1$ while warlord, being an *upstart*, has a pre-game budget of $Y_B = 0$. The number of warriors hired under the Nash equilibrium now are

$$W_A^* = W_B^* = \frac{\sigma}{c_w + \sigma} + \frac{1}{2} \cdot \left(\frac{1}{c_w + \sigma} \right) - 1,$$

and the amount of capital investment is

$$K_A^* = \frac{1}{2 \cdot c_k} + \frac{c_w + \sigma}{c_k};$$

$$K_B^* = \frac{c_w + \sigma}{c_k} - \frac{1}{2 \cdot c_k}.$$

For the equilibrium conditions from Theorem 1 to be satisfied, the wage paid to warriors cannot be greater than $\frac{1}{2}$. That is,

$$W_A^* = W_B^* = \begin{cases} \frac{\sigma}{c_w+\sigma} + \frac{1}{2} \cdot \left(\frac{1}{c_w+\sigma} \right) - 1 & \text{if} \quad c_w < \frac{1}{2}; \\ 0 & \text{if} \quad c_w = \frac{1}{2}. \end{cases}$$

While it may be intuitive as to how warlord A can afford to invest in warfare, one maybe perplexed as to how warlord B can afford to hire warriors now when Y_B is still equal to zero. The answer lies within equation (3.21) and (3.22), which state that any increase in either Y_A and/or Y_B will increase both W_A^* and W_B^* proportionately. Therefore, both warlords will hire the same number warriors such that $\pi_{AB} = \pi_{BA} =$

$\frac{1}{2}$. To afford the increase in warrior purchasing, warlord B will decrease the amount of goods and services he produces while warlord A will increase his production of goods and services. By definition, the equilibrium level of capital investment by warlord A satisfies the equilibrium conditions of Theorem 1: from $m_R > c_E$, $K_A^* > 0$. For warlord B's equilibrium level of capital investment to be positive, $\sigma \geq \frac{1}{2} - c_w$; that is,

$$K_B^* = \begin{cases} \frac{c_w+\sigma}{c_k} - \frac{1}{2 \cdot c_k} & \text{if} \quad \sigma > \frac{1}{2} - c_w, c_w < \frac{1}{2}; \\ \\ \frac{\sigma}{c_k} - \frac{1}{2 \cdot c_k} & \text{if} \quad c_w = 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

The above example illustrates that when one warlord is endowed with a pre-game budget and one is not, both warlords will begin to put resources toward conflict while total production within the economy will stay unchanged.

3.3.1.3 Two Established Warlords

Finally, suppose that both warlord A and warlord B are *established* and each have a pre-game budget of $Y_A = Y_B = 1$. The number of warriors hired under the Nash equilibrium are

$$W_A^* = W_B^* = \frac{\sigma}{c_w + \sigma} + \frac{1}{c_w + \sigma} - 1,$$

where W_A^* and W_B^* are positive when the wage paid to warriors is less than 1 and equal to zero when the wage is exactly 1; that is, $W_A^*, W_B^* > 0$ when $1 > c_w$ and $W_A^* = W_B^* = 0$ when $1 = c_w$. Each warlord then invests in capital levels of

$$K_A^* = K_B^* = \frac{c_w + \sigma}{c_k},$$

where $K_A^* = K_B^* \geq 0$ when $\sigma \geq -c_w$. From the assumptions that $c_w \geq 0$ and $m_R > c_E$, K_A^* and K_B^* will always be positive.

The final example presented shows that when both warlords have equal pre-game budgets that are greater than zero, warfare will take place as well as the production of goods and services.

3.3.2 Simplified Example with Distance Effects

Let both warlord A's and warlord B's population sizes still be generalized to 1; that is, $N_A = N_B = 1$. For added simplicity, let $c_w^A = c_w^B = c_w$, $c_{E_A} = c_{E_B} = c_E$, the mass-effect variable again be equal to one, $\alpha = 1$, and the effect of distance on each warrior, ϕ, be equal to $\frac{1}{2}$. From equations (3.21) and (3.22), the number of warriors hired by each warlord is

$$W_A^* = \frac{\sigma}{c_w + \sigma} + \frac{Y_A + Y_B}{2 \cdot (c_w + \sigma)} + \frac{\hat{\ell}}{4} - 1;$$

$$W_B^* = \frac{\sigma}{c_w + \sigma} + \frac{Y_A + Y_B}{2 \cdot (c_w + \sigma)} - \frac{\hat{\ell}}{4} - 1.$$

Given the above equations, the contest success functions π_{AB} and π_{BA} are equal at $\frac{1}{2}$. From equations (3.25), the total amount of goods and services produced within the economy is

$$K^* = 2 \cdot \frac{c_w + \sigma}{c_k},$$

where, from equations (3.23) and (3.24),

$$K_A^* = \frac{Y_A - Y_B}{2 \cdot c_k} + \frac{c_w + \sigma}{2 \cdot c_k} \cdot \left(2 - \frac{\hat{\ell}}{2}\right);$$

$$K_B^* = \frac{Y_B - Y_A}{2 \cdot c_w} + \frac{c_k + \sigma}{2 \cdot c_k} \cdot \left(2 + \frac{\hat{\ell}}{2}\right).$$

3.3.2.1 No Pre-Game Budgets

Suppose, initially, that each warlord has a pre-game budget of 0 such that $Y_A = Y_B = 0$. The number of warriors hired under the Nash Equilibrium are

$$W_A^* = \frac{\sigma}{c_w + \sigma} + \frac{\hat{\ell}}{4} - 1;$$

$$W_B^* = \frac{\sigma}{c_w + \sigma} - \frac{\hat{\ell}}{4} - 1.$$

and the amount of capital investment is

$$K_A^* = \frac{c_w + \sigma}{2 \cdot c_k} \cdot \left(2 - \frac{\hat{\ell}}{2}\right);$$

$$K_B^* = \frac{c_w + \sigma}{2 \cdot c_k} \cdot \left(2 + \frac{\hat{\ell}}{2}\right).$$

Theorem 1 states that for a Nash equilibrium to be found, $W_A^*, W_B^* \geq 0$. For this condition to be satisfied, the wage offered to warriors by warlord A must be

$$W_A^* \geq 0 \quad \rightarrow \quad (\sigma) \cdot \left(\frac{\ell_c - \frac{1}{2}}{\frac{5}{2} - \ell_c}\right) \geq c_w,$$

and the wage offered to warriors by warlord B must be

$$W_B^* \geq 0 \quad \rightarrow \quad (\sigma) \cdot \left(\frac{\frac{1}{2} - \ell_c}{\frac{3}{2} + \ell_c}\right) \geq c_w.$$

In the extreme case where $\ell_c = \ell_A = 0$, warlord A's equilibrium condition from equation (3.20) will not hold unless the wage paid to warriors, c_w, is negative which, by definition, is not possible. In the polar extreme case where $\ell_c = \ell_B = 1$, warlord B's equilibrium condition will not hold unless c_w is negative. In fact, warlord A's equilibrium condition will only be satisfied when $\ell_c \leq \frac{1}{2}$ and warlord B's equilibrium condition will only be satisfied when $\ell_c \geq \frac{1}{2}$. Therefore, the only outcome that is possible is when the point of conflict is $\ell_c = \frac{1}{2}$ and, hence, the wage paid to warriors is $c_w = 0$ and there are no warriors are hired by either warlord; that is, $W_A^* = W_B^* = 0$. As a result of hiring no warriors, each warlord invests in capital levels of

$$K_A^* = K_B^* = \frac{\sigma}{c_k},$$

which is greater than zero by definition. The total production within the economy is then $K^* = 2 \cdot \frac{\sigma}{c_k}$.

The above example illustrates that when both warlords are not endowed with a pre-game budget and are left with only their current set of resources, the only equilibrium outcome occurs when the point of conflict is the half way point between the two warlord strongholds and neither warlord is able to afford putting any resources toward conflict unless the wage paid to warriors is negative. Therefore, each warlord will dedicate his resources toward the extraction of natural resources and the production of goods and services.

3.3.2.2 An Established Warlord versus an Upstart

Now suppose that warlord A is an *established* warlord that has a pre-game budget of $Y_A = 1$ while warlord, being an *upstart*, has a pre-game budget of $Y_B = 0$. The number of warriors hired by warlord A under the Nash equilibrium now is

$$W_A^* = \frac{\sigma}{c_w + \sigma} + \frac{1}{2} \cdot \left(\frac{1}{c_w + \sigma} \right) + \frac{\hat{\ell}}{4} - 1,$$

and the number hired by warlord B is

$$W_B^* = \frac{\sigma}{c_w + \sigma} + \frac{1}{2} \cdot \left(\frac{1}{c_w + \sigma} \right) - \frac{\hat{\ell}}{4} - 1.$$

The level of capital investment by each warlord is thus

$$K_A^* = \frac{1}{2 \cdot c_k} + \frac{c_w + \sigma}{c_k} \cdot \left(2 - \frac{\hat{\ell}}{2}\right);$$

$$K_B^* = \frac{c_w + \sigma}{c_k} \cdot \left(2 + \frac{\hat{\ell}}{2}\right) - \frac{1}{2 \cdot c_k}.$$

For the conditions from Theorem 1 to be satisfied, the wage paid to warriors by warlord A must be

$$W_A^* \geq 0 \quad \rightarrow \quad (\sigma) \cdot \left(\frac{\ell_c - \frac{1}{2}}{\frac{5}{2} - \ell_c}\right) + \frac{1}{\frac{5}{2} - \ell_c} \geq c_w,$$

and the wage paid to warriors by warlord B must be

$$W_B^* \geq 0 \quad \rightarrow \quad (\sigma) \cdot \left(\frac{\frac{1}{2} - \ell_c}{\frac{3}{2} + \ell_c}\right) + \frac{1}{\frac{3}{2} + \ell_c} \geq c_w.$$

If the point of conflict is equidistant from each warlord's stronghold, $\ell_c = 1/2$, the same conditions hold as the did within the above example without the effects of distance. If the point of conflict is located at warlord A's stronghold, $\ell_c = 0$, warlord B's above condition is satisfied when $2 \geq 3 \cdot c_w - (\sigma)$, while warlord A's equilibrium condition is satisfied only if $2 \geq 5 \cdot c_w + (\sigma)$; hence,

$$2 \geq \max\{3 \cdot c_w - (\sigma), 5 \cdot c_w + (\sigma)\}.$$

Likewise, if the point of conflict is located at warlord B's stronghold, $\ell_c = 1$, warlord A's equilibrium condition is satisfied when $2 \geq 3 \cdot c_w - (\sigma)$, while warlord B's condition is satisfied when $2 \geq 3 \cdot c_w - (\sigma)$. Therefore, the above conditions are satisfied when

$$c_w \leq \min\left\{ (\sigma) \cdot \left(\frac{\ell_c - \frac{1}{2}}{\frac{5}{2} - \ell_c}\right) + \frac{1}{\frac{5}{2} - \ell_c}, (\sigma) \cdot \left(\frac{\frac{1}{2} - \ell_c}{\frac{3}{2} + \ell_c}\right) + \frac{1}{\frac{3}{2} + \ell_c} \right\}$$

The equilibrium conditions from Theorem 1 also require that both $K_A^* \geq 0$ and $K_B^* \geq 0$. Specifically for warlord A, the existence of an equilibrium requires

$$K_A^* \geq 0 \quad \rightarrow \quad \frac{1}{2 \cdot \ell_c - 5} - \sigma \leq c_w.$$

The above condition itself will be satisfied when $1 \geq \sigma \cdot (2 \cdot \ell_c - 5)$, which, for any value of $\ell_c \in [0, 1]$, will always be satisfied because of the assumption that $m_R > c_E$. Likewise for warlord B,

$$K_B^* \geq 0 \quad \rightarrow \quad \frac{1}{3 + 2 \cdot \ell_c} + \sigma \geq -c_w,$$

which, for any $\ell_c \in [0, 1]$, is satisfied.

3.3.3 Summary

The examples illustrated above show that the value of each warlord's pre-game budget is crucially important for the game to have a solution.

When the point of conflict is located at an equidistant point of the two strongholds, $\ell_c = \frac{1}{2}$, and the pre-game is equal to zero, Theorem (1) requires each warlord pay each warrior a wage of 0 — which causes each warlord to hire zero warriors and dedicate all their resources toward production of goods and services — for an equilibrium to exist. As pre-game budgets start to increase, the wage paid to each warrior that results in an equilibrium increases. This fact only requires that the total pre-game budget amount increases and not each warlord's own budget; that is, it is possible for an equilibrium to be found that includes warfare even when one warlord has a positive pre-game budget while the opposing warlord has a pre-game budget of zero.

By allowing the point of conflict to vary between zero and one, the above examples show that, again, the existence of pre-game budgets are a necessary component to the game having a solution. More so, the examples imply that as the point of conflict moves closer to one's stronghold, the wage paid to each warrior must decrease to satisfy equation (3.20). Since the wage paid to warriors is the same for both warlords, the wage paid to warriors must decrease as the point of conflict varies from the midpoint for an equilibrium to exist.

3.4 Multi-Warlord Conflict

Consider an economy where $J \geq 2$ warlords, $J = \{1, 2, \ldots, j\}$, are each in control of a distinct territory. Within each territory of warlord $i \in J$, there is a population of loyal subjects, $N_i \in \mathbf{R}_+$ and a cache of unextracted natural resources, $R_i \in \mathbf{R}_+$. Warlord j's population size N_j is understood to be continuous. Each member of the population N_i $\forall i \in J$ is endowed with a single unit of resource that can be used toward one and only one economic activity.

Each warlord's strategy set includes two economic activities: producing goods and services and appropriating goods and services produced by another warlord through force. All decisions by the warlords are made simultaneously during a one-stage game. In addition, it is assumed that all warlords have complete information in that each warlord knows with full certainty both the game structure as well as his own and the other warlords' payoff structure and abundance of resources.

Production of Goods and Services

The production of goods and services by warlord i, denoted by Q_i, is a function of the level of capital invested into production by the warlord i, denoted by $K_i \in \mathbf{R}_+$. This paper interprets capital as a basic input into the production of goods and services and does not consider intertemporal issues.

To help facilitate production within his territory, warlord i can invest in the capital stock at a price of $c_k > 0$ per unit of capital K_i. Assuming linearity, Q_i is defined by,

$$Q_i = \theta_i \cdot K_i, \text{with } \theta_i > 0, \qquad \forall i \in J \tag{3.34}$$

where θ_i represents the effectiveness that each unit of capital invested by warlord i has on the total quantity produced. θ_i is assumed to be equal to 1; that is, $Q_i = K_i$ $\forall i \in J$. It is assumed that the goods and services produced by all warlords are sold to an external *purchaser*, who pays a fixed exogenous price of $m \in \mathbf{R}_+$ per each unit of Q_i.

Resources

Each warlord is able to increase his chances within the conflict by employing *Warriors*, denoted by $W_i \in \mathbf{R}_+$ $\forall i \in J$. Warriors hired by warlord i are hired from his population N_i and paid a compensating wage of c_w^i. The compensating wages are taken from the gains made by the selling of a warlord's extracted natural resource stock $R_i \in \mathbf{R}_{++}$ and some pre-existing cache of monetary resources $Y_i \in \mathbf{R}_+$. Let $E_i \in \mathbf{R}_+$ denote the total number of natural resource extractors hired by warlord i. Each warlord $i \in J$ pays each extractor an exogenously set wage of $c_E^i \in \mathbf{R}_+$. Let $\hat{R}_i \in \mathbf{R}_+$ be the amount of natural resources that warlord i chooses to extract.

Assumption 4. *Each warlord $i \in J$ is incapable of extracting all of the natural resources such that,*

$$\hat{R}_i < R_i, \qquad i \in J \tag{3.35}$$

Equation (3.35) states that each warlord's level of extracted natural resources is not constrained by the total amount that is endowed within his given territory.

Assumption 5. *Each unit of natural resources extracted requires a population unit of extractors. That is, for all $i \in J$, $\hat{R}_i = E_i$.*

The goal of each warlord is to maximize his own income subject to two constraints.

The first is labeled the *Population Constraint* and it is

$$N_i = W_i + E_i, \qquad \forall i \in J. \tag{3.36}$$

Equation (3.36) states that each warlord i's decision on how to allocate his population amongst the two economic activities is restricted by the total population within his stronghold. Similarly, equation (3.36) states that everyone within warlord i's populace will be economically active. The second constraint is a *Resource Constraint*:

$$m_R \cdot \hat{R}_i + Y_i = c_k \cdot K_i + c_w^i \cdot W_i + c_E^i \cdot E, \qquad \forall i \in J \tag{3.37}$$

where $m_R \in \mathbf{R}_+$ is the exogenously set price paid, by some external buyer, to each warlord for a single unit of natural resource extracted and sold.

Assumption 6. *The price for a unit of natural resource is greater than the cost to extract; that is,* $m_R > c_E^i$ *for all* $i \in J$.

Given the relationship between the variables $\hat{R}_i$ and E_i from Assumption 5, the population and resource constraints are simplified into a single equation. For simplicity, let the profit each warlord $i \in J$ earns from the sale of each unit of extracted natural resources be denoted by $\sigma_i = m_R - c_E^i$. Substituting the budget constraint from equation (3.37) into the population constraint from equation (3.36), warlord i's income maximization decision is constrained by the total population at his disposal, N_i, such that

$$N_i + \frac{Y_i}{\sigma_i} = \left(\frac{c_w^i}{\sigma_i} + 1 \right) \cdot W_i + \frac{c_k}{\sigma_i} \cdot K_i. \tag{3.38}$$

Geography of the Economy

Within each territory, the prevailing warlord has an established stronghold where all economic operations take place. Let the location of each warlord i's stronghold be denoted by ℓ_i. Each warlord's stronghold is connected to a point of conflict ℓ_c, which is exogenously determined, by a line of fixed length. Figure 3.4 illustrates the economy.

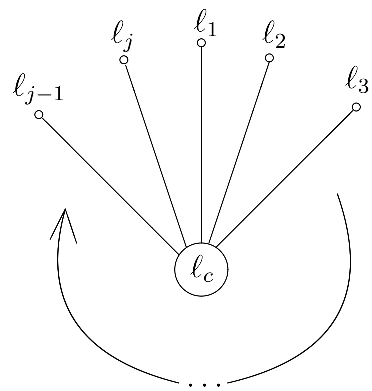


Figure 3.2: Geography of the Multi-warlord Economy

Appropriation through Force

Following Buhaug et al (2009), the impact function I_i $\forall i \in J$ is explicitly defined, using the base model form, as

$$I_i\left(W_i; \ell_i, \ell_c, \phi\right) = W_i - \phi \cdot \left(\ell_c - \ell_i\right)^2, \tag{3.39}$$

where ϕ is an exogenous scalar which represents the implicit costs of geographical distance, such that $1 > \phi > 0$. Equations (3.39) states that the impact function of warlord i's effort toward the contest against all other warlords dependents upon the amount of warriors hired and the distance between warlord i's stronghold and the area of conflict.

As in the previous chapter, each warlord also has the ability to take revenues earned by an opposing warlord through force. The difference form CSF, presented by Hirshleifer (1989), is again used to model situations where contests can result in both contestants surviving but with different portions of the prize.

Definition 5. *For each warlord $i \in J$, let $\pi_i : \mathbf{R}_+^2 \to \mathbf{R}_+$ be the CSF for the conflict between warlord i and all other warlords $h \in J$. From Hirshleifer (1989), the explicit form of π_i is defined as*

$$\pi_i = \frac{e^{\alpha \cdot I_i}}{\sum_{i=1}^{J} e^{I_i}} = \frac{1}{1 + \sum_{h \in J, h \neq i}^{J} e^{\alpha \cdot (I_h - I_i)}}, \tag{3.40}$$

where $0 < \alpha < 1$ is an exogenous mass effect *variable and $\pi_i = 1 - \sum_{h \neq i}^{J} \pi_h$.*

From Definition 5 and using equation (3.39), the CSF π_i is defined as

$$\pi_i = \frac{1}{1 + \Sigma_{h \in J, h \neq i}^{J} e^{\alpha \cdot \left(W_h - W_i + \phi \cdot (\ell_c - \ell_i)^2 - \phi \cdot (\ell_c - \ell_j)^2\right)}} \qquad \forall i \in J, \tag{3.41}$$

Let $J/i = \{1, 2, \dots, i-1, i+1, \dots, j\}$. As intuition would predict, π_i is a function increasing in W_i and decreasing in W_h $\forall h \in J/i$. In addition, as the distance between warlord i's stronghold and the conflict zone location ℓ_c increases, π_i decreases.

Income Gained from Conflict

The income gained by each warlord is the total amount of production profits he is able to defend and the amount he is able to take from the opposing warlords. Let $\mathcal{W} = \{W_1, W_2, \dots, W_j\}$ and $\mathcal{K} = \{K_1, K_2, \dots, K_j\}$. In addition, let $\mathcal{W}/i = \{W_1, W_2, \dots, W_{i-1}, W_{i+1}, \dots, W_j\}$ and $\mathcal{K}/i = \{K_1, K_2, \dots, K_{i-1}, K_{i+1}, \dots, K_j\}$.

Definition 6. *Let V_i be the income gained by each warlord $i \in J$ from both production and warfare. Warlord i's income is*

$$V_i\left(W_i, K_i; \mathcal{W}/i, \mathcal{K}/i\right) = \pi_i \cdot m \cdot \sum_{i=1}^{J} K_i, \qquad \forall i \in J. \tag{3.42}$$

Given $\mathcal{W}/i$ and $\mathcal{K}/i$, warlord i optimizes W_i and K_i to maximizes V_i subject to the constraint (3.38).

Each warlord $i \in J$ then must decide on the optimal levels of W_i and K_i to maximize his income subject to the single constraint from equation (3.38). Using equations (3.42) and (3.38), each warlord's maximization problem is fully constructed.

Assume that each warlord pays identical wages to each extractor and warrior; that is, $c_w^i = c_w^h\ \forall i, h \in J$ and $c_E^i = c_E^h\ \forall i, h \in J$. Then given $\mathcal{W}/i$ and $\mathcal{K}/i$, warlord i seeks to solve his share of income made on the total production within the economy subject to his population size such that

$$\begin{aligned} \max_{W_i, K_i} V_i &= \left(\frac{1}{1 + \sum_{h \in J, h \neq i}^{J} e^{\alpha \cdot \left(W_h - W_i + \phi \cdot (\ell_c - \ell_i)^2 - \phi \cdot (\ell_c - \ell_j)^2\right)}} \right) \cdot m \cdot \sum_{i=1}^{J} K_i \\ \text{s.t.} \qquad N_i + \frac{Y_i}{\sigma} &= \left(\frac{c_w}{\sigma} + 1 \right) \cdot W_i + \frac{c_k}{\sigma} \cdot K_i. \end{aligned} \tag{3.43}$$

Solving maximization problem (3.43) for each warlord $i \in J$ for the choice variables and rearranging the appropriate variables gives rise to the following equilibrium result.

Theorem 3. *Let* $\aleph_i = \left(N_i + \frac{Y_i}{\sigma}\right)$ *for all* $i \in J$. *Suppose assumptions* 1, 2 *and* 3 *with the following condition,*

$$\sum_{i=1}^{J} \aleph_i > \left(\frac{c_w + \sigma}{\sigma} \right) \cdot \Omega_i > \left(\frac{1}{J} \cdot \sum_{i=1}^{J} \aleph_i \right) - \aleph_i \qquad \forall i \in J, \tag{3.44}$$

where

$$\Omega_i = \frac{1}{\alpha \cdot (J-1)^2} + \phi \cdot \left(\frac{\sum_{i=1}^{J} (\ell_c - \ell_i)^2}{J} - (\ell_c - \ell_i)^2 \right).$$

Then a pure strategy Nash equilibrium exists. Additionally, this equilibrium is characterized as follows.

1. *Each warlord* $i \in J$ *hires a number of warriors equal to*

$$W_i^* = \left(\frac{\sigma}{c_w + \sigma} \right) \cdot \frac{\sum_{i=1}^{J} \aleph_i}{J} - \Omega_i \qquad \forall i \in J. \tag{3.45}$$

2. Each warlord $i \in J$ will invest in capital levels of

$$K_i^* = \left(\frac{\sigma}{c_k}\right) \cdot \left(\aleph_i - \frac{\sum_{i=1}^{J} \aleph_i}{J}\right) + \left(\frac{c_w + \sigma}{c_k}\right) \cdot \Omega_i \qquad \forall i \in J. \tag{3.46}$$

3. the total production of goods and services within the economy is

$$K^* = \Sigma_{i=1}^{J} K_i = \left(\frac{c_w + \sigma}{\sigma}\right) \cdot \frac{J}{(J-1)^2} \tag{3.47}$$

Proof. See page 126 in the appendix. □

As in the previous chapter when the base form of the impact function was used, Theorem 3 shows that a change in a warlord's (exogenously given) resources has the expected effect. Any increase in $\aleph_i$ $\forall i \in J$ will cause every warlord to increase his hiring of warriors while an increase in $\aleph_h$ $\forall h \neq i \in J$ will make warlord i decrease his capital investment; that is, the *paradox of power* is still present when there are $J \geq 2$ warlords present. Likewise, the farther warlord i's stronghold is from the point of conflict, the more he will choose to spend on hiring warriors and less on the production of goods and services. As another warlord $h \neq i$ is located further away from the point of conflict, warlord i will need to hire fewer warriors and, therefore, increase his investment in capital.

The more interesting results of Theorem 3 come with an increase in the number of warlords J. Equation (3.45) demonstrates that each individual warlord will put increasing amounts of resources toward warfare as the number of warlords involved gets larger. As a result, which is confirmed by both equation (3.46) and equation (3.47), the increasing dedication to conflict by each warlord decreases his own production of goods and services.

The economy is then worse off, if the social welfare is being defined as increasing with production of goods and services and decreasing with warfare, when there are fewer warlords present. More so, it is better for the economy as a whole to have warlords closer to the point of conflict than further away. In other words, theorem 3 elucidates that conflicts will be more intense when the number of combatants are many and further away from each other. Conflict will de-escalate when either fewer combatants are present and/or closer to each other.

By substituting the above equilibrium equations into the contest success function defined in equation (3.40), each warlord i receives a proportion of K^* dependent only upon the total number of warlords involved within the conflict.

Theorem 4. *Suppose assumptions* 1, 2 *and* 3 *and the condition in equation* (3.44). *The proportions of* K^* *that each warlord receives in equilibrium are dependent the number of warlords involved in the game:*

$$\pi_i^* = \frac{1}{J}. \tag{3.48}$$

Given that warrior and extractor wages are held to be symmetric between warlords, theorem 4 simply states that each warlord $i \in J$ obtains an equal division of the economy's total production and its division decreases as the number of warlords increases.

Proof. See page 133 in the appendix. □

3.5 Gates-Logit Model

Returning to the original two warlord case, the impact function from Gates (2002) can be constructed and implemented into the above guns-and-butter model that again has the properties of $\frac{\partial I_A}{\partial W_A} > 0$, $\frac{\partial I_B}{\partial W_B} > 0$, $\frac{\partial I_A}{\partial \ell_c} < 0$ and $\frac{\partial I_B}{\partial \ell_c} > 0$. Recalling $\ell_A = 0$ and

$\ell_B = 1$, the newly constructed impact functions I_{AB} and I_{BA} are explicitly defined as

$$I_A = \alpha + \ln(W_A) - \phi \cdot (\ell_c - \ell_A)^2 = \alpha + \ln(W_A) - \phi \cdot \ell_c^2; \tag{3.49}$$

$$I_B = \alpha + \ln(W_B) - \phi \cdot (\ell_c - \ell_B)^2 = \alpha + \ln(W_B) - \phi \cdot (\ell_c - 1)^2. \tag{3.50}$$

where ϕ is again an exogenous scalar such that $1 > \phi > 0$ and $\ell_c \in [0, 1]^2$.

From equations (3.49), (3.50) and (3.3), the CSF π_A for the conflict between warlord A and B equals

$$\begin{aligned} \pi_A &= \frac{1}{1 + e^{I_B - I_A}} \\ &= \frac{1}{1 + e^{\ln(W_B) - \ln(W_A) + \phi \cdot (\ell_c^2 - (\ell_c - 1)^2)}} \\ &= \frac{1}{1 + \frac{W_B}{W_A} \cdot e^{\phi \cdot (2\ell_c - 1)}} \end{aligned} \tag{3.51}$$

and the CSF π_{BA} for the conflict between warlord B and A equals

$$\pi_B = 1 - \pi_A = \frac{1}{1 + \frac{W_A}{W_B} \cdot \frac{1}{e^{\phi(2\ell_c - 1)}}} \tag{3.52}$$

[2]Gates (2002) uses slightly modified versions of equations (3.49) and (3.50) to include two stochastic elements such that

$$\begin{aligned} I_A &= \alpha + \ln(W_A) - (\ell_c - \ell_A)^2 + \eta_A; \\ I_B &= \alpha + \ln(W_B) - (\ell_c - \ell_B)^2 + \eta_B. \end{aligned}$$

To derive the logit success function, the stochastic element of the impact function is featured. The cumulative density function of the difference between the two stochastic elements, $F(\eta_B - \eta_A)$ is assumed to have the following logistic form,

$$F(\eta_B - \eta_A) = \frac{e^{\eta_B - \eta_A}}{1 + e^{\eta_B - \eta_A}}.$$

The contest success function is derived by applying the above impact functions into the cumulative density function:

$$F(\eta_B - \eta_A) = \pi_A = \frac{\frac{W_A}{W_B}}{\frac{W_A}{W_B} + e^{(\ell_c - \ell_A)^2 - (\ell_c - \ell_B)^2}},$$

which is very similar to the formula in equation (3.51).

As stated previously, the new impact functions still result in π_A being an increasing function in W_A and a decreasing function in W_B. In addition, as the distance between warlord A's stronghold and the conflict zone location ℓ_c increases, π_A decreases. As the distance between warlord B's stronghold and the conflict zone location ℓ_c increases, π_A increases.

Let Assumptions 1 through 3 still hold as well as the structure of game Γ defined above. Solving maximization problems (3.18) and (3.19), given the alternative forms of π_{AB} and π_{BA} defined in equations (3.51) and (3.52), for the choice variables and rearranging the appropriate variables gives rise to the following equilibrium result.

Theorem 5. *Let $\hat{\ell} = \phi \cdot (2 \cdot \ell_c - 1)$. In the game Γ defined above, given assumptions 1 through 3 hold and the following condition is satisfied,*

$$\left(1 + \frac{2}{\sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)}}\right) > \left(\frac{\sigma_B}{\sigma_A}\right)\left(\frac{\aleph_B}{\aleph_A}\right) > \frac{1}{1 + 2 \cdot \sqrt{e^{\hat{\ell}}\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)}} \tag{3.53}$$

an interior pure strategy Nash equilibrium exists where,

1. *Warlord A and warlord B hire warrior numbers of*

$$W_A^* = \frac{1}{2} \cdot \left(\frac{1}{c_w^A + \sigma_A}\right) \cdot \left(\frac{(\sigma_A \cdot \aleph_A) + (\sigma_B \cdot \aleph_B)}{1 + \left(\frac{1}{\sqrt{e^{\hat{\ell}}}}\right) \cdot \sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}}\right); \tag{3.54}$$

$$W_B^* = \frac{1}{2} \cdot \left(\frac{1}{c_w^B + \sigma_B}\right) \cdot \left(\frac{(\sigma_A \cdot \aleph_A) + (\sigma_B \cdot \aleph_B)}{1 + \sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}}\right). \tag{3.55}$$

2. Warlord A and B will invest in capital levels of

$$K_A^* = \frac{1}{2 \cdot c_k} \cdot \left(\frac{\left(1 + \frac{2}{\sqrt{e^{\hat{\ell}}}} \cdot \sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}\right) \cdot (\sigma_A \cdot \aleph_A) - \sigma_B \cdot \aleph_B}{1 + \left(\frac{1}{\sqrt{e^{\hat{\ell}}}}\right) \cdot \sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}} \right); \quad (3.56)$$

$$K_B^* = \frac{1}{2 \cdot c_k} \cdot \left(\frac{\left(1 + 2 \cdot \sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}\right) \cdot (\sigma_B \cdot \aleph_B) - \sigma_A \cdot \aleph_A}{1 + \sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}} \right). \quad (3.57)$$

3. the total production of goods and services within the economy is

$$K^* = \frac{1}{2} \cdot \left(\frac{\sigma_A \cdot \aleph_A + \sigma_B \cdot \aleph_B}{c_k} \right). \quad (3.58)$$

Proof. See page 133 in the Appendix. □

To illustrate a few possible Nash equilibria, let both population sizes be normalized to one, such that $N_A = N_B = 1$, and let each warlord have a pre-game budget of one, $Y_A = Y_B = 1$. In addition, let $c_w^A = c_w^B = c_w$ and $c_E^A = c_E^B = c_E$ where both the cost of hiring a warrior and the cost per unit of capital equal 1/10 of the profit made from the extraction and selling of a natural resource; that is, $c_w = c_k = (1/10) \cdot (\sigma)$. Finally, let the importance of location be $\phi = 1/4$. Figure 3.2 illustrates the results of Theorem 5 for the three points of conflict of $\ell_A = 0$, $\frac{1}{2}$ and $\ell_B = 1$.

Remark Through a series of examples, the existence of a Nash equilibrium in the Base model was shown to be dependent upon the the total pre-game budget being greater than zero. Interestingly, the existence of a Nash equilibrium in the Gates-logit Model is not fully dependent upon positive pre-game budgets. It can further be shown that equilibria can exist where both warlords have a pre-game budget of zero.

Let $N_A = N_B = 1$, $Y_A = Y_B = 0$, $\alpha = 1$ and $\phi = 1/2$. In addition, let $c_w^A = c_w^B = c_w$

Table 3.2: Three Simple Examples for the Logit-CSF

	$\ell_c = 0$	$\ell_c = \frac{1}{2}$	$\ell_c = 1$
Equ. Con.	$3.27 \geq 1 \geq 0.36$	$3 \geq 1 \geq 0.33$	$2.77 \geq 1 \geq 0.31$
π_A	0.532	0.5	0.469
π_B	0.469	0.5	0.532
W_A^*	$(0.43)\left(1+\frac{1}{\sigma}\right)$	$(0.45)\left(1+\frac{1}{\sigma}\right)$	$(0.47)\left(1+\frac{1}{\sigma}\right)$
W_B^*	$(0.47)\left(1+\frac{1}{\sigma}\right)$	$(0.5)\left(1+\frac{1}{\sigma}\right)$	$(0.43)\left(1+\frac{1}{\sigma}\right)$
W^*	$(0.5)\left(1+\frac{1}{\sigma}\right)$	$(0.5)\left(1+\frac{1}{\sigma}\right)$	$(0.5)\left(1+\frac{1}{\sigma}\right)$
K_A^*	$(5.33)\left(1+\frac{1}{\sigma}\right)$	$(5)\left(1+\frac{1}{\sigma}\right)$	$(4.67)\left(1+\frac{1}{\sigma}\right)$
K_B^*	$(4.67)\left(1+\frac{1}{\sigma}\right)$	$(5)\left(1+\frac{1}{\sigma}\right)$	$(5.33)\left(1+\frac{1}{\sigma}\right)$
K^*	$(10)\left(1+\frac{1}{\sigma}\right)$	$(10)\left(1+\frac{1}{\sigma}\right)$	$(10)\left(1+\frac{1}{\sigma}\right)$

and $c_E^A = c_E^B = c_E$. From equation (3.53), an interior Nash equilibrium exists when,

$$\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\right) \geq 1 \geq \left(\frac{1}{1+2\cdot\sqrt{e^{\hat{\ell}}}}\right).$$

Assume at first that the point of conflict is set at $\ell_c = 0$ such that $\sqrt{e^{\hat{\ell}}} = 0.78$. Therefore,

$$3.56 \geq 1 \geq 0.39.$$

Adjusting the point of conflict to the polar opposite position of $\ell_c = 1$ such that $\sqrt{e^{\hat{\ell}}} = 1.28$:

$$2.56 \geq 1 \geq 0.28.$$

From the above simplified examples, it is possible to find an equilibrium when both warlords have pre-game budgets equal to zero. These examples give different results from those examples shown for the Base model with the same specifications; that is, positive pre-game budgets are not required for a Nash equilibrium being found.

By substituting equation (A.47) into the contest success function defined in equation (3.69), warlord A and B each receive a proportion of K^* dependent upon the point of conflict ℓ_c.

Theorem 6. *In the game* Γ *defined above, given assumptions through* 1 *and* 3 *hold and the condition in equation* (3.53) *is satisfied, the proportions of* K^* *are divided as such*

$$\pi_A^* = \frac{1}{1+\sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}} \tag{3.59}$$

$$\pi_B^* = 1-\pi_A^* = \frac{1}{1+\left(\frac{1}{\sqrt{e^{\hat{\ell}}}}\right) \cdot \sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}} \tag{3.60}$$

Proof. See page 144 in the Appendix. □

Like in the Base model, Theorem 6 affirms that the proportion of goods and services won by each warlord after conflict is dependent on both the location of the point of conflict and the wages paid per worker and extractor hired. In terms of geographical distance, a warlord gains a greater proportion of total goods and services won when the location of the point of conflict is closer to his stronghold.

Corollary 4. *Given assumptions* 1 *through* 3 *hold and the condition in equation* (3.53) *is satisfied, warlord* B*'s proportion of total goods and services gained through conflict increases while warlord* A*'s proportion decreases when the point of conflict* *moves away form $\ell_A = 0$ and closer to $\ell_c = 1$:*

$$\frac{\partial \pi_A}{\partial \ell_c} = \frac{-\phi \cdot \sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}}{\left(1 + \sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}\right)^2} < 0; \tag{3.61}$$

$$\frac{\partial \pi_B}{\partial \ell_c} = \frac{\phi \cdot \sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}}{\sqrt{e^{\hat{\ell}}}\left(1 + \frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}\right)^2} > 0. \tag{3.62}$$

As in the Base model, an increase in a warlord's wage paid per warrior hired will decrease the proportion of total goods and services awarded after conflict and increases the opposing warlord's proportion. Likewise, an increase in a warlord's wage paid per extractor hired will increase the proportion of total goods and services won through conflict and decreases the opposing warlord's proportion.

Corollary 5. *Given assumptions* 1 *through* 3 *hold and the condition in equation* (3.53) *is satisfied,*

1. *an increase in a warlord's wage paid per warrior hired will decrease the proportion of total goods and services won after conflict;*

2. *the proportion of goods and services that a warlord wins after conflict increases when the opposing warlord's wage paid per warrior hired increases;*

3. *an increase in a warlord's wage paid per extractor hired will increase the proportion of total goods and services won after conflict;*

4. *the proportion of goods and services that a warlord wins after conflict decrease when the opposing warlord's wage paid per extractor hired increases.*

Proof. See page 145 in the Appendix. □

3.5.1 Comparative Statics

3.5.1.1 Varying Population Sizes and Pre-Game Budgets

An increase in either warlord's population size and/or pre-game budget affects each warlord's individual decisions as well as the aggregate. Using the results in both Theorem 5 and 6, an increase in either population size, N_A and/or N_B, or pre-game budget, Y_A and/or Y_B, increases the number of warriors hired by both warlords:

$$\frac{\partial W_A^*}{\partial N_A} = \frac{\sigma_A}{2\left(c_w^A + \sigma_A\right)}\left(1 - \pi_A\right) > 0 \qquad \frac{\partial W_A^*}{\partial N_B} = \frac{\sigma_B}{2(c_w^A + \sigma_A)}\left(1 - \pi_A\right) > 0$$

$$\frac{\partial W_B^*}{\partial N_A} = \frac{\sigma_A}{2\left(c_w^B + \sigma_B\right)}\left(\pi_A\right) > 0 \qquad \frac{\partial W_B^*}{\partial N_B} = \frac{\sigma_B}{2(c_w^B + \sigma_B)}\left(\pi_A\right) > o$$

$$\frac{\partial W_A^*}{\partial Y_A} = \frac{\partial W_A^*}{\partial Y_B} = \frac{1}{2\left(c_w^A + \sigma_A\right)}\left(1 - \pi_A\right) > 0 \qquad \frac{\partial W_B^*}{\partial Y_A} = \frac{\partial W_B^*}{\partial Y_B} = \frac{1}{2(c_w^A + \sigma_A)}\left(1 - \pi_A\right) > 0$$

In regards to capital investment, an increase in either N_A or Y_A will cause an increase in K_A^* and a decrease in K_B^*,

$$\frac{\partial K_A^*}{\partial N_A} = \frac{\sigma_A}{2c_k}\left(1 - \pi_A\right)\left(1 + \frac{2}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}\right) > 0; \qquad \frac{\partial K_B^*}{\partial N_A} = (-1)\left(\frac{\sigma_A}{2c_k}\right)\left(\pi_A\right) < 0;$$

$$\frac{\partial K_A^*}{\partial Y_A} = \frac{1}{2c_k}\left(1 - \pi_A\right)\left(1 + \frac{2}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}\right) > 0; \qquad \frac{\partial K_B^*}{\partial Y_A} = (-1)\left(\frac{1}{2c_k}\right)\left(\pi_A\right) < 0,$$

while an increase in either N_B or Y_B causes an increase in K_B^* and a decrease in K_A^*:

$$\frac{\partial K_A^*}{\partial N_B} = (-1)\left(\frac{\sigma_B}{2c_k}\right)\left(1 - \pi_A\right) > 0; \qquad \frac{\partial K_B^*}{\partial N_B} = \frac{\sigma_A}{2c_k}\left(\pi_A\right)\left(1 + 2\sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}\right) < 0;$$

$$\frac{\partial K_A^*}{\partial Y_B} = (-1)\left(\frac{1}{2c_k}\right)\left(1 - \pi_A\right) > 0; \qquad \frac{\partial K_B^*}{\partial Y_B} = \frac{1}{2c_k}\left(\pi_A\right)\left(1 + 2\sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}\right) < 0.$$

As opposed to the results of the Base model, population sizes and pre-game budgets do have an affect on the total amount of goods and services. Equation (3.58) shows that any increase in either warlord's population sizes or pre-game budget will cause K^* to increase:

$$\frac{\partial K^*}{\partial N_A} = \frac{\sigma_A}{2c_k} > 0; \qquad \frac{\partial K^*}{\partial N_B} = \frac{\sigma_B}{2c_k} > 0;$$

$$\frac{\partial K^*}{\partial Y_A} = \frac{1}{2c_k} > 0; \qquad \frac{\partial K^*}{\partial Y_B} = \frac{1}{2c_k} > 0.$$

Therefore, even though an increase in a variable such as N_A will cause an increase in K_A^* and a decrease in K_B^*, the increase in warlord A's population sizes will have an increasing affect on the total amount of goods and services produced within the economy.

3.5.1.2 Varying Point of Conflict and Importance of Geographical Distance

From Theorem 6, the effect that the variable ϕ has on the proportion of goods and services each warlord receives after conflict depends on the location of the point of conflict. From equation (3.59),

$$\frac{\partial \pi_A}{\partial \phi} = \frac{-\left(\ell_c - \frac{1}{2}\right) \cdot \sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}}{\left(1 + \sqrt{e^{\hat{\ell}} \cdot \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}}\right)^2}, \tag{3.63}$$

which is positive when the location of the point of conflict is less than 1/2 and negative when the location of the point of conflict is greater than 1/2. From equation (3.60),

$$\frac{\partial \pi_B}{\partial \phi} = \frac{\left(\ell_c - \frac{1}{2}\right) \cdot \sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}}{\sqrt{e^{\hat{\ell}}} \left(1 + \frac{1}{\sqrt{e^{\hat{\ell}}}} \sqrt{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}}\right)^2}, \tag{3.64}$$

which is positive when the location of the point of conflict is greater than 1/2 and negative when the location of the point of conflict is less than 1/2. The above two equations demonstrate that when the point of conflict is past the midpoint and closer to $\ell_B = 1$, an increase in the effects of geographical distance, ϕ, will decrease the proportion of goods and services awarded to warlord A and increases the amount to warlord B after conflict. Oppositely, when the point of conflict is located before the midpoint and closer to $\ell_A = 0$, an increase in ϕ will increase the proportion of goods and services awarded to warlord A and decrease the size won by warlord B after conflict.

From equations (3.54) and (3.60), the equilibrium number of warriors hired by warlord A is

$$W_A^* = \frac{\sigma_A \cdot \aleph_A + \sigma_B \cdot \aleph_B}{2\left(c_w^A + \sigma_A\right)} \cdot \pi_B, \tag{3.65}$$

which, from equation (3.62), increases as the point of conflict moves away from his stronghold, $\ell_A = 0$, and closer to warlord B's stronghold, $\ell_B = 1$. Furthermore, from equations (3.55) and (3.61), the equilibrium number of warriors hired by warlord B is

$$W_B^* = \frac{\sigma_A \cdot \aleph_A + \sigma_B \cdot \aleph_B}{2\left(c_w^B + \sigma_B\right)} \cdot \pi_A, \tag{3.66}$$

which, from equation (3.61), decreases as the point of conflict moves away from warlord A's stronghold and closer his own. As in the Base model, the above two equations illustrate that as the point of conflict moves away from a warlord's stronghold, the greater the distance that needs to be traveled by each warrior hired and, hence, the less effective each warrior is against the opposing warlord's hired army. Therefore, the further the point of conflict is from a warlord's, the greater number of warriors he needs to hired.

The effect that the variable ϕ has on the number of warriors hired by both warlord A and B is also dependent upon the location of the point of conflict. Again, from equations (3.65) and (3.64),

$$\frac{\partial W_A^*}{\partial \phi} = \begin{cases} > 0 & \text{if} \quad \ell_c > \frac{1}{2}; \\ < 0 & \text{if} \quad \ell_c < \frac{1}{2}, \end{cases}$$

and from equations (3.66) and (3.63),

$$\frac{\partial W_B^*}{\partial \phi} = \begin{cases} > 0 & \text{if} \quad \ell_c < \frac{1}{2}; \\ < 0 & \text{if} \quad \ell_c > \frac{1}{2}. \end{cases}$$

In other words, when the point of conflict is located closer to warlord A's stronghold, an increase in the effect of the point of conflict, ϕ, will cause warlord B to increase the number of warriors hired — due to the increasing negative affect that distance has on warlord B's warriors — and warlord A to decrease the number of warriors hired. On the other hand, when the point of conflict is located closer to warlord B's stronghold, an increase in ϕ will cause warlord B to decrease the number of warriors hired while warlord A increases the number.

The relationship between ℓ_c and a warlord's investment in capital is quite the opposite of its relationship with the number of warriors hired. From equation (3.56),

$$\begin{aligned}
\frac{\partial K_A^*}{\partial \ell_c} &= \left(\frac{\phi\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{\sqrt{e^{\hat{\ell}}}\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)^2}\right)\left(\left(1+\frac{2}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)(\sigma_A)(\aleph_A)\right) \\
&\quad -\left(\frac{\phi\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{\sqrt{e^{\hat{\ell}}}\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)^2}\right)\left((\sigma_B)\left(N_B-\frac{Y_B}{\sigma_B}\right)\right) \\
&\quad -\left(\frac{1}{1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}\right)\cdot\left(\frac{2\phi}{\sqrt{e^{\hat{\ell}}}\sqrt{\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)(\sigma_A)(\aleph_A)\gtreqless 0 \\
&= \left(\frac{\phi\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{\sqrt{e^{\hat{\ell}}}\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)}\right)\left(\left(1+\frac{2}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)(\sigma_A)(\aleph_A)\right) \\
&\quad -\left(\frac{\phi\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{\sqrt{e^{\hat{\ell}}}\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)}\right)\left((\sigma_B)\left(N_B-\frac{Y_B}{\sigma_B}\right)\right) \\
&\quad -\left(\frac{2\phi}{\sqrt{e^{\hat{\ell}}}\sqrt{\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)(\sigma_A)(\aleph_A)\gtreqless 0 \\
&= \left(1+\frac{2}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)(\sigma_A)(\aleph_A)-(\sigma_B)\left(N_B-\frac{Y_B}{\sigma_B}\right) \\
&\quad -2\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)(\sigma_A)(\aleph_A)\gtreqless 0 \\
&= -\sqrt{e^{\hat{\ell}}(c_w^A+\sigma_A)}(\sigma_A)\left(N_A+\frac{Y_A}{\sigma_A}\right)-\sqrt{e^{\hat{\ell}}(c_w^A+\sigma_A)}(\sigma_B)(\aleph_B)<0.
\end{aligned}$$

As the point of conflict moves from $\ell_A = 0$ to $\ell_B = 1$, the amount of goods and services produced by warlord A decreases. From equation (3.57),

$$\frac{\partial K_B^*}{\partial \ell_c} = -\left(\frac{\phi \cdot \sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}{\left(1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)^2}\right)\left(1+2\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)(\sigma_B)(\aleph_B)$$

$$+\left(\frac{1}{1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)\left(2\phi \cdot \sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)(\sigma_B)(\aleph_B)$$

$$+\left(\frac{\phi \cdot \sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}{\left(1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)^2}\right)(\sigma_A)\left(N_A+\frac{Y_B}{\sigma_A}\right) \gtreqless 0$$

$$= -\left(1+2\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)(\sigma_B)(\aleph_B)$$

$$+2\left(1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)(\sigma_B)(\aleph_B)$$

$$+(\sigma_A)(\aleph_A) \gtreqless 0$$

$$= (\sigma_B)(\aleph_B)+(\sigma_A)(\aleph_A)>0.$$

As the point of conflict moves away from $\ell_A = 0$ and closer $\ell_B = 1$, warlord B increases his capital investment and produces more goods and services. As in the Base model, the partial derivatives $\frac{\partial W_A^*}{\partial \ell_c}$, $\frac{\partial W_B^*}{\partial \ell_c}$, $\frac{\partial K_A^*}{\partial \ell_c}$ and $\frac{\partial K_B^*}{\partial \ell_c}$ show that as the point of conflict moves away from a warlord's stronghold, more resources must be spent on hiring warriors and less on the production of goods and services. As the point of conflict moves closer to a warlord's stronghold, less resources are needed for hiring warriors and more is shifted to the production of goods and services.

3.5.1.3 Exogenous Prices

The effect of the exogenous prices are similar to those found in the previous section. Because each warlord pays the same price in capital, an increase in the price per unit of capital will result in both K_A^* and K_B^* to decrease; that is,

$$\frac{\partial K_A^*}{\partial c_k} = (-1)\left(\frac{1}{c_k}\right) \cdot K_A^* < 0;$$

$$\frac{\partial K_B^*}{\partial c_k} = (-1)\left(\frac{1}{c_k}\right) \cdot K_B^* < 0.$$

Since an increase in c_k decreases the production of goods and services for both warlords, it is not surprising that the total amount of goods and services produced within the economy also decreases when the price per unit of capital rises

$$\frac{\partial K^*}{\partial c_k} = (-1)\frac{1}{2 \cdot c_k^2} \cdot \left((\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)\right) < 0.$$

Conversely, each warlord does not necessarily pay the same per warrior wage as the other. As a result, the wage per warrior hired affects many important aspects the game. Inspecting the number of warriors hired by each warlord in equilibrium, an increase in the wage per warrior hired by warlord A will decrease the amount of warriors he hires,

$$
\begin{aligned}
\frac{\partial W_A^*}{\partial c_w^A} &= \left(\frac{-1}{c_w^A+\sigma_A}\right) W_A^* + \left(\frac{\frac{1}{2\cdot\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+m_R-c_w^A}}}{(c_w^A+\sigma_A)\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)}\right) W_A^* \gtrless 0 \\
&= -1 + \left(\frac{\frac{1}{2\cdot\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}\right)}\right) \gtrless 0 \\
&= \left(\frac{-1-\frac{1}{2\cdot\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}\right) \gtrless 0,
\end{aligned}
$$

which is negative by definition. An increase in c_w^B will also lead to a decrease in the number of warriors hired by warlord A

$$
\frac{\partial W_A^*}{\partial c_w^B} = W_A^* \cdot \left(\frac{-\frac{1}{2\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{1}{(c_w^A+\sigma_A)(c_w^B+\sigma_B)}}}{1+\frac{1}{sqrte^{\hat{\ell}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}\right) < 0.
$$

Analogously for warlord B, an increase in c_w^B will decrease the number of warriors warlord B will choose to hire,

$$
\begin{aligned}
\frac{\partial W_B^*}{\partial c_w^B} &= \left(\frac{-1}{c_w^B+\sigma_B}\right) W_B^* + \left(\frac{\frac{\sqrt{e^{\hat{\ell}}}}{2}\sqrt{\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}{(c_w^B+\sigma_B)\left(1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_A}}\right)}\right) W_B^* \gtrless 0 \\
&= -1 + \left(\frac{\frac{\sqrt{e^{\hat{\ell}}}}{2}\sqrt{\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}{(c_w^B+\sigma_B)\left(1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_A}}\right)}\right) \gtrless 0 \\
&= \left(\frac{-1-\frac{1}{2}\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}{1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right) \gtrless 0,
\end{aligned}
$$

which is negative by definition. An increase c_w^A will also cause warlord B to decrease the number of warriors to hire,

$$\frac{\partial W_B^*}{\partial c_w^A} = W_B^* \left(\frac{-\frac{1}{2}\sqrt{e^{\hat{\ell}} \frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{1+\sqrt{e^{\hat{\ell}} \frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}} \right) < 0.$$

The above equations illustrate an interesting relationship between wages paid to warriors hired and the number of warriors hired by each warlord. When c_w^A increases, warlord A cannot afford to hire as many warriors and warlord B now does not need to hire as many to remain competitive within the conflict. Likewise, an increase in c_w^B will decrease the number of warriors that warlord B can hire and warlord A has the incentive to decrease the number of warriors he hires.

Expanding on this point, it can be recalled from Corollary 5 that an increase in c_w^A will decrease the proportion of goods and services gained by warlord A after conflict takes place and an increase in c_w^B will increase the proportion. Therefore, an increase c_w^A will cause warlord A to decrease the number of warriors he hires and be awarded less of the total production of goods and services within the economy after conflict takes place. Warlord B can now also decrease the number of warriors hired and still gain an increased proportion of post-conflict goods and services won. A similar analysis holds for an increase in c_w^B.

While equation (3.58) from Theorem 5 reveals that an increase in either c_w^A or c_w^B will not affect the total amount of goods and services produced within the economy, both warrior wages do affect the individual warlord's capital investment decision. Therefore, it should be expected that any affect that an increase in one warlord's wage per warrior hired has on his own capital investment decision will have the opposite affect on the opposing warlord's investment; that is, from equation (3.56), an increase in c_w^A will cause K_A^* to decrease:

$$\frac{\partial K_A^*}{\partial c_w^A} = \left(\frac{\frac{1}{2\sqrt{e^{\hat{\ell}}}} \sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{(c_w^A+\sigma_A)\left(1+\frac{1}{e^{\hat{\ell}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)} \right) K_A^*$$

$$-\left(\frac{1}{2c_k\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)} \right) \left(\frac{1}{\sqrt{e^{\hat{\ell}}}} \sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}} \right) \left(\frac{(\sigma_A)(\aleph_A)}{c_w^A+\sigma_A} \right) \gtreqless 0$$

$$= K_A^* - 2(\sigma_A)(\aleph_A)\frac{1}{2c_k} \gtreqless 0$$

$$= \left(1+\frac{2}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)((\sigma_A)(\aleph_A))-(\sigma_B)(\aleph_B)$$

$$-\left(2+\frac{2}{\sqrt{e^{\hat{\ell}}}}\sqrt{\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}\right)((\sigma_A)(\aleph_A)) \gtreqless 0$$

$$= -(\sigma_A)(\aleph_A)-(\sigma_B)(\aleph_B)<0,$$

and from equation (3.57), an increase in c_w^A will result in a increase in K_B^*:

$$\frac{\partial K_B^*}{\partial c_w^A} = \left(\frac{\sqrt{\frac{e^{\hat{\ell}}}{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}}}{2\left(1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)} \right) K_B^*$$

$$+\sqrt{\frac{e^{\hat{\ell}}}{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}} \cdot \left(\frac{1}{1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}} \right) \left(\frac{(\sigma_B)(\aleph_B)}{2c_k} \right) > 0.$$

In words, an increase in c_w^A will cause warlord A to decrease the number of warriors hired but the increased wage decreases the amount of resources available for capital investment and, hence, K_A^* decreases as well. Similarly for c_w^B, an increase in the wage that warlord B pays per warrior hired will cause result in a decrease in the level of capital investment by warlord B and an increase the production of goods and services by warlord A.

3.6 Ratio Model

The impact functions defined above in the both the Base and Gates-logit models interpret the location of the point of conflict as having a linear affect on each warlord's decision. An alternative form of the impact function, known as the Ratio model, can be constructed that still upholds the properties $\frac{\partial I_A}{\partial W_A} > 0$, $\frac{\partial I_B}{\partial W_B} > 0$, $\frac{\partial I_A}{\partial \ell_c} < 0$ and $\frac{\partial I_B}{\partial \ell_c} > 0$. Recalling again $\ell_A = 0$ and $\ell_B = 1$, the newly constructed impact functions I_A and I_B are therefore explicitly defined as

$$I_A(\ell_c, \ell_A; W_A) = \frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} = \frac{W_A}{\phi \cdot \ell_c^2}; \tag{3.67}$$

$$I_B(\ell_c, \ell_B; W_B) = \frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} = \frac{W_A}{\phi \cdot (\ell_c - 1)^2}, \tag{3.68}$$

where ϕ is again an exogenous scalar such that $1 > \phi > 0$ and $0 < \ell_c < 1$. Equations (3.67) and (3.68) state that the impact function of warlord A's effort toward the contest against warlord B is still dependent upon the amount of warriors hired and the distance between warlord A's stronghold and the area of conflict.

From equations (3.67), (3.68) and (3.3), the CSF π_A for the conflict between warlord A and B equals

$$\pi_A = \frac{1}{1 + e^{\alpha \cdot \left(\frac{W_B}{\phi \cdot (\ell_c - 1)^2} - \frac{W_A}{\phi \cdot \ell_c^2} \right)}} \tag{3.69}$$

and the CSF π_B for the conflict between warlord B and A equals

$$\pi_B = 1 - \pi_{AB} = \frac{1}{1 + e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot \ell_c^2} - \frac{W_B}{\phi \cdot (\ell_c - 1)^2} \right)}} \tag{3.70}$$

As stated previously, the new impact functions still result in π_A being an increasing function in W_A and a decreasing function in W_B. In addition, as the distance between warlord A's stronghold and the conflict zone location ℓ_c increases, π_A decreases. As the distance between warlord B's stronghold and the conflict zone location ℓ_c increases, π_A increases.

Let Assumptions 1 through 3 still hold as well as the structure of game Γ defined above. Solving maximization problems (3.18) and (3.19), given the alternative forms of π_{AB} and π_{BA} defined in equations (3.69) and (3.70), for the choice variables and rearranging the appropriate variables gives rise to the following equilibrium result.

Theorem 7. *In the game* Γ *defined above, given assumptions* 1 *through* 3 *hold and the following two conditions are satisfied,*

$$\left.\begin{aligned} \sigma_A \cdot \aleph_A + \sigma_B \cdot \aleph_B > \quad & \frac{\phi}{\alpha}\Omega_A > \sigma_B \cdot \aleph_B - \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \left(\frac{\ell_c - 1}{\ell_c} \right)^2 \right) \cdot (\sigma_A \cdot \aleph_A) \\ \sigma_A \cdot \aleph_A + \sigma_B \cdot \aleph_B > \quad & \frac{\phi}{\alpha}\Omega_A > \sigma_A \cdot \aleph_B - \left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B} \left(\frac{\ell_c}{\ell_c - 1} \right)^2 \right) \cdot (\sigma_A \cdot \aleph_A) \end{aligned}\right\} \quad (3.71)$$

where

$$\begin{aligned} \Omega_A &= \frac{\alpha}{\phi} \left(\left(c_w^A + \sigma_A \right) \ell_c^2 + \left(c_w^B + \sigma_B \right) (\ell_c - 1)^2 \left(1 - \ln \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \left(\frac{\ell_c - 1}{\ell_c} \right)^2 \right) \right) \right); \\ \Omega_B &= \frac{\alpha}{\phi} \left(\left(c_w^B + \sigma_B \right) (\ell_c - 1) + \left(c_w^A + \sigma_A \right) \ell_c^2 \left(1 - \ln \left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B} \left(\frac{\ell_c}{\ell_c - 1} \right)^2 \right) \right) \right), \end{aligned}$$

an interior pure strategy Nash equilibrium exists where,

1. *Warlord A and warlord B, where $\ell_A = 0$ and $\ell_B = 1$, hire warrior numbers of*

$$W_A^* = \ell_c^2 \cdot \left(\frac{(\sigma_A \cdot \aleph_A) + (\sigma_B \cdot \aleph_B)}{\left(c_w^A + \sigma_A\right)\ell_c^2 + \left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2} - \frac{\phi}{\alpha} \right)$$
$$+ \frac{\phi}{\alpha} \left(\frac{\left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2 \ell_c^2 \ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \left(\frac{\ell_c - 1}{\ell_c} \right)^2 \right)}{\left(c_w^A + \sigma_A\right)\ell_c^2 + \left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2} \right); \quad (3.72)$$

$$W_B^* = (\ell_c - 1)^2 \cdot \left(\frac{(\sigma_A \cdot \aleph_A) + (\sigma_B \cdot \aleph_B)}{\left(c_w^A + \sigma_A\right)\ell_c^2 + \left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2} - \frac{\phi}{\alpha} \right)$$
$$+ \frac{\phi}{\alpha} \left(\frac{\left(c_w^A + \sigma_A\right)\left(\ell_c - 1\right)^2 \ell_c^2 \ln\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B} \left(\frac{\ell_c}{\ell_c - 1} \right)^2 \right)}{\left(c_w^A + \sigma_A\right)\ell_c^2 + \left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2} \right). \quad (3.73)$$

2. *Warlord A and B will invest in capital levels of*

$$K_A^* = \frac{1}{c_k} \left(\frac{\left(c_w^B + \sigma_B\right)(\sigma_A \cdot \aleph_A)\left(\ell_c - 1\right)^2 - \left(c_w^A + \sigma_A\right)(\sigma_B \cdot \aleph_B)\ell_c^2}{\left(c_w^A + \sigma_A\right)\ell_c^2 + \left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2} \right)$$
$$+ \frac{\phi}{\alpha c_k} \left(\left(\frac{\left(c_w^A + \sigma_A\right)\left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2 \ell_c^2}{\left(c_w^A + \sigma_A\right)\left(\ell_c - 1\right)^2 + \left(c_w^B + \sigma_B\right)\ell_c^2} \right) \cdot \ln\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B} \left(\frac{\ell_c}{\ell_c - 1} \right)^2 \right) \right)$$
$$+ \frac{\phi}{\alpha c_k} \left(\left(c_w^A + \sigma_A\right)\ell_c^2 \right); \quad (3.74)$$

$$K_B^* = \frac{1}{c_k} \left(\frac{\left(c_w^A + \sigma_A\right)(\sigma_B \cdot \aleph_B)\ell_c^2 - \left(c_w^B + \sigma_B\right)(\sigma_A \cdot \aleph_A)\left(\ell_c - 1\right)^2}{\left(c_w^A + \sigma_A\right)\ell_c^2 + \left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2} \right)$$
$$+ \frac{\phi}{\alpha c_k} \left(\left(\frac{\left(c_w^A + \sigma_A\right)\left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2 \ell_c^2}{\left(c_w^A + \sigma_A\right)\left(\ell_c - 1\right)^2 + \left(c_w^B + \sigma_B\right)\ell_c^2} \right) \cdot \ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \left(\frac{\ell_c - 1}{\ell_c} \right)^2 \right) \right)$$
$$+ \frac{\phi}{\alpha c_k} \left(\left(c_w^B + \sigma_B\right)\left(\ell_c - 1\right)^2 \right). \quad (3.75)$$

3. the total production of goods and services within the economy is

$$K^* = \frac{\phi}{\alpha c_k}\left(\left(c_w^A + \sigma_A\right) \cdot \ell_c^2 + \left(c_w^B + \sigma_B\right) \cdot (\ell_c - 1)^2\right). \tag{3.76}$$

Proof. See page 146 in the Appendix. □

By substituting equation (A.47) into the contest success function defined in equation (3.69), warlord A and B each receive a proportion of K^* dependent upon the point of conflict ℓ_c.

Theorem 8. *In the game* Γ *defined above, given assumptions* 1 *through* 3 *hold and the two conditions in equation* (3.71) *are satisfied, the proportion of* K^* *that each warlord receives after conflict is based upon the location of the point of conflict and the wages each warlord pays per warrior and extractor hired; that is,*

$$\pi_A^* = \frac{1}{1 + \left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right) \cdot \left(\frac{\ell_c}{\ell_c - 1}\right)^2} \tag{3.77}$$

$$\pi_B^* = 1 - \pi_A^* = \frac{1}{1 + \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right) \cdot \left(\frac{\ell_c - 1}{\ell_c}\right)^2} \tag{3.78}$$

Proof. See page 155 in the Appendix. □

Theorem 8 states that the proportion of goods and services each warlord will attain within equilibrium is dependent upon both warlord A and warlord B's wage per warrior hired, the wage per extractor hired and the point of conflict ℓ_c.

3.6.1 Comparative Statics

3.6.1.1 Population Sizes and Pre-Game Budgets

The effect that the size of each warlord's population has on the equilibrium number of warriors hired and level of capital produced is identical to both the Base and Gates-logit models presented above; that is, for warlord A

$$\frac{\partial W_A^*}{\partial N_A}=\left(\frac{(\sigma_A)\cdot\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)>0 \qquad \frac{\partial W_A^*}{\partial N_B}=\left(\frac{(\sigma_B)\cdot\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)>0,$$

and for warlord B

$$\frac{\partial W_B^*}{\partial N_A}=\left(\frac{(\sigma_A)\cdot(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)>0 \qquad \frac{\partial W_B^*}{\partial N_B}=\left(\frac{(\sigma_B)\cdot(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)>0.$$

Again, as in both the Base and Gates-logit model, any increase in the size of either warlord's population is going to raise the number of warriors hired by each warlord. The same holds true with increases in Y_A and Y_B such that

$$\frac{\partial W_A^*}{\partial Y_A} = \frac{\partial W_A^*}{\partial Y_B} = \left(\frac{\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)\left(\ell_c-1\right)^2}\right) > 0;$$

$$\frac{\partial W_B^*}{\partial Y_A} = \frac{\partial W_B^*}{\partial Y_B} = \left(\frac{\left(\ell_c-1\right)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)\left(\ell_c-1\right)^2}\right) > 0.$$

Population sizes and pre-game budgets again have the same effects on the equilibrium investment level of capital as in the first two model. Any increase in either N_A or Y_A will cause an increase in warlord A's investment in capital and decrease warlord B's investment; that is,

$$\frac{\partial K_A^*}{\partial N_A}=\left(\frac{\left(c_w^B+\sigma_B\right)\left(\frac{\sigma_A}{c_k}\right)(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)>0 \qquad \frac{\partial K_A^*}{\partial Y_A}=\left(\frac{\left(\frac{c_w^B+\sigma_B}{c_k}\right)(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)>0$$

$$\frac{\partial K_B^*}{\partial N_A}=\left(\frac{(-1)\cdot\left(c_w^B+\sigma_B\right)\left(\frac{\sigma_A}{c_k}\right)(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)<0 \qquad \frac{\partial K_A^*}{\partial Y_A}=\left(\frac{(-1)\cdot\left(\frac{c_w^B+\sigma_B}{c_k}\right)(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)<0.$$

Likewise, any increase in either N_B or Y_B will cause a decrease warlord A's investment in capital and increase warlord B's; that is,

$$\frac{\partial K_A^*}{\partial N_B}=\left(\frac{(-1)\cdot\left(c_w^A+\sigma_A\right)\left(\frac{\sigma_B}{c_k}\right)\cdot\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)<0 \qquad \frac{\partial K_A^*}{\partial Y_B}=\left(\frac{(-1)\cdot\left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)<0$$

$$\frac{\partial K_B^*}{\partial N_B}=\left(\frac{\left(c_w^A+\sigma_A\right)\left(\frac{\sigma_B}{c_k}\right)\cdot\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)>0 \qquad \frac{\partial K_A^*}{\partial Y_B}=\left(\frac{\left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)<0.$$

3.6.1.2 Location of the Point of Conflict

To help get some clear results, it is assumed that $c_w^A = c_w^B = c_w$ and $c_E^A = c_E^B = c_E$. Comparing equations (3.25) and (3.76), it stands out that the the total amount of goods and services produced within the Ratio model is strictly dependent upon the point of conflict. Starting at the midpoint of $\ell_c = 1/2$, any variation either toward $\ell_A = 0$ or $\ell_B = 1$ will cause an increase in the total production of goods and services. Specifically, by taking the partial derivative of K^*,

$$\frac{\partial K^*}{\partial \ell_c} = \begin{cases} < 0 & \text{if} \quad \ell_c < \frac{1}{2}; \\ \\ > 0 & \text{if} \quad \ell_c > \frac{1}{2}. \end{cases}$$

The above equation states that as the point of conflict moves away from warlord A's stronghold, the amount of goods and services produced within the economy decreases. Once the point of conflict crosses the midpoint and moves toward warlord B's stronghold, the amount of goods and services produced within the economy starts to increase again. This implies that the worst outcome, in terms of the production of goods and services, is when the war is fought at an equidistant location between the two warlords.

Similarly, using equations (3.72) and (3.73), the total number of warriors dedicated toward warfare is

$$W^* = \left(\frac{\sigma}{c_w + \sigma}\right) \cdot (\aleph_A + \aleph_B) - \frac{\phi}{\alpha} \cdot \left(2 \cdot \ell_c^2 - 2 \cdot \ell_c + 1\right),$$

in which,

$$\frac{\partial W^*}{\partial \ell_c} = \begin{cases} > 0 & \text{if} \quad \ell_c < \frac{1}{2}; \\ \\ < 0 & \text{if} \quad \ell_c > \frac{1}{2}. \end{cases}$$

The partial derivative states that as the point of conflict moves away from $\ell_A = 0$ toward the midpoint, the amount of warriors dedicated for warfare increases. As the point of conflict moves away from the midpoint and toward $\ell_B = 1$, the amount of warriors dedicated for warfare decreases. Therefore, warfare is at its height when the point of conflict is at the midpoint between the two strongholds.

Finally, the equilibrium CSFs found in equations (3.77) and (3.78) state that each warlord gains a greater proportion of the total production of goods and services when the point of conflict is close to his stronghold. Specifically,

$$\frac{\partial \pi_A^*}{\partial \ell_c} = \ell_c - 1 < 0 \qquad \frac{\partial \pi_B^*}{\partial \ell_c} = 1 - \ell_c > 0.$$

In other words, as the point of conflict moves away from $\ell_c = 0$ toward $\ell_B = 1$, the proportion of goods and services attained and retained by warlord A decreases while warlord B's proportion increases.

3.6.1.3 Model Comparison

These results are in stark contrast to those of the Base model which state that the total production of goods and services and the total amount of warriors hired are unaffected by the point of conflict. The reason being that any shift in resources by one warlord was made up by the opposing warlord.

The Ratio model seems to suggest a different scenario. Assuming warrior and extractor wages are the same for both warlord, when the point of conflict is set at the midpoint between the two strongholds, each warlord wins half the goods and services produced within the economy while production is at its lowest and warfare its highest. When the point of conflict moves closer to warlord A's stronghold, the production of goods and services increases while warfare decreases and warlord A wins a larger proportion of goods and services. When the point of conflict moves closer to warlord B's stronghold, the production of goods and services again will increase while warfare decreases and warlord B now wins the larger proportion of goods and services.

3.6.2 Remark

Through a series of examples, the existence of a Nash equilibrium in the Base model was shown to be dependent upon the the total pre-game budget being greater than zero while such restrictions are not necessarily needed for an equilibrium to be found in the Gates-logit model. Like the Gates-logit model, the existence of a Nash equilibrium in the Ratio Model is not fully dependent upon positive pre-game budgets. It can further be shown that equilibria can exist where both warlords have a pre-game budget of zero.

Let $N_A = N_B = 1$, $Y_A = Y_B = 0$, $\alpha = 1$ and $\phi = 1/2$. In addition, let $c_w^A = c_w^B = c_w$ and $c_E^A = c_E^B = c_E$.Assume at first that the point of conflict is set at $\ell_c = 1/2$. The equilibrium values of the decision variables are thus

$$W_A^* = W_B^* = \left(\frac{\sigma}{c_w + \sigma}\right) - \frac{1}{8};$$

$$K_A^* = K_B^* = \left(\frac{c_w + \sigma}{c_k}\right) \cdot \left(\frac{1}{8}\right).$$

Substituting these values into the two conditional equations found in Theorem 7 leads to an equilibrium as long as $7 \cdot (\sigma) \geq c_w$.

Adjusting the point of conflict to $\ell_c = 1/4$ results in the equilibrium values W_A^* and W_B^* of

$$W_A^* = \left(\frac{\sigma}{c_w + \sigma}\right) \cdot \left(\frac{1}{5}\right) + 0.031;$$

$$W_B^* = \left(\frac{\sigma}{c_w + \sigma}\right) \cdot \left(\frac{9}{5}\right) - 0.343,$$

and equilibrium capital investment levels of

$$K_A^* = \left(\frac{\sigma}{c_k}\right) \cdot \left(\frac{4}{5}\right) - (0.031) \cdot \left(\frac{c_w + \sigma}{c_k}\right);$$

$$K_B^* = (0.343) \cdot \left(\frac{c_w + \sigma}{c_k}\right) - \left(\frac{\sigma}{c_k}\right) \cdot \left(\frac{4}{5}\right)$$

Substituting the level of capital investment and number of warrior hired by each warlord into equation (3.71) results in a Nash equilibrium when

$$(4.25) \cdot (\sigma) \geq c_w \geq (1.33) \cdot (\sigma).$$

The same holds true when the point of conflict is located at $\ell_c = 3/4$.

From the above simplified examples, it is possible to have an equilibrium where there exist no pre-game budgets. These examples give different results from those examples shown for the original model with the same specifications; that is, positive pre-game budgets are not required for a Nash equilibrium being found.

CHAPTER 4

Asymmetric Rates of Seizure

The model introduced in Chapter 3 was based on the assumption that the rate of seizure of a warlord's production of goods and services was symmetric and equal to 1; that is, when $J = 2$, the total production of goods and services and the total number of goods and services being fought over is $K^* = K_A + K_B$. One can interpret this symmetric assumption as the production of goods and services by both warlords taking place at the same location — namely, the point of conflict ℓ_c. While a symmetric model can represent some, if not many, modern civil conflicts, the asymmetric scenario is also prevalent and studied.

The model is now altered to have each warlord's production of goods and services taking place at his respective territory or stronghold, ℓ_A and ℓ_B respectively, instead of the point of conflict ℓ_c.

Definition 7. *The* Geography *of the economy is defined as a line of fixed length on an interval* $[0, 1]$ *where* $\ell_A = 0$ *and* $\ell_B = 1$*. Figure* 7 *illustrates the basics of the economy's geography.*

Once again, it is assumed that the location of conflict ℓ_c is exogenously determined

Assuming each warlord produces goods and services at her own stronghold location, the point of conflict now also affects the amount of each warlord's production available for appropriation. Recalling K^* as denoting the total amount of goods and

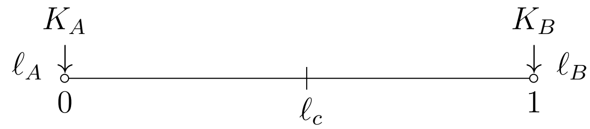


Figure 4.1: Geography of the Asymmetric Economy

services produced by both warlords, it now does not accurately represent the total amount of production that is being fought over between warlords A and B. Therefore, let $\widehat{K}$ be the amount of goods and services on the table for appropriation such that

$$\widehat{K} = K_A \cdot (\ell_B - \ell_c) \cdot \beta + K_B \cdot (\ell_c - \ell_A) \cdot \beta = K_A \cdot (1 - \ell_c) \cdot \beta + K_B \cdot \ell_c \cdot \beta, \qquad (4.1)$$

where $\beta > 0$ is a scalar representing the effect that terrain and distance have on the rate of seizure. Equation (4.1) asserts that as ℓ_c moves closer to warlord A's stronghold $\ell_A = 0$, more of K_A and less of K_B is subject to appropriation through conflict. Likewise, as ℓ_c moves toward warlord B's stronghold $\ell_B = 1$, less of K_A and more of K_B is subject to appropriation through force.

Holding assumptions 1, 2 and 3 from Chapter 3, the income gained by warlord A after conflict is constructed from equations (3.39), (3.41) and (4.1) where $J = \{A, B\}$:

$$\begin{aligned} V_A(W_A, K_A; W_B, K_B) &= (m \cdot \pi_A) \cdot \widehat{K} + m \cdot K_A \cdot (1 - (\ell_B - \ell_c) \cdot \beta) \\ &= (m \cdot \pi_A) \cdot \widehat{K} + m \cdot K_A \cdot (1 - (1 - \ell_c) \cdot \beta), \qquad (4.2) \end{aligned}$$

and for warlord B,

$$\begin{aligned} V_B(W_B, K_B; W_A, K_A) &= (m \cdot \pi_B) \cdot \widehat{K} + m \cdot K_B \cdot (1 - (\ell_c - \ell_A) \cdot \beta) \\ &= (m \cdot \pi_B) \cdot \widehat{K} + m \cdot K_B \cdot (1 - \ell_c \cdot \beta). \qquad (4.3) \end{aligned}$$

For simplicity, it is again assumed that both warlords pay the same warrior and extractor wages; that is, $c_w^A = c_w^B = c_w$, $c_E^A = c_E^B = c_E$ and $\sigma = m_R - c_E$. Therefore, given W_B and K_B, warlord A optimizes W_A and K_A to maximize equation (4.2) subject to equation (3.38). The same holds for warlord B given W_A and K_A.

4.1 Equilibrium

Solving maximization problems (4.2) and (4.3) for the choice variables and rearranging the appropriate variables gives rise to the following equilibrium result.

Theorem 9. *Let* $\hat{\ell} = (\ell_c - \ell_A)^2 - (\ell_c - \ell_B)^2$, $\aleph_A = \left(N_A + \frac{Y_A}{\sigma}\right)$ *and* $\aleph_B = \left(N_B + \frac{Y_B}{\sigma}\right)$. *Suppose assumptions* 1, 2, 3 *and*

$$(\beta - 2) + \sqrt{4 + \beta^2} > (2 \cdot \beta) \cdot \ell_c > (2 + \beta) - \sqrt{4 + \beta^2} \tag{4.4}$$

with the following two conditions,

$$\left.\begin{aligned} \aleph_A \cdot (1 - \ell_c) + \aleph_B \cdot \ell_c \geq \left(\frac{c_w + \sigma}{\alpha \cdot \sigma}\right) \cdot \Omega_A \geq \ell_c \cdot (\aleph_B - \aleph_A) \\ \aleph_A \cdot (1 - \ell_c) + \aleph_B \cdot \ell_c \geq \left(\frac{c_w + \sigma}{\alpha \cdot \sigma}\right) \cdot \Omega_B \geq (1 - \ell_c) \cdot (\aleph_A - \aleph_B) \end{aligned}\right\} \tag{4.5}$$

where

$$\begin{aligned} \Omega_A &= \left(\frac{1 - \ell_c \cdot (1 - \ell_c) \cdot \beta}{\ell_c \cdot (1 - \ell_c) \cdot \beta} + \ell_c \cdot \ln\left(\frac{\ell_c}{1 - \ell_c}\right) - \alpha \cdot \widehat{\ell_c} \cdot \ell_c\right); \\ \Omega_B &= \left(\frac{1 - \ell_c \cdot (1 - \ell_c) \cdot \beta}{\ell_c \cdot (1 - \ell_c) \cdot \beta} + (1 - \ell_c) \cdot \ln\left(\frac{1 - \ell_c}{\ell_c}\right) + \alpha \cdot \widehat{\ell_c} \cdot (1 - \ell_c)\right), \end{aligned}$$

a pure strategy Nash equilibrium exists where,

1. *Warlord A and warlord B hire warrior numbers of*

$$W_A^* \;=\; \left(\frac{\sigma}{c_w+\sigma}\right)\cdot((1-\ell_c)\cdot\aleph_A+\ell_c\cdot\aleph_B)-\Omega_A \tag{4.6}$$

$$W_B^* \;=\; \left(\frac{\sigma}{c_w+\sigma}\right)\cdot((1-\ell_c)\cdot\aleph_A+\ell_c\cdot\aleph_B)-\Omega_B. \tag{4.7}$$

2. *Warlord A and B will invest in capital levels of*

$$K_A^* \;=\; \left(\frac{\sigma\cdot\ell_c}{c_k}\right)\cdot(\aleph_A-\aleph_B)+\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)\cdot\Omega_A; \tag{4.8}$$

$$K_B^* \;=\; \left(\frac{\sigma\cdot(1-\ell_c)}{c_k}\right)\cdot(\aleph_B-\aleph_A)+\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)\cdot\Omega_B. \tag{4.9}$$

3. *the total production of goods and services within the economy is*

$$\begin{aligned} K^* \;=\;& \left(\frac{\sigma}{c_k}\right)\cdot\widehat{\ell}\cdot(\aleph_A-\aleph_B)+2\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\frac{1-\ell_c\cdot(1-\ell_c)\cdot\beta}{(1-\ell_c)\cdot\ell_c\cdot\beta}\right) \\ &+\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)\cdot\widehat{\ell_c}\cdot\left(\ln\left(\frac{\ell_c}{1-\ell_c}\right)-\alpha\cdot\phi\cdot\widehat{\ell_c}\right), \end{aligned} \tag{4.10}$$

while the number of goods and services available for appropriation through conflict is

$$\widehat{K}=\left(\frac{1}{\ell_c(1-\ell_c)}-\beta\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right). \tag{4.11}$$

Proof. See page 156 in the Appendix. □

Similar to the results of the symmetric base model in the previous chapter, Theorem 9 shows that the *paradox of power* is still present in the current model with asymmetric rates of seizure. From equations (4.6) and (4.7), an increase in either $\aleph_A$

and/or $\aleph_B$ will cause both warlords to increase the number of warriors they hire; that is, if either warlord is given (or acquires) more pre-game exogenous resources, both warlords will increase their level of warfare. On the other hand, equations (4.8) and (4.9) show that an increase in a warlord's pre-game exogenous resources will cause the warlord to increase spending on the production of goods and services while also causing the opposing warlord to decrease his investment into production.

By substituting the above equilibrium equations into the contest success function defined in equation (3.41), warlord A and B each receive a proportion of $\widehat{K}$ dependent upon the location of conflict ℓ_c.

Theorem 10. *Suppose assumptions* 1, 2, 3, *equation* (4.4) *and the two conditions in equation* (4.5). *The proportions of* $\widehat{K}$ *that each warlord receive in equilibrium are dependent the geography of the economy:*

$$\pi_A^* = \frac{1}{1+\frac{\ell_c}{1-\ell_c}} = 1-\ell_c \tag{4.12}$$

$$\pi_B^* = 1-\pi_A^* = \ell_c \tag{4.13}$$

Proof. See page 166 in the Appendix. □

According to Theorem 10, each warlord is able to acquire more of $\widehat{K}$ when the point of conflict is closer to his own stronghold.

4.2 The Effect of Geography: Examples

The most prudent way to show the affect the point of conflict and the variable β have on the equilibrium is through a series of examples. Let $c_w = (1/10)\cdot\sigma$ and $c_k = (1/5)\cdot\sigma$. In addition, let $N_A = N_B = Y_A = Y_B = 1$ and $(m_R - c_E) = (0.05)\cdot Y_A =$

$(0.05) \cdot Y_B$ such that $\aleph_A = \aleph_B = 21$. Finally, let it be that $\phi = 0.5$ and $\alpha = 1$. Given these specifications, an equilibrium can be found when, from the two conditions in (4.5),

$$19.11 \geq \Omega_A, \Omega_B \geq 0$$

where the equilibrium number of warriors hired and capital invested by each warlord are

$$W_A^* = 19.11 - \Omega_A \qquad W_B^* = 19.11 - \Omega_B;$$

$$K_A^* = (2.2) \cdot \Omega_A \qquad K_B^* = (2.2) \cdot \Omega_B.$$

In addition, the total amount of goods and services produced within the economy is

$$K^* = (2.2) \cdot (\Omega_A + \Omega_B)$$

and the amount of goods and services produced that are available for appropriation through conflict equals

$$\widehat{K} = (2.2) \cdot \left(\frac{1}{\ell_c \cdot (1 - \ell_c)} - \beta \right).$$

Since the equilibrium values, and the existence of the equilibrium itself, are dependent upon β and ℓ_c, a series of four examples are explored below. For means of analysis and comparison, let $\widetilde{K}$ be equal to the percentage of goods and services produced that are actually being fought over; that is,

$$\widetilde{K} = \left(\frac{\widehat{K}}{K^*} \right) \cdot 100.$$

4.2.1 Example A: $\beta = 0.25$

When the affect that geography has on the seizure rate of production is $\beta = 0.25$, equation (4.4) shows that an equilibrium can be found when $0.54 > \ell_c > 0.46$; that is, an equilibrium solution will exist somewhere very close to $\ell_c = 0.5$, the midpoint between the two warlords' strongholds. Table 4.1 illustrates the corresponding values for the variables in equilibrium. The blue shaded cells in Table 4.1 show decreasing values and the green cells show increasing values when ℓ_c moves from 0.46 to 0.54.

Table 4.1: Example A: $\beta = 0.25$

ℓ_c	Ω_A	Ω_B	W_A^*	W_B^*	W^*	K_A^*	K_B^*	K^*	$\widehat{K}$
0.53	15.12	15.02	4.00	4.09	8.09	33.23	33.03	66.26	8.28
0.52	15.06	15.00	4.05	4.11	8.16	33.13	32.99	66.12	8.26
0.51	15.02	14.99	4.09	4.12	8.21	33.05	32.98	66.03	8.254
0.50	15.00	15.00	4.11	4.11	8.22	33.00	33.00	66.00	8.25
0.49	14.99	15.02	4.12	4.09	8.21	32.98	33.05	66.03	8.254
0.48	15.00	15.06	4.11	4.05	8.16	32.99	33.13	66.12	8.26
0.47	15.02	15.12	4.09	4.00	8.09	33.03	33.23	66.26	8.28

Immediately, the first two columns of Table 4.1 show that both Ω_A and Ω_B are less than 19.11 for $0.54 > \ell_c > 0.47$ and, hence, the two conditions found in equation (4.5) are satisfied. Contrary to the base model results found in the previous chapter, warlord A will hire more warriors than warlord B when the point of conflict is closer to his stronghold. When the point of conflict is closer to $\ell_B = 1$, warlord B will hire more warriors than warlord A. It should also be noted that for both warlords, more warriors are hired as the point of conflict approaches the midpoint of $\ell_c = 1/2$.

The opposite is found with each warlord's decision to invest into capital. Warlord A tends to invest less than warlord B when the point of conflict is closer to $\ell_A = 0$ and the contrast is true when the point of conflict approaches $\ell_B = 1$. As a result, the total production of goods and services within the economy increases as the point of conflict moves away from $\ell_c = 1/2$ toward either warlord's stronghold. Finally, the last two columns of Table 4.1 show that when $\beta = 0.25$, only a very small percentage of the total number of goods and services produced are available for appropriation through conflict; that is, $\widetilde{K} = 12.5\%$ when $0.54 > \ell_c > 0.48$.

4.2.2 Example B: $\beta = 0.5$

When the effect that geography has on the seizure rate of production increases to $\beta = 0.5$, equation (4.4) shows that an equilibrium can now be found when $0.56 > \ell_c > 0.44$. The region where an equilibrium solution can be found increases marginally from when $\beta = 0.25$ and is still very close to the midpoint between the two warlords' strongholds. Table 4.2 illustrates the corresponding values for the variables in equilibrium. Again, the blue shaded cells in Table 4.2 show decreasing values and the green cells show increasing values when ℓ_c moves from 0.44 to 0.56.

Table 4.2: Example B: $\beta = 0.5$

ℓ_c	Ω_A	Ω_B	W_A^*	W_B^*	W^*	K_A^*	K_B^*	K^*	$\widehat{K}$
0.55	7.16	7.01	11.95	12.10	24.04	15.76	15.43	31.19	7.79
0.54	7.12	7.00	12.00	12.11	24.11	15.66	15.39	31.05	7.76
0.53	7.08	6.99	12.03	12.12	24.15	15.57	15.37	30.94	7.73
0.52	7.04	6.98	12.07	12.13	24.19	15.50	15.36	30.86	7.71
0.51	7.02	6.99	12.09	12.12	24.21	15.44	15.37	30.82	7.704
0.50	7.00	7.00	12.11	12.11	24.22	15.40	15.40	30.80	7.70
0.49	6.99	7.02	12.12	12.09	24.21	15.37	15.44	30.82	7.704
0.48	6.98	7.04	12.13	12.07	24.19	15.36	15.50	30.86	7.71
0.47	6.99	7.08	12.12	12.03	24.15	15.37	15.57	30.94	7.73
0.46	7.00	7.12	12.11	12.00	24.11	15.39	15.66	31.05	7.76
0.45	7.01	7.16	12.10	11.95	24.04	15.43	15.76	31.19	7.79

Again, the first two columns of Table 4.2 show that both Ω_A and Ω_B are less than 19.11 for $0.54 > \ell_c > 0.47$ and, hence, the two conditions found in equation (4.5) are satisfied. The geographical elasticity of the decision variables hold the same pattern as in the previous example, yet the total numbers differ. Now that the effect of geography has increased, both warlord A and warlord B increase the amount of resources dedicated to the hiring of warriors and decrease the amount of capital invested. Interestingly, the percentage of goods and services that are subject to appropriation through conflict, $\widetilde{K}$, increases. When $\beta = 0.25$, only around 12% of the economy's total production is being fought over. As β increases to 0.5, both K^* and $\widehat{K}$ decrease but $\widetilde{K} = 25\%$ when $0.62 > \ell_c > 0.38$.

4.2.3 Example C: $\beta = 1$

Increasing the effect that geography has on the seizure rate of production to $\beta = 1$, equation (4.4) shows that an equilibrium can now be found when $0.62 > \ell_c > 0.38$. The region where an equilibrium solution can be found increases further from, but still very close to, the midpoint between the two warlords' strongholds. Table 4.3 illustrates the corresponding values for the variables in equilibrium. Again, the blue shaded cells in Table 4.3 show decreasing values and the green cells show increasing values when ℓ_c moves from 0.38 to 0.62.

The first two columns of Table 4.3 show that, again, both Ω_A and Ω_B are less than 19.11 for $0.62 > \ell_c > 0.38$ and, hence, the two conditions found in equation (4.5) are satisfied. As when β equals 0.25 and/or 0.5, the geographical elasticity of the decision variables hold the same patterns and, yet again, warlord A and warlord B increase the amount of resources dedicated to the hiring of warriors and decrease the amount of capital invested. The percentage of goods and services that subject to appropriation through conflict increases even further as β increases from 0.5 to 1; that is, both K^* and $\widehat{K}$ decrease but, roughly, $\widetilde{K} = 50\%$.

4.2.4 Example D: $\beta = 2$

Increasing the effect that geography has on the seizure rate of production to $\beta = 2$, equation (4.4) shows that an equilibrium can now be found when $1 > \ell_c > 0$. The region where an equilibrium solution can be found increases further from the

Table 4.3: Example C: $\beta = 1$

ℓ_c	Ω_A	Ω_B	W_A^*	W_B^*	W^*	K_A^*	K_B^*	K^*	$\widehat{K}$
0.61	3.41	3.07	15.70	16.04	31.74	7.50	6.76	14.26	7.05
0.60	3.35	3.04	15.76	16.07	31.83	7.34	6.70	14.07	6.97
0.59	3.30	3.02	15.81	16.09	31.90	7.25	6.65	13.90	6.89
0.58	3.25	3.00	15.86	16.11	31.97	7.14	6.61	13.75	6.83
0.57	3.20	2.99	15.91	16.12	32.03	7.04	6.58	13.62	6.78
0.56	3.16	2.98	15.95	16.13	32.08	6.95	6.55	13.50	6.73
0.55	3.12	2.972	16.00	16.137	32.137	6.87	6.54	13.41	6.69
0.54	3.09	2.970	16.02	16.40	32.42	6.80	6.53	13.33	6.66
0.53	3.06	2.972	16.05	16.137	32.187	6.74	6.54	13.28	6.63
0.52	3.04	2.98	16.07	16.13	32.20	6.68	6.55	13.23	6.61
0.51	3.02	2.99	16.09	16.12	32.21	6.64	6.57	13.21	6.603
0.50	3.00	3.00	16.11	16.11	32.22	6.60	6.60	13.20	6.60
0.49	2.99	3.02	16.12	16.09	32.21	6.57	6.64	13.21	6.603
0.48	2.98	3.04	16.13	16.07	32.20	6.55	6.68	13.23	6.61
0.47	2.972	3.06	16.137	16.05	32.187	6.54	6.74	13.28	6.63
0.46	2.970	3.09	16.14	16.02	32.42	6.53	6.80	13.33	6.66
0.45	2.972	3.12	16.137	16.00	32.137	6.54	6.87	13.41	6.69
0.44	2.98	3.16	16.13	15.95	32.08	6.55	6.95	13.50	6.73
0.43	2.99	3.20	16.12	15.91	32.03	6.58	7.04	13.62	6.78
0.42	3.00	3.25	16.11	15.86	31.97	6.61	7.14	13.75	6.83
0.41	3.02	3.30	16.09	15.81	31.90	6.65	7.25	13.90	6.89
0.40	3.04	3.35	16.07	15.76	31.83	6.70	7.34	14.07	6.97
0.39	3.07	3.41	16.04	15.70	31.74	6.76	7.50	14.26	7.04

Table 4.4: Example D: $\beta = 2$

ℓ_c	Ω_A	Ω_B	W_A^*	W_B^*	W^*	K_A^*	K_B^*	K^*	$\widehat{K}$	$\widetilde{K}$
0.97	19.10	16.09	0.01	3.02	3.03	42.02	35.40	77.42	71.20	91.97%
0.96	14.63	11.91	4.48	7.20	11.68	32.20	26.21	58.39	52.89	90.58%
0.95	11.90	9.40	7.21	9.71	16.92	26.17	20.68	46.85	41.92	89.46%
0.90	6.17	4.38	12.94	14.73	27.67	13.58	9.63	23.21	20.04	86.37%
0.85	4.10	2.71	15.01	16.40	31.41	9.02	5.97	14.99	12.85	85.77%
0.80	3.00	1.91	16.12	17.20	33.32	6.59	4.20	10.79	9.35	86.70
0.75	2.30	1.45	16.81	17.66	34.46	5.07	3.20	8.27	7.33	88.71%
0.70	1.83	1.19	17.28	17.92	35.20	4.03	2.61	6.64	6.08	91.43%
0.65	1.50	1.03	17.61	18.08	35.68	3.31	2.27	5.58	5.27	94.45%
0.60	1.27	0.96	17.84	18.15	35.99	2.79	2.11	4.90	4.77	97.26%
0.55	1.10	0.95	18.01	18.16	36.16	2.43	2.10	4.53	4.49	99.23%
0.50	1.00	1.00	18.11	18.11	36.22	2.20	2.20	4.40	4.40	100%
0.45	0.95	1.10	18.16	18.01	36.17	2.10	2.43	4.53	4.49	99.27%
0.40	0.96	1.27	18.15	17.84	35.99	2.11	2.79	4.90	4.77	97.23%
0.35	1.03	1.50	18.08	17.61	35.68	2.27	3.31	5.58	5.27	94.45%
0.30	1.19	1.83	17.92	17.28	35.20	2.61	4.03	6.64	6.08	91.43%
0.25	1.45	2.30	17.66	16.81	34.46	3.20	5.07	8.27	7.33	88.71%
0.20	1.91	3.00	17.20	16.12	33.32	4.20	6.59	10.79	9.35	86.70%
0.15	2.71	4.10	16.40	15.01	31.41	5.97	9.02	14.99	12.85	85.77%
0.10	4.38	6.17	14.73	12.94	27.67	9.63	13.58	23.21	20.04	86.37%
0.05	9.40	11.90	9.71	7.21	16.92	20.68	26.17	46.85	41.92	89.46%
0.04	11.91	14.63	7.20	4.48	11.68	26.21	32.19	58.39	52.89	90.58%
0.03	16.09	19.10	3.02	0.01	3.03	35.40	42.02	77.42	71.20	91.97%

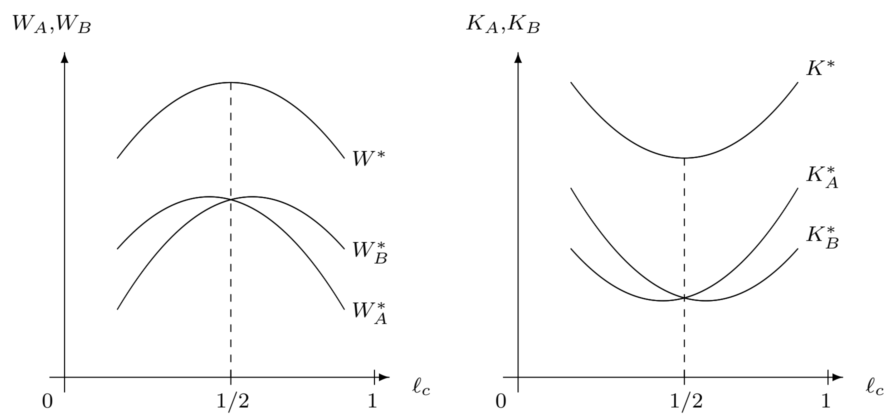
midpoint between the two warlords' strongholds and almost encompasses the entire geographical landscape of the economy. Table 4.4 illustrates the corresponding values for the variables in equilibrium. Again, the blue shaded cells in Table 4.4 show decreasing values and the green cells show increasing values when ℓ_c moves from 0 to 1.

From the first two columns of Table 4.4, Ω_A and Ω_B are again less than 19.11 for $0.97 > \ell_c > 0.3$ and, hence, the two conditions found in equation (4.5) are satisfied. As in the previous examples, the increase in β causes an increase in the hiring of warriors and a further decrease in capital investment. The last column shows that $\widetilde{K}$

ranges from 85.77% to a full 100%; that is, the increase in the value of β decreases total production in the economy but increase the number of these goods and services that will be seized during conflict such that $\widetilde{K} = 100\%$ when $\ell_c = 1/2$

4.3 Summary

In all of the above four examples, the midpoint $\ell_c = 1/2$ is a focal point in comparing the decisions made by the warlords. At $\ell_c = 1/2$, warlords hire the same number of warriors and invest in the same amount of capital; that is, $W_A^* = W_B^*$ and $K_A^* = K_B^*$. As the point of conflict moves closer to $\ell_A = 0$, warlord A will hire more warriors and invest in less capital than warlord B. Likewise, warlord A will hire less warriors and invest in more capital than warlord B as the point of conflict moves toward $\ell_B = 1$. The two graphs in Figure 4.3 illustrate the effect that the distance from the point of conflict has on warlord decision making.



These two graphs tell an interesting story in relationship with the asymmetric rate of seizure model. As the point of conflict moves away from $\ell_A = 0$ toward $\ell_B = 1$, warlord A briefly increases his participation in warfare but then quickly begins to decrease the hiring of warriors. Why? The answer lies on the adjacent graph. When the point of conflict is closer to warlord A, more of his production is being fought over and, hence, more warriors are needed to help secure as much of K_A^* as possible.

As ℓ_c moves away from $\ell_A = 0$, less of his production is being fought over. Since less of K_A^* is vulnerable, warlord A increases his production and has no real incentive to put up a fight against warlord B; that is, warlord A loses the urge to fight because the conflict is no longer a matter of protection as it is attrition. From this angle, warlord A winning less of $\widehat{K}$ as the point of conflict becomes more distant is because he chooses to fight less and yet is able to keep more of his own production. The same analysis holds for warlord B.

The above four basic examples elucidate the importance that β and ℓ_c have on both the value of the decision variables in equilibrium and on the geographical area in which an equilibrium can be found. As the value of β increase, so does the area where an equilibrium can be found. To reframe, as the level of seizure upon the warlords' production increases, the incentives are right for each warlord to venture past the midpoint into the opposing warlord's territory.

While the area in which an equilibrium can be found widens as β increases, it can also be observed that the total amount of production in the economy decreases. Similarly, as K^* decreases, so does $\widehat{K}$ — albeit at different rates. This decrease in K^* and $\widehat{K}$ are primarily due to increasing resources being spent on the hiring of warriors; that is, as W^* increases as β gets larger. The soaring level of $\widetilde{K}$ due to the increase in β should not be overstated or read into too much. Given that the elasticity of K^* in relation to β is substantially greater than the elasticity of $\widehat{K}$, the increase in $\widetilde{K}$ is more a result of decreasing production than an increase in $\widehat{K}$.

CHAPTER 5

Conclusion

I presented in the analysis above three models where two non-parasitic warlords, who are geographically connected, must decide on the amount of resources to be dedicated for conflict and the amount to be dedicated toward production.

Within the Base model, an equilibrium is found which depends upon a positive pre-game budget. In equilibrium, the exogenously chosen location of conflict affects each warlord's individual level of conflict and production, but has no effect on the aggregate number of warriors hired and the total quantity of goods and services produced. As a result, the equilibrium contest success function is independent of the location of the point of conflict but does depend upon the wages each warlord pays warriors and extractors hired. In addition, as the number of warlords participating in the economy increases, each warlord hires increasing number of warriors and invests in less capital and production of goods and services.

Under the Gates-logit and Ratio model, an interior equilibrium can be found that does not depend on positive pre-game budgets. In the Ratio model, the point of conflict affects both individual warlord levels of conflict and production as well as the aggregate number of warriors hired and quantity of goods and services produced. Unlike the Base and Ratio models, the total production of goods and services produced in the Gates-logit model increases when either warlord's populations size grows or is given a large pre-game budget. According to the Ratio model, the total production of goods and services produced is dependent upon the location of the point of conflict. Specifically, when the point of conflict is at the midpoint, the total production of goods and services is at its lowest and the total number of warriors hired is at its highest. As the point of conflict moves toward either warlord A's stronghold or warlord B's stronghold, the total production of goods and services within the economy increases and the total number of warriors hired decreases. In addition, the equilibrium contest success functions in both the Gates-logit and Ratio models are dependent upon the point of conflict and the wages paid to both warriors and extractors by each warlord. As the point of conflict gets closer to warlord A's stronghold, warlord A wins a larger share of the total production. As the point of conflict moves away from warlord A's stronghold and toward warlord B's, warlord A wins less and warlord B gains more of the total production.

Using the Base model formulation of the contest success function, I extend the original model by abandoning the assumption that all of a warlord's production of goods and services are available for appropriation. Instead, only a portion — dependent upon the location of the point of conflict — of a warlord's production is under threat of appropriation. In contrast to the original model, this new approach finds that a warlord hires more warriors and invests in less capital when the point of conflict is closer to his stronghold. This is in direct contrast to the results above.

BIBLIOGRAPHY

BIBLIOGRAPHY

Acemoglu, D., and J. A. Robinson (2006), *Economic Origins of Dictatorship and Democracy*, Cambridge University Press.

Ali, T. M., and R. O. Matthews (Eds.) (1999), *Civil Wars in Africa: Roots and Resolution*, McGill-Queen's University Press.

Arbatskaya, M., and H. M. Mialon (2008), Multi-Activity Contests, *Economic Theory*, pp. 1–21.

Bester, H., and K. A. Konrad (2004), Delay in Contests, *European Economic Review*, *48*, 1169–1178.

Bester, H., and K. Wärneryd (2006), Conflict and the Social Contract, *Scandinavian Journal of Economics*, *108*(2), 231–249.

Biak, K. H. (2008), Contests with Group-Specific Public-Good Prizes, *Social Choice Welfare*, *30*, 103–117.

Blavatskyy, P. (2010), Contest Success Function with the Possibility of a Draw: Axiomatization, *Journal of Mathematical Economics*, *46*(2), 267–276.

Brito, D. L. (1988), A Predator-Prey Model of Guerrilla Warfare, *Synthese*, *76*(2), 235–244.

Brito, D. L., and M. D. Intriligator (1990), An Economic Model of Guerrilla Warfare, *International Interactions: Empirical and Theoretical Research in International Relations*, *15*(3-4), 319–329.

Brito, D. L., and M. D. Intriligator (1992), Narco-Traffic and Guerrilla Warfare: A New Symboisis, *Defense Economics*, *3*(4), 263–274.

Buhaug, H., and S. Gates (2002), The Geography of Civil War, *Journal of Peace Research*, *39*(4), 417–433.

Buhaug, H., and J. K. Rød (2006), Local Determinants of African Civil Wars, 1971-2001, *Political Geography*, *25*(3), 315–335.

Buhaug, H., S. Gates, and P. Lujala (2009), Geography, Rebel Capability and the Duration of Civil Conflict, *Journal of Conflict Resolution*, *53*(4), 544–569.

Clark, D., and C. Raiis (1998), Contest Success Functions: An Extension, *Economic Theory*, *11*, 201–204.

Collier, P., and A. Hoeffler (2004), Greed and Grievance in Civil War, *Oxford Economic Papers*, *56*(4), 663–695.

Collier, P., and N. Sambanis (Eds.) (2005a), *Understanding Civil War: Evidence and Analysis*, vol. 1: Afica, The International Bank for Reconstruction and Development/ The World Bank.

Collier, P., and N. Sambanis (Eds.) (2005b), *Understanding Civil War: Evidence and Analysis*, vol. 2: Europe, Central Asia and Other Regions, The International Bank for Reconstruction and Development/ The World Bank.

Corchon, L. C. (2007), A Theory of Contests: A Survey, *Review of Economic Design*, *11*(2), 69–100.

Dixit, A. (1987), Strategic Behavior in Contests, *The American Economic Review*, *77*(5), 891–898.

Fearon, J. D. (1995), Rationalist Explanations of War, *Internal Organization*, *49*, 379–414.

Fearon, J. D., and D. Laitin (2003), Ethnicity, Insurgency and Civil War, *American Political Science Review*, *97*(1), 75–90.

Ferrero, M. (2005), Radicalization as a Reaction to Failure: An Economic Model of Islamic Extremism, *Public Choice*, *122*, 199 – 220.

Ferrero, M. (2008), The Triumph of Christianity in the Roman Empire: An Economic Interpretation, *European Journal of Political Economy*, *24*, 73 – 87.

Findlay, R. (1996), *The Political Economy of Conflict and Appropriation*, chap. 3: Toward a Model of Territorial Expansion and the Limits of Empire, pp. 41–56, 1 ed., Cambridge University Press.

Garfinkel, M. R. (2004), Stable Alliance Formation in Distributional Conflict, *European Journal of Political Economy*, *20*, 829–852.

Garfinkel, M. R., and S. Skaperdas (Eds.) (1996), *The Political Economy of Conflict and Appropriation*, 1 ed., Cambridge University Press.

Garfinkel, M. R., and S. Skaperdas (2007), *Handbook of Defense Economics*, vol. 2, 1 ed., Elsvier.

Gates, S. (2002), Recruitment and Allegiance: The Microfoundations of Rebellion, *Journal of Conflict Resolution*, *46*(1), 111–130.

Grossman, H. I. (1991), A General Equilibrium Model of Insurrections, *The American Economic Review*, *81*(4), 912 – 921.

Grossman, H. I. (1995), Robin Hood and the Redistribution of Property Income, *European Journal of Political Economy*, *11*, 399 – 410.

Grossman, H. I. (1999), Kleptocracy and Revolutions, *Oxford Economic Papers*, *51*(2), 267–283.

Grossman, H. I. (2002), "Make Us a King":Anarchy, Predation and the State, *European Journal of Political Economy*, *18*, 31–46.

Grossman, H. I., and M. Kim (1995), Swords or Plowshares? A Theory of the Security of Claims to Property, *The Journal of Political Economy*, *103*(6), 1275–1288.

Grossman, H. I., and M. Kim (2000), Predators, Moral Decay and Moral Rivals, *European Journal of Political Economy*, *16*, 173–187.

Haavelmo, T. (1954), A Study in the Theory of Economic Evolution.

Hartley, K., and T. Sandler (1995), *Handbook of Defense Economics*, vol. 1, 1 ed., Elsvier.

Hirshleifer, J. (1988), The Analytics of Continuing Conflict, *Synthese*, *76*(2), 201–233.

Hirshleifer, J. (1989), Conflict and Rent-Seeking Success Functions: Ratio vs. Difference Models of Relative Success, *Public Choice*, *63*, 101–112.

Hirshleifer, J. (1991), The Technology of Conflict as an Economic Activity, *American Economic Review*, *8*(2), 130–134.

Hirshleifer, J. (1995), Anarchy and its Breakdown, *Journal of Political Economy*, *103*(1), 26–52.

Hirshleifer, J. (2001), *The Dark Side of the Force: Economic Foundations of Conflict Theory*, Cambridge University Press.

Jai, H., S. Skaperdas, and S. Vaidya (2012), Contest Functions: Theoretical Foundations and Issues in Estimation, `http://www.economics.uci.edu/files/economics/docs/workingpapers/2011-2012/skaperdas-14.pdf`. Accessed on 06/19/2013.

Jordan, J. (2006), Pillage and Property, *Journal of Economic Theory*, *131*, 26–44.

Jung, H. M. (2009), Spatial Pillage Game, *Journal of Mathematical Economics*, *45*, 701–707.

Konrad, K., and S. Skaperdas (2012), The Market for Protection and the Origin of the State, *Economic Theory*, *50*(2), 417–443.

Konrad, K. A. (2004), Bidding in Hierarchies, *European Economic Review*, *48*, 1301–1308.

Konrad, K. A. (2007), Strategy in contests: An introduction, wZB-Markets and Politics Working Paper N0. SP II 2007-01, `http://econstor.eu/bitstream/10419/51106/1/565198513.pdf`. Accessed on 06/19/2013.

Kuran, T. (1989), Sparks and Prairie Fires: A Theory of Unanticipated Political Revolution, *Public Choice*, *61*, 41 – 74.

McBride, M., and S. Skaperdas (2005), Explaining Conflict in Low-Income Countries: Incomplete Contracting in the Shadow of the Future, university of California-Irvine Working Papers Number 050606, `http://www.economics.uci.edu/files/economics/docs/workingpapers/2005-06/McBride-06.pdf`. Accessed on 06/19/2013.

Mehlum, H., K. OveMoene, and R. Torvik (2003), Parasites, memorandum, `http://www.sv.uio.no/econ/english/research/memorandum/pdf-files/2003/Memo-16-2003.pdf`. Accessed on 06/19/2013.

Moldovanu, B., and A. Sela (2001), The Optimal Allocation of Prizes in Contests, *American Economic Review*, *91*(3), 542–558.

Munster, J. (2009), Group Contest Success Functions, *Economic Theory*, *41*, 345–357.

Noh, S. J. (2002), Resource Distribution and Stable Alliances with Endogenous Sharing Rules, *European Journal of Political Economy*, *18*, 129 – 151.

Olsson, O. (2007), Conflict Diamonds, *Journal of Development Economics*, *82*, 267–286.

Piccione, M., and A. Rubinstein (2007), Equilibrium in the Jungle, *The Economic Journal*, *117*, 883–896.

Powell, R. (1993), Guns, Butter and Anarchy, *American Political Science Review*, *87*, 115–132.

Rai, B. K., and R. Sarin (2009), Generalized Contest Success Functions, *Economic Theory*, *40*, 139–149.

Reno, W. (1998), *Warlord Politics and African States*, Lynne Rienner Publishers.

Roemer, J. E. (1985), Rationalizing Revolutionary Ideology, *Econometrica*, *53*(1), 85 – 108.

Sambanis, N. (2002), A Review of Recent Advances and Future Directions in the Quantitative Literature on Civil War, *Defence and Peace Economics*, *13*(3), 215–243.

Sanchez-Pages, S. (2004), The Use of Conflict as a Bargaining Tool Against Unsophisticated Opponents, eSE Discussion Papers Number 99, `http://www.econ.ed.ac.uk/papers/unsophisticated.pdf`. Accessed on 06/19/2013.

Sandler, T. (1999), Formation, Alliance Expansion and the Core, *The Journal of Conflict Resolution*, *43*(6), 727–747.

Sandler, T., and K. Hartley (2001), Economics of Alliances: The Lessons for Collective Action, *Journal of Economic Literature*, *39*, 869–896.

Skaperdas, S. (1992), Cooperation, Conflict and Power in the Absence of Property Rights, *American Economic Review*, *82*(4), 720–739.

Skaperdas, S. (1996), Contest Success Functions, *Economic Theory*, *7*(2), 283–290.

Skaperdas, S. (1998), On the Formation of Alliances in Conflict and Contests, *Public Choice*, *96*, 25–42.

Skaperdas, S. (2001), The Political Economy of Organized Crime: Providing Protection When the State Does Not, *Economics of Governance*, *2*, 173–202.

Skaperdas, S. (2002), Warlord Competition, *Journal of Peace Research*, *39*(4), 435–446.

Skaperdas, S. (2008), An Economic Approach to Analyzing Civil Wars, *Economics of Governance*, *9*, 25–44.

Szymanski, S. (2003), The Economic Design of Sporting Contests, *Journal of Economic Literature*, *41*(4), 1137–1187.

Tullock, G. (1980), Efficient Rent Seeking, in *Toward a Theory of the Rent-Seeking Society*, edited by J. Buchanan, R. Tollison, and G. Tullock, pp. 97–112, Texas A&M University Press.

Usher, D. (1989), The Dynastic Cycle and the Stationary State, *American Economic Review*, *79*(5), 1031–1044.

Wittman, D. (2000), The Wealth and Size of Nations, *The Journal of Conflict Resolution*, *44*(6), 868–884.

APPENDIX

$\diamond$ Proof for Theorem 1:

The proof for Theorem 1 begins by maximizing warlord A and B's optimization problems for the two choice variables W and K. Let λ_A and λ_B be the associated Lagrangian multipliers for maximization problems (3.18) and (3.19). The Lagrangian equations for warlords A and B are

$$\mathscr{L}_A = \left(\frac{m\left(K_A+K_B\right)}{1+e^{\alpha\cdot\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)+\lambda_A\left(\aleph_A-\left(\frac{c_w^A+\sigma_A}{\sigma_A}\right)W_A-\left(\frac{c_k}{\sigma_A}\right)K_A\right) \quad (A.1)$$

$$\mathscr{L}_B = \left(\frac{m\left(K_A+K_B\right)}{1+e^{\alpha\cdot\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)+\lambda_B\left(\aleph_B-\left(\frac{c_w^B+\sigma_B}{\sigma_B}\right)W_B-\left(\frac{c_k}{\sigma_B}\right)K_B\right) \quad (A.2)$$

Therefore:

$$\frac{\partial\mathscr{L}_A}{\partial W_A}=0 \;\Rightarrow\; \frac{\alpha\cdot m\left(K_A+K_B\right)\cdot e^{\alpha\cdot\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}-\left(\frac{c_w^A}{\sigma_A}+1\right)\cdot\lambda_A=0 \quad (A.3)$$

$$\frac{\partial\mathscr{L}_A}{\partial K_A}=0 \;\Rightarrow\; \frac{m}{1+e^{\alpha\cdot\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}-\frac{c_k}{\sigma_A}\cdot\lambda_A=0 \quad (A.4)$$

$$\frac{\partial\mathscr{L}_B}{\partial W_B}=0 \;\Rightarrow\; \frac{\alpha\cdot m\left(K_A+K_B\right)\cdot e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\cdot\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}-\left(\frac{c_w^B}{\sigma_B}+1\right)\cdot\lambda_B=0 \quad (A.5)$$

$$\frac{\partial\mathscr{L}_B}{\partial K_B}=0 \;\Rightarrow\; \frac{m}{1+e^{\alpha\cdot\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}-\frac{c_k}{\sigma_B}\cdot\lambda_B=0. \quad (A.6)$$

Using equations (A.3) and (A.4),

$$
\begin{aligned}
\left((\alpha\cdot m)\cdot\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\ell\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}\right)\cdot(K_A+K_B)\cdot\left(\frac{\sigma_A}{c_w^A+\sigma_A}\right) &= \left(\frac{m}{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(\frac{\sigma_A}{c_k}\right)\\
\left(\alpha\cdot\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\ell\right)}}{1+e^{\alpha\left(W_B-W_A+\phi\cdot\ell\right)}}\right)\cdot(K_A+K_B)\cdot\left(\frac{\sigma_A}{c_w^A+\sigma_A}\right) &= \left(\frac{\sigma_A}{c_k}\right)\\
\left(\alpha\cdot\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\ell\right)}}{1+e^{\alpha\left(W_B-W_A+\phi\cdot\ell\right)}}\right)\cdot(K_A+K_B)\cdot\left(\frac{c_k}{c_w^A+\sigma_A}\right) &= 1\\
\left(e^{\alpha\left(W_B-W_A+\phi\cdot\ell\right)}\right)\cdot(K_A+K_B)\cdot\left(\alpha\cdot\frac{c_k}{c_w^A+\sigma_A}\right) &= 1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\\
\left(e^{\alpha\left(W_B-W_A+\phi\cdot\ell\right)}\right)\cdot\left((K_A+K_B)\cdot\left(\alpha\cdot\frac{c_k}{c_w^A+\sigma_A}\right)\right)-e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)} &= 1\\
\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot\left((K_A+K_B)\cdot\left(\alpha\cdot\frac{c_k}{c_w^A+\sigma_A}\right)-1\right) &= 1\\
e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)} &= \left(\frac{c_w^A+\sigma_A}{(\alpha\cdot c_k)\cdot(K_A+K_B)-\left(c_w^A+\sigma_A\right)}\right)\\
W_B-W_A+\phi\cdot\hat{\ell} &= \frac{1}{\alpha}\cdot\ln\left(\frac{\left(\frac{c_w^A}{\sigma_A}+1\right)}{\frac{\alpha\cdot c_k}{m_R-c_E}(K_A+K_B)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right)\\
W_B+\phi\cdot\hat{\ell}-\frac{1}{\alpha}\cdot\ln\left(\frac{\left(\frac{c_w^A}{\sigma_A}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_A}(K_A+K_B)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right) &= W_A. \qquad \text{(A.7)}
\end{aligned}
$$

Solving warlord B's optimization problem from equation (3.19) and performing similar substitutions,

$$\begin{aligned}
\left((\alpha\cdot m)\cdot\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}\right)\cdot(K_A+K_B)\cdot\left(\frac{\sigma_B}{c_w^B+\sigma_B}\right) &= \left(\frac{m}{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(\frac{\sigma_B}{c_k}\right)\\
\left(\alpha\cdot\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot(K_A+K_B)\cdot\left(\frac{\sigma_B}{c_w^B+\sigma_B}\right) &= \left(\frac{\sigma_B}{c_k}\right)\\
\left(\alpha\cdot\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot(K_A+K_B)\cdot\left(\frac{c_k}{c_w^B+\sigma_B}\right) &= 1\\
\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot(K_A+K_B)\cdot\left(\frac{\alpha\cdot c_k}{c_w^B+\sigma_B}\right) &= 1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\\
\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot\left((K_A+K_B)\cdot\left(\frac{\alpha\cdot c_k}{c_w^B+\sigma_B}\right)\right)-e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)} &= 1\\
\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot\left((K_A+K_B)\cdot\left(\frac{\alpha\cdot c_k}{c_w^B+\sigma_B}\right)-1\right) &= 1\\
e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)} &= \left(\frac{c_w^B+\sigma_B}{(\alpha\cdot c_k)\cdot(K_A+K_B)-\left(c_w^B+\sigma_B\right)}\right)\\
W_A-W_B-\phi\cdot\hat{\ell} &= \frac{1}{\alpha}\cdot\ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_B}(K_A+K_B)-\left(\frac{c_w^B}{\sigma_B}+1\right)}\right)\\
W_A-\phi\cdot\hat{\ell}-\frac{1}{\alpha}\cdot\ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_B}(K_A+K_B)-\left(\frac{c_w^B}{\sigma_B}+1\right)}\right) &= W_B. \qquad \text{(A.8)}
\end{aligned}$$

By substituting equation (A.7) into (A.8),

$$
\begin{aligned}
W_A &= W_B + \phi\cdot\hat{\ell} - \tfrac{1}{\alpha}\cdot\ln\left(\frac{\left(\frac{c_w}{\sigma_A}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_A}(K_A+K_B)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right)\\
W_A &= \left(W_A-\phi\cdot\hat{\ell}-\tfrac{1}{\alpha}\cdot\ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_B}(K_A+K_B)-\left(\frac{c_w^B}{\sigma_B}+1\right)}\right)\right)+\phi\cdot\hat{\ell}\\
&\qquad -\tfrac{1}{\alpha}\cdot\ln\left(\frac{\left(\frac{c_w^A}{\sigma_A}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_A}(K_A+K_B)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right)\\
0 &= -\ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_B}(K_A+K_B)-\left(\frac{c_w^B}{\sigma_B}+1\right)}\right)\\
&\qquad -\ln\left(\frac{\left(\frac{c_w^A}{\sigma_A}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_A}(K_A+K_B)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right)\\
\ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_B}(K_A+K_B)-\left(\frac{c_w^B}{\sigma_B}+1\right)}\right) &= -\ln\left(\frac{\left(\frac{c_w^A}{\sigma_A}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_A}(K_A+K_B)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right)\\
\ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\frac{\alpha\cdot c_k}{\sigma_B}(K_A+K_B)-\left(\frac{c_w^B}{\sigma_B}+1\right)}\right) &= \ln\left(\frac{\left(\frac{\alpha\cdot c_k}{\sigma_A}\right)(K_A+K_B)-\left(\frac{c_w^A}{\sigma_A}+1\right)}{\frac{c_w^A}{\sigma_A}+1}\right)\\
\left(\frac{c_w^B}{\sigma_B}+1\right)\left(\frac{c_w^A}{\sigma_A}+1\right) &= (\alpha\cdot c_k)^2\cdot\left(\frac{(K_A+K_B)^2}{\sigma_A\cdot\sigma_B}\right)+\left(\frac{c_w^A}{\sigma_A}+1\right)\left(\frac{c_w^B}{\sigma_B}+1\right)-\left(\frac{\alpha\cdot c_k(K_A+K_B)}{\sigma_A\cdot\sigma_B}\right)\left(c_w^A+c_w^B+\sigma_A+\sigma_B\right)\\
(\alpha\cdot c_k)^2\cdot\left(\frac{(K_A+K_B)^2}{\sigma_A\cdot\sigma_B}\right) &= \left(\frac{\alpha\cdot c_k(K_A+K_B)}{\sigma_A\cdot\sigma_B}\right)\left(c_w^A+c_w^B+\sigma_A+\sigma_B\right)\\
K_A+K_B &= \frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\alpha\cdot c_k}. \qquad \text{(A.9)}
\end{aligned}
$$

Substituting equation (A.9) into (A.7) and (A.8),

$$\begin{aligned}
W_A &= W_B + \phi \cdot \hat{\ell} - \frac{1}{\alpha} \cdot \ln\left(\frac{\left(\frac{c_w^A}{\sigma_A}+1\right)}{\frac{\alpha \cdot c_k}{\sigma_A}\left(K_A+K_B\right)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right) \\
&= W_B + \phi \cdot \hat{\ell} - \frac{1}{\alpha} \cdot \ln\left(\frac{\left(\frac{c_w^A}{\sigma_A}+1\right)}{\frac{\alpha \cdot c_k}{\sigma_A}\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\alpha \cdot c_k}\right)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right) \\
&= W_B + \phi \cdot \hat{\ell} - \frac{1}{\alpha} \cdot \ln\left(\frac{\left(\frac{c_w^A}{\sigma_A}+1\right)}{\left(\frac{c_w^A}{\sigma_A}+1\right)+\left(\frac{c_w^B+\sigma_B}{\sigma_A}\right)-\left(\frac{c_w^A}{\sigma_A}+1\right)}\right) \\
W_A &= W_B + \phi \cdot \hat{\ell} - \frac{1}{\alpha} \cdot \ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right), \qquad \text{(A.10)}
\end{aligned}$$

and

$$\begin{aligned}
W_B &= W_A - \phi \cdot \hat{\ell} - \frac{1}{\alpha} \cdot \ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\frac{\alpha \cdot c_k}{\sigma_B}\left(K_A+K_B\right)-\left(\frac{c_w}{\sigma_B}+1\right)}\right) \\
&= W_A - \phi \cdot \hat{\ell} - \frac{1}{\alpha} \cdot \ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\frac{\alpha \cdot c_k}{\sigma_B}\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\alpha \cdot c_k}\right)-\left(\frac{c_w^B}{\sigma_B}+1\right)}\right) \\
&= W_A - \phi \cdot \hat{\ell} - \frac{1}{\alpha} \cdot \ln\left(\frac{\left(\frac{c_w^B}{\sigma_B}+1\right)}{\left(\frac{c_w^B+\sigma_B}{\sigma_B}+1\right)+\left(\frac{c_w^A+\sigma_A}{\sigma_B}\right)-\left(\frac{c_w^B}{\sigma_B}+1\right)}\right) \\
W_B &= W_A - \phi \cdot \hat{\ell} - \frac{1}{\alpha} \cdot \ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right). \qquad \text{(A.11)}
\end{aligned}$$

Using warlord A's constraint from equation (3.8) and the above equations, the equilibrium level of capital investment for warlord B is found; that is,

$$\begin{aligned}
\aleph_A &= \left(\frac{c_w^A}{\sigma_A}+1\right)W_A+\frac{c_k}{\sigma_A}K_A \\
&= \left(\frac{c_w^A}{\sigma_A}\right)\left(W_B+\phi\cdot\hat{\ell}-\frac{1}{\alpha}\cdot\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)\right)+\left(\frac{c_k}{\sigma_A}+1\right)\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\alpha\cdot c_k}-K_B\right) \\
&= \left(\frac{c_w^A}{\sigma_A}+1\right)\left(\frac{N_B}{\frac{c_w^B}{\sigma_B}+1}+\frac{Y_B}{c_w^B+\sigma_B}-\frac{c_k}{c_w^B+\sigma_B}K_B\right)+\left(\frac{c_w^A}{\sigma_A}+1\right)\left(\phi\cdot\hat{\ell}-\frac{1}{\alpha}\cdot\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)\right) \\
&\quad +\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\alpha(\sigma_A)}\right)-\left(\frac{c_k}{\sigma_A}\right)K_B
\end{aligned}$$

$$\left(\frac{\sigma_A}{c_w^A+\sigma_A}\right)\cdot\aleph_A = \left(\frac{\sigma_B}{c_w^B+\sigma_B}\right)\cdot\aleph_B+\left(\phi\cdot\hat{\ell}-\frac{1}{\alpha}\cdot\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)\right)+\frac{1}{\alpha}\left(1+\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)-\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}\right)K_B\cdot c_k$$

$$\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}\right)K_B\cdot c_k = \left(\frac{\sigma_B}{c_w^B+\sigma_B}\right)\cdot\aleph_A-\left(\frac{\sigma_A}{c_w^A+\sigma_A}\right)\cdot\aleph_B+\frac{1}{\alpha}\left(1+\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)+\left(\phi\cdot\hat{\ell}-\frac{1}{\alpha}\cdot\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)\right)$$

which equals the equilibrium level of K_B^* in equation (3.24). Similarly for warlord A,

$$\begin{aligned}
\aleph_B &= \left(\frac{c_w^B}{\sigma_B}+1\right)W_B+\frac{c_k}{\sigma_B}K_B \\
&= \left(\frac{c_w^B}{\sigma_B}\right)\left(W_A-\phi\cdot\hat{\ell}-\frac{1}{\alpha}\cdot\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)\right)+\left(\frac{c_k}{\sigma_B}+1\right)\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\alpha\cdot c_k}-K_A\right) \\
&= \left(\frac{c_w^B}{\sigma_B}+1\right)\left(\frac{N_A}{\frac{c_w^A}{\sigma_A}+1}+\frac{Y_A}{c_w^A+\sigma_A}-\frac{c_k}{c_w^A+\sigma_A}K_A\right) \\
&\quad -\left(\frac{c_w^B}{\sigma_B}+1\right)\left(\phi\cdot\hat{\ell}+\frac{1}{\alpha}\cdot\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)\right)+\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\alpha(\sigma_B)}\right)-\left(\frac{c_k}{\sigma_B}\right)K_A \\
\left(\frac{\sigma_B}{c_w^B+\sigma_B}\right)\cdot\aleph_B &= \left(\frac{\sigma_A}{c_w^A+\sigma_A}\right)\cdot\aleph_A-\left(\phi\cdot\hat{\ell}+\frac{1}{\alpha}\cdot\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)\right)+\frac{1}{\alpha}\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)-\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}\right)K_A\cdot c_k \\
\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}\right)K_A\cdot c_k &= \left(\frac{\sigma_A}{c_w^A+\sigma_A}\right)\cdot\aleph_A-\left(\frac{\sigma_B}{c_w^B+\sigma_B}\right)\cdot\aleph_B+\frac{1}{\alpha}\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)-\left(\phi\cdot\hat{\ell}+\frac{1}{\alpha}\cdot\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)\right)
\end{aligned}$$

where, again, rearranging the variables leads to the equilibrium level of K_A^* in equation (3.23).

Substituting the equilibrium level of capital K_A^*, from equation (3.23), into warlord A's constraint from equation (3.8) leads to the equilibrium number of warriors W_A^*,

found in equation (3.21):

$$W_A^* \cdot \left(\frac{c_w^A}{\sigma_A} + 1 \right) = \aleph_A - \left(\frac{c_k}{\sigma_A} \right) \cdot K_A$$

$$= \aleph_A - \frac{1}{\sigma_A} \left(\frac{\left(c_w^A + \sigma_A \right)\left(c_w^B + \sigma_B \right)\left(\frac{1}{\alpha} \ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \right) - \phi \cdot \hat{\ell} \right)}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) - \left(\frac{c_w^A + \sigma_A}{\alpha \cdot (\sigma_A)} \right)$$

$$- \frac{1}{\sigma_A} \left(\frac{\left(c_w^B + \sigma_B \right)(\sigma_A) \cdot \aleph_A - \left(c_w^A + \sigma_A \right)(\sigma_B) \cdot \aleph_B}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right)$$

$$W_A^* = \left(\frac{\sigma_A}{c_w^A + \sigma_A} \right) \cdot \aleph_A - \left(\frac{\left(c_w^B + \sigma_B \right)\left(\frac{1}{\alpha} \ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \right) - \phi \cdot \hat{\ell} \right)}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) - \left(\frac{\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \right)(\sigma_A) \cdot \aleph_A - (\sigma_B) \cdot \aleph_B}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) - \frac{1}{\alpha}$$

$$= \left(\frac{(\sigma_A)(\aleph_A)\left(\frac{c_w^A + c_w^B + \sigma_A + \sigma_B - \left(c_w^B + \sigma_B \right)}{c_w^A + \sigma_A} \right)}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) - \left(\frac{(\sigma_B) \cdot \aleph_B}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) - \left(\frac{\left(c_w^B + \sigma_B \right)\left(\frac{1}{\alpha} \ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \right) - \phi \cdot \hat{\ell} \right)}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) - \frac{1}{\alpha}$$

$$W_A^* = \frac{(\sigma_A) \cdot \aleph_A + (\sigma_B) \cdot \aleph_B}{c_w^A + c_w^B + \sigma_A + \sigma_B} + \left(\frac{c_{w_B} + \sigma_B}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) \cdot \left(\phi \cdot \hat{\ell} + \frac{1}{\alpha} \ln\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B} \right) \right) - \frac{1}{\alpha}.$$

Similarly, substituting the equilibrium level of capital K_B^*, from equation (3.24), into warlord B's constraint from equation (3.9) leads to the equilibrium number of warriors

W_B^*, found in equation (3.22):

$$W_B^* \cdot \left(\frac{c_w^B}{\sigma_B}+1\right) = \aleph_B - \left(\frac{c_k}{\sigma_B}\right)\cdot K_B$$

$$= \aleph_B - \frac{1}{\sigma_B}\left(\frac{\left(c_w^B+\sigma_B\right)\left(c_w^A+\sigma_A\right)\left(\frac{1}{\alpha}\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)+\phi\cdot\hat{\ell}\right)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right) - \left(\frac{c_w^B+\sigma_B}{\alpha\cdot(\sigma_B)}\right)$$

$$-\frac{1}{\sigma_B}\left(\frac{\left(c_w^A+\sigma_A\right)(\sigma_B)\cdot\aleph_B-\left(c_w^B+\sigma_B\right)(\sigma_A)\cdot\aleph_A}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)$$

$$W_B^* = \left(\frac{\sigma_B}{c_w^B+\sigma_B}\right)\cdot\aleph_B - \left(\frac{\left(c_w^A+\sigma_A\right)\left(\frac{1}{\alpha}\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)-\phi\cdot\hat{\ell}\right)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right) - \left(\frac{\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)\cdot\aleph_B-(\sigma_A)\cdot\aleph_A}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right) - \frac{1}{\alpha}$$

$$= \left(\frac{(\sigma_B)(\aleph_B)\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B-\left(c_w^A+\sigma_A\right)}{c_w^B+\sigma_B}\right)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right) - \left(\frac{(\sigma_A)\cdot\aleph_A}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right) - \left(\frac{\left(c_w^A+\sigma_A\right)\left(\frac{1}{\alpha}\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)+\phi\cdot\hat{\ell}\right)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right) - \frac{1}{\alpha}$$

$$W_B^* = \frac{(\sigma_B)\cdot\aleph_B+(\sigma_A)\cdot\aleph_A}{c_w^A+c_w^B+\sigma_A+\sigma_B} + \left(\frac{c_{w_A}+\sigma_A}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)\cdot\left(\frac{1}{\alpha}\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)-\phi\cdot\hat{\ell}\right)-\frac{1}{\alpha}.$$

To show the second-order conditions, the bordered Hessian for warlord A is

$$\mathcal{H}_A^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w^A}{\sigma_A}+1\right) & -\frac{c_k}{\sigma_A} \\ -\left(\frac{c_w^A}{\sigma_A}+1\right) & \frac{\partial^2\mathscr{L}_A}{\partial W_A^2} & \frac{\partial^2\mathscr{L}_A}{\partial W_A K_A} \\ -\frac{c_k}{\sigma_A} & \frac{\partial^2\mathscr{L}_A}{\partial K_A W_A} & \frac{\partial^2\mathscr{L}_A}{\partial K_A^2} \end{pmatrix} \tag{A.12}$$

$$= \begin{pmatrix} 0 & -\left(\frac{c_w^A}{\sigma_A}+1\right) & -\frac{c_k}{\sigma_A} \\ -\left(\frac{c_w^A}{\sigma_A}+1\right) & \alpha^2\cdot\frac{\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}-1\right)}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^3}\cdot m\cdot(K_A+K_B) & \frac{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}\cdot(\alpha\cdot m) \\ -\frac{c_k}{\sigma_A} & \frac{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}\cdot(\alpha\cdot m) & 0 \end{pmatrix}$$

and is satisfied when the determinant, $|\mathcal{H}_A^{\mathcal{B}}|$, is greater than zero; that is,

$$|\mathcal{H}_A^{\mathcal{B}}|>0 \quad \rightarrow \quad \left(\frac{c_w^A}{\sigma_A}+1\right)\cdot\left(\frac{c_k}{\sigma_A}\right)\cdot\left(\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}\cdot(\alpha\cdot m)\right)$$

$$-\left(\frac{c_k}{\sigma_A}\right)^2\cdot\left(\alpha^2\cdot\frac{\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}-1\right)}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^3}\cdot m\cdot(K_A+K_B)\right)$$

$$+\left(\frac{c_k}{\sigma_A}\right)\cdot\left(\left(\frac{c_w}{\sigma_A}+1\right)\cdot\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}\cdot(\alpha\cdot m)\right)>0$$

$$\rightarrow \quad \left(\frac{c_w^A}{\sigma_A}+1\right)\left(\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}\right)$$

$$-\left(\alpha\cdot\frac{\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}-1\right)}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^3}\cdot(K_A+K_B)\right)\left(\frac{c_k}{\sigma_A}\right)$$

$$+\left(\frac{c_w^A}{\sigma_A}+1\right)\cdot\left(\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}\right)>0$$

$$|\mathcal{H}_A^{\mathcal{B}}|>0 \quad \rightarrow \quad \frac{2(c_w^A+\sigma_A)}{c_k(K_A+K_B)}>\alpha\cdot\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}-1}{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}. \tag{A.13}$$

Similarly, the bordered Hessian for warlord B is

$$\mathcal{H}_B^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w^B}{\sigma_B}+1\right) & -\frac{c_k}{\sigma_B} \\ -\left(\frac{c_w^B}{\sigma_B}+1\right) & \frac{\partial^2 \mathscr{L}_B}{\partial W_B^2} & \frac{\partial^2 \mathscr{L}_B}{\partial W_B K_B} \\ -\frac{c_k}{\sigma_B} & \frac{\partial^2 \mathscr{L}_B}{\partial K_B W_B} & \frac{\partial^2 \mathscr{L}_B}{\partial K_B^2} \end{pmatrix} \tag{A.14}$$

$$= \begin{pmatrix} 0 & -\left(\frac{c_w^B}{\sigma_B}+1\right) & -\frac{c_k}{\sigma_B} \\ -\left(\frac{c_w^B}{\sigma_B}+1\right) & \alpha^2 \cdot \frac{\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}-1\right)}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^3}\cdot m\cdot(K_A+K_B) & \frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}\cdot(\alpha\cdot m) \\ -\frac{c_k}{\sigma_B} & \frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}\cdot(\alpha\cdot m) & 0 \end{pmatrix}$$

and is satisfied when the determinant, $|\mathcal{H}_B^{\mathcal{B}}|$, is greater than zero; that is,

$$|\mathcal{H}_A^{\mathcal{B}}|>0 \quad \rightarrow \quad \left(\tfrac{c_w^B}{\sigma_B}+1\right)\cdot\left(\tfrac{c_k}{\sigma_B}\right)\cdot\left(\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}\cdot(\alpha\cdot m)\right)$$

$$-\left(\tfrac{c_k}{\sigma_B}\right)^2\cdot\left(\alpha^2\cdot\frac{\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}-1\right)}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^3}\cdot m\cdot(K_A+K_B)\right)$$

$$+\left(\tfrac{c_k}{\sigma_B}\right)\cdot\left(\left(\tfrac{c_w^B}{\sigma_B}+1\right)\cdot\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}\cdot(\alpha\cdot m)\right)>0$$

$$\rightarrow \quad \left(\tfrac{c_w^B}{\sigma_B}+1\right)\left(\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}\right)$$

$$-\left(\alpha\cdot\frac{\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}-1\right)}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^3}\cdot(K_A+K_B)\right)\left(\tfrac{c_k}{\sigma_B}\right)$$

$$+\left(\tfrac{c_w^B}{\sigma_B}+1\right)\cdot\left(\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}\right)>0$$

$$|\mathcal{H}_B^{\mathcal{B}}|>0 \quad \rightarrow \quad \alpha\cdot\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}-1}{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}<\frac{2(c_w^B+\sigma_B)}{c_k(K_A+K_B)}. \tag{A.15}$$

Substituting the equilibrium levels of W_A^*, W_B^*, K_A^* and K_B^* into equation (A.13):

$$\alpha \cdot \frac{e^{\alpha \cdot \left(W_B - W_A + \phi \cdot \hat{\ell}\right)} - 1}{1 + e^{\alpha \cdot \left(W_B - W_A + \phi \cdot \hat{\ell}\right)}} \quad < \quad \frac{2\left(c_w^A + \sigma_A\right)}{c_k(K_A + K_B)}$$

$$\alpha \cdot \frac{e^{\left(\ln\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)\right)} - 1}{1 + e^{\left(\ln\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)\right)}} \quad < \quad \frac{2\left(c_w^A + \sigma_A\right)}{\left(\frac{c_w^A + c_w^B + \sigma_A + \sigma_B}{\alpha}\right)}$$

$$\frac{\frac{c_w^A + \sigma_A - \left(c_w^B + \sigma_B\right)}{c_w^B + \sigma_B}}{\frac{c_w^A + c_w^B + \sigma_A + \sigma_B}{c_w^B + \sigma_B}} \quad < \quad \frac{2\left(c_w^A + \sigma_A\right)}{c_w^A + c_w^B + \sigma_A + \sigma_B}$$

$$c_w^A + \sigma_A - \left(c_w^B + \sigma_B\right) \quad < \quad 2 \cdot \left(c_w^A + \sigma_A\right)$$

$$0 \quad < \quad c_w^A + c_w^B + \sigma_A + \sigma_B.$$

To check that the second-order condition is satisfied for warlord B, the equilibrium levels W_A^*, W_B^*, K_A^* and K_B^* are substituted into equation (A.15):

$$\alpha \cdot \frac{e^{\alpha \cdot \left(W_A - W_B - \phi \cdot \hat{\ell}\right)} - 1}{1 + e^{\alpha \cdot \left(W_A - W_B - \phi \cdot \hat{\ell}\right)}} \quad < \quad \frac{2\left(c_w^B + \sigma_B\right)}{c_k(K_A + K_B)}$$

$$\alpha \cdot \frac{e^{\left(\ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right)\right)} - 1}{1 + e^{\left(\ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right)\right)}} \quad < \quad \frac{2\left(c_w^B + \sigma_B\right)}{\left(\frac{c_w^A + c_w^B + \sigma_A + \sigma_B}{\alpha}\right)}$$

$$\frac{\frac{c_w^B + \sigma_B - \left(c_w^A + \sigma_A\right)}{c_w^A + \sigma_A}}{\frac{c_w^A + c_w^B + \sigma_A + \sigma_B}{c_w^A + \sigma_A}} \quad < \quad \frac{2\left(c_w^B + \sigma_B\right)}{c_w^A + c_w^B + 2 \cdot m_R - (c_{E_A} + c_{E_B})}$$

$$c_w^B + \sigma_B - \left(c_w^A + \sigma_A\right) \quad < \quad 2 \cdot \left(c_w^B + \sigma_B\right)$$

$$0 \quad < \quad c_w^A + c_w^B + \sigma_A + \sigma_B.$$

Finally, it needs to be shown that W_A^*, W_B^*, K_A^* and K_B^* are positive; that is, it needs to be shown that [1]

$$W_A^*, W_B^*, K_A^*, K_B^* > 0.$$

From equation (3.21), W_A^* is greater than or equal to zero when

$$\frac{(\sigma_A)\cdot\aleph_A+(\sigma_B)\cdot\aleph_B}{c_w^B+\sigma_B}>\frac{1}{\alpha}\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)-\phi\hat{\ell}+\frac{1}{\alpha}\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{c_w^B+\sigma_B}\right),$$

and from equation (3.23), K_A^* is greater than or equal to zero when

$$\frac{1}{\alpha}\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)-\phi\hat{\ell}+\frac{1}{\alpha}\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{c_w^B+\sigma_B}\right)>\frac{(\sigma_B)\cdot\aleph_B}{c_w^B+\sigma_B}-\frac{(\sigma_A)\cdot\aleph_A}{c_w^A+\sigma_A}.$$

The above two equations lead to the first equilibrium condition found. Likewise, the feasibility condition for the decision variables W_B^* and K_B^* are found in a similar way. From equation (3.22), W_B^* is greater than or equal to zero when

$$\frac{(\sigma_A)\cdot\aleph_A+(\sigma_B)\cdot\aleph_B}{c_w^A+\sigma_A}>\frac{1}{\alpha}\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)+\phi\hat{\ell}+\frac{1}{\alpha}\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{c_w^A+\sigma_A}\right),$$

and from equation (3.24), K_B^* is greater than or equal to zero when

$$\frac{1}{\alpha}\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)+\phi\hat{\ell}+\frac{1}{\alpha}\left(\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{c_w^A+\sigma_A}\right)>\frac{(\sigma_A)\cdot\aleph_A}{c_w^A+\sigma_A}-\frac{(\sigma_B)\cdot\aleph_B}{c_w^B+\sigma_B}.$$

⋄ Proof for Theorem 2:

Equation (3.26) is derived by substituting equations (3.21) and (3.22) into the CSF

[1] Showing that K_A^* and K_B^* are greater than zero is equivalent to proving that W_A^* and W_B^* are not greater than $\aleph_A$ and $\aleph_B$, respectively, due to the constraints found in equations (3.8) and (3.9).

found above in equation (3.3). Explicitly,

$$\begin{aligned}
\pi_A^* &= \frac{1}{1+e^{\alpha\left(W_B^*-W_A^*+\phi\cdot\hat{\ell}\right)}}\\
&= \frac{1}{1+e^{\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\\
&= \frac{1}{1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}.
\end{aligned}$$

Equation (3.27) follows by

$$\begin{aligned}
\pi_B^* &= 1-\pi_A^*\\
&= 1-\frac{c_w^B+\sigma_B}{c_w^A+c_w^B+\sigma_A+\sigma_B} = \frac{1}{1+\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}.
\end{aligned}$$

◇ Proof for Corollary 1:

1. Using warlord A and B's equilibrium capital investment decision equations (3.23) and (3.24), an increase in either N_A and/or Y_A will cause an increase the equilibrium levels of capital investment by warlord A and decrease it for warlord B:

$$\frac{\partial K_A^*}{\partial N_A}=\frac{1}{c_k}\left(\frac{\left(c_w^B+\sigma_B\right)(\sigma_A)}{c_w^A+\sigma_A+\sigma_B}\right)>0 \qquad \frac{\partial K_A^*}{Y_A}=\frac{1}{c_k}\left(\frac{c_w^B+\sigma_B}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)>0$$

$$\frac{\partial K_B^*}{\partial N_A}=\frac{-1}{c_k}\left(\frac{\left(c_w^B+\sigma_B\right)(\sigma_A)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)<0 \qquad \frac{\partial K_B^*}{Y_A}=\frac{-1}{c_k}\left(\frac{c_w^B+\sigma_B}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)<0.$$

Similarly, an increase in either N_B and/or Y_B will cause an increase in the equilibrium levels of capital investment by warlord B and decrease it for warlord A:

$$\frac{\partial K_A^*}{\partial N_B}=\frac{-1}{c_k}\left(\frac{\left(c_w^A+\sigma_A\right)(\sigma_B)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)<0 \qquad \frac{\partial K_A^*}{Y_B}=\frac{-1}{c_k}\left(\frac{c_w^A+\sigma_A}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)<0$$

$$\frac{\partial K_B^*}{\partial N_B}=\frac{1}{c_k}\left(\frac{\left(c_w^A+\sigma_A\right)(\sigma_B)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)>0 \qquad \frac{\partial K_B^*}{Y_B}=\frac{1}{c_k}\left(\frac{c_w^A+\sigma_A}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)>0.$$

2. This result follows from equation (3.25) and its subsequent proof. Specifically,

$$K^*=\frac{c_w^A+c_w^B+\sigma_A+\sigma_B}{\alpha\cdot c_k} \quad \Rightarrow \quad \frac{\partial K^*}{\partial N_A}=\frac{\partial K^*}{\partial N_B}=\frac{\partial K^*}{\partial Y_A}=\frac{\partial K^*}{\partial Y_B}=0.$$

3. From equations (3.21) and (3.22), an increase in either warlord's population size and/or pre-game budget will increase the number of warriors hired by both warlords:

$$\frac{\partial W_A^*}{\partial N_A}=\frac{\partial W_B^*}{\partial N_A}=\frac{\sigma_A}{\left(c_w^A+c_w^B+\sigma_A+\sigma_B\right)^2}>0 \qquad \frac{\partial W_A^*}{\partial Y_A}=\frac{\partial W_B^*}{\partial Y_A}=\frac{1}{\left(c_w^A+c_w^B+\sigma_A+\sigma_B\right)^2}>0$$

$$\frac{\partial W_B^*}{\partial N_B}=\frac{\partial W_A^*}{\partial N_B}=\frac{\sigma_B}{\left(c_w^A+c_w^B+\sigma_A+\sigma_B\right)^2}>0 \qquad \frac{\partial W_B^*}{\partial Y_B}=\frac{\partial W_A^*}{\partial Y_B}=\frac{1}{\left(c_w^A+c_w^B+\sigma_A+\sigma_B\right)^2}>0.$$

$\diamond$ Proof for Corollary 2:

1. From definition of $\hat{\ell}$, an increase of ℓ_c will cause $\hat{\ell}$ to decrease:

$$\frac{\partial\hat{\ell}}{\partial\ell_c}=2\cdot(\ell_c-\ell_A)-2\cdot(\ell_c-\ell_B)=2\cdot(\ell_c-0)-2\cdot(\ell_c-1)=2\cdot\ell_c-2\cdot\ell_c+2=2>0.$$

Given $\partial\hat{\ell}/\partial\ell_c>0$, an increase of the conflict location from 0 to 1 will result in an increase in K_B^* and a decrease in K_A^*:

$$\frac{\partial K_A^*}{\partial\ell_c}=\left(\frac{-2\cdot\phi}{c_k}\right)\left(\frac{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)<0; \qquad \frac{\partial K_B^*}{\partial\ell_c}=\left(\frac{2\cdot\phi}{c_k}\right)\left(\frac{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}{c_w^A+c_w^B+\sigma_A+\sigma_B}\right)>0.$$

2. This result follows from equation (3.25) and its subsequent proof. Specifically,

$$K^* = \frac{c_w^A + c_w^B + \sigma_A + \sigma_B}{\alpha \cdot c_k} \quad \Rightarrow \quad \frac{\partial K^*}{\partial \ell_c} = 0.$$

3. Using the equilibrium amount of hired warriors found in equations (3.21) and (3.22),

$$\frac{\partial W_A^*}{\partial \ell_c} = (2 \cdot \phi) \left(\frac{c_w^B + \sigma_B}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) > 0; \qquad \frac{\partial W_B^*}{\partial \ell_c} = (-2 \cdot \phi) \left(\frac{c_w^A + \sigma_A}{c_w^A + c_w^B + \sigma_A + \sigma_B} \right) < 0.$$

4. From equation (3.26),

$$\pi_A = \frac{1}{1 + \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}} \quad \Rightarrow \quad \frac{\partial \pi_A}{\partial \ell_c} = 0.$$

Likewise with equation (3.27),

$$\pi_B = \frac{1}{1 + \frac{c_w^B + \sigma_B}{c_w^A + \sigma_B}} \quad \Rightarrow \quad \frac{\partial \pi_B}{\partial \ell_c} = 0.$$

⋄ Proof for Corollary 3.2.1.3:

1. From equation (3.26), an increase in c_w^A will decrease π_{AB}:

$$\frac{\partial \pi_A}{\partial c_w^A} = \frac{\frac{-1}{c_w^B + \sigma_B}}{\left(1 + \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)^2} < 0,$$

and from equation (3.27), an increase in c_w^B will decrease π_B:

$$\frac{\partial \pi_B}{\partial c_w^B} = \frac{\frac{-1}{c_w^A + \sigma_A}}{\left(1 + \frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right)^2} < 0.$$

2. From equation (3.26), an increase in c_w^B will increase π_A:

$$\frac{\partial \pi_A}{\partial c_w^B} = \frac{\frac{c_w^A+\sigma_A}{(c_w^B+\sigma_B)^2}}{\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)^2} > 0.$$

Likewise, from equation (3.27), an increase in c_w^A will increase π_B:

$$\frac{\partial \pi_B}{\partial c_w^A} = \frac{\frac{c_w^B+\sigma_B}{(c_w^A+\sigma_A)^2}}{\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)^2} > 0.$$

3. From equation (3.26), an increase in c_E^A will increase π_A:

$$\frac{\partial \pi_A}{\partial c_E^A} = \frac{\frac{1}{c_w^B+\sigma_B}}{\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)^2} > 0.$$

Likewise, from equation (3.27), an increase in c_E^B will increase π_B:

$$\frac{\partial \pi_B}{\partial c_E^B} = \frac{\frac{1}{c_w^A+\sigma_A}}{\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)^2} > 0.$$

4.From equation (3.26), an increase in c_E^B will decrease π_A:

$$\frac{\partial \pi_A}{\partial c_E^B} = \frac{(-1)\cdot\frac{c_w^A+\sigma_A}{(c_w^B+\sigma_B)^2}}{\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)^2} < 0.$$

Likewise, from equation (3.27), an increase in c_E^A will decrease π_B:

$$\frac{\partial \pi_B}{\partial c_E^A} = \frac{(-1)\cdot\frac{c_w^B+\sigma_B}{(c_w^A+\sigma_A)^2}}{\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)^2} < 0.$$

⋄ Proof for Theorem 3:

The proof for Theorem 3 begins by maximizing $V_i\left(W_i, K_i; \mathcal{W}/i, \mathcal{K}/i\right)$ subject to the constraint (3.38) for each warlord $i \in J$. Let λ_i be the associated Lagrangian multiplier. The Lagrangian equation for each warlord $i \in J$ is

$$\mathscr{L}_i = \left(\frac{m \cdot \sum_{i=1}^{J} K_i}{1 + \sum_{h \in J, h \neq i}^{J} e^{\alpha \cdot (I_h - I_i)}}\right) + \lambda_i \left(\aleph_i - \left(\frac{c_w + \sigma}{\sigma}\right) W_i - \left(\frac{c_k}{\sigma}\right) K_i\right). \quad \text{(A.16)}$$

Therefore, for all $i \in J$

$$\frac{\partial V_i}{\partial W_i} = 0 \quad \Rightarrow \quad \frac{(\alpha \cdot m) \cdot (J-1) \left(\sum_{i=1}^{J} K_i\right) \cdot \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}}{\left(1 + \sum_{h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}\right)^2} - \left(\frac{c_w}{\sigma} + 1\right) \cdot \lambda_A = 0 \quad \text{(A.17)}$$

$$\frac{\partial V_i}{\partial K_i} = 0 \quad \Rightarrow \quad \frac{m}{1 + \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}} - \frac{c_k}{\sigma} \cdot \lambda_A = 0.. \quad \text{(A.18)}$$

Using equations (A.17) and (A.18),

$$\frac{(\alpha \cdot m) \cdot (J-1) \left(\sum_{i=1}^{J} K_i\right) \cdot \sum_{h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}}{\left(1 + \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}\right)^2} \cdot \left(\frac{\sigma}{c_w + \sigma}\right) = \frac{m}{1 + \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}} \cdot \left(\frac{\sigma}{c_k}\right)$$

$$\frac{\alpha \cdot (J-1) \left(\sum_{i=1}^{J} K_i\right) \cdot \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}}{\left(1 + \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}\right)^2} \cdot \left(\frac{c_k}{c_w + \sigma}\right) = \frac{1}{1 + \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}}$$

$$\alpha \cdot (J-1) \left(\sum_{i=1}^{J} K_i\right) \cdot \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)} \cdot \left(\frac{c_k}{c_w + \sigma}\right) = 1 + \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}$$

$$\alpha \cdot (J-1) \left(\sum_{i=1}^{J} K_i\right) \cdot \left(\frac{c_k}{c_w + \sigma}\right) = \frac{1 + \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}}{\sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}}$$

$$\left(\frac{c_w + \sigma}{\alpha \cdot (J-1) \cdot c_k}\right) \cdot \left(\frac{1 + \sum_{h \in J, h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}}{\sum_{h \in J, h \neq i}^{J} e^{\alpha \cdot \left(I_h - I_i\right)}}\right) = \sum_{i=1}^{J} K_i, \quad \text{(A.19)}$$

for all $i, h \in J$. Since equation (A.19) holds for all warlords in J, the following extension is made for all $z \in J, z \neq i$:

$$\begin{aligned}
\left(\frac{c_w+\sigma}{\alpha\cdot(J-1)\cdot c_k}\right)\cdot\left(\frac{1+\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot\left(I_h-I_z\right)}}{\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot\left(I_h-I_z\right)}}\right) &= \sum_{i=1}^{J} K_i = \left(\frac{c_w+\sigma}{\alpha\cdot(J-1)\cdot c_k}\right)\cdot\left(\frac{1+\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot\left(I_h-I_i\right)}}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot\left(I_h-I_i\right)}}\right) \\
\left(\frac{1+\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot\left(I_h-I_z\right)}}{\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot\left(I_h-I_z\right)}}\right) &= \left(\frac{1+\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot\left(I_h-I_i\right)}}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot\left(I_h-I_i\right)}}\right) \\
\left(\frac{1}{\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot\left(I_h-I_z\right)}}\right)\cdot\left(1+\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot\left(I_h-I_z\right)}\right) &= \left(\frac{1}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot\left(I_h-I_i\right)}}\right)\cdot\left(1+\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot\left(I_h-I_i\right)}\right) \\
\left(\frac{e^{\alpha\cdot I_z}}{\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot I_h}}\right)\cdot\left(1+\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot\left(I_h-I_z\right)}\right) &= \left(\frac{e^{\alpha\cdot I_i}}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot I_h}}\right)\cdot\left(1+\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot\left(I_h-I_i\right)}\right) \\
\left(\frac{e^{\alpha\cdot I_z}}{\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot I_h}}\right)\cdot\left(1+\frac{\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot I_h}}{e^{\alpha\cdot I_z}}\right) &= \left(\frac{e^{\alpha\cdot I_i}}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot I_h}}\right)\cdot\left(1+\frac{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot I_h}}{e^{\alpha\cdot I_i}}\right) \\
\left(\frac{1}{\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot I_h}}\right)\cdot\left(e^{\alpha\cdot I_z}+\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot I_h}\right) &= \left(\frac{1}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot I_h}}\right)\cdot\left(e^{\alpha\cdot I_i}+\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot I_h}\right) \\
\left(\frac{1}{\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot I_h}}\right)\cdot\left(\sum_{i}^{J} e^{\alpha\cdot I_i}\right) &= \left(\frac{1}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot I_h}}\right)\cdot\left(\sum_{i}^{J} e^{\alpha\cdot I_i}\right) \\
\sum_{h\in J,h\neq z}^{J} e^{\alpha\cdot I_h} &= \sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot I_h} \\
\sum_{h\in J,h\neq z,i}^{J} e^{\alpha\cdot I_h}+e^{\alpha\cdot I_i} &= \sum_{h\in J,h\in J,h\neq i,z}^{J} e^{\alpha\cdot I_h}+e^{\alpha\cdot I_z} \\
e^{\alpha\cdot I_i} &= e^{\alpha\cdot I_z} \\
W_i-\phi\cdot(\ell_c-\ell_i)^2 &= W_z-\phi\cdot(\ell_c-\ell_z)^2, \qquad \forall i,z\in J. \qquad \text{(A.20)}
\end{aligned}$$

Equation (A.20) states that the impact function of every warlord $i \in J$ is identical and, hence,

$$e^{\alpha\cdot(I_h - I_i)} = e^{\alpha\cdot 0} = e^0 = 1 \qquad \forall i, h \in J.$$

Therefore, using equations (A.19) and (A.20), the total amount of goods and services produced within equilibrium, equation (3.47), is found:

$$\begin{aligned}
\textstyle\sum_{i=1}^{J} K_i &= \left(\frac{c_w+\sigma}{\alpha\cdot(J-1)\cdot c_k}\right)\cdot\left(\frac{1+\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot(I_h-I_i)}}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{\alpha\cdot(I_h-I_i)}}\right)\\
&= \left(\frac{c_w+\sigma}{\alpha\cdot(J-1)\cdot c_k}\right)\cdot\left(\frac{1+\sum_{h\in J,h\in J,h\neq i}^{J} e^{0}}{\sum_{h\in J,h\in J,h\neq i}^{J} e^{0}}\right)\\
&= \left(\frac{c_w+\sigma}{\alpha\cdot(J-1)\cdot c_k}\right)\cdot\left(\frac{1+\sum_{h\in J,h\in J,h\neq i}^{J} 1}{\sum_{h\in J,h\in J,h\neq i}^{J} 1}\right)\\
&= \left(\frac{c_w+\sigma}{\alpha\cdot(J-1)\cdot c_k}\right)\cdot\left(\frac{1+(J-1)}{(J-1)}\right)\\
&= \left(\frac{c_w+\sigma}{\alpha\cdot(J-1)\cdot c_k}\right)\cdot\left(\frac{J}{(J-1)}\right)\\
\textstyle\sum_{i=1}^{J} K_i &= \left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\frac{J}{(J-1)^2}\right).
\end{aligned}$$

Using warlord i's constraint from equation (3.38), (A.20) and the above equation, the equilibrium level of capital investment for warlord i is found; that is, for all $i \in J$,

$$\begin{aligned}
\textstyle\sum_{i=1}^{J} K_i &= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right) \\
K_i &= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right)-\textstyle\sum_{h\in J,h\in J,h\neq i}^{J} K_h \\
&= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right)-\textstyle\sum_{h\in J,h\in J,h\neq i}^{J}\left(\tfrac{\sigma}{c_k}\cdot\aleph_h-\tfrac{c_W+\sigma}{c_k}\cdot W_h\right) \\
&= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right)-\left(\tfrac{\sigma}{c_k}\cdot\textstyle\sum_{h\in J,h\neq i}^{J}\aleph_h\right)+\left(\tfrac{c_W+\sigma}{c_k}\right)\cdot\Sigma_{h\in J,h\in J,h\neq i}^{J}W_h \\
&= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right)-\left(\tfrac{\sigma}{c_k}\cdot\textstyle\sum_{h\in J,h\neq i}^{J}\aleph_h\right)+\left(\tfrac{c_W+\sigma}{c_k}\right)\cdot\Sigma_{h\in J,h\neq i}^{J}\left(W_i+\phi\cdot\left((\ell_c-\ell_h)^2-(\ell_c-\ell_i)^2\right)\right) \\
&= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right)-\left(\tfrac{\sigma}{c_k}\cdot\textstyle\sum_{h\in J,h\neq i}^{J}\aleph_h\right)+\left(\tfrac{c_W+\sigma}{c_k}\right)\cdot(J-1)\cdot W_i+\left(\tfrac{\phi\cdot(c_W+\sigma)}{c_k}\right)\cdot\Sigma_{h\in J,h\neq i}^{J}\left((\ell_c-\ell_h)^2-(\ell_c-\ell_i)^2\right) \\
&= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right)-\left(\tfrac{\sigma}{c_k}\cdot\textstyle\sum_{h\in J,h\neq i}^{J}\aleph_h\right)+\left(\tfrac{\sigma}{c_k}\right)\cdot(J-1)\cdot\left(\aleph_i-\tfrac{c_k}{\sigma}\cdot K_i\right) \\
&\quad +\left(\tfrac{\phi\cdot(c_W+\sigma)}{c_k}\right)\cdot\left(\Sigma_{h\in J,h\neq i}^{J}(\ell_c-\ell_h)^2-(J-1)\cdot(\ell_c-\ell_i)^2\right) \\
K_i\cdot(1+(J-1)) &= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right)+\left(\tfrac{\sigma}{c_k}\right)\cdot\left((J-1)\cdot\aleph_i-\textstyle\sum_{h\in J,h\neq i}^{J}\aleph_h\right)+\left(\tfrac{\phi\cdot(c_W+\sigma)}{c_k}\right)\cdot\left(\Sigma_{i=1}^{J}(\ell_c-\ell_i)^2-J\cdot(\ell_c-\ell_i)^2\right) \\
K_i\cdot J &= \left(\tfrac{c_W+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\tfrac{J}{(J-1)^2}\right)+\left(\tfrac{\sigma}{c_k}\right)\cdot\left(J\cdot\aleph_i-\textstyle\sum_{i=1}^{J}\aleph_i\right)+\left(\tfrac{\phi\cdot(c_W+\sigma)}{c_k}\right)\cdot\left(\Sigma_{i=1}^{J}(\ell_c-\ell_i)^2-J\cdot(\ell_c-\ell_i)^2\right)
\end{aligned}$$

which equals the equilibrium level of K_i^* in equation (3.46).

Substituting the equilibrium level of capital K_i^*, from equation (3.46), into warlord i's constraint from equation (3.38) leads to the equilibrium number of warriors W_i^*,

found in equation (3.45):

$$W_i^* = \left(\frac{\sigma}{c_w+\sigma}\right)\cdot\aleph_i - \left(\frac{c_k}{c_w+\sigma}\right)\cdot K_i$$

$$= \left(\frac{\sigma}{c_w+\sigma}\right)\cdot\aleph_i - \left(\frac{c_k}{c_w+\sigma}\right)\cdot\left(\left(\frac{c_w+\sigma}{\alpha\cdot c_k\cdot(J-1)^2}\right)+\left(\frac{\sigma}{c_k}\right)\cdot\left(\aleph_i - \frac{\sum_{i=1}^J \aleph_i}{J}\right)+\left(\frac{\phi\cdot(c_w+\sigma)}{c_k}\right)\cdot\left(\Sigma_{i=1}^J\frac{(\ell_c-\ell_i)^2}{J}-(\ell_c-\ell_i)^2\right)\right)$$

$$= \left(\frac{\sigma}{c_w+\sigma}\right)\cdot\aleph_i - \frac{1}{\alpha\cdot(J-1)^2} - \left(\frac{\sigma}{c_w+\sigma}\right)\cdot\left(\aleph_i - \frac{\sum_{i=1}^J \aleph_i}{J}\right) - \phi\cdot\left(\Sigma_{i=1}^J\frac{(\ell_c-\ell_i)^2}{J}-(\ell_c-\ell_i)^2\right)$$

$$W_i^* = \left(\frac{\sigma}{c_w+\sigma}\right)\cdot\frac{\sum_{i=1}^J \aleph_i}{J}+\phi\cdot\left((\ell_c-\ell_i)^2-\Sigma_{i=1}^J\frac{(\ell_c-\ell_i)^2}{J}\right)-\frac{1}{\alpha\cdot(J-1)^2}.$$

To show the second-order conditions, the bordered Hessian for each warlord $i \in J$

is

$$
\mathcal{H}_i^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w}{\sigma}+1\right) & -\frac{c_k}{\sigma} \\ -\left(\frac{c_w}{\sigma}+1\right) & \frac{\partial^2 \mathscr{L}_i}{\partial W_i^2} & \frac{\partial^2 \mathscr{L}_i}{\partial W_i K_i} \\ -\frac{c_k}{\sigma} & \frac{\partial^2 \mathscr{L}_i}{\partial K_i W_i} & \frac{\partial^2 \mathscr{L}_i}{\partial K_i^2} \end{pmatrix} \tag{A.21}
$$

$$
= \begin{pmatrix} 0 & -\left(\frac{c_w}{\sigma}+1\right) & -\frac{c_k}{\sigma} \\ -\left(\frac{c_w}{\sigma}+1\right) & (\alpha m(J-1))^2 \frac{\left(\Sigma_{h\in J/i}^J e^{\alpha(I_h-I_i)}\right)\left(\Sigma_{h\in J/i}^J e^{\alpha(I_h-I_i)}-1\right)}{\left(1+\Sigma_{h\in J/i}^J e^{\alpha(I_h-I_i)}\right)^3}\left(\Sigma_{i=1}^J K_i\right) & (\alpha m(J-1))\frac{\Sigma_{h\in J,h\neq i}^J e^{\alpha(I_h-I_i)}}{\left(1+\Sigma_{h\in J/i}^J e^{\alpha(I_h-I_i)}\right)^2} \\ -\frac{c_k}{\sigma} & (\alpha\cdot m\cdot(J-1))\cdot\frac{\Sigma_{h\in J,h\neq i}^J e^{\alpha\cdot(I_h-I_i)}}{\left(1+\Sigma_{h\in J,h\neq i}^J e^{\alpha\cdot(I_h-I_i)}\right)^2} & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -\left(\frac{c_w}{\sigma}+1\right) & -\frac{c_k}{\sigma} \\ -\left(\frac{c_w}{\sigma}+1\right) & (\alpha\cdot m\cdot(J-1))^2\cdot\frac{(J-1)\cdot(J-2)}{J^3}\cdot\left(\Sigma_{i=1}^J K_i\right) & (\alpha\cdot m\cdot(J-1))\cdot\frac{J-1}{J^2} \\ -\frac{c_k}{\sigma} & (\alpha\cdot m\cdot(J-1))\cdot\frac{J-1}{J^2} & 0 \end{pmatrix} \tag{A.22}
$$

and is satisfied when the determinant, $|\mathcal{H}_i^{\mathcal{B}}|$, is greater than zero; that is,

$$
\begin{aligned}
|\mathcal{H}_i^{\mathcal{B}}| \quad &\rightarrow \quad \left(\tfrac{cw}{\sigma}+1\right)\cdot\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\frac{\partial^2 V_i}{\partial W_i K_i}\right)-\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\frac{\partial^2 V_i}{\partial W_i^2}\right)-\left(\tfrac{cw}{\sigma}+1\right)\cdot\left(\frac{\partial^2 V_i}{\partial K_i W_i}\right)\right)>0 \\
&\rightarrow \quad \left(\tfrac{cw}{\sigma}+1\right)\cdot\left(\frac{\partial^2 V_i}{\partial W_i K_i}\right)-\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\frac{\partial^2 V_i}{\partial W_i^2}\right)+\left(\tfrac{cw}{\sigma}+1\right)\cdot\left(\frac{\partial^2 V_i}{\partial K_i W_i}\right)>0 \\
&\rightarrow \quad 2\cdot\left(\tfrac{cw}{\sigma}+1\right)\cdot\left(\frac{\partial^2 V_i}{\partial W_i K_i}\right)-\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\frac{\partial^2 V_i}{\partial W_i^2}\right)>0 \\
&\rightarrow \quad 2\cdot\left(\tfrac{cw}{\sigma}+1\right)\cdot\left((\alpha\cdot m\cdot(J-1))\cdot\tfrac{J-1}{J^2}\right)-\left(\tfrac{c_k}{\sigma}\right)\cdot\left((\alpha\cdot m\cdot(J-1))^2\cdot\tfrac{(J-1)\cdot(J-2)}{J^3}\cdot\left(\Sigma_{i=1}^{J}K_i\right)\right)>0 \\
&\rightarrow \quad 2\cdot\left(\tfrac{cw}{\sigma}+1\right)\cdot\left((\alpha\cdot m\cdot(J-1))\cdot\tfrac{J-1}{J^2}\right)-\left(\tfrac{c_k}{\sigma}\right)\cdot\left((\alpha\cdot m\cdot(J-1))^2\cdot\tfrac{(J-1)\cdot(J-2)}{J^3}\cdot\left(\tfrac{cw+\sigma}{c_k}\cdot\tfrac{J}{(J-1)^2}\right)\right)>0 \\
&\rightarrow \quad 2\cdot\left(\tfrac{cw}{\sigma}+1\right)\cdot\left(\alpha\cdot m\cdot(J-1)^2\right)-\left(\tfrac{\alpha\cdot m\cdot(J-1)^3\cdot(J-2)}{J}\cdot\left(\tfrac{cw+\sigma}{\sigma}\cdot\tfrac{J}{(J-1)^2}\right)\right)>0 \\
&\rightarrow \quad 2-\tfrac{(J-1)\cdot(J-2)}{(J-1)^2}>0 \\
&\rightarrow \quad 2\cdot(J-1)>J-2 \\
&\rightarrow \quad 2\cdot J-2>J-2 \\
&\rightarrow \quad J>0,
\end{aligned}
$$

which holds by definition.

◇ Proof for Theorem 4:

Equation (3.48) is found by substituting in equation (A.20), found in the above proof, into equation (3.40). Explicitly,

$$\begin{aligned}
\pi_A^* &= \frac{1}{1+\Sigma_{h\in J,h\neq i}^{J} e^{\alpha\cdot(I_h-I_i)}} \\
&= \frac{1}{1+\Sigma_{h\in J,h\neq i}^{J} e^{0}} \\
&= \frac{1}{1+J-1} \\
&= \frac{1}{J}; \qquad \forall i\in J.
\end{aligned}$$

$\diamond$ Proof for Theorem 5:

Let $\hat{\ell}=\phi\cdot((\ell_c-\ell_A)^2-(\ell_c-\ell_B)^2)=\phi\cdot(2\cdot\ell_c-1)$. Let λ_A and λ_B again be the associated Lagrangian multipliers for the maximization problems of warlord A and warlord B. The Lagrangian equations for warlords A and B are

$$\mathscr{L}_A = \left(\frac{m\left(K_A+K_B\right)}{1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}}\right)+\lambda_A\left(\aleph_A-\left(\frac{c_w^A+\sigma_A}{\sigma_A}\right)W_A-\left(\frac{c_k}{\sigma_A}\right)K_A\right); \quad \text{(A.23)}$$

$$\mathscr{L}_B = \left(\frac{m\left(K_A+K_B\right)}{1+\frac{W_A}{W_B}\cdot\frac{1}{e^{\hat{\ell}}}}\right)+\lambda_B\left(\aleph_B-\left(\frac{c_w^B+\sigma_B}{\sigma_B}\right)W_B-\left(\frac{c_k}{\sigma_B}\right)K_B\right). \quad \text{(A.24)}$$

Solving warlord A and B's optimization problem for the two choice variables W and

K:

$$\frac{\partial \mathscr{L}_A}{\partial W_A} = 0 \quad \Rightarrow \quad \frac{W_B \cdot e^{\hat{\ell}}}{W_A^2 \cdot \left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^2}\cdot(m)\cdot(K_A+K_B)-\lambda_A\cdot\left(\frac{c_w^A}{\sigma_A}+1\right)=0 \tag{A.25}$$

$$\frac{\partial \mathscr{L}_A}{\partial K_A} = 0 \quad \Rightarrow \quad \frac{1}{\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)}\cdot m-\lambda_A\cdot\left(\frac{c_k}{\sigma_A}\right)=0 \tag{A.26}$$

$$\frac{\partial \mathscr{L}_B}{\partial W_B} = 0 \quad \Rightarrow \quad \frac{W_A \cdot \frac{1}{e^{\hat{\ell}}}}{W_B^2 \cdot \left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right)^2}\cdot(m)\cdot(K_A+K_B)-\lambda_B\cdot\left(\frac{c_w^B}{\sigma_B}+1\right)=0 \tag{A.27}$$

$$\frac{\partial \mathscr{L}_B}{\partial K_B} = 0 \quad \Rightarrow \quad \frac{1}{\left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right)}\cdot m-\lambda_B\cdot\left(\frac{c_k}{\sigma_B}\right)=0. \tag{A.28}$$

From equations (A.25) and (A.26),

$$\frac{W_B \cdot e^{\hat{\ell}}}{W_A^2 \cdot \left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^2}\cdot(m)\cdot(K_A+K_B)\cdot\left(\frac{\sigma_A}{c_w^A+\sigma_A}\right) = \frac{1}{\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)}\cdot m\cdot\left(\frac{\sigma_A}{c_k}\right)$$

$$\frac{W_B \cdot e^{\hat{\ell}}}{W_A^2 \cdot \left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^2}\cdot(K_A+K_B)\cdot\left(\frac{c_k}{c_w^A+\sigma_A}\right) = \frac{1}{\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)}$$

$$(K_A+K_B)\cdot\left(\frac{c_k}{c_w^A+\sigma_A}\right) = \frac{W_A^2\cdot\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)}{W_B\cdot e^{\hat{\ell}}} \tag{A.29}$$

and from equations (A.27) and (A.28),

$$\frac{W_A \cdot \frac{1}{e^{\hat{\ell}}}}{W_B^2 \cdot \left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right)^2}\cdot(m)\cdot(K_A+K_B)\cdot\left(\frac{\sigma_B}{c_w^B+\sigma_B}\right) = \frac{1}{\left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right)}\cdot m\cdot\left(\frac{\sigma_B}{c_k}\right)$$

$$\frac{W_A \cdot \frac{1}{e^{\hat{\ell}}}}{W_B^2 \cdot \left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right)^2}\cdot(K_A+K_B)\cdot\left(\frac{c_k}{c_w^B+\sigma_B}\right) = \frac{1}{\left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right)}$$

$$(K_A+K_B)\cdot\left(\frac{c_k}{c_w^B+\sigma_B}\right) = \frac{W_B^2\cdot\left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right)}{W_A\cdot \frac{1}{e^{\hat{\ell}}}} \tag{A.30}$$

Equating equations (A.29) and (A.30) leads to the following relationship:

$$\frac{W_A^2 \cdot \left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)}{W_B \cdot e^{\hat{\ell}}} \cdot \left(\frac{c_w^A+\sigma_A}{c_k}\right) = \frac{W_B^2 \cdot \left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right)}{W_A \cdot \frac{1}{e^{\hat{\ell}}}} \cdot \left(\frac{c_w^B+\sigma_B}{c_k}\right)$$

$$\frac{W_A}{e^{\hat{\ell}}} \cdot W_A^2 \cdot \left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right) = \left(W_B \cdot e^{\hat{\ell}}\right) \cdot W_B^2 \cdot \left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right) \cdot \left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)$$

$$W_A \cdot W_A^2 \cdot \left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right) = \left(W_B \cdot e^{2\cdot\hat{\ell}}\right) \cdot W_B^2 \cdot \left(1+\frac{W_A}{W_B}\cdot \frac{1}{e^{\hat{\ell}}}\right) \cdot \left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)$$

$$W_A^2 \cdot \left(W_A+W_B \cdot e^{\hat{\ell}}\right) = \left(e^{2\cdot\hat{\ell}}\right) \cdot W_B^2 \cdot \left(W_B+W_A \cdot \frac{1}{e^{\hat{\ell}}}\right) \cdot \left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)$$

$$W_A^2 \cdot \left(W_A+W_B \cdot e^{\hat{\ell}}\right) = \left(e^{\hat{\ell}}\right) \cdot W_B^2 \cdot \left(W_B \cdot e^{\hat{\ell}}+W_A\right) \cdot \left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)$$

$$W_A^2 = e^{\hat{\ell}} \cdot W_B^2 \cdot \left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)$$

$$W_A = W_B \cdot \sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)} \tag{A.31}$$

and

$$\frac{W_A}{\sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)}} = W_B. \tag{A.32}$$

By substituting (A.32) into (A.29),

$$
\begin{aligned}
(K_A+K_B)\cdot\left(\frac{c_k}{c_w^A+\sigma_A}\right) &= \frac{W_A^2\cdot\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)}{W_B\cdot e^{\hat{\ell}}} \\
(K_A+K_B)\cdot\left(\frac{c_k}{c_w^A+\sigma_A}\right) &= \frac{W_A^2\cdot\left(1+\frac{e^{\hat{\ell}}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)}}\right)}{\left(W_A\cdot\frac{e^{\hat{\ell}}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)}}\right)} \\
(K_A+K_B)\cdot\left(\frac{c_k}{c_w^A+\sigma_A}\right) &= \frac{W_A\cdot\left(1+\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}\right)}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}} \\
(K_A+K_B) &= \left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\left(W_A\cdot\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right)\right) \\
K_A+K_B &= \left.\begin{cases} \frac{W_A}{c_k}\cdot\left(\left(c_w^A+\sigma_A\right)+\frac{\sqrt{\left(c_w^A+\sigma_A\right)\left(c_w^B+\sigma_B\right)}}{\sqrt{e^{\hat{\ell}}}}\right) \\ \\ \frac{W_B}{c_k}\cdot\left(\left(c_w^B+\sigma_B\right)+\sqrt{e^{\hat{\ell}}(c_w^A+\sigma_A)(c_w^B+\sigma_B)}\right) \end{cases}\right\}
\end{aligned}
\tag{A.33}
$$

Using the equations (3.8), (3.9), (A.31), (A.32) and (A.33)

$$
\begin{aligned}
\left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\left(W_A\cdot\left(\frac{1+\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right)\right)-K_B &= K_A \\
\left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\left(W_A\cdot\left(\frac{1+\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right)\right)-K_B &= \left(\frac{\sigma_A}{c_k}\right)\cdot\left(\aleph_A-\left(\frac{c_w}{\sigma_A}+1\right)\cdot W_A\right) \\
\left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\left(W_A\cdot\left(\frac{1+2\cdot\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right)\right) &= \left(\frac{\sigma_A}{c_k}\right)\cdot(\aleph_A)+K_B \\
\left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\left(W_A\cdot\left(\frac{1+\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right)\right) &= \left(\frac{\sigma_A}{c_k}\right)\cdot(\aleph_A)+\left(\frac{\sigma_B}{c_k}\right)\cdot\left(\aleph_B-\left(\frac{c_w}{\sigma_B}+1\right)\cdot W_B\right) \\
\left(c_w^A+\sigma_A\right)\cdot\left(W_A\cdot\left(\frac{1+\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right)\right) &= (\sigma_A)\cdot(\aleph_A)+(\sigma_B)\cdot(\aleph_B)-\left(c_w^B+\sigma_B\right)\left(\frac{W_A}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)}}\right) \\
\left(c_w^A+\sigma_A\right)\cdot\left(W_A\cdot\left(\frac{1+\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right)\right) &= (\sigma_A)\cdot(\aleph_A)+(\sigma_B)\cdot(\aleph_B)-\left(c_w^A+\sigma_A\right)\left(\frac{W_A}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right) \\
\left(c_w^A+\sigma_A\right)\cdot\left(W_A\cdot\left(\frac{2+2\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}{\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}}\right)\right) &= (\sigma_A)(\aleph_A)+(\sigma_B)(\aleph_B),
\end{aligned}
$$

which equals the equilibrium level of warriors hired by warlord A found in equation (3.54). By using W_A^* and equation (A.31), the equilibrium level of warriors hired by warlord B is found:

$$W_B^* \cdot \sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right)} = W_A^*$$

$$W_B^* \cdot \sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right)} = \left(\frac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)}{2\left(c_w^A + \sigma_A\right)}\right) \cdot \frac{\sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)}}{1 + \sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)}}$$

$$W_B^* = \left(\frac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)}{2\left(c_w^A + \sigma_A\right)}\right) \cdot \frac{\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)}{1 + \sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right)}}$$

$$W_B^* = \left(\frac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)}{2\left(c_w^B + \sigma_B\right)}\right) \cdot \frac{1}{1 + \sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A}\right)}}$$

Substituting the W_A^*, from equation (3.54) into (A.33) and applying equations

(3.8), (3.9) and (3.55):

$$
\begin{aligned}
K_A + K_B &= \left(\frac{c_w^A + \sigma_A}{c_k}\right) \cdot \left(W_A \cdot \left(1 + \frac{1}{\sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B}\right)}} \right) \right) \\
K_A + K_B &= \left(\frac{c_w^A + \sigma_A}{c_k}\right) \left(\frac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)}{2\left(c_w^A + \sigma_A\right)} \right) \\
K_A &= \left(\frac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)}{2 \cdot c_k} \right) - K_B \\
&= \left(\frac{(\sigma_A)(\aleph_A) + (\sigma_B)(\aleph_B)}{2 \cdot c_k} \right) - \left(\frac{\sigma_B}{c_k}\right) \cdot \left(\aleph_B - \left(\frac{c_w^B}{\sigma_B} + 1 \right) \cdot W_B \right) \\
&= \left(\frac{(\sigma_A)(\aleph_A) - (\sigma_B)(\aleph_B)}{2 \cdot c_k} \right) + \left(\frac{c_w^B + \sigma_B}{c_k} \right) \cdot \frac{W_A}{\sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \right)}} \\
&= \left(\frac{(\sigma_A)(\aleph_A) - (\sigma_B)(\aleph_B)}{2 \cdot c_k} \right) + \left(\frac{c_w^B + \sigma_B}{c_k} \right) \cdot \left(\frac{\sigma_A}{c_w^A + \sigma_A} \right) \cdot \frac{\left(\aleph_A - \frac{c_k}{\sigma_A} K_A \right)}{\sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \right)}} \\
K_A \left(1 + \frac{1}{\sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B} \right)}} \right) &= \left(\frac{\sigma_A}{2 \cdot c_k} \right) (\aleph_A) \left(1 + \frac{2}{\sqrt{e^{\hat{\ell}} \cdot \left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \right)}} \right) - \left(\frac{\sigma_B}{2 \cdot c_k} \right) (\aleph_B),
\end{aligned}
$$

which is equal to the equilibrium level of capital investment by warlord A found in equation (3.56). Similarly for warlord B,

$$K_A+K_B = \left(\frac{c_w^B+\sigma_B}{c_k}\right)\cdot\left(W_B\cdot\left(1+\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}\right)\right)$$

$$K_B = \frac{(\sigma_A)(\aleph_A)+(\sigma_B)(\aleph_B)}{2\cdot c_k}-K_A$$

$$= \frac{(\sigma_A)(\aleph_A)+(\sigma_B)(\aleph_B)}{2\cdot c_k}-\frac{\sigma_A}{c_k}\left(\aleph_A-\frac{c_w^A+\sigma_A}{\sigma_A}\cdot W_A\right)$$

$$= \frac{(\sigma_B)(\aleph_B)-(\sigma_A)(\aleph_A)}{2\cdot c_k}+\left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\left(W_B\cdot\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)}\right)$$

$$= \frac{(\sigma_B)(\aleph_B)-(\sigma_A)(\aleph_A)}{2\cdot c_k}+\left(\frac{c_w^A+\sigma_A}{c_k}\right)\cdot\left(\left(\frac{\sigma_B}{c_w^B+\sigma_B}\right)\left(\aleph_B-\frac{c_k}{\sigma_B}\cdot K_B\right)\right)\cdot\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)}$$

$$K_B\left(1+\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}\right) = \left(\frac{\sigma_B}{2\cdot c_k}\right)(\aleph_B)\left(1+2\cdot\sqrt{e^{\hat{\ell}}\cdot\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)}\right)-(\sigma_A)(\aleph_A)$$

To show the second-order conditions, the bordered Hessian for warlord A again is

$$\mathcal{H}_A^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w}{\sigma_A}+1\right) & -\frac{c_k}{\sigma_A} \\ -\left(\frac{c_w}{\sigma_A}+1\right) & \frac{\partial^2\mathscr{L}_A}{\partial W_A^2} & \frac{\partial^2\mathscr{L}_A}{\partial W_A K_A} \\ -\frac{c_k}{\sigma_A} & \frac{\partial^2\mathscr{L}_A}{\partial K_A W_A} & \frac{\partial^2\mathscr{L}_A}{\partial K_A^2} \end{pmatrix} \tag{A.34}$$

where $\frac{\partial^2\mathscr{L}_A}{\partial K_A^2} = 0$ and is satisfied when the determinant, $|\mathcal{H}_A^{\mathcal{B}}|$, is greater than zero;

that is,

$$|\mathcal{H}_A^{\mathcal{B}}| = \left(\frac{c_w^A}{\sigma_A}+1\right)\cdot\left(\frac{c_k}{\sigma_A}\right)\cdot\left(\frac{\partial^2 V_A}{\partial W_A K_A}\right)-\left(\frac{c_k}{\sigma_A}\right)\cdot\left(\left(\frac{c_k}{\sigma_A}\right)\cdot\left(\frac{\partial^2 V_A}{\partial W_A^2}\right)-\left(\frac{c_w^A}{\sigma_A}+1\right)\cdot\left(\frac{\partial^2 V_A}{\partial K_A W_A}\right)\right)>0$$

$$= \left(\frac{c_w^A}{\sigma_A}+1\right)\cdot\left(\frac{\partial^2 V_A}{\partial W_A K_A}\right)-\left(\frac{c_k}{\sigma_A}\right)\cdot\left(\frac{\partial^2 V_A}{\partial W_A^2}\right)+\left(\frac{c_w^A}{\sigma_A}+1\right)\cdot\left(\frac{\partial^2 V_A}{\partial K_A W_A}\right)>0$$

$$= 2\cdot\left(\frac{c_w^A}{\sigma_A}+1\right)\cdot\left(\frac{\partial^2 V_A}{\partial W_A K_A}\right)-\left(\frac{c_k}{\sigma_A}\right)\cdot\left(\frac{\partial^2 V_A}{\partial W_A^2}\right)>0 \tag{A.35}$$

From equation (A.25)

$$\frac{\partial^2 V_A}{\partial W_A^2} = \left(\frac{-2\cdot W_A\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^2+2\cdot W_A^2\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)\left(\frac{W_B\cdot e^{\hat{\ell}}}{W_A^2}\right)}{W_A^4\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^4}\right)(K_A+K_B)\cdot m$$

$$= \left(\frac{2\cdot\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)\left(W_B\cdot e^{\hat{\ell}}\right)-2\cdot W_A\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^2}{W_A^4\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^4}\right)(K_A+K_B)\cdot m$$

$$= \left(\frac{2\left(W_B\cdot e^{\hat{\ell}}\right)-2\cdot W_A-2\left(W_B\cdot e^{\hat{\ell}}\right)}{W_A^4\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^3}\right)\cdot(K_A+K_B)\cdot m$$

$$= -\left(\frac{2}{W_A^3\left(1+\frac{W_B}{W_A}\cdot e^{\hat{\ell}}\right)^3}\right)\cdot\left(\left(c_w^B+\sigma_B\right)\cdot\left(W_B\cdot\left(1+\sqrt{e^{\hat{\ell}}\cdot\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)\right)\right)\cdot m$$

$$= -\left(\frac{2\left(c_w^B+\sigma_B\right)}{W_A^3\left(1+\sqrt{e^{\hat{\ell}}\cdot\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)^2}\right)\cdot\frac{W_B\cdot m}{c_k}$$

$$= -\left(\frac{2\left(c_w^B+\sigma_B\right)}{W_A^2\left(1+\sqrt{e^{\hat{\ell}}\cdot\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)^2}\right)\cdot\left(\frac{m}{c_k\cdot\sqrt{e^{\hat{\ell}}\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}\right)<0,$$

and from equations (A.25) and(A.26),

$$\frac{\partial^2 V_A}{\partial W_A \partial K_A} = \frac{\partial^2 V_A}{\partial K_A \partial W_A} = \frac{W_B \cdot e^{\hat{\ell}}}{W_A^2 \cdot \left(1 + \frac{W_B}{W_A} \cdot e^{\hat{\ell}}\right)^2} \cdot (m)$$

$$= \frac{m \cdot \sqrt{e^{\hat{\ell}}}}{W_A \cdot \left(1 + \sqrt{e^{\hat{\ell}}}\right)^2} > 0.$$

The above two equations show that $\frac{\partial^2 V_A}{\partial W_A \partial K_A}$ is positive while $\frac{\partial^2 V_A}{\partial W_A^2}$ is negative. Therefore, equation (A.35) is positive for all value of ℓ_c ranging from 0 to 1.

To check that the second-order condition is satisfied for warlord B,

$$\mathcal{H}_B^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w^B}{\sigma_B} + 1\right) & -\frac{c_k}{\sigma_B} \\ -\left(\frac{c_w^B}{\sigma_B} + 1\right) & \frac{\partial^2 V_B}{\partial W_B^2} & \frac{\partial^2 V_B}{\partial W_B K_B} \\ -\frac{c_k}{\sigma_B} & \frac{\partial^2 V_B}{\partial K_B W_B} & \frac{\partial^2 V_B}{\partial K_B^2} \end{pmatrix} \tag{A.36}$$

where $\frac{\partial^2 V_B}{\partial K_B^2} = 0$ and is satisfied when the determinant, $|\mathcal{H}_B^{\mathcal{B}}|$, is greater than zero; that is,

$$|\mathcal{H}_B^{\mathcal{B}}| = \left(\frac{c_w^B}{\sigma_B}+1\right) \cdot \left(\frac{c_k}{\sigma_B}\right) \cdot \left(\frac{\partial^2 V_B}{\partial W_B K_B}\right) - \left(\frac{c_k}{\sigma_B}\right) \cdot \left(\left(\frac{c_k}{\sigma_B}\right) \cdot \left(\frac{\partial^2 V_B}{\partial W_B^2}\right) - \left(\frac{c_w^B}{\sigma_B}+1\right) \cdot \left(\frac{\partial^2 V_B}{\partial K_B W_B}\right)\right) > 0$$

$$= \left(\frac{c_w^B}{\sigma_B}+1\right) \cdot \left(\frac{\partial^2 V_B}{\partial W_B K_B}\right) - \left(\frac{c_k}{\sigma_B}\right) \cdot \left(\frac{\partial^2 V_B}{\partial W_B^2}\right) + \left(\frac{c_w^B}{\sigma_B}+1\right) \cdot \left(\frac{\partial^2 V_B}{\partial K_B W_B}\right) > 0$$

$$= 2 \cdot \left(\frac{c_w^B}{\sigma_B}+1\right) \cdot \left(\frac{\partial^2 V_B}{\partial W_B K_B}\right) - \left(\frac{c_k}{\sigma_B}\right) \cdot \left(\frac{\partial^2 V_B}{\partial W_B^2}\right) > 0 \tag{A.37}$$

From equation (A.27)

$$
\begin{aligned}
\frac{\partial^2 V_B}{\partial W_B^2} &= \left(\frac{-2\cdot W_B\left(1+\frac{W_A}{W_B\cdot e^{\hat{\ell}}}\right)^2+2\cdot W_B^2\left(1+\frac{W_A}{W_B\cdot e^{\hat{\ell}}}\right)\left(\frac{W_A}{W_B^2\cdot e^{\hat{\ell}}}\right)}{W_B^4\left(1+\frac{W_A}{W_B\cdot e^{\hat{\ell}}}\right)^4}\right)(K_A+K_B)\cdot m \\
&= \left(\frac{2\cdot\left(1+\frac{W_A}{W_B\cdot e^{\hat{\ell}}}\right)\left(\frac{W_A}{e^{\hat{\ell}}}\right)-2\cdot W_B\left(1+\frac{W_A}{W_B\cdot e^{\hat{\ell}}}\right)^2}{W_B^4\left(1+\frac{W_A}{W_B\cdot e^{\hat{\ell}}}\right)^4}\right)(K_A+K_B)\cdot m \\
&= \left(\frac{2\left(\frac{W_A}{e^{\hat{\ell}}}\right)-2\cdot W_B-2\left(\frac{W_A}{e^{\hat{\ell}}}\right)}{W_B^4\left(1+\frac{W_A}{W_B\cdot e^{\hat{\ell}}}\right)^3}\right)\cdot(K_A+K_B)\cdot m \\
&= -\left(\frac{2}{W_B^3\left(1+\frac{W_A}{W_B\cdot e^{\hat{\ell}}}\right)^3}\right)\cdot\left(\left(c_w^A+\sigma_A\right)\cdot\left(W_A\cdot\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}\cdot\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)\right)\right)\cdot m \\
&= -\left(\frac{2\left(c_w^A+\sigma_A\right)}{W_B^3\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}\cdot\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)^2}\right)\cdot\frac{W_A\cdot m}{c_k} \\
&= -\left(\frac{2\left(c_w^A+\sigma_A\right)}{W_B^2\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}\cdot\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)^2}\right)\cdot\left(\frac{m\sqrt{e^{\hat{\ell}}\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}}}{c_k}\right)<0,
\end{aligned}
$$

and from equations (A.27) and(A.28),

$$
\begin{aligned}
\frac{\partial^2 V_B}{\partial W_B\partial K_B}=\frac{\partial^2 V_B}{\partial K_B\partial W_B} &= \frac{W_A\cdot\frac{1}{e^{\hat{\ell}}}}{W_B^2\cdot\left(1+\frac{W_A}{W_B}\cdot\frac{1}{e^{\hat{\ell}}}\right)^2}\cdot(m) \\
&= \frac{m\cdot\frac{1}{\sqrt{e^{\hat{\ell}}}}}{W_B\cdot\left(\frac{1+\sqrt{e^{\hat{\ell}}}}{\sqrt{e^{\hat{\ell}}}}\right)^2} \\
&= \frac{m\cdot\sqrt{e^{\hat{\ell}}}}{W_B\cdot\left(1+\sqrt{e^{\hat{\ell}}}\right)^2}.
\end{aligned}
$$

The above two equations show that $\frac{\partial^2 V_B}{\partial W_B\partial K_B}$ is positive while $\frac{\partial^2 V_B}{\partial W_B^2}$ is negative. Therefore, equation (A.37) is positive for all value of ℓ_c ranging from 0 to 1.

Finally, it needs to be shown that W_A^*, W_B^*, K_A^* and K_B^* are non-negative; that is, it needs to be shown that

$$W_A^*, W_B^*, K_A^*, K_B^* \geq 0.$$

Equations (3.54) and (3.55) show that W_A^* and W_B^* are positive by definition when the point of conflict is between the values of 0 and 1. From equation (3.56), the equilibrium level of investment into capital by warlord A is greater than zero when

$$\left(\left(1+\frac{2}{\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)\cdot(\sigma_A)(\aleph_A)-(\sigma_B)(\aleph_B)\right) > 0$$

$$\left(1+\frac{2}{\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)\cdot(\sigma_A)(\aleph_A) > (\sigma_B)(\aleph_B)$$

$$1+\frac{2}{\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}} > \left(\frac{(\sigma_B)(\aleph_B)}{(\sigma_A)(\aleph_A)}\right).$$

Likewise from equation (3.57), the equilibrium level of investment into capital by warlord B is greater than zero when

$$\left(\left(1+2\cdot\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)\cdot(\sigma_B)(\aleph_B)-(\sigma_A)(\aleph_A)\right) > 0$$

$$\left(1+2\cdot\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)\cdot(\aleph_B)-(\aleph_A) > 0$$

$$\left(\frac{(\sigma_B)(\aleph_B)}{(\sigma_A)(\aleph_A)}\right) > \frac{1}{1+2\cdot\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}.$$

◇ Proof for Theorem 6:

Equation (3.59) is derived by substituting equation (A.31) into the CSF found above in equation (3.3). Explicitly,

$$\begin{aligned}\pi_A^* &= \frac{1}{1+\frac{W_B}{W_A}e^{\hat{\ell}}}\\ &= \frac{1}{1+\frac{1}{\sqrt{e^{\hat{\ell}}\frac{c_w^B+\sigma_B}{c_w^A+\sigma_B}}}e^{\hat{\ell}}}\\ &= \frac{1}{1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}.\end{aligned}$$

Equation (3.60) follows by

$$\begin{aligned}\pi_B^* &= 1-\pi_A^*\\ &= 1-\frac{1}{1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\\ &= \frac{\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}{1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}} = \frac{1}{1+\frac{1}{\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}}.\end{aligned}$$

◇ Proof for Corollary 5:

1. From equation (3.26), an increase in c_w^A will decrease π_A:

$$\frac{\partial\pi_A}{\partial c_w^A} = \left(\frac{\left(-\frac{1}{2}\sqrt{\frac{e^{\hat{\ell}}}{(c_w^A+\sigma_A)(c_w^B+\sigma_B)}}\right)}{\left(1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)^2}\right) < 0,$$

and from equation (3.27), an increase in c_w^B will decrease π_B:

$$\frac{\partial\pi_B}{\partial c_w^B} = \frac{\frac{-1}{2\sqrt{e^{\hat{\ell}}(c_w^A+\sigma_A)(c_w^B+\sigma_B)}}}{\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)^2} < 0.$$

2. From equation (3.26), an increase in c_w^B will increase π_A:

$$\frac{\partial \pi_A}{\partial c_w^B} = -\left(\frac{\partial \pi_B}{\partial c_w^B}\right) > 0.$$

Likewise, from equation (3.27), an increase in c_w^A will increase π_B:

$$\frac{\partial \pi_B}{\partial c_w^A} = -\left(\frac{\partial \pi_A}{\partial c_w^A}\right) > 0.$$

3. From equation (3.26), an increase in c_E^A will increase π_A:

$$\frac{\partial \pi_A}{\partial c_E^A} = \left(\frac{\left(\frac{1}{2}\sqrt{\frac{e^{\hat{\ell}}}{(c_w^A+\sigma_A)(c_w^B+\sigma_B)}}\right)}{\left(1+\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}\right)^2}\right) > 0.$$

Likewise, from equation (3.27), an increase in c_E^B will increase π_B:

$$\frac{\partial \pi_B}{\partial c_E^B} = \frac{\frac{1}{2\sqrt{e^{\hat{\ell}}(c_w^A+\sigma_A)(c_w^B+\sigma_B)}}}{\left(1+\frac{1}{\sqrt{e^{\hat{\ell}}\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}}}\right)^2} > 0.$$

4.From equation (3.26), an increase in c_E^B will decrease π_A:

$$\frac{\partial \pi_A}{\partial c_E^B} = -\left(\frac{\partial \pi_B}{\partial c_E^B}\right) < 0.$$

Likewise, from equation (3.27), an increase in c_E^A will decrease π_B:

$$\frac{\partial \pi_B}{\partial c_E^A} = -\left(\frac{\partial \pi_A}{\partial c_E^A}\right) > 0 < 0.$$

◇ Proof for Theorem 7:

The proof and its structure of Theorem 7 is similar to the proof of Theorem 1.

Let λ_A and λ_B again be the associated Lagrangian multipliers for the maximization problems of warlord A and warlord B. The Lagrangian equations for warlords A and B are

$$\mathscr{L}_A = \left(\frac{m(K_A+K_B)}{1+e^{\alpha \cdot \left(\frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} - \frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} \right)}} \right) + \lambda_A \left(\aleph_A - \left(\frac{c_w^A + \sigma_A}{\sigma_A} \right) W_A - \left(\frac{c_k}{\sigma_A} \right) K_A \right); \tag{A.38}$$

$$\mathscr{L}_B = \left(\frac{m(K_A+K_B)}{1+e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} - \frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} \right)}} \right) + \lambda_B \left(\aleph_B - \left(\frac{c_w^B + \sigma_B}{\sigma_B} \right) W_B - \left(\frac{c_k}{\sigma_B} \right) K_B \right). \tag{A.39}$$

Solving warlord A and B's optimization problem for the two choice variables W and K:

$$\frac{\partial \mathscr{L}_A}{\partial W_A} = 0 \;\Rightarrow\; \frac{\frac{(\alpha \cdot m)(K_A+K_B)}{\phi \cdot (\ell_c - \ell_A)^2} \cdot \left(e^{\alpha \cdot \left(\frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} - \frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} \right)} \right)}{\left(1+e^{\alpha \cdot \left(\frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} - \frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} \right)} \right)^2} = \left(\frac{c_w^A}{\sigma_A} + 1 \right) \cdot \lambda_A \tag{A.40}$$

$$\frac{\partial \mathscr{L}_A}{\partial K_A} = 0 \;\Rightarrow\; \frac{m}{1+e^{\alpha \cdot \left(\frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} - \frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} \right)}} = \frac{c_k}{\sigma_A} \cdot \lambda_A \tag{A.41}$$

$$\frac{\partial \mathscr{L}_B}{\partial W_B} = 0 \;\Rightarrow\; \frac{\frac{(\alpha \cdot m)(K_A+K_B)}{\phi \cdot (\ell_c - \ell_B)^2} \cdot \left(e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} - \frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} \right)} \right)}{\left(1+e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} - \frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} \right)} \right)^2} = \left(\frac{c_w^B}{\sigma_B} + 1 \right) \cdot \lambda_B \tag{A.42}$$

$$\frac{\partial \mathscr{L}_B}{\partial K_B} = 0 \;\Rightarrow\; \frac{m}{1+e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} - \frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} \right)}} = \frac{c_k}{\sigma_B} \cdot \lambda_B. \tag{A.43}$$

Using equations (A.40) and (A.41),

$$\left(\frac{\frac{(\alpha\cdot m)(K_A+K_B)}{\phi\cdot(\ell_c-\ell_A)^2}\cdot\left(e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)}\right)}{\left(1+e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)}\right)^2}\right)\cdot\left(\frac{\sigma_A}{c_w^A+\sigma_A}\right) = \left(\frac{m}{1+e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)}}\right)\cdot\left(\frac{\sigma_A}{c_k}\right)$$

$$\frac{\alpha\cdot(K_A+K_B)}{\phi\cdot(\ell_c-\ell_A)^2}\cdot\left(e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)}\right)\cdot\left(\frac{c_k}{c_w^A+\sigma_A}\right) = 1+e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)}$$

$$\left(e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)}\right)\cdot\left(\frac{\alpha\cdot(K_A+K_B)}{\phi\cdot(\ell_c-\ell_A)^2}\cdot\frac{c_k}{c_w^A+\sigma_A}-1\right) = 1$$

$$e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)} = \left(\frac{\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}\right)$$

$$\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right) = \ln\left(\frac{\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}\right)$$

$$\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}\right) = \frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}. \tag{A.44}$$

Performing similar substitutions for warlord B using equations (A.42) and (A.43),

$$\left(\frac{\frac{(\alpha\cdot m)(K_A+K_B)}{\phi\cdot(\ell_c-\ell_B)^2}\cdot\left(e^{\alpha\cdot\left(\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}\right)}\right)}{\left(1+e^{\alpha\cdot\left(\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}\right)}\right)^2}\right)\cdot\left(\frac{\sigma_B}{c_w^B+\sigma_B}\right) = \left(\frac{m}{1+e^{\alpha\cdot\left(\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}\right)}}\right)\cdot\left(\frac{\sigma_B}{c_k}\right)$$

$$\frac{\alpha\cdot(K_A+K_B)}{\phi\cdot(\ell_c-\ell_B)^2}\cdot\left(e^{\alpha\cdot\left(\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}\right)}\right)\cdot\left(\frac{c_k}{c_w^B+\sigma_B}\right) = 1+e^{\alpha\cdot\left(\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}\right)}$$

$$\left(e^{\alpha\cdot\left(\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}\right)}\right)\cdot\left(\frac{\alpha\cdot(K_A+K_B)}{\phi\cdot(\ell_c-\ell_B)^2}\cdot\frac{c_k}{c_w^B+\sigma_B}-1\right) = 1$$

$$e^{\alpha\cdot\left(\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}\right)} = \left(\frac{\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}\right)$$

$$\alpha\cdot\left(\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}\right) = \ln\left(\frac{\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}\right)$$

$$\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}\right) = \frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}. \tag{A.45}$$

By substituting equation (A.44) into (A.45),

$$\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}\right) = \frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}$$

$$-\frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}\right) = \frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}\right)$$

$$\frac{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}{\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)} = \frac{\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}$$

$$\begin{aligned} K_A+K_B &= \frac{\phi}{\alpha\cdot c_k}\cdot\left(c_w^A+\sigma_B\right)\cdot(\ell_c-\ell_A)^2 \\ &\quad +\frac{\phi}{\alpha\cdot c_k}\left(c_w^B+\sigma_B\right)(\ell_c-\ell_B)^2. \end{aligned} \tag{A.46}$$

Substituting equation (A.46) into (A.44),

$$\frac{w_A}{\phi\cdot(\ell_c-\ell_A)^2} = \frac{w_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}\right)$$

$$\frac{w_A}{\phi\cdot(\ell_c-\ell_A)^2} = \frac{w_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}{\phi\left(\left(c_w^A+\sigma_A\right)(\ell_c-\ell_A)^2+\left(c_w^B+\sigma_B\right)(\ell_c-\ell_B)^2\right)-\phi\cdot(\ell_c-\ell_A)^2\cdot\left(c_w^A+\sigma_A\right)}\right)$$

$$\frac{w_A}{\phi\cdot(\ell_c-\ell_A)^2} = \frac{w_B}{\phi\cdot(\ell_c-\ell_B)^2}+\frac{1}{\alpha}\ln\left(\frac{\left(c_w^B+\sigma_B\right)(\ell_c-\ell_B)^2}{\left(c_w^A+\sigma_A\right)(\ell_c-\ell_A)^2}\right), \tag{A.47}$$

and substituting equation (A.46) into (A.45),

$$\frac{w_B}{\phi\cdot(\ell_c-\ell_B)^2} = \frac{w_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}{c_k\cdot\alpha\cdot(K_A+K_B)-\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}\right)$$

$$\frac{w_B}{\phi\cdot(\ell_c-\ell_B)^2} = \frac{w_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{1}{\alpha}\cdot\ln\left(\frac{\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}{\phi\left(\left(c_w^A+\sigma_A\right)(\ell_c-\ell_A)^2+\left(c_w^B+\sigma_B\right)(\ell_c-\ell_B)^2\right)-\phi\cdot(\ell_c-\ell_B)^2\cdot\left(c_w^B+\sigma_B\right)}\right)$$

$$\frac{w_B}{\phi\cdot(\ell_c-\ell_B)^2} = \frac{w_A}{\phi\cdot(\ell_c-\ell_A)^2}-\frac{1}{\alpha}\ln\left(\frac{\left(c_w^B+\sigma_B\right)(\ell_c-\ell_B)^2}{\left(c_w^A+\sigma_A\right)(\ell_c-\ell_A)^2}\right). \tag{A.48}$$

Using the above equations and recalling $\ell_A = 0$ and $\ell_B = 1$, the equilibrium level of capital investment for warlord A is found; that is,

$$K_A = \frac{\phi}{\alpha \cdot c_k}\Big(\big(c_w^A+\sigma_A\big)\cdot \ell_c^2+\big(c_w^B+\sigma_B\big)(\ell_c-1)\Big)-K_B$$

$$= \frac{\phi}{\alpha \cdot c_k}\Big(\big(c_w^A+\sigma_A\big)\cdot \ell_c^2+\big(c_w^B+\sigma_B\big)(\ell_c-1)\Big)-\frac{\sigma_B}{c_k}\left(\aleph_B-\left(\frac{c_w^B}{\sigma_B}+1\right)W_B\right)$$

$$= \frac{\phi}{\alpha \cdot c_k}\Big(\big(c_w^A+\sigma_A\big)\cdot \ell_c^2+\big(c_w^B+\sigma_B\big)(\ell_c-1)\Big)-\frac{\sigma_B}{c_k}(\aleph_B)$$

$$-\left(\frac{c_w^B+\sigma_B}{c_k}\right)\left(\frac{\phi}{\alpha}\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\Big(\frac{\ell_c}{\ell_c-1}\Big)^2\right)+\Big(\frac{\ell_c-1}{\ell_c}\Big)^2 W_A\right)$$

$$= \frac{\phi}{\alpha \cdot c_k}\Big(\big(c_w^A+\sigma_A\big)\cdot \ell_c^2+\big(c_w^B+\sigma_B\big)(\ell_c-1)\Big)-\frac{\sigma_B}{c_k}(\aleph_B)-\left(\frac{c_w^B+\sigma_B}{c_k}\right)\left(\frac{\phi}{\alpha}\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\Big(\frac{\ell_c}{\ell_c-1}\Big)^2\right)\right)$$

$$+\frac{\sigma_A}{c_k}\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\right)\Big(\frac{\ell_c-1}{\ell_c}\Big)^2\left(\aleph_A-\frac{c_k}{\sigma_A}K_A\right)$$

$$c_k\cdot K_A\left(1+\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\Big(\frac{\ell_c-1}{\ell_c}\Big)^2\right) = \frac{\big(c_w^B+\sigma_B\big)(\sigma_A)(\aleph_A)}{c_w^A+\sigma_A}-\frac{\big(c_w^A+\sigma_A\big)(\sigma_B)\big(N_B+\frac{Y_B}{\sigma_B}\big)}{c_w^A+\sigma_A}+\frac{\phi}{\alpha\cdot c_k}\Big(\big(c_w^A+\sigma_A\big)\cdot\ell_c^2+\big(c_w^B+\sigma_B\big)(\ell_c-1)\Big)$$

$$-\left(\frac{c_w^B+\sigma_B}{c_k}\right)\left(\frac{\phi}{\alpha}\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\Big(\frac{\ell_c}{\ell_c-1}\Big)^2\right)\right)$$

and rearranging the variables leads to the equilibrium level of K_A^* in equation (3.74).

Similarly for warlord B,

$$K_B = \frac{\phi}{\alpha \cdot c_k}\left(\left(c_w^A+\sigma_A\right)\cdot\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)\right)-K_A$$

$$= \frac{\phi}{\alpha \cdot c_k}\left(\left(c_w^A+\sigma_A\right)\cdot\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)\right)-\frac{\sigma_A}{c_k}\left(\aleph_A-\left(\frac{c_w^A}{\sigma_A}+1\right)W_A\right)$$

$$= \frac{\phi}{\alpha \cdot c_k}\left(\left(c_w^A+\sigma_A\right)\cdot\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)\right)-\frac{\sigma_A}{c_k}(\aleph_A)$$

$$-\left(\frac{c_w^A+\sigma_A}{c_k}\right)\left(\frac{\phi}{\alpha}\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\left(\frac{\ell_c-1}{\ell_c}\right)^2\right)+\left(\frac{\ell_c}{\ell_c-1}\right)^2 W_B\right)$$

$$= \frac{\phi}{\alpha \cdot c_k}\left(\left(c_w^A+\sigma_A\right)\cdot\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)\right)-\frac{\sigma_A}{c_k}(\aleph_A)-\left(\frac{c_w^A+\sigma_A}{c_k}\right)\left(\frac{\phi}{\alpha}\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\left(\frac{\ell_c-1}{\ell_c}\right)^2\right)\right)$$

$$+\frac{\sigma_B}{c_k}\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\right)\left(\frac{\ell_c}{\ell_c-1}\right)^2\left(\aleph_B-\frac{c_k}{\sigma_B}K_B\right)$$

$$c_k \cdot K_B\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\left(\frac{\ell_c-1}{\ell_c}\right)^2\right) = \frac{\left(c_w^B+\sigma_B\right)(\sigma_B)(\aleph_B)}{c_w^B+\sigma_B}-\frac{\left(c_w^B+\sigma_B\right)(\sigma_A)\left(N_A+\frac{Y_A}{\sigma_A}\right)}{c_w^B+\sigma_B}+\frac{\phi}{\alpha \cdot c_k}\left(\left(c_w^A+\sigma_A\right)\cdot\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)\right)$$

$$-\left(\frac{c_w^A+\sigma_A}{c_k}\right)\left(\frac{\phi}{\alpha}\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\left(\frac{\ell_c-1}{\ell_c}\right)^2\right)\right)$$

where, again, rearranging the variables leads to the equilibrium level of K_B^* in equation (3.75).

Substituting the equilibrium level of capital K_A^*, from equation (3.74), into warlord A's constraint from equation (3.8) leads to the equilibrium number of warriors W_A^*,

found in equation (3.72):

$$
\begin{aligned}
W_A^* &= \left(\frac{\sigma_A}{c_w^A+\sigma_A}\right)(\aleph_A)-\left(\frac{c_k}{c_w^A+\sigma_A}\right)\cdot K_A^* \\
&= \left(\frac{\sigma_A}{c_w^A+\sigma_A}\right)(\aleph_A)-\frac{\phi}{\alpha}\left(\ell_c^2+\left(\frac{\left(c_w^B+\sigma_B\right)\ell_c^2(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\left(\frac{\ell_c}{\ell_c-1}\right)^2\right)\right) \\
&\quad -\left(\frac{1}{c_w^A+\sigma_A}\right)\left(\frac{\left(c_w^B+\sigma_B\right)(\sigma_A)(\aleph_A)(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)+\left(\frac{1}{c_w^A+\sigma_A}\right)\left(\frac{\left(c_w^A+\sigma_A\right)(\sigma_B)(\aleph_B)\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right) \\
&= \left(\frac{\ell_c^2}{c_w^A+\sigma_A}\right)\left(\frac{\left(c_w^A+\sigma_A\right)(\sigma_A)(\aleph_A)+\left(c_w^A+\sigma_A\right)(\sigma_B)(\aleph_B)}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)-\frac{\phi}{\alpha}\left(\ell_c^2+\left(\frac{\left(c_w^B+\sigma_B\right)\ell_c^2(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)\ln\left(\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\left(\frac{\ell_c}{\ell_c-1}\right)^2\right)\right).
\end{aligned}
$$

Similarly, substituting the equilibrium level of capital K_B^*, from equation (3.75), into warlord B's constraint from equation (3.9) leads to the equilibrium number of warriors W_B^*, found in equation (3.73):

$$
\begin{aligned}
W_B^* &= \left(\frac{\sigma_B}{c_w^B+\sigma_B}\right)(\aleph_B)-\left(\frac{c_k}{c_w^B+\sigma_B}\right)\cdot K_B^* \\
&= \left(\frac{\sigma_B}{c_w^B+\sigma_B}\right)(\aleph_B)-\frac{\phi}{\alpha}\left((\ell_c-1)^2+\left(\frac{\left(c_w^A+\sigma_A\right)\ell_c^2(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\left(\frac{\ell_c-1}{\ell_c}\right)^2\right)\right) \\
&\quad -\left(\frac{1}{c_w^B+\sigma_B}\right)\left(\frac{\left(c_w^A+\sigma_A\right)(\sigma_B)(\aleph_B)\ell_c^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)+\left(\frac{1}{c_w^B+\sigma_B}\right)\left(\frac{\left(c_w^B+\sigma_B\right)(\sigma_A)(\aleph_A)(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right) \\
&= \left(\frac{(\ell_c-1)^2}{c_w^B+\sigma_B}\right)\left(\frac{\left(c_w^A+\sigma_A\right)(\sigma_A)(\aleph_A)+\left(c_w^A+\sigma_A\right)(\sigma_B)(\aleph_B)}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)-\frac{\phi}{\alpha}\left((\ell_c-1)^2+\left(\frac{\left(c_w^A+\sigma_A\right)\ell_c^2(\ell_c-1)^2}{\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2}\right)\ln\left(\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\left(\frac{\ell_c-1}{\ell_c}\right)^2\right)
\end{aligned}
$$

To check that the second-order condition is satisfied for warlord A, the bordered

Hessian for warlord A is

$$\mathcal{H}_A^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w^A}{\sigma_A}+1\right) & -\frac{c_k}{\sigma_A} \\ -\left(\frac{c_w^A}{\sigma_A}+1\right) & \frac{\partial^2 V_A}{\partial W_A^2} & \frac{\partial^2 V_A}{\partial W_A K_A} \\ -\frac{c_k}{\sigma_A} & \frac{\partial^2 V_A}{\partial K_A W_A} & \frac{\partial^2 V_A}{\partial K_A^2} \end{pmatrix} \tag{A.49}$$

and is satisfied when the determinant, $|\mathcal{H}_A^{\mathcal{B}}|$, is greater than zero; that is,

$$|\mathcal{H}_A^{\mathcal{B}}| > 0 \quad \rightarrow \quad \alpha \cdot \frac{e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)}-1}{\left(1+e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-\ell_B)^2}-\frac{W_A}{\phi\cdot(\ell_c-\ell_A)^2}\right)}\right)\cdot\phi\cdot(\ell_c-\ell_A)^2} < \frac{2\left(c_w^A+\sigma_A\right)}{c_k(K_A+K_B)}. \tag{A.50}$$

Substituting the equilibrium levels of W_A^*, W_B^*, K_A^* and K_B^* into equation (A.50):

$$\alpha \cdot \frac{e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-1)^2}-\frac{W_A}{\phi\cdot\ell_c^2}\right)}-1}{\left(1+e^{\alpha\cdot\left(\frac{W_B}{\phi\cdot(\ell_c-1)^2}-\frac{W_A}{\phi\cdot\ell_c^2}\right)}\right)\cdot\phi\cdot\ell_c^2} \quad < \quad \frac{2(c_w^A+\sigma_A)}{c_k(K_A+K_B)}$$

$$\alpha \cdot \frac{\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\left(\frac{\ell_c}{\ell_c-1}\right)^2-1}{\left(1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\left(\frac{\ell_c}{\ell_c-1}\right)^2\right)\cdot\phi\cdot\ell_c^2} \quad < \quad \frac{2\left(c_w^A+\sigma_A\right)}{\frac{\phi}{\alpha}\cdot\left(\left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2\right)}$$

$$\left(c_w^A+\sigma_A\right)\ell_c^2-\left(c_w^B+\sigma_B\right)(\ell_c-1)^2 \quad < \quad 2\left(c_w^A+\sigma_A\right)\cdot\ell_c^2$$

$$0 \quad < \quad \left(c_w^A+\sigma_A\right)\ell_c^2+\left(c_w^B+\sigma_B\right)(\ell_c-1)^2.$$

To check that the second-order condition is satisfied for warlord B, the bordered Hessian for warlord B is

$$\mathcal{H}_B^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w^B}{\sigma_B}+1\right) & -\frac{c_k}{\sigma_B} \\ -\left(\frac{c_w^B}{\sigma_B}+1\right) & \frac{\partial^2 V_B}{\partial W_B^2} & \frac{\partial^2 V_B}{\partial W_B K_B} \\ -\frac{c_k}{\sigma_B} & \frac{\partial^2 V_B}{\partial K_B W_B} & \frac{\partial^2 V_B}{\partial K_B^2} \end{pmatrix} \tag{A.51}$$

and is satisfied when the determinant, $|\mathcal{H}_B^{\mathcal{B}}|$, is greater than zero; that is,

$$|\mathcal{H}_B^{\mathcal{B}}| > 0 \quad \rightarrow \quad \alpha \cdot \frac{e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} - \frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2}\right)} - 1}{\left(1 + e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot (\ell_c - \ell_A)^2} - \frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2}\right)}\right) \cdot \phi \cdot (\ell_c - \ell_B)^2} < \frac{2\left(c_w^B + \sigma_B\right)}{c_k (K_A + K_B)}. \tag{A.52}$$

Substituting the equilibrium levels W_A^*, W_B^*, K_A^* and K_B^* into equation (A.52):

$$\begin{aligned}
\alpha \cdot \frac{e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot \ell_c^2} - \frac{W_A}{\phi \cdot (\ell_c - 1)^2}\right)} - 1}{\left(1 + e^{\alpha \cdot \left(\frac{W_A}{\phi \cdot \ell_c^2} - \frac{W_B}{\phi \cdot (\ell_c - 1)^2}\right)}\right) \cdot \phi \cdot (\ell_c - 1)^2} &< \frac{2(c_w^B + \sigma_B)}{c_k (K_A + K_B)} \\
\alpha \cdot \frac{\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \left(\frac{\ell_c - 1}{\ell_c}\right)^2 - 1}{\left(1 + \frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \left(\frac{\ell_c - 1}{\ell_c}\right)^2\right) \cdot \phi \cdot (\ell_c - 1)^2} &< \frac{2\left(c_w^B + \sigma_B\right)}{\frac{\phi}{\alpha} \cdot \left(\left(c_w^A + \sigma_A\right)\ell_c^2 + \left(c_w^B + \sigma_B\right)(\ell_c - 1)^2\right)} \\
\left(c_w^B + \sigma_B\right)(\ell_c - 1)^2 - \left(c_w^A + \sigma_A\right)\ell_c^2 &< 2\left(c_w^B + \sigma_B\right)(\ell_c - 1)^2 \\
0 &< \left(c_w^A + \sigma_A\right)\ell_c^2 + \left(c_w^B + \sigma_B\right)(\ell_c - 1)^2.
\end{aligned}$$

◇ Proof for Theorem 8:

Equation (3.77) is derived by substituting equations (3.72) and (3.73) into the CSF found above in equation (3.3). Explicitly,

$$\begin{aligned}
\pi_A^* &= \frac{1}{1 + e^{\alpha \cdot \left(\frac{W_B}{\phi \cdot (\ell_c - \ell_B)^2} - \frac{W_A}{\phi \cdot (\ell_c - \ell_B)^2}\right)}} \\
&= \frac{1}{1 + e^{-\ln\left(\frac{c_w^B + \sigma_B}{c_w^A + \sigma_A} \frac{(\ell_c - 1)^2}{\ell_c^2}\right)}} \\
&= \frac{1}{1 + e^{\ln\left(\frac{c_w^A + \sigma_A}{c_w^B + \sigma_B} \frac{\ell_c^2}{(\ell_c - 1)^2}\right)}} = \frac{1}{1 + \frac{c_w^A + \sigma_A}{c_w^B + \sigma_B} \left(\frac{\ell_c}{\ell_c - 1}\right)^2}.
\end{aligned}$$

Equation (3.78) follows by

$$\begin{aligned}
\pi_B^* &= 1-\pi_A^* \\
&= 1-\frac{1}{1+\frac{c_w^A+\sigma_A}{c_w^B+\sigma_B}\left(\frac{\ell_c}{\ell_c-1}\right)^2} \\
&= \frac{1}{1+\frac{c_w^B+\sigma_B}{c_w^A+\sigma_A}\left(\frac{\ell_c-1}{\ell_c}\right)^2}.
\end{aligned}$$

◇ Proof for Theorem 9:

The proof for Theorem 9 begins by maximizing warlord A and B's optimization problems for the two choice variables W and K. Let λ_A and λ_B be the associated Lagrangian multipliers for maximization problems (4.2) and (4.3). The Lagrangian equations for warlords A and B are

$$\mathscr{L}_A = \left(\frac{m\cdot\widehat{K}}{1+e^{\alpha\cdot(W_B-W_A+\phi\cdot\hat{\ell})}}\right)+(mK_A)\cdot(1-(1-\ell_c)\beta)+\lambda_A\left(\aleph_A-\left(\frac{c_w^A+\sigma_A}{\sigma_A}\right)W_A-\left(\frac{c_k}{\sigma_A}\right)K_A\right) \quad (A.53)$$

$$\mathscr{L}_B = \left(\frac{m\cdot\widehat{K}}{1+e^{\alpha\cdot(W_A-W_B-\phi\cdot\hat{\ell})}}\right)+(mK_B)\cdot(1-\ell_c\beta)+\lambda_B\left(\aleph_B-\left(\frac{c_w^B+\sigma_B}{\sigma_B}\right)W_B-\left(\frac{c_k}{\sigma_B}\right)K_B\right) \quad (A.54)$$

Therefore:

$$\frac{\partial\mathscr{L}_A}{\partial W_A}=0 \quad\Rightarrow\quad \frac{\alpha\cdot m\cdot\widehat{K}\cdot e^{\alpha\cdot(W_B-W_A+\phi\cdot\hat{\ell})}}{\left(1+e^{\alpha(W_B-W_A+\phi\cdot\hat{\ell})}\right)^2}=\left(\frac{c_w}{\sigma}+1\right)\cdot\lambda_A \quad (A.55)$$

$$\frac{\partial\mathscr{L}_A}{\partial K_A}=0 \quad\Rightarrow\quad \frac{m\cdot(1-\ell_c)\cdot\beta}{1+e^{\alpha\cdot(W_B-W_A+\phi\cdot\hat{\ell})}}+m\cdot(1-(1-\ell_c)\cdot\beta)=\frac{c_k}{\sigma}\cdot\lambda_A \quad (A.56)$$

$$\frac{\partial\mathscr{L}_B}{\partial W_B}=0 \quad\Rightarrow\quad \frac{\alpha\cdot m\cdot\widehat{K}\cdot e^{\alpha(W_A-W_B-\phi\cdot\hat{\ell})}}{\left(1+e^{\alpha\cdot(W_A-W_B-\phi\cdot\hat{\ell})}\right)^2}=\left(\frac{c_w}{\sigma}+1\right)\cdot\lambda_B \quad (A.57)$$

$$\frac{\partial\mathscr{L}_B}{\partial K_B}=0 \quad\Rightarrow\quad \frac{m\cdot\ell_c}{1+e^{\alpha\cdot(W_A-W_B-\phi\cdot\hat{\ell})}}+m\cdot(1-\ell_c\cdot\beta)=\frac{c_k}{\sigma}\cdot\lambda_B. \quad (A.58)$$

From equations (A.55) and (A.56),

$$
\begin{aligned}
\left(\frac{m\cdot(1-\ell_c)\cdot\beta}{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}+m\cdot(1-(1-\ell_c)\cdot\beta)\right)\cdot\left(\frac{\sigma}{c_k}\right) &= \left(\frac{\alpha\cdot m\cdot\widehat{K}\cdot\sigma}{c_w+\sigma}\right)\cdot\frac{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2} \\
\left(\frac{1}{\alpha\cdot m}\cdot\frac{\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)^2}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(\frac{m\cdot(1-\ell_c)\cdot\beta}{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}+m\cdot(1-(1-\ell_c)\cdot\beta)\right)\cdot\left(\frac{\sigma}{c_k}\right) &= \frac{\widehat{K}\cdot\sigma}{c_w+\sigma} \\
\left(\frac{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left((1-\ell_c)\cdot\beta+\left(1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot(1-(1-\ell_c)\cdot\beta)\right)\cdot\left(\frac{\sigma}{\alpha\cdot c_k}\right) &= \frac{\widehat{K}\cdot\sigma}{c_w+\sigma} \\
\left(\frac{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot(1-(1-\ell_c)\cdot\beta)\right)\cdot\left(\frac{\sigma}{\alpha\cdot c_k}\right) &= \frac{\widehat{K}\cdot\sigma}{c_w+\sigma} \\
\left(\frac{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot(1-(1-\ell_c)\cdot\beta)\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right) &= \widehat{K}, \qquad \text{(A.59)}
\end{aligned}
$$

and from equations (A.57) and (A.58),

$$
\begin{aligned}
\left(\frac{m\cdot\ell_c\cdot\beta}{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}+m\cdot(1-\ell_c\cdot\beta)\right)\cdot\left(\frac{\sigma}{c_k}\right) &= \left((\alpha\cdot m)\cdot\frac{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}\right)\cdot\left(\frac{\widehat{K}\cdot\sigma}{c_w+\sigma}\right) \\
\left(\frac{1}{\alpha\cdot m}\cdot\frac{\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)^2}{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(\frac{m\cdot\ell_c\cdot\beta}{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}+m\cdot(1-\ell_c\cdot\beta)\right)\cdot\left(\frac{\sigma}{c_k}\right) &= \frac{\widehat{K}\cdot\sigma}{c_w+\sigma} \\
\left(\frac{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(\ell_c\cdot\beta+\left(1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot(1-\ell_c\cdot\beta)\right)\cdot\left(\frac{\sigma}{\alpha\cdot c_k}\right) &= \frac{\widehat{K}\cdot\sigma}{c_w+\sigma} \\
\left(\frac{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot(1-\ell_c\cdot\beta)\right)\cdot\left(\frac{\sigma}{\alpha\cdot c_k}\right) &= \frac{\widehat{K}\cdot\sigma}{c_w+\sigma} \\
\left(\frac{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot(1-\ell_c)\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right) &= \widehat{K}. \qquad \text{(A.60)}
\end{aligned}
$$

Equating equations (A.59) and (A.60) leads to the following relationship:

$$\left(\frac{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot(1-(1-\ell_c)\cdot\beta)\right)=\left(\frac{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot(1-\ell_c\cdot\beta)\right).$$

From the definitions of π_A and π_B found in equation (3.41),

$$e^{\alpha\cdot\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}=e^{\alpha\cdot(-1)\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}=\frac{1}{e^{\alpha\cdot\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}.$$

Substituting the above equation into (A.61),

$$\left(\frac{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot(1-(1-\ell_c)\cdot\beta)\right)$$

$$=\left(\frac{1+e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}\right)\cdot(1-\ell_c\cdot\beta)\right)$$

$$\left(\frac{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot(1-(1-\ell_c)\cdot\beta)\right)$$

$$=\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot\left(\frac{1+e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)\cdot\left(1+\frac{1-\ell_c\cdot\beta}{e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}}\right)$$

$$1+\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot(1-(1-\ell_c)\cdot\beta)=\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)+1-\ell_c\cdot\beta$$

$$\ell_c\cdot\beta=\left(e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}\right)\cdot(1-\ell_c)\cdot\beta$$

$$e^{\alpha\left(W_B-W_A+\phi\cdot\hat{\ell}\right)}=\frac{\ell_c}{1-\ell_c}\quad\text{and}\quad e^{\alpha\cdot\left(W_A-W_B-\phi\cdot\hat{\ell}\right)}=\frac{1-\ell_c}{\ell_c}\tag{A.61}$$

in which,

$$W_A=W_B+\phi\cdot\hat{\ell}-\frac{1}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)\tag{A.62}$$

and

$$W_B = W_A - \phi \cdot \hat{\ell} + \frac{1}{\alpha} \cdot \ln\left(\frac{\ell_c}{1-\ell_c}\right). \tag{A.63}$$

Substituting (A.61) into either (A.59) or (A.60),

$$\begin{aligned}
\widehat{K} &= \left(\frac{1+e^{\alpha\left(W_B - W_A + \phi\cdot\hat{\ell}\right)}}{e^{\alpha\left(W_B - W_A + \phi\cdot\hat{\ell}\right)}}\right) \cdot \left(1 + \left(e^{\alpha\left(W_B - W_A + \phi\cdot\hat{\ell}\right)}\right) \cdot (1-(1-\ell_c)\cdot\beta)\right) \cdot \left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right) \\
&= \left(\frac{1+\frac{\ell_c}{1-\ell_c}}{\frac{\ell_c}{1-\ell_c}}\right) \cdot \left(1 + \left(\frac{\ell_c}{1-\ell_c}\right) \cdot (1-(1-\ell_c)\cdot\beta)\right) \cdot \left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right) \\
&= \left(\frac{\ell_c+(1-\ell_c)}{\ell_c}\right) \cdot \left(1 + \frac{\ell_c}{1-\ell_c} - \ell_c\cdot\beta\right) \cdot \left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right) \\
&= \left(\frac{1}{\ell_c}\right) \cdot \left(\frac{1}{1-\ell_c} - \ell_c\cdot\beta\right) \cdot \left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right) \\
\widehat{K} &= \left(\frac{1}{\ell_c\cdot(1-\ell_c)} - \beta\right) \cdot \left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right). && \text{(A.64)}
\end{aligned}$$

Using the equations (3.38), (A.62), (A.63) and (A.64)

$$\widehat{K} = \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)$$

$$K_A\cdot(1-\ell_c)\cdot\beta+K_B\cdot\ell_c\cdot\beta = \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)$$

$$K_A\cdot(1-\ell_c)\cdot\beta = \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)-K_B\cdot\ell_c\cdot\beta$$

$$K_A = \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{1}{(1-\ell_c)\cdot\beta}\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)-K_B\cdot\frac{\ell_c}{1-\ell_c}$$

$$= \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{1}{(1-\ell_c)\cdot\beta}\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)-\left(\frac{\sigma}{c_k}\right)\cdot\left(\frac{\ell_c}{1-\ell_c}\right)\cdot\left(\aleph_B-\left(\frac{c_w}{\sigma}+1\right)\cdot W_B\right)$$

$$= \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{1}{(1-\ell_c)\cdot\beta}\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)$$

$$-\left(\frac{m_R-c_E}{c_k}\right)\cdot\left(\frac{\ell_c}{1-\ell_c}\right)\cdot\left(\aleph_B-\left(\frac{c_w}{\sigma}+1\right)\cdot\left(W_A-\phi\cdot\hat{\ell}+\frac{1}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)\right)\right)$$

$$= \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{1}{(1-\ell_c)\cdot\beta}\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)-\left(\frac{\sigma}{c_k}\right)\cdot\left(\frac{\ell_c}{1-\ell_c}\right)\cdot\left(\aleph_A+\frac{c_k}{\sigma}\cdot K_A\right)$$

$$-\left(\frac{\sigma}{c_k}\right)\cdot\left(\frac{\ell_c}{1-\ell_c}\right)\cdot\left(\aleph_B-\left(\frac{c_w}{\sigma}+1\right)\cdot\left(\frac{1}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)-\phi\cdot\hat{\ell}\right)\right)$$

$$K_A\cdot\left(1+\frac{\ell_c}{1-\ell_c}\right) = \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{1}{(1-\ell_c)\cdot\beta}\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)-\left(\frac{\sigma}{c_k}\right)\cdot\left(\frac{\ell_c}{1-\ell_c}\right)\cdot(\aleph_B-\aleph_A)$$

$$+\left(\frac{\sigma}{c_k}\right)\cdot\left(\frac{\ell_c}{1-\ell_c}\right)\cdot\left(\frac{c_w}{\sigma}+1\right)\cdot\left(\frac{1}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)-\phi\cdot\hat{\ell}\right)$$

$$K_A^* = (\ell_c)\cdot\left(\frac{\sigma}{c_k}\right)\cdot(\aleph_A-\aleph_B)+\left(\frac{c_w+\sigma}{c_k}\right)\cdot\left(\left(\frac{1}{\alpha}\right)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)+\frac{\ell_c}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)-(\ell_c\phi)\cdot\hat{\ell}\right).$$

Similarly for warlord B,

$$\begin{aligned}
K_B &= \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{1}{\ell_c\cdot\beta}\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)-K_A\cdot\left(\frac{1-\ell_c}{\ell_c}\right) \\
&= \left(\frac{1}{\ell_c\cdot(1-\ell_c)}-\beta\right)\cdot\left(\frac{1}{\ell_c\cdot\beta}\right)\cdot\left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)-\left(\frac{1-\ell_c}{\ell_c}\right)\cdot(\ell_c)\cdot(\aleph_A-\aleph_B)\cdot\left(\frac{\sigma}{c_k}\right) \\
&\quad -\left(\frac{1-\ell_c}{\ell_c}\right)\cdot\left(\frac{c_w+\sigma}{c_k}\right)\cdot\left(\left(\frac{1}{\alpha}\right)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)+\frac{\ell_c}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)-(\ell_c\phi)\cdot\hat{\ell}\right) \\
&= \left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\alpha\cdot(1-\ell_c)\cdot\left(\phi\cdot\hat{\ell}+\ln\left(\frac{1-\ell_c}{\ell_c}\right)\right)+\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)\cdot\left(\frac{1}{\ell_c}-\frac{1-\ell_c}{\ell_c}\right)\right)+(1-\ell_c)\cdot\left(\frac{\sigma}{c_k}\right)\cdot(\aleph_B-\aleph_A) \\
K_B^* &= \left(\frac{c_w+\sigma}{\alpha\cdot c_k}\right)\cdot\left(\alpha\cdot(1-\ell_c)\cdot\left(\phi\cdot\hat{\ell}+\ln\left(\frac{1-\ell_c}{\ell_c}\right)\right)+\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)\right)+(1-\ell_c)\cdot\left(\frac{\sigma}{c_k}\right)\cdot(\aleph_B-\aleph_A).
\end{aligned}$$

Substituting the equilibrium level of capital K_A^*, from equation (4.8), into warlord A's constraint from equation (3.38) leads to the equilibrium number of warriors W_A^*, found in equation (4.6):

$$\begin{aligned}
W_A^*\cdot\left(\frac{c_w}{\sigma}+1\right) &= \aleph_A-\left(\frac{c_k}{\sigma}\right)\cdot K_A \\
&= \aleph_A-\ell_c\cdot(\aleph_A-\aleph_B)+\left(\frac{c_w+\sigma}{\sigma}\right)\cdot\left(\left(\frac{1}{\alpha}\right)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)+\frac{\ell_c}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)-(\ell_c\phi)\cdot\hat{\ell}\right) \\
&= \aleph_A\cdot(1-\ell_c)+\aleph_B\cdot\ell_c-\left(\frac{c_w}{\sigma}+1\right)\cdot\left(\left(\frac{1}{\alpha}\right)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)+\frac{\ell_c}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)-(\ell_c\phi)\cdot\hat{\ell}\right) \\
W_A^* &= \left(\frac{\sigma}{c_w+\sigma}\right)\cdot(\aleph_A\cdot(1-\ell_c)+\aleph_B\cdot\ell_c)-\left(\left(\frac{1}{\alpha}\right)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)+\frac{\ell_c}{\alpha}\cdot\ln\left(\frac{\ell_c}{1-\ell_c}\right)-(\ell_c\phi)\cdot\hat{\ell}\right).
\end{aligned}$$

Similarly, substituting the equilibrium level of capital K_B^*, from equation (4.9), into warlord B's constraint from equation (3.38) leads to the equilibrium number of warriors W_B^*, found in equation (4.7):

$$W_B^* \cdot \left(\frac{c_w}{\sigma}+1\right) = \aleph_B - \left(\frac{c_k}{\sigma}\right) \cdot K_B$$

$$= \aleph_B - \left(\frac{c_w+\sigma}{\alpha \cdot \sigma}\right) \cdot \left(\alpha \cdot (1-\ell_c) \cdot \left(\phi \cdot \hat{\ell} + \ln\left(\frac{1-\ell_c}{\ell_c}\right)\right) + \left(\frac{1}{\ell_c \cdot (1-\ell_c) \cdot \beta} - 1\right)\right) + (1-\ell_c) \cdot (\aleph_B - \aleph_A)$$

$$= \aleph_A \cdot (1-\ell_c) + \aleph_B \cdot \ell_c - \left(\frac{c_w+\sigma}{\alpha \cdot \sigma}\right) \cdot \left(\alpha \cdot (1-\ell_c) \cdot \left(\phi \cdot \hat{\ell} + \ln\left(\frac{1-\ell_c}{\ell_c}\right)\right) + \left(\frac{1}{\ell_c \cdot (1-\ell_c) \cdot \beta} - 1\right)\right)$$

$$W_B^* = \left(\frac{\sigma}{c_w+\sigma}\right) \cdot (\aleph_A \cdot (1-\ell_c) + \aleph_B \cdot \ell_c) - \frac{1}{\alpha} \cdot \left(\alpha \cdot (1-\ell_c) \cdot \left(\phi \cdot \hat{\ell} + \ln\left(\frac{1-\ell_c}{\ell_c}\right)\right) + \left(\frac{1}{\ell_c \cdot (1-\ell_c) \cdot \beta} - 1\right)\right).$$

To show the second-order conditions, the bordered Hessian for warlord A again is

$$\mathcal{H}_A^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w}{\sigma}+1\right) & -\frac{c_k}{\sigma} \\ -\left(\frac{c_w}{\sigma}+1\right) & \frac{\partial^2 V_A}{\partial W_A^2} & \frac{\partial^2 V_A}{\partial W_A K_A} \\ -\frac{c_k}{\sigma} & \frac{\partial^2 V_A}{\partial K_A W_A} & \frac{\partial^2 V_A}{\partial K_A^2} \end{pmatrix} \tag{A.65}$$

and is satisfied when the determinant, $|\mathcal{H}_A^{\mathcal{B}}|$, is greater than zero; that is,

$$|\mathcal{H}_A^{\mathcal{B}}| = \left(\frac{c_w}{\sigma}+1\right) \cdot \left(\frac{c_k}{\sigma}\right) \cdot \left(\frac{\partial^2 V_A}{\partial W_A K_A}\right) - \left(\frac{c_k}{\sigma}\right) \cdot \left(\left(\frac{c_k}{\sigma}\right) \cdot \left(\frac{\partial^2 V_A}{\partial W_A^2}\right) - \left(\frac{c_w}{\sigma}+1\right) \cdot \left(\frac{\partial^2 V_A}{\partial K_A W_A}\right)\right) > 0$$

$$= \left(\frac{c_w}{\sigma}+1\right) \cdot \left(\frac{\partial^2 V_A}{\partial W_A K_A}\right) - \left(\frac{c_k}{\sigma}\right) \cdot \left(\frac{\partial^2 V_A}{\partial W_A^2}\right) + \left(\frac{c_w}{\sigma}+1\right) \cdot \left(\frac{\partial^2 V_A}{\partial K_A W_A}\right) > 0$$

$$= 2 \cdot \left(\frac{c_w}{\sigma}+1\right) \cdot \left(\frac{\partial^2 V_A}{\partial W_A K_A}\right) - \left(\frac{c_k}{\sigma}\right) \cdot \left(\frac{\partial^2 V_A}{\partial W_A^2}\right) > 0$$

Using equations (A.55), (A.56), (A.62), (A.63), taking the appropriate derivatives and substituting into the above equation

$$
\begin{aligned}
|\mathcal{H}_A^{\mathcal{B}}| &= 2\cdot\left(\frac{c_w}{\sigma}+1\right)\cdot\left(\frac{e^{\alpha\cdot(W_B-W_A+\phi\ell_c)}}{\left(1+e^{\alpha\cdot(W_B-W_A+\phi\ell_c)}\right)^2}\right)\cdot(\alpha\cdot m)\cdot(1-\ell_c)\cdot\beta \\
&\quad -\left(\frac{c_k}{\sigma}\right)\cdot\left(\frac{e^{\alpha\cdot(W_B-W_A+\phi\ell_c)}\left(e^{\alpha\cdot(W_B-W_A+\phi\ell_c)}-1\right)}{\left(1+e^{\alpha\cdot(W_B-W_A+\phi\ell_c)}\right)^3}\right)\cdot(\alpha\cdot m)\cdot\widehat{K}>0 \\
&= 2\cdot\left(\frac{c_w}{\sigma}+1\right)\cdot(\alpha\cdot m)\cdot(1-\ell_c)\cdot\beta-\left(\frac{e^{\alpha\cdot(W_B-W_A+\phi\ell_c)}-1}{e^{\alpha\cdot(W_B-W_A+\phi\ell_c)}+1}\right)\cdot(\alpha\cdot m)\cdot\widehat{K}>0 \\
&= 2\cdot\left(\frac{c_w}{\sigma}+1\right)\cdot(1-\ell_c)\cdot\beta-\left(\frac{c_k}{\sigma}\right)\cdot\left(\frac{\frac{\ell_c}{1-\ell_c}-1}{\frac{\ell_c}{1-\ell_c}+1}\right)\cdot\left(\frac{c_w+\sigma}{c_k}\right)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)>0 \\
&= 2\cdot(1-\ell_c)\cdot\beta-(2\ell_c-1)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)>0 \\
&= 2\cdot\beta-2\cdot\ell_c\cdot\beta-\frac{2\cdot\ell_c-1}{\ell_c\cdot(1-\ell_c)}+2\cdot\ell_c\cdot\beta-\beta>0 \\
&= \beta>\frac{2\cdot\ell_c-1}{\ell_c\cdot(1-\ell_c)} \\
&= \ell_c\cdot\beta-\ell_c^2\cdot\beta>2\cdot\ell_c-1 \\
&= 0>\ell_c^2\cdot\beta+\ell_c\cdot(2-\beta)-1,
\end{aligned}
$$

where $\ell_c^2\cdot\beta+\ell_c\cdot(2-\beta)-1=0$ when

$$\ell_c=\frac{\beta-2\pm\sqrt{(2-\beta)^2+4\cdot\beta}}{2\cdot\beta}=\frac{\beta-2\pm\sqrt{4+\beta^2}}{2\cdot\beta}.$$

Therefore, $|\mathcal{H}_A^{\mathcal{B}}|>0$ when $\beta-2\pm\sqrt{4+\beta^2}>\ell_c\cdot(2\cdot\beta)$.

To check that the second-order condition is satisfied for warlord B,

$$\mathcal{H}_B^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w}{\sigma}+1\right) & -\frac{c_k}{\sigma} \\ -\left(\frac{c_w}{\sigma}+1\right) & \frac{\partial^2 V_B}{\partial W_B^2} & \frac{\partial^2 V_B}{\partial W_B K_B} \\ -\frac{c_k}{\sigma} & \frac{\partial^2 V_B}{\partial K_B W_B} & \frac{\partial^2 V_B}{\partial K_B^2} \end{pmatrix} \tag{A.66}$$

and is satisfied when the determinant, $|\mathcal{H}_B^{\mathcal{B}}|$, is greater than zero; that is,

$$\begin{aligned} |\mathcal{H}_B^{\mathcal{B}}| &= \left(\tfrac{c_w}{\sigma}+1\right)\cdot\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\frac{\partial^2 V_B}{\partial W_B K_B}\right)-\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\frac{\partial^2 V_B}{\partial W_B^2}\right)-\left(\tfrac{c_w}{\sigma}+1\right)\cdot\left(\frac{\partial^2 V_B}{\partial K_B W_B}\right)\right)>0 \\ &= \left(\tfrac{c_w}{\sigma}+1\right)\cdot\left(\frac{\partial^2 V_B}{\partial W_B K_B}\right)-\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\frac{\partial^2 V_B}{\partial W_B^2}\right)+\left(\tfrac{c_w}{\sigma}+1\right)\cdot\left(\frac{\partial^2 V_B}{\partial K_B W_B}\right)>0 \\ &= 2\cdot\left(\tfrac{c_w}{\sigma}+1\right)\cdot\left(\frac{\partial^2 V_B}{\partial W_B K_B}\right)-\left(\tfrac{c_k}{\sigma}\right)\cdot\left(\frac{\partial^2 V_B}{\partial W_B^2}\right)>0 \end{aligned}$$

Using equations (A.57), (A.58), (A.62), (A.63), taking the appropriate derivatives and substituting into the above equation

$$\begin{aligned}
|\mathcal{H}_B^{\mathcal{B}}| &= 2\cdot\left(\frac{c_w}{\sigma}+1\right)\cdot\left(\frac{e^{\alpha\cdot(W_A-W_B-\phi\ell_c)}}{\left(1+e^{\alpha\cdot(W_A-W_B-\phi\ell_c)}\right)^2}\right)\cdot(\alpha\cdot m)\cdot\ell_c\cdot\beta \\
&\quad -\left(\frac{c_k}{\sigma}\right)\cdot\left(\frac{e^{\alpha\cdot(W_A-W_B-\phi\ell_c)}\left(e^{\alpha\cdot(W_A-W_B-\phi\ell_c)}-1\right)}{\left(1+e^{\alpha\cdot(W_A-W_B-\phi\ell_c)}\right)^3}\right)\cdot(\alpha\cdot m)\cdot\widehat{K}>0 \\
&= 2\cdot\left(\frac{c_w}{\sigma}+1\right)\cdot(\alpha\cdot m)\cdot\ell_c\cdot\beta-\left(\frac{e^{\alpha\cdot(W_A-W_B-\phi\ell_c)}-1}{e^{\alpha\cdot(W_A-W_B-\phi\ell_c)}+1}\right)\cdot(\alpha\cdot m)\cdot\widehat{K}>0 \\
&= 2\cdot\left(\frac{c_w}{\sigma}+1\right)\cdot\ell_c\cdot\beta-\left(\frac{c_k}{\sigma}\right)\cdot\left(\frac{\frac{1-\ell_c}{\ell_c}-1}{\frac{1-\ell_c}{\ell_c}+1}\right)\cdot\left(\frac{c_w+\sigma}{c_k}\right)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)>0 \\
&= 2\cdot\ell_c\cdot\beta-(1-2\ell_c)\cdot\left(\frac{1}{\ell_c\cdot(1-\ell_c)\cdot\beta}-1\right)>0 \\
&= 2\cdot\ell_c\cdot\beta-\frac{1-2\cdot\ell_c}{\ell_c\cdot(1-\ell_c)}+\beta\cdot(1-2\cdot\ell_c)>0 \\
&= \beta\cdot\ell_c-\ell_c^2\cdot\beta>1-2\cdot\ell_c \\
&= -\ell_c^2\cdot\beta+\ell_c\cdot(2+\beta)-1>0,
\end{aligned}$$

where $-\ell_c^2\cdot\beta+\ell_c\cdot(2+\beta)-1=0$ when

$$\ell_c=\frac{-(2+\beta)\pm\sqrt{(2+\beta)-4\cdot\beta}}{-2\cdot\beta}=\frac{(2+\beta)\pm(-1)\cdot\sqrt{(2+\beta)-4\cdot\beta}}{2\cdot\beta}.$$

Therefore, $|\mathcal{H}_A^{\mathcal{B}}|>0$ when $\ell_c\cdot(2\cdot\beta)>\frac{(2+\beta)\pm(-1)\cdot\sqrt{(2+\beta)-4\cdot\beta}}{2\cdot\beta}$. From the above analysis, the second-order conditions for warlord A and warlord B are both satisfied when the location of the point of conflict satisfies equation (4.4).

◇ Proof for Theorem 10:

Equation (4.12) is found by substituting in equation (A.61), found in the above proof, into equation (3.41). Explicitly,

$$\begin{aligned}
\pi_A^* &= \frac{1}{1+e^{\alpha\cdot\left(W_B^*-W_A^*+\phi\cdot\hat{\ell_c}\right)}} \\
&= \frac{1}{1+\frac{\ell_c}{1-\ell_c}} \\
&= \frac{1-\ell_c}{1-\ell_c+\ell_c} \\
&= 1-\ell_c.
\end{aligned}$$

Equation (4.13) follows by

$$\begin{aligned}
\pi_B^* &= 1-\pi_A^* \\
&= 1-(1-\ell_c) \\
&= \ell_c.
\end{aligned}$$