***N*-PLAYER STATISTICAL NASH GAME CONTROL: *M*-TH COST CUMULANT OPTIMIZATION**

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ABSTRACT

Game theory is the study of tactical interactions involving conflicts and cooperations among multiple decision makers called players with applications in diverse disciplines such as economics, biology, management, communication networks, electric power systems and control. This dissertation studies a statistical differential game problem where finite N players optimize their system performance by shaping the distribution of their cost function through cost cumulants. This research integrates game theory with statistical optimal control theory and considers a statistical Nash non-cooperative nonzero-sum game for a nonlinear dynamic system with nonquadratic cost functions. The objective of the statistical Nash game is to find the equilibrium solution where no player has the incentive to deviate once other players maintain their equilibrium strategy. The necessary condition for the existence of the Nash equilibrium solution is given for the m -th cumulant cost optimization using the Hamilton-Jacobi-Bellman (HJB) equations. In addition, the sufficient condition which is the verification theorem for the existence of Nash equilibrium solution is given for the m -th cumulant cost optimization using the Hamilton-Jacobi-Bellman (HJB) equations. However, solving the HJB equations even for relatively low dimensional game problem is not trivial, we propose to use neural network approximate method to find the solution of the HJB partial differential equations for the statistical game problem. Convergence proof of the neural network approximate method solution to exact solution is given. In addition, numerical examples are provided for the statistical game to demonstrate the applicability of the proposed theoretical developments.

This dissertation is dedicated to Mum and Dad, for instilling the importance of perseverance,
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PREFACE

“Far and away the best prize that life offers is the chance to work hard at work worth doing”.

Theodore Roosevelt, 1858-1919.

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CHAPTER 1

INTRODUCTION

This chapter presents the problem statement, motivation and related background of this dissertation. A literature review of the related work to this research is presented in Section 1.2 while the outline for the remaining parts of this dissertation is given in Section 1.3.

1.1 Problem Statement and Motivation

In this dissertation, we investigate statistical differential game using neural network approximate method and illustrate specific applications. Consider a system consisting of interacting controllers or players coupled with Gaussian noise otherwise referred to as stochastic dynamic system. To mathematically model this stochastic system, we utilize Ito-Sense stochastic differential equation. The performance measure of the system is represented by the quadratic cost function. Since the dynamic system is represented by a stochastic differential equation and the performance index by the quadratic system cost function, the system cost function is viewed as a random variable. Thus, the question arises, how can we optimize system performance in the presence of the interacting players that may exhibit different conflicting or cooperating objectives. Statistical game in particular statistical non-cooperative game or statistical cooperative game provides a solution.

The goal of statistical game is to enhance system performance by optimizing the system cost function constrained on the system dynamic equation. In fact, statistical game controls the performance of the system of interacting players by shaping the distribution of the system cost function through cumulants see 1.1. Cumulants are descriptive parameters of a probability distribution that are useful for measuring its properties. Statistical game is related to stochastic game in the following aspect: in stochastic game; the mean or first cumulant of the distribution

of system cost function is optimized while statistical game extends the optimization of the first cumulant of the system cost function distribution to include higher-order cumulants of the system cost function distribution. Some of the higher-order properties of the system cost function distribution are second cumulant or variance which measures the variation of the system cost function distribution around its mean value, the third cumulant or skewness which measures the departure from symmetry of system cost function distribution around its mean value, and fourth cumulant or kurtosis which measures the peakedness (width of peak) of the system cost function distribution around its mean value. In this research, we are interested in finding a statistical dif-

Figure 1.1: Density Function $F(J)$ of the Cost (J) .

ferential game method which minimizes m arbitrary cumulants of the system cost function for a system of finite N players. Statistical game involves integrating game theory with statistical optimal control theory. This research is formulated as a cost cumulant minimization problem which is based on solving a system of coupled HJB equations of the m -th cost cumulant function of the N players. With renewed interests in formation control of robots, laser pointing control systems, unmanned vehicle, sensor networks, satellite altitude control systems and military surveillance problems which can be cast as differential game problems, we are motivated to find a general solution to the coupled HJB equations for the game theoretical framework.

With the exception of linear quadratic (LQ) systems which are linear system with quadratic cost function, the analytical solutions of HJB equations are difficult to find requiring the need for numerical methods.

In the next section of this chapter, a review of the related research in game theory, statistical game and methods for solving coupled HJB equations are presented.

1.2 Literature Review

In this section, the current contributions to game theory, statistical game with related concepts and methods for solving HJB equations are reviewed.

1.2.1 Game Theory

Game theory is the study of tactical interactions involving conflicts and cooperations among multiple decision makers called players with applications in diverse disciplines such as economics, biology, management, communication networks, electric power systems and control [1], [2], [3], [4], [5], [6]. Differential game theory, which was first studied by R. Isaacs [7] in conflict-related scenario, is an aspect of decision theory in state-space form related to the modeling and analysis of conflict and cooperative problems in systems driven by differential equations. In differential game, the state of the players evolve continuously with time. Differential games can be classified as cooperative game and non-cooperative game. In cooperative differential game, the players cooperate to reach a common goal while in a non-cooperative differential game, each player acts independently to optimize his/her cost function. Non-cooperative differential game [8] may further be classified as zero-sum game or nonzero-sum game. In non-cooperative zero-sum game, the sum of the cost function of the game is zero since a gain by a player is equal to a loss by other players in the game and vice-versa. Solutions of non-cooperative zero-sum game are often referred to as Minimax or saddle point solution. For a two-player non-cooperative zero-sum game, both players have opposing goals and a single cost function which one player attempts to maximize while the other player attempts to minimize it.

In non-cooperative nonzero-sum game, the sum of the cost function of the game is not nec-

essarily zero nor a constant since the gain of one player does not equal the loss of other players and vice-versa. Non-cooperative nonzero-sum game solutions are referred to as Nash equilibrium solutions since the solution is optimal to all players in the game and no player benefit by unilaterally deviating from his/her equilibrium strategy while other players are maintaining their equilibrium strategies. This dissertation concentrates on non-cooperative, nonzero-sum or non-cooperative Nash game. An algorithm for computing the non-cooperative game equilibrium solutions in a scalar multiple-player linear quadratic cost system was proposed in [9]. In [10], the authors investigated control formulations for a linear deterministic multi-body spacecraft of quadratic cost using Nash strategy. A finite horizon, non-cooperative game approach to formation control in a linear deterministic robotic system was studied in [11]. An infinite horizon, full-state feedback non-cooperative game was analyzed for a large-scale weakly coupled linear system of N -players in [12]. A nonzero-sum game with cooperative control was investigated for a linear deterministic system and quadratic cost consisting of multiple pursuits and single invader in [13]. In [14], the authors analyzed an infinite horizon, nonzero-sum non-cooperative game using online adaptive algorithm where the system was nonlinear and deterministic. In the above-cited references the authors did not consider stochastic systems.

Stochastic differential game results from strategic interactions among players in a random dynamic system [15]. The Nash equilibrium for stochastic differential game for a linear quadratic system with state and control dependent noise was studied in [16]. Stochastic differential games are closely related to stochastic optimal control. In stochastic optimal control, there is a single player and a single cost function to be optimized while in stochastic non-cooperative differential games, there are multiple players and separate cost function to be optimized by each player. A stochastic game control can be formulated as mixed H_2/H_∞ control [17], where the control wishes to minimize an H_2 norm while keeping the H_∞ norm constrained and the Nash game approach to the controller problem is given in [18]. Klompston [19], extended risk-sensitive control to discrete time game theory and solved the Nash equilibrium for fully observed and partially observed state for a 2-player game, which is a special case of statistical game where the cost function of the players is a denumerable sum of cost cumulants. Nguyen *et*

al. [20], applied stochastic game theory to network security application using graph theory but did not consider higher-order statistics of cost function. The necessary and sufficient conditions for a nonzero sum open-loop stochastic game using maximum principle was given in [21] with application to financial problem. In this dissertation, Hamilton-Jacobi-Bellman equation will be utilized to as the necessary and sufficient conditions for the existence of equilibrium solution in the statistical game. See [22] for comprehensive treatment of dynamic non-cooperative game.

1.2.2 Statistical Game

In this research, we will consider statistical Nash differential game for a system of finite N players on a fixed time interval. Statistical Nash game can be viewed as a generalization of risk-sensitive dynamic Nash game. One of the earliest consideration of the risk-sensitive cost criterion was by Jacobson [23]. In [23], stochastic optimal control problem was analyzed and related to linear differential game for a full-state feedback system. Speyer *et al.* [24] studied a stochastic optimal control problem with an exponential performance index for both continuous and discrete linear systems and showed that separation theorem [25] but not certainty equivalence principle did apply to such systems when only the terminal state appeared in the cost function. In certainty equivalence principle [26], the control gain operating on the current state estimate is independent of the noise characteristics. Whittle [27] further analyzed risk-sensitive linear quadratic gaussian control for completely and partially observed discrete systems. Risk-neutral, risk-averse and risk-prefer aspects of the risk-sensitive cost criterion were stated with the formal proof of certainty equivalence principle for partially observed linear system.

Large deviation principle [28] was applied to analyze both finite and infinite horizon, risk-sensitive optimal control problem with relations to stationary dynamic game in [29], [30], [31], [32]. Large deviation theory is concerned with exponential decay of probability measures of tail events and generalizes the notion of convergence of probability measures. The systems considered in [29], [30] were nonlinear completely and partially observed systems while Charalambous [33], [34] utilized sufficient statistics (information state) approach to relate nonlinear risk-sensitive control to output feedback differential game. Information state approach is an al-

ternate approach to certainty equivalence principle which involves converting partially-observed control problem to an equivalent completely observed control problem using the new information state. In [35], [36], a risk-sensitive maximum principle was applied to solve optimal control and dynamic games problems while in [37], a combined risk-sensitive maximum and dynamic programming principles was utilized to solve a stochastic optimal control problem. In above-cited references, the authors parameterized the cost function, apply log transformation to the cost function and then studied the limits of the transformation as the parameters tend to zero. In this dissertation, we will generalize our cost function and thus, do not need to consider any limits of cost transformation.

Statistical Nash differential game involves simultaneous optimization of the higher-order statistical properties of each individual player's cost function distribution through cumulants [38]. The optimization of cost function distribution through cost cumulant was initiated by Sain [39], [40]. Sain *et al.* [41] analyzed the minimum cost variance control as an approximation of Risk Sensitive control where the cost variance is optimized at a specified cost mean. Won *et al.* [38], extended the theory of cost cumulant to second, third and fourth cumulants for a nonlinear system with nonquadratic cost and derived the corresponding HJB equations. Statistical optimal control was applied to hybrid systems in [42]. Cumulant control as a performance shaping concept with a linear system example was addressed in [43]. Kang *et al.* [44] applied induction method to prove the necessary and sufficient condition for higher-order statistical control for nonlinear system and provided a statistical control example. Pham [45], analyzed multi-objective zero-sum game for a 2-players linear stochastic system based on cost cumulant function. A multi-player noncooperative game for a linear stochastic system was given in [46] while a 2-player higher-order statistical game with application to a linearized satellite system with quadratic cost function was investigated in [47]. In [47], the statistical game problem was constrained on the first cumulant.

Consider a risk-sensitive dynamic game cost function J_{RS}^k defined on finite time interval

$[t, t_F]$ for k -th player given as:

$$\begin{aligned}
J_{RS}^k(\theta^k) &= -\theta^{-k} \ln \left(E \left\{ \exp \left(-\theta^k J^k \right) \right\} \right), \\
J^k(t, x(t), \mu_1, \mu_2, \dots, \mu_N) &= \int_t^{t_F} L^k(s, x(s), \mu_1, \mu_2, \dots, \mu_N) ds + \psi^k(x(t_F), t_F), \\
\phi^k(\theta^k) &= E \left\{ \exp \left(-\theta^k J^k \right) \right\}, \\
J_{RS}^k(\theta^k) &= -\theta^{-k} \ln \phi^k(\theta^k) = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \beta_i(J^k) (\theta^k)^i, \\
J_{RS}^k(\theta^k) &= \beta_1(J^k) - \frac{\theta^k}{4} \beta_2(J^k) + \mathcal{O}\left((\theta^k)^2\right),
\end{aligned} \tag{1.1}$$

where θ^k is the k -th player risk-sensitive parameter, J^k is the k -th player cost function, $x(t)$ is the state, $\mu_1, \mu_2, \dots, \mu_N$ are the players feedback strategy defined on the state $x(t)$, L^k is the k -th player running cost, ψ^k is the k -th player terminal cost, ϕ^k is the k -th player first characteristic function of J^k , $\beta_i(J^k)$ are the i -th cumulants of J^k and $\mathcal{O}\left((\theta^k)^2\right)$ is the higher-order terms. If J_{RS}^k is approximated to second order on the fifth line of (1.1), then for small θ^k , J_{RS}^k becomes equivalent to first cumulant $\beta_1(J^k)$ of J^k . The fourth line of (1.1) has been derived from the relationship between moments and cumulants in [48]. In general, if the cumulants $\beta_i(J^k)$ of J^k exist, then the statistical game cost criterion can be viewed as an infinite linear combination of cost cumulants without restricting the weighting coefficients of the cumulants to the single risk-sensitive parameter θ^k . This shows that risk-sensitive game is optimizing a denumerable sum of all cost cumulants. On the other hand, statistical game control optimizes with respect to any cost cumulant. So in a way, statistical game control generalizes risk-sensitive idea.

1.2.3 Solutions to HJB Equations

The statistical game solutions are derived using HJB equations, which are nonlinear partial differential equations. With respect to specific systems such as low dimensional linear system with quadratic cost function, analytical solutions can be found. These analytical solutions are solutions to the resulting coupled matrix Riccatic differential equations. Numerical solutions to statistical game problem, just like stochastic optimal control problem, suffer from *Curse of Dimensionality*; which is the exponential increase in computational memory requirement due to

increase in dimension of the input state variables, increase in dimension of the control variables and increase in dimension of the output state variables. However, the effect of the *Curse of Dimensionality* can be more pronounced in some game problems than others and there is no systematic numerical approach for solving the coupled HJB equations. In this dissertation, an approximate approach is utilized to solve the coupled HJB equations in the statistical game of a nonlinear system and nonquadratic system cost function.

Power series approximate method for the value function approximation in the HJB equation, was first considered in [49] for the stabilization problem of nonlinear deterministic systems. Garrand *et al.* further applied a truncated Taylor's series of the value function in the HJB equation for the optimal feedback control of a nonlinear deterministic system [50]. Some implicit sufficient conditions were given in [51] for the existence of a solution for a coupled Riccati equation associated with minimal variance control. The power series method was also applied to solve a HJB equation described by a time-varying differential algebraic equation for a nonlinear deterministic system in [52]. Successive Gelarkin Approximation method consisting of Gelarkin spectral method and policy iteration was utilized to solve a generalized HJB for nonlinear optimal control in [53]. Kumar *et al.* applied successive policy iterative method and spectral method to solve nonlinear stochastic system [54]. Fuzzy model applications to solving HJB equations for nonlinear stochastic system have been reported in [55]. The Fuzzy model involves linearizing the nonlinear stochastic system, thus resulting in a simplified HJB equation. For non-nominal state conditions, the linearized model behavior will be different from the nonlinear system behavior.

Dynamic approximate method involving conversion of partial differential equations (PDEs) to algebraic equations through application of dynamic extensions to state equation was utilized to solve the HJB equation associated with nonlinear system optimal control problem in [56] and nonlinear differential game problem in [57]. Neural network approximate method was applied to solve the HJB equation in a finite-horizon optimal control problem for a nonlinear continuous-time deterministic system in [58], [59]. Medagam *et al.* [60] applied neural network method and adaptive filter to solve the nonlinear optimal control problem for an unknown sys-

tem. The application of neural network method to optimal control of nonlinear discrete-time systems was reported in [61].

The above cited-references to the solutions of HJB equations with the exception of [51] did not consider higher-order cost cumulants, we prove the necessary and sufficient condition for the existence of Nash equilibrium solution using the HJB equations. The HJB equations are then solved using neural network approximate method following [58], [59]. This research is the first approach to statistical Nash game with nonlinear system applications where the cost function of the players when expressed as function of normal vectors is governed by a nonsymmetric noncentral generalized Chi-square distribution [62], [63].

1.3 Outline of Dissertation

The rest of this dissertation is organized as follows. Chapter 2, reviews the mathematical preliminaries including the assumptions needed to layout the statistical game and then formulates an N player, m -th cost cumulant statistical game as a cost cumulant optimization problem. Chapter 3, presents and proves the necessary condition for the existence of equilibrium solutions for the full-state feedback statistical game. Chapter 4, presents and proves the sufficient condition for the existence of equilibrium solutions for the full-state feedback statistical game. The sufficient condition is also the verification theorem for the existence of equilibrium solutions for the statistical game. From the sufficient condition, the players strategies are derived. Neural network approximate method is reviewed and applied to solve the coupled HJB equations in Chapter 5. In addition, the proof of the convergence of the neural network approximate HJB equations to exact solution and a summary of the full-state feedback statistical game algorithm are also given in Chapter 5. Chapter 6, presents and discusses the simulation results of the full-state feedback statistical game control problem for some linear and nonlinear dynamical system while Chapter 7, discusses ad hoc output feedback statistical game control including a summary of the output feedback statistical game algorithm. Finally, we give the contributions of this dissertation, conclude this research and present future research directions in the last chapter. Moreover, relevant proofs and lemmas are provided in the appendices.

CHAPTER 2

STATISTICAL GAME PROBLEM FORMULATION

This chapter formulates the statistical game control based on cost cumulant optimization. In order to describe a game, the players, their performance indices, their strategy, any relationship among the players and how the game evolves need to be known. In the first section of this chapter, the mathematical preliminaries needed to formulate the statistical game are given. In the next section, the statistical game problem is formulated as a multi-player optimization problem.

2.1 Mathematical Preliminaries

Consider an N player state dynamics described by the nonlinear Ito-sense stochastic differential equation.

$$dx(t) = f(t, x(t), u_1(t), u_2(t), \dots, u_N(t))dt + \sigma(t, x(t))dw(t), \quad (2.1)$$

where $t \in [t_0, t_F] = T$, $x(t) \in \mathbb{R}^n$ is the state and $x(t_0) = x_0$ is the initial state, $u_k(t) \in U_k \subset \mathbb{R}^m$ is the k -th player strategy, $k = 1, 2, \dots, N$ and $dw(t)$ is a Gaussian random process of dimension d with zero mean, covariance of $W(t)dt$ defined on a probability space (Ω_0, F, P) where Ω_0 is a nonempty set, F is a σ -algebra of Ω_0 and P is a probability measure on (Ω_0, F) . Let $Q_0 = [t_0, t_F] \times \mathbb{R}^n$, $\bar{Q}_0$ denote the closure of Q_0 , i.e $\bar{Q}_0 = T \times \mathbb{R}^n$. Assume $f : \bar{Q}_0 \times U_1 \times U_2 \times \dots \times U_N \rightarrow \mathbb{R}^n$ is $C^1(\bar{Q}_0 \times U_1 \times U_2 \times \dots \times U_N)$, and $\sigma : \bar{Q}_0 \rightarrow \mathbb{R}^{n \times d}$ is $C^1(\bar{Q}_0)$. In addition, f and σ satisfy the following conditions. There exists a constant C where

$$\|f(t, x(t), u_1(t), u_2(t), \dots, u_N(t))\| \leq C \left(1 + \|x\| + \sum_{k=1}^N \|u_k\| \right), \quad (2.2)$$

$$\|\sigma(t, x)\| \leq C(1 + \|x\|),$$

with $(t, x(t), u_1(t), u_2(t), \dots, u_N(t)) \in \bar{Q}_0 \times U_1 \times U_2 \times \dots \times U_N \rightarrow \mathbb{R}^n$, $(t, x(t)) \in \bar{Q}_0$ and $\|\cdot\|$ is the Euclidean norm. The above condition (2.2) represents the linear growth condition, which

prevents the state x from going unbounded in finite time [64].

Also, there exist a constant $\mathbb{K}$ where

$$\begin{aligned} & \|f(t, x(t), u_1(t), u_2(t), \dots, u_N(t)) - f(t, \tilde{x}(t), \tilde{u}_1(t), \tilde{u}_2(t), \dots, \tilde{u}_N(t))\| \\ & \leq \mathbb{K} \left(\|x - \tilde{x}\| + \sum_{k=1}^N \|u_k - \tilde{u}_k\| \right), \\ & \|\sigma(t, x) - \sigma(t, \tilde{x})\| \leq K \|x - \tilde{x}\|, \end{aligned} \quad (2.3)$$

with $t \in T$; $x, \tilde{x} \in \mathbb{R}^n$, $u_k, \tilde{u}_k \in U_k$. The above condition (2.3) represents the Lipschitz condition. A memoryless state feedback *strategy* is introduced as $u_k(t) = \mu_k(t, x(t))$, $t \in T$, where $\mu_k(t, x)$ is non-random function with random arguments and $\mu_k(t, x)$ satisfies Lipschitz and linear growth condition as shown below

$$\begin{aligned} & \|\mu_k(t, x_1) - \mu_k(t, x_2)\| \leq \mathbb{K}_1 (\|x_1 - x_2\|), \\ & \|\mu_k(t, x)\| \leq \mathbb{K}_2 (1 + \|x\|), \end{aligned} \quad (2.4)$$

where $\mathbb{K}_1, \mathbb{K}_2$ are constants and $\|\cdot\|$ is the Euclidean norm. Then, if $E\|x(t)\|^2$ is finite, there is a unique solution to (2.1) which is a Markov diffusion process [65]. Furthermore, we utilize the backward evolution operator, $\mathcal{O}^k(\mu_1, \mu_2, \dots, \mu_N)$, as defined in [41]:

$$\begin{aligned} \mathcal{O}^k(\mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) &= \frac{\partial}{\partial t} + f'(t, x, \mu_1, \mu_2, \dots, \mu_N) \frac{\partial}{\partial x} + \frac{1}{2} \text{tr} \left(\sigma W(t) \sigma' \frac{\partial^2}{\partial x^2} \right) \\ &= \mathcal{O}_1^k(\mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) + \mathcal{O}_2^k(\mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N), \\ \mathcal{O}_1^k(\mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) &= \frac{\partial}{\partial t} + f'(t, x, \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \frac{\partial}{\partial x}, \\ \mathcal{O}_2^k(\mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) &= \frac{1}{2} \text{tr} \left(\sigma(t, x) W(t) \sigma'(t, x) \frac{\partial^2}{\partial x^2} \right), \end{aligned} \quad (2.5)$$

where in (2.5), tr denotes trace operator. The backward evolution operator $\mathcal{O}^k$ is defined since our resulting HJB equations are solved backwards in time according to dynamic programming principles.

In the N player nonzero-sum differential game, there are N performance indices, one for each player, and each player minimizes his performance index when other players have played

their equilibrium solution. The cost function (J^k) for the k -th player is given as:

$$J^k(t, x(t), \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) = \int_t^{t_F} L^k(s, x(s), \mu_1, \mu_2, \dots, \mu_N) ds + \psi^k(x(t_F), t_F), \quad (2.6)$$

where $k = 1, 2, \dots, N$, $x(t_F) = x_F$. $L^k : \bar{Q}_0 \times U_1 \times U_2 \times \dots \times U_N \rightarrow \mathbb{R}^+$ is $C(\bar{Q}_0 \times U_1 \times U_2 \times \dots \times U_N)$ and $\psi^k : \bar{Q}_0 \rightarrow \mathbb{R}^+$ is $C(\bar{Q}_0)$ with both satisfying polynomial growth condition,

$$\begin{aligned} \|L^k(t, x, \mu_1, \mu_2, \dots, \mu_N)\| &\leq c1 \left(1 + \|x\| + \|\mu_1\| + \|\mu_2\| + \dots + \|\mu_N\|\right)^{c2}, \\ \|\psi^k\| &\leq c1(1 + \|x\|)^{c2}, \end{aligned} \quad (2.7)$$

where $c1, c2$ are constants. L^k is the running cost and ψ^k is the terminal cost. These conditions allow the mean of the cost function to be finite [65].

The i -th moment cost function M_i^k of the k -th player is defined by [41],

$$\begin{aligned} M_i^k(t, x, \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) &= E \left\{ (J^k)^i(t, x, \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) | x(t) = x \right\} \\ &= E_{tx} \left\{ (J^k)^i(t, x, \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \right\}, \end{aligned} \quad (2.8)$$

where $i = 1, 2, \dots, m$, $k = 1, 2, \dots, N$ and $E_{tx} = E\{ \cdot | x(t) = x \}$ is the conditional expectation.

Similarly, m -th cumulant cost function V_m^k of the k -th player is defined by [66],

$$V_m^k(t, x) = M_m^k(t, x) - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k(t, x) V_{i+1}^k(t, x), \quad (2.9)$$

where $k = 1, 2, \dots, N$, $t \in T = [t_0, t_F]$, $x(t_0) = x_0$, $x(t) \in \mathbb{R}^n$, M_i^k is the i -th moment cost function and V_i^k is the i -th cumulant cost function with $i \leq m$. Equation (2.9) gives the mathematical relationship between moment and cumulant. Cumulants instead of moments will be the reference in this dissertation, since in second cumulant control optimization for a linear system with quadratic cost function, a linear controller always results while in second moment control optimization for a linear system with quadratic cost function, a nonlinear controller may result [67]. Thus, higher cumulants control optimization is more effective than higher moments control optimization.

2.2 Statistical Game Problem Formulation

In this section, the statistical game is formulated using cost moment-cumulant concepts. The relevant definitions are given followed by a formal description of the differential game problem.

The definitions are based on the statistical optimal control approach in [38].

2.2.1 N -Player Minimal 2-nd Cumulant Statistical Game

Definition 2.1: A function $M_1^k : \bar{Q}_0 \rightarrow \mathbb{R}^+$ is an admissible 1-st moment cost function if there exists a strategy μ_k such that

$$M_1^k(t, x) = M_1^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N), \quad (2.10)$$

for $t \in T, x \in \mathbb{R}^n$, $M_1^k \in C^{1,2}(\bar{Q}_0)$ and $k = 1, 2, \dots, N$. Also, V_1^k is the admissible 1-st cumulant cost function for the k -th player related to the moment function through the moment-cumulant relation (2.9). In addition, $\mu_k \in U_{M^k} \subset \mathbb{R}^m$, $V_1^k(t, x) = V_1^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N)$.

Definition 2.2: A class of admissible strategy law U_{M^k} is defined such that if $\mu_k \in U_{M^k}$ then μ_k satisfies the equality of Definition 2.1 for M_0^k, M_1^k . It should be noted that first moment M_1^k is the same as first cumulant V_1^k , $M_0^k = 1$ and $V_0^k = 0$.

Definition 2.3: Let V_1^k be the k -th player admissible 1-st cumulant cost functions. The player strategy μ_k^* is the k -th player equilibrium solution if it is such that

$$M_1^{k*}(t, x) = M_1^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq M_1^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*), \quad (2.11)$$

for all $\mu_k \in U_{M^k}$. Similarly, the V_1^k cumulant cost function is given as

$$V_1^{k*}(t, x) = V_1^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq V_1^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*). \quad (2.12)$$

where the set $\{\mu_1^*, \mu_2^*, \dots, \mu_N^*\}$ is a Nash equilibrium solution and the set $\{V_1^{1*}, V_1^{2*}, \dots, V_1^{N*}\}$ is the Nash equilibrium value set.

Problem Definition: Consider an open set $Q \subset Q_0$ and let the k -th player cost cumulant function $V_1^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible cumulant function where the set $C_p^{1,2}(Q) \cap C(\bar{Q})$ means that the function V_1^k satisfy polynomial growth condition and is continuous in the first and second derivatives of Q , and continuous on the closure of Q . Assume the existence of an optimal strategy $\mu_k^* \in U_{M^k}$ and optimal value function $V_2^{k*}(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ for the k -th player. Thus, the minimal second cumulant game problem is to find the full-state-feedback

Nash equilibrium strategy $\mu_k^*(t, x)$ for $k = 1, 2, \dots, N$ which results in the minimal 2-nd value function $V_2^{k*}(t, x)$ given as

$$V_2^{k*}(t, x) = \min_{\mu_k \in U_{M^k}} \left\{ V_2^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \right\}, \quad (2.13)$$

for $k = 1, 2, \dots, N$.

Remark: U_{M^k} is the set of all admissible controllers constrained in $M = V_1^k$, where μ_k is then found among this set of controllers so that V_2^k minimal. This μ_k is the optimal control strategy.

2.2.2 N -Player Minimal 3-rd Cumulant Statistical Game

Definition 2.4: A function $M_i^k : \bar{Q}_0 \rightarrow \mathbb{R}^+$ is an admissible i -th moment cost function if there exists a strategy μ_k such that

$$M_i^k(t, x) = M_i^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N), \quad (2.14)$$

for $t \in T, x \in \mathbb{R}^n, M_i^k \in C^{1,2}(\bar{Q}_0), i = 1, 2$ and $k = 1, 2, \dots, N$. Also, V_i^k is the admissible i -th cumulant cost function for the k -th player related to the moment function through the moment-cumulant relation (2.9). In addition, $\mu_k \in U_{M^k} \subset \mathbb{R}^m, V_i^k(t, x) = V_i^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N)$.

Definition 2.5: A class of admissible strategy U_{M^k} is defined such that if $\mu_k \in U_{M^k}$ then μ_k satisfies the equality of Definition 2.4 for M_0^k, M_1^k, M_2^k . It should be noted that first moment M_1^k is the same as first cumulant $V_1^k, M_0^k = 1$ and $V_0^k = 0$.

Definition 2.6: Let $V_1^k, \dots, V_{i-1}^k$ be the k -th player admissible cumulant cost functions. The player strategy μ_k^* is the k -th player equilibrium solution if it is such that

$$M_i^{k*}(t, x) = M_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq M_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*), \quad (2.15)$$

for all $\mu_k \in U_{M^k}$. Similarly, the V_i^k cumulant cost function is given as

$$V_i^{k*}(t, x) = V_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq V_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*). \quad (2.16)$$

where the set $\{\mu_1^*, \mu_2^*, \dots, \mu_N^*\}$ is a Nash equilibrium solution and the set $\{V_i^{1*}, V_i^{2*}, \dots, V_i^{N*}\}$ is the Nash equilibrium value set.

Problem Definition: Consider an open set $Q \subset Q_0$ and let the k -th player cost cumulant functions $V_i^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ for $i = 1, 2$ be the admissible cumulant function where the set $C_p^{1,2}(Q) \cap C(\bar{Q})$ means that the function V_i^k satisfy polynomial growth condition and is continuous in the first and second derivatives of Q , and continuous on the closure of Q . Assume the existence of an optimal strategy $\mu_k^* \in U_{M^k}$ and optimal value function $V_3^{k*}(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ for the k -th player. Thus, the minimal third cumulant game problem is to find the full-state-feedback Nash equilibrium strategy $\mu_k^*(t, x)$ for $k = 1, 2, \dots, N$ which results in the minimal 3-rd value function $V_3^{k*}(t, x)$ given as

$$V_3^{k*}(t, x) = \min_{\mu_k \in U_{M^k}} \left\{ V_3^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \right\}, \quad (2.17)$$

for $k = 1, 2, \dots, N$.

Remark: U_{M^k} is the set of all admissible controllers constrained in M which is a set of V_1^k, V_2^k , where μ_k is then found among this set of controllers so that V_3^k minimal. This μ_k is the optimal control strategy.

2.2.3 N -Player Minimal m -th Cumulant Statistical Game

Definition 2.7: A function $M_i^k : \bar{Q}_0 \rightarrow \mathbb{R}^+$ is an admissible i -th moment cost function if there exists a strategy μ_k such that

$$M_i^k(t, x) = M_i^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N), \quad (2.18)$$

for $t \in T, x \in \mathbb{R}^n, M_i^k \in C^{1,2}(\bar{Q}_0), i = 1, 2, \dots, m-1$ and $k = 1, 2, \dots, N$. Also, V_i^k is the admissible i -th cumulant cost function for the k -th player related to the moment function through the moment-cumulant relation (2.9). In addition, $\mu_k \in U_{M^k} \subset \mathbb{R}^m, V_i^k(t, x) = V_i^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N)$.

Definition 2.8: A class of admissible strategy U_{M^k} is defined such that if $\mu_k \in U_{M^k}$ then μ_k satisfies the equality of Definition 2.7 for $M_0^k, M_1^k, M_2^k, \dots, M_i^k$. It should be noted that first moment M_1^k is the same as first cumulant $V_1^k, M_0^k = 1$ and $V_0^k = 0$.

Definition 2.9: Let $V_1^k, \dots, V_{i-1}^k$ be the k -th player admissible cumulant cost functions. The player strategy μ_k^* is the k -th player equilibrium solution if it is such that

$$M_i^{k*}(t, x) = M_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq M_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*), \quad (2.19)$$

for all $\mu_k \in U_{M^k}$. Similarly, the V_i^k cumulant cost function is given as

$$V_i^{k*}(t, x) = V_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq V_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*). \quad (2.20)$$

where the set $\{\mu_1^*, \mu_2^*, \dots, \mu_N^*\}$ is a Nash equilibrium solution and the set $\{V_i^{1*}, V_i^{2*}, \dots, V_i^{N*}\}$ is the Nash equilibrium value set.

Problem Definition: Consider an open set $Q \subset Q_0$ and let the k -th player cost cumulant functions $V_i^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ for $i = 1, 2, \dots, m-1$ be the admissible cumulant function where the set $C_p^{1,2}(Q) \cap C(\bar{Q})$ means that the function V_i^k satisfy polynomial growth condition and is continuous in the first and second derivatives of Q , and continuous on the closure of Q . Assume the existence of an optimal strategy for the k -th player $\mu_k^* \in U_{M^k}$ and optimal value function $V_m^{k*}(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ for the k -th player. Thus, the minimal m -th cumulant statistical game problem is to find the full-state-feedback Nash equilibrium strategy $\mu_k^*(t, x)$ for $k = 1, 2, \dots, N$ which results in the minimal m -th value function $V_m^{k*}(t, x)$ given as

$$V_m^{k*}(t, x) = \min_{\mu_k \in U_{M^k}} \left\{ V_m^k(t, x; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \right\}, \quad (2.21)$$

for $k = 1, 2, \dots, N$.

Remarks: It should be noted that to find the Nash equilibrium strategies $\mu_k^*(t, x)$, we constrain the candidates of the optimal players strategy to U_{M^k} , and the optimal value function $V_m^{k*}(t, x)$ is found with the assumption that lower order cumulants, V_i^k , are admissible for $i = 1, 2, \dots, m-1$. For instance, in a second cumulant statistical game, all the strategies that give an admissible specified first cumulant are found and within this set, the strategies that minimizes the second cumulant value function for the game are found. In a third cumulant statistical game, all the strategies that give an admissible specified first and second cumulants are found and within this set, the strategies that minimizes the third cumulant value function for the

game are found. In the Nash equilibrium state, no player has the incentive to unilaterally deviate from their strategy $\mu_k^*(t, x)$ once other players maintain their equilibrium strategies, since doing so will not be advantageous to that player [22].

CHAPTER 3

HJB EQUATIONS: NECESSARY CONDITION

In this chapter, the necessary condition for the existence of equilibrium solutions of N -player, m -th cost cumulant statistical game is stated and proved. The relevant assumptions and lemmas needed for proving the necessary condition for the existence of equilibrium solution in N -player, m -th cost cumulant statistical game are given.

3.1 Mathematical Preliminaries

Consider the N -player continuous-time differential game (2.1) with cost functional (2.6) of fixed duration $[t_0, t_F]$. Let $V_1^k(t, x), V_2^k(t, x), \dots, V_{m-1}^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible cost cumulant function for the k -th player.

Assumption 3.1: The function $f(t, x(t), u_1(t), u_2(t), \dots, u_N(t))$ in (2.1) is given as

$$f(t, x(t), u_1(t), u_2(t), \dots, u_N(t)) = g(t, x(t)) + \sum_{i=1}^N B_i(t, x(t))u_i(t), \quad (3.1)$$

where $g : \bar{Q}_0 \rightarrow \mathbb{R}^n$ is $C^1(\bar{Q}_0)$ and $B_i(t, x(t))$ are real matrices of appropriate dimensions which are continuous for all $t \in T$. The form in (3.1) makes the system (2.1) a control affine nonlinear system, which means that the system is linear with respect to control (players strategy) and nonlinear in state.

Assumption 3.2: The $L^k(s, x(s), \mu_1, \mu_2, \dots, \mu_N)$ in (2.6) is given as

$$L^k = l(x) + \sum_{i=1}^N \mu_i'(x) R_{ki}(t) \mu_i(x), \quad (3.2)$$

where $l(x) : \bar{Q}_0 \rightarrow \mathbb{R}^+$ is $C(\bar{Q}_0)$ and $R_{ki}(t) > 0, k = i$ and $R_{ki}(t) \geq 0, k \neq i$ are symmetric matrices as a noncooperative nonzero-sum game is considered. From (3.2), the running cost is nonquadratic in state and quadratic in control. $l(x)$ is the state penalty function for the cost function (2.6).

Assumption 3.3: The value functions $V_j^k(t, x)$ are twice continuously differentiable. For non-differentiable functions, weak or viscosity solution [68] can be applied to solve the HJB equations. In this research, the twice continuously differentiability assumption is made on the value functions $V_j^k(t, x)$ for $j = 1, 2, \dots, m$.

Lemma 3.1: Let $M_i^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be the admissible moment cost function, if there exists an optimal control strategy $\mu_k^*(t, x)$ such that $M_i^{k*}(t, x) = M_i^k(t, x; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*)$, $t \in T$ then

$$\mathcal{O}^k \left[M_i^{k*}(t, x) \right] + i M_{i-1}^k(t, x) L^k(t, x; \mu_1, \mu_2, \dots, \mu_N) = 0, \quad (3.3)$$

where the terminal condition is $M_i^k(t_F, x_F) = \psi_i^k(x(t_F))$, and $i = 1, 2, \dots, m$.

Remark: This lemma is an extension of Theorem 3.1 in [38] which only considered a single player version. This lemma relates k -th player arbitrary order of moments to lower moments and running cost (L^k). This lemma is utilized in deriving m -th cumulant Hamilton-Jacobi-Bellman equations.

Lemma 3.2: Consider two functions $M_i^k(t, x), V_j^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$, where i and j are non-negative integers and $k = 1, 2, \dots, N$, then

$$\begin{aligned} & \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[M_i^k(t, x) V_j^k(t, x) \right] \\ &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[M_i^k(t, x) \right] V_j^k(t, x) + \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_j^k(t, x) \right] M_i^k(t, x) \\ &+ \left(\frac{\partial M_i^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_j^k(t, x)}{\partial x} \right). \end{aligned} \quad (3.4)$$

Proof: From the backward evolution operator (2.5), the first line of (3.4) becomes

$$\begin{aligned} & \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[M_i^k(t, x) V_j^k(t, x) \right] = \mathcal{O}_1^k \left[M_i^k(t, x) V_j^k(t, x) \right] + \mathcal{O}_2^k \left[M_i^k(t, x) V_j^k(t, x) \right] \\ &= \mathcal{O}_1^k \left[M_i^k(t, x) \right] V_j^k(t, x) + \mathcal{O}_1^k \left[V_j^k(t, x) \right] M_i^k(t, x) + \mathcal{O}_2^k \left[M_i^k(t, x) V_j^k(t, x) \right]. \end{aligned} \quad (3.5)$$

Then, expanding the third term of (3.5) further using chain rule of differentiation gives

$$\begin{aligned}
\mathcal{O}_2^k[M_i^k(t, x)V_j^k(t, x)] &= \frac{1}{2}\text{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\frac{\partial^2}{\partial x^2}\left[M_i^k(t, x)V_j^k(t, x)\right]\right), \\
\mathcal{O}_2^k[\cdot] &= \frac{1}{2}\text{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left[M_i^k(t, x)V_j^k(t, x)\right]\right), \\
\mathcal{O}_2^k[\cdot] &= \frac{1}{2}\text{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\frac{\partial}{\partial x}\left[\left(\frac{\partial M_i^k(t, x)}{\partial x}\right)V_j^k(t, x) + \left(\frac{\partial V_j^k(t, x)}{\partial x}\right)M_i^k(t, x)\right]\right), \\
\mathcal{O}_2^k[\cdot] &= \frac{1}{2}\text{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\left[\left(\frac{\partial^2 M_i^k}{\partial x^2}\right)V_j^k + \left(\frac{\partial M_i^k}{\partial x}\right)\left(\frac{\partial V_j^k}{\partial x}\right)' \right. \right. \\
&\quad \left. \left. + \left(\frac{\partial V_j^k}{\partial x}\right)\left(\frac{\partial M_i^k}{\partial x}\right)' + \left(\frac{\partial^2 V_j^k}{\partial x^2}\right)M_i^k\right]\right), \\
\mathcal{O}_2^k[\cdot] &= \frac{1}{2}\text{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\left(\frac{\partial^2 M_i^k}{\partial x^2}\right)V_j^k\right) + \frac{1}{2}\text{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\left(\frac{\partial^2 V_j^k}{\partial x^2}\right)M_i^k\right) \\
&\quad + \frac{1}{2}\text{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\left(\frac{\partial M_i^k}{\partial x}\right)\left(\frac{\partial V_j^k}{\partial x}\right)'\right) \\
&\quad + \frac{1}{2}\text{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\left(\frac{\partial V_j^k}{\partial x}\right)\left(\frac{\partial M_i^k}{\partial x}\right)'\right), \\
\mathcal{O}_2^k[\cdot] &= V_j^k\mathcal{O}_2^k[M_i^k(t, x)] + M_i^k\mathcal{O}_2^k[V_j^k(t, x)] + \left(\frac{\partial M_i^k(t, x)}{\partial x}\right)'\sigma(t, x)W\sigma(t, x)'\left(\frac{\partial V_j^k(t, x)}{\partial x}\right). \tag{3.6}
\end{aligned}$$

From (3.5), substituting for $\mathcal{O}_2^k$ using (3.6) results in

$$\begin{aligned}
&\mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N)\left[M_i^k(t, x)V_j^k(t, x)\right] \\
&= V_j^k(t, x)\mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N)\left[M_i^k(t, x)\right] + M_i^k(t, x)\mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N)\left[V_j^k(t, x)\right] \\
&\quad + \left(\frac{\partial M_i^k(t, x)}{\partial x}\right)'\sigma(t, x)W\sigma(t, x)'\left(\frac{\partial V_j^k(t, x)}{\partial x}\right).
\end{aligned}$$

The lemma is proved. $\square$

Remark: This lemma is needed in the necessary and sufficient condition proof. It relates the moment and cumulant to the diffusion and covariance term in the system state equation when both cumulant and moment functions are jointly operated on by the backward evolution operator.

Lemma 3.3: Let $V_1^k(t, x), \dots, V_{m-1}^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be admissible value functions

with k as a positive integer, then we have that the following relation holds

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^{m-1} \frac{m!}{j!(m-j)!} \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{m-j}^k}{\partial x} \right) \\ &= \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial V_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right). \end{aligned} \quad (3.7)$$

Proof: Let $j = i + 1$, the right hand side of (3.7) becomes

$$\sum_{j=1}^{m-1} \frac{(m-1)!}{(j-1)!(m-j)!} \left(\frac{\partial V_{m-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_j^k}{\partial x} \right). \quad (3.8)$$

Then, when m is odd, there are even number of terms in the above summation, (3.8). We can rewrite the above summation, (3.8) in pairs. For example, the first term and last term, the second term and second to last term, and so on. We represent the sum of each pair as follows:

$$\begin{aligned} & \frac{(m-1)!}{(j-1)!(m-j)!} \left(\frac{\partial V_{m-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_j^k}{\partial x} \right) + \frac{(m-1)!}{(m-j-1)!(j)!} \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{m-j}^k}{\partial x} \right) \\ &= \frac{j m!}{m j!(m-j)!} \left(\frac{\partial V_{m-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_j^k}{\partial x} \right) + \frac{(m-j) m!}{m(m-j)!(j)!} \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{m-j}^k}{\partial x} \right) \\ &= \left(\frac{j m!}{m j!(m-j)!} + \frac{(m-j) m!}{m(m-j)!(j)!} \right) \left(\frac{\partial V_{m-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_j^k}{\partial x} \right) \\ &= \frac{m!}{j!(m-j)!} \left(\frac{\partial V_{m-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_j^k}{\partial x} \right). \end{aligned}$$

Thus, for $j = 1$ to $m-1$,

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^{m-1} \frac{m!}{j!(m-j)!} \left(\frac{\partial V_{m-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_j^k}{\partial x} \right) \\ &= \sum_{j=1}^{m-1} \frac{(m-1)!}{(j-1)!(m-j)!} \left(\frac{\partial V_{m-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_j^k}{\partial x} \right). \end{aligned} \quad (3.9)$$

When m is even, we have the same result except for the term when $j = m/2$. In this case, we rewrite the right-hand side of (3.9) as

$$\frac{(m-1)!}{(m/2-1)!(m-m/2)!} \left(\frac{\partial V_{m/2}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{m/2}^k}{\partial x} \right).$$

Thus,

$$\begin{aligned} & \frac{(m-1)!}{(m/2-1)!(m-m/2)!} \left(\frac{\partial V_{m/2}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{m/2}^k}{\partial x} \right) \\ &= \frac{1}{2} \frac{m(m-1)!}{(m/2)!(m/2)!} \left(\frac{\partial V_{m/2}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{m/2}^k}{\partial x} \right). \end{aligned}$$

The lemma is proved. $\square$

Remarks: This relation is needed in the proof of the necessary condition for the existence of solution for the statistical game.

3.2 Necessary Condition for the HJB equations

3.2.1 N -Player Minimal 2-nd Cumulant Statistical Game

Theorem 3.1: Consider the N -player continuous-time differential game (2.1) with cost functional (2.6) of fixed duration $[t_0, t_F]$. Let $V_1^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible value function for the k -th player. Assume the existence of optimal player strategy $\mu_k^*(t, x)$. Then, the minimal 2-nd value function $V_2^{k*}(t, x)$ satisfies the following coupled HJB equations for the first player to N -th player.

$$\begin{aligned} 0 &= \min_{\mu_1 \in U_{M1}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k^*, \dots, \mu_N^*) [V_2^{1*}(t, x)] \right. \\ &\quad \left. + \left(\frac{\partial V_1^1(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^1(t, x)}{\partial x} \right) \right\}, \\ &\vdots \\ 0 &= \min_{\mu_N \in U_{MN}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k^*, \dots, \mu_N) [V_2^{N*}(t, x)] \right. \\ &\quad \left. + \left(\frac{\partial V_1^N(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^N(t, x)}{\partial x} \right) \right\}. \end{aligned} \tag{3.10}$$

With the terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2$ and $(t, x) \in Q$. In compact form, the minimal 2-nd value function $V_2^{k*}(t, x)$ satisfies the following HJB equation

for the k -th player.

$$0 = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_2^{k*}(t, x)] \right. \\ \left. + \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^k(t, x)}{\partial x} \right) \right\}, \quad (3.11)$$

for $(t, x) \in Q$, with the terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2$.

Proof: Let V_2^k be a class of $C_p^{1,2}(Q) \cap C(\bar{Q})$. From the cumulant-moment relationship (2.9),

$$V_2^k(t, x) = M_2^k(t, x) - (V_1^k)^2(t, x), \quad (3.12)$$

where $M_2^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$. Then $M_2^k(t, x)$ from (3.3) is the solution to

$$\mathcal{O}^k [M_2^{k*}(t, x)] + 2M_1^k(t, x)L^k(t, x, \mu_1, \dots, \mu_k, \dots, \mu_N) = 0. \quad (3.13)$$

Applying the backward evolution operator (2.5) to (3.12) gives

$$\mathcal{O}^k [V_2^{k*}(t, x)] = \mathcal{O}^k [M_2^k(t, x)] - \mathcal{O}^k [(V_1^k)^2(t, x)]. \quad (3.14)$$

Using (3.13) in (3.14) gives

$$\mathcal{O}^k [V_2^{k*}(t, x)] + \mathcal{O}^k [(V_1^k)^2(t, x)] + 2M_1^k L^k = 0. \quad (3.15)$$

Manipulating (3.15) with $M_1^k = V_1^k$ gives

$$\mathcal{O}^k [V_2^{k*}(t, x)] + \mathcal{O}^k [(V_1^k)^2(t, x)] + 2V_1^k L^k = 0. \quad (3.16)$$

From the backward operator given in (2.5), expanding $\mathcal{O}^k (V_1^k)^2(t, x)$ gives

$$\mathcal{O}^k [(V_1^k)^2] = \left(\frac{\partial (V_1^k)^2}{\partial t} \right) + f' \left(\frac{\partial (V_1^k)^2}{\partial x} \right) + \frac{1}{2} \text{tr} \left[\sigma W \sigma' \left(\frac{\partial^2 (V_1^k)^2}{\partial x^2} \right) \right]. \quad (3.17)$$

Differentiating the terms in (3.17) gives

$$\mathcal{O}^k [(V_1^k)^2] = 2V_1^k \left(\frac{\partial V_1^k}{\partial t} \right) + 2V_1^k f' \left(\frac{\partial V_1^k}{\partial x} \right) + \frac{1}{2} \text{tr} \left[\sigma W \sigma' \frac{\partial}{\partial x} \frac{\partial}{\partial x} ((V_1^k)^2) \right]. \quad (3.18)$$

Differentiating the last term of (3.18) gives

$$\mathcal{O}^k [(V_1^k)^2] = 2V_1^k \left(\frac{\partial V_1^k}{\partial t} \right) + 2V_1^k f' \left(\frac{\partial V_1^k}{\partial x} \right) + \frac{1}{2} \text{tr} \left[\sigma W \sigma' \frac{\partial}{\partial x} \left[2V_1^k \frac{\partial V_1^k}{\partial x} \right] \right]. \quad (3.19)$$

Further differentiating the last term of (3.19) gives

$$\begin{aligned}\mathcal{O}^k \left[(V_1^k)^2 \right] &= 2V_1^k \left(\frac{\partial V_1^k}{\partial t} \right) + 2V_1^k f' \left(\frac{\partial V_1^k}{\partial x} \right) \\ &\quad + \frac{1}{2} \text{tr} \left[\sigma W \sigma' \left[2 \left(\frac{\partial V_1^k}{\partial x} \right)^2 + 2V_1^k \frac{\partial^2 V_1^k}{\partial x^2} \right] \right].\end{aligned}\quad (3.20)$$

Re-arranging the last term of (3.20) and applying trace identity gives

$$\begin{aligned}\mathcal{O}^k \left[(V_1^k)^2 \right] &= 2V_1^k \left(\frac{\partial V_1^k}{\partial t} \right) + 2V_1^k f' \left(\frac{\partial V_1^k}{\partial x} \right) \\ &\quad + \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x} \right) + V_1^k \text{tr} \left[\sigma W \sigma' \frac{\partial^2 V_1^k}{\partial x^2} \right].\end{aligned}\quad (3.21)$$

Factorizing (3.21) gives

$$\begin{aligned}\mathcal{O}^k \left[(V_1^k)^2 \right] &= 2V_1^k \left[\left(\frac{\partial V_1^k}{\partial t} \right) + f' \left(\frac{\partial V_1^k}{\partial x} \right) + \frac{1}{2} \text{tr} \left(\sigma W \sigma' \frac{\partial^2 V_1^k}{\partial x^2} \right) \right] \\ &\quad + \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x} \right).\end{aligned}\quad (3.22)$$

Applying the evolution operation identity (2.5) in (3.22) gives

$$\mathcal{O}^k \left[(V_1^k)^2 \right] = 2V_1^k \mathcal{O}^k \left[V_1^k \right] + \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x} \right). \quad (3.23)$$

Substituting (3.23) in (3.16) gives

$$\mathcal{O}^k \left[V_2^{k*}(t, x) \right] + 2V_1^k \mathcal{O}^k \left[V_1^k \right] + \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x} \right) + 2V_1^k L^k = 0. \quad (3.24)$$

Simplifying (3.24), since $\mathcal{O}^k \left[V_1^k \right] = -L^k$ (3.3) gives

$$\begin{aligned}0 &= \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) \left[V_2^{k*}(t, x) \right] \right. \\ &\quad \left. + \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^k(t, x)}{\partial x} \right) \right\},\end{aligned}\quad (3.25)$$

The theorem is proved. $\square$

Remarks: The HJB equation (3.25) provides a necessary condition for the existence of equilibrium solution of the N -player, 2-nd cost cumulant statistical game. The equilibrium solution is achieved under the constraints that $V_1^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$ is the admissible value function.

3.2.2 N -Player Minimal 3-rd Cumulant Statistical Game

Theorem 3.2: Consider the N -player continuous-time differential game (2.1) with cost functional (2.6) of fixed duration $[t_0, t_F]$. Let $V_1^k(t, x), V_2^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible value function for the k -th player. Assume the existence of optimal player strategy $\mu_k^*(t, x)$. Then, the minimal 3-rd value function $V_3^{k*}(t, x)$ satisfies the following coupled HJB equations for the first player to N -th player.

$$\begin{aligned}
0 &= \min_{\mu_1 \in U_{M^1}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k^*, \dots, \mu_N^*) [V_3^{1*}(t, x)] \right. \\
&\quad \left. + 3 \left(\frac{\partial V_1^1(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_2^1(t, x)}{\partial x} \right) \right\}, \\
&\vdots \\
0 &= \min_{\mu_N \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k^*, \dots, \mu_N) [V_3^{N*}(t, x)] \right. \\
&\quad \left. + 3 \left(\frac{\partial V_1^N(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_2^N(t, x)}{\partial x} \right) \right\}.
\end{aligned} \tag{3.26}$$

With the terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2, 3$ and $(t, x) \in Q$. In compact form, the minimal 3-rd value function $V_3^{k*}(t, x)$ satisfies the following HJB equation for the k -th player.

$$\begin{aligned}
0 &= \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_3^{k*}(t, x)] \right. \\
&\quad \left. + 3 \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_2^k(t, x)}{\partial x} \right) \right\},
\end{aligned} \tag{3.27}$$

for $(t, x) \in Q$, with the terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2, 3$.

Proof: Let V_3^k be a class of $C_p^{1,2}(Q) \cap C(\bar{Q})$. From the cumulant-moment relationship (2.9),

$$V_3^k(t, x) = M_3^k(t, x) - 3V_2^k(t, x)V_1^k(t, x) - (V_1^k)^3(t, x), \tag{3.28}$$

where $M_3^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$. Then $M_3^k(t, x)$ from (3.3) is the solution to

$$\mathcal{O}^k [M_3^{k*}(t, x)] + 3M_2^k(t, x)L^k(t, x, \mu_1, \dots, \mu_k, \dots, \mu_N) = 0. \tag{3.29}$$

Applying the backward evolution operator (2.5) to (3.28) gives

$$\mathcal{O}^k[V_3^{k*}(t, x)] = \mathcal{O}^k[M_3^{k*}(t, x)] - \mathcal{O}^k[3V_2^k(t, x)V_1^k(t, x)] - \mathcal{O}^k[(V_1^k)^3(t, x)]. \quad (3.30)$$

Using (3.29) in (3.30) gives

$$\mathcal{O}^k[V_3^{k*}(t, x)] + \mathcal{O}^k[3V_2^k(t, x)V_1^k(t, x)] + \mathcal{O}^k[(V_1^k)^3(t, x)] + 3M_2^k L^k = 0. \quad (3.31)$$

Manipulating (3.31) with $M_2^k = (V_2^k + (V_1^k)^2)$ gives

$$\mathcal{O}^k[V_3^{k*}(t, x)] + \mathcal{O}^k[3V_2^k(t, x)V_1^k(t, x)] + \mathcal{O}^k[(V_1^k)^3(t, x)] + 3(V_2^k + (V_1^k)^2)L^k = 0. \quad (3.32)$$

From the backward operator given in (2.5), expanding $\mathcal{O}^k(3V_2^k V_1^k + (V_1^k)^3)$ gives

$$\begin{aligned} & \mathcal{O}^k[3V_2^k(t, x)V_1^k(t, x) + (V_1^k)^3(t, x)] = \\ & 3\left[\mathcal{O}^k[V_2^k]V_1^k + \mathcal{O}^k[V_1^k]V_2^k + \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_2^k}{\partial x}\right)\right] + \\ & 3\left[\mathcal{O}^kV_1^k^2 + V_1^k \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right)\right]. \end{aligned} \quad (3.33)$$

Expanding terms in (3.33) using result in (3.24) gives

$$\begin{aligned} & \mathcal{O}^k[3V_2^k(t, x)V_1^k(t, x) + (V_1^k)^3(t, x)] = \\ & 3\left[-V_1^k \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right) - V_2^k L^k + \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_2^k}{\partial x}\right)\right] + \\ & 3\left[-(V_1^k)^2 L^k + V_1^k \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right)\right]. \end{aligned} \quad (3.34)$$

Combining terms in (3.34) with (3.32) gives

$$\begin{aligned} \mathcal{O}^k[V_3^{k*}] &= \left[3V_1^k \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right) + 3V_2^k L^k - 3\left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_2^k}{\partial x}\right)\right] + \\ & \left[+ 3(V_1^k)^2 L^k - 3V_1^k \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right)\right] - 3(V_2^k + (V_1^k)^2)L^k. \end{aligned} \quad (3.35)$$

Manipulating and reducing terms in (3.35) gives

$$\mathcal{O}^k[V_3^{k*}] = -3\left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_2^k}{\partial x}\right). \quad (3.36)$$

Re-arranging (3.36) gives

$$0 = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_3^{k*}(t, x)] \right. \\ \left. + 3 \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_2^k(t, x)}{\partial x} \right) \right\}, \quad (3.37)$$

The theorem is proved. $\square$

Remarks: The HJB equation (3.37) provides a necessary condition for the existence of equilibrium solution of the N -player, 3-rd cost cumulant statistical game. The equilibrium solution is achieved under the constraints that $V_1^k, V_2^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$ are admissible value function.

3.2.3 N -Player Minimal m -th Cumulant Statistical Game

Theorem 3.3: Consider the N -player continuous-time differential game (2.1) with cost functional (2.6) of fixed duration $[t_0, t_F]$. Let $V_1^k(t, x), V_2^k(t, x), V_3^k(t, x), \dots, V_{m-1}^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible value function for the k -th player. Assume the existence of optimal player strategy $\mu_k^*(t, x)$. Then, the minimal m -th value function $V_m^{k*}(t, x)$ satisfies the following coupled HJB equations for the first player to N -th player.

$$0 = \min_{\mu_1 \in U_{M^1}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k^*, \dots, \mu_N^*) [V_m^{1*}(t, x)] \right. \\ \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{n!}{i!(m-i)!} \left(\frac{\partial V_i^1(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-i}^1(t, x)}{\partial x} \right) \right\}, \\ \vdots \quad (3.38) \\ 0 = \min_{\mu_N \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k^*, \dots, \mu_N) [V_m^{N*}(t, x)] \right. \\ \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^N(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-i}^N(t, x)}{\partial x} \right) \right\}.$$

With the terminal condition $V_j^k(t_F, x_F) = 0, k = 1, 2, \dots, N, j = 1, 2, \dots, m$ and $(t, x) \in Q$. In compact form, the minimal m -th value function $V_m^{k*}(t, x)$ satisfies the following HJB

equation for the k -th player.

$$0 = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_m^{k*}(t, x)] + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-i}^k(t, x)}{\partial x} \right) \right\}, \quad (3.39)$$

for $(t, x) \in Q$, with the terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2, \dots, m$.

Proof: Let V_m^k be a class of $C_p^{1,2}(Q) \cap C(\bar{Q})$. From the cumulant-moment relationship (2.9),

$$V_m^k(t, x) = M_m^k(t, x) - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k(t, x) V_{i+1}^k(t, x), \quad (3.40)$$

where $M_m^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$.

From (3.3), the moment function M_m^k and running cost function L^k satisfy the following relationship (where the backward evolution operator arguments are now suppressed)

$$\min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k [M_m^{k*}(t, x)] + m M_{m-1}^k(t, x) L^k(t, x, \mu_1, \dots, \mu_k, \dots, \mu_N) \right\} = 0. \quad (3.41)$$

Then $M_m^k(t, x)$ is the solution to

$$\mathcal{O}^k [M_m^{k*}(t, x)] + m M_{m-1}^k(t, x) L^k(t, x, \mu_1, \dots, \mu_k, \dots, \mu_N) = 0. \quad (3.42)$$

Applying the backward evolution operator (2.5) to (3.40) gives

$$\mathcal{O}^k [V_m^{k*}(t, x)] = \mathcal{O}^k [M_m^{k*}(t, x)] - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \mathcal{O}^k [M_{m-1-i}^k(t, x) V_{i+1}^k(t, x)]. \quad (3.43)$$

Rearranging (3.43) gives

$$\mathcal{O}^k [M_m^{k*}(t, x)] = \mathcal{O}^k [V_m^{k*}(t, x)] + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \mathcal{O}^k [M_{m-1-i}^k(t, x) V_{i+1}^k(t, x)]. \quad (3.44)$$

Replacing the left-hand side of (3.42) with (3.44) gives

$$\mathcal{O}^k [V_m^{k*}(t, x)] + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \mathcal{O}^k [M_{m-1-i}^k(t, x) V_{i+1}^k(t, x)] + m M_{m-1}^k L^k = 0. \quad (3.45)$$

Applying lemma 3.2 (3.4) to the second term of (3.45) gives

$$\begin{aligned} \mathcal{O}^k[V_m^{k*}] + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & \left[M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] + V_{i+1}^k \mathcal{O}^k[M_{m-1-i}^k] \right. \\ & \left. + \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k = 0. \end{aligned} \quad (3.46)$$

Expanding the third term of (3.46) using (3.42) gives

$$\begin{aligned} \mathcal{O}^k[V_m^{k*}] + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & \left[M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] - (m-1-i) V_{i+1}^k M_{m-2-i}^k L^k \right. \\ & \left. + \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k = 0. \end{aligned} \quad (3.47)$$

Further expansion of the second to fourth term of (3.47) gives

$$\begin{aligned} \mathcal{O}^k[V_m^{k*}] + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\ - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & (m-1-i) V_{i+1}^k M_{m-2-i}^k L^k \\ + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k = 0. \end{aligned} \quad (3.48)$$

Splitting the second term of (3.48) into $i = 0$ and $i = 1$ to $m-2$ gives

$$\begin{aligned} \mathcal{O}^k[V_m^{k*}] + \frac{(m-1)!}{0!(m-1-0)!} & M_{m-1-0}^k \mathcal{O}^k[V_{0+1}^k] \\ + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} (m-1-i) V_{i+1}^k M_{m-2-i}^k L^k \\ + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k = 0. \end{aligned} \quad (3.49)$$

Simplifying (3.49) gives

$$\begin{aligned} \mathcal{O}^k[V_m^{k*}] + M_{m-1}^k \mathcal{O}^k[V_1^k] + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\ - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & (m-1-i) V_{i+1}^k M_{m-2-i}^k L^k \\ + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} & \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k = 0. \end{aligned} \quad (3.50)$$

where $M_0^k = 1$ and $V_1^k = M_1^k$.

Simplifying the second term of (3.50) using (3.42) gives

$$\begin{aligned} & \mathcal{O}^k[V_m^{k*}] - L^k M_{m-1}^k + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\ & - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} (m-1-i) V_{i+1}^k M_{m-2-i}^k L^k \\ & + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k = 0. \end{aligned} \quad (3.51)$$

Rearrange (3.51) by moving similar terms close to each other to give

$$\begin{aligned} & \mathcal{O}^k[V_m^{k*}] + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\ & - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} (m-1-i) V_{i+1}^k M_{m-2-i}^k L^k \\ & + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k - L^k M_{m-1}^k = 0. \end{aligned} \quad (3.52)$$

Split the third term of (3.52) into $i = 0$ to $m-3$ and $i = m-2$ to give

$$\begin{aligned} & \mathcal{O}^k[V_m^{k*}] + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\ & - \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-1-i)!} (m-1-i) V_{i+1}^k M_{m-2-i}^k L^k \\ & - \frac{(m-1)!}{(m-2)!(m-1-(m-2))!} (m-1-(m-2)) V_{m-2+1}^k M_{m-2-(m-2)}^k L^k \\ & + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k - L^k M_{m-1}^k = 0. \end{aligned} \quad (3.53)$$

Simplify (3.53) to give

$$\begin{aligned} & \mathcal{O}^k[V_m^{k*}] + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\ & - \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-1-i)!} (m-1-i) V_{i+1}^k M_{m-2-i}^k L^k - (m-1) V_{m-1}^k M_0^k L^k \\ & + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + m M_{m-1}^k L^k - L^k M_{m-1}^k = 0. \end{aligned} \quad (3.54)$$

where $M_0^k = 1$.

Simplify and rearrange (3.54) to give

$$\begin{aligned}
& \mathcal{O}^k[V_m^{k*}] + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] \\
& - \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-2-i)!} V_{i+1}^k M_{m-2-i}^k L^k - (m-1) V_{m-1}^k L^k + m M_{m-1}^k L^k - L^k M_{m-1}^k = 0.
\end{aligned} \tag{3.55}$$

Further rearranging the fourth to the last term in (3.55) gives

$$\begin{aligned}
& \mathcal{O}^k[V_m^{k*}] + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] \\
& + (m-1) L^k \left\{ M_{m-1}^k - \sum_{i=0}^{m-3} \frac{(m-2)!}{i!(m-2-i)!} V_{i+1}^k M_{m-2-i}^k - V_{m-1}^k \right\} = 0.
\end{aligned} \tag{3.56}$$

Comparing the last term of (3.56) with (3.40) gives

$$\begin{aligned}
& \mathcal{O}^k[V_m^{k*}] + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] + (m-1) L^k(0) = 0.
\end{aligned} \tag{3.57}$$

Simplifying (3.57) gives

$$\begin{aligned}
& \mathcal{O}^k[V_m^{k*}] + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] = 0.
\end{aligned} \tag{3.58}$$

Apply (3.39), (since (3.39) holds for $m-1$ to lower order cumulant functions) to the second

term of (3.58) gives

$$\begin{aligned} \mathcal{O}^k[V_m^{k*}] &+ \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \right. \\ &- \frac{1}{2} \sum_{i=1}^{j=1} \frac{(i+1)!}{j!(i+1-j)!} \left(\frac{\partial V_j^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{i+1-j}^k(t, x)}{\partial x} \right) \left. \right\} \\ &+ \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] = 0. \end{aligned} \quad (3.59)$$

Apply (3.7) to the second term of (3.59) gives

$$\begin{aligned} \mathcal{O}^k[V_m^{k*}] &- \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \right. \\ &\sum_{i=1}^{j=0} \frac{(i)!}{(i-j)!} \left(\frac{\partial V_{i-j}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j+1}^k(t, x)}{\partial x} \right) \left. \right\} \\ &+ \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] = 0. \end{aligned} \quad (3.60)$$

From the formula in [48] that

$$\frac{\partial M_i^k}{\partial V_j^k} = \sum_{j=1}^i \frac{i!}{j!(i-j)!} M_{i-j}^k. \quad (3.61)$$

Applying the function of function differentiation in the last term of (3.60) gives

$$\frac{\partial M_i^k}{\partial x} = \frac{\partial M_i^k}{\partial V_j^k} \times \frac{\partial V_j^k}{\partial x}. \quad (3.62)$$

Using (3.61) in (3.62) gives

$$\frac{\partial M_i^k}{\partial x} = \sum_{j=1}^i \frac{i!}{j!(i-j)!} M_{i-j}^k \frac{\partial V_j^k}{\partial x}. \quad (3.63)$$

Apply (3.63) to last term of (3.60) gives

$$\begin{aligned} \mathcal{O}^k[V_m^{k*}] &- \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \right. \\ &\sum_{i=1}^{j=0} \frac{(i)!}{(i-j)!} \left(\frac{\partial V_{i-j}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j+1}^k(t, x)}{\partial x} \right) \left. \right\} \\ &+ \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\sum_{j=1}^{m-1-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] = 0. \end{aligned} \quad (3.64)$$

Split the last term of (3.64) into $j = 1$ to $m - 2 - i$ and $j = m - 1 - i$ to give

$$\begin{aligned}
& \mathcal{O}^k[V_m^{k*}] - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \right. \\
& \sum_{j=1}^{m-2-i} \frac{(i-j)!}{(i-j)!} \left(\frac{\partial V_{i-j}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j+1}^k(t, x)}{\partial x} \right) \left. \right\} \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\sum_{j=1}^{m-2-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right. \\
& \left. + \frac{(m-1-i)!}{(m-1-i)!(m-1-i-(m-1-i))!} M_{m-1-i-(m-1-i)}^k \left(\frac{\partial V_{(m-1-i)}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] = 0.
\end{aligned} \tag{3.65}$$

Simplify (3.65) to give

$$\begin{aligned}
& \mathcal{O}^k[V_m^{k*}] - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \right. \\
& \sum_{j=1}^{m-2-i} \frac{(i-j)!}{(i-j)!} \left(\frac{\partial V_{i-j}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j+1}^k(t, x)}{\partial x} \right) \left. \right\} \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\sum_{j=1}^{m-2-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right. \\
& \left. + M_0^k \left(\frac{\partial V_{(m-1-i)}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] = 0.
\end{aligned} \tag{3.66}$$

Further simplification of (3.66) gives

$$\begin{aligned}
& \mathcal{O}^k[V_m^{k*}] - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \right. \\
& \sum_{j=1}^{m-2-i} \frac{(i-j)!}{(i-j)!} \left(\frac{\partial V_{i-j}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j+1}^k(t, x)}{\partial x} \right) \left. \right\} \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\sum_{j=1}^{m-2-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right. \\
& \left. + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial V_{(m-1-i)}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] = 0,
\end{aligned} \tag{3.67}$$

where $M_0^l = 1$.

Rearranging (3.67) gives

$$\begin{aligned}
& \mathcal{O}^k[V_m^{k*}] + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial V_{(m-1-i)}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \sum_{j=1}^{m-2-i} \frac{(i)!}{(i-j)!} \left(\frac{\partial V_{i-j}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j+1}^k(t, x)}{\partial x} \right) \right\} \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[\sum_{j=1}^{m-2-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] = 0.
\end{aligned} \tag{3.68}$$

Let $m = 3$, then substituting in third term of (3.68) and manipulating (3.68) gives

$$G_{11}[\cdot] = 2M_1^k \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x} \right). \tag{3.69}$$

Similarly, substituting $m = 3$ in fourth term of (3.68) and manipulating (3.68) gives

$$G_{12}[\cdot] = 2M_1^k \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x} \right). \tag{3.70}$$

Also, let $m = 4$, then substituting in third term of (3.68) and manipulating (3.68) gives

$$G_{21}[\cdot] = 9M_1^k \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_2^k}{\partial x} \right) + 3M_2^k \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x} \right). \tag{3.71}$$

Similarly, substituting $m = 4$, in fourth term of (3.68) and manipulating (3.68) gives

$$G_{22}[\cdot] = 9M_1^k \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_2^k}{\partial x} \right) + 3M_2^k \left(\frac{\partial V_1^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x} \right). \tag{3.72}$$

Comparing terms from G_{11} in (3.69) with G_{12} in (3.70) and G_{21} in (3.71) with G_{22} in (3.72) show that (3.68) equal to zero. In general, for all positive integer values of $m \geq 2$, the summation of third and fourth terms in (3.68) equals zero. Thus, (3.68) simplifies to

$$\mathcal{O}^k[V_m^k] + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial V_{(m-1-i)}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0. \tag{3.73}$$

Apply lemma 3.3 (3.7), to second term of (3.73) gives

$$\begin{aligned}
& \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_m^{k*}(t, x)] \right. \\
& \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-i}^k(t, x)}{\partial x} \right) \right\} = 0.
\end{aligned} \tag{3.74}$$

The theorem is proved. □

Remarks: The HJB equation (3.74) satisfies the necessary condition for the existence of equilibrium solutions of the N -player, m -th cost cumulant statistical game. The equilibrium solution is achieved under the constraints that $V_1^k, V_2^k, \dots, V_{m-1}^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$ are admissible value functions. In the next chapter, the verification theorem for the existence of equilibrium solutions for the game will be stated and proved. From the classical differential game equation (N -Player, minimal 1-st cumulant statistical game or linear quadratic gaussian (LQG) game) [16], the complexity in the derivative of the higher-order moment cost with respect to state does not allow for simple replacement of the N -Player, minimal 1-st cumulant statistical game cost function with the higher-order cost function to get the necessary equation (3.74). The higher-order cumulant statistical game control inherently has complexity due to the m -th moment appearing in the equations. So it cannot be simply generated from N -Player, minimal 1-st cumulant statistical game result. In fact, statistical game can be viewed as a generalization of linear quadratic gaussian (LQG) game and risk-sensitive game and thus needs to be derived from first principles.

CHAPTER 4

HJB EQUATIONS VERIFICATION THEOREM: SUFFICIENT CONDITION

In this chapter, the sufficient condition for the existence of equilibrium solution of N -player, m -th cost cumulant statistical game is stated and proved. The sufficient condition is also the verification theorem for the existence of equilibrium solution of N -player, m -th cost cumulant statistical game.

4.1 Mathematical Preliminaries

All the mathematical formulations, lemmas and assumptions in Chapter 3 that are applicable are referenced accordingly.

4.2 Sufficient Condition for the HJB equations

4.2.1 N -Player Minimal 2-nd Cumulant Statistical Game

Theorem 4.1: Consider the N -player continuous-time differential game (2.1) with cost functional (2.6) of fixed duration $[t_0, t_F]$. Let $V_1^k(t, x), M_1^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible cost cumulant and moment function for the k -th player. Then, value function $V_2^{k*}(t, x)$

satisfies the following coupled HJB equations for the first player to the N -th player.

$$\begin{aligned}
0 &= \min_{\mu_1 \in U_{M^1}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k^*, \dots, \mu_N^*) [V_2^{1*}(t, x)] \right. \\
&\quad \left. + \left(\frac{\partial V_1^1(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^1(t, x)}{\partial x} \right) \right\}, \\
&\vdots \\
0 &= \min_{\mu_N \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k^*, \dots, \mu_N) [V_2^{N*}(t, x)] \right. \\
&\quad \left. + \left(\frac{\partial V_1^N(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^N(t, x)}{\partial x} \right) \right\}.
\end{aligned} \tag{4.1}$$

With the terminal condition $V_j^k(t_F, x) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2$ and $(t, x) \in Q$. In compact form, the minimal 2-nd value function $V_2^{k*}(t, x)$ satisfies the following HJB equation for the k -th player.

$$\begin{aligned}
0 &= \min_{\mu_k \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_2^{k*}(t, x)] \right. \\
&\quad \left. + \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^k(t, x)}{\partial x} \right) \right\},
\end{aligned} \tag{4.2}$$

with terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2$, then

$$V_2^{k*}(t, x, \mu_1^*, \dots, \mu_k, \dots, \mu_N^*) \leq V_2^k(t, x, \mu_1^*, \dots, \mu_k, \dots, \mu_N^*), \tag{4.3}$$

for all $\mu_k \in U_{M^k}$ and $(t, x) \in Q$. If, in addition, there exist a strategy $\mu_k^*(t, x)$ that satisfies the following equation:

$$\mathcal{O}^k[V_2^{k*}(t, x)] = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) V_2^k(t, x) \right\}, \tag{4.4}$$

then $\mu_k^*(t, x)$ is the optimal k -th player strategy.

Proof: Let V_2^k be a class of $C_p^{1,2}(Q) \cap C(\bar{Q})$. From the cumulant-moment relationship (2.9),

$$V_2^k(t, x) = M_2^k(t, x) - (V_1^k)^2(t, x), \tag{4.5}$$

where $M_2^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$.

Re-arranging (4.5) gives

$$M_2^k(t, x) = V_2^k(t, x) + (V_1^k)^2(t, x), \quad (4.6)$$

Applying the backward evolution operator (2.5) to (4.6) gives

$$\mathcal{O}^k[M_2^{k*}(t, x)] = \mathcal{O}^k[V_2^k(t, x)] + \mathcal{O}^k[(V_1^k)^2(t, x)]. \quad (4.7)$$

From (3.23), (4.1) substituting in (4.7) and expanding gives

$$\mathcal{O}^k[M_2^{k*}(t, x)] = -\left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right) + 2V_1^k \mathcal{O}^k[V_1^k] + \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right). \quad (4.8)$$

Then $M_1^k(t, x)$ from (3.3) is the solution to

$$\mathcal{O}^k[M_1^k(t, x)] + M_0^k(t, x) L^k(t, x, \mu_1, \dots, \mu_k, \dots, \mu_N) = 0. \quad (4.9)$$

Applying (4.9) in first term of (4.8) gives

$$\mathcal{O}^k[M_2^{k*}(t, x)] = -2V_1^k L^k + \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right) - \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right). \quad (4.10)$$

Simplifying terms in (4.10) gives

$$\min_{\mu_k \in U_{M^1}} \left\{ \mathcal{O}^k[M_2^{k*}(t, x)] + 2M_1^k L^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \right\} = 0. \quad (4.11)$$

The theorem is proved. $\square$

Remarks: The HJB equation (4.11) satisfies the sufficient condition for the existence of equilibrium solution of the N -player, 2-nd cost cumulant statistical game in (4.2). The equilibrium solution $V_2^{k*}(t, x)$ is achieved under the constraints that $M_1^k, V_1^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$ exist and are admissible moment and cumulant functions.

4.2.2 N -Player Minimal 3-rd Cumulant Statistical Game

Theorem 4.2: Consider the N -player continuous-time differential game (2.1) with cost functional (2.6) of fixed duration $[t_0, t_F]$. Let $V_1^k(t, x), M_1^k(t, x), V_2^k(t, x), M_2^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible cost cumulant and moment function for the k -th player. Then, value

function $V_3^{k*}(t, x)$ satisfies the following coupled HJB equations for the first player to the N -player.

$$\begin{aligned}
0 &= \min_{\mu_1 \in U_{M^1}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k^*, \dots, \mu_N^*) [V_3^{1*}(t, x)] \right. \\
&\quad \left. + 3 \left(\frac{\partial V_1^1(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_2^1(t, x)}{\partial x} \right) \right\}, \\
&\vdots \\
0 &= \min_{\mu_N \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k^*, \dots, \mu_N) [V_3^{N*}(t, x)] \right. \\
&\quad \left. + 3 \left(\frac{\partial V_1^N(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_2^N(t, x)}{\partial x} \right) \right\}.
\end{aligned} \tag{4.12}$$

With the terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2, 3$ and $(t, x) \in Q$. In compact form, the minimal 3-rd value function $V_3^{k*}(t, x)$ satisfies the following HJB equation for the k -th player.

$$\begin{aligned}
0 &= \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_3^{k*}(t, x)] \right. \\
&\quad \left. + 3 \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_2^k(t, x)}{\partial x} \right) \right\},
\end{aligned} \tag{4.13}$$

with terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2, 3$, then

$$V_3^{k*}(t, x, \mu_1^*, \dots, \mu_k, \dots, \mu_N^*) \leq V_3^k(t, x, \mu_1^*, \dots, \mu_k, \dots, \mu_N^*), \tag{4.14}$$

for all $\mu_k \in U_{M^k}$ and $(t, x) \in Q$. If, in addition, there exist a strategy $\mu_k^*(t, x)$ that satisfies the following equation:

$$\mathcal{O}^k[V_3^{k*}(t, x)] = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) V_3^k(t, x) \right\}, \tag{4.15}$$

then $\mu_k^*(t, x)$ is the optimal k -th player strategy.

Proof: Let V_3^k be a class of $C_p^{1,2}(Q) \cap C(\bar{Q})$. From the cumulant-moment relationship (2.9),

$$V_3^k(t, x) = M_3^k(t, x) - 3M_2^k(t, x)M_1^k(t, x) + 2(M_1^k)^3(t, x), \tag{4.16}$$

where $M_3^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$.

Re-arranging (4.16) gives

$$M_3^k(t, x) = V_3^k(t, x) + 3M_2^k(t, x)M_1^k(t, x) - 2(M_1^k)^3(t, x), \quad (4.17)$$

Applying the backward evolution operator (2.5) to (4.17) gives

$$\mathcal{O}^k[M_3^{k*}(t, x)] = \mathcal{O}^k[V_3^{k*}(t, x)] + \mathcal{O}^k[3M_2^k(t, x)M_1^k(t, x)] - \mathcal{O}^k[2(M_1^k)^3(t, x)]. \quad (4.18)$$

From (3.4), (4.13) substituting in (4.18) and expanding gives

$$\begin{aligned} \mathcal{O}^k[M_3^{k*}(t, x)] &= \mathcal{O}^k[V_3^{k*}(t, x)] + 3M_1^k \mathcal{O}^k[M_2^k] + 3M_2^k \mathcal{O}^k[M_1^k] + 3\left(\frac{\partial M_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_2^k}{\partial x}\right) \\ &- 6(M_1^k)^2 \mathcal{O}^k[M_1^k] - 6M_1^k \left(\frac{\partial M_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_1^k}{\partial x}\right). \end{aligned} \quad (4.19)$$

Then $M_j^k(t, x)$ from (3.3) is the solution to

$$\mathcal{O}^k[M_j^k(t, x)] + jM_{j-1}^k(t, x)L^k(t, x, \mu_1, \dots, \mu_k, \dots, \mu_N) = 0. \quad (4.20)$$

Applying (4.20) in second and third right-side terms of (4.19) gives

$$\begin{aligned} \mathcal{O}^k[M_3^{k*}(t, x)] &= \mathcal{O}^k[V_3^{k*}(t, x)] - 6(M_1^k)^2 L^k - 3M_2^k L^k + 3\left(\frac{\partial M_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_2^k}{\partial x}\right) \\ &+ 6(M_1^k)^2 L^k - 6M_1^k \left(\frac{\partial M_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_1^k}{\partial x}\right). \end{aligned} \quad (4.21)$$

Substitute for $\mathcal{O}^k[V_3^k]$ from (4.13) gives

$$\begin{aligned} \mathcal{O}^k[M_3^{k*}(t, x)] &= -3\left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_2^k}{\partial x}\right) - 6(M_1^k)^2 L^k - 3M_2^k L^k \\ &+ 3\left(\frac{\partial M_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_2^k}{\partial x}\right) + 6(M_1^k)^2 L^k - 6M_1^k \left(\frac{\partial M_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_1^k}{\partial x}\right). \end{aligned} \quad (4.22)$$

Using the relation between $\frac{\partial V_j^k}{\partial x}$ and $\frac{\partial M_j^k}{\partial x}$ in [44], and updating the fourth-term of the left-hand side of (4.22) gives

$$\begin{aligned} \mathcal{O}^k[M_3^{k*}(t, x)] &= -3\left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_2^k}{\partial x}\right) - 6(M_1^k)^2 L^k - 3M_2^k L^k \\ &+ 6M_1^k \left(\frac{\partial M_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_1^k}{\partial x}\right) + 3\left(\frac{\partial V_2^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_1^k}{\partial x}\right) \\ &+ 6(M_1^k)^2 L^k - 6M_1^k \left(\frac{\partial M_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial M_1^k}{\partial x}\right). \end{aligned} \quad (4.23)$$

Simplifying terms in (4.23) gives

$$\min_{\mu_1 \in U_{M^1}} \left\{ \mathcal{O}^k \left[M_3^{k*}(t, x) \right] + 3M_2^k L^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \right\} = 0. \quad (4.24)$$

The theorem is proved. $\square$

Remarks: The HJB equation (4.24) satisfies the sufficient condition for the existence of equilibrium solution of the N -player, 3-rd cost cumulant statistical game in (4.13). The equilibrium solution $V_1^k(t, x)$ is achieved under the constraints that $V_1^k, V_2^k, M_1^k, M_2^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$ exist and are admissible cumulant and moment functions.

4.2.3 N -Player Minimal m -th Cumulant Statistical Game

Theorem 4.3: Consider the N -player continuous-time differential game (2.1) with cost functional (2.6) of fixed duration $[t_0, t_F]$. Let $V_1^k(t, x), \dots, V_{m-1}^k(t, x), M_1^k(t, x), \dots, M_{m-1}^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible cost cumulant and moment function for the k -th player. Then, value function $V_m^{k*}(t, x)$ satisfies the following coupled HJB equations for the first player to the N -player.

$$\begin{aligned} 0 &= \min_{\mu_1 \in U_{M^1}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k^*, \dots, \mu_N^*) \left[V_m^{1*}(t, x) \right] \right. \\ &\quad \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^1(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-i}^1(t, x)}{\partial x} \right) \right\}, \\ &\quad \vdots \\ 0 &= \min_{\mu_N \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k^*, \dots, \mu_N) \left[V_m^{N*}(t, x) \right] \right. \\ &\quad \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^N(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-i}^N(t, x)}{\partial x} \right) \right\}. \end{aligned} \quad (4.25)$$

With the terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2, \dots, m$ and $(t, x) \in Q$. In compact form, the minimal m -th value function $V_m^{k*}(t, x)$ satisfies the following HJB

equation for the k -th player.

$$0 = \min_{\mu_k \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_m^{k*}(t, x)] \right. \\ \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-i}^k(t, x)}{\partial x} \right) \right\}, \quad (4.26)$$

with terminal condition $V_j^k(t_F, x_F) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2, \dots, m$, then

$$V_m^{k*}(t, x, \mu_1^*, \dots, \mu_k, \dots, \mu_N^*) \leq V_m^k(t, x, \mu_1^*, \dots, \mu_k, \dots, \mu_N^*), \quad (4.27)$$

for all $\mu_k \in U_{M^k}$ and $(t, x) \in Q$. If, in addition, there exist a strategy $\mu_k(t, x)$ that satisfies the following equation:

$$\mathcal{O}^k[V_m^{k*}(t, x)] = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) V_m^k(t, x) \right\}, \quad (4.28)$$

then $\mu_k^*(t, x)$ is the optimal k -th player strategy.

Proof: Let $V_m^{k*}(t, x)$ be a class of $C_p^{1,2}(Q) \cap C(\bar{Q})$. Apply (3.7) to (4.26) to give

$$0 = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_m^{k*}(t, x)] \right. \\ \left. + \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{i+1}^k(t, x)}{\partial x} \right) \right\}. \quad (4.29)$$

Substitute the moment-cumulant relationship (2.9) to (4.29) which gives

$$\mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [M_m^{k*}] - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \mathcal{O}^k[M_{m-1-i}^k V_{i+1}^k] \\ + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0. \quad (4.30)$$

Apply lemma 3.2 (3.4) to the second term of (4.30) which gives

$$\mathcal{O}^k[M_m^{k*}] - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left[M_{m-1-i}^k \mathcal{O}^k[V_{i+1}^k] + V_{i+1}^k \mathcal{O}^k[M_{m-1-i}^k] \right. \\ \left. + \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right] \\ + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0. \quad (4.31)$$

Expanding (4.31) gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} V_{i+1}^k \mathcal{O}^k \left[M_{m-1-i}^k \right] \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k \left[V_{i+1}^k \right] \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.32}$$

Split the second term of (4.32) from $i = 0$ to $m-3$ and $i = m-2$ and split the third term of (4.32) from $i = 0$ and $i = 1$ to $i = m-2$ to give

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] - \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-1-i)!} V_{i+1}^k \mathcal{O}^k \left[M_{m-1-i}^k \right] \\
& - \frac{(m-1)!}{(m-2)!(m-1-(m-2))!} V_{m-2+1}^k \mathcal{O}^k \left[M_{m-1-(m-2)}^k \right] \\
& - \frac{(m-1)!}{0!(m-1-0)!} M_{m-1-0}^k \mathcal{O}^k \left[V_{0+1}^k \right] - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k \left[V_{i+1}^k \right] \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.33}$$

Simplify (4.33) to give

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] - \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-1-i)!} V_{i+1}^k \mathcal{O}^k \left[M_{m-1-i}^k \right] - (m-1) V_{m-1}^k \mathcal{O}^k \left[M_1^k \right] \\
& - M_{m-1}^k \mathcal{O}^k \left[V_1^k \right] - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k \left[V_{i+1}^k \right] \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.34}$$

Using (3.42) moment-cumulant relation, substitute in the second term of (4.34) gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] - \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-1-i)!} - (m-1-i)V_{i+1}^k M_{m-2-i}^k L^k - (m-1)V_{m-1}^k \mathcal{O}^k \left[M_1^k \right] \\
& - M_{m-1}^k \mathcal{O}^k \left[V_1^k \right] - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k \left[V_{i+1}^k \right] \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k(t, x)}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.35}$$

Simplify (4.35) gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-2-i)!} V_{i+1}^k M_{m-2-i}^k L^k - (m-1)V_{m-1}^k \mathcal{O}^k \left[M_1^k \right] \\
& - M_{m-1}^k \mathcal{O}^k \left[V_1^k \right] - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k \left[V_{i+1}^k \right] \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.36}$$

Using (3.42), substitute in the third and fourth terms of (4.34) to give

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-2-i)!} V_{i+1}^k M_{m-2-i}^k L^k \\
& + (m-1)V_{m-1}^k L^k + M_{m-1}^k L^k - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k \left[V_{i+1}^k \right] \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.37}$$

From (3.40), the second to fourth terms in (4.37) can be rearranged as

$$(m-1)L^k V_{m-1}^k + L^k M_{m-1}^k + \sum_{i=0}^{m-3} \frac{(m-1)!}{i!(m-2-i)!} M_{m-2-i}^k V_{i+1}^k = m L^k M_{m-1}^k. \tag{4.38}$$

Substitute (4.38) for the second to fourth terms in (4.37) gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + m M_{m-1}^k L^k - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \mathcal{O}^k \left[V_{i+1}^k \right] \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.39}$$

Apply (4.26) to the third term of (4.39), since (4.26) holds for $m-1$ to lower order cumulant functions, gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + m M_{m-1}^k L^k - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \frac{1}{2} \sum_{j=1}^i \frac{(i+1)!}{j!(i+1-j)!} \right. \\
& \left. \left(\frac{\partial V_i^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1-j}^k}{\partial x} \right) \right\} \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.40}$$

Using (3.7) to the third term of (4.40) gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + m M_{m-1}^k L^k - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \sum_{j=0}^{i-1} \frac{i!}{j!(i-j)!} \right. \\
& \left. \left(\frac{\partial V_{i-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{j+1}^k}{\partial x} \right) \right\} \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial M_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.41}$$

Apply (3.63) to the fourth term of (4.41) gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + m M_{m-1}^k L^k \\
& - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \sum_{j=0}^{i-1} \frac{i!}{j!(i-j)!} \left(\frac{\partial V_{i-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{j+1}^k}{\partial x} \right) \right\} \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \sum_{j=0}^{m-1-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.42}$$

Split the fourth term of (4.42) from $j = 0$ to $m-2-i$ and $j = m-1-i$ to give

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + m M_{m-1}^k L^k \\
& - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \sum_{j=0}^{i-1} \frac{i!}{j!(i-j)!} \left(\frac{\partial V_{i-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{j+1}^k}{\partial x} \right) \right\} \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left\{ \sum_{j=0}^{m-2-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right. \\
& + \left. \frac{(m-1-i)!}{(m-1-i)!(m-1-i-(m-1-i))!} M_{m-1-i-(m-1-i)}^k \left(\frac{\partial V_{(m-1-i)}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right\} \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.43}$$

Simplify (4.43) gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + m M_{m-1}^k L^k \\
& - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \sum_{j=0}^{i-1} \frac{i!}{j!(i-j)!} \left(\frac{\partial V_{i-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{j+1}^k}{\partial x} \right) \right\} \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left\{ \sum_{j=0}^{m-2-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right. \\
& + \left. \frac{(m-1-i)!}{(m-1-i)!0!} M_0^k \left(\frac{\partial V_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \right\} \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.44}$$

Further simplification and expansion of (4.44) gives

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + m M_{m-1}^k L^k \\
& - \sum_{i=1}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k \left\{ \sum_{j=0}^{i-1} \frac{i!}{j!(i-j)!} \left(\frac{\partial V_{i-j}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{j+1}^k}{\partial x} \right) \right\} \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \sum_{j=0}^{m-2-i} \frac{(m-1-i)!}{j!(m-1-i-j)!} M_{m-1-i-j}^k \left(\frac{\partial V_j^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial V_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.45}$$

From (3.68), (3.74) the third and fourth terms are equal, thus (4.45) becomes

$$\begin{aligned}
& \mathcal{O}^k \left[M_m^{k*} \right] + m M_{m-1}^k L^k - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} \left(\frac{\partial V_{m-1-i}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) \\
& + \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-i-1)!} \left(\frac{\partial V_{m-i-1}^k}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{i+1}^k}{\partial x} \right) = 0.
\end{aligned} \tag{4.46}$$

Further simplification of (4.46) gives

$$\min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) M_m^{k*} + m M_{m-1}^k L^k \right\} = 0. \tag{4.47}$$

The theorem is proved. $\square$

Remarks: The HJB equation (4.47) satisfies the sufficient condition for the existence of equilibrium solution of the N -player, m -th cost cumulant statistical game. The equilibrium solution is achieved under the constraints that $V_1^k, V_2^k, \dots, V_{m-1}^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$ are admissible cumulant functions and $M_1^k, M_2^k, \dots, M_{m-1}^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$ are admissible moment functions. From the classical differential game equation (N -Player minimal 1-st cumulant statistical game or linear quadratic gaussian (LQG) game) [16], the complexity in the derivative of the higher-order moment cost with respect to state does not allow for simple replacement of the N -Player minimal 1-st cumulant statistical game cost function with the higher-order cost function to get the sufficient equation (4.47). The higher-order cumulant statistical game control inherently has complexity due to the m -th moment appearing in the equations. So it cannot

be simply generated from N -Player minimal 1-st cumulant statistical game result. In fact, statistical game can be viewed as a generalization of linear quadratic gaussian (LQG) game and risk-sensitive game and thus needs to be derived from first principles.

Theorem 4.4: Let $V_1^k(t, x), \dots, V_{m-1}^k(t, x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible value function for the k -th player. The k -th player state-feedback optimal strategy is given as

$$\mu_k^*(t, x) = -\frac{1}{2}R_{kk}^{-1}B_k' \left(\frac{\partial V_1^k(t, x)}{\partial x} + \gamma_{2k}(t) \frac{\partial V_2^k(t, x)}{\partial x} + \dots + \gamma_{mk}(t) \frac{\partial V_m^{k*}(t, x)}{\partial x} \right), \quad (4.48)$$

for $(t, x) \in Q$, with terminal condition $V_i^k(t_F, x_F) = 0$ where $i = 1, 2, \dots, m$ and $\gamma_{2k}(t), \dots, \gamma_{mk}(t)$ are the Lagrange multipliers.

Proof: The N -player minimal m -th value function $V_m^{k*}(t, x)$ satisfies the following coupled HJB equation

$$\begin{aligned} 0 = \min_{\mu_k \in U_{M^k}} & \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_m^{k*}(t, x)] \right. \\ & \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-i}^k(t, x)}{\partial x} \right) \right\}, \end{aligned} \quad (4.49)$$

with the constraint condition that $V_1^k(t, x), V_2^k(t, x), \dots, V_{m-1}^k(t, x)$ are admissible value functions. Then, value functions $V_j^k(t, x)$ satisfy the following coupled partial differential equations for $j = 1$,

$$\mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_j^k(t, x)] + L^k(t, x, \mu_1, \mu_2, \dots, \mu_N) = 0, \quad (4.50)$$

and for $j = 2, 3, \dots, m$,

$$\begin{aligned} & \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) [V_{j-1}^k(t, x)] \\ & + \frac{1}{2} \sum_{s=1}^{j-2} \frac{(j-1)!}{s!(j-s-1)!} \left(\frac{\partial V_s^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j-s-1}^k(t, x)}{\partial x} \right), \end{aligned} \quad (4.51)$$

with the terminal condition $V_i^k(t_F, x_F) = 0$ with $i = 1, 2, \dots, m$ and $k = 1, 2, \dots, N$.

Using the method of Lagrange multipliers and suppressing the arguments in $\mathcal{O}^k$, let a new function $G_k(t, x, \mu_1, \mu_2, \dots, \mu_N)$ with $k = 1, 2, \dots, N$ be formulated by converting the con-

strained coupled HJB equation (4.49) to unconstrained coupled HJB equations as follows

$$\begin{aligned}
G_k = & \mathcal{O}^k \left[V_m^{k*}(t, x) \right] + \frac{1}{2} \sum_{s=1}^{m-1} \frac{m!}{s!(m-s)!} \left(\frac{\partial V_s^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-s}^k(t, x)}{\partial x} \right) \\
& + \lambda_{1k}(t) \left(\mathcal{O}^k \left[V_1^k(t, x) \right] + L^k(t, x, \mu_1, \mu_2, \dots, \mu_N) \right) \\
& + \lambda_{2k}(t) \left(\mathcal{O}^k \left[V_2^k(t, x) \right] + \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^k(t, x)}{\partial x} \right) \right) + \dots \\
& + \lambda_{(m-1)k}(t) \left(\mathcal{O}^k \left[V_{m-1}^k(t, x) \right] + \frac{1}{2} \sum_{s=1}^{m-2} \frac{(m-1)!}{s!(m-s-1)!} \right. \\
& \left. \left(\frac{\partial V_s^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-s-1}^k(t, x)}{\partial x} \right) \right),
\end{aligned} \tag{4.52}$$

where $\lambda_{1k}(t), \lambda_{2k}(t), \dots, \lambda_{(m-1)k}(t)$ are time-varying Lagrange multipliers. At equilibrium, the stationary conditions are given by the partial derivative of $G_k(t, x, \mu_1, \mu_2, \dots, \mu_N)$ in (4.52) with respect to $\mu_k, \lambda_{1k}(t), \lambda_{2k}(t), \dots, \lambda_{(m-1)k}(t)$, which is zero. These partial derivatives are

$$\begin{aligned}
\frac{\partial G_k(t, x, \mu_1, \mu_2, \dots, \mu_N)}{\partial \mu_k} = & B'_k \frac{\partial V_m^{k*}(t, x)}{\partial x} + 2\lambda_{1k}(t) R_{kk} \mu_k + \lambda_{1k}(t) B'_k \frac{\partial V_1^k(t, x)}{\partial x} \\
& + \lambda_{2k}(t) B'_k \frac{\partial V_2^k(t, x)}{\partial x} + \dots + \lambda_{(m-1)k}(t) B'_k \frac{\partial V_{m-1}^k(t, x)}{\partial x} = 0.
\end{aligned} \tag{4.53}$$

$$\frac{\partial G_k(t, x, \mu_1, \mu_2, \dots, \mu_N)}{\partial \lambda_{1k}(t)} = \mathcal{O}^k \left[V_1^k(t, x) \right] + L^k(t, x, \mu_1, \mu_2, \dots, \mu_N) = 0. \tag{4.54}$$

$$\frac{\partial G_k(t, x, \mu_1, \mu_2, \dots, \mu_N)}{\partial \lambda_{2k}(t)} = \mathcal{O}^k \left[V_2^k(t, x) \right] + \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_1^k(t, x)}{\partial x} \right) = 0. \tag{4.55}$$

⋮

$$\begin{aligned}
\frac{\partial G_k(t, x, \mu_1, \mu_2, \dots, \mu_N)}{\partial \lambda_{(m-1)k}}(t) = & \mathcal{O}^k \left[V_{m-1}^k(t, x) \right] + \frac{1}{2} \sum_{s=1}^{m-2} \frac{(m-1)!}{s!(m-s-1)!} \\
& \left(\frac{\partial V_s^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{m-s-1}^k(t, x)}{\partial x} \right) = 0.
\end{aligned} \tag{4.56}$$

From (4.53), the optimal k -th player strategy $\mu_k^*(t, x)$ is

$$\mu_k^*(t, x) = -\frac{1}{2} R_{kk}^{-1} B'_k \left(\frac{\partial V_1^k(t, x)}{\partial x} + \frac{\lambda_{2k}(t)}{\lambda_{1k}(t)} \frac{\partial V_2^k(t, x)}{\partial x} \dots + \frac{1}{\lambda_{1k}(t)} \frac{\partial V_m^{k*}(t, x)}{\partial x} \right). \tag{4.57}$$

Now, let the Lagrange multipliers in (4.57) be defined as

$$\gamma_{2k}(t) = \frac{\lambda_{2k}(t)}{\lambda_{1k}(t)}, \gamma_{3k}(t) = \frac{\lambda_{3k}(t)}{\lambda_{1k}(t)}, \dots, \gamma_{(m-1)k}(t) = \frac{\lambda_{(m-1)k}(t)}{\lambda_{1k}(t)}, \gamma_{mk}(t) = \frac{1}{\lambda_{1k}(t)}. \quad (4.58)$$

Then, substituting (4.58) in (4.57) gives

$$\mu_k^*(t, x) = -\frac{1}{2}R_{kk}^{-1}B_k' \left(\frac{\partial V_1^k(t, x)}{\partial x} + \gamma_{2k}(t) \frac{\partial V_2^k(t, x)}{\partial x} \dots + \gamma_{mk}(t) \frac{\partial V_m^{k*}(t, x)}{\partial x} \right). \quad (4.59)$$

Thus, substituting for $\mu_1^*, \mu_2^*, \dots, \mu_{N-1}^*, \mu_N^*$ to the N -th player m -th cumulant cost HJB equations (4.25), (4.26) gives the closed loop system form of the statistical game.

The theorem is proved. $\square$

Remarks: The coupled cost cumulant HJB equation (4.25), (4.26) provides the sufficiency condition for the Nash equilibrium solution of the N -player m -th cumulant statistical game. The equilibrium solution is achieved under the constraints that $V_1^k, V_2^k, \dots, V_{m-1}^k \in C_p^{1,2}(Q) \cap C(\bar{Q})$, $\dots, V_1^N, V_2^N, \dots, V_{m-1}^N \in C_p^{1,2}(Q) \cap C(\bar{Q})$ are admissible. The Nash equilibrium strategies lead to the different shapes of the distribution of the players cost function. The optimal strategies for the N -player m -th cumulant cost statistical game can be found by solving these corresponding coupled HJB equations (4.25), (4.26). For example, a 3-player 2^{nd} cumulant or minimum cost variance (MCV) game leads to six (6) coupled HJB equations to be formulated.

The second-order partial derivatives of (4.52), give

$$\begin{aligned} \frac{\partial^2 G_k}{\partial \mu_k^2} &= 2\lambda_{1k}(t)R_{kk} > 0, \frac{\partial^2 G_k}{\partial \lambda_{1k}^2}(t) = 0, \frac{\partial^2 G_k}{\partial \lambda_{2k}^2}(t) = 0, \\ \frac{\partial^2 G_k}{\partial \lambda_{3k}^2}(t) &= 0, \frac{\partial^2 G_k}{\partial \lambda_{4k}^2}(t) = 0, \dots, \frac{\partial^2 G_k}{\partial \lambda_{(n-1)k}^2}(t) = 0. \end{aligned} \quad (4.60)$$

Thus, the function (4.52) attains a minimum with respect to the player's strategy $\mu_k(t, x)$. Using μ_k^* (4.59), the closed loop system for the system equation (2.1) with assumption (3.1) is given as

$$\begin{aligned} dx(t) &= g(x(t))dt - \frac{1}{2} \sum_{i=1}^N B_i(x)R_{ii}^{-1}B_i'(x) \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right) dt \\ &+ \sigma(t, x(t))dw(t). \end{aligned} \quad (4.61)$$

Corollary 4.1: The statistical game in closed-loop form is obtained by substituting (4.59)

into (4.26) to give the Nm -coupled HJB equations for $j = 1$

$$\begin{aligned}
0 = & \frac{\partial V_1^k(t, x)}{\partial t} + \left(g'(x) - \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, x)}{\partial x} \right. \right. \\
& \left. \left. + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \right) \left(\frac{\partial V_1^k(t, x)}{\partial x} \right) + l \\
& + \frac{1}{4} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, x)}{\partial x} + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right)' \\
& B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, x)}{\partial x} + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right) \\
& + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_1^k(t, x)}{\partial x^2} \right), \tag{4.62}
\end{aligned}$$

and then for $j = 2, 3, \dots, m$ gives the following

$$\begin{aligned}
0 = & \frac{\partial V_j^k(t, x)}{\partial t} + \left(g'(x) - \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, x)}{\partial x} \right. \right. \\
& \left. \left. + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \right) \left(\frac{\partial V_j^k(t, x)}{\partial x} \right) + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_j^k(t, x)}{\partial x^2} \right) \\
& + \frac{1}{2} \sum_{s=1}^{j-1} \frac{j!}{s!(j-s)!} \left(\frac{\partial V_s^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j-s}^k(t, x)}{\partial x} \right), \tag{4.63}
\end{aligned}$$

with the terminal condition $V_j^k(t_F, x_F) = 0$ and $j = 1, 2, \dots, m$, $k = 1, 2, \dots, N$. The Nm coupled equations (4.62) and (4.63) are second-order nonlinear partial differential equation which do not have analytical solutions. The minimal positive semi-definite solution of $V_m^{k*}(t, x)$ gives the Nash value.

CHAPTER 5

NEURAL NETWORK SOLUTIONS OF HJB EQUATIONS

This chapter describes the neural network approximate method which is utilized to solve the coupled HJB equations of the statistical game. Section 5.1 gives an introduction to the concept of neural networks while Section 5.2 details the neural network method applied to solve the N -player, m -th cumulant statistical game problem. The proof of convergence of the neural network approximate method solution to exact solution is given in Section 5.3 while the full-state feedback statistical game algorithm is given in the last section.

5.1 Introduction to Neural Networks Method

The analytical solutions of nonlinear HJB equations (4.62) and (4.63) are difficult to find for nonlinear systems. In fact, closed-form solutions for HJB equations are difficult to find except for linear quadratic (LQ) system problems which are linear system with quadratic cost function in state and control. Thus, to solve the HJB equations, we utilize the neural network method in [58] to approximate the players value functions V_j^k , for $k = 1, 2, \dots, N$ and $j = 1, 2, \dots, m$ and then find the solution of the resulting coupled HJB equations. Early neural network applications in control systems were reported in [69] and [70]. Neural networks have been applied successfully in the identification and control of dynamic systems [71]. The nonlinear function approximation properties of neural network makes it a good choice in nonlinear controller design. Chen *et al.* [61] used neural network to solve a generalized HJB equation for a discrete-time nonlinear system with quadratic cost function. Medagam *et al.* [60] applied neural network method and adaptive filter to solve the nonlinear optimal control problem for an unknown system. In [14], the authors applied neural network method to solve an infinite horizon, nonzero-sum non-cooperative game using online adaptive algorithm. We are inspired by

the work in [58], to extend neural network approach to statistical game. In addition, Sandberg showed that uniform continuous time-varying functions can be approximated by neural network functions [72]. Since our value functions $V_j^k(t, x)$ are assumed $C_p^{1,2}(Q) \cap C(\bar{Q})$, the approach in [72] can be utilized to uniformly approximate our value functions in time.

In this dissertation, neural network approach is used as a function approximator to the HJB equation. The value functions are assumed to be smooth positive semi-definite functions with continuous first derivative in time and continuous first and second derivative in state. Thus, an even polynomial activation function will be utilized in this dissertation to satisfy the value function requirements. The approximate value functions are equal to sum of the independent state-based set functions multiplied by their corresponding time-varying weights. In general, the selection of the neural network set functions is based on experience.

5.2 Neural Network Approximate Solution

Consider the first player value function V_j^1 which is assumed $C_p^{1,2}(Q) \cap C(\bar{Q})$. The function V_j^1 for $j = 1, 2, \dots, m$ is approximated as

$$V_j^1(t, x) \approx V_{jL}^1(t, x) = \mathbf{w}'_{j1L}(t) \mathbf{\Lambda}_{j1L}(x) = \sum_{i=1}^L w_{j1i}(t) \Lambda_{j1i}(x), \quad (5.1)$$

where $\mathbf{w}_{j1L}(t)$ and $\mathbf{\Lambda}_{j1L}(x)$ are vectors, and $\mathbf{w}_{j1L}(t) = \{w_{j11}(t), \dots, w_{j1L}(t)\}'$ and $\mathbf{\Lambda}_{j1L}(x) = \{\Lambda_{j11}(x), \dots, \Lambda_{j1L}(x)\}'$ is the neural network activation function vector and L is the number of the hidden layer neurons that models the independent set function $\Lambda_{j1i}(x) : \Omega \rightarrow \mathbb{R}$. For the N -th player, the value function V_j^N for $j = 1, 2, \dots, m$ is approximated as

$$V_j^N(t, x) \approx V_{jL}^N(t, x) = \mathbf{w}'_{jNL}(t) \mathbf{\Lambda}_{jNL}(x) = \sum_{i=1}^L w_{jNi}(t) \Lambda_{jNi}(x), \quad (5.2)$$

where $\mathbf{w}_{jNL}(t)$ and $\mathbf{\Lambda}_{jNL}(x)$ are vectors, and $\mathbf{w}_{jNL}(t) = \{w_{jN1}(t), \dots, w_{jNL}(t)\}'$ and $\mathbf{\Lambda}_{jNL}(x) = \{\Lambda_{jN1}(x), \dots, \Lambda_{jNL}(x)\}'$ is the neural network activation function vector. We use notation $V_j^k(t, x)$ instead of $V_j^k(t, x)$ to emphasize the difference between the approximated value function and the original value function for the k -th player. An even polynomial activation function related to polynomial series expansion will be utilized in this research. Because the polynomial

series expansion is pre-defined and known, the only unknown in the approximated cumulant HJB equation is the weight or coefficients of the series expansion. These weights are time dependent, thus solving the coupled HJB partial differential equation problem becomes finding the weights of the series expansion problem. See Fig. 5.1 for the neural network approximation model. To distinguish the weights for the different value functions, we define the weights

Figure 5.1: Neural Network Approximation Model.

for $V_{jL}^k(t, x)$ as $\mathbf{w}_{jkL}(t)$. For example, we use $V_{1L}^k(t, x) = \mathbf{w}'_{1kL}(t)\Lambda_{1kL}(x)$ to approximate $V_1^k(t, x)$, $V_{2L}^k(t, x) = \mathbf{w}'_{2kL}(t)\Lambda_{2kL}(x)$ to approximate $V_2^k(t, x)$, $V_{3L}^k(t, x) = \mathbf{w}'_{3kL}(t)\Lambda_{3kL}(x)$ to approximate $V_3^k(t, x)$, $\dots$, $V_{mL}^k(t, x) = \mathbf{w}'_{mkL}(t)\Lambda_{mkL}(x)$ to approximate $V_m^{k*}(t, x)$. Then, we use the neural network approximations and substitute $V_{1L}^k(t, x) = \mathbf{w}'_{1kL}(t)\Lambda_{1kL}(x)$, $V_{2L}^k(t, x) = \mathbf{w}'_{2kL}(t)\Lambda_{2kL}(x)$, $V_{3L}^k(t, x) = \mathbf{w}'_{3kL}(t)\Lambda_{3kL}(x)$, $\dots$, $V_{mL}^k(t, x) = \mathbf{w}'_{mkL}(t)\Lambda_{mkL}(x)$ in (4.62)-(4.63).

However, $e_{jkL}(t, x)$ is defined as the j -th cumulant HJB residual equation error resulting from k -th player value function approximation in (4.62)-(4.63), with $j = 1, 2, \dots, m$. Thus,

approximating the k -th player value functions in the cumulant HJB equations (4.62)-(4.63) give for $j = 1$,

$$\begin{aligned}
e_{1kL} = & \frac{\partial V_{1L}^k(t, x)}{\partial t} + g'(x) \left(\frac{\partial V_{1L}^k(t, x)}{\partial x} \right) \\
& - \sum_{i=1}^N \frac{1}{2} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \left(\frac{\partial V_{1L}^k(t, x)}{\partial x} \right) \\
& + \sum_{i=1}^N \frac{1}{4} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \\
& \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right) \\
& + l + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_{1L}^k(t, x)}{\partial x^2} \right), \tag{5.3}
\end{aligned}$$

and for $j = 2, 3, \dots, m$,

$$\begin{aligned}
e_{jkL} = & \frac{\partial V_{jL}^k(t, x)}{\partial t} + g'(x) \left(\frac{\partial V_{jL}^k(t, x)}{\partial x} \right) \\
& - \sum_{i=1}^N \frac{1}{2} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \left(\frac{\partial V_{jL}^k(t, x)}{\partial x} \right) \\
& + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_{jL}^k(t, x)}{\partial x^2} \right) \\
& + \frac{1}{2} \sum_{s=1}^{j-1} \frac{j!}{s!(j-s)!} \left(\frac{\partial V_{sL}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{(j-s)L}^k(t, x)}{\partial x} \right), \tag{5.4}
\end{aligned}$$

with the terminal condition $V_{jL}^k(t_F, x_F) = 0$ and $j = 1, 2, \dots, m, k = 1, 2, \dots, N$.

The neural network-based players strategy from (4.59) is given as

$$\mu_k^*(t, x) = -\frac{1}{2} R_{kk}^{-1} B_k' \left(\frac{\partial V_{1L}^k(t, x)}{\partial x} + \gamma_{2k}(t) \frac{\partial V_{2L}^k(t, x)}{\partial x} + \dots + \gamma_{mk}(t) \frac{\partial V_{mL}^{k*}(t, x)}{\partial x} \right), \tag{5.5}$$

where $k = 1, 2, \dots, N$.

The closed loop system neural network-based equation is given as

$$\begin{aligned}
dx(t) = & g(x(t))dt - \frac{1}{2} \sum_{i=1}^N B_i(x) R_{ii}^{-1} B_i'(x) \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi}(t) \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right) dt \\
& + \sigma(t, x(t))dw(t). \tag{5.6}
\end{aligned}$$

The method of weighted residuals [73] is applied to $e_{jkL}(t, x)$ to find the weight coefficients of the value functions that will minimize the approximation errors $e_{jkL}(t, x)$ over the domain or

set Ω of the value functions. The method of weighted residuals involves forcing the $e_{jkL}(t, x)$ to zero in the least-square sense with respect to the basis function $\Lambda_{jkL}(x)$. In fact, the form of the weighted-residuals is given as

$$\left\langle e_{jkL}(t, x), \Lambda_{jkL}(x) \right\rangle_{\Omega} = 0. \quad (5.7)$$

where $\langle, \rangle$ is an inner product and $\Omega = C_p^{1,2}(Q) \cap C(\bar{Q})$ is the domain where $V_{jL}^i(t, x)$ exists.

Introducing the following relationships and using (5.1),

$$\frac{\partial V_{mL}^i(t, x)}{\partial x} = \frac{\partial \Lambda'_{miL}(x)}{\partial x} \mathbf{w}_{miL}(t) \equiv \nabla \Lambda'_{miL}(x) \mathbf{w}_{miL}(t). \quad (5.8)$$

$$\frac{\partial V_{mL}^i(t, x)}{\partial t} = \frac{\partial \mathbf{w}'_{miL}(t)}{\partial t} \Lambda_{miL}(x) \equiv \dot{\mathbf{w}}'_{miL}(t) \Lambda_{miL}(x). \quad (5.9)$$

$$\left\langle \mathbf{w}'_{miL}(t) \Lambda_{miL}(x), \Lambda_{miL}(x) \right\rangle_{\Omega} \equiv \left\langle \Lambda_{miL}(x), \Lambda_{miL}(x) \right\rangle_{\Omega} \mathbf{w}_{miL}(t). \quad (5.10)$$

For derivation of (5.10), see Appendix A, Lemma A.1. Thus, substitute (5.3) into (5.7), utilizing (5.8) to (5.10) and obtain the following after some algebraic manipulations for $j = 1$

$$\begin{aligned} \dot{\mathbf{w}}_{1kL}(t) = & - \left\langle \Lambda_{1kL}(x), \Lambda_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle \nabla \Lambda_{1kL}(x) g(x), \Lambda_{1kL}(x) \right\rangle_{\Omega} \mathbf{w}_{1kL}(t) \\ & - \left\langle \Lambda_{1kL}(x), \Lambda_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle l, \Lambda_{1kL}(x) \right\rangle_{\Omega} \\ & - \frac{1}{2} \left\langle \Lambda_{1kL}(x), \Lambda_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle \text{tr} \left(\sigma W \sigma' \nabla \left(\nabla \Lambda'_{1kL}(x) \mathbf{w}_{1kL}(t) \right) \right), \Lambda_{1kL}(x) \right\rangle_{\Omega} \\ & + \frac{1}{2} \sum_{i=1}^N \left\langle \Lambda_{1kL}(x), \Lambda_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle \left(\mathbf{w}'_{1iL}(t) \nabla \Lambda_{1iL} + \dots + \gamma_{mi} \mathbf{w}'_{miL}(t) \nabla \Lambda_{miL} \right) \right. \\ & \left. B_i R_{ii}^{-1} B_i' \right) \nabla \Lambda'_{1kL}(x) \mathbf{w}_{1kL}(t), \Lambda_{1kL}(x) \right\rangle_{\Omega} \\ & - \frac{1}{4} \sum_{i=1}^N \left\langle \Lambda_{1kL}(x), \Lambda_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle \left(\mathbf{w}'_{1iL}(t) \nabla \Lambda_{1iL} + \dots + \gamma_{mi} \mathbf{w}'_{miL}(t) \nabla \Lambda_{miL} \right) \right. \\ & \left. B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \left(\mathbf{w}'_{1iL}(t) \nabla \Lambda_{1iL} + \dots + \gamma_{mi} \mathbf{w}'_{miL}(t) \nabla \Lambda_{miL} \right) \right), \Lambda_{1kL}(x) \right\rangle_{\Omega}. \end{aligned} \quad (5.11)$$

Similarly, for $j = 2, 3, \dots, m$, we substitute (5.4) into (5.7), utilizing (5.8) to (5.10) and obtain,

$$\begin{aligned}
\dot{\mathbf{w}}_{jkL}(t) = & - \left\langle \Lambda_{jkL}(x), \Lambda_{jkL}(x) \right\rangle_{\Omega}^{-1} \left\langle \nabla \Lambda_{jkL}(x) g(x), \Lambda_{jkL}(x) \right\rangle_{\Omega} \mathbf{w}_{jkL}(t) \\
& - \frac{1}{2} \left\langle \Lambda_{jkL}(x), \Lambda_{jkL}(x) \right\rangle_{\Omega}^{-1} \left\langle \text{tr} \left(\sigma W \sigma' \nabla \left(\nabla \Lambda'_{jkL}(x) \mathbf{w}_{jkL}(t) \right) \right), \Lambda_{jkL}(x) \right\rangle_{\Omega} \\
& - \frac{1}{2} \sum_{s=1}^{j-1} \frac{j!}{s!(j-s)!} \left\langle \Lambda_{jkL}(x), \Lambda_{jkL}(x) \right\rangle_{\Omega}^{-1} \left\langle \mathbf{w}'_{skL}(t) \nabla \Lambda_{skL}(x) \sigma(t, x) W \sigma(t, x)' \right. \\
& \left. \nabla \Lambda'_{(j-s)kL}(x) \mathbf{w}_{(j-s)kL}(t), \Lambda_{jkL}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \sum_{i=1}^N \left\langle \Lambda_{jkL}(x), \Lambda_{jkL}(x) \right\rangle_{\Omega}^{-1} \left\langle \left(\mathbf{w}'_{1iL}(t) \nabla \Lambda_{1iL} + \dots + \gamma_{ji} \mathbf{w}'_{jiL}(t) \nabla \Lambda_{jiL} \right) \right. \\
& \left. B_i R_{ii}^{-1} B_i' \right) \nabla \Lambda'_{jkL}(x) \mathbf{w}_{jkL}(t), \Lambda_{jkL}(x) \right\rangle_{\Omega},
\end{aligned} \tag{5.12}$$

where the terminal conditions $\mathbf{w}_{1kL}(t_F), \mathbf{w}_{2kL}(t_F), \dots, \mathbf{w}_{mkL}(t_F), k = 1, 2, \dots, N$ are assumed to be zero. Therefore, by knowing the terminal conditions of the above ordinary differential equations (5.11)-(5.12), we solve the k -th player Nash equilibrium m -th cumulant neural network approximate solution along with other $(m-1)$ cumulant equations. $\square$

Remarks: In order to solve the k -th player Nash equilibrium m -th cumulant neural network equation, we need to simultaneously solve all players Nash equilibrium m -th cumulant neural network equations. By using neural network approximate method, we convert the partial differential equations (4.62)-(4.63) into neural network ordinary differential equations (5.11)-(5.12). Then, we solve the approximated ordinary differential equations for $\mathbf{w}_{1kL}(t), \mathbf{w}_{2kL}(t), \dots, \mathbf{w}_{mkL}(t)$ and determine the corresponding Lagrange multipliers $\gamma_{2i}(t), \gamma_{3i}(t), \dots, \gamma_{mi}(t)$.

5.3 Convergence of Neural Network Approximate HJB Equations

In this section, the convergence results of the neural network approximate HJB equation to the exact solution of the coupled HJB equation as the neural network activation size L increases is shown.

Assumption 5.1: $\{\Lambda_{ikj}(x)\}_{j=1}^{\infty}$, form a set of orthonormal functions which are continuous in state x and exist in the domain $\Omega = C_p^{1,2}(Q) \cap C(\bar{Q})$ and $L_2(\Omega)$ with $i = 1, 2, \dots, m$ and $k = 1, 2, \dots, N$.

Assumption 5.2: $V_{iL}^k(t, x) = C_p^{1,2}(Q) \cap C(\bar{Q})$ is approximated by $\{w_{ikj}\Lambda_{ikj}(x)\}_{j=1}^L$.

Assumption 5.3: $V_i^k(t, x)$ satisfies (4.62)-(4.63) for $i = 1, 2, \dots, m$ and $k = 1, 2, \dots, N$.

Assumption 5.4: The approximated neural network value function $V_{iL}^k(t, x)$ satisfies the following condition, $\langle e_{ikL}(t, x), \Lambda_{ikL}(x) \rangle_\Omega = [0]_{L \times 1}$, where e_{ikL} is the HJB approximation error due to neural network method with $i = 1, 2, \dots, m$ and $k = 1, 2, \dots, N$.

Lemma 5.1: From assumption 5.1, $\{\Lambda_{ikj}(x)\}_1^\infty$ on Ω form an orthonormal basis functions. For a function $\psi_1(t, x) \in L_2$ on domain Ω then the following property holds,

$$\psi_1(t, x) = \sum_{j=1}^{\infty} \langle \psi_1(t, x), \Lambda_{ikj}(x) \rangle_\Omega \Lambda_{ikj}(x),$$

where the series converges pointwise, thus for $\varepsilon > 0$, $t_0 \in [t_0, t_F]$ and $x \in \Omega$, there exists a large positive value L , such that

$$\left| \sum_{j=L+1}^{\infty} \langle \psi_1(t, x), \Lambda_{ikj}(x) \rangle_\Omega \Lambda_{ikj}(x) \right| < \varepsilon.$$

Proof: See [74], [53].

Theorem 5.1: Let assumption 5.1 to assumption 5.4 be satisfied. Assume the neural network approximation for the k -th player Nash equilibrium first cumulant HJB given in (5.3). Then, as L approaches infinity, $|e_{1kL}(t, x)| \rightarrow 0$.

Proof: The first cumulant HJB neural network approximation (5.3) for the k -th player-strategy Nash equilibrium is given as

$$\begin{aligned} e_{1kL} = & \left(\frac{\partial V_{1L}^k(t, x)}{\partial t} \right) + g'(x) \left(\frac{\partial V_{1L}^k(t, x)}{\partial x} \right) - \sum_{i=1}^N \frac{1}{2} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots \right. \\ & + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \Big)' B_i R_{ii}^{-1} B_i' \left(\frac{\partial V_{1L}^k(t, x)}{\partial x} \right) \\ & + \sum_{i=1}^N \frac{1}{4} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \\ & \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right) + l + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_{1L}^k(t, x)}{\partial x^2} \right). \end{aligned} \quad (5.13)$$

From Lemma 5.1, we have the following

$$\begin{aligned}
e_{1kL}(t, x) &= \sum_{j=1}^{\infty} \left\langle e_{1kL}(t, x), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \\
&= \sum_{j=1}^L \left\langle e_{1kL}(t, x), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) + \sum_{j=L+1}^{\infty} \left\langle e_{1kL}(t, x), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x).
\end{aligned} \tag{5.14}$$

Substituting $V_{1kL}(t, x) = \mathbf{w}'_{1kL}(t) \Lambda_{1kL}(x)$ defined in (5.1) into (5.14) and applying assumptions 5.2 and assumption 5.4 to (5.13), we have the following

$$\begin{aligned}
e_{1kL}(t, x) &= 0 + \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \Lambda_{1kp}(x), \Lambda_{1kj}(x) \right\rangle_{\Omega} \dot{w}_{1kp}(t) \right) \Lambda_{1kj}(x) \\
&\quad + \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \nabla \Lambda_{1kp}(x) g(x), \Lambda_{1kj}(x) \right\rangle_{\Omega} w_{1kp}(t) \right) \Lambda_{1kj}(x) \\
&\quad + \sum_{j=L+1}^{\infty} \left\langle l, \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \\
&\quad + \frac{1}{2} \sum_{j=L+1}^{\infty} \left\langle \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \nabla \left(\nabla \Lambda'_{1kL}(x) \mathbf{w}_{1kL}(t) \right) \right), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \\
&\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j=L+1}^{\infty} \sum_{p=1}^L \left\langle \left(\left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{mi} w'_{mip}(t) \nabla \Lambda_{mip} \right) \right. \right. \\
&\quad \left. \left. B_i(t, x) R_{ii}^{-1} B'_i(t, x) \right) \nabla \Lambda'_{1kp}(x) w_{1kp}(t), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \\
&\quad + \frac{1}{4} \sum_{i=1}^N \sum_{j=L+1}^{\infty} \sum_{p=1}^L \left\langle \left(\left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{ni} w'_{nip}(t) \nabla \Lambda_{nip} \right) \right. \right. \\
&\quad \left. \left. B_i(t, x) R_{ii}^{-1} R_{ki} R_{ii}^{-1} B'_i(t, x) \right. \right. \\
&\quad \left. \left. \left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{mi} w'_{mip}(t) \nabla \Lambda_{mip} \right) \right), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x).
\end{aligned} \tag{5.15}$$

With $\left\langle \Lambda_{1kp}, \Lambda_{1kj} \right\rangle_{\Omega} = 0$ because Λ_{1kp} and Λ_{1kj} are orthonormal according to Assumption 5.1 thus,

$$\sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \Lambda_{1kp}(x), \Lambda_{1kj}(x) \right\rangle_{\Omega} \dot{w}_{1kp}(t) \right) \Lambda_{1kj}(x) = 0.$$

Then according to triangle inequality, we have

$$\begin{aligned}
|e_{1kL}(t, x)| \leq & \left| \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \nabla \Lambda_{1kp}(x) g(x), \Lambda_{1kj}(x) \right\rangle_{\Omega} w_{1kp}(t) \right) \Lambda_{1kj}(x) \right| \\
& + \left| \sum_{j=L+1}^{\infty} \left\langle l, \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \right| \\
& + \left| \frac{1}{2} \sum_{i=1}^N \sum_{j=L+1}^{\infty} \sum_{p=1}^L \left\langle \left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{mi} w'_{mip}(t) \nabla \Lambda_{mip} \right) \right. \right. \\
& \left. \left. B_i(t, x) R_{ii}^{-1} B'_i(t, x) \right) \nabla \Lambda'_{1kp}(x) w_{1kp}(t), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \right| \\
& + \left| \frac{1}{4} \sum_{i=1}^N \sum_{j=L+1}^{\infty} \sum_{p=1}^L \left\langle \left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{mi} w'_{mip}(t) \nabla \Lambda_{mip} \right) B_i(t, x) R_{ii}^{-1} \right. \right. \\
& \left. \left. R_{ki} R_{ii}^{-1} B'_i(t, x) \left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{mi} w'_{mip}(t) \nabla \Lambda_{mip} \right) \right), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \right| \\
& + \left| \frac{1}{2} \sum_{j=L+1}^{\infty} \left\langle \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \nabla \left(\nabla \Lambda'_{1kL}(x) \mathbf{w}_{1kL}(t) \right) \right), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \right|.
\end{aligned} \tag{5.16}$$

The first four terms on the right-hand side of (5.16), are the same for deterministic system [58]. The results from [58] show that for deterministic systems, as $L \rightarrow \infty$, these six terms converge to zero. However, the last term on the right-hand side of (5.16), exists only in stochastic systems. We prove that this last term converges to zero as $L \rightarrow \infty$. Thus (5.16), becomes

$$|e_{1kL}(t, x)| = \left| \frac{1}{2} \sum_{j=L+1}^{\infty} \left\langle \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \nabla \left(\nabla \Lambda'_{1kL}(x) \mathbf{w}_{1kL}(t) \right) \right), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \right|. \tag{5.17}$$

Using Lemma 5.1 and the trace formula, (5.17) becomes

$$|e_{1kL}(t, x)| = \left| \frac{1}{2} \sum_{j=L+1}^{\infty} \left\langle \left(\sigma(t, x) W \sigma(t, x)' \nabla^2 \Lambda'_{1kL}(x) \mathbf{w}_{1kL}(t) \right), \Lambda_{1kj}(x) \right\rangle_{\Omega} \Lambda_{1kj}(x) \right|. \tag{5.18}$$

□

Remark: Therefore (5.18) tends to zero as L approaches infinity using Lemma 5.1. We have $|e_{1kL}(t, x)| \rightarrow 0$ as $L \rightarrow \infty$ with $\nabla^2 \Lambda'_{1kL}(x) \in L_2(\Omega)$. Therefore the approximated neural

network equation converges to the original first cumulant HJB equation.

Theorem 5.2: Let assumption 5.1 to assumption 5.4 be satisfied. Assume the neural network approximation for the k -th player Nash equilibrium m -th cumulant HJB given in (5.4). Then, as L approaches infinity, $|e_{mkL}(t, x)| \rightarrow 0$.

Proof: The m -th cumulant HJB neural network approximation (5.4) for the k -th player-strategy Nash equilibrium is given as

$$\begin{aligned}
e_{mkL} = & \left(\frac{\partial V_{mL}^k(t, x)}{\partial t} \right) + g'(x) \left(\frac{\partial V_{mL}^k(t, x)}{\partial x} \right) - \sum_{i=1}^N \frac{1}{2} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots \right. \\
& + \left. \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \left(\frac{\partial V_{mL}^k(t, x)}{\partial x} \right) \\
& + \frac{1}{2} \text{tr} \left(\sigma W \sigma' \frac{\partial^2 V_{mL}^k(t, x)}{\partial x^2} \right) + \frac{1}{2} \sum_{s=1}^{m-1} \frac{m!}{s!(m-s)!} \left(\frac{\partial V_{sL}^k(t, x)}{\partial x} \right)' \sigma W \sigma' \left(\frac{\partial V_{(m-s)L}^k(t, x)}{\partial x} \right).
\end{aligned} \tag{5.19}$$

From Lemma 5.1, we have the following

$$\begin{aligned}
e_{mkL}(t, x) &= \sum_{j=1}^{\infty} \left\langle e_{mkL}(t, x), \Lambda_{mkj}(x) \right\rangle_{\Omega} \Lambda_{mkj}(x) \\
&= \sum_{j=1}^L \left\langle e_{mkL}(t, x), \Lambda_{mkj}(x) \right\rangle_{\Omega} \Lambda_{mkj}(x) + \sum_{j=L+1}^{\infty} \left\langle e_{mkL}(t, x), \Lambda_{mkj}(x) \right\rangle_{\Omega} \Lambda_{mkj}(x).
\end{aligned} \tag{5.20}$$

Substituting $V_{mL}^k(t, x) = \mathbf{w}'_{mkL}(t) \Lambda_{mkL}(x)$ into (5.20), and applying assumption 5.2 and as-

sumption 5.4 to (5.20), we have the following

$$\begin{aligned}
e_{mkL}(t, x) = & 0 + \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \nabla \Lambda_{mkp}(x) g(x), \Lambda_{mkj}(x) \right\rangle_{\Omega} w_{mkp}(t) \right) \Lambda_{mkj}(x) \\
& + \frac{1}{2} \sum_{j=L+1}^{\infty} \left\langle \text{tr} \left(\sigma W \sigma' \nabla \left(\nabla \Lambda'_{mkL}(x) \mathbf{w}_{mkL}(t) \right) \right), \Lambda_{mkj}(x) \right\rangle_{\Omega} \Lambda_{mkj}(x) \\
& + \frac{1}{2} \sum_{s=1}^{m-1} \frac{m!}{s!(m-s)!} \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle w'_{skp}(t) \nabla \Lambda_{skp}(x) \sigma(t, x) W \sigma(t, x)' \right. \right. \\
& \left. \left. \nabla \Lambda'_{(m-s)kp}(x) w_{(m-s)kp}(t), \Lambda_{mkj}(x) \right\rangle_{\Omega} \right) \Lambda_{mkj}(x) \\
& + \frac{1}{2} \sum_{i=1}^N \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{mi} w'_{mip}(t) \nabla \Lambda_{mip} \right) \right. \right. \\
& \left. \left. B_i(t, x) R_{ii}^{-1} B_i'(t, x) \right) \nabla \Lambda'_{mkp}(x) w_{mkp}(t), \Lambda_{mkj}(x) \right\rangle_{\Omega} \right) \Lambda_{mkj}(x)
\end{aligned} \tag{5.21}$$

Then, we have by triangle inequality,

$$\begin{aligned}
|e_{mkL}(t, x)| \leq & \left| \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \nabla \Lambda_{mkp}(x) g(x), \Lambda_{mkj}(x) \right\rangle_{\Omega} w_{mkp}(t) \right) \Lambda_{mkj}(x) \right| \\
& + \left| \frac{1}{2} \sum_{j=L+1}^{\infty} \left\langle \text{tr} \left(\sigma W \sigma' \nabla \left(\nabla \Lambda'_{mkL}(x) \mathbf{w}_{mkL}(t) \right) \right), \Lambda_{mkj}(x) \right\rangle_{\Omega} \Lambda_{mkj}(x) \right| \\
& + \left| \frac{1}{2} \sum_{s=1}^{m-1} \frac{m!}{s!(m-s)!} \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle w'_{skp}(t) \nabla \Lambda_{skp}(x) \sigma(t, x) W \sigma(t, x)' \right. \right. \right. \\
& \left. \left. \nabla \Lambda'_{(m-s)kp}(x) w_{(m-s)kp}(t), \Lambda_{mkj}(x) \right\rangle_{\Omega} \right) \Lambda_{mkj}(x) \right| \\
& + \left| \frac{1}{2} \sum_{i=1}^N \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{mi} w'_{mip}(t) \nabla \Lambda_{mip} \right) \right. \right. \right. \\
& \left. \left. B_i(t, x) R_{ii}^{-1} B_i'(t, x) \right) \nabla \Lambda'_{mkp}(x) w_{mkp}(t), \Lambda_{mkj}(x) \right\rangle_{\Omega} \right) \Lambda_{mkj}(x) \Big|,
\end{aligned} \tag{5.22}$$

which can be further written as

$$\begin{aligned}
|e_{mkL}(t, x)| \leq & \max_{1 \leq p \leq L} |w_{mkp}(t)| \sup_{(t, x) \in \Omega} \left| \sum_{p=1}^L \left(\sum_{j=L+1}^{\infty} \left\langle \nabla \Lambda_{mkp}(x) g(x), \Lambda_{mkj}(x) \right\rangle_{\Omega} \Lambda_{mkj}(x) \right) \right| \\
& + \left| \frac{1}{2} \sum_{j=L+1}^{\infty} \left\langle \text{tr} \left(\sigma W \sigma' \nabla \left(\nabla \Lambda'_{mkL}(x) \mathbf{w}_{mkL}(t) \right) \right), \Lambda_{mkj}(x) \right\rangle_{\Omega} \Lambda_{mkj}(x) \right| \\
& + \frac{1}{2} \sum_{s=1}^{m-1} \frac{m!}{s!(m-s)!} \max_{1 \leq p \leq L} |w'_{skp}(t) w_{(m-s)kp}(t)| \sup_{(t, x) \in \Omega} \left| \sum_{p=1}^L \right. \\
& \left. \left(\sum_{j=L+1}^{\infty} \left\langle \nabla \Lambda_{skp}(x) \sigma(t, x) W \sigma(t, x)' \nabla \Lambda'_{(m-s)kp}(x), \Lambda_{mkj}(x) \right\rangle_{\Omega} \Lambda_{mkj}(x) \right) \right| \\
& + \left| \frac{1}{2} \sum_{i=1}^N \sum_{j=L+1}^{\infty} \left(\sum_{p=1}^L \left\langle \left(w'_{1ip}(t) \nabla \Lambda_{1ip} + \dots + \gamma_{ni} w'_{mip}(t) \nabla \Lambda_{mip} \right) \right. \right. \right. \\
& \left. \left. \left. B_i(t, x) R_{ii}^{-1} B_i'(t, x) \right) \nabla \Lambda'_{mkp}(x) w_{mkp}(t), \Lambda_{mkj}(x) \right\rangle_{\Omega} \right) \Lambda_{mkj}(x) \right|.
\end{aligned} \tag{5.23}$$

Because $e_{mkL}(t, x) \in L_2$, then $\nabla \Lambda_{mkp}(x) g(x), \text{tr} \left(\sigma W \sigma' \nabla \left(\nabla \Lambda'_{mkL}(x) \mathbf{w}_{mkL}(t) \right) \right),$

$$\nabla \Lambda_{mip}(x) B_i R_{ii}^{-1} B_i' \nabla \Lambda_{mkp}(x), \nabla \Lambda_{skp}(x) \sigma W \sigma' \nabla \Lambda_{(m-s)kp}(x)$$

belong to L_2 . Then according to assumption 5.1 and Lemma 5.1, we know that all terms on the right hand side of (5.23) converge to zero when L approaches infinity.

Thus theorem is proved. $\square$

Remark: This theorem shows that the approximation error between k -th player Nash equilibrium neural network approximation and the original cumulant HJB equations (4.62)-(4.63) can be made infinitesimally small by properly choosing the approximation order L . This choice of L involves experience.

5.4 Full-State Feedback Statistical Game Algorithm

In this section, a summary of the full-state feedback statistical game algorithm for computing the value functions, state and control strategies is given as follows:

Step 1: Define a mathematical model for the N -player system (2.1), (3.1), define the cost (2.6) and running cost (3.2) for the system and the m -th value function for the N -player system

(2.9). See equations below.

$$dx(t) = f(t, x(t), u_1(t), u_2(t), \dots, u_N(t))dt + \sigma(t, x(t))dw(t), \quad (5.24)$$

$$f(t, x(t), u_1(t), u_2(t), \dots, u_N(t)) = g(t, x(t)) + \sum_{i=1}^N B_i(t, x(t))u_i(t), \quad (5.25)$$

$$J^k(t, x(t), \mu_1, \mu_2, \dots, \mu_N) = \int_t^{t_F} L^k(s, x(s), \mu_1, \mu_2, \dots, \mu_N)ds + \psi^k(x(t_F), t_F), \quad (5.26)$$

$$L^k = l(x) + \sum_{i=1}^N \mu'_i(x) R_{ki}(t) \mu_i(x), \quad (5.27)$$

$$V_m^k(t, x) = M_m^k(t, x) - \sum_{i=0}^{m-2} \frac{(m-1)!}{i!(m-1-i)!} M_{m-1-i}^k(t, x) V_{i+1}^k(t, x), \quad (5.28)$$

Step 2: Formulate the closed-loop m -th cost cumulant equations for the N -player system (4.62)-(4.63). See equations below.

$$\begin{aligned} 0 = & \frac{\partial V_1^k(t, x)}{\partial t} + \left(g'(x) - \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, x)}{\partial x} \right. \right. \\ & \left. \left. + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \right) \left(\frac{\partial V_1^k(t, x)}{\partial x} \right) + l \\ & + \frac{1}{4} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, x)}{\partial x} + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right)' \\ & B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, x)}{\partial x} + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right) \\ & + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_1^k(t, x)}{\partial x^2} \right), \end{aligned} \quad (5.29)$$

and then for $j = 2, 3, \dots, m$ gives the following

$$\begin{aligned} 0 = & \frac{\partial V_j^k(t, x)}{\partial t} + \left(g'(x) - \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, x)}{\partial x} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, x)}{\partial x} \right. \right. \\ & \left. \left. + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \right) \left(\frac{\partial V_j^k(t, x)}{\partial x} \right) + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_j^k(t, x)}{\partial x^2} \right) \\ & + \frac{1}{2} \sum_{s=1}^{j-1} \frac{j!}{s!(j-s)!} \left(\frac{\partial V_s^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{j-s}^k(t, x)}{\partial x} \right), \end{aligned} \quad (5.30)$$

Step 3: Formulate the neural network approximating functions for the value functions for the N -player system (5.1), (5.2). See equations below.

$$V_j^1(t, x) \approx V_{jL}^1(t, x) = \mathbf{w}'_{j1L}(t) \mathbf{\Lambda}_{j1L}(x) = \sum_{i=1}^L w_{j1i}(t) \Lambda_{j1i}(x), \quad (5.31)$$

$$V_j^N(t, x) \approx V_{jL}^N(t, x) = \mathbf{w}'_{jNL}(t) \mathbf{\Lambda}_{jNL}(x) = \sum_{i=1}^L w_{jNi}(t) \Lambda_{jNi}(x). \quad (5.32)$$

Step 4: Formulate the m -th cost cumulant equations using the neural network approximating functions for the value functions for the N -player system (5.3)-(5.4). See equations below.

$$\begin{aligned} e_{1kL} = & \frac{\partial V_{1L}^k(t, x)}{\partial t} + g'(x) \left(\frac{\partial V_{1L}^k(t, x)}{\partial x} \right) \\ & - \sum_{i=1}^N \frac{1}{2} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \left(\frac{\partial V_{1L}^k(t, x)}{\partial x} \right) \\ & + \sum_{i=1}^N \frac{1}{4} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \\ & \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right) \\ & + l + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_{1L}^k(t, x)}{\partial x^2} \right), \end{aligned} \quad (5.33)$$

and for $j = 2, 3, \dots, m$,

$$\begin{aligned} e_{jkL} = & \frac{\partial V_{jL}^k(t, x)}{\partial t} + g'(x) \left(\frac{\partial V_{jL}^k(t, x)}{\partial x} \right) \\ & - \sum_{i=1}^N \frac{1}{2} \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi} \frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right)' B_i R_{ii}^{-1} B_i' \left(\frac{\partial V_{jL}^k(t, x)}{\partial x} \right) \\ & + \frac{1}{2} \text{tr} \left(\sigma(t, x) W \sigma(t, x)' \frac{\partial^2 V_{jL}^k(t, x)}{\partial x^2} \right) \\ & + \frac{1}{2} \sum_{s=1}^{j-1} \frac{j!}{s!(j-s)!} \left(\frac{\partial V_{sL}^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_{(j-s)L}^k(t, x)}{\partial x} \right). \end{aligned} \quad (5.34)$$

Step 5: Apply weighted-residuals method (5.7) to the m -th cost cumulant neural-network-based equations (5.3)-(5.4) for the N -player system to form the following equations (5.11)-(5.12). See equations (5.7), (5.11)-(5.12) below, (5.3)-(5.4) were given in step 4.

$$\left\langle e_{jkL}(t, x), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega} = 0. \quad (5.35)$$

$$\begin{aligned}
\dot{\mathbf{w}}_{1kL}(t) = & - \left\langle \mathbf{\Lambda}_{1kL}(x), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle \nabla \mathbf{\Lambda}_{1kL}(x) g(x), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega} \mathbf{w}_{1kL}(t) \\
& - \left\langle \mathbf{\Lambda}_{1kL}(x), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle l, \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega} \\
& - \frac{1}{2} \left\langle \mathbf{\Lambda}_{1kL}(x), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle \text{tr} \left(\sigma W \sigma' \nabla \left(\nabla \mathbf{\Lambda}'_{1kL}(x) \mathbf{w}_{1kL}(t) \right) \right), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \sum_{i=1}^N \left\langle \mathbf{\Lambda}_{1kL}(x), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle \left(\mathbf{w}'_{1iL}(t) \nabla \mathbf{\Lambda}_{1iL} + \dots + \gamma_{mi} \mathbf{w}'_{miL}(t) \nabla \mathbf{\Lambda}_{miL} \right) \right. \\
& \left. B_i R_{ii}^{-1} B_i' \right\rangle \nabla \mathbf{\Lambda}'_{1kL}(x) \mathbf{w}_{1kL}(t), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega} \\
& - \frac{1}{4} \sum_{i=1}^N \left\langle \mathbf{\Lambda}_{1kL}(x), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega}^{-1} \left\langle \left(\mathbf{w}'_{1iL}(t) \nabla \mathbf{\Lambda}_{1iL} + \dots + \gamma_{mi} \mathbf{w}'_{miL}(t) \nabla \mathbf{\Lambda}_{miL} \right) \right. \\
& \left. B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \left(\mathbf{w}'_{1iL}(t) \nabla \mathbf{\Lambda}_{1iL} + \dots + \gamma_{mi} \mathbf{w}'_{miL}(t) \nabla \mathbf{\Lambda}_{miL} \right), \mathbf{\Lambda}_{1kL}(x) \right\rangle_{\Omega}.
\end{aligned} \tag{5.36}$$

and then for $j = 2, 3, \dots, m$ gives the following

$$\begin{aligned}
\dot{\mathbf{w}}_{jkL}(t) = & - \left\langle \mathbf{\Lambda}_{jkL}(x), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega}^{-1} \left\langle \nabla \mathbf{\Lambda}_{jkL}(x) g(x), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega} \mathbf{w}_{jkL}(t) \\
& - \frac{1}{2} \left\langle \mathbf{\Lambda}_{jkL}(x), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega}^{-1} \left\langle \text{tr} \left(\sigma W \sigma' \nabla \left(\nabla \mathbf{\Lambda}'_{jkL}(x) \mathbf{w}_{jkL}(t) \right) \right), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega} \\
& - \frac{1}{2} \sum_{s=1}^{j-1} \frac{j!}{s!(j-s)!} \left\langle \mathbf{\Lambda}_{jkL}(x), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega}^{-1} \left\langle \mathbf{w}'_{skL}(t) \nabla \mathbf{\Lambda}_{skL}(x) \sigma(t, x) W \sigma(t, x)' \right. \\
& \left. \nabla \mathbf{\Lambda}'_{(j-s)kL}(x) \mathbf{w}_{(j-s)kL}(t), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \sum_{i=1}^N \left\langle \mathbf{\Lambda}_{jkL}(x), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega}^{-1} \left\langle \left(\mathbf{w}'_{1iL}(t) \nabla \mathbf{\Lambda}_{1iL} + \dots + \gamma_{ji} \mathbf{w}'_{jiL}(t) \nabla \mathbf{\Lambda}_{jiL} \right) \right. \\
& \left. B_i R_{ii}^{-1} B_i' \right\rangle \nabla \mathbf{\Lambda}'_{jkL}(x) \mathbf{w}_{jkL}(t), \mathbf{\Lambda}_{jkL}(x) \right\rangle_{\Omega},
\end{aligned} \tag{5.37}$$

Step 6: Solve equations (5.11)-(5.12) for the weights of the value functions of the N -player system. If the weights are not converging go to step 3. However, if the weights converge, increase the size L of the neural network function and compare convergence time for different sizes of L .

Step 7: Using the weights in step 6, compute the value functions trajectory (5.1), (5.2) for the N -player system.

Step 8: Form the players strategies using (4.61). See equation below.

$$\mu_k^*(t, x) = -\frac{1}{2}R_{kk}^{-1}B'_k \left(\frac{\partial V_{1L}^k(t, x)}{\partial x} + \gamma_{2k}(t)\frac{\partial V_{2L}^k(t, x)}{\partial x} + \dots + \gamma_{mk}(t)\frac{\partial V_{mL}^{k*}(t, x)}{\partial x} \right), \quad (5.38)$$

Step 9: Form the system closed loop equation from (5.6). See equation below.

$$\begin{aligned} dx(t) = & g(x(t))dt - \frac{1}{2} \sum_{i=1}^N B_i(x)R_{ii}^{-1}B'_i(x) \left(\frac{\partial V_{1L}^i(t, x)}{\partial x} + \dots + \gamma_{mi}(t)\frac{\partial V_{mL}^{i*}(t, x)}{\partial x} \right) dt \\ & + \sigma(t, x(t))dw(t). \end{aligned} \quad (5.39)$$

CHAPTER 6

FULL-STATE FEEDBACK APPLICATION EXAMPLES

In this chapter, N -player, m -th cost cumulant statistical differential game problem will be solved using the neural network approximation method. In these illustrative examples, the system dynamics are challenging due to the nonlinearity and uncertainty associated with the dynamics.

6.1 Linear Systems

6.1.1 Simple Linear System

Consider a linear dynamic equation given in [75] as

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) + 2x_2(t) + u_1(t) + 0.5u_2(t) + u_3(t), \\ \dot{x}_2(t) &= x_1(t) + u_1(t) + u_2(t) + 0.5u_3(t), \\ \dot{x}_3(t) &= x_2(t) - x_3(t),\end{aligned}\tag{6.1}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the states and $u_1(t)$, $u_2(t)$, $u_3(t)$ are the controls (first, second and third players). The initial state condition is given as $x(t_0) = x(0) = x_0 = [0.5 \ 0.5 \ 0.5]'$. The system (6.1) is a deterministic system, thus we introduce Gaussian noise into the system as process noise. The stochastic system is then represented as

$$dx(t) = Ax(t)dt + B_1u_1(t)dt + B_2u_2(t)dt + B_3u_3(t)dt + Edw(t),\tag{6.2}$$

where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix},$$

and the state variable $x(t)$ is defined as: $x(t) = [x_1(t) \ x_2(t) \ x_3(t)]'$. We assume that E in (6.2) is a 3×1 constant vector given as $E = [1 \ 1 \ 1]'$ and $dw(t)$ in (6.2) as a Gaussian process with mean $E\{dw(t)\} = 0$ and covariance $E\{dw(t)dw(t)'\} = 0.01$. In this example, we study a 3-player 2^{nd} cumulant full-state feedback Nash game with the aim to stabilize the system. Here, we compute the equilibrium solution for the players through solving the minimal second cumulant game problem constrained on the first cumulant cost function.

The first player cost function J^1 is

$$J^1(t_0, x(t), u_1(t), u_2(t), u_3(t)) = \int_{t_0}^{t_F} \{x_1^2(t) + x_2^2(t) + x_3^2(t) + 2u_1^2(t) + 0.5u_2^2(t) + 0.5u_3^2(t)\}dt + \psi^1(x(t_F), t_F), \quad (6.3)$$

where $\psi^1(x(t_F), t_F) = 0$ is the terminal cost and the second player cost function J^2 is

$$J^2(t_0, x(t), u_1(t), u_2(t), u_3(t)) = \int_{t_0}^{t_F} \{x_1^2(t) + x_2^2(t) + x_3^2(t) + u_1^2(t) + 2u_2^2(t) + u_3^2(t)\}dt + \psi^2(x(t_F), t_F), \quad (6.4)$$

where $\psi^2(x(t_F), t_F) = 0$ is the terminal cost and the third player cost function J^3 is

$$J^3(t_0, x(t), u_1(t), u_2(t), u_3(t)) = \int_{t_0}^{t_F} \{x_1^2(t) + x_2^2(t) + x_3^2(t) + 0.5u_1^2(t) + 0.5u_2^2(t) + u_3^2(t)\}dt + \psi^3(x(t_F), t_F), \quad (6.5)$$

where $\psi^3(x(t_F), t_F) = 0$ is the terminal cost. The input functions $\Lambda_L(x)$ for the value functions of the players are the same and based on [61] which are formulated as

$$\Lambda_L(x) = \sum_{i=1}^{\frac{M}{2}} \left(\sum_{j=1}^n x_j \right)^{2i}. \quad (6.6)$$

Where in (6.6), M is an even-order of the approximation, L is the number of neural network neurons, n is the system dimension.

The input function $\Lambda_L(x)$ is

$$\Lambda_L = \{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3\}' . \quad (6.7)$$

The first player value functions V_{1L}^1, V_{2L}^1 are

$$V_{1L}^1(x_1, x_2, x_3) = w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + w_4x_1x_2 + w_5x_1x_3 + w_6x_2x_3. \quad (6.8)$$

$$V_{2L}^1(x_1, x_2, x_3) = w_7x_1^2 + w_8x_2^2 + w_9x_3^2 + w_{10}x_1x_2 + w_{11}x_1x_3 + w_{12}x_2x_3. \quad (6.9)$$

Similarly, the second player value functions V_{1L}^2, V_{2L}^2 are

$$V_{1L}^2(x_1, x_2, x_3) = w_{13}x_1^2 + w_{14}x_2^2 + w_{15}x_3^2 + w_{16}x_1x_2 + w_{17}x_1x_3 + w_{18}x_2x_3. \quad (6.10)$$

$$V_{2L}^2(x_1, x_2, x_3) = w_{19}x_1^2 + w_{20}x_2^2 + w_{21}x_3^2 + w_{22}x_1x_2 + w_{23}x_1x_3 + w_{24}x_2x_3. \quad (6.11)$$

Finally, the third player value functions V_{1L}^3, V_{2L}^3 are

$$V_{1L}^3(x_1, x_2, x_3) = w_{25}x_1^2 + w_{26}x_2^2 + w_{27}x_3^2 + w_{28}x_1x_2 + w_{29}x_1x_3 + w_{30}x_2x_3. \quad (6.12)$$

$$V_{2L}^3(x_1, x_2, x_3) = w_{31}x_1^2 + w_{32}x_2^2 + w_{33}x_3^2 + w_{34}x_1x_2 + w_{35}x_1x_3 + w_{36}x_2x_3. \quad (6.13)$$

We selected as polynomial function (6.7) of up to second-order ($M = 2$) of approximation in the state variable (i.e x is 2^{nd} order) with length of the neural network neurons $L = 6$. Higher order polynomial did not provide significant improvement in the approximation accuracy. The first, second and third player value functions are substituted to (5.3)-(5.4), see Appendix B, Lemma B.3 for the expanded six coupled equations that are solved for the value functions. In the simulation, the asymptotic stability region for states was arbitrarily chosen as $-1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1$ and $-1 \leq x_3 \leq 1$. The final time t_F was 10 s and $\mathbf{w}'_{11L}(t_F) - \mathbf{w}'_{21L}(t_F) = \{0\}$, $\mathbf{w}'_{12L}(t_F) - \mathbf{w}'_{22L}(t_F) = \{0\}$ and $\mathbf{w}'_{13L}(t_F) - \mathbf{w}'_{23L}(t_F) = \{0\}$ and we integrate backwards in time to solve for the optimal weights.

From Figs. 6.1(a) to 6.1(f), the neural network weights converge to constants, which are the optimal weights when integrated backwards in (5.3)-(5.4). Figs. 6.2(a) to 6.2(f) show the first and second value cumulant functions for all players. From Figs. 6.2(a), it is observed that the value function V_1^1 decreases with increases in γ_{23} up to 15 and then increases with an

(a) 1st Player Neural Network First Cumulant Weights

(b) 1st Player Neural Network Second Cumulant Weights

(c) 2nd Player Neural Network First Cumulant Weights

(d) 2nd Player Neural Network Second Cumulant Weights

(e) 3rd Player Neural Network First Cumulant Weights

(f) 3rd Player Neural Network Second Cumulant Weights

Figure 6.1: Three Player Linear Nash Game System Neural Network Weights.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{23}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{23}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{23}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{23}

(e) 3rd Player First Cumulant Value Function - V_1^3 versus γ_{23}

(f) 3rd Player Second Cumulant Value Function - V_2^3 versus γ_{23}

Figure 6.2: Three Player Linear Nash Game System Value Functions.

(a) State trajectory at initial condition $x_0 = (0.5, 0.5, 0.5)$ (b) Control trajectory at initial condition $x_0 = (0.5, 0.5, 0.5)$

Figure 6.3: Three Player Linear Nash Game System States and Optimal Control Strategy Trajectory.

increase in γ_{23} . From Figs 6.2(c) and 6.2(e), it is observed that the value functions V_1^2 and V_1^3 slightly increases when the value of γ_{23} increases. From Figs. 6.2(b), 6.2(d) and 6.2(f) the value functions show a decrease with increase in γ_{23} for the higher-order cumulants. Fig. 6.3(a) shows the state trajectory for noise influence with variance $\sigma^2 = 0.01$. The states are bounded and converged to value close to the origin. The Lagrange multipliers γ_{xx} were selected as constants.

The Nash equilibrium controls, u_1 , u_2 and u_3 are shown in Fig. 6.3(b). It should be noted from Fig. 6.3(b), that the Nash equilibrium controls for the three players were solved for the 2^{nd} cumulant game by selecting γ_{21} , γ_{22} , γ_{23} where the value functions are minimum which in our case were $\gamma_{21} = 0.1$, $\gamma_{22} = 0.1$ and $\gamma_{23} = 30$ as shown in Figs. 6.2(b), 6.2(d) and 6.2(f). In addition, we have the design freedom in γ_{xx} values selection to enhance system performance.

Remark: The second cumulant Nash strategy is found within all admissible first cumulant strategy. The states and controls are obtained by a forward integration of the state equation using the optimal weights. The neural network functions size was varied until convergence in the weights was observed. To find the first cumulant Nash strategy, set $\gamma_{21} = \gamma_{22} = \gamma_{23} = 0$ in (5.3)-(5.4).

6.1.2 Laser Pointing System

In this example, the statistical game approach is studied for a linearized stochastic dynamic system model given in [76]. The pitch axis model which is a part of the two-axis gimbal dynamics for the laser system that is based on rigid body dynamics is considered. In the linear laser model, the servo-motor torque input is modeled as the first player while the disturbance torque input due to friction, cable restraint, mass and base imbalances is modeled as the second player. Thus, the state-space model is given as

$$dx(t) = Ax(t)dt + B_1u_1(t)dt + B_2u_2(t)dt + Edw(t), \quad (6.14)$$

where

$$A = \begin{bmatrix} 0 & C_y & 0 & 0 \\ -\frac{K_{ew}}{C_y J_{ey}} & -\frac{K_{ef}}{J_{ey}} & \frac{k_b N}{J_{ey}} & 0 \\ 0 & 0 & -\frac{1}{\tau_{iEL}} & \frac{K_{iEL}}{\tau_{iEL}} \\ 0 & 0 & 0 & -\frac{1}{\tau_{sEL}} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_{sEL} C_{\omega EL}}{\tau_{sEL}} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 10 \\ 0 \\ 0 \end{bmatrix},$$

and the state variable $x(t)$ is defined as: $x(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]'$. The state variables; x_1, x_2, x_3, x_4 are position, angular rate, armature current and reference current. The first player is u_1 and second player is u_2 . The linear laser pointing model parameter names and values used in simulation are given in Table 6.1. In this example, we study a 2-player, 2nd cumulant full-state feedback Nash game with the aim to stabilize the system. We assume that E in (6.14) is a 4×1 constant vector given as $E = [1 \ 1 \ 1 \ 1]'$ and $dw(t)$ in (6.14) as a Gaussian process with mean $E\{dw(t)\} = 0$ and covariance $E\{dw(t)dw(t)'\} = 0.01$. The initial state condition is given as $x(t_0) = x(0) = x_0 = [2 \ 0 \ 0 \ 2]'$.

The first player cost function J^1 is

$$J^1(t_0, x(t), u_1(t), u_2(t)) = \int_{t_0}^{t_F} [x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + 0.01u_1^2(t)] dt + \psi^1(x(t_F), t_F), \quad (6.15)$$

where $\psi^1(x(t_F), t_F) = 0$ is the terminal cost and the second player cost function J^2 is

$$J^2(t_0, x(t), u_1(t), u_2(t)) = \int_{t_0}^{t_F} [x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + u_2^2(t)] dt + \psi^2(x(t_F), t_F), \quad (6.16)$$

where $\psi^2(x(t_F), t_F) = 0$ is the terminal cost.

Here, we compute the Nash equilibrium solution for the players through solving the minimal second cumulant game problem constrained on the first cumulant cost. To find the Nash strategies, we solve the HJB equations (5.3)-(5.4) with N equal to 2.

However, we utilize the neural network method described in Section 5 to approximate the first player value functions V_1^1, V_2^1 as $V_{1L}^1(t, x) = \mathbf{w}'_{11L}(t)\mathbf{\Lambda}_{11L}(x)$, $V_{2L}^1(t, x) = \mathbf{w}'_{21L}(t)\mathbf{\Lambda}_{21L}(x)$ and the second player value functions V_1^2, V_2^2 as $V_{1L}^2(t, x) = \mathbf{w}'_{12L}(t)\mathbf{\Lambda}_{12L}(x)$, $V_{2L}^2(t, x) = \mathbf{w}'_{22L}(t)\mathbf{\Lambda}_{22L}(x)$.

The input function $\mathbf{\Lambda}_L(x)$ (6.6) is

$$\begin{aligned} \mathbf{\Lambda}_L = \{ & x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^2, x_1^2x_3^2, x_1^2x_4^2, x_1x_2^3, x_1x_3^3, x_1x_4^3, x_2^4, x_2^3x_3, x_2^3x_4, x_2^2x_3^2, \\ & x_2^2x_4^2, x_2x_3^3, x_2x_4^3, x_3^4, x_3^3x_4, x_3^2x_4^2, x_3x_4^3, x_4^4, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_3x_4, x_1x_2^2x_3, x_1x_2^2x_4, \\ & x_1x_3^2x_4, x_1x_3x_4^2, x_1x_2x_3^2, x_1x_2x_4^2, x_1x_2x_3x_4, x_2^2x_3x_4, x_2x_3^2x_4, x_2x_3x_4^2, x_1^2, x_1x_2, x_1x_3, x_1x_4, \\ & x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2 \}' . \end{aligned} \quad (6.17)$$

We selected as polynomial function (6.17) of up to fourth-order ($M = 4$) of approximation in the state variable (i.e x is 4th order) with length of the neural network neurons $L = 45$. Higher order polynomial did not provide significant improvement in the approximation accuracy.

The first player value functions V_{1L}^1, V_{2L}^1 are

$$\begin{aligned}
V_{1L}^1(x_1, x_2, x_3, x_4) = & w_1x_1^4 + w_2x_1^3x_2 + w_3x_1^3x_3 + w_4x_1^3x_4 + w_5x_1^2x_2^2 + w_6x_1^2x_3^2 + w_7x_1^2x_4^2 \\
& + w_8x_1x_2^3 + w_9x_1x_3^3 + w_{10}x_1x_4^3 + w_{11}x_2^4 + w_{12}x_2^3x_3 + w_{13}x_2^3x_4 \\
& + w_{14}x_2^2x_3^2 + w_{15}x_2^2x_4^2 + w_{16}x_2x_3^3 + w_{17}x_2x_4^3 + w_{18}x_3^4 + w_{19}x_3^3x_4 \\
& + w_{20}x_3^2x_4^2 + w_{21}x_3x_4^3 + w_{22}x_4^4 + w_{23}x_1^2x_2x_3 + w_{24}x_1^2x_2x_4 \\
& + w_{25}x_1^2x_3x_4 + w_{26}x_1x_2^2x_3 + w_{27}x_1x_2^2x_4 + w_{28}x_1x_3^2x_4 + w_{29}x_1x_3x_4^2 \\
& + w_{30}x_1x_2x_3^2 + w_{31}x_1x_2x_4^2 + w_{32}x_1x_2x_3x_4 + w_{33}x_2^2x_3x_4 + w_{34}x_2x_3^2x_4 \\
& + w_{35}x_2x_3x_4^2 + w_{36}x_1^2 + w_{37}x_1x_2 + w_{38}x_1x_3 + w_{39}x_1x_4 + w_{40}x_2^2 \\
& + w_{41}x_2x_3 + w_{42}x_2x_4 + w_{43}x_3^2 + w_{44}x_3x_4 + w_{45}x_4^2.
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
V_{2L}^1(x_1, x_2, x_3, x_4) = & w_{46}x_1^4 + w_{47}x_1^3x_2 + w_{48}x_1^3x_3 + w_{49}x_1^3x_4 + w_{50}x_1^2x_2^2 + w_{51}x_1^2x_3^2 \\
& + w_{52}x_1^2x_4^2 + w_{53}x_1x_2^3 + w_{54}x_1x_3^3 + w_{55}x_1x_4^3 + w_{56}x_2^4 + w_{57}x_2^3x_3 \\
& + w_{58}x_2^3x_4 + w_{59}x_2^2x_3^2 + w_{60}x_2^2x_4^2 + w_{61}x_2x_3^3 + w_{62}x_2x_4^3 \\
& + w_{63}x_3^4 + w_{64}x_3^3x_4 + w_{65}x_3^2x_4^2 + w_{66}x_3x_4^3 + w_{67}x_4^4 + w_{68}x_1^2x_2x_3 \\
& + w_{69}x_1^2x_2x_4 + w_{70}x_1^2x_3x_4 + w_{71}x_1x_2^2x_3 + w_{72}x_1x_2^2x_4 \\
& + w_{73}x_1x_3^2x_4 + w_{74}x_1x_3x_4^2 + w_{75}x_1x_2x_3^2 + w_{76}x_1x_2x_4^2 \\
& + w_{77}x_1x_2x_3x_4 + w_{78}x_2^2x_3x_4 + w_{79}x_2x_3^2x_4 + w_{80}x_2x_3x_4^2 \\
& + w_{81}x_1^2 + w_{82}x_1x_2 + w_{83}x_1x_3 + w_{84}x_1x_4 + w_{85}x_2^2 \\
& + w_{86}x_2x_3 + w_{87}x_2x_4 + w_{88}x_3^2 + w_{89}x_3x_4 + w_{90}x_4^2.
\end{aligned} \tag{6.19}$$

Similarly, the second player value functions V_{1L}^2, V_{2L}^2 are

$$\begin{aligned}
V_{1L}^2(x_1, x_2, x_3, x_4) = & w_{91}x_1^4 + w_{92}x_1^3x_2 + w_{93}x_1^3x_3 + w_{94}x_1^3x_4 + w_{95}x_1^2x_2^2 + w_{96}x_1^2x_3^2 \\
& + w_{97}x_1^2x_4^2 + w_{98}x_1x_2^3 + w_{99}x_1x_3^3 + w_{100}x_1x_4^3 + w_{101}x_2^4 \\
& + w_{102}x_2^3x_3 + w_{103}x_2^3x_4 + w_{104}x_2^2x_3^2 + w_{105}x_2^2x_4^2 + w_{106}x_2x_3^3 \\
& + w_{107}x_2x_4^3 + w_{108}x_3^4 + w_{109}x_3^3x_4 + w_{110}x_3^2x_4^2 + w_{111}x_3x_4^3 \\
& + w_{112}x_4^4 + w_{113}x_1^2x_2x_3 + w_{114}x_1^2x_2x_4 + w_{115}x_1^2x_3x_4 + w_{116}x_1x_2^2x_3 \\
& + w_{117}x_1x_2^2x_4 + w_{118}x_1x_3^2x_4 + w_{119}x_1x_3x_4^2 + w_{120}x_1x_2x_3^2 + w_{121}x_1x_2x_4^2 \\
& + w_{122}x_1x_2x_3x_4 + w_{123}x_2^2x_3x_4 + w_{124}x_2x_3^2x_4 + w_{125}x_2x_3x_4^2 + w_{126}x_1^2 \\
& + w_{127}x_1x_2 + w_{128}x_1x_3 + w_{129}x_1x_4 + w_{130}x_2^2 + w_{131}x_2x_3 + w_{132}x_2x_4 \\
& + w_{133}x_3^2 + w_{134}x_3x_4 + w_{135}x_4^2.
\end{aligned} \tag{6.20}$$

$$\begin{aligned}
V_{2L}^2(x_1, x_2, x_3, x_4) = & w_{136}x_1^4 + w_{137}x_1^3x_2 + w_{138}x_1^3x_3 + w_{139}x_1^3x_4 + w_{140}x_1^2x_2^2 + w_{141}x_1^2x_3^2 \\
& + w_{142}x_1^2x_4^2 + w_{143}x_1x_2^3 + w_{144}x_1x_3^3 + w_{145}x_1x_4^3 + w_{146}x_2^4 + w_{147}x_2^3x_3 \\
& + w_{148}x_2^3x_4 + w_{149}x_2^2x_3^2 + w_{150}x_2^2x_4^2 + w_{151}x_2x_3^3 + w_{152}x_2x_4^3 \\
& + w_{153}x_3^4 + w_{154}x_3^3x_4 + w_{155}x_3^2x_4^2 + w_{156}x_3x_4^3 + w_{157}x_4^4 + w_{158}x_1^2x_2x_3 \\
& + w_{159}x_1^2x_2x_4 + w_{160}x_1^2x_3x_4 + w_{161}x_1x_2^2x_3 + w_{162}x_1x_2^2x_4 + w_{163}x_1x_3^2x_4 \\
& + w_{164}x_1x_3x_4^2 + w_{165}x_1x_2x_3^2 + w_{166}x_1x_2x_4^2 + w_{167}x_1x_2x_3x_4 + w_{168}x_2^2x_3x_4 \\
& + w_{169}x_2x_3^2x_4 + w_{170}x_2x_3x_4^2 + w_{171}x_1^2 + w_{172}x_1x_2 + w_{173}x_1x_3 + w_{174}x_1x_4 \\
& + w_{175}x_2^2 + w_{176}x_2x_3 + w_{177}x_2x_4 + w_{178}x_3^2 + w_{179}x_3x_4 + w_{180}x_4^2.
\end{aligned} \tag{6.21}$$

The first and second player value functions are substituted to (5.3)-(5.4). See Appendix B, Lemma B.1 for the expanded four coupled equations that are solved for the value functions. In the simulation, the asymptotic stability region for states was arbitrarily chosen as $-2 \leq x_1 \leq 2, -2 \leq x_2 \leq 2, -2 \leq x_3 \leq 2$ and $-2 \leq x_4 \leq 2$. The final time t_F was 1000 ms and $\mathbf{w}'_{11L}(t_F) - \mathbf{w}'_{21L}(t_F) = \{0\}$ and $\mathbf{w}'_{12L}(t_F) - \mathbf{w}'_{22L}(t_F) = \{0\}$ and we integrate backwards in time to solve for the optimal weights.

(a) 1st Player Neural Network First Cumulant Weights

(b) 1st Player Neural Network Second Cumulant Weights

(c) 2nd Player Neural Network First Cumulant Weights

(d) 2nd Player Neural Network Second Cumulant Weights

Figure 6.4: Two Player Linear Nash Game System Neural Network Weights.

From Figs. 6.4(a) to 6.4(d), the neural network weights converge to constants, which are the optimal weights when integrated backwards in time in (5.3)-(5.4). Figs. 6.5(a) to 6.5(d) show the first and second value cumulant functions for both players. From Fig. 6.5(a), it is observed that the value function increases with increases in γ_{21} . From Figs. 6.5(b) to 6.5(d) the value functions show a decrease with increase in γ_{21} . Fig. 6.6(a) shows the state trajectory for noise influence with variance $\sigma^2 = 0.01$. The states are bounded and converged to value close to the origin. The Lagrange multipliers γ_{xx} were selected to be constants.

The Nash equilibrium controls, u_1 and u_2 are shown in Figs. 6.6(b) and 6.6(c). It should

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{21} (b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{21}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{21} (d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{21}

Figure 6.5: Two Player Linear Nash Game System Value Functions.

be noted from Figs. 6.6(b) and 6.6(c), that the Nash equilibrium strategy for the two players were solved for the second cumulant game by selecting γ_{21} , γ_{22} , where the value functions are minimum which in our case were $\gamma_{21} = 100$, $\gamma_{22} = 100$ as shown in Figs. 6.5(b) and 6.5(d). In addition, we have the design freedom in γ_{xx} values selection to enhance system performance.

Remark: The second cumulant Nash strategy is found within all admissible first cumulant strategy. The states and controls are obtained by a forward integration of the state equation using the optimal weights. The neural network functions size was varied until convergence

(a) State trajectory at initial condition $x_0 = (2,0,0,2)$

(b) First Player Control trajectory at initial condition $x_0 = (2,0,0,2)$

(c) Second Player Control trajectory at initial condition $x_0 = (2,0,0,2)$

Figure 6.6: Two Player Linear Nash Game System States and Optimal Control Strategy Trajectory.

in the weights was observed. To find the first cumulant Nash strategy, set $\gamma_{21} = \gamma_{22} = 0$ in (5.3)-(5.4). From the convergence of the weights, the simulation time for the state and control trajectory was limited to 500 ms.

6.2 Nonlinear Systems

6.2.1 Simple Nonlinear System

Consider the nonlinear dynamic equation in [77] given as

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -x_2(t) - \frac{1}{2}x_1(t) + \frac{1}{4}x_2(t)(\cos(2x_1(t)) + 2)^2 + \frac{1}{4}x_2(t)(\sin(4x_1^2(t)) + 2)^2 \\ &\quad + (\cos(2x_1(t)) + 2)u(t) + (\sin(4x_1^2(t)) + 2)v(t),\end{aligned}\quad (6.22)$$

where $x_1(t)$ and $x_2(t)$ are set of state vectors and $u(t)$, $v(t)$ are the controls (players). The initial state condition is given as $x(0) = [0.5 \ 0.5]'$. The system (6.22) is a deterministic system, thus we introduce Gaussian noise into the system to add the process noise. The stochastic system is represented as

$$dx(t) = g(x(t))dt + B(x(t))u(t)dt + D(x(t))v(t)dt + Edw(t), \quad (6.23)$$

where

$$g(x(t)) = \begin{bmatrix} x_2(t) \\ -x_2(t) - \frac{1}{2}x_1(t) + \frac{1}{4}x_2(t)(\cos(2x_1(t)) + 2)^2 + \frac{1}{4}x_2(t)(\sin(4x_1^2(t)) + 2)^2 \end{bmatrix},$$

$$B(x(t)) = \begin{bmatrix} 0 \\ \cos(2x_1(t)) + 2 \end{bmatrix},$$

$$D(x(t)) = \begin{bmatrix} 0 \\ \sin(4x_1^2(t)) + 2 \end{bmatrix},$$

and the state variables $\bar{x}$ are defined as: $\bar{x}(t) = [x_1(t) \ x_2(t)]'$. We assume that E in (6.23) is a 3×1 constant vector and $dw(t)$ in (6.23) as a Gaussian process with mean $E\{dw(t)\} = 0$, and covariance $E\{dw(t)dw(t)'\} = 0.01$. In this example, we study a 2-player 3^{rd} cumulant full-state feedback Nash game with the aim to stabilize the system.

The first player cost function J^1 is

$$J^1(t_0, x(t), u(t), v(t)) = \int_{t_0}^{t_F} [x_1^2(t) + x_2^2(t) + 0.01u^2(t)] dt + \psi^1(x(t_F), t_F), \quad (6.24)$$

where $\psi^1(x(t_F), t_F) = 0$ is the terminal cost and the second player cost function J^2 is

$$J^2(t_0, x(t), u(t), v(t)) = \int_{t_0}^{t_F} [x_1^2(t) + x_2^2(t) + 0.01v^2(t)] dt + \psi^2(x(t_F), t_F), \quad (6.25)$$

where $\psi^2(x(t_F), t_F) = 0$ is the terminal cost.

The input function $\Lambda_L(x)$ (6.6) is

$$\Lambda_L = \{x_1^4, x_2^4, x_1^2, x_2^2, x_1^2x_2^2, x_1x_2, x_1^3x_2, x_1x_2^3, x_1^6, x_2^6, x_1^5x_2, x_1^4x_2^2, x_1^3x_2^3, x_2^5x_1, x_2^4x_1^2\}'. \quad (6.26)$$

The first player value functions $V_{1L}^1, V_{2L}^1, V_{3L}^1$ are

$$\begin{aligned} V_{1L}^1(x_1, x_2) = & w_1x_1^4 + w_2x_2^4 + w_3x_1^2 + w_4x_2^2 + w_5x_1^2x_2^2 + w_6x_1x_2 \\ & + w_7x_1^3x_2 + w_8x_1x_2^3 + w_9x_1^6 + w_{10}x_2^6 + w_{11}x_1^5x_2 \\ & + w_{12}x_1^4x_2^2 + w_{13}x_1^3x_2^3 + w_{14}x_2^5x_1 + w_{15}x_2^4x_1^2. \end{aligned} \quad (6.27)$$

$$\begin{aligned} V_{2L}^1(x_1, x_2) = & w_{16}x_1^4 + w_{17}x_2^4 + w_{18}x_1^2 + w_{19}x_2^2 + w_{20}x_1^2x_2^2 + w_{21}x_1x_2 \\ & + w_{22}x_1^3x_2 + w_{23}x_1x_2^3 + w_{24}x_1^6 + w_{25}x_2^6 + w_{26}x_1^5x_2 \\ & + w_{27}x_1^4x_2^2 + w_{28}x_1^3x_2^3 + w_{29}x_2^5x_1 + w_{30}x_2^4x_1^2. \end{aligned} \quad (6.28)$$

$$\begin{aligned} V_{3L}^1(x_1, x_2) = & w_{31}x_1^4 + w_{32}x_2^4 + w_{33}x_1^2 + w_{34}x_2^2 + w_{35}x_1^2x_2^2 + w_{36}x_1x_2 \\ & + w_{37}x_1^3x_2 + w_{38}x_1x_2^3 + w_{39}x_1^6 + w_{40}x_2^6 + w_{41}x_1^5x_2 \\ & + w_{42}x_1^4x_2^2 + w_{43}x_1^3x_2^3 + w_{44}x_2^5x_1 + w_{45}x_2^4x_1^2. \end{aligned} \quad (6.29)$$

Similarly, the second player value functions $V_{1L}^2, V_{2L}^2, V_{3L}^2$ are

$$\begin{aligned} V_{1L}^2(x_1, x_2) = & w_{46}x_1^4 + w_{47}x_2^4 + w_{48}x_1^2 + w_{49}x_2^2 + w_{50}x_1^2x_2^2 + w_{51}x_1x_2 \\ & + w_{52}x_1^3x_2 + w_{53}x_1x_2^3 + w_{54}x_1^6 + w_{55}x_2^6 + w_{56}x_1^5x_2 \\ & + w_{57}x_1^4x_2^2 + w_{58}x_1^3x_2^3 + w_{59}x_2^5x_1 + w_{60}x_2^4x_1^2. \end{aligned} \quad (6.30)$$

$$\begin{aligned}
V_{2L}^2(x_1, x_2) = & w_{61}x_1^4 + w_{62}x_2^4 + w_{63}x_1^2 + w_{64}x_2^2 + w_{65}x_1^2x_2^2 + w_{66}x_1x_2 \\
& + w_{67}x_1^3x_2 + w_{68}x_1x_2^3 + w_{69}x_1^6 + w_{70}x_2^6 + w_{71}x_1^5x_2 \\
& + w_{72}x_1^4x_2^2 + w_{73}x_1^3x_2^3 + w_{74}x_2^5x_1 + w_{75}x_2^4x_1^2.
\end{aligned} \tag{6.31}$$

$$\begin{aligned}
V_{3L}^2(x_1, x_2) = & w_{76}x_1^4 + w_{77}x_2^4 + w_{78}x_1^2 + w_{79}x_2^2 + w_{80}x_1^2x_2^2 + w_{81}x_1x_2 \\
& + w_{82}x_1^3x_2 + w_{83}x_1x_2^3 + w_{84}x_1^6 + w_{85}x_2^6 + w_{86}x_1^5x_2 \\
& + w_{87}x_1^4x_2^2 + w_{88}x_1^3x_2^3 + w_{89}x_2^5x_1 + w_{90}x_2^4x_1^2.
\end{aligned} \tag{6.32}$$

We selected as polynomial function (6.26) of up to sixth-order ($M = 6$) of approximation in the state variable (i.e x is 6th order) with length of the neural network neurons $L = 15$. Higher order polynomial did not provide significant improvement in the approximation accuracy. The first and second player value functions are substituted to (5.3)-(5.4) and then the six coupled equations are numerically solved for the value functions, see Appendix B, Lemma B.2 for the coupled equations. In the simulation, the asymptotic stability region for states was arbitrarily chosen as $-1 \leq x_1 \leq 1$ and $-1 \leq x_2 \leq 1$. The final time t_F was 10 s and $\mathbf{w}'_{1L}(t_F) - \mathbf{w}'_{3L}(t_F) = \{0\}$, $\bar{\mathbf{w}}'_{1L}(t_F) - \bar{\mathbf{w}}'_{3L}(t_F) = \{0\}$, and we integrate backwards in time to solve for the optimal weights.

From Figs. 6.7(a) and 6.7(f), the neural network weights converge to constants, which are the optimal weights when integrated backwards in (5.3)-(5.4) together with similar equations for the second player. Figs. 6.8(a) to 6.8(f) show the first, second and third value cumulant functions for both players. From Figs. 6.8(a) and 6.8(b), it is observed that the value function V_1^1 and V_1^2 slightly increases when the value of γ_{32} increases. From Figs. 6.8(c) to 6.8(f), the value functions show a decrease with increase in γ_{32} for the higher-order cumulants. Fig. 6.9(a) shows the state trajectory for noise influence with variance $\sigma^2 = 0.01$. The states are bounded and converged to value close to the origin.

The Nash equilibrium controls, u and v are shown in Fig. 6.9(b). It should be noted from Fig. 6.9(b), that the Nash equilibrium controls for the two player is solved by selecting γ_{32} , γ_{31} , γ_{22} and γ_{21} where the value functions are minimum which in our case were $\gamma_{21} = 10$, $\gamma_{31} = 10$, $\gamma_{22} = 10$ and $\gamma_{32} = 100$ as shown in Figs. 6.8(a) to 6.8(f). In addition, we have the design

(a) 1st Player Neural Network First Cumulant Weights

(b) 2nd Player Neural Network First Cumulant Weights

(c) 1st Player Neural Network Second Cumulant Weights

(d) 2nd Player Neural Network Second Cumulant Weights

(e) 1st Player Neural Network Third Cumulant Weights

(f) 2nd Player Neural Network Third Cumulant Weights

Figure 6.7: Two Player Nash Game System Neural Network Weights.

(a) 1st Player First Cumulant Value Function - V_1 versus γ_{32} (b) 2nd Player First Cumulant Value Function - V_1 versus γ_{32}

(c) 1st Player Second Cumulant Value Function - V_2 versus γ_{32} (d) 2nd Player Second Cumulant Value Function - V_2 versus γ_{32}

(e) 1st Player Third Cumulant Value Function - V_3 versus γ_{32} (f) 2nd Player Third Cumulant Value Function - V_3 versus γ_{32}

Figure 6.8: Two Player Nash Game System Neural Network Weights and Value Functions.

(a) State trajectory at initial condition $x_0 = (0.5, 0.5)$

(b) Control trajectory at initial condition $x_0 = (0.5, 0.5)$

Figure 6.9: Two Player Nash Game System States and Optimal Control Strategy Trajectory.

freedom in γ_{xx} values selection to enhance system performance.

Remark: The states and controls are obtained by a forward integration of the state equation using the optimal weights. The third cumulant Nash strategy is found within all admissible first and second cumulant strategy. To find the first cumulant Nash strategy, set $\gamma_{21} = \gamma_{22} = \gamma_{31} = \gamma_{32} = 0$ and similar computation steps for the third cumulant Nash strategy are followed. To find the second cumulant Nash strategy, set $\gamma_{31} = \gamma_{32} = 0$ and similar computation steps for the third cumulant Nash strategy are followed. The neural network functions size was varied until convergence in the weights was observed.

6.2.2 Electric Power System

In this example, the statistical game approach is studied for a three-machine interconnected power system is given in [55]. A fuzzy controller design was utilized in [55] to minimize individual performance to achieve synchronization. Consider the nonlinear system dynamics given as

$$\begin{aligned}
 dx_1(t) &= x_2(t)dt + dw_1(t), \\
 dx_2(t) &= f_2(t)dt + dw_2(t), \\
 dx_3(t) &= x_4(t)dt + dw_3(t), \\
 dx_4(t) &= f_4(t)dt + dw_4(t), \\
 dx_5(t) &= x_6(t)dt + dw_5(t), \\
 dx_6(t) &= f_6(t)dt + dw_6(t),
 \end{aligned} \tag{6.33}$$

where

$$\begin{aligned}
f_2(t) &= -\frac{D_1}{M_1}x_2(t) + \frac{1}{M_1}u_1(t) + \frac{E_1E_2Y_{12}}{M_1} \times [(\cos(\delta_{12}^0 - \theta_{12}) - \cos(\theta_{12} - x_1(t) \\
&\quad + x_3(t))) + \frac{E_1E_3Y_{13}}{M_1} \times [(\cos(\delta_{13}^0 - \theta_{13}) - \cos(\theta_{13} - x_1(t) + x_5(t)))] \\
f_4(t) &= -\frac{D_2}{M_2}x_4(t) + \frac{1}{M_2}u_2(t) + \frac{E_2E_1Y_{21}}{M_2} \times [(\cos(\delta_{21}^0 - \theta_{21}) - \cos(\theta_{21} - x_3(t) \\
&\quad + x_1(t))) + \frac{E_2E_3Y_{23}}{M_2} \times [(\cos(\delta_{23}^0 - \theta_{23}) - \cos(\theta_{23} - x_3(t) + x_5(t)))] \\
f_6(t) &= -\frac{D_3}{M_3}x_6(t) + \frac{1}{M_3}u_3(t) + \frac{E_3E_1Y_{31}}{M_3} \times [(\cos(\delta_{31}^0 - \theta_{31}) - \cos(\theta_{31} - x_5(t) \\
&\quad + x_1(t))) + \frac{E_3E_2Y_{32}}{M_3} \times [(\cos(\delta_{32}^0 - \theta_{32}) - \cos(\theta_{32} - x_5(t) + x_3(t))],
\end{aligned}$$

where $x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)$ and $x_6(t)$ are set of state vectors with $x_1(t), x_2(t), x_3(t)$ as the absolute rotor angles of the 1st, 2nd and 3rd machine while $x_4(t), x_5(t), x_6(t)$ gives the absolute angular velocity of the 1st, 2nd and 3rd machine and $u_1(t), u_2(t), u_3(t)$ are the controls (players). The three-machine interconnected system parameter names and values used in simulation are given in Table 6.2. In this example, we study a 3-player 2nd cumulant full-state feedback Nash game with the aim to stabilize or synchronize the system. The players $u_i, i = 1, 2, 3$ are used to control and eventually eliminate the transient phenomenon of the multimachine system to maintain network synchronization. The transient phenomenon can be due to short circuit or sudden increment of power load on the interconnected power system. Whenever network synchronization is achieved, all the u_i are always zero indicating steady state. The initial state condition is given as $x(0) = [1 \ 0 \ 1 \ 0 \ 1 \ 0]'$. The system (6.33) is a stochastic system and it is represented in general form as

$$dx(t) = g(x(t))dt + B_1(x(t))u_1(t)dt + B_2(x(t))u_2(t)dt + B_3(x(t))u_3(t)dt + Edw(t), \tag{6.34}$$

where

$$g(x(t)) = \begin{bmatrix} x_2 \\ -\frac{D_1}{M_1}x_2(t) + \frac{E_1E_2Y_{12}}{M_1} \times [(\cos(\delta_{12}^0 - \theta_{12}) - \cos(\theta_{12} - x_1(t) + x_3(t))) + \frac{E_1E_3Y_{13}}{M_1} \times [(\cos(\delta_{13}^0 - \theta_{13}) - \cos(\theta_{13} - x_1(t) + x_5(t)))] \\ x_4 \\ -\frac{D_2}{M_2}x_4(t) + \frac{E_2E_1Y_{21}}{M_2} \times [(\cos(\delta_{21}^0 - \theta_{21}) - \cos(\theta_{21} - x_3(t) + x_1(t))) + \frac{E_2E_3Y_{23}}{M_2} \times [(\cos(\delta_{23}^0 - \theta_{23}) - \cos(\theta_{23} - x_3(t) + x_5(t)))] \\ x_6 \\ -\frac{D_3}{M_3}x_6(t) + \frac{E_3E_1Y_{31}}{M_3} \times [(\cos(\delta_{31}^0 - \theta_{31}) - \cos(\theta_{31} - x_5(t) + x_1(t))) + \frac{E_3E_2Y_{32}}{M_3} \times [(\cos(\delta_{32}^0 - \theta_{32}) - \cos(\theta_{32} - x_5(t) + x_3(t)))] \end{bmatrix},$$

$$B_1(x(t)) = \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_2(x(t)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \\ 0 \\ 0 \end{bmatrix}, B_3(x(t)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{M_3} \end{bmatrix},$$

and the state variables x are defined as: $x(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t) \ x_5(t) \ x_6(t)]'$. We assume that E in (6.34) is a 6×1 constant vector and $dw(t)$ in (6.34) as a Gaussian process with mean $E\{dw(t)\} = 0$, and covariance $E\{dw(t)dw(t)'\} = 0.1$.

The first player cost function J^1 is

$$J^1(t_0, x(t), u_1(t), u_2(t), u_3(t)) = \int_{t_0}^{t_F} [x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + x_5^2(t) + x_6^2(t) + 0.001u_1^2(t)] dt + \psi^1(x(t_F), t_F), \quad (6.35)$$

where $\psi^1(x(t_F), t_F) = 0$ is the terminal cost and the second player cost function J^2 is

$$J^2(t_0, x(t), u_1(t), u_2(t), u_3(t)) = \int_{t_0}^{t_F} [x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + x_5^2(t) + x_6^2(t) + 0.001u_2^2(t)] dt + \psi^2(x(t_F), t_F), \quad (6.36)$$

where $\psi^2(x(t_F), t_F) = 0$ is the terminal cost and the third player cost function J^3 is

$$J^3(t_0, x(t), u_1(t), u_2(t), u_3(t)) = \int_{t_0}^{t_F} [x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + x_5^2(t) + x_6^2(t) + 0.001u_3^2(t)] dt + \psi^3(x(t_F), t_F), \quad (6.37)$$

where $\psi^3(x(t_F), t_F) = 0$ is the terminal cost.

The input function $\Lambda_L(x)$ (6.6) is

$$\Lambda_L = \{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3x_4, x_3x_5, x_3x_6, x_4x_5, x_4x_6, x_5x_6\}'. \quad (6.38)$$

The first player value functions V_{1L}^1, V_{2L}^1 are

$$\begin{aligned} V_{1L}^1 = & w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + w_4x_4^2 + w_5x_5^2 + w_6x_6^2 + w_7x_1x_2 + w_8x_1x_3 + w_9x_1x_4 \\ & + w_{10}x_1x_5 + w_{11}x_1x_6 + w_{12}x_2x_3 + w_{13}x_2x_4 + w_{14}x_2x_5 + w_{15}x_2x_6 + w_{16}x_3x_4 \\ & + w_{17}x_3x_5 + w_{18}x_3x_6 + w_{19}x_4x_5 + w_{20}x_4x_6 + w_{21}x_5x_6. \end{aligned} \quad (6.39)$$

$$\begin{aligned} V_{2L}^1 = & w_{22}x_1^2 + w_{23}x_2^2 + w_{24}x_3^2 + w_{25}x_4^2 + w_{26}x_5^2 + w_{27}x_6^2 + w_{28}x_1x_2 + w_{29}x_1x_3 \\ & + w_{30}x_1x_4 + w_{31}x_1x_5 + w_{32}x_1x_6 + w_{33}x_2x_3 + w_{34}x_2x_4 + w_{35}x_2x_5 + w_{36}x_2x_6 \\ & + w_{37}x_3x_4 + w_{38}x_3x_5 + w_{39}x_3x_6 + w_{40}x_4x_5 + w_{41}x_4x_6 + w_{42}x_5x_6. \end{aligned} \quad (6.40)$$

Similarly, the second player value functions V_{1L}^2, V_{2L}^2 are

$$\begin{aligned} V_{1L}^2 = & w_{43}x_1^2 + w_{44}x_2^2 + w_{45}x_3^2 + w_{46}x_4^2 + w_{47}x_5^2 + w_{48}x_6^2 + w_{49}x_1x_2 + w_{50}x_1x_3 \\ & + w_{51}x_1x_4 + w_{52}x_1x_5 + w_{53}x_1x_6 + w_{54}x_2x_3 + w_{55}x_2x_4 + w_{56}x_2x_5 \\ & + w_{57}x_2x_6 + w_{58}x_3x_4 + w_{59}x_3x_5 + w_{60}x_3x_6 + w_{61}x_4x_5 + w_{62}x_4x_6 + w_{63}x_5x_6. \end{aligned} \quad (6.41)$$

$$\begin{aligned} V_{2L}^2 = & w_{64}x_1^2 + w_{65}x_2^2 + w_{66}x_3^2 + w_{67}x_4^2 + w_{68}x_5^2 + w_{69}x_6^2 + w_{70}x_1x_2 + w_{71}x_1x_3 \\ & + w_{72}x_1x_4 + w_{73}x_1x_5 + w_{74}x_1x_6 + w_{75}x_2x_3 + w_{76}x_2x_4 + w_{77}x_2x_5 + w_{78}x_2x_6 \\ & + w_{79}x_3x_4 + w_{80}x_3x_5 + w_{81}x_3x_6 + w_{82}x_4x_5 + w_{83}x_4x_6 + w_{84}x_5x_6. \end{aligned} \quad (6.42)$$

Finally, the third player value functions V_{1L}^3, V_{2L}^3 are

$$\begin{aligned}
V_{1L}^3 = & w_{85}x_1^2 + w_{86}x_2^2 + w_{87}x_3^2 + w_{88}x_4^2 + w_{89}x_5^2 + w_{90}x_6^2 + w_{91}x_1x_2 + w_{92}x_1x_3 \\
& + w_{93}x_1x_4 + w_{94}x_1x_5 + w_{95}x_1x_6 + w_{96}x_2x_3 + w_{97}x_2x_4 + w_{98}x_2x_5 + w_{99}x_2x_6 \\
& + w_{100}x_3x_4 + w_{101}x_3x_5 + w_{102}x_3x_6 + w_{103}x_4x_5 + w_{104}x_4x_6 + w_{105}x_5x_6.
\end{aligned} \tag{6.43}$$

$$\begin{aligned}
V_{2L}^3 = & w_{106}x_1^2 + w_{107}x_2^2 + w_{108}x_3^2 + w_{109}x_4^2 + w_{110}x_5^2 + w_{111}x_6^2 + w_{112}x_1x_2 \\
& + w_{113}x_1x_3 + w_{114}x_1x_4 + w_{115}x_1x_5 + w_{116}x_1x_6 + w_{117}x_2x_3 + w_{118}x_2x_4 \\
& + w_{119}x_2x_5 + w_{120}x_2x_6 + w_{121}x_3x_4 + w_{122}x_3x_5 + w_{123}x_3x_6 + w_{124}x_4x_5 \\
& + w_{125}x_4x_6 + w_{126}x_5x_6.
\end{aligned} \tag{6.44}$$

We selected as polynomial function (6.38) of up to second-order ($M = 2$) of approximation in the state variable (i.e x is 2^{nd} order) with length of the neural network neurons $L = 21$. Higher order polynomial did not provide significant improvement in the approximation accuracy. The first, second and third player value functions are substituted to (5.3)-(5.4). Then, the six coupled equations are numerically solved for the value functions, see Appendix B, Lemma B.3 for the coupled equations. In the simulation, the asymptotic stability region for states was arbitrarily chosen as $-pi/2 \leq x_1 \leq pi/2$, $-pi/2 \leq x_2 \leq pi/2$, $-pi/2 \leq x_3 \leq pi/2$, $-pi/2 \leq x_4 \leq pi/2$, $-pi/2 \leq x_5 \leq pi/2$ and $-pi/2 \leq x_6 \leq pi/2$. The final time t_F was 10 s and $\mathbf{w}'_{11L}(t_F) - \mathbf{w}'_{21L}(t_F) = \{0\}$, $\mathbf{w}'_{12L}(t_F) - \mathbf{w}'_{22L}(t_F) = \{0\}$ and $\mathbf{w}'_{13L}(t_F) - \mathbf{w}'_{23L}(t_F) = \{0\}$ and we integrate backwards in time to solve for the optimal weights.

From Figs. 6.10(a) to 6.10(f), the neural network weights converge to constants, which are the optimal weights when integrated backwards in (5.3)-(5.4). Figs. 6.11(a) to 6.11(f) show the first and second value cumulant functions for all players. From Figs. 6.11(a), 6.11(c) and 6.11(e), it is observed that the value function V_1^1, V_1^2 and V_1^3 slightly increases when the value of γ_{23} increases. From Figs. 6.11(b), 6.11(d) and 6.11(f) the value functions show a decrease with increase in γ_{23} for the higher-order cumulants. Figs. 6.12(a) and 6.12(b) show the state trajectory for noise influence with variance $\sigma^2 = 0.01$. The states are bounded and converged to value close to the origin.

The Nash equilibrium controls, u_1 , u_2 and u_3 are shown in Fig. 6.12(c). It should be noted

(a) 1st Player Neural Network First Cumulant Weights

(b) 1st Player Neural Network Second Cumulant Weights

(c) 2nd Player Neural Network First Cumulant Weights

(d) 2nd Player Neural Network Second Cumulant Weights

(e) 3rd Player Neural Network First Cumulant Weights

(f) 3rd Player Neural Network Second Cumulant Weights

Figure 6.10: Three Player Nonlinear Nash Game System Neural Network Weights.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{23}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{23}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{23}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{23}

(e) 3rd Player First Cumulant Value Function - V_1^3 versus γ_{23}

(f) 3rd Player Second Cumulant Value Function - V_2^3 versus γ_{23}

Figure 6.11: Three Player Nonlinear Nash Game System Value Functions.

(a) State trajectory at initial condition $x_0 = (1,0,1,0,1,0)$

(b) State trajectory at initial condition $x_0 = (1,0,1,0,1,0)$

(c) Control trajectory at initial condition $x_0 = (1,0,1,0,1,0)$

Figure 6.12: Three Player Nonlinear Nash Game System States and Optimal Control Strategy Trajectory.

from Fig. 6.12(c), that the Nash equilibrium controls for the three players were solved by selecting $\gamma_{21}(t)$, $\gamma_{22}(t)$, $\gamma_{23}(t)$ to be constants where the value functions are minimum which in our case were $\gamma_{21} = 0.1$, $\gamma_{22} = 0.1$ and $\gamma_{23} = 1$ as shown in Figs. 6.11(b), 6.11(d) to 6.11(f). Result from the control strategy trajectory plot show that system are matched as expected from problem formulation. In addition, we have the design freedom in γ_{xx} values selection to enhance system performance. Results show that steady-state can be quickly achieved and as expected

all players are at same strategy levels.

Remark: The second cumulant Nash strategy is found within all admissible first cumulant strategy. The states and controls are obtained by a forward integration of the state equation using the optimal weights. To find the first cumulant Nash strategy, set $\gamma_{21} = \gamma_{22} = \gamma_{23} = 0$ and similar computation steps for the second cumulant Nash strategy are followed. The neural network functions size was varied until convergence in the weights was observed.

6.2.3 Nonholonomic Robot System

In this example, the statistical game approach is studied for a nonholonomic dynamic mobile robot [78]. Nonholonomic control systems have been studied in [79] and by Brockett [80], who investigated stabilization issues for the nonholonomic system by applying continuously differentiable time invariant state feedback law. Applying game theoretical and neural network approach, the nonholonomic system can be stabilized.

The nonlinear dynamic equation is

$$\begin{aligned}\dot{x}_1(t) &= u_1(t), \\ \dot{x}_2(t) &= u_2(t), \\ \dot{x}_3(t) &= x_1(t)u_2(t),\end{aligned}\tag{6.45}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are set of state vectors and $u_1(t)$, $u_2(t)$ are the controls (first and second player). The initial state condition is given as $x(0) = x_0 = [1 \ 1 \ 1]'$. The system in (6.45) is a deterministic system [58], thus we introduce Gaussian noise into the system to add the process noise. The stochastic system is represented in general form as

$$dx(t) = g(x(t))dt + B_1(x(t))u_1(t)dt + B_2(x(t))u_2(t)dt + Edw(t),\tag{6.46}$$

where

$$g(x(t)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, B_1(x(t)) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_2(x(t)) = \begin{bmatrix} 0 \\ 0 \\ x_1(t) \end{bmatrix},$$

and the state variables x are defined as: $x(t) = [x_1(t), x_2(t), x_3(t)]'$. We assume that E in (6.46) is a 3×1 constant vector and $dw(t)$ in (6.46) as a Gaussian process with mean $E\{dw(t)\} = 0$,

and covariance $E\{dw(t)dw(t)'\} = 0.01$. In this example, we study a 2-player 2^{nd} cumulant full-state feedback Nash game with the aim to stabilize the system.

The first player cost function J^1 is

$$J^1(t_0, x(t), u_1(t), u_2(t)) = \int_{t_0}^{t_F} [0.25x_2^2(t) + 0.25x_3^2(t) + x_1^2(t) + x_1^4(t) + 2u_1^2(t) + u_2^2(t)]dt + \psi^1(x(t_F), t_F), \quad (6.47)$$

where $\psi^1(x(t_F), t_F) = 0$ is the terminal cost and the second player cost function J^2 is

$$J^2(t_0, x(t), u_1(t), u_2(t)) = \int_{t_0}^{t_F} [0.25x_2^2(t) + 0.25x_3^2(t) + x_1^2(t) + x_1^4(t) + u_1^2(t) + 2u_2^2(t)]dt + \psi^2(x(t_F), t_F), \quad (6.48)$$

where $\psi^2(x(t_F), t_F) = 0$ is the terminal cost.

The input function $\Lambda_L(x)$ (6.6) is

$$\Lambda_L = \{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3, x_1^4, x_1^3x_2, x_1^3x_3, x_1^2x_2x_3, x_1^2x_2^2, x_1^2x_3^2, x_2^4, x_2^3x_1, x_2^3x_3, x_2^2x_1x_3, x_2^2x_3^2, x_3^4, x_3^3x_1, x_3^3x_2, x_3^2x_1x_2\}'. \quad (6.49)$$

The first player value functions V_{1L}^1, V_{2L}^1 are

$$\begin{aligned} V_{1L}^1(x_1, x_2, x_3) = & w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + w_4x_1x_2 + w_5x_1x_3 + w_6x_2x_3 \\ & + w_7x_1^4 + w_8x_1^3x_2 + w_9x_1^3x_3 + w_{10}x_1^2x_2x_3 + w_{11}x_1^2x_2^2 \\ & + w_{12}x_1^2x_3^2 + w_{13}x_2^4 + w_{14}x_2^3x_1 + w_{15}x_2^3x_3 + w_{16}x_2^2x_1x_3 \\ & + w_{17}x_2^2x_3^2 + w_{18}x_3^4 + w_{19}x_3^3x_1 + w_{20}x_3^3x_2 + w_{21}x_3^2x_1x_2. \end{aligned} \quad (6.50)$$

$$\begin{aligned} V_{2L}^1(x_1, x_2, x_3) = & w_{22}x_1^2 + w_{23}x_2^2 + w_{24}x_3^2 + w_{25}x_1x_2 + w_{26}x_1x_3 + w_{27}x_2x_3 \\ & + w_{28}x_1^4 + w_{29}x_1^3x_2 + w_{30}x_1^3x_3 + w_{31}x_1^2x_2x_3 + w_{32}x_1^2x_2^2 \\ & + w_{33}x_1^2x_3^2 + w_{34}x_2^4 + w_{35}x_2^3x_1 + w_{36}x_2^3x_3 + w_{37}x_2^2x_1x_3 \\ & + w_{38}x_2^2x_3^2 + w_{39}x_3^4 + w_{40}x_3^3x_1 + w_{41}x_3^3x_2 + w_{42}x_3^2x_1x_2. \end{aligned} \quad (6.51)$$

Similarly, the second player value functions V_{1L}^2, V_{2L}^2 are

$$\begin{aligned}
V_{1L}^2(x_1, x_2, x_3) = & w_{43}x_1^2 + w_{44}x_2^2 + w_{45}x_3^2 + w_{46}x_1x_2 + w_{47}x_1x_3 + w_{48}x_2x_3 \\
& + w_{49}x_1^4 + w_{50}x_1^3x_2 + w_{51}x_1^3x_3 + w_{52}x_1^2x_2x_3 + w_{53}x_1^2x_2^2 \\
& + w_{54}x_1^2x_3^2 + w_{55}x_2^4 + w_{56}x_2^3x_1 + w_{57}x_2^3x_3 + w_{58}x_2^2x_1x_3 \\
& + w_{59}x_2^2x_3^2 + w_{60}x_3^4 + w_{61}x_3^3x_1 + w_{62}x_3^3x_2 + w_{63}x_3^2x_1x_2.
\end{aligned} \tag{6.52}$$

$$\begin{aligned}
V_{2L}^2(x_1, x_2, x_3) = & w_{64}x_1^2 + w_{65}x_2^2 + w_{66}x_3^2 + w_{67}x_1x_2 + w_{68}x_1x_3 + w_{69}x_2x_3 \\
& + w_{70}x_1^4 + w_{71}x_1^3x_2 + w_{72}x_1^3x_3 + w_{73}x_1^2x_2x_3 + w_{74}x_1^2x_2^2 \\
& + w_{75}x_1^2x_3^2 + w_{76}x_2^4 + w_{77}x_2^3x_1 + w_{78}x_2^3x_3 + w_{79}x_2^2x_1x_3 \\
& + w_{80}x_2^2x_3^2 + w_{81}x_3^4 + w_{82}x_3^3x_1 + w_{83}x_3^3x_2 + w_{84}x_3^2x_1x_2.
\end{aligned} \tag{6.53}$$

We selected as polynomial function (6.49) of up to fourth-order ($M = 4$) of approximation in the state variable (i.e x is 4th order) with length of the neural network neurons $L = 21$. Higher order polynomial did not provide significant improvement in the approximation accuracy. The first and second player value functions are substituted to (5.3)-(5.4). Then, the four coupled equations are numerically solved for the value functions, see Appendix B, Lemma B.1 for the coupled equations. In the simulation, the asymptotic stability region for states was arbitrarily chosen the asymptotic stability region for states was arbitrarily chosen as $-1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1$ and $-1 \leq x_3 \leq 1$. The final time t_F was 30 s and $\mathbf{w}'_{11L}(t_F) - \mathbf{w}'_{21L}(t_F) = \{0\}$ and $\mathbf{w}'_{12L}(t_F) - \mathbf{w}'_{22L}(t_F) = \{0\}$ and we integrate backwards in time to solve for the optimal weights.

From Figs. 6.13(a) to 6.13(d), the neural network weights converge to constants, which are the optimal weights when integrated backwards in (5.3)-(5.4). Figs. 6.14(a) to 6.14(d) show the first and second value cumulant functions for all players. From Figs. 6.14(a) and 6.14(c), it is observed that the value function V_1^1 and V_1^2 slightly increases when the value of γ_{22} increases. From Figs. 6.14(b) and 6.14(d), the value functions show a decrease with increase in γ_{22} for the higher-order cumulants. Figs. 6.15(a) shows the state trajectory for noise influence with variance $\sigma^2 = 0.01$. The states are bounded and converged to value close to the origin.

(a) 1st Player Neural Network First Cumulant Weights

(b) 1st Player Neural Network Second Cumulant Weights

(c) 2nd Player Neural Network First Cumulant Weights

(d) 2nd Player Neural Network Second Cumulant Weights

Figure 6.13: Two Player Nonlinear Nash Game System Neural Network Weights.

The Nash equilibrium controls, u_1 and u_2 are shown in Fig. 6.15(b). It should be noted from Fig. 6.15(d), that the Nash equilibrium controls for the two players were solved by selecting $\gamma_{21}(t)$, $\gamma_{22}(t)$ to be constants where the value functions are minimum which in our case were $\gamma_{21} = 0.01$ and $\gamma_{22} = 0.01$ as shown in Figs. 6.14(b) and 6.14(d). In addition, we have the design freedom in γ_{xx} values selection to enhance system performance.

Remark: A nonquadratic cost function [81], [82] was utilized in this case due to the stronger nonlinearity of the system. The second cumulant Nash strategy is found within all admissible first cumulant strategy. To find the first cumulant Nash strategy, set $\gamma_{21} = \gamma_{22} = 0$ and similar

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{22}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{22}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{22}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{22}

Figure 6.14: Two Player Nonlinear Nash Game System Value Functions.

computation steps for the second cumulant Nash strategy are followed. The states and controls are obtained by a forward integration of the state equation using the optimal weights. The neural network functions size was varied until convergence in the weights was observed.

6.3 Value Function Analysis

In this section, the value functions for the game problem in Section 6.2.2 are analyzed through different combination of values for the Lagrange multipliers in the coupled HJB equa-

(a) State trajectory at initial condition $x_0 = (1,1,1)$

(b) Control trajectory at initial condition $x_0 = (1,1,1)$

Figure 6.15: Two Player Nonlinear Nash Game System States and Optimal Control Strategy Trajectory.

tions. The aim of this section is to show the effect of Lagrange multiplier selection on the computation of the value functions. In Fig. 6.16, the Lagrange multipliers were $\gamma_{21} = 0.01$, $\gamma_{22} = 0.1$ and γ_{23} varies from 0 to 1. The subplots of Fig. 6.16, show the first and second value functions for the first and second cost cumulant for the three players. In all subplots, the first cumulant value functions increase with respect to γ_{23} while the second cumulant value functions decrease with respect to γ_{23} . In Fig. 6.17, the Lagrange multipliers were $\gamma_{21} = 0.05$, $\gamma_{22} = 0.1$ and γ_{23} varies from 0 to 1. The subplots of Fig. 6.17, show the first and second value functions for the first and second cost cumulant for the three players. In all subplots, the first cumulant value functions increase with respect to γ_{23} while the second cumulant value functions decrease with respect to γ_{23} .

In Fig. 6.18, the Lagrange multipliers were $\gamma_{21} = 1$, $\gamma_{22} = 0.1$ and γ_{23} varies from 0 to 1. The subplots of Fig. 6.18, show the first and second value functions for the first and second cost cumulant for the three players. In all subplots, the first cumulant value functions also increase with respect to γ_{23} while the second cumulant value functions do decrease with respect to γ_{23} . Similarly, in Fig 6.19, the Lagrange multipliers were $\gamma_{21} = 0.1$, $\gamma_{22} = 0.01$ and γ_{23} varies from 0 to 1, while in Fig 6.20, the Lagrange multipliers were $\gamma_{21} = 0.1$, $\gamma_{22} = 0.05$ and γ_{23} varies from 0 to 1 and in Fig 6.21, the Lagrange multipliers were $\gamma_{21} = 0.1$, $\gamma_{22} = 1$ and γ_{23} varies from 0 to 1.

From the above plots, it is seen that the trajectories of the value functions for each player varies with different combinational value of the Lagrange multipliers γ_{2x} . From these Figures, the first value functions are monotonously increasing with respect to γ_{2x} while the second value functions are monotonously decreasing with respect to γ_{2x} . The extent of coupling among the players contribute to these value functions which in turn affect the players strategies - controllers for the system equilibrium. The value functions are uniquely determined once the Lagrange multipliers are set.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{23}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{23}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{23}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{23}

(e) 3rd Player First Cumulant Value Function - V_1^3 versus γ_{23}

(f) 3rd Player Second Cumulant Value Function - V_2^3 versus γ_{23}

Figure 6.16: Three Player Nonlinear Nash Game System Value Functions with $\gamma_{21} = 0.01$.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{23}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{23}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{23}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{23}

(e) 3rd Player First Cumulant Value Function - V_1^3 versus γ_{23}

(f) 3rd Player Second Cumulant Value Function - V_2^3 versus γ_{23}

Figure 6.17: Three Player Nonlinear Nash Game System Value Functions with $\gamma_{21} = 0.05$.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{23}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{23}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{23}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{23}

(e) 3rd Player First Cumulant Value Function - V_1^3 versus γ_{23}

(f) 3rd Player Second Cumulant Value Function - V_2^3 versus γ_{23}

Figure 6.18: Three Player Nonlinear Nash Game System Value Functions with $\gamma_{21} = 1$.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{23}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{23}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{23}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{23}

(e) 3rd Player First Cumulant Value Function - V_1^3 versus γ_{23}

(f) 3rd Player Second Cumulant Value Function - V_2^3 versus γ_{23}

Figure 6.19: Three Player Nonlinear Nash Game System Value Functions with $\gamma_{22} = 0.01$.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{23}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{23}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{23}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{23}

(e) 3rd Player First Cumulant Value Function - V_1^3 versus γ_{23}

(f) 3rd Player Second Cumulant Value Function - V_2^3 versus γ_{23}

Figure 6.20: Three Player Nonlinear Nash Game System Value Functions with $\gamma_{22} = 0.05$.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{23}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{23}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{23}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{23}

(e) 3rd Player First Cumulant Value Function - V_1^3 versus γ_{23}

(f) 3rd Player Second Cumulant Value Function - V_2^3 versus γ_{23}

Figure 6.21: Three Player Nonlinear Nash Game System Value Functions with $\gamma_{22} = 1$.

Table 6.1: Laser Pointing System Parameters

Parameter	Value	Description (unit)
K_{ew}	0.000113	pitch cable restraint co-efficient (Nm/rad)
K_{ef}	0.0006328	pitch viscous friction co-efficient (N/m ² s)
k_b	0.0678	damping co-efficient of motor
N	90	gear ratio
K_{iEL}	0.12456	server motor current loop gain
t_{iEL}	0.00099	server motor current loop time constant (s)
J_{ey}	0.036725	pitch inertia tensor diagonal element
C_y	94.237	scale factor for conversion of angular pitch cm/(rad/s)
t_{sEL}	0.0052	server motor speed loop time constant
K_{sEL}	10	server motor speed loop gain constant
C_{wEL}	0.0198	scale factor position conversion

Table 6.2: Electric Power System Parameters

Parameter	Value	Description
M_1	1.03	inertial coefficient of machine 1
M_2	1.25	inertial coefficient of machine 2
M_3	1.4	inertial coefficient of machine 3
D_1	0.8	damping co-efficient of machine 1
D_2	1.2	damping co-efficient of machine 2
D_3	1.1	damping co-efficient of machine 3
E_1	1.017	internal voltage of machine 1
E_2	1.005	internal voltage of machine 2
E_3	1.033	internal voltage of machine 3
Y_{12}	1.98	modulus of transfer admittance between machine 1 and 2
Y_{13}	1.14	modulus of transfer admittance between machine 1 and 3
Y_{23}	1.06	modulus of transfer admittance between machine 2 and 3
θ_{12}	1.5	phase angle of transfer admittance between machine 1 and 2
θ_{13}	1.55	phase angle of transfer admittance between machine 1 and 3
θ_{23}	1.56	phase angle of transfer admittance between machine 2 and 3
δ_{12}^0	0.5	absolute rotor angle between machine 1 and 2
δ_{13}^0	0.18	absolute rotor angle between machine 1 and 3
δ_{23}^0	-0.32	absolute rotor angle between machine 2 and 3

CHAPTER 7

AD HOC OUTPUT FEEDBACK STATISTICAL GAME FOR LINEAR SYSTEMS

This chapter describes the N -player, m -th cost cumulant, ad hoc output feedback statistical differential game problem for a linear system with quadratic cost function. The output feedback method is termed ad hoc because we assume certainty equivalence principle for higher-order statistical control. The first section of this chapter reviews output feedback game control, while the second section gives the lemmas for formulating the N -player, m -th cost cumulant ad hoc output feedback statistical game. The third section formulates the ad hoc output feedback statistical game while the ad hoc output feedback statistical game algorithm is given in section four. An application example is given in the last section.

7.1 Introduction

In most practical control engineering applications, not all the states are measurable. The system model may consists of unknown disturbances usually expressed as process noise while the inaccuracies in measurement are usually expressed as measurement noise. An approach to account for the unmeasurable states is to estimate those states using an estimator before utilizing the states in a controller in a feedback control system. This approach is part of a generalized method to analyzing linear stochastic systems by applying the concept of *Certainty Equivalence Principle* [26] and related *Separation Theorem* [25]. In this chapter, we will consider ad hoc output feedback statistical Nash differential game for a system of finite N players on a fixed time interval.

Statistical Nash game can be viewed as a generalization of risk-sensitive dynamic Nash game for both full-state feedback and output feedback systems. Bensoussan *et al.* [83] investigated the stochastic optimal control problem for partially-observed system with exponential cost

criterion and proved that separation theorem does not hold for such scenario. Dia *et al.* [84] applied a change of probability using Gisanov theorem to solve an output feedback risk-sensitive optimal control problem. A risk-sensitive certainty equivalence principle for a linear quadratic gaussian (LQG) problem was deduced by Whittle [85]. Risk-sensitive filtering and smoothing issues for a nonlinear discrete system with relation to H_2 & H_∞ estimation problem were studied in [86] while in [87], a linear system was employed and the risk-sensitive filtering problem related to robust/optimal filtering. In [31], risk-sensitive estimation problem was studied for a linear system and the resulting solution applied to an equivalent differential game problem. Zheng *et al.* [88] investigated both optimal and suboptimal approach to output feedback control for a linear system with quadratic cost function while the solvability of the necessary and sufficient conditions for the existence of a stabilizing output feedback solution for a continuous-time linear systems was studied in [89].

The dynamic output feedback multi-objective control problem for a mixed H_2 & H_∞ design consisting of a linear system at finite and infinite horizon was solved in [90] while a suboptimal dynamic H_2 & H_∞ output feedback control was analyzed in [91]. The static output feedback control for linear time invariant system including the necessary and sufficient conditions for the existence of solution was investigated as a suboptimal problem in [92]. Applying backstepping design technique and estimation of unmeasured states for a stochastic strict feedback system, an output feedback controller was designed in [93], [94]. A suboptimal output feedback solution for stochastic linear system with state and control dependent noise was analyzed in [95] while Boukas [96] applied linear matrix inequality (LMI) method to the static output feedback solution for stochastic hybrid linear systems with no measurement noise. Aberkane *et al.* [97] investigated the output feedback solution for generalized stochastic hybrid linear systems and provided a dynamic system practical example. The infinite-horizon output feedback Nash game for a stochastic weakly-coupled system with state-dependent noise was studied in [98]. In addition, the necessary conditions for the existence of Nash equilibrium were given in [98]. Klompston [19], extended risk-sensitive control to discrete time game theory and solved the Nash equilibrium for partially observed state for a 2-player game. The output feedback multi-

objective control with applications to structural application was investigated in [99], where completion of square method was utilized to find the strategy of a player, while the output feedback solution for an infinite-horizon linear quadratic system was considered in [100].

Currently, separation theorem allows the feedback stabilization of a linear stochastic system with unobservable states and corrupted by gaussian noise to be analyzed as a linear estimator using Kalman filter and linear quadratic regulator (LQR) to form linear quadratic gaussian (LQG) control. However, there is no separation theorem result for the second cumulant and higher-order statistical output feedback game control. Therefore, an optimal higher-order statistical output feedback game control strategy is currently not within reach. This chapter investigates a suboptimal output feedback game strategy, termed ad hoc output feedback game strategy where the unmeasurable states are estimated using a Kalman filter.

7.2 Mathematical Preliminaries

Consider an N -player state dynamics and measured output described by the linear Ito-sense stochastic differential equation.

$$dx(t) = A(t)x(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right) dt + G(t)dw_1(t), \quad (7.1)$$

$$dy(t) = C(t)x(t)dt + dw_2(t), \quad (7.2)$$

where $t \in [t_0, t_F] = T$, $x(t) \in \mathbb{R}^n$ is the state, $u_k(t) \in U_k \subset \mathbb{R}^m$ is the k -th player strategy, $k = 1, 2, \dots, N$ and $w_1(t)$, $w_2(t)$ are Gaussian random process defined on a probability space (Ω_0, F, P) where Ω_0 is a nonempty set, F is a σ -algebra of Ω_0 and P is a probability measure on (Ω_0, F) . $x(t_0) = x_0$ is the initial state vector with covariance matrix P_0 . The Gaussian random process $w_1(t)$ has zero mean and covariance of $E(dw_1(t)dw_1'(t)) = W_1(t)dt$ and similarly the Gaussian random process $w_2(t)$ has zero mean and covariance of $E(dw_2(t)dw_2'(t)) = W_2(t)dt$. The noise processes $w_1(t)$ and $w_2(t)$ are uncorrelated with $E(dw_1(t)dw_2'(t)) = E(dw_2(t)dw_1'(t)) = 0$. Let $Q_0 = [t_0, t_F) \times \mathbb{R}^n$, $\bar{Q}_0$ denote the closure

of Q_0 , i.e $\bar{Q}_0 = T \times \mathbb{R}^n$. Assume there exist constants $c_1, c_2 > 0 \in \mathbb{R}$ such that

$$\|A(t)\| + \sum_{k=1}^N \|B_k\| \leq c_1, \quad (7.3)$$

$$\|G(t)\| \leq c_2,$$

and $A(\cdot), B_k(\cdot), C(\cdot), G(\cdot)$ are elements of $C^1([t_0, t_F])$. Let a feedback strategy law be defined as $u_k(t) = \mu_k(t, x(t)), t \in T$. Then, (7.1) can be written as

$$dx(t) = f^{\mu_k}(x(t))dt + G(t)dw_1(t), x(t_0) = x_0, \quad (7.4)$$

where $f^{\mu_k}(x)$ denotes $A(t)x(t) + \sum_{k=1}^N B_k(t)u_k(t)$. There exist a bounded, Borel measurable feedback strategy $\mu_k(x) : \mathbb{R}^n \rightarrow U_k$ such that $\mu_k(x)$ satisfies a global Lipschitz condition: i.e there exists a constant c_1 such that

$$\|\mu_k(x_1) - \mu_k(x_2)\| \leq c_1 \|x_1 - x_2\|, \quad (7.5)$$

$\|\cdot\|$ is the Euclidean norm and $x_1, x_2 \in \mathbb{R}^n$. Also, $\mu_k(x)$ satisfies linear growth condition

$$\|\mu_k(x)\| \leq c_2(1 + \|x\|). \quad (7.6)$$

Then, if $E\|x(t)\|^2$ is finite, there is a unique solution to (7.1) which is a Markov diffusion process on $\mathbb{R}^n$ [65]. In order to assess performance of (7.1), consider the cost function (J^k) for the k -th player given as:

$$\begin{aligned} J^k(t, x(t), \mu_1, \mu_2, \dots, \mu_N) = & \int_t^{t_F} \left[x'(s)Q(s)x(s) + \sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s) \right] ds \\ & + x'(t_F)Q(t_F)x(t_F), \end{aligned} \quad (7.7)$$

where $k = 1, 2, \dots, N, x(t_F) = x_f, Q(\cdot)$ is positive semi-definite and $R_{ki}(\cdot)$ is positive definite.

Let the state estimate be $\hat{x}(t)$ and the state estimate error be $\bar{x}(t)$ where $x(t)$ is the state true value. Then, the state estimation error $\bar{x}(t)$, is given as

$$\bar{x}(t) = x(t) - \hat{x}(t). \quad (7.8)$$

Lemma 7.1: The expected value of the cost function (7.7) conditioned on the σ -algebra generated by the measured output (7.2) can be rewritten as

$$\begin{aligned} E\left\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\right\} &= \int_t^{t_F} \left[E\left(\hat{x}'(s)Q(s)\hat{x}(s)\right) + \text{tr}\left(Q(s)P(s)\right) \right] ds \\ &\quad + \int_t^{t_F} \left[\sum_{i=1}^N \mu'_i(s)R_{ki}(s)\mu_i(s) \right] ds \\ &\quad + E\left\{\hat{x}'(t_F)Q(t_F)\hat{x}(t_F)\right\} + \text{tr}\left(Q(t_F)P(t_F)\right), \end{aligned} \quad (7.9)$$

where $k = 1, 2, \dots, N$, $\hat{x}(t_F) = \hat{x}_f$, $Q(\cdot), P(\cdot)$ are positive semi-definite, $R_{ki}(s)$ is positive definite for $k = i$ and positive semi-definite for $k \neq i$.

Proof: From (7.7), the system cost function is

$$\begin{aligned} J^k(t, x(t), \mu_1, \mu_2, \dots, \mu_N) &= \int_t^{t_F} \left[x'(s)Q(s)x(s) + \sum_{i=1}^N \mu'_i(s)R_{ki}(s)\mu_i(s) \right] ds \\ &\quad + x'(t_F)Q(t_F)x(t_F). \end{aligned} \quad (7.10)$$

The expected value of the cost function is $V_1^k = E(J^k)$, which is given as

$$\begin{aligned} E\left\{J^k(t, x(t), \mu_1, \mu_2, \dots, \mu_N)\right\} &= E\left\{ \int_t^{t_F} \left[x'(s)Q(s)x(s) + \sum_{i=1}^N \mu'_i(s)R_{ki}(s)\mu_i(s) \right] ds \right. \\ &\quad \left. + x'(t_F)Q(t_F)x(t_F) \right\}. \end{aligned} \quad (7.11)$$

From (7.8) in (7.11) gives

$$\begin{aligned} E\left\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\right\} &= E\left\{ \int_t^{t_F} \left[\left(\hat{x}(s) + \bar{x}(s)\right)' Q(s) \left(\hat{x}(s) + \bar{x}(s)\right) \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^N \mu'_i(s)R_{ki}(s)\mu_i(s) \right] ds + x'(t_F)Q(t_F)x(t_F) \right\}. \end{aligned} \quad (7.12)$$

Further expansion of the state variables in (7.12) gives

$$\begin{aligned}
E\left\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\right\} = & E\left\{\int_t^{t_F} \left[\hat{x}'(s)Q(s)\hat{x}(s) + \hat{x}'(s)Q(s)\bar{x}(s) \right. \right. \\
& \left. \left. + \bar{x}'(s)Q(s)\hat{x}(s) + \bar{x}'(s)Q(s)\bar{x}(s)\right] ds\right\} \\
& + \int_t^{t_F} E\left[\sum_{i=1}^N \mu'_i(s)R_{ki}(s)\mu_i(s)\right] ds \\
& + E\{x'(t_F)Q(t_F)x(t_F)\}.
\end{aligned} \tag{7.13}$$

Adding the state quadratic terms in (7.13) gives

$$\begin{aligned}
E\left\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\right\} = & E\left\{\int_t^{t_F} \left[\hat{x}'(s)Q(s)\hat{x}(s) + 2\hat{x}'(s)Q(s)\bar{x}(s) \right. \right. \\
& \left. \left. + \bar{x}'(s)Q(s)\bar{x}(s)\right] ds\right\} \\
& + \int_t^{t_F} E\left[\sum_{i=1}^N \mu'_i(s)R_{ki}(s)\mu_i(s)\right] ds \\
& + E\{x'(t_F)Q(t_F)x(t_F)\}.
\end{aligned} \tag{7.14}$$

Applying trace formula to the state terms in (7.14) gives

$$\begin{aligned}
E\left\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\right\} = & E\left\{\int_t^{t_F} \left[\hat{x}'(s)Q(s)\hat{x}(s) + 2\text{tr}\left(Q(s)\bar{x}(s)\hat{x}'(s)\right) \right. \right. \\
& \left. \left. + \text{tr}\left(Q(s)\bar{x}(s)\bar{x}'(s)\right)\right] ds\right\} \\
& + \int_t^{t_F} E\left[\sum_{i=1}^N \mu'_i(s)R_{ki}(s)\mu_i(s)\right] ds \\
& + E\{x'(t_F)Q(t_F)x(t_F)\}.
\end{aligned} \tag{7.15}$$

From projection theorem [101], the optimal state error $\bar{x}(t)$ is orthogonal to the optimal state

estimate $\hat{x}(t)$, thus $E(\hat{x}'(t)\bar{x}(t)) = E(\hat{x}(t)\bar{x}'(t)) = 0$, therefore manipulating (7.15) gives

$$\begin{aligned} E\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\} &= E\left\{\int_t^{t_F} \left[\hat{x}'(s)Q(s)\hat{x}(s) + \text{tr}\left(Q(s)\bar{x}(s)\bar{x}'(s)\right)\right] ds\right\} \\ &\quad + \int_t^{t_F} E\left[\sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s)\right] ds \\ &\quad + E\{x'(t_F)Q(t_F)x(t_F)\}. \end{aligned} \quad (7.16)$$

Switching the expectation and integral of the first line of (7.16) gives

$$\begin{aligned} E\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\} &= \int_t^{t_F} \left[E(\hat{x}'(s)Q(s)\hat{x}(s)) + E(\text{tr}(Q(s)\bar{x}(s)\bar{x}'(s)))\right] ds \\ &\quad + \int_t^{t_F} E\left[\sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s)\right] ds \\ &\quad + E\{x'(t_F)Q(t_F)x(t_F)\}. \end{aligned} \quad (7.17)$$

Switching the expectation and trace operator of the second term of the integral on the right-hand side of (7.17) becomes

$$\begin{aligned} E\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\} &= \int_t^{t_F} \left[E(\hat{x}'(s)Q(s)\hat{x}(s)) + \text{tr}\left(Q(s)E(\bar{x}(s)\bar{x}'(s))\right)\right] ds \\ &\quad + \int_t^{t_F} E\left[\sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s)\right] ds \\ &\quad + E\{x'(t_F)Q(t_F)x(t_F)\}. \end{aligned} \quad (7.18)$$

Substituting for $P(t) = E(\bar{x}(t)\bar{x}'(t))$ in the first line of (7.18) gives

$$\begin{aligned} E\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\} &= \int_t^{t_F} \left[E(\hat{x}'(s)Q(s)\hat{x}(s)) + \text{tr}(Q(s)P(s))\right] ds \\ &\quad + \int_t^{t_F} E\left[\sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s)\right] ds \\ &\quad + E\{x'(t_F)Q(t_F)x(t_F)\}. \end{aligned} \quad (7.19)$$

Replacing the last term of (7.19) with the optimal estimate state variable $\hat{x}(t_F)$ similar to

the state $x(t)$ replacement in the cost-to-go term leads to

$$\begin{aligned} E\left\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\right\} &= \int_t^{t_F} \left[E\left(\hat{x}'(s)Q(s)\hat{x}(s)\right) + \text{tr}\left(Q(s)P(s)\right) \right] ds \\ &+ \int_t^{t_F} E\left[\sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s)\right] ds \\ &+ E\left\{\hat{x}'(t_F)Q(t_F)\hat{x}(t_F)\right\} + \text{tr}\left(Q(t_F)P(t_F)\right). \end{aligned} \quad (7.20)$$

Further manipulations of the expectation E operator terms in (7.20) gives

$$\begin{aligned} E\left\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\right\} &= \int_t^{t_F} \left[E\left(\hat{x}'(s)Q(s)\hat{x}(s)\right) + \text{tr}\left(Q(s)P(s)\right) \right] ds \\ &+ \int_t^{t_F} \left[\sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s) \right] ds \\ &+ E\left\{\hat{x}'(t_F)Q(t_F)\hat{x}(t_F)\right\} + \text{tr}\left(Q(t_F)P(t_F)\right). \end{aligned} \quad (7.21)$$

The lemma is proved. $\square$

Remarks: This lemma shows that minimizing the expected value of the original cost function (7.7) is equivalent to minimizing the same expected value of cost function subject to (7.1) with state estimate $\hat{x}(t)$.

Lemma 7.2: The filtered state estimate $\hat{x}(t)$ of the true state $x(t)$ and output estimate are given as

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t)\right)dt + K(t)\left(dy(t) - d\hat{y}(t)\right). \quad (7.22)$$

$$d\hat{y}(t) = C(t)\hat{x}(t)dt, \quad (7.23)$$

where $K(t) = P(t)C(t)'W_2(t)^{-1}$ is the Kalman gain and $P(t)$ is the state estimate error covariance satisfying the following continuous Riccatic equation:

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)' - K(t)W_2(t)K(t)' + G(t)W_1(t)G(t)', \quad (7.24)$$

where $P(t_0) = P_0 = E\left\{\bar{x}(t_0)\bar{x}(t_0)'\right\}$ is the state estimate error covariance at time t_0 .

Proof: From (7.1) and (7.2), the state dynamics and output equations are given as

$$dx(t) = A(t)x(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t)\right)dt + G(t)dw_1(t). \quad (7.25)$$

$$dy(t) = C(t)x(t)dt + dw_2(t). \quad (7.26)$$

The state estimate equation is given as

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right)dt + K(t) \left(dy(t) - d\hat{y}(t) \right), \quad (7.27)$$

where the $K(t)$ is the Kalman gain.

Substituting for $dy(t)$ from (7.26) and $d\hat{y}(t)$ from (7.23) into (7.27) gives

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right)dt + K(t) \left(C(t)x(t)dt + dw_2(t) - C(t)\hat{x}(t)dt \right). \quad (7.28)$$

Subtracting (7.28) from (7.25) gives

$$\begin{aligned} dx(t) - d\hat{x}(t) = & A(t) \left(x(t) - \hat{x}(t) \right)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right)dt - \sum_{k=1}^N \left(B_k(t)u_k(t) \right)dt \\ & - K(t) \left(C(t)x(t)dt + dw_2(t) - C(t)\hat{x}(t)dt \right) + G(t)dw_1(t). \end{aligned} \quad (7.29)$$

Simplifying (7.29) gives

$$\begin{aligned} dx(t) - d\hat{x}(t) = & A(t) \left(x(t) - \hat{x}(t) \right)dt - K(t) \left(C(t)x(t)dt + dw_2(t) - C(t)\hat{x}(t)dt \right) \\ & + G(t)dw_1(t). \end{aligned} \quad (7.30)$$

Where $\bar{x}(t) = \left(x(t) - \hat{x}(t) \right)$ is the state estimation error, which is assumed to be uncorrelated to dw_1, dw_2 . Further simplification of (7.30) gives

$$d\bar{x}(t) = A(t)\bar{x}(t)dt - K(t)C(t)\bar{x}(t)dt - K(t)dw_2(t) + G(t)dw_1(t). \quad (7.31)$$

Factorizing terms in (7.31) gives

$$d\bar{x}(t) = \left(A(t) - K(t)C(t) \right) \bar{x}(t)dt + G(t)dw_1(t) - K(t)dw_2(t). \quad (7.32)$$

In compact form (7.32) is given as

$$d\bar{x}(t) = \hat{A}(t)\bar{x}(t)dt + G(t)dw_1(t) - K(t)dw_2(t), \quad (7.33)$$

with $\hat{A}(t) = \left(A(t) - K(t)C(t) \right)$.

Using the matrix exponential method, the state solution for (7.32) is given as

$$\bar{x}(t) = \Phi(t, t_0)\bar{x}(t_0) + \int_{t_0}^t \Phi(t, \tau) \left(G(\tau)dw_1(\tau) - K(\tau)dw_2(\tau) \right). \quad (7.34)$$

where $\Phi(\cdot)$ is the state transition matrix corresponding to $\dot{\bar{x}}(t) = \left(A(t) - K(t)C(t) \right)\bar{x}$

The transpose of (7.34) is given as

$$\bar{x}(t)' = \bar{x}(t_0)'\Phi(t, t_0)' + \int_{t_0}^t \left(dw_1(\tau)'G(\tau)' - dw_2(\tau)'K(\tau)' \right)\Phi(t, \tau)'. \quad (7.35)$$

However, the covariance $P(t)$ of the state estimate error ($\bar{x}$) is given as $P(t) = E\left(\bar{x}(t)\bar{x}(t)'\right)$, thus expanding $P(t)$ from (7.34) and (7.35) gives

$$P(t) = E\left(\bar{x}(t)\bar{x}(t)'\right) = E\left\{ \left(\Phi(t, t_0)\bar{x}(t_0) + \int_{t_0}^t \Phi(t, \tau) \left(G(\tau)dw_1(\tau) - K(\tau)dw_2(\tau) \right) \right) \left(\bar{x}(t_0)'\Phi(t, t_0)' + \int_{t_0}^t \left(dw_1(\tau)'G(\tau)' - dw_2(\tau)'K(\tau)' \right)\Phi(t, \tau)' \right) \right\}. \quad (7.36)$$

Multiplying out the terms in (7.36) gives

$$\begin{aligned} P(t) = E \left\{ \right. & \Phi(t, t_0)\bar{x}(t_0)\bar{x}(t_0)'\Phi(t, t_0)' \\ & + \int_{t_0}^t \Phi(t, t_0)\bar{x}(t_0) \left(dw_1(\tau)'G(\tau)' - dw_2(\tau)'K(\tau)' \right)\Phi(t, \tau)' \\ & + \int_{t_0}^t \Phi(t, \tau) \left(G(\tau)dw_1(\tau) - K(\tau)dw_2(\tau) \right) \bar{x}(t_0)'\Phi(t, t_0)' \\ & + \int_{t_0}^t \Phi(t, \tau) \left(G(\tau)dw_1(\tau) - K(\tau)dw_2(\tau) \right) \times \\ & \left. \int_{t_0}^t \left(dw_1(\tau)'G(\tau)' - dw_2(\tau)'K(\tau)' \right)\Phi(t, \tau)' \right\}. \end{aligned} \quad (7.37)$$

Readjusting the last term on the right-hand side of (7.37) gives

$$\begin{aligned}
P(t) = & E \left\{ \Phi(t, t_0) \bar{x}(t_0) \bar{x}(t_0)' \Phi(t, t_0)' \right. \\
& + \int_{t_0}^t \Phi(t, t_0) \bar{x}(t_0) \left(dw_1(\tau)' G(\tau)' - dw_2(\tau)' K(\tau)' \right) \Phi(t, \tau)' \\
& + \int_{t_0}^t \Phi(t, \tau) \left(G(\tau) dw_1(\tau) - K(\tau) dw_2(\tau) \right) \bar{x}(t_0)' \Phi(t, t_0)' \\
& + \int_{t_0}^t \int_{t_0}^t \Phi(t, \tau) \left(G(\tau) dw_1(\tau) - K(\tau) dw_2(\tau) \right) \times \\
& \left. \left(dw_1(\tau)' G(\tau)' - dw_2(\tau)' K(\tau)' \right) \Phi(t, \tau)' \right\}. \tag{7.38}
\end{aligned}$$

Expanding the expectation operation in (7.38) gives

$$\begin{aligned}
P(t) = & E \left\{ \Phi(t, t_0) \bar{x}(t_0) \bar{x}(t_0)' \Phi(t, t_0)' \right\} \\
& + E \left\{ \int_{t_0}^t \Phi(t, t_0) \bar{x}(t_0) \left(dw_1(\tau)' G(\tau)' - dw_2(\tau)' K(\tau)' \right) \Phi(t, \tau)' \right\} \\
& + E \left\{ \int_{t_0}^t \Phi(t, \tau) \left(G(\tau) dw_1(\tau) - K(\tau) dw_2(\tau) \right) \bar{x}(t_0)' \Phi(t, t_0)' \right\} \\
& + E \left\{ \int_{t_0}^t \int_{t_0}^t \Phi(t, \tau) \left(G(\tau) dw_1(\tau) - K(\tau) dw_2(\tau) \right) \times \right. \\
& \left. \left(dw_1(\tau)' G(\tau)' - dw_2(\tau)' K(\tau)' \right) \Phi(t, \tau)' \right\}. \tag{7.39}
\end{aligned}$$

Readjusting the integrals in the last term on the right-hand side of (7.39) gives

$$\begin{aligned}
P(t) = & E \left\{ \Phi(t, t_0) \bar{x}(t_0) \bar{x}(t_0)' \Phi(t, t_0)' \right\} \\
& + E \left\{ \int_{t_0}^t \Phi(t, t_0) \bar{x}(t_0) \left(dw_1(\tau)' G(\tau)' - dw_2(\tau)' K(\tau)' \right) \Phi(t, \tau)' \right\} \\
& + E \left\{ \int_{t_0}^t \Phi(t, \tau) \left(G(\tau) dw_1(\tau) - K(\tau) dw_2(\tau) \right) \bar{x}(t_0)' \Phi(t, t_0)' \right\} \\
& + E \left\{ \int_{t_0}^t \Phi(t, \tau) \left(G(\tau) dw_1(\tau) - K(\tau) dw_2(\tau) \right) \times \right. \\
& \left. \left(dw_1(\tau)' G(\tau)' - dw_2(\tau)' K(\tau)' \right) \Phi(t, \tau)' \right\}. \tag{7.40}
\end{aligned}$$

Simplifying (7.40) further gives

$$\begin{aligned}
P(t) = & E \left\{ \Phi(t, t_0) \bar{x}(t_0) \bar{x}(t_0)' \Phi(t, t_0)' \right\} \\
& + E \left\{ \int_{t_0}^t \Phi(t, \tau) K(\tau) dw_2(\tau) dw_2(\tau)' K(\tau)' \Phi(t, \tau)' \right\} \\
& - E \left\{ \int_{t_0}^t \Phi(t, \tau) K(\tau) dw_2(\tau) dw_1(\tau)' G(\tau)' \Phi(t, \tau)' \right\} \\
& - E \left\{ \int_{t_0}^t \Phi(t, \tau) G(\tau) dw_1(\tau) dw_2(\tau)' K(\tau)' \Phi(t, \tau)' \right\} \\
& + E \left\{ \int_{t_0}^t \Phi(t, \tau) G(\tau) dw_1(\tau) dw_1(\tau)' G(\tau)' \Phi(t, \tau)' \right\},
\end{aligned} \tag{7.41}$$

where $E(dw_2(\tau)) = E(dw_1(\tau)) = 0$.

Manipulating the expectation operator in (7.41) gives

$$\begin{aligned}
P(t) = & \Phi(t, t_0) E \left\{ \bar{x}(t_0) \bar{x}(t_0)' \right\} \Phi(t, t_0)' \\
& + \int_{t_0}^t \Phi(t, \tau) K(\tau) E \left(dw_2(\tau) dw_2(\tau)' \right) K(\tau)' \Phi(t, \tau)' \\
& - \int_{t_0}^t \Phi(t, \tau) K(\tau) E \left(dw_2(\tau) dw_1(\tau)' \right) G(\tau)' \Phi(t, \tau)' \\
& - \int_{t_0}^t \Phi(t, \tau) G(\tau) E \left(dw_1(\tau) dw_2(\tau)' \right) K(\tau)' \Phi(t, \tau)' \\
& + \int_{t_0}^t \Phi(t, \tau) G(\tau) E \left(dw_1(\tau) dw_1(\tau)' \right) G(\tau)' \Phi(t, \tau)'.
\end{aligned} \tag{7.42}$$

Further manipulation of (7.42) gives

$$\begin{aligned}
P(t) = & \Phi(t, t_0) P(t_0) \Phi(t, t_0)' + \int_{t_0}^t \Phi(t, \tau) K(\tau) W_2(\tau) K(\tau)' \Phi(t, \tau)' d\tau \\
& + \int_{t_0}^t \Phi(t, \tau) G(\tau) W_1(\tau) G(\tau)' \Phi(t, \tau)' d\tau,
\end{aligned} \tag{7.43}$$

where $E(dw_2(\tau) dw_1'(\tau)) = E(dw_1(\tau) dw_2'(\tau)) = 0$ and $P(t_0) = E \left\{ \bar{x}(t_0) \bar{x}(t_0)' \right\}$.

Taking the derivative of (7.43) using Leibnitz rule for the integrals gives

$$\begin{aligned}
\dot{P}(t) = & \frac{\partial \Phi(t, t_0)}{\partial t} P(t_0) \Phi(t, t_0)' + \Phi(t, t_0) \frac{\partial P(t_0)}{\partial t} \Phi(t, t_0)' + \Phi(t, t_0) P(t_0) \frac{\partial \Phi(t, t_0)'}{\partial t} \\
& + \int_{t_0}^t \frac{\partial \Phi(t, \tau)}{\partial t} K(\tau) W_2(\tau) K(\tau)' \Phi(t, \tau)' d\tau \\
& + \int_{t_0}^t \Phi(t, \tau) K(\tau) W_2(\tau) K(\tau)' \frac{\partial \Phi(t, \tau)'}{\partial t} d\tau \\
& + \int_{t_0}^t \frac{\partial \Phi(t, \tau)}{\partial t} G(\tau) W_1(\tau) G(\tau)' \Phi(t, \tau)' d\tau \\
& + \int_{t_0}^t \Phi(t, \tau) G(\tau) W_1(\tau) G(\tau)' \frac{\partial \Phi(t, \tau)'}{\partial t} d\tau \\
& + \Phi(t, t) \left(K(t) W_2(t) K(t)' + G(t) W_1(t) G(t)' \right) \Phi(t, t)'.
\end{aligned} \tag{7.44}$$

From (7.33), $\hat{A}(t) = A(t) - K(t)C(t)$, substituting for the derivatives in (7.44) using matrix exponential properties gives

$$\begin{aligned}
\dot{P}(t) = & \hat{A}(t) \Phi(t, t_0) P(t_0) \Phi(t, t_0)' + \Phi(t, t_0) P(t_0) \Phi(t, t_0)' \hat{A}(t)' \\
& + \int_{t_0}^t \hat{A}(t) \Phi(t, \tau) K(\tau) W_2(\tau) K(\tau)' \Phi(t, \tau)' d\tau \\
& + \int_{t_0}^t \Phi(t, \tau) K(\tau) W_2(\tau) K(\tau)' \Phi(t, \tau)' \hat{A}(t)' d\tau \\
& + \int_{t_0}^t \hat{A}(t) \Phi(t, \tau) G(\tau) W_1(\tau) G(\tau)' \Phi(t, \tau)' d\tau \\
& + \int_{t_0}^t \Phi(t, \tau) G(\tau) W_1(\tau) G(\tau)' \Phi(t, \tau)' \hat{A}(t)' d\tau \\
& + \Phi(t, t) \left(K(t) W_2(t) K(t)' + G(t) W_1(t) G(t)' \right) \Phi(t, t)'.
\end{aligned} \tag{7.45}$$

Rearranging (7.45) gives

$$\begin{aligned}
\dot{P}(t) = & \hat{A}(t) \left[\Phi(t, t_0) P(t_0) \Phi(t, t_0)' + \int_{t_0}^t \Phi(t, \tau) K(\tau) W_2(\tau) K(\tau)' \Phi(t, \tau)' d\tau \right. \\
& \left. + \int_{t_0}^t \Phi(t, \tau) G(\tau) W_1(\tau) G(\tau)' \Phi(t, \tau)' d\tau \right] \\
& + \left[\Phi(t, t_0) P(t_0) \Phi(t, t_0)' + \int_{t_0}^t \Phi(t, \tau) K(\tau) W_2(\tau) K(\tau)' \Phi(t, \tau)' d\tau \right. \\
& \left. + \int_{t_0}^t \Phi(t, \tau) G(\tau) W_1(\tau) G(\tau)' \Phi(t, \tau)' d\tau \right] \hat{A}(t)' \\
& + \Phi(t, t) \left(K(t) W_2(t) K(t)' + G(t) W_1(t) G(t)' \right) \Phi(t, t)'.
\end{aligned} \tag{7.46}$$

Simplifying (7.46) using (7.43) gives

$$\dot{P}(t) = \hat{A}(t)P(t) + P(t)\hat{A}(t)' + K(t)W_2(t)K(t)' + G(t)W_1(t)G(t)', \quad (7.47)$$

where $\Phi(t, t)$ is an identity matrix.

Further simplifying of (7.47) using $\hat{A}(t) = A(t) - K(t)C(t)$ gives

$$\begin{aligned} \dot{P}(t) = & \left(A(t) - K(t)C(t) \right) P(t) + P(t) \left(A(t)' - C(t)'K(t)' \right) \\ & + K(t)W_2(t)K(t)' + G(t)W_1(t)G(t)'. \end{aligned} \quad (7.48)$$

Minimizing $\dot{P}(t)$ in (7.48) with respect to $K(t)$ to determine the gain $K(t)$ gives

$$\begin{aligned} \frac{\partial \dot{P}(t)}{\partial K(t)} &= -C(t)P(t) - P(t)C(t)' + 2K(t)W_2(t) = 0, \\ K(t) &= P(t)C(t)'W_2(t)^{-1}. \end{aligned} \quad (7.49)$$

Substituting the value of $K(t)$ from (7.49) into (7.48) gives

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)' - P(t)C(t)'W_2(t)^{-1}C(t)P(t) + G(t)W_1(t)G(t)'. \quad (7.50)$$

which is same as

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)' - K(t)W_2(t)K(t)' + G(t)W_1(t)G(t)'. \quad (7.51)$$

The lemma is proved. □

Remarks: Equation (7.50) is the continuous Riccatic equation for the solving the Kalman filter error covariance. The Kalman gain does not depend on the player strategy parameters. The Kalman gain only depends on the difference between the measured and estimated output.

Lemma 7.3: Consider the system equation (7.1), (7.2) and the cost functions (7.7), (7.9). Assume $V_1^k(t, \hat{x}) \in C_p^{1,2}(\hat{Q}) \cap C(\bar{\bar{Q}})$. Let V_1^k be an admissible first cumulant cost function with $\mu_k \in U_{M_k}$ as the player strategy corresponding to V_1^k . Then,

$$\begin{aligned} & \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + \hat{x}'(t)Q(t)\hat{x}(t) + \text{tr} \left(Q(t)P(t) \right) \\ & + \sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s) = 0, \end{aligned} \quad (7.52)$$

where $V_1^k(t_F, \hat{x}) = \Psi_1^k(t_F, \hat{x})$, $P(t) = E\{\bar{x}(t)\bar{x}(t)'\}$, $Q(t)$ is a semi-positive definite matrix.

Proof: From Dynkin's formula [68] for Markov diffusion process,

$$\Phi(t, \hat{x}) = E \left\{ \int_{t_0}^{t_F} -\mathcal{O}^k \Phi(s, \hat{x}) ds + \Phi(t_F, \hat{x}(t_F)) | \hat{x}(t) = \hat{x} \right\}. \quad (7.53)$$

assuming $\Phi(t, \hat{x}) \in C_p^{1,2}(\bar{Q}_0)$, μ_k is admissible and $E\{|\hat{x}(s)|^n | \hat{x}(t) = \hat{x}\}$ is bounded for $n = 1, 2, \dots, m$ ensures term in the right-hand side of (7.53) are finite and $t \in T$.

From the first cumulant cost function $V_1^k(t, \hat{x})$, applying Dynkin formula gives

$$V_1^k(t, \hat{x}) = E \left\{ \int_{t_0}^{t_F} -\mathcal{O}^k V_1^k(s, \hat{x}) ds + V_1^k(t_F, \hat{x}(t_F)) | \hat{x}(t) = \hat{x} \right\}. \quad (7.54)$$

However, the first cumulant cost function for (7.1) and (7.2) is given as (7.9). From certainty equivalence principle and the cost-moment formulations in [38], the moment function $M_1^k = V_1^k$ and running cost function L^k for the k -th player satisfy the following relationship

$$\mathcal{O}^k [M_1^k(t, \hat{x})] + M_0^k(t, \hat{x}) L^k(\mu_1, \dots, \mu_k, \dots, \mu_N) = 0. \quad (7.55)$$

where $L^k = (\hat{x}'(s)Q(s)\hat{x}(s)) + \text{tr}(Q(s)P(s)) + \sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s)$ (7.9) and $V_1^k(t_F, \hat{x}) = \Psi_1^k(t_F, \hat{x})$. But $M_1^k(t, \hat{x}) = V_1^k(t, \hat{x})$, thus

$$\mathcal{O}^k [V_1^k(t, \hat{x})] + M_0^k(t, \hat{x}) L^k(\mu_1, \dots, \mu_k, \dots, \mu_N) = 0. \quad (7.56)$$

Substituting for L^k in (7.56) gives

$$\begin{aligned} & \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) [V_1^k(t, \hat{x})] + \hat{x}'(t)Q(t)\hat{x}(t) + \text{tr}(Q(t)P(t)) \\ & + \sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s) = 0, \end{aligned} \quad (7.57)$$

where $V_1^k(t_F, \hat{x}) = \Psi_1^k(t_F, \hat{x}) = 0$ and $M_0^k = 1$.

The lemma is proved. $\square$

Remarks: Certainty equivalence principle is assumed in the derivation by relating (7.56) to the completely observed system case.

Lemma 7.4: The optimal feedback player strategy $\mu_k^*(t, \hat{x}(t))$ is given as

$$\mu_k^*(t, \hat{x}(t)) = I_k(t) \left(\frac{\partial V_1^{k*}(t, \hat{x})}{\partial \hat{x}} \right)' \quad (7.58)$$

which is same as

$$\mu_k^*(t, \hat{x}(t)) = -R_{kk}^{-1}(t)B_k(t)'Q_k(t)\hat{x}(t), \quad (7.59)$$

where $I_k(t) = -\frac{1}{2}R_{kk}^{-1}(t)B_k(t)'$, is the feedback gain, $k = 1, 2, \dots, N$ and $V_1^{k*}(t, \hat{x})$ is the first-cumulant value function of the players.

Proof: Let $v(t)$ be innovative process [102], [101] corresponding to the system (7.1), (7.2) with orthogonal increments. Thus, $v(t)$ is a white-noise process with same covariance as $dw_2(t)$. Then, $dv(t)$ is defined as

$$dv(t) \triangleq dy(t) - d\hat{y}(t) \triangleq dy(t) - C\hat{x}(t)dt. \quad (7.60)$$

From (7.2), equation (7.60) becomes

$$dv(t) \triangleq C(t)x(t)dt + dw_2(t) - C\hat{x}(t)dt. \quad (7.61)$$

From (7.27) the state estimate equation of the system (7.1), (7.2) is,

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right)dt + K(t) \left(dy(t) - d\hat{y}(t) \right). \quad (7.62)$$

Substituting (7.60) for the last term of (7.62) gives

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right)dt + K(t)dv(t). \quad (7.63)$$

where $K(t) = P(t)C(t)'W_2(t)^{-1}$ from (7.49). Equation (7.62) is the generalized system state estimate dynamics.

Substituting for $K(t)$ in (7.63) gives

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right)dt + P(t)C(t)'W_2(t)^{-1}dv(t). \quad (7.64)$$

From (7.9), the expected value of the cost function of the system (7.1), (7.2) is

$$\begin{aligned} E\left\{ J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N) \right\} &= \int_t^{t_F} \left[E\left(\hat{x}'(s)Q(s)\hat{x}(s) \right) + \text{tr}\left(Q(s)P(s) \right) \right] ds \\ &\quad + \int_t^{t_F} \left[\sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s) \right] ds \\ &\quad + E\left\{ \hat{x}'(t_F)Q(t_F)\hat{x}(t_F) \right\} + \text{tr}\left(Q(t_F)P(t_F) \right). \end{aligned} \quad (7.65)$$

where $k = 1, 2, \dots, N$, $\hat{x}(t_F) = \hat{x}_f$, $Q(\cdot)$ is positive semi-definite and $R_{ki}(t)$ is positive definite for $k = i$ and positive-semi definite for $k \neq i$.

The k -th player first cumulant value function V_1^{k*} is given as

$$V_1^{k*}(t, \hat{x}) = \min_{\mu_k \in U_{M_k}} E \left\{ J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N) \right\}, \quad (7.66)$$

where $\mu_k^*(t, \hat{x})$ is the k -th player optimal strategy.

From Lemma 7.3 (7.57), the Hamilton-Jacobi-Bellman equation (HJB) [65] for the k -th player strategy satisfy, the following equation

$$\begin{aligned} \min_{\mu_k \in U_{M_k}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) [V_1^k(t, \hat{x})] + \hat{x}'(t)Q(t)\hat{x}(t) + \text{tr}(Q(t)P(t)) \right. \\ \left. + \sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s) \right\} = 0, \end{aligned} \quad (7.67)$$

where $V_1^k(t_F, x) = \Psi_1^k(t_F, \hat{x}) = 0$.

Using the backward operator definition (2.5), (7.63) and expanding (7.67) gives

$$\begin{aligned} 0 = - \min_{\mu_k \in U_{M_k}} \left\{ \left(\frac{\partial V_1^k(t, \hat{x})}{\partial t} \right) + \hat{x}'(t)Q(t)\hat{x}(t) + \text{tr}(Q(t)P(t)) + \sum_{i=1}^N \mu_i(t)R_{ki}(t)\mu_i'(t) \right. \\ \left. + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \left\{ \hat{x}(t)'A(t)' + \sum_{i=1}^N \mu_i(t)'B_i(t)' \right\} \right. \\ \left. + \frac{1}{2} \text{tr} \left(K(t)W_2(t)K(t)' \left(\frac{\partial^2 V_1^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) \right\}. \end{aligned} \quad (7.68)$$

Rearranging of (7.68) gives

$$\begin{aligned} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial t} \right) = - \min_{\mu_k \in U_{M_k}} \left\{ \hat{x}'(t)Q(t)\hat{x}(t) + \text{tr}(Q(t)P(t)) + \sum_{i=1}^N \mu_i(t)R_{ki}(t)\mu_i'(t) \right. \\ \left. + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \left\{ \hat{x}(t)'A(t)' + \sum_{i=1}^N \mu_i(t)'B_i(t)' \right\} \right. \\ \left. + \frac{1}{2} \text{tr} \left(K(t)W_2(t)K(t)' \left(\frac{\partial^2 V_1^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) \right\}. \end{aligned} \quad (7.69)$$

Substituting for $K(t)$ in the trace term in (7.69) gives

$$\begin{aligned} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial t} \right) = & - \min_{\mu_k \in U_{M_k}} \left\{ \hat{x}'(t) Q(t) \hat{x}(t) + \text{tr} \left(Q(t) P(t) \right) + \sum_{i=1}^N \mu_i(t) R_{ki}(t) \mu_i'(t) \right. \\ & + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \left\{ \hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right\} \\ & \left. + \frac{1}{2} \text{tr} \left(P(t) C(t)' W_2(t)^{-1} \left(\frac{\partial^2 V_1^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) \right\}. \end{aligned} \quad (7.70)$$

The optimal k -th player strategy is found by minimizing (7.70) with respect to $\mu_k(t, \hat{x})$ to give

$$\begin{aligned} 0 = & 2\mu_k(t, \hat{x}) R_{kk}(t) + \left(\frac{\partial V_1^{k*}(t, \hat{x})}{\partial \hat{x}} \right) B_k(t)', \\ \mu_k^*(t, \hat{x}) = & -\frac{1}{2} R_{kk}^{-1}(t) B_k(t)' \left(\frac{\partial V_1^{k*}(t, \hat{x})}{\partial \hat{x}} \right). \end{aligned} \quad (7.71)$$

Let the value function $V_1^{k*}(t, \hat{x})$ which is quadratic in state for the k -th player be defined as

$$V_1^{k*}(t, \hat{x}) = \hat{x}'(t) Q_k(t) \hat{x}(t) + \eta(t). \quad (7.72)$$

Differentiating (7.72) with respect to state estimate $\hat{x}(t)$ gives

$$\left(\frac{\partial V_1^{k*}(t, \hat{x}(t))}{\partial \hat{x}(t)} \right) = 2Q_k(t) \hat{x}(t). \quad (7.73)$$

Substituting (7.73) into (7.71) becomes

$$\mu_k^*(t, \hat{x}) = -R_{kk}^{-1}(t) B_k(t)' Q_k(t) \hat{x}(t). \quad (7.74)$$

The lemma is proved. $\square$

Remark: The feedback gain $I_k(t)$ does not depend on the noise parameters. The certainty equivalence principle applies to the first cumulant game as the state $x(t)$ in deterministic game can be replaced with state estimate $\hat{x}(t)$ to compute the players strategies.

7.3 Ad hoc Output feedback Statistical Game

7.3.1 N -player Minimal 2-nd Cumulant Statistical Game

In this section, the second cumulant statistical linear output feedback Nash game will be formulated using cost cumulant concept. Some relevant definitions and assumption will be stated and utilized in the second cumulant game analysis.

Definition 7.1: A function $M_1^k : \bar{Q}_0 \rightarrow \mathbb{R}^+$ is an admissible first moment cost function if there exists a strategy μ_k such that

$$M_1^k(t, \hat{x}) = M_1^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N), \quad (7.75)$$

for $t \in T, x \in \mathbb{R}^n, M_1^k \in C^{1,2}(\bar{Q}_0)$. Also, V_1^k is the admissible first cumulant cost function for the k -th player related to the moment function through the moment-cumulant relation (2.9). In addition, $\mu_k \in U_{M_k}, V_1^k(t, \hat{x}) = V_1^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N)$.

Definition 7.2: A class of admissible strategy U_{M_k} is defined such that if $\mu_k \in U_{M_k}$ then μ_k satisfies the equality of Definition 7.1 for M_0^k, M_1^k . It should be noted that first moment M_1^k is the same as first cumulant $V_1^k, M_0^k = 1$ and $V_0^k = 0$.

Definition 7.3: Let V_1^k be the k -th player admissible cumulant cost functions. The player strategy μ_k^* is the k -th player equilibrium solution if it is such that

$$V_2^{k*}(t, \hat{x}) = V_2^k(t, \hat{x}; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq V_2^k(t, \hat{x}; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*). \quad (7.76)$$

for all $\mu_k \in U_{M_k}$ where the set $\{\mu_1^*, \mu_2^*, \dots, \mu_N^*\}$ is a Nash equilibrium solution and the set $\{V_2^{1*}, V_2^{2*}, \dots, V_2^{N*}\}$ is the Nash equilibrium value set.

Problem Definition: Consider an open set $\hat{Q} \subset Q_0$ and let the k -th player cost cumulant functions $V_1^k(t, \hat{x}) \in C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ be an admissible cumulant function where the set $C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ means that the function V_1^k satisfy polynomial growth condition and is continuous in the first and second derivatives of $\hat{Q}$, and continuous on the closure of $\hat{Q}$. Assume the existence of a suboptimal strategy $\mu_k^* \in U_{M_k}$ and suboptimal value function $V_2^{k*}(t, \hat{x}) \in C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ for the k -th player. Thus, the N -player second cumulant output feedback game problem is to find the Nash equilibrium strategy $\mu_k^*(t, \hat{x})$ for the partially-observed state system with $k = 1, 2, \dots, N$ which results in the suboptimal 2-nd value function $V_2^{k*}(t, x)$ given as

$$V_2^{k*}(t, \hat{x}) = \min_{\mu_k \in U_{M_k}} \left\{ V_2^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \right\}, \quad (7.77)$$

for $k = 1, 2, \dots, N$.

Theorem 7.1: Under the definitions and Lemmas 7.1 - 7.4, let the solutions to the k -th player second cumulant output feedback game be given by

$$\mu_k^*(t)(t, \hat{x}) = -\frac{1}{2}R_{kk}^{-1}(t)B_k(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} + \gamma_{2k} \frac{\partial V_2^{k*}(t, \hat{x})}{\partial \hat{x}} \right). \quad (7.78)$$

where γ_{2k} is the Lagrange multiplier and V_1^k, V_2^k are the first, second cumulant cost functions and solutions of the following coupled Riccatic equations:

$$\begin{aligned} 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(t, \hat{x}, \mu_1, \dots, \mu_k, \dots, \mu_N), \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right), \end{aligned} \quad (7.79)$$

where $M_0^k = 1$ and $\mathcal{O}^k(\cdot)$ is the backward operator (2.5) and the state equation is (7.63).

Proof: From the system equation (2.1) and the k -th player second cumulant full-state feedback game (4.2), the minimal 2-nd value function $V_2^{k*}(t, \hat{x})$ satisfies the following HJB equation for the k -th player.

$$\begin{aligned} 0 &= \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) \left[V_2^{k*}(t, x) \right] + \right. \\ &\quad \left. \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_1^k(t, x)}{\partial x} \right) \right\}, \end{aligned} \quad (7.80)$$

with terminal condition $V_j^k(t_F, x) = 0, k = 1, 2, \dots, N, j = 1, 2$, then

$$\begin{aligned} 0 &= \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) \left[V_2^{k*}(t, \hat{x}) \right] + \right. \\ &\quad \left. \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \right\}, \end{aligned} \quad (7.81)$$

where $K(t)$ is the Kalman filter defined in (7.49) and the certainty equivalence principle has been assumed.

Since the first cost cumulant function V_1^k is admissible, the following coupled equations are satisfied

$$\begin{aligned} 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(t, \hat{x}, \mu_1, \dots, \mu_k, \dots, \mu_N), \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right), \end{aligned} \quad (7.82)$$

where $M_0^k = 1$ and $\mathcal{O}^k(\cdot)$ is the backward operator (2.5) and the state equation (7.63).

Minimizing (7.82) with respect to player strategy $\mu_k(t, \hat{x})$ gives

$$0 = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(t, \hat{x}, \mu_1, \dots, \mu_k, \dots, \mu_N) \right. \\ \left. + \gamma_{2k} \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \gamma_{2k} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial x} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \right\}, \quad (7.83)$$

where γ_{2k} is the Lagrange multiplier.

Using the backward operator definition (2.5) and expanding (7.83) gives

$$\min_{\mu_k \in U_{M^k}} \left\{ \left(\frac{\partial V_1^k(t, \hat{x})}{\partial t} \right) + \hat{x}'(t) Q(t) \hat{x}(t) + \text{tr} \left(Q(t) P(t) \right) + \sum_{i=1}^N \mu_i(t) R_{ki}(t) \mu_i'(t) \right. \\ \left. + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \left(\hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right) + \frac{1}{2} \text{tr} \left(K(t) W_2 K(t)' \left(\frac{\partial^2 V_1^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) \right. \\ \left. + \gamma_{2k} \left(\frac{\partial V_2^{k*}(t, \hat{x})}{\partial t} \right) + \gamma_{2k} \left(\frac{\partial V_2^{k*}(t, \hat{x})}{\partial \hat{x}} \right) \left(\hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right) \right. \\ \left. + \frac{\gamma_{2k}}{2} \text{tr} \left(K(t) W_2 K(t)' \left(\frac{\partial^2 V_2^{k*}(t, \hat{x})}{\partial \hat{x}^2} \right) \right) + \gamma_{2k} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \right\} = 0. \quad (7.84)$$

Minimizing (7.84) with respect to $\mu_k(t, \hat{x})$ gives

$$2\mu_k(t) R_{kk}(t) + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) B_k(t)' + \gamma_{2k} \left(\frac{\partial V_2^{k*}(t, \hat{x})}{\partial \hat{x}} \right) B_k(t)' = 0. \quad (7.85)$$

Further simplification of (7.85) gives

$$\mu_k^*(t, \hat{x}) = -\frac{1}{2} R_{kk}^{-1}(t) B_k(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} + \gamma_{2k} \frac{\partial V_2^{k*}(t, \hat{x})}{\partial \hat{x}} \right). \quad (7.86)$$

Remark: The minimum strategy for the k -th player $\mu_k^*(t, \hat{x})$ is the suboptimal strategy derived from the output feedback coupled equation (7.82). In (7.82), the certainty equivalent principle has been extended to the second cumulant output feedback game where a Kalman filter is used for state estimation. The strategy is suboptimal since it was derived from suboptimal cost cumulants.

7.3.2 N -player Minimal 3-rd Cumulant Statistical Game

In this section, the third cumulant statistical linear output feedback Nash game will be formulated using cost cumulant concept. Some relevant definitions and assumption will be stated and utilized in the third cumulant game analysis.

Definition 7.4: A function $V_1^k, V_2^k : \bar{Q}_0 \rightarrow \mathbb{R}^+$ are admissible first and second cumulant cost function if there exists a strategy μ_k such that

$$V_i^k(t, \hat{x}) = V_i^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N), \quad (7.87)$$

for $t \in T, \hat{x} \in \mathbb{R}^n, i = 1, 2$. $V_i^k \in C^{1,2}(\bar{Q}_0)$. Also, V_i^k are the admissible i -th cumulant cost function for the k -th player related to the through the moment-cumulant relation (2.9). In addition, $\mu_k \in U_{M_k}, V_i^k(t, \hat{x}) = V_i^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N)$.

Definition 7.5: A class of admissible strategy U_{M_k} is defined such that if $\mu_k \in U_{M_k}$ then μ_k satisfies the equality of Definition 7.4 for M_0^k, V_1^k, V_2^k . It should be noted that first moment M_1^k is the same as first cumulant $V_1^k, M_0^k = 1$ and $V_0^k = 0$.

Definition 7.6: Let V_i^k be the k -th player admissible cumulant cost functions. The player strategy μ_k^* is the k -th player equilibrium solution if it is such that

$$V_3^{k*}(t, \hat{x}) = V_3^k(t, \hat{x}; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq V_3^k(t, \hat{x}; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*), \quad (7.88)$$

for all $\mu_k \in U_{M_k}$, where the set $\{\mu_1^*, \mu_2^*, \dots, \mu_N^*\}$ is a Nash equilibrium solution and the set $\{V_2^{1*}, V_2^{2*}, \dots, V_2^{N*}\}$ is the Nash equilibrium value set.

Problem Definition: Consider an open set $\hat{Q} \subset Q_0$ and let the k -th player cost cumulant functions $V_i^k(t, \hat{x}) \in C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ be an admissible cumulant function where the set $C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ means that the function V_i^k satisfy polynomial growth condition, are continuous in the first and second derivatives of $\hat{Q}$, and continuous on the closure of $\hat{Q}$. Assume the existence of an suboptimal strategy $\mu_k^* \in U_{M_k}$ and suboptimal value function $V_3^{k*}(t, \hat{x}) \in C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ for the k -th player. Thus, the N -player third cumulant output feedback game problem is to find the Nash equilibrium strategy $\mu_k^*(t, \hat{x})$ for the partially-observed state system

with $k = 1, 2, \dots, N$ which results in the suboptimal 3-rd value function $V_3^{k*}(t, x)$ given as

$$V_3^{k*}(t, \hat{x}) = \min_{\mu_k \in U_{M_k}} \left\{ V_3^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \right\}, \quad (7.89)$$

for $k = 1, 2, \dots, N$.

Theorem 7.2: Under the definitions and Lemmas 7.1 - 7.4, let the solutions to the k -th player third cumulant output feedback game be given by

$$\mu_k^*(t)(t, \hat{x}) = -\frac{1}{2} R_{kk}^{-1}(t) B_k(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} + \gamma_{2k} \frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} + \gamma_{3k} \frac{\partial V_3^{k*}(t, \hat{x})}{\partial \hat{x}} \right). \quad (7.90)$$

where γ_{2k}, γ_{3k} are the Lagrange multipliers. $V_i^k, i = 1, 2, 3$ is the i -th cumulant cost function and solutions of the following coupled Riccatic equations:

$$\begin{aligned} 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(t, \hat{x}, \mu_1, \dots, \mu_k, \dots, \mu_N), \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right), \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_3^k(t, \hat{x}) \right] + 3 \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} \right), \end{aligned} \quad (7.91)$$

where $M_0^k = 1$ and $\mathcal{O}^k(\cdot)$ is the backward operator (2.5) and the state equation (7.63).

Proof: From the system equation (2.1) and the k -th player third cumulant full-state feedback game (4.2), the minimal 3-rd value function $V_3^{k*}(t, x)$ satisfies the following HJB equation for the k -th player.

$$\begin{aligned} 0 &= \min_{\mu_k \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) \left[V_3^{k*}(t, x) \right] \right. \\ &\quad \left. + 3 \left(\frac{\partial V_1^k(t, x)}{\partial x} \right)' \sigma(t, x) W \sigma(t, x)' \left(\frac{\partial V_2^k(t, x)}{\partial x} \right) \right\}, \end{aligned} \quad (7.92)$$

with terminal condition $V_j^k(t_F, x) = 0, k = 1, 2, \dots, N, j = 1, 2, 3$, then

$$\begin{aligned} 0 &= \min_{\mu_k \in U_{M^N}} \left\{ \mathcal{O}^k(\mu_1^*, \dots, \mu_k, \dots, \mu_N^*) \left[V_3^{k*}(t, \hat{x}) \right] \right. \\ &\quad \left. + 3 \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} \right) \right\}, \end{aligned} \quad (7.93)$$

where $K(t)$ is the Kalman filter defined in (7.49) and the certainty equivalence principle has been assumed.

Since the first cost cumulant function V_1^k is admissible, the following coupled equations are satisfied

$$\begin{aligned} 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(t, \hat{x}, \mu_1, \dots, \mu_k, \dots, \mu_N), \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right), \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_3^k(t, \hat{x}) \right] + 3 \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} \right), \end{aligned} \quad (7.94)$$

where $M_0^k = 1$ and $\mathcal{O}^k(\cdot)$ is the backward operator (2.5) and the state equation (7.63).

Minimizing (7.94) with respect to player strategy μ_k gives

$$\begin{aligned} 0 &= \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(t, \hat{x}, \mu_1, \dots, \mu_k, \dots, \mu_N) \right. \\ &\quad + \gamma_{2k} \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \gamma_{2k} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \\ &\quad \left. + \gamma_{3k} \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_3^{k*}(t, \hat{x}) \right] + 3\gamma_{3k} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} \right) \right\}, \end{aligned} \quad (7.95)$$

where γ_{2k}, γ_{3k} are the lagrange multipliers.

Using the backward operator definition (2.5) and expanding (7.95) gives

$$\begin{aligned} \min_{\mu_k \in U_{M^k}} &\left\{ \left(\frac{\partial V_1^k(t, \hat{x})}{\partial t} \right) + \hat{x}'(t) Q(t) \hat{x}(t) + \text{tr} \left(Q(t) P(t) \right) + \sum_{i=1}^N \mu_i(t) R_{ki}(t) \mu_i'(t) \right. \\ &+ \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \left(\hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right) + \frac{1}{2} \text{tr} \left(K(t) W_2 K(t)' \left(\frac{\partial^2 V_1^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) \\ &+ \gamma_{2k} \left(\frac{\partial V_2^k(t, \hat{x})}{\partial t} \right) + \gamma_{2k} \left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} \right) \left(\hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right) \\ &+ \frac{\gamma_{2k}}{2} \text{tr} \left(K(t) W_2 K(t)' \left(\frac{\partial^2 V_2^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) + \gamma_{2k} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \\ &+ \gamma_{3k} \left(\frac{\partial V_3^{k*}(t, \hat{x})}{\partial t} \right) + \gamma_{3k} \left(\frac{\partial V_3^{k*}(t, \hat{x})}{\partial \hat{x}} \right) \left(\hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right) \\ &\left. + \frac{\gamma_{3k}}{2} \text{tr} \left(K(t) W_2 K(t)' \left(\frac{\partial^2 V_3^{k*}(t, \hat{x})}{\partial \hat{x}^2} \right) \right) + 3\gamma_{3k} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} \right) \right\} = 0. \end{aligned} \quad (7.96)$$

Minimizing (7.96) gives

$$2\mu_k(t)R_{kk}(t) + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}}\right)B_k(t)' + \gamma_{2k}\left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}}\right)B_k(t)' + \gamma_{3k}\left(\frac{\partial V_3^{k*}(t, \hat{x})}{\partial \hat{x}}\right)B_k(t)' = 0. \quad (7.97)$$

Further simplification of (7.85) gives

$$\mu_k^*(t, \hat{x}) = -\frac{1}{2}(t)R_{kk}^{-1}(t)B_k(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} + \gamma_{2k}\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} + \gamma_{3k}\frac{\partial V_3^{k*}(t, \hat{x})}{\partial \hat{x}} \right). \quad (7.98)$$

Remark: The minimum strategy for the k -th player $\mu_k^*(t, \hat{x})$ is the suboptimal strategy derived from the output feedback coupled equation (7.94). In (7.94), the certainty equivalent principle has been extended to the third cumulant output feedback game where a Kalman filter is used for state estimation. The strategy is suboptimal since it was derived from suboptimal cumulants.

7.3.3 N -player Minimal m -th Cumulant Statistical Game

In this section, the m -th cumulant statistical linear output feedback Nash game will be formulated using cost cumulant concept. Some relevant definitions and assumption will be stated and utilized in the m -th cumulant game analysis.

Definition 7.7: A function $V_i^k : \bar{Q}_0 \rightarrow \mathbb{R}^+$ be admissible i -th cumulant cost function if there exists a strategy μ_k such that

$$V_i^k(t, \hat{x}) = V_i^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N), \quad (7.99)$$

for $t \in T, x \in \mathbb{R}^n, i = 1, 2, \dots, m, V_i^k \in C^{1,2}(\bar{Q}_0)$. Also, V_i^k is the admissible i -th cumulant cost function for the k -th player related to the through the moment-cumulant relation (2.9). In addition, $\mu_k \in U_{M_k}, V_i^k(t, \hat{x}) = V_i^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N)$.

Definition 7.8: A class of admissible strategy U_{M_k} is defined such that if $\mu_k \in U_{M_k}$ then μ_k satisfies the equality of Definition 7.7 for $M_0^k, V_i^k, i = 1, 2, \dots, m - 1$.

Definition 7.9: Let V_i^k be the k -th player admissible cumulant cost functions. The player strategy μ_k^* is the k -th player equilibrium solution if it is such that

$$V_m^{k*}(t, \hat{x}) = V_m^k(t, \hat{x}; \mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_N^*) \leq V_m^k(t, \hat{x}; \mu_1^*, \mu_2^*, \dots, \mu_k, \dots, \mu_N^*), \quad (7.100)$$

for all $\mu_k \in U_{M_k}$, where the set $\{\mu_1^*, \mu_2^*, \dots, \mu_N^*\}$ is a Nash equilibrium solution and the set $\{V_m^{1*}, V_m^{2*}, \dots, V_m^{N*}\}$ is the Nash equilibrium value set.

Problem Definition: Consider an open set $\hat{Q} \subset Q_0$ and let the k -th player cost cumulant functions $V_i^k(t, \hat{x}) \in C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ be an admissible cumulant function where the set $C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ means that the function V_i^k satisfy polynomial growth condition, are continuous in the first and second derivatives of $\hat{Q}$, and continuous on the closure of $\hat{Q}$. Assume the existence of an suboptimal strategy $\mu_k^* \in U_{M_k}$ and suboptimal value function $V_m^{k*}(t, \hat{x}) \in C_p^{1,2}(\hat{Q}) \cap C(\bar{\hat{Q}})$ for the k -th player. Thus, the N -player m -th cumulant output feedback game problem is to find the Nash equilibrium strategy $\mu_k^*(t, \hat{x})$ for the partially-observed state system with $k = 1, 2, \dots, N$ which results in the suboptimal m -th value function $V_m^{k*}(t, x)$ given as

$$V_m^{k*}(t, \hat{x}) = \min_{\mu_k \in U_{M_k}} \left\{ V_m^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \right\}, \quad (7.101)$$

for $k = 1, 2, \dots, N$.

Theorem 7.3: Under the definitions and Lemmas 7.1 - 7.4, let the solutions to the k -th player m -th cumulant output feedback game be given by

$$\mu_k^*(t, \hat{x}) = -\frac{1}{2} R_{kk}^{-1}(t) B_k(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} + \gamma_{2k} \frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{mk} \frac{\partial V_m^{k*}(t, \hat{x})}{\partial \hat{x}} \right). \quad (7.102)$$

where γ_{mk} is the Lagrange multiplier. $V_i^k, i = 1, 2, \dots, m$ is the i -th cumulant cost function and solutions of the following coupled Riccatic equations:

$$\begin{aligned} 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(\mu_1, \dots, \mu_k, \dots, \mu_N), \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right), \\ &\vdots \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_m^{k*}(t, \hat{x}) \right] \\ &\quad + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_{m-i}^k(t, \hat{x})}{\partial \hat{x}} \right), \end{aligned} \quad (7.103)$$

where $M_0^k = 1$ and $\mathcal{O}^k(\cdot)$ is the backward operator (2.5) and state equation (7.63).

Proof: From the system equation (2.1) and the k -th player m -th cumulant full-state feedback game (4.2), the minimal m -th value function $V_m^{k*}(t, x)$ satisfies the following HJB equation for the k -th player.

$$0 = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) [V_m^k(t, x)] \right. \\ \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, x)}{\partial x} \right)' \sigma(t, x) W_2 \sigma(t, x)' \left(\frac{\partial V_{m-i}^k(t, x)}{\partial x} \right) \right\}, \quad (7.104)$$

with terminal condition $V_j^k(t_F, x) = 0$, $k = 1, 2, \dots, N$, $j = 1, 2, \dots, m$, then

$$0 = \min_{\mu_k \in U_{M^k}} \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) [V_m^{k*}(t, \hat{x})] \right. \\ \left. + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_{m-i}^k(t, \hat{x})}{\partial \hat{x}} \right) \right\}, \quad (7.105)$$

where $K(t)$ is the Kalman filter defined in (7.49) and the certainty equivalence principle has been assumed.

Since the first cost cumulant function V_1^k is admissible, the following coupled equations are satisfied

$$0 = \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) [V_1^k(t, \hat{x})] + M_0^k(t, \hat{x}) L^k(t, \hat{x}, \mu_1, \dots, \mu_k, \dots, \mu_N), \\ 0 = \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) [V_2^k(t, \hat{x})] + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right), \\ \vdots \\ 0 = \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) [V_m^k(t, \hat{x})] \\ + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_{m-i}^k(t, \hat{x})}{\partial \hat{x}} \right), \quad (7.106)$$

where $M_0^k = 1$ and $\mathcal{O}^k(\cdot)$ is the backward operator (2.5) and the state equation (7.63).

Minimizing (7.106) with respect to player strategy μ_k gives

$$\begin{aligned}
0 = \min_{\mu_k \in U_{M^k}} & \left\{ \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(t, \hat{x}, \mu_1, \dots, \mu_k, \dots, \mu_N) \right. \\
& + \gamma_{2k} \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \gamma_{2k} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \\
& \vdots \\
& + \gamma_{mk} \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_m^{k*}(t, \hat{x}) \right] \\
& \left. + \gamma_{mk} \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_{m-i}^k(t, \hat{x})}{\partial \hat{x}} \right) \right\},
\end{aligned} \tag{7.107}$$

where $j = 1, 2, \dots, m$, γ_{jk} are Lagrange multipliers.

Using the backward operator definition (2.5) and expanding (7.107) gives

$$\begin{aligned}
\min_{\mu_k \in U_{M^k}} & \left\{ \left(\frac{\partial V_1^k(t, \hat{x})}{\partial t} \right) + \hat{x}'(t) Q(t) \hat{x}(t) + \text{tr} \left(Q(t) P(t) \right) + \sum_{i=1}^N \mu_i(t) R_{ki}(t) \mu_i'(t) \right. \\
& + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \left(\hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right) + \frac{1}{2} \text{tr} \left(K(t) W_2 K(t)' \left(\frac{\partial^2 V_1^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) \\
& + \gamma_{2k} \left(\frac{\partial V_2^k(t, \hat{x})}{\partial t} \right) + \gamma_{2k} \left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} \right) \left(\hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right) \\
& + \frac{\gamma_{2k}}{2} \text{tr} \left(K(t) W_2 K(t)' \left(\frac{\partial^2 V_2^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) + \gamma_{2k} \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) \\
& \vdots \\
& + \gamma_{mk} \left(\frac{\partial V_m^k(t, \hat{x})}{\partial t} \right) + \gamma_{mk} \left(\frac{\partial V_m^k(t, \hat{x})}{\partial \hat{x}} \right) \left(\hat{x}(t)' A(t)' + \sum_{i=1}^N \mu_i(t)' B_i(t)' \right) \\
& + \frac{\gamma_{mk}}{2} \text{tr} \left(K(t) W_2 K(t)' \left(\frac{\partial^2 V_m^k(t, \hat{x})}{\partial \hat{x}^2} \right) \right) \\
& \left. + \gamma_{mk} \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_{m-i}^k(t, \hat{x})}{\partial \hat{x}} \right) \right\} = 0.
\end{aligned} \tag{7.108}$$

Minimizing (7.108) gives

$$\begin{aligned}
& 2\mu_k R_{kk}(t) + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right) B_k(t)' + \gamma_{2k} \left(\frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} \right) B_k(t)' + \dots \\
& + \dots + \gamma_{mk} \left(\frac{\partial V_m^{k*}(t, \hat{x})}{\partial \hat{x}} \right) B_k(t)' = 0.
\end{aligned} \tag{7.109}$$

Further simplification of (7.109) gives

$$\mu_k^*(t, \hat{x}) = -\frac{1}{2}R_{kk}^{-1}(t)B_k(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} + \gamma_{2k} \frac{\partial V_2^k(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{mk} \frac{\partial V_m^{k*}(t, \hat{x})}{\partial \hat{x}} \right). \quad (7.110)$$

Remark: The minimum strategy for the k -th player $\mu_k^*(t, \hat{x})$ is the suboptimal strategy derived from the output feedback coupled equation (7.106). In (7.106), the certainty equivalent principle has been extended to the m -th cumulant output feedback game where a Kalman filter is used for state estimation. The strategy is suboptimal since it was derived from suboptimal cumulants.

Using μ_k (7.110), the closed loop system for the system equation (7.63) is given as

$$d\hat{x}(t) = A(t)\hat{x}(t)dt - \frac{1}{2} \sum_{i=1}^N B_i(t)R_{ii}^{-1}B_i'(t) \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \gamma_{2k} \frac{\partial V_2^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right) dt + K(t)dv(t). \quad (7.111)$$

Corollary 7.1: The statistical game in closed-loop form is obtained by substituting (7.110) into (7.103) to give the Nm -coupled HJB equations for $j = 1$

$$\begin{aligned} 0 = & \frac{\partial V_j^k(t, \hat{x})}{\partial t} + \left(\hat{x}(t)'A(t)' - \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, \hat{x})}{\partial \hat{x}} \right. \right. \\ & \left. \left. + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right)' B_i R_{ii}^{-1} B_i' \right) \left(\frac{\partial V_j^k(t, \hat{x})}{\partial \hat{x}} \right) + \hat{x}'(t)Q(t)\hat{x}(t) + \text{tr}(Q(t)P(t)) \\ & + \frac{1}{4} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, \hat{x})}{\partial \hat{x}} + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right)' \times \\ & B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, \hat{x})}{\partial \hat{x}} + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right) \\ & + \frac{1}{2} \text{tr} \left(K(t)W_2K(t)' \frac{\partial^2 V_j^k(t, \hat{x})}{\partial \hat{x}^2} \right), \end{aligned} \quad (7.112)$$

and then for $j = 2, 3, \dots, m$ gives the following

$$\begin{aligned}
0 = & \frac{\partial V_j^k(t, \hat{x})}{\partial t} + \left(\hat{x}(t)' A(t)' - \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, \hat{x})}{\partial \hat{x}} \right. \right. \\
& \left. \left. + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right)' B_i R_{ii}^{-1} B_i' \right) \left(\frac{\partial V_j^k(t, \hat{x})}{\partial \hat{x}} \right) + \frac{1}{2} \text{tr} \left(K(t) W_2 K(t)' \frac{\partial^2 V_j^k(t, \hat{x})}{\partial \hat{x}^2} \right) \\
& + \frac{1}{2} \sum_{s=1}^{j-1} \frac{j!}{s!(j-s)!} \left(\frac{\partial V_s^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_{j-s}^k(t, \hat{x})}{\partial \hat{x}} \right),
\end{aligned} \tag{7.113}$$

with the terminal condition $V_j^k(t_F, \hat{x}) = 0$ and $j = 1, 2, \dots, m$, $k = 1, 2, \dots, N$. The Nm coupled equations (7.112) and (7.113) are second-order nonlinear partial differential equation.

7.4 Ad hoc Output Feedback Statistical Game Algorithm

In this section, a summary of the ad hoc output feedback statistical game algorithm for computing the value functions, state and control strategies is given. The referenced equations are also stated:

Step 1: Define a mathematical model for the N -player system (7.1),(7.2) and cost (7.7), (7.9) for the system.

$$dx(t) = A(t)x(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right) dt + G(t)dw_1(t), \tag{7.114}$$

$$dy(t) = C(t)x(t)dt + dw_2(t), \tag{7.115}$$

$$\begin{aligned}
J^k(t, x(t), \mu_1, \mu_2, \dots, \mu_N) = & \int_t^{t_F} \left[x'(s)Q(s)x(s) + \sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s) \right] ds \\
& + x'(t_F)Q(t_F)x(t_F),
\end{aligned} \tag{7.116}$$

$$\begin{aligned}
E\{J^k(t, \hat{x}(t), \mu_1, \mu_2, \dots, \mu_N)\} = & \int_t^{t_F} \left[E\left(\hat{x}'(s)Q(s)\hat{x}(s)\right) + \text{tr}\left(Q(s)P(s)\right) \right] ds \\
& + \int_t^{t_F} \left[\sum_{i=1}^N \mu_i'(s)R_{ki}(s)\mu_i(s) \right] ds \\
& + E\{\hat{x}'(t_F)Q(t_F)\hat{x}(t_F)\} + \text{tr}\left(Q(t_F)P(t_F)\right).
\end{aligned} \tag{7.117}$$

Step 2: Transform the mathematical system model to an innovation model (7.63) from Lemma 7.4.

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right) dt + K(t)dv(t). \quad (7.118)$$

Step 3: Define the m -th value function for the N -player system (7.101).

$$V_m^{k*}(t, \hat{x}) = \min_{\mu_k \in U_{M_k}} \left\{ V_m^k(t, \hat{x}; \mu_1, \mu_2, \dots, \mu_k, \dots, \mu_N) \right\}. \quad (7.119)$$

Step 4: State the m -th cost cumulant equations for the N -player system (7.103).

$$\begin{aligned} 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_1^k(t, \hat{x}) \right] + M_0^k(t, \hat{x}) L^k(\mu_1, \dots, \mu_k, \dots, \mu_N), \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_2^k(t, \hat{x}) \right] + \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_1^k(t, \hat{x})}{\partial \hat{x}} \right), \\ &\vdots \\ 0 &= \mathcal{O}^k(\mu_1, \dots, \mu_k, \dots, \mu_N) \left[V_m^{k*}(t, \hat{x}) \right] \\ &\quad + \frac{1}{2} \sum_{i=1}^{m-1} \frac{m!}{i!(m-i)!} \left(\frac{\partial V_i^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_{m-i}^k(t, \hat{x})}{\partial \hat{x}} \right), \end{aligned} \quad (7.120)$$

Step 5: Formulate and solve the Kalman filter parameter for the system using (7.50) from Lemma 7.2.

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)' - K(t)W_2(t)K(t)' + G(t)W_1(t)G(t)'. \quad (7.121)$$

Step 6: Formulate the neural network approximating functions for the value functions as in full-state feedback game section for (7.112)-(7.113).

$$\begin{aligned}
0 = & \frac{\partial V_j^k(t, \hat{x})}{\partial t} + \left(\hat{x}(t)' A(t)' - \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, \hat{x})}{\partial \hat{x}} \right. \right. \\
& \left. \left. + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right)' B_i R_{ii}^{-1} B_i' \right) \left(\frac{\partial V_j^k(t, \hat{x})}{\partial \hat{x}} \right) + \hat{x}'(t) Q(t) \hat{x}(t) + \text{tr} \left(Q(t) P(t) \right) \\
& + \frac{1}{4} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, \hat{x})}{\partial \hat{x}} + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right)' \times \\
& B_i R_{ii}^{-1} R_{ki} R_{ii}^{-1} B_i' \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, \hat{x})}{\partial \hat{x}} + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right) \\
& + \frac{1}{2} \text{tr} \left(K(t) W_2 K(t)' \frac{\partial^2 V_j^k(t, \hat{x})}{\partial \hat{x}^2} \right),
\end{aligned} \tag{7.122}$$

and then for $j = 2, 3, \dots, m$ gives the following

$$\begin{aligned}
0 = & \frac{\partial V_j^k(t, \hat{x})}{\partial t} + \left(\hat{x}(t)' A(t)' - \frac{1}{2} \sum_{i=1}^N \left(\frac{\partial V_1^i(t, \hat{x})}{\partial \hat{x}} + \dots + \gamma_{(m-1)i}(t) \frac{\partial V_{m-1}^i(t, \hat{x})}{\partial \hat{x}} \right. \right. \\
& \left. \left. + \gamma_{mi}(t) \frac{\partial V_m^{i*}(t, \hat{x})}{\partial \hat{x}} \right)' B_i R_{ii}^{-1} B_i' \right) \left(\frac{\partial V_j^k(t, \hat{x})}{\partial \hat{x}} \right) + \frac{1}{2} \text{tr} \left(K(t) W_2 K(t)' \frac{\partial^2 V_j^k(t, \hat{x})}{\partial \hat{x}^2} \right) \\
& + \frac{1}{2} \sum_{s=1}^{j-1} \frac{j!}{s!(j-s)!} \left(\frac{\partial V_s^k(t, \hat{x})}{\partial \hat{x}} \right)' K(t) W_2 K(t)' \left(\frac{\partial V_{j-s}^k(t, \hat{x})}{\partial \hat{x}} \right),
\end{aligned} \tag{7.123}$$

Step 7: Solve the neural network form equations of (7.112)-(7.113) for the weights of the value functions of the N -player system. If the weights are not converging go to step 6. However, if the weights converge, increase the size L of the neural network function and compare convergence time for different sizes of L .

Step 8: Using the weights in step 7, compute the value functions trajectory for the neural network form of (7.112)-(7.113) for the N -player system.

Step 9: Form the state and players strategies trajectory using the state equation (7.63).

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + \sum_{k=1}^N \left(B_k(t)u_k(t) \right) dt + K(t)dv(t). \tag{7.124}$$

7.5 Application Example

In this example, the aim is to illustrate the application of the ad hoc output feedback statistical game approach to a linear system example in [88]. Consider a linear dynamic equation given as

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) + 3x_3(t) + 2x_4(t) + u_1(t), \\ \dot{x}_2(t) &= 4x_1(t) - 2x_2(t) + x_3(t) + 3x_4(t) + u_2(t), \\ \dot{x}_3(t) &= 2x_1(t) + 3x_2(t) - 3x_3(t) + 4x_4(t) + u_1(t), \\ \dot{x}_4(t) &= -2x_4(t) + u_2(t),\end{aligned}\tag{7.125}$$

where $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ are the states and $u_1(t), u_2(t)$ are the controls (first and second players). The initial state condition is given as $x(t_0) = x(0) = x_0 = [1 \ 1 \ 1 \ 1]'$. The output equation for the system is given as

$$\begin{aligned}y_1(t) &= x_1(t) + 0.1x_4(t), \\ y_2(t) &= x_2(t) + 0.1x_4(t), \\ y_3(t) &= x_3(t) + x_4(t).\end{aligned}\tag{7.126}$$

The system represented by (7.125), (7.126) is a deterministic system, thus we introduce Gaussian noise into the system as process and measurement noise. The stochastic system is then represented as

$$\begin{aligned}dx(t) &= Ax(t)dt + B_1u_1(t)dt + B_2u_2(t)dt + Gdw_1(t), \\ dy(t) &= Cx(t)dt + dw_2(t),\end{aligned}\tag{7.127}$$

where

$$A = \begin{bmatrix} -2 & 0 & 3 & 2 \\ 4 & -2 & 1 & 3 \\ 2 & 3 & -3 & 4 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and the state variable $x(t)$ is defined as: $x(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]'$. We assume that G in (7.127) is a 4×1 constant vector given as $G = [1 \ 1 \ 1 \ 1]'$ and $dw_1(t), dw_2(t)$ in (7.127) as a Gaussian process with mean $E\{dw_1(t)\} = E\{dw_2(t)\} = 0$ and covariance $E\{dw_1(t)dw_1(t)'\} = 0.1$ and $E\{dw_2(t)dw_2(t)'\} = 0.1$. In this example, we study a 2-player 2nd cumulant ad hoc output feedback Nash game with the aim to stabilize the system. Here, we compute the suboptimal solution for the player strategy through solving the output feedback second cumulant game problem constraint on the first cumulant cost function.

The first player cost function J^1 is

$$J^1(t_0, x(t), u_1(t), u_2(t)) = \int_{t_0}^{t_F} \{x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + u_1^2(t) + u_2^2(t)\} dt + \psi^1(x(t_F), t_F), \quad (7.128)$$

where $\psi^1(x(t_F), t_F) = 0$ is the terminal cost and the second player cost function J^2 is

$$J^2(t_0, x(t), u_1(t), u_2(t)) = \int_{t_0}^{t_F} \{x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + u_1^2(t) + u_2^2(t)\} dt + \psi^2(x(t_F), t_F), \quad (7.129)$$

where $\psi^2(x(t_F), t_F) = 0$ is the terminal cost.

The input function $\Lambda_L(x)$ (6.6) is

$$\Lambda_L = \{x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4\}'. \quad (7.130)$$

The first player value functions V_{1L}^1, V_{2L}^1 are

$$V_{1L}^1(x_1, x_2, x_3, x_4) = w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + w_4x_4^2 + w_5x_1x_2 + w_6x_1x_3 + w_7x_1x_4 + w_8x_2x_3 + w_9x_2x_4 + w_{10}x_3x_4. \quad (7.131)$$

$$V_{2L}^1(x_1, x_2, x_3, x_4) = w_{11}x_1^2 + w_{12}x_2^2 + w_{13}x_3^2 + w_{14}x_4^2 + w_{15}x_1x_2 + w_{16}x_1x_3 + w_{17}x_1x_4 + w_{18}x_2x_3 + w_{19}x_2x_4 + w_{20}x_3x_4. \quad (7.132)$$

Similarly, the second player value functions V_{1L}^2, V_{2L}^2 are

$$\begin{aligned} V_{1L}^2(x_1, x_2, x_3, x_4) = & w_{21}x_1^2 + w_{22}x_2^2 + w_{23}x_3^2 + w_{24}x_4^2 + w_{25}x_1x_2 + w_{26}x_1x_3 \\ & + w_{27}x_1x_4 + w_{28}x_2x_3 + w_{29}x_2x_4 + w_{30}x_3x_4. \end{aligned} \quad (7.133)$$

$$\begin{aligned} V_{2L}^2(x_1, x_2, x_3, x_4) = & w_{31}x_1^2 + w_{32}x_2^2 + w_{33}x_3^2 + w_{34}x_4^2 + w_{35}x_1x_2 + w_{36}x_1x_3 \\ & + w_{37}x_1x_4 + w_{38}x_2x_3 + w_{39}x_2x_4 + w_{40}x_3x_4. \end{aligned} \quad (7.134)$$

We transform the problem in terms of state estimate and then apply the algorithm in Section 7.4 to analyze the game problem. We selected as polynomial function (7.130) of up to second-order ($M = 2$) of approximation in the state variable (i.e x is 2^{nd} order) with length of the neural network input function $L = 10$. Higher order polynomial did not provide significant improvement in the approximation accuracy. The first and second player value functions are substituted to (7.112)-(7.113). In the simulation, the asymptotic stability region for states was arbitrarily chosen as $-5 \leq x_1 \leq 5, -5 \leq x_2 \leq 5, -5 \leq x_3 \leq 5$ and $-5 \leq x_4 \leq 5$. The final time t_F was 20 s and $\mathbf{w}'_{11L}(t_F) - \mathbf{w}'_{21L}(t_F) = \{0\}$, $\mathbf{w}'_{12L}(t_F) - \mathbf{w}'_{22L}(t_F) = \{0\}$ and $\mathbf{w}'_{13L}(t_F) - \mathbf{w}'_{23L}(t_F) = \{0\}$ and we integrate backwards in time to solve for the suboptimal weights.

From Figs. 7.1(a) to 7.1(d), the neural network weights converge to constants, which are the suboptimal weights. Plots 7.2(a) to 7.2(d) show the first and second value cumulant functions for both players. From Figs. 7.2(a) and 7.2(c), it is observed that the value function V_1^1 and V_1^2 increase with increases in γ_{21} while from Figs 7.2(b) and 7.2(d), it is observed that the value functions V_2^1 and V_2^2 decrease as γ_{21} increases. Figs. 7.3(a) and 7.3(b) show the state trajectory for noise influence with variance $\sigma^2 = 0.1$. The estimated states from the Kalman filter are bounded and converged to value close to the origin. The Lagrange multipliers γ_{xx} were selected as constants.

The Nash suboptimal controls, u_1 and u_2 are shown in Fig. 7.3(c). It should be noted from Fig. 7.3(c), that the Nash suboptimal controls for the two players were solved for the 2^{nd} cumulant game by selecting γ_{21}, γ_{22} where the value functions are minimum which in our case were $\gamma_{21} = 0.001, \gamma_{22} = 0.001$ as shown in Figs. 7.1(b), 7.1(d). In addition, we have the design

(a) 1st Player Neural Network First Cumulant Weights

(b) 1st Player Neural Network Second Cumulant Weights

(c) 2nd Player Neural Network First Cumulant Weights

(d) 2nd Player Neural Network Second Cumulant Weights

Figure 7.1: Two Player Linear Output Feedback Nash Game System Neural Network Weights.

(a) 1st Player First Cumulant Value Function - V_1^1 versus γ_{21}

(b) 1st Player Second Cumulant Value Function - V_2^1 versus γ_{21}

(c) 2nd Player First Cumulant Value Function - V_1^2 versus γ_{21}

(d) 2nd Player Second Cumulant Value Function - V_2^2 versus γ_{21}

Figure 7.2: Two Player Linear Output Feedback Nash Game System Value Functions.

(a) State trajectory at initial condition $x_0 = (1,1,1,1)$

(b) State trajectory at initial condition $x_0 = (1,1,1,1)$

(c) Control trajectory at initial condition $x_0 = (1,1,1,1)$

Figure 7.3: Two Player Linear Output Feedback Nash Game System States and Suboptimal Control Strategy Trajectory.

freedom in γ_{xx} values selection to enhance system performance.

Remark: The ad hoc output feedback second cumulant Nash strategy is found within all admissible first cumulant strategy. The neural network functions size was varied until close convergence in the weights was observed. To find the first cumulant output feedback Nash strategy, set $\gamma_{21} = \gamma_{22} = 0$ in (7.112)-(7.113). From the results, we cannot conclude that this algorithm will yield interesting result for more complex problems as it only an approximate approach. The certainty equivalence approach for output feedback statistical game control is reserved for future study.

CHAPTER 8

CONTRIBUTIONS, FUTURE WORK AND CONCLUSIONS

In this dissertation, we have investigated the continuous statistical game control for dynamic systems with full-state and output feedback information. Statistical game control integrates game theory with statistical optimal control theory. The statistical game control problem was analyzed as a cost cumulant optimization problem. The statistical game control assumed that all lower-order cost cumulants has been specified for each player and then seeks an optimal player strategy that minimizes the higher-order cost cumulants for all the players constrained on a set of all lower-order cost cumulants. If the lower-order cumulants are not constrained then a degenerate solution results. The necessary and sufficient conditions for the existence of equilibrium solutions were derived on assuming the twice differentiability of the value function and from the sufficient condition proof, the optimal player strategy was derived.

For most practical control systems except for linear quadratic (LQ) systems, which are linear systems with quadratic cost function, analytical or closed-form solutions are difficult to find, consequently we utilized a numerical solution method using neural networks approximation to solve the nonlinear coupled Hamilton-Jacobi-Bellman (HJB) equations. The statistical game control problem was analyzed in finite-time horizon for both full-state feedback and output feedback system. Simulation results show that our cost cumulant optimization approach allows a system of interacting players to be controlled.

8.1 Contributions

The contributions of this dissertation are as follows:

8.1.1 Necessary Condition for the HJB Equations

The necessary condition for the existence of equilibrium solutions of N -player full-state feedback statistical game was proved in Chapter 3. The necessary condition for the existence of equilibrium solutions is given by the Hamilton-Jacobi-Bellman equation. The necessary condition for optimality assumes that an optimal value function and optimal player strategy are given that satisfy the HJB equation for the statistical game. We proved the necessary conditions for the N -player, 2-nd cumulant; N -player, 3-rd cumulant and N -player, m -th cumulant statistical game cases having nonquadratic cost function. An important assumption in this dissertation was the twice continuously differentiability of the value functions to satisfy the HJB equations.

8.1.2 Sufficient Condition for the HJB Equations

The sufficient condition for the existence of equilibrium solutions of N -player full-state feedback statistical game was proved in Chapter 4. The sufficient condition is also the verification theorem for the existence of equilibrium solutions of N -player statistical game. The sufficient condition for the existence of equilibrium solutions is given by Hamilton-Jacobi-Bellman equation. The sufficient condition for optimality assumes that if an optimal value function satisfies the HJB equation for the statistical game, then if a player strategy satisfies the HJB equation, the player strategy is optimal. We proved the sufficient conditions for the N -player, 2-nd cumulant; N -player, 3-rd cumulant and N -player, m -th cumulant statistical game cases having nonquadratic cost function. From the sufficient condition for the N -player, m -th cumulant statistical game, the optimal player strategy was derived.

8.1.3 Solutions of HJB Equations

The neural network approximate method was utilized to solve the coupled HJB equations for the cost cumulant statistical game as analytical or closed-form solution are difficult to find for the nonlinear partial differential equation resulting from the problem formulation in Chapter 5. The proof of convergence of the neural network approximate method solution to exact solution showed the feasibility of the approach. Linear and nonlinear practical system examples involv-

ing laser pointing control, electric power system control and robotic control were simulated based in Chapter 6. The illustrated examples show the application of neural network approximate method to solving coupled HJB equations for different types of systems. The flexibility in selection of the Lagrange multipliers helped improve system performance.

8.1.4 Ad hoc Output feedback Statistical Game

The N -player, m -th cost cumulant ad hoc output feedback statistical game problem for a linear system with quadratic cost function was analyzed and the suboptimal player strategy was derived in Chapter 7. Certainty equivalence principle was assumed for the statistical game control where the state estimate was determined using a Kalman filter and then utilized as the true state to solve for the suboptimal player strategy. We derived the suboptimal strategy for the N -player, 2-nd cumulant; N -player, 3-rd cumulant and N -player, m -th cumulant statistical game cases having quadratic cost function. The suboptimal strategy appears as an attractive strategy for ad hoc output feedback statistical game for linear systems.

8.2 Future Work

The suggestions for future research are as follows:

8.2.1 Mean Field Game

We considered a finite N -player statistical game control in this dissertation and proved the necessary and sufficiency conditions for the existence of equilibrium solutions for the full-state feedback m -th cumulant statistical game. In mean field game, the finite number of players is replaced with a density function to indicate a large number of players [103], [104]. A mean field game and relationship to statistical game control is a possible extension of this research.

8.2.2 Infinite Horizon

The statistical game control problem solved in this dissertation considered finite horizon where the final time $t_F < \infty$. A possible extension of this research is to consider the effect of infinite horizon where the final time $t_F = \infty$.

8.2.3 Output Feedback Statistical Game

The N -player, m -th cost cumulant ad hoc output feedback statistical differential game problem for a linear system with quadratic cost function was analyzed in Chapter 7. Currently, there is no proof/counterproof of certainty equivalence for higher-order output feedback statistical game. This issue is an open research area.

8.2.4 Adaptive Neural Network Approximate Method

The neural network approximate method is utilized to solve the coupled HJB equations for the cost cumulant statistical game. Jiang *et al.* [105] have shown the feasibility of adaptive method based on reinforcement learning to stochastic optimal control. An adaptive approach to solving HJB equations is another area for research.

8.3 Conclusions

For a stochastic complex system, it is possible to analyze the system as a statistical noncooperative nonzero-sum game consisting of finite number of players and find the optimal player feedback strategy. From the results, we conclude the following: statistical game control minimizes the different orders of the cost cumulant to improve system performance; neural network method can be applied to solve HJB equations for linear and nonlinear dynamical systems; more neurons improve the state convergence for higher-order systems; flexibility in Lagrange multipliers selection enhances system performance. In the statistical game examples, the states were bounded. The use of sophisticated neural network design with multi-layered neurons is an option for solving HJB equations for more practical nonlinear systems. This dissertation shows the steps to analyze complex systems of multiple decision makers/players for a wide-range of engineering applications.

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APPENDIX A

VECTOR INNER PRODUCT

Lemma A.1: Let the following vectors be defined,

$$\mathbf{w}_{mL}(t) = [w_{m1}(t)w_{m2}(t) \dots w_{mL}(t)]',$$

$$\mathbf{\Lambda}_{mL}(x) = [\Lambda_{m1}(x)\Lambda_{m2}(x) \dots \Lambda_{mL}(x)]'.$$

Assume $\mathbf{w}_{mL} : \mathbb{R}^n \rightarrow \mathbb{R}$, then define

$$\langle \mathbf{w}'_{mL}, \mathbf{\Lambda}_{mL} \rangle_{\Omega} \equiv \left(\langle \mathbf{w}_{mL}, \Lambda_{m1}(x) \rangle_{\Omega}, \dots, \langle \mathbf{w}_{mL}, \Lambda_{mL}(x) \rangle_{\Omega} \right)', \quad (\text{A.1})$$

where Ω is the interval the vectors are defined. Then

$$\langle \mathbf{w}'_{mL}(t) \mathbf{\Lambda}_{mL}(x), \mathbf{\Lambda}_{mL}(x) \rangle_{\Omega} \equiv \langle \mathbf{\Lambda}_{mL}(x), \mathbf{\Lambda}_{mL}(x) \rangle_{\Omega} \mathbf{w}_{mL}(t). \quad (\text{A.2})$$

Proof:

$$\begin{aligned}
& \langle \mathbf{w}'_{mL}(t) \mathbf{\Lambda}_{mL}(x), \mathbf{\Lambda}_{mL}(x) \rangle_{\Omega} \\
&= \left\langle [w_{m1}(t) \dots w_{mL}(t)] [\mathbf{\Lambda}_{m1}(x) \dots \mathbf{\Lambda}_{mL}(x)]', [\mathbf{\Lambda}_{m1}(x) \dots \mathbf{\Lambda}_{mL}(x)]' \right\rangle_{\Omega}, \\
&= \left\langle [w_{m1} \mathbf{\Lambda}_{m1}(x) + \dots + w_{mL}(t) \mathbf{\Lambda}_{mL}(x)], [\mathbf{\Lambda}_{m1}(x) \dots \mathbf{\Lambda}_{mL}(x)]' \right\rangle_{\Omega}, \\
&= \begin{bmatrix} \left\langle [w_{m1} \mathbf{\Lambda}_{m1}(x) + \dots + w_{mL}(t) \mathbf{\Lambda}_{mL}(x)], [\mathbf{\Lambda}_{m1}(x)] \right\rangle_{\Omega} \\ \left\langle [w_{m1} \mathbf{\Lambda}_{m1}(x) + \dots + w_{mL}(t) \mathbf{\Lambda}_{mL}(x)], [\mathbf{\Lambda}_{m2}(x)] \right\rangle_{\Omega} \\ \vdots \\ \left\langle [w_{m1} \mathbf{\Lambda}_{m1}(x) + \dots + w_{mL}(t) \mathbf{\Lambda}_{mL}(x)], [\mathbf{\Lambda}_{m(L-1)}(x)] \right\rangle_{\Omega} \\ \left\langle [w_{m1} \mathbf{\Lambda}_{m1}(x) + \dots + w_{mL}(t) \mathbf{\Lambda}_{mL}(x)], [\mathbf{\Lambda}_{mL}(x)] \right\rangle_{\Omega} \end{bmatrix}, \\
&= \begin{bmatrix} w_{m1} \mathbf{\Lambda}_{m1}(x) \mathbf{\Lambda}_{m1}(x) + \dots + w_{mL} \mathbf{\Lambda}_{mL}(x) \mathbf{\Lambda}_{m1}(x) \\ w_{m1} \mathbf{\Lambda}_{m1}(x) \mathbf{\Lambda}_{m2}(x) + \dots + w_{mL} \mathbf{\Lambda}_{mL}(x) \mathbf{\Lambda}_{m2}(x) \\ \vdots \\ w_{m1} \mathbf{\Lambda}_{m1}(x) \mathbf{\Lambda}_{m(L-1)}(x) + \dots + w_{mL} \mathbf{\Lambda}_{mL}(x) \mathbf{\Lambda}_{m(L-1)}(x) \\ w_{m1} \mathbf{\Lambda}_{m1}(x) \mathbf{\Lambda}_{mL}(x) + \dots + w_{mL} \mathbf{\Lambda}_{mL}(x) \mathbf{\Lambda}_{mL}(x) \end{bmatrix}, \tag{A.3} \\
&= \begin{bmatrix} \langle \mathbf{\Lambda}_{m1}(x), \mathbf{\Lambda}_{m1}(x) \rangle_{\Omega} \dots \langle \mathbf{\Lambda}_{mL}(x), \mathbf{\Lambda}_{m1}(x) \rangle_{\Omega} \\ \langle \mathbf{\Lambda}_{m1}(x), \mathbf{\Lambda}_{m2}(x) \rangle_{\Omega} \dots \langle \mathbf{\Lambda}_{mL}(x), \mathbf{\Lambda}_{m2}(x) \rangle_{\Omega} \\ \vdots \\ \langle \mathbf{\Lambda}_{m1}(x), \mathbf{\Lambda}_{m(L-1)}(x) \rangle_{\Omega} \dots \langle \mathbf{\Lambda}_{mL}(x), \mathbf{\Lambda}_{m(L-1)}(x) \rangle_{\Omega} \\ \langle \mathbf{\Lambda}_{m1}(x), \mathbf{\Lambda}_{mL}(x) \rangle_{\Omega} \dots \langle \mathbf{\Lambda}_{mL}(x), \mathbf{\Lambda}_{mL}(x) \rangle_{\Omega} \end{bmatrix} \begin{bmatrix} w_{m1} \\ w_{m2} \\ \vdots \\ w_{m(L-1)} \\ w_{mL} \end{bmatrix}, \\
&\equiv \left\langle \mathbf{\Lambda}_{mL}(x), \mathbf{\Lambda}_{mL}(x) \right\rangle_{\Omega} \mathbf{w}_{mL}(t).
\end{aligned}$$

□

APPENDIX B

COUPLED NEURAL NETWORK HJB EQUATIONS

Lemma B.1: From section 5.1, we expand on the four (4) coupled system of equations for a 2-player : 2^{nd} cost cumulant statistical game formation.

$$\begin{aligned}
\dot{\mathbf{w}}_{11L}(t) = & - \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{11L}(x) g(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \mathbf{w}_{11L}(t) \\
& - \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t))), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \\
& B_1(t, x) R_{11}^{-1} R_{11} B'_1(t, x) (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \\
& B_2(t, x) R_{22}^{-1} R_{12} B'_2(t, x) (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{21L}(t) = & -\langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{21L}(x) g(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \mathbf{w}_{21L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t))), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& - \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{12L}(t) = & -\langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{12L}(x) g(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \mathbf{w}_{12L}(t) \\
& - \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t))), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \\
& B_1(t, x) R_{11}^{-1} R_{21} R_{11}^{-1} B'_1(t, x) \left(\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \\
& B_2(t, x) R_{22}^{-1} R_{22} R_{22}^{-1} B'_2(t, x) \left(\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{22L}(t) = & -\langle \Lambda_{22L}(x), \Lambda_{22L}(x) \rangle_{\Omega}^{-1} \langle \nabla \Lambda_{22L}(x) g(x), \Lambda_{22L}(x) \rangle_{\Omega} \mathbf{w}_{22L}(t) \\
& - \frac{1}{2} \langle \Lambda_{22L}(x), \Lambda_{22L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \Lambda'_{22L}(x) \mathbf{w}_{22L}(t))), \Lambda_{22L}(x) \rangle_{\Omega} \\
& - \langle \Lambda_{22L}(x), \Lambda_{22L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{12L}(t) \nabla \Lambda_{12L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \Lambda'_{12L}(x) \mathbf{w}_{12L}(t), \Lambda_{22L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \Lambda_{22L}(x), \Lambda_{22L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \Lambda_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \Lambda_{21L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \Lambda'_{22L}(x) \mathbf{w}_{22L}(t), \Lambda_{22L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \Lambda_{22L}(x), \Lambda_{22L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \Lambda_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \Lambda_{22L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \Lambda'_{22L}(x) \mathbf{w}_{22L}(t), \Lambda_{22L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.4}$$

Lemma B.2: From section 5.1, we expand on the six (6) coupled system of equations for a 2-player : 3rd cost cumulant statistical game formation.

$$\begin{aligned}
\dot{\mathbf{w}}_{11L}(t) = & -\langle \Lambda_{11L}(x), \Lambda_{11L}(x) \rangle_{\Omega}^{-1} \langle \nabla \Lambda_{11L}(x) g(x), \Lambda_{11L}(x) \rangle_{\Omega} \mathbf{w}_{11L}(t) \\
& - \langle \Lambda_{11L}(x), \Lambda_{11L}(x) \rangle_{\Omega}^{-1} \langle l, \Lambda_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \Lambda_{11L}(x), \Lambda_{11L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \Lambda'_{11L}(x) \mathbf{w}_{11L}(t))), \Lambda_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \Lambda_{11L}(x), \Lambda_{11L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \Lambda_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \Lambda_{21L} \\
& + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \Lambda_{31L}) B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \Lambda'_{11L}(x) \mathbf{w}_{11L}(t), \Lambda_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \Lambda_{11L}(x), \Lambda_{11L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \Lambda_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \Lambda_{22L} \\
& + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \Lambda_{32L}) B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \Lambda'_{11L}(x) \mathbf{w}_{11L}(t), \Lambda_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \Lambda_{11L}(x), \Lambda_{11L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \Lambda_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \Lambda_{21L} \\
& + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \Lambda_{31L}) B_1(t, x) R_{11}^{-1} R_{11} R_{11}^{-1} B'_1(t, x) (\mathbf{w}'_{11L}(t) \nabla \Lambda_{11L} \\
& + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \Lambda_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \Lambda_{31L}) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \Lambda_{11L}(x), \Lambda_{11L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \Lambda_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \Lambda_{22L} \\
& + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \Lambda_{32L}) B_2(t, x) R_{22}^{-1} R_{12} R_{22}^{-1} B'_2(t, x) (\mathbf{w}'_{12L}(t) \nabla \Lambda_{12L} \\
& + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \Lambda_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \Lambda_{32L}) \rangle_{\Omega}
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{21L}(t) = & -\langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{21L}(x) g(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \mathbf{w}_{21L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t))), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& - \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{31L}(t) = & -\langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{31L}(x) g(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \mathbf{w}_{31L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{31L}(x) \mathbf{w}_{31L}(t))), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \\
& - 3 \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{31L}(x) \mathbf{w}_{31L}(t), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{31L}(x) \mathbf{w}_{31L}(t), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{12L}(t) = & -\langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{12L}(x) g(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \mathbf{w}_{12L}(t) \\
& - \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t))), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} R_{21} R_{11}^{-1} B'_1(t, x) \left(\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} \right. \\
& \left. + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L} \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} R_{22} R_{22}^{-1} B'_2(t, x) \left(\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} \right. \\
& \left. + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L} \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{22L}(t) = & -\langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{22L}(x) g(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \mathbf{w}_{22L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t))), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \\
& - \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{32L}(t) = & -\langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{32L}(x) g(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \mathbf{w}_{32L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{32L}(x) \mathbf{w}_{32L}(t))), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \\
& - 3 \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{32L}(x) \mathbf{w}_{32L}(t), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{32L}(x) \mathbf{w}_{32L}(t), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.10}$$

Lemma B.3: From section 5.1, we expand on the six (6) coupled system of equations for a 3-player : 2^{nd} cost cumulant statistical game formation.

$$\begin{aligned}
\dot{\mathbf{w}}_{11L}(t) = & - \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{11L}(x) g(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \mathbf{w}_{11L}(t) \\
& - \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t))), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \\
& \left. B_1(t, x) R_{11}^{-1} B'_1(t, x) \right) \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \\
& \left. B_2(t, x) R_{22}^{-1} B'_2(t, x) \right) \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \\
& \left. B_3(t, x) R_{33}^{-1} B'_3(t, x) \right) \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \\
& \left. B_1(t, x) R_{11}^{-1} R_{11} R_{11}^{-1} B'_1(t, x) (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \\
& \left. B_2(t, x) R_{22}^{-1} R_{12} R_{22}^{-1} B'_2(t, x) (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \\
& \left. B_3(t, x) R_{33}^{-1} R_{13} R_{33}^{-1} B'_3(t, x) (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{21L}(t) = & - \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{21L}(x) g(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \mathbf{w}_{21L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t))), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& - \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L}(x) \sigma(t, x) W \sigma(t, x)' \\
& \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \\
& \left. B_1(t, x) R_{11}^{-1} B'_1(t, x) \right) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \\
& \left. B_2(t, x) R_{22}^{-1} B'_2(t, x) \right) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \\
& \left. B_3(t, x) R_{33}^{-1} B'_3(t, x) \right) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.12}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{12L}(t) = & - \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{12L}(x) g(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \mathbf{w}_{12L}(t) \\
& - \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t))), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \\
& \left. B_1(t, x) R_{11}^{-1} B'_1(t, x) \right) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \\
& \left. B_2(t, x) R_{22}^{-1} B'_2(t, x) \right) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \\
& \left. B_3(t, x) R_{33}^{-1} B'_3(t, x) \right) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \\
& \left. B_1(t, x) R_{11}^{-1} R_{21} R_{11}^{-1} B'_1(t, x) (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \\
& \left. B_2(t, x) R_{22}^{-1} R_{22} R_{22}^{-1} B'_2(t, x) (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \\
& \left. B_3(t, x) R_{33}^{-1} R_{23} R_{33}^{-1} B'_3(t, x) (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& \tag{B.13}
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{22L}(t) = & - \left\langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega}^{-1} \left\langle \nabla \mathbf{\Lambda}_{22L}(x) g(x), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega} \mathbf{w}_{22L}(t) \\
& - \frac{1}{2} \left\langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega}^{-1} \left\langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t))), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega} \\
& - \left\langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega}^{-1} \left\langle \mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L}(x) \sigma(t, x) W \sigma(t, x)' \right. \\
& \left. \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \left\langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega}^{-1} \left\langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \right. \\
& \left. \left. B_1(t, x) R_{11}^{-1} B'_1(t, x) \right) \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \left\langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega}^{-1} \left\langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \right. \\
& \left. \left. B_2(t, x) R_{22}^{-1} B'_2(t, x) \right) \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \left\langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega}^{-1} \left\langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \right. \\
& \left. \left. B_3(t, x) R_{33}^{-1} B'_3(t, x) \right) \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{22L}(x) \right\rangle_{\Omega}
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{13L}(t) = & - \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{13L}(x) g(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \mathbf{w}_{13L}(t) \\
& - \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t))), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \\
& \left. B_1(t, x) R_{11}^{-1} B'_1(t, x) \right) \nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \\
& \left. B_2(t, x) R_{22}^{-1} B'_2(t, x) \right) \nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \\
& \left. B_3(t, x) R_{33}^{-1} B'_3(t, x) \right) \nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \\
& \left. B_1(t, x) R_{11}^{-1} R_{31} R_{11}^{-1} B'_1(t, x) (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \\
& \left. B_2(t, x) R_{22}^{-1} R_{32} R_{22}^{-1} B'_2(t, x) (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \\
& \left. B_3(t, x) R_{33}^{-1} R_{33} R_{33}^{-1} B'_3(t, x) (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& \tag{B.15}
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{23L}(t) = & - \left\langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega}^{-1} \left\langle \nabla \mathbf{\Lambda}_{23L}(x) g(x), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega} \mathbf{w}_{23L}(t) \\
& - \frac{1}{2} \left\langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega}^{-1} \left\langle \text{tr}(\sigma(t, x) W \sigma(t, x)') \nabla (\nabla \mathbf{\Lambda}'_{23L}(x) \mathbf{w}_{23L}(t)), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega} \\
& - \left\langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega}^{-1} \left\langle \mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L}(x) \sigma(t, x) W \sigma(t, x)' \right. \\
& \left. \nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \left\langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega}^{-1} \left\langle \left((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L}) \right. \right. \\
& \left. \left. B_1(t, x) R_{11}^{-1} B'_1(t, x) \right) \nabla \mathbf{\Lambda}'_{23L}(x) \mathbf{w}_{23L}(t), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \left\langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega}^{-1} \left\langle \left((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L}) \right. \right. \\
& \left. \left. B_2(t, x) R_{22}^{-1} B'_2(t, x) \right) \nabla \mathbf{\Lambda}'_{23L}(x) \mathbf{w}_{23L}(t), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega} \\
& + \frac{1}{2} \left\langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega}^{-1} \left\langle \left((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L}) \right. \right. \\
& \left. \left. B_3(t, x) R_{33}^{-1} B'_3(t, x) \right) \nabla \mathbf{\Lambda}'_{23L}(x) \mathbf{w}_{23L}(t), \mathbf{\Lambda}_{23L}(x) \right\rangle_{\Omega}
\end{aligned} \tag{B.16}$$

Lemma B.4: From section 5.1, we expand on the nine (9) coupled system of equations for a 3-player : 3^{rd} cost cumulant statistical game formation.

$$\begin{aligned}
\dot{\mathbf{w}}_{11L}(t) = & - \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{11L}(x) g(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \mathbf{w}_{11L}(t) \\
& - \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t))), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} \\
& + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) B_1(t, x) R_{11}^{-1} B'_1(t, x)) \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} \\
& + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) B_2(t, x) R_{22}^{-1} B'_2(t, x)) \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} \\
& + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) B_3(t, x) R_{33}^{-1} B'_3(t, x)) \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} \\
& + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) B_1(t, x) R_{11}^{-1} R_{11} R_{11}^{-1} B'_1(t, x) (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} \\
& + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L})) , \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} \\
& + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) B_2(t, x) R_{22}^{-1} R_{12} R_{22}^{-1} B'_2(t, x) (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} \\
& + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L})) , \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{11L}(x), \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} \\
& + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) B_3(t, x) R_{33}^{-1} R_{13} R_{33}^{-1} B'_3(t, x) (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} \\
& + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L})) , \mathbf{\Lambda}_{11L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.17}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{21L}(t) = & -\langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{21L}(x) g(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \mathbf{w}_{21L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t))), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& - \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{11L}(x) \mathbf{w}_{11L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{21L}(x), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} B'_3(t, x) \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{21L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.18}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{31L}(t) = & -\langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{31L}(x) g(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \mathbf{w}_{31L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{31L}(x) \mathbf{w}_{31L}(t))), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \\
& - 3 \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{21L}(x) \mathbf{w}_{21L}(t), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{31L}(x) \mathbf{w}_{31L}(t), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{31L}(x) \mathbf{w}_{31L}(t), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{31L}(x), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} B'_3(t, x) \nabla \mathbf{\Lambda}'_{31L}(x) \mathbf{w}_{31L}(t), \mathbf{\Lambda}_{31L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.19}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{12L}(t) = & -\langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{12L}(x) g(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \mathbf{w}_{12L}(t) \\
& - \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t))), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} B'_3(t, x) \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} R_{21} R_{11}^{-1} B'_1(t, x) \left(\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} \right. \\
& \left. + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L} \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} R_{22} R_{22}^{-1} B'_2(t, x) \left(\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} \right. \\
& \left. + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L} \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{12L}(x), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}^{-1} \langle ((\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} R_{23} R_{33}^{-1} B'_3(t, x) \left(\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} \right. \\
& \left. + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L} \right), \mathbf{\Lambda}_{12L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.20}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{22L}(t) = & -\langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{22L}(x) g(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \mathbf{w}_{22L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t))), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \\
& - \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{22L}(x), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} B'_3(t, x) \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{22L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.21}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{32L}(t) = & -\langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{32L}(x) g(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \mathbf{w}_{32L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{32L}(x) \mathbf{w}_{32L}(t))), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \\
& - 3 \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{32L}(x) \mathbf{w}_{32L}(t), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{32L}(x) \mathbf{w}_{32L}(t), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{32L}(x), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} B'_3(t, x) \nabla \mathbf{\Lambda}'_{32L}(x) \mathbf{w}_{32L}(t), \mathbf{\Lambda}_{32L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.22}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{13L}(t) = & -\langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{13L}(x) g(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \mathbf{w}_{13L}(t) \\
& - \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle l, \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t))), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} B'_3(t, x) \nabla \mathbf{\Lambda}'_{13L}(x) \mathbf{w}_{13L}(t), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} R_{21} R_{11}^{-1} B'_1(t, x) (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} \\
& + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} R_{22} R_{22}^{-1} B'_2(t, x) (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} \\
& + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega} \\
& - \frac{1}{4} \langle \mathbf{\Lambda}_{13L}(x), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} R_{23} R_{33}^{-1} B'_3(t, x) (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} \\
& + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}), \mathbf{\Lambda}_{13L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.23}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{23L}(t) = & -\langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{23L}(x) g(x), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega} \mathbf{w}_{23L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{23L}(x) \mathbf{w}_{23L}(t))), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega} \\
& - \langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{12L}(x) \mathbf{w}_{12L}(t), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{23L}(x) \mathbf{w}_{23L}(t), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{23L}(x) \mathbf{w}_{23L}(t), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{23L}(x), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} B'_3(t, x) \nabla \mathbf{\Lambda}'_{23L}(x) \mathbf{w}_{23L}(t), \mathbf{\Lambda}_{23L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
\dot{\mathbf{w}}_{33L}(t) = & -\langle \mathbf{\Lambda}_{33L}(x), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega}^{-1} \langle \nabla \mathbf{\Lambda}_{33L}(x) g(x), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega} \mathbf{w}_{33L}(t) \\
& - \frac{1}{2} \langle \mathbf{\Lambda}_{33L}(x), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega}^{-1} \langle \text{tr}(\sigma(t, x) W \sigma(t, x)' \nabla (\nabla \mathbf{\Lambda}'_{33L}(x) \mathbf{w}_{33L}(t))), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega} \\
& - 3 \langle \mathbf{\Lambda}_{33L}(x), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega}^{-1} \langle \mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L}(x) \sigma(t, x) W \sigma(t, x)' \nabla \mathbf{\Lambda}'_{22L}(x) \mathbf{w}_{22L}(t), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{33L}(x), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{11L}(t) \nabla \mathbf{\Lambda}_{11L} + \gamma_{21} \mathbf{w}'_{21L}(t) \nabla \mathbf{\Lambda}_{21L} + \gamma_{31} \mathbf{w}'_{31L}(t) \nabla \mathbf{\Lambda}_{31L}) \\
& B_1(t, x) R_{11}^{-1} B'_1(t, x) \nabla \mathbf{\Lambda}'_{33L}(x) \mathbf{w}_{33L}(t), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{33L}(x), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{12L}(t) \nabla \mathbf{\Lambda}_{12L} + \gamma_{22} \mathbf{w}'_{22L}(t) \nabla \mathbf{\Lambda}_{22L} + \gamma_{32} \mathbf{w}'_{32L}(t) \nabla \mathbf{\Lambda}_{32L}) \\
& B_2(t, x) R_{22}^{-1} B'_2(t, x) \nabla \mathbf{\Lambda}'_{33L}(x) \mathbf{w}_{33L}(t), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega} \\
& + \frac{1}{2} \langle \mathbf{\Lambda}_{33L}(x), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega}^{-1} \langle (\mathbf{w}'_{13L}(t) \nabla \mathbf{\Lambda}_{13L} + \gamma_{23} \mathbf{w}'_{23L}(t) \nabla \mathbf{\Lambda}_{23L} + \gamma_{33} \mathbf{w}'_{33L}(t) \nabla \mathbf{\Lambda}_{33L}) \\
& B_3(t, x) R_{33}^{-1} B'_3(t, x) \nabla \mathbf{\Lambda}'_{33L}(x) \mathbf{w}_{33L}(t), \mathbf{\Lambda}_{33L}(x) \rangle_{\Omega}
\end{aligned} \tag{B.25}$$